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RECTANGULAR AXIOMS, PERFECT SET PROPERTIES AND DECOMPOSITION[1]

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A b s t r a c t. We consider three combinatorial topics appearing in Gödel's manuscript Some considerations leading to the probable conclusion that the true power of the continuum is $\aleph_2$. *These statements concern rectangular functions, perfect set properties, and covering properties of sets of reals. We consider these statements in light of more recent work on the set theory of the reals.*

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1. *Introduction*

In 1970, Kurt Gödel circulated a manuscript in which he presented four axioms with the aim of bounding the size of the continuum. The history of

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this manuscript and the argument it contains have been discussed by Moore [13] and Solovay [14]. The work in this paper began by trying to understand these axioms and the corresponding argument. We isolated three statements which appear implicitly in his manuscript, and found that taken together these statements do indeed put a bound on the continuum. These statements concern dominating sequences for functions of the form $f : \kappa^+ \to \kappa$, perfect set axioms, and decompositions of sets of reals. In the forms we consider here, these areas are still not well understood. A special role is played by the $G_{\aleph_1}$ sets, those sets which can be represented as an intersection of $\aleph_1$ many open sets. In particular, perhaps the most quotable result presented here is the fact that the perfect set property for intersections of κ many open sets is equivalent to the statement that $\mathfrak{d} > \kappa$ for any cardinal κ (Theorem 5.6). Considerable attention is also given to perfect set properties that hold in the model obtained after adding ω_2 many Sacks reals to a model of CH (see Theorem 5.11).

As for the idea of settling the continuum from reasonable hypotheses, in the thirty years since Gödel produced his argument several axioms have been studied which do imply $\mathfrak{c} \leq \aleph_2$. These include the Proper Forcing Axiom [8], the Mapping Reflection Principle [24], Stationary Reflection at ω_2 [8], Rado's Conjecture [29], Martin's Maximum [15], ψ_{AC} [33], Bounded Martin's Maximum [31] and others. These statements are not independent of one another, but the point is that several different proofs correspond to them. None of them is in the spirit of Gödel's approach, however. It remains to be seen whether Gödel's original idea, decomposing the reals into small, simple sets, can give us similar evidence as to the size of the continuum.

2. *Notation*

Except where noted, the reals are considered to be the set 2^ω. As usual, $\mathbb{R}^+$ and $\mathbb{Q}^+$ are the sets of positive reals and rationals, respectively. Lebesgue measure is denoted by μ.

Modifying notation in [28], we let $g(\kappa, \lambda)$ be the least η such that there is a family of functions $F \subset \lambda^\kappa$ such that every such function is everywhere dominated by some member of F, and such that $|\{f \restriction \gamma \mid \gamma < \kappa \wedge f \in F\}| = \eta$.

For κ a cardinal and $\Gamma \subset \mathcal{P}(\mathbb{R})$, we let $\mathrm{PSP}(\kappa, \Gamma)$ denote the statement that every member of Γ of cardinality κ or greater contains a perfect set. A set of reals is in G_κ if it is the intersection of κ many open sets, and F_κ if it is the union of κ many closed sets.

For $\Gamma \subset \mathcal{P}(\mathbb{R})$, $Cov(\Gamma)$ is the least κ such that there exists a subset of Γ of cardinality κ whose union is all the reals. $Non(\Gamma)$ is the least κ such that there exists a set of reals not in Γ of cardinality κ. We let $\mathcal{N}$ denote the collection of subsets of the reals of measure zero, $\mathcal{M}$ the meager sets, and $\mathcal{SN}$ the sets of strong measure zero.

The cardinal invariant $\mathfrak{d}$ is the cardinality of the smallest set of functions from ω to ω such that every such function is dominated mod finite by a member of the set. The bounding number $\mathfrak{b}$ is the cardinality of the smallest set of functions from ω to ω such that no such function dominates mod finite every member of the set. The cardinality of the continuum is denoted by $\mathfrak{c}$.

Given a function $g : \omega \to \mathbb{R}^+$, a *g-set* is a set of the form $\bigcap_{i<\omega} \bigcup_{j>i} O_i$, where each O_i is an interval of width $g(i)$. We say that a set $A \subset \mathbb{R}$ can be *g-covered* or is *g-coverable* if A is contained in a g-set. For sets $A \subset B$, g *separates* A from B if A can be g-covered and B cannot. This induces the partial ordering $\lhd$ on sets of reals $A \subset B$ defined by letting $A \lhd B$ if there is a function separating A from B.

3. *A version of Gödel's argument*

Gödel began his paper by presenting the statements G1-G4 below. Axiom G4 says that there are no (ω_1, ω_1)-gaps in the scale from G3. Hausdorff showed that the existence of such a scale implies that $2^\omega = 2^{\omega_1}$. It is still not known, however, whether the existence of such a scale is consistent with ZFC. Martin and Solovay showed [14] that that G1-G3 together do not put a bound in the size of the continuum.

- G1. There exists a scale of functions $\omega_n \to \omega_n$ of type ω_{n+1} majorizing by end pieces every such function. It follows that there exists a set M of power $\aleph_{n+1}$ majorizing everywhere every such function.
- G2. The total number of initial segments of all the functions in this scale and in M is $\aleph_n$.
- G3. There exists a complete scale in $\mathbb{R}^\omega$ such that all increasing or decreasing sequences in this scale have cofinality at most ω_1.
- G4. The Hausdorff continuity axiom for this scale.

We work with the following axioms, in addition to ZFC. Each of these axioms, or something stronger, appeared explicitly or implicitly in Gödel's argument. In particular, Gödel claimed, incorrectly (see Sections 4 and 5), to derive our Axioms 1 and 2 from G1 and G2. Axiom 3 below plays the role of G3.

Axiom 1. $g(\omega_2, \omega_1) = \aleph_2$.

Axiom 2. $\mathrm{PSP}(\aleph_3, G_{\aleph_1})$.

Axiom 3. There exists a set H of functions from ω to $\mathbb{R}^+$ such that the following hold:

(a) All sequences from H which are increasing or decreasing in the mod-finite domination ordering have cofinality ω_1 or less.

(b) For any set $A \subset \mathbb{R}$ not of strong measure zero there exists a sequence
$$\langle (B_\alpha, g_\alpha) \in \mathcal{P}(\mathbb{R}) \times H : \alpha < \omega_1 \rangle$$
such that the B_α's are nondecreasing under inclusion with union containing A, each B_α is a g_α-set, and each g_α is mod-finite less than each $f \in H$ for which A is f-coverable.

Theorem 3.1 below is proved by Gödel's argument. The main line of the argument is showing that Axioms 1 and 3 together imply that the reals are the union of $\aleph_2$ many $G_{\aleph_1}$ sets of strong measure zero. By Axiom 2 these sets must each have cardinality less than $\aleph_3$.

Theorem 3.1. *Axioms 1-3 together imply that* $\mathfrak{c} \leq \aleph_2$.

P r o o f. Let $F \subset \omega_1^{\omega_2}$ be as given by Axiom 1, and let
$$\mathcal{F} = \{f{\restriction}\gamma : f \in F \wedge \gamma < \omega_2\}.$$
Fix a wellordering $E : \omega_2 \to \mathcal{F}$ such that for all $\sigma \subset \tau \in \mathcal{F}$, $E^{-1}(\sigma) \leq E^{-1}(\tau)$.

Construct a matrix
$$\langle (A_{\alpha,\beta}, g_{\alpha,\beta}) \in \mathcal{P}(\mathbb{R}) \times H : \alpha < \omega_2, \beta < \omega_1 \rangle$$
and a sequence
$$\langle B_\alpha \subset \mathbb{R} : \alpha < \omega_2 \rangle$$
of $G_{\aleph_1}$ sets with the following properties.

- Each $A_{\alpha,\beta}$ is a $g_{\alpha,\beta}$-set.

- $B_0 = \mathbb{R}$, and for $\alpha \in \omega_2 \setminus \{0\}$,

$$B_\alpha = \bigcap\{A_{E^{-1}(E(\alpha)\restriction\gamma),E(\alpha)(\gamma)} : \gamma \in dom(E(\alpha))\}.$$

- For all $\alpha < \omega_2$, $\langle A_{\alpha,\beta} : \beta < \omega_1\rangle$ is nondecreasing in the subset order and has union containing B_α.

- For all $\alpha < \omega_2$, if B_α is not of strong measure zero, then for each $\beta < \omega_1$ and $\gamma \in dom(E(\alpha))$, $g_{\alpha,\beta}$ is mod-finite less than $g_{E^{-1}(E(\alpha)\restriction\gamma),E(\alpha)(\gamma)}$.

The construction of the matrix is straightforward, using Axiom 3 to define each column by recursion.

Given such a matrix, $\mathfrak{c} \leq \aleph_2$ as follows. For each $f \in F$, consider the sequence $\langle B_{E^{-1}(f\restriction\gamma)} : \gamma < \omega_2\rangle$. The sets in this sequence must eventually be of strong measure zero, since otherwise one gets an ω_2-sequence of members of H which is decreasing in the mod-finite domination ordering. But by our perfect set property for $G_{\aleph_1}$ sets, this strong measure zero set cannot be of cardinality greater than $\aleph_2$, since perfect sets cannot be of strong measure zero. Lastly, each real defines a function from ω_2 to ω_1 by where it first appears in each column (letting the value of the function at α be 0 if $x \notin B_\alpha$). This function is everywhere dominated by some $f \in F$. Since f everywhere dominates the function determined by x, x is in all the B_α's corresponding to f, and so x is in the strong measure zero set corresponding to f restricted to some $\gamma < \omega_2$. Thus $\mathbb{R}$ is the union of ω_2-many sets of cardinality less than or equal to $\aleph_2$, and so $\mathfrak{c} \leq \aleph_2$. □

Theorem 3.1 shows that a substantial part of Gödel's argument is correct. The rest of the paper analyzes Axioms 1-3.

4. *Axiom 1 and Rectangles*

In this section we consider variations of Axiom 1. We have not resolved whether some weakening of Axiom 1 is sufficient for putting a bound on the continuum, or even whether Axiom 1 can be removed altogether.

4.1 Question. Do axioms 2 and 3 together imply $\mathfrak{c} \leq \aleph_2$?

4.2 Remark. A diagonal argument shows that if F is a set of functions from ω_2 to ω_1 dominating every such function everywhere, then $|F| \geq \aleph_3$.

The cardinal invariant $\mathfrak{d}_1$ is the natural generalization of $\mathfrak{d}$ to ω_1, that is, the least κ such that there exists a set of functions from ω_1 to ω_1 of cardinality κ dominating every such function mod countable. Likewise, $\mathfrak{b}_1$ is the least κ such that there exists a set of functions from ω_1 to ω_1 of cardinality κ such that no such function dominates every member of the set mod countable. In the definitions of $\mathfrak{d}$ and $\mathfrak{d}_1$, 'mod finite' and 'mod countable' can be replaced by 'everywhere.' This isn't so for $\mathfrak{b}$ and $\mathfrak{b}_1$. The statement $\mathfrak{d}_1 = \aleph_2$ is a natural weakening of Axiom 1. We shall see that it is properly weaker.

4.3 Remark. Another weakening of Axiom 1 results from letting F be an eventually dominating scale. The existence of such an F with just $\aleph_2$ many initial segments is easily seen to imply Axiom 1, however, since it implies $\mathfrak{d}_1 = \aleph_2$, and a witness for Axiom 1 can be constructed replacing the initial segments of the functions in F with the rearranged members of the witness for $\mathfrak{d}_1 = \aleph_2$.

The following proof appears in [28] with a slightly different presentation. It is essentially the same proof as Gödel's for putting a bound on the continuum; one could make the claim that this is the natural theorem for his argument.

Theorem 4.4. *([28]) If $2^{<\kappa} < cof(\lambda)$ and $g(\lambda, cof(\kappa)) = \lambda$, then $2^\kappa \leq \lambda$.*

P r o o f. Let 2^κ have the initial segment topology, and let F be as given by the fact that $g(\lambda, cof(\kappa)) = \lambda$. Let $\mathcal{F} = \{f{\restriction}\beta : f \in F \wedge \beta < \lambda\}$ and fix a wellordering $E : \lambda \to \mathcal{F}$ such that for all $\sigma \subset \tau \in \mathcal{F}$, $E^{-1}(\sigma) \leq E^{-1}(\tau)$.

Construct a matrix

$$\langle A_{\alpha,\beta} \subset 2^\kappa : \alpha < \lambda, \beta < cof(\kappa)\rangle$$

of closed sets and a sequence

$$\langle B_\alpha \subset 2^\kappa : \alpha < \lambda\rangle$$

of closed sets with the following properties.

1. $B_0 = 2^\kappa$, and for $\alpha \in \lambda \setminus \{0\}$,

$$B_\alpha = \bigcap\{A_{E^{-1}(E(\alpha){\restriction}\eta), E(\alpha)(\eta)} : \eta \in dom(E(\alpha))\}.$$

2. For all $\alpha < \lambda, \beta < \beta' < cof(\kappa)$, $A_{\alpha,\beta} \subset A_{\alpha,\beta'}$.

3. For all $\alpha < \lambda$, $\bigcup\{A_{\alpha,\beta} : \beta < cof(\kappa)\} = B_\alpha$.

4. For all $\alpha < \lambda$, if B_α has cardinality λ or greater, then for each $\beta < \kappa$ $A_{\alpha,\beta}$ is a proper subset of B_α.

The construction is straightforward, using the fact that intersections of closed sets are closed, and that since $2^{<\kappa} < cof(\lambda)$, closed sets in 2^κ of cardinality λ or more can be written as the union of an increasing $cof(\kappa)$-sequence of closed sets, since there must be a point which is not $<\lambda$-isolated. Given such a matrix, $2^\kappa \leq \lambda$ as follows. For each $f \in F$, consider the sequence $\langle B_{E^{-1}(f\restriction\eta)} : \eta < \lambda\rangle$. The sets in this sequence must eventually be of cardinality less than λ, since 2^κ has a basis with $2^{<\kappa} < cof(\lambda)$ many members. Lastly, each element x of 2^κ defines a function from λ to $cof(\kappa)$ by where it first appears in each column (letting the value of the function at α be 0 if $x \notin B_\alpha$). This function is everywhere dominated by some $f \in F$. Since f everywhere dominates the function determined by x, x is in all the B_α's corresponding to f, and so x is in the set of cardinality less than or equal to λ corresponding to f restricted to some $\gamma < \lambda$. Thus 2^κ is the union of λ-many sets of cardinality less than or equal to λ, and so $2^\kappa \leq \lambda$. □

Corollary 4.5. $\forall\kappa < \gamma(g(\kappa^+, cof(\kappa)) = \kappa^+)$ *implies* $\forall\kappa < \gamma(2^\kappa = \kappa^+)$. In particular, CH + Axiom 1 implies $2^{\omega_1} = \omega_2$.

Theorem 4.6. *If CH +* $2^{\omega_1} = \omega_3$ *holds, then there is a forcing extension in which Axiom 1 fails, but CH and* $\mathfrak{d}_1 = \aleph_2$ *hold.*

4.7 Remark Similar statements are true on other cardinals. For instance, one can force $\mathfrak{d} = \aleph_1$ + "if $\mathcal{F} \subset \omega_1^\omega$ is such that for every $g \in \omega_1^\omega$ there exists $f \in \mathcal{F}$ dominating g everywhere then for some $\alpha < \omega_1$, $|\{f\restriction\alpha : f \in \mathcal{F}\}| \geq \aleph_2$", since the second statement follows from the failure of CH.

Theorem 4.6 follows from Lemma 4.8 below. Let $\mathbb{D}^{\omega_1}$ be Hechler forcing on ω_1. Conditions are of the form (s, f), where $f \in \omega_1^{\omega_1}$, $s \in \omega_1^{<\omega_1}$ and $s \subset f$. $(s, f) \leq (t, g)$ iff $t \subset s$ and for all α, $f(\alpha) \geq g(\alpha)$. This forcing is well known and has been used for instance in [11].

Assuming CH, which will be true in the ground model in the proof of Theorem 4.6, $\mathbb{D}^{\omega_1}$ is ω_2-c.c. and σ-closed, so it preserves cardinals. Let $\mathbb{D}^{\omega_1}_{\omega_2}$ be the countable support iteration of $\mathbb{D}^{\omega_1}$ of length ω_2. The following are standard facts about $\mathbb{D}^{\omega_1}_{\omega_2}$.

Lemma 4.8. *(CH)*

1. $\mathbb{D}^{\omega_1}_{\omega_2}$ *is* σ*-closed.*

2. $\mathbb{D}^{\omega_1}_{\omega_2}$ *is* ω_2*-c.c.*

3. $\mathbb{D}^{\omega_1}_{\omega_2}$ *forces* $\mathfrak{d}_1 = \aleph_2$.

P r o o f. Part 1 of the lemma is well known, and part 3 is trivial. We sketch a proof of part 2. Let $p \in \mathbb{D}^{\omega_1}_{\omega_2}$. For $\alpha \in supp(p)$, $p(\alpha) = (\dot{s}^p_\alpha, \dot{f}^p_\alpha)$. First, we may assume that the $\dot{s}^p_\alpha$ are not names but partial functions s^p_α in the ground model. To see this, given p, construct conditions p_n, finite sets A_n and partial functions s^n_α such that

- $A_n \subset A_{n+1}$, $A_n \subset supp(p_n)$, $\cup_{n<\omega} A_n = \cup_{n<\omega} supp(p_n)$,
- $\forall \alpha \in A_n$, $p_n \restriction \alpha \Vdash \dot{s}^{p_n}_\alpha = s^n_\alpha$,
- $p_{n+1} \leq p_n \leq p$.

This is possible because $\mathbb{D}^{\omega_1}_{\omega_2}$ is σ-closed. For $\alpha \in \cup_{n<\omega} A_n$, let

$$s^\omega_\alpha = \cup\{s^n_\alpha \mid \alpha \in A_n\}$$

and define a condition p_ω with support $\cup_{n<\omega} A_n$ such that

- for all $\alpha \in \cup_{n<\omega} A_n$ $p_\omega \restriction \alpha \Vdash$ "$\dot{s}^{p_\omega}_\alpha = s^\omega_\alpha$ and $\dot{f}^{p_\omega}_\alpha \geq \dot{f}^{p_n}_\alpha$ everywhere."

Then clearly $p_\omega \leq p_n < p$ for all n. Call such p decided.

Now the rest of the proof is standard. Given an ω_2 sequence of decided conditions, we can find a subset of size ω_2 for which the supports form a Δ-system with root r. By CH, there are just ω_1 many decided conditions with support r, and so our original sequence cannot have been an antichain. □

5. *Axiom 2 and Perfect Set Axioms*

As shown in [17], [12] and Theorem 4.6, $g(\omega_1, \omega) = \omega_1$ implies CH, and so does not follow from axioms G1 and G2. At the time he produced his manuscript, Gödel had not realized this implication, and the use of such a scale in his proof is to derive that $G_{\aleph_1} = F_{\aleph_1}$.[2] Since an $F_{\aleph_1}$ set of

[2]Briefly, the argument is that given a $G_{\aleph_1}$-representation we can write each open set as an increasing ω-sequence of closed sets, and associate each function from ω_1 to ω to the corresponding intersection of closed sets. Each point in the intersection defines a function from ω_1 to ω, and each intersection is eventually constant. The $F_{\aleph_1}$ set then is the set of intersections which are defined by initial segments of the dominating scale and contained in the $G_{\aleph_1}$ set.

cardinality $\aleph_2$ must contain an uncountable closed set, $G_{\aleph_1} = F_{\aleph_1}$ implies PSP$(\aleph_2, G_{\aleph_1})$.

The $\aleph_3$ in Axiom 2 appears to be arbitrarily chosen to link the reals to $\aleph_2$. By contrast, Axiom 3 refers only to ω_1, and Axiom 1 does not mention the reals at all. As far as we know, replacing $\aleph_3$ with κ^+ in Axiom 2 gives only κ as an upper bound on the continuum by the proof of Theorem 3.1. Further, Axioms 1-3 together imply that Axiom 2 is vacuously true. The weakest nonvacuous version of the axiom is PSP$(\mathfrak{c}, G_{\aleph_1})$. Using this instead of Axiom 2, the proof of Theorem 3.1 gives that the cofinality of the continuum is ω_2 or less. We haven't resolved whether Axioms 1 and 3 plus PSP$(\mathfrak{c}, G_{\aleph_1})$ is consistent with the continuum being $\aleph_{\omega_1}$ or $\aleph_{\omega_2}$. Replacing Axiom 2 with PSP$(cof(\mathfrak{c}), G_{\aleph_1})$ or $\exists\kappa < \mathfrak{c}(\text{PSP}(\kappa, G_{\aleph_1}))$, the argument that $\mathfrak{c} \leq \aleph_2$ still goes through.

Axiom 2 fails after adding $\aleph_3$ many random reals.

Theorem 5.1. *PSP$(\aleph_1 + \kappa, G_{\aleph_1})$ is false after adding κ random reals to a model of CH.*

We use a lemma from the proof of the Cichoń-Mokobodzki Theorem [3] to prove Theorem 5.1. This theorem says that adding random reals does not add a perfect set of random reals. During the proof the following is established.

Lemma 5.2. *Let $A \subset 2^\omega \times 2^\omega$ be a Borel set such that $\{x \in 2^\omega \mid A_x$ is perfect $\}$ has measure 1. Then there exists an F_σ null set B such that*

$$\{x \in 2^\omega \mid \exists y \in B\ \langle x, y\rangle \in A\}$$

has measure 1.

Here $A_x = \{y \mid \langle x, y\rangle \in A\}$. Except for B being F_σ, this is Lemma 3.2.20 of [3]. However, it is clear from the proof of Lemma 3.2.19 there that B may be taken to be F_σ.

P r o o f of **Theorem 5.1**: The interesting case is when $\kappa \geq \omega_2$. Let C be the intersection of all G_δ measure one sets coded in the ground model. By CH, C is a $G_{\aleph_1}$ set. Also $|C| = \mathfrak{c}$ in the extension since each random real belongs to C. Assume that P is a perfect set in the extension. Then P is added by adjoining one random real r, and in fact there is a Borel set $A \subset 2^\omega \times 2^\omega$ in the ground model such that $P = A_r$. Note that $r \in \{x \mid A_x$ is perfect$\}$ and without loss of generality we may assume that $\{x \mid$

A_x is perfect$\}$ has measure 1. By Lemma 5.2, there is an F_σ null set B in the ground model such that $\{x \mid \exists y \in B\ \langle x, y\rangle \in A\}$ has measure 1. Then $r \in \{x \mid \exists y \in B\ \langle x, y\rangle \in A\}$, i.e., $P \cap B = A_r \cap B \neq \emptyset$. Thus $P \not\subset C$. $\square$

Instead of CH, the proof of Theorem 5.1 requires just that $Cof(\mathcal{M}) = \aleph_1$, where $Cof(\mathcal{M})$ is the cardinality of the smallest basis for the meager ideal. This is so because $Cof(\mathcal{M}) = Cof(\mathcal{E})$ in ZFC ([3], Theorem 2.6.17), where $\mathcal{E}$ is the σ-ideal generated by the closed null sets. More generally, the proof gives that PSP($\mathfrak{c}^{V[G]}, G_{Cof(\mathcal{M})^V}$) fails after adding one or more random reals. Some assumption is necessary, however, since if $\mathfrak{d} > \aleph_1$ in the ground model then the forcing will preserve this, and so by Theorem 5.6 PSP($\aleph_1, G_{\aleph_1}$) will hold.

5.1. *Perfect set axioms and the dominating number*

The following was pointed out to us by Hugh Woodin.

Proposition 5.3. *There exists an ω_1-sequence of F_σ sets whose intersection has size ω_1.*

Proof. Fix a bijection $b: \omega \to \omega \times \omega$, and for each $i < \omega$ let $b_i: \omega^\omega \to \omega^\omega$ be such that $b_i(x)(j) = x(b^{-1}(i, j))$. Let $\langle a_\alpha: \omega \to \alpha \mid \alpha < \omega_1\rangle$ be a set of bijections. Let $\langle x_\alpha : \alpha < \omega_1\rangle$ be such that for each infinite $\alpha < \omega_1$,

$$\{x_\beta : \beta < \alpha\} = \{b_i(x_\alpha) : i < \omega\}.$$

Now for $\alpha < \omega_1$ and $i < \omega$ let $A_{\alpha,i}$ be the set of $x \in \omega^\omega$ such that either $x \in \{x_\beta : \beta < \alpha\}$ or there exists $j < \omega$ such that $b_j(x) = x_{a_\alpha(i)}$. Note that each $A_{\alpha,i}$ is an F_σ set, and that $\{x_\alpha : \alpha < \omega_1\} \subset \bigcap\{A_{\alpha,i} : \alpha < \omega_1, i < \omega\}$.

Now say that $x \in \bigcap\{A_{\alpha,i} : \alpha < \omega_1, i < \omega\}$. We need to see that x is equal to some x_α. Since $\{b_i(x) : i < \omega\}$ is countable, there is an α such that $\{x_\beta : \beta < \alpha\} \not\subset \{b_i(x) : i < \omega\}$. Since $x \in \bigcap_{i<\omega} A_{\alpha,i}$, then, x must be equal to some x_β, $\beta < \alpha$. $\square$

Corollary 5.4. *There exists an ω_1-sequence of F_σ sets whose intersection does not contain a perfect set.*

Another example is given in the first section of [32]. A sequence of functions $\langle \rho_{0\beta} : \beta < \omega_1\rangle$ is presented, each $\rho_{0\beta}$ being an increasing function from β to $\mathbb{Q} \cap (0, 1)$. This sequence has the property that

$$T(\rho_0) = \{\rho_{0\beta}{\restriction}\alpha \mid \alpha \le \beta < \omega_1\}$$

under the extension ordering is a special Aronszajn tree. Each function $\rho_{0\beta}$ induces the function $x_\beta \in 2^{(\mathbb{Q}\cap(0,1))}$ given by the range of $\rho_{0,\beta}$. Then the set $\{x_\beta : \beta < \omega_1\}$ is the intersection of ω_1 many F_σ sets in $2^{(\mathbb{Q}\cap(0,1))}$ as follows. For each $t \in T(\rho_0)$, let

$$P_t = \{x \subset \mathbb{Q} \cap (0,1) \mid x \cap sup(range(t)) = range(t)\}.$$

Each P_t is a perfect subset of $2^{(\mathbb{Q}\cap(0,1))}$. For each $\beta < \omega_1$, let

$$G_\beta = \bigcap\{\bar{P}_t \mid lev_T(t) = \beta\} \setminus \{x_\alpha : \alpha < \beta\},$$

where $\bar{P}_t$ is the complement of P_t. Then $\langle G_\alpha : \alpha < \omega_1\rangle$ is an increasing sequence of G_δ subsets of $2^{(\mathbb{Q}\cap(0,1))}$, and since there are no cofinal paths through $T(\rho_0)$,

$$\bigcup_{\alpha<\omega_1} G_\alpha = 2^{(\mathbb{Q}\cap(0,1))} \setminus \{x_\beta : \beta < \omega_1\}.$$

Lastly, it is shown in [32] that $\{x_\beta : \beta < \omega_1\}$ is of universal measure zero, and so cannot contain a perfect set.

Lemma 5.5. *Every analytic set of reals can be written as the union of a family of compact sets of size at most $\mathfrak{d}$.*

P r o o f. Let $f : \omega^\omega \to \mathbb{R}$ be continuous with $range(f) = A$. For each $x \in \omega^\omega$, let $C_x = \{y \in \omega^\omega \mid \forall n < \omega\ y(n) \leq x(n)\}$. Then each C_x is compact, and so each $D_x = f[C_x]$ is compact as well. If $\mathcal{D} \subset \omega^\omega$ is a dominating family of size $\mathfrak{d}$, then $\{D_x : x \in \mathcal{D}\}$ is a family of $\mathfrak{d}$-many compact sets whose union is A. □

The following theorem subsumes the well known fact that $Cov(\mathcal{M}) \geq \aleph_2$ implies PSP$(\aleph_1, G_{\aleph_1})$.

Theorem 5.6. *For any cardinal κ, $\mathfrak{d} > \kappa \Leftrightarrow PSP(\aleph_1, G_\kappa)$.*

P r o o f. For the reverse direction, by Proposition 5.3, all we need to see is that every F_σ set is in $G_\mathfrak{d}$. By Lemma 5.5, every co-analytic set is in $G_\mathfrak{d}$.

For the other direction, we work in 2^ω. Define $T \subset 2^{<\omega}$ by

$$s \in T \Leftrightarrow [s] \cap \bigcap_{\alpha<\kappa} U_\alpha \neq \emptyset,$$

where $\{U_\alpha : \alpha < \kappa\}$ is a sequence of open sets such that $\bigcap_{\alpha<\kappa} U_\alpha$ is uncountable, and $[s]$ indicates the set of extensions of s. Note that each U_α is open dense in T, i.e., for all $\alpha < \kappa$, $s \in T$ there is a $t \in T$ such that $s \subset t$ and $[t] \subset U_\alpha$. Also, $\bigcap_{\alpha<\kappa} U_\alpha \subset [T]$ and $\overline{\bigcap_{\alpha<\kappa} U_\alpha} = [T]$. Since $\bigcap_{\alpha<\kappa} U_\alpha$ is uncountable, $[T]$ contains a perfect set, so we can assume that T is a perfect tree.

Choose recursively

$$\{x_\sigma \in \bigcap_{\alpha<\kappa} U_\alpha : \sigma \in \omega^{<\omega}\}, \{k_{\sigma,n} \in \omega : \sigma \in \omega^{<\omega}, n \in \omega\}$$

such that

- each $k_{\sigma,n}$ is the largest integer k such that $x_{\sigma^\frown\langle n\rangle}\restriction k = x_\sigma\restriction k$,
- for each σ the $k_{\sigma,n}$'s form an increasing sequence,
- for all σ, n, $k_{\sigma^\frown\langle n\rangle,0} > k_{\sigma,n}$.

Then the sequences $x_{\sigma^\frown\langle n\rangle}\restriction(k_{\sigma,n}+1)$ form a perfect tree. We now use $\mathfrak{d} > \kappa$ to find a perfect subtree all of whose branches are in $\bigcap_{\alpha<\kappa} U_\alpha$.

For each $\alpha < \kappa$, define $\phi_\alpha : \omega^{<\omega} \to \omega$ by

$$\phi_\alpha(\sigma) = min\{k \mid [x_\sigma\restriction k] \subset U_\alpha\}.$$

Now let M be a model of set theory of size κ containing everything so far, and assume $f \in \omega^\omega$ is unbounded over M, and that for all n, $f(n+1) \geq f(n)+2$.

For $\alpha < \kappa$, let $g_\alpha(n) = max\{\phi_\alpha(\sigma) \mid |\sigma| = n$ and $\forall i < n\ \sigma(i) \leq f(i)+1\}$. We claim that for all α there are infinitely many n such that $f(n) > g_\alpha(n)$. To see this, assume that for all $n \geq n_0$, $f(n) \leq g_\alpha(n)$. Define recursively $\bar{g}_\alpha \in \omega^\omega$ by

- $\bar{g}_\alpha\restriction n_0 = f\restriction n_0$,
- $\bar{g}_\alpha(n) = max\{\phi_\alpha(\sigma) \mid |\sigma| = n$ and $\forall i < n\ \sigma(i) \leq \bar{g}_\alpha(i)+1\}$.

Note that $\bar{g}_\alpha(n_0) = g_\alpha(n_0)$. Then $\bar{g}_\alpha \in M$, so there is a minimal $n > n_0$ with $f(n) > \bar{g}_\alpha(n)$. Since $f(i) \leq \bar{g}_\alpha(i)$ for all $i < n$, we have $g_\alpha(n) \leq \bar{g}_\alpha(n)$, a contradiction. This proves the claim.

Next, define recursively

$$\{s_t : t \in 2^{<\omega}\} \subset T,\ \{\sigma_t \in \omega^{<\omega} : t \in 2^{<\omega}\}$$

such that

1. $\sigma_{\langle\rangle} = \langle\rangle$, $s_{\langle\rangle} = \langle\rangle$,

2. $\sigma_{t^\frown\langle 0\rangle} = \sigma_t^\frown\langle f(|t|)\rangle$, $\sigma_{t^\frown\langle 1\rangle} = \sigma_t^\frown\langle f(|t|)+1\rangle$,

3. $s_{t^\frown\langle 0\rangle} = x_{\sigma_{t^\frown\langle 0\rangle}}\restriction(k_{\sigma_t, f(|t|)}+1)$, $s_{t^\frown\langle 1\rangle} = x_{\sigma_{t^\frown\langle 1\rangle}}\restriction(k_{\sigma_t, f(|t|)+1}+1)$,

We have that $s_t \subset s_{t^\frown\langle i\rangle}$ and $s_{t^\frown\langle 0\rangle} \neq s_{t^\frown\langle 1\rangle}$. This means that

$$P = \{\bigcup_{i<\omega} s_{h\restriction i} : h \in 2^\omega\}$$

is a perfect subset of T. We check $P \subset \bigcap_{\alpha<\kappa} U_\alpha$ by showing that for all $i < \omega, \alpha < \kappa$, if $f(i) > g_\alpha(i)$ and $t \in 2^i$, then $[s_{t^\frown\langle 0\rangle}], [s_{t^\frown\langle 1\rangle}] \subset U_\alpha$. This follows from the fact that for all $j < i$, $\sigma_t(j) \leq f(j)+1$. By the definition of g_α, then, $f(i) > \phi_\alpha(\sigma_t)$, so we are done. □

For the following, recall that a set of reals is *dense in itself* if it has no isolated points.

Corollary 5.7. $\mathfrak{d}$ *is equal to each of the following.*

1. *The least κ such that there is an uncountable G_κ set which does not contain a perfect set.*

2. *The least κ such that there is a G_κ set which is dense in itself and does not contain a perfect set.*

P r o o f. Call the first κ_1 and the second κ_2. Theorem 5.6 says just that $\kappa_1 = \mathfrak{d}$. Since every uncountable G_κ set contains a G_κ set which is dense in itself, $\kappa_1 \geq \kappa_2$. That $\kappa_2 \geq \mathfrak{d}$ follows from a straightforward generalization of the proof of Theorem 5.6. □

5.8 Conjecture. $\mathrm{PSP}(\mathfrak{d}, G_\mathfrak{d})$ is false.

If Conjecture 5.8 is correct, then the next step is to analyze the axioms $\mathrm{PSP}(\mathfrak{d}^+, G_\mathfrak{d})$ and $\mathrm{PSP}(\mathfrak{d}^{++}, G_\mathfrak{d})$ (see [30]).

As we shall see, $\mathrm{PSP}(\aleph_3, G_{\aleph_1})$ does not follow from $\mathfrak{d}_1 = \aleph_2$ and $\mathfrak{d} = \aleph_1$. Furthermore, Axioms 1 and 3 together do not imply a bound on the continuum. Theorem 5.9 also shows that Axioms 1-3 don't imply CH.

Theorem 5.9. *Forcing to add ω_1 many Hechler reals with finite support preserves Axiom 1 and makes $\mathfrak{d} = Cov(\mathcal{SN}) = \aleph_1$, and so forces Axiom 3.*

To see that $Cov(\mathcal{SN}) = \aleph_1$ after adding ω_1-many Hechler reals, note that Hechler forcing makes the reals of the ground model f-coverable for every $f : \omega \to \mathbb{Q}^+$ in the ground model. Actually, the Hechler reals are not needed; since the iteration is by finite support, Cohen reals are added, and Cohen forcing also makes the ground model reals f-coverable for every f in the ground model. For Hechler reals, and many other kinds of reals, something stronger is true, that finite support iterations of any length with uncountable cofinality force $Cov(\mathcal{SN}) = \aleph_1$. This argument is much more difficult, see [25] or Theorem 8.4.5 of [3].

5.2. *Perfect set axioms in the iterated Sacks model*

Sacks forcing $\mathbb{S}$ is the set of perfect subtrees of $2^{<\omega}$ ordered by inclusion. We let $\mathbb{S}_\alpha$ be the α-length iteration of $\mathbb{S}$ with countable support. For $p \in \mathbb{S}_\alpha$, $supp(p)$ is the support of p.

We let $B_{\aleph_1}$ denote the sets which can be represented as an intersection of $\aleph_1$ many Borel sets.

Theorem 5.10. *(CH) If $G \subset \mathbb{S}_{\omega_2}$ is V-generic, then $V[G] \models PSP(\aleph_2, B_{\aleph_1})$.*

It is well known (see [2]) that $\mathfrak{d} = Non(\mathcal{M}) = \aleph_1$ holds in $V[G]$ as above, and so $\mathrm{PSP}(\aleph_1, G_{\aleph_1})$ fails there - see Theorem 5.6.

Given a set of reals A in $B_{\aleph_1}$ and a forcing extension $V[G]$, we let $A^{V[G]}$ be the interpretation of A in $V[G]$.

Theorem 5.11. *Let $A \subset 2^\omega$ be a $B_{\aleph_1}$ set in V. If $G \subset \mathbb{S}_{\omega_2}$ is V-generic, then either $A^{V[G]} = A$ or there is a perfect set $P \in V[G]$ such that $P \subset A^{V[G]}$.*

The idea behind Theorem 5.11 is contained in the one-step argument.

Proposition 5.12. *Let $A \subset 2^\omega$ be a $B_{\aleph_1}$ set in V. If G is $\mathbb{S}$-generic over V then either $A^{V[G]} = A$ or there is a perfect set $P \in V[G]$ such that $P \subset A^{V[G]}$.*

P r o o f. Assume that the first alternative fails. Let $\dot{f}$ be an $\mathbb{S}$-name for a member of 2^ω such that $1_{\mathbb{S}} \Vdash \dot{f} \in A^{V[\dot{G}]} \setminus A$. Given $S \in \mathbb{S}$, it is straightforward to construct $T \leq S$, a perfect set P and a homeomorphism $F : [T] \to P$ such that

$$T \Vdash \dot{f} = F(\dot{s}),$$

where $\dot{s}$ is the canonical name for the $\mathbb{S}$-generic real (see [27]). Now work in $V[G]$, assuming $T \in G$. From [27], we have that every new real is $\mathbb{S}$-generic.

Also note that for each perfect tree coded in V, there is a perfect subtree coded in $V[G]$ all of whose branches are new reals; this is easy to see and always true when new reals are added. So T contains a perfect subtree T', all of whose branches are Sacks-generic. Now $P' = F''[T']$ is a perfect subset of P and we claim that $P' \subset A^{V[G]}$. For indeed, if $x \in P'$, then $x = F(s)$ for some Sacks generic real $s \in [T']$. Let G' be the corresponding generic filter. So $x = F(s) = F(\dot{s}_{G'}) = \dot{f}_{G'}$. Since $s \in [T'] \subset [T]$, $T \in G'$ follows. As $T \Vdash \dot{f} = F(\dot{s}) \in A^{V[\dot{G}]}$, $x \in A^{V[G]}$ follows. $\square$

5.13 Remark. The pointclass $B_{\aleph_1}$ can be increased to the class of ω_1-Borel sets as defined in [33] in the statement of Proposition 5.12. We don't know if this is true for Theorems 5.10 and 5.11. Roughly, the ω_1-Borel sets are those sets of reals which have descriptions of size $\aleph_1$. The issue is that unlike for $B_{\aleph_1}$ sets, the statement that a perfect set is contained in a given ω_1-Borel set is not necessarily upwards absolute; if one real is added to a model of CH, for example, then the reals of the ground model are an $F_{\aleph_1}$ set not containing a perfect set, even though they trivially contain a perfect set in the ground model. The absoluteness of the statement that a given perfect set is contained in a given $B_{\aleph_1}$ set is key to the proofs in this section.

Ciesielski and Pawlikowski have recently [10] produced a shorter proof of Theorem 5.11, using their axiom $\mathrm{CPA}_{\mathrm{prism}}$. In that paper they also prove the following theorem, refuting a conjecture in an earlier version of this paper (and negatively answering a question whose positive answer implied the conjecture).

Theorem 5.14. *[10] If CH holds and $G \subset \mathbb{S}_{\omega_2}$ is V-generic, then $B_{\aleph_1} \neq F_{\aleph_1}$ in $V[G]$.*

It is easy to see that $B_{\aleph_1} = F_{\aleph_1}$ is equivalent to the statement $G_{\aleph_1} = F_{\aleph_1}$, which Gödel mistakenly claimed to derive from G1 and G2. That $G_{\aleph_1} = F_{\aleph_1}$ follows trivially from CH, and implies $\neg\mathrm{PSP}(\aleph_1, G_{\aleph_1})$ and $\mathrm{PSP}(\aleph_2, G_{\aleph_1})$. To see $\neg\mathrm{PSP}(\aleph_1, F_{\aleph_1})$, note that ZFC implies the existence of an $F_{\aleph_1}$ set of cardinality $\aleph_1$ which does not contain a perfect set. The statement $\mathrm{PSP}(\aleph_2, F_{\aleph_1})$ follows from the fact that uncountable closed sets contain perfect sets.

The following questions remain open for lack of models that would show consistency.

5.15 Question. Does $G_{\aleph_1} = F_{\aleph_1}$ imply CH?

5.16 Question. Does $\mathfrak{d} = \aleph_1 \wedge$ $\mathrm{PSP}(\aleph_2, G_{\aleph_1})$ imply $\mathfrak{c} \leq \aleph_2$?

P r o o f of Theorem 5.11: By Proposition 5.12, we need to consider only limit stages α where α has cofinality ω. Given a condition $p_0 \in \mathbb{S}_\alpha$ and a name $\dot{f}$ for a member of 2^ω, we shall construct:

Step 1. a condition $p \leq p_0$ and a perfect tree T such that $p \Vdash_\alpha \dot{f} \in [T]$ in a canonical way,

Step 2. a canonical name $\dot{S}$ such that $p \Vdash_\alpha$ "$\dot{S} \subset T$, $\dot{S}$ is a perfect tree, and

$$[\dot{S}] \subset V[\dot{G}_\alpha] \setminus \bigcup_{\beta<\alpha} V[\dot{G}_\beta],"$$

i.e., all branches have the same constructibility degree as $\dot{f}$.

Then given a name $\dot{g}$ and a condition $q_0 \leq p$ such that $q_0 \Vdash_\alpha \dot{g} \in [\dot{S}]$, we construct:

Step 3. a condition $q \leq q_0$ and a perfect tree U such that $q \Vdash_\alpha \dot{g} \in [U]$ in a canonical way, and a condition $r \leq p$ such that canonically $r \Vdash_\alpha \dot{f} \in [U]$, and further, whenever G_α is generic, $q \in G_\alpha$, then there is G'_α generic, $r \in G'_\alpha$ such that

$$\dot{g}_{G_\alpha} = \dot{f}_{G'_\alpha}.$$

More explicitly, there will be a canonical isomorphism $\pi : \mathbb{S}_\alpha \restriction q \to \mathbb{S}_\alpha \restriction r$ with $\pi(q) = r$ such that whenever G_α is generic, $q \in G_\alpha$, then $\dot{g}_{G_\alpha} = \dot{f}_{\pi[G_\alpha]}$.

From Step 3 we then have that p forces that all branches of $\dot{S}$ are generic in the same sense as $\dot{f}$, so whatever p forces about $\dot{f}$ will be true about all members of $[\dot{S}]$, so we will be done.

Step 1: We construct a fusion sequence $\langle p_n : n < \omega \rangle$. The given condition is p_0; p will be the result of the fusion. We also construct for each $\sigma \in 2^{<\omega}$ a sequence $S_{|\sigma|,\sigma} \in 2^{<\omega}$ and a condition $p_{|\sigma|}\langle\sigma\rangle \leq p_{|\sigma|}$ such that

- $\{S_{|\sigma|,\sigma} \mid \sigma \in 2^{<\omega}\}$ forms a perfect tree, with extension and incompatibility in accordance with the corresponding σ's,
- for each n, $\{p_n\langle\sigma\rangle \mid \sigma \in 2^n\}$ forms a maximal antichain below p_n,
- for each $\sigma \in 2^{<\omega}$, $p_{|\sigma|}\langle\sigma\rangle \Vdash S_{|\sigma|,\sigma} \subset \dot{f}$.

Then

$$T = \{s \in 2^{<\omega} \mid \exists \sigma \in 2^{<\omega}(s \subset S_{|\sigma|,\sigma})\}$$

will be the desired tree.

Fix a function $F : \omega \to \omega$ such that

(*) $F^{-1}(\{n\})$ is infinite for all n.

(**) $n < m \Rightarrow min(F^{-1}(n)) < min(F^{-1}(m))$.

Note then that $F(n) \leq n$ for all n. Our construction will also fix ordinals α_n such that $supp(p) = \{\alpha_n : n \in \omega\}$. We suppress the bookkeeping of which α_n's arise when, except for the stipulation that $\alpha_n \in supp(p_n)$ for each n. The fusion condition will be that for each α_n and each $m \geq min\{i \mid F(i) = n\}$, $1_{\mathbb{S}_{\alpha_n}}$ forces that the first $|\{i \leq m \mid F(i) = n\}|$ splitting levels of $p_m(\alpha_n)$ and $p_{m+1}(\alpha_n)$ will be the same. We will use the following sets, where $n \in \omega$ and $\gamma < \alpha$:

$$A_n^\gamma = \{i < n \mid \alpha_{F(i)} \leq \gamma\}, B_n^\gamma = \{i < n \mid \alpha_{F(i)} < \gamma\}$$

(so $A_n^\gamma \setminus B_n^\gamma = \{i < n \mid \alpha_{F(i)} = \gamma\}$). During our construction, we will produce sequences $S^\gamma_{n,\sigma\restriction A_n^\gamma} \in 2^{<\omega}$, where $n \in \omega$, $\sigma \in 2^n$ and $\gamma \in \{\alpha_{F(i)} : i < n\}$, such that

(a) for $\gamma \in \{\alpha_{F(i)} : i < n\}$ and $\sigma\colon B_n^\gamma \to 2$, there exists $(p_n\restriction\gamma)\langle\sigma\rangle \leq p_n\restriction\gamma$ such that $(p_n\restriction\gamma)\langle\sigma\rangle$ forces

- $\{S^\gamma_{n,\tau} \mid \sigma \subset \tau \wedge dom(\tau) = A_n^\gamma\} \subset p_n(\gamma)$,
- $\forall S \in p_n(\gamma)\exists\tau \in 2^{A_n^\gamma}(\sigma \subset \tau \wedge (S \subset S^\gamma_{n,\tau} \vee S^\gamma_{n,\tau} \subset S))$,

(b) fixing $n \in \omega$ and $\gamma \in \{\alpha_{F(i)} : i < n\}$, for all distinct $\sigma, \sigma' \in 2^{A_n^\gamma}$, $S^\gamma_{n,\sigma}$ and $S^\gamma_{n,\sigma'}$ are incompatible,

(c) fixing $n \in \omega$ and $\gamma \in \{\alpha_{F(i)} : i < n\}$, for all $\sigma \in 2^{A^\gamma_{n+1}}$,

$$S^\gamma_{n,\sigma\restriction A_n^\gamma} \subset S^\gamma_{n+1,\sigma}.$$

Therefore, if $(p_n\restriction\gamma)\langle\sigma\rangle \in G_\gamma$ then the $S^\gamma_{n,\tau}$ with $\sigma \subset \tau$ canonically define the $|A_n^\gamma \setminus B_n^\gamma|$-th splitting level of $p_n(\gamma)$. Note that if $\alpha_{F(n)} > \gamma$, then for all $\sigma \in 2^{n+1}$ we may choose

$$S^\gamma_{n+1,\sigma\restriction A^\gamma_{n+1}} = S^\gamma_{n,\sigma\restriction A_n^\gamma}.$$

The $(p_n\restriction\gamma)\langle\sigma\rangle$ will satisfy the following recursively defined conditions for $\gamma \in \{\alpha_{F(i)} : i < n\}$ and σ with $dom(\sigma) = B_n^\gamma$. The same construction will be used in Step 3 (without repeating the details).

- If $\gamma = min\{\alpha_{F(i)} : i < n\}$, then $B_n^\gamma = \emptyset$ and $(p_n{\restriction}\gamma)\langle\sigma\rangle = p_n{\restriction}\gamma\langle\rangle = p_n{\restriction}\gamma$.

- Assume $(p_n{\restriction}\gamma)\langle\sigma{\restriction}B_n^\gamma\rangle$ is defined for some $\gamma \in \{\alpha_{F(i)} : i < n\}$ and σ with $dom(\sigma) = A_n^\gamma$. Let $\delta = min\{\alpha_{F(i)} \mid i < n \wedge \alpha_{F(i)} > \gamma\}$. Note that $B_n^\delta = A_n^\gamma$. Then

$$(p_n{\restriction}\delta)\langle\sigma\rangle = (p_n{\restriction}\gamma)\langle\sigma{\restriction}B_n^\gamma\rangle^\frown(p_n(\gamma)_{S^\gamma_{n,\sigma}})^\frown p_n{\restriction}[\gamma+1,\delta),$$

where as usual $1_{\mathbb{S}_\alpha}\Vdash \dot{p}_s = \{\dot{t} \in \dot{p} \mid s \subset \dot{t} \vee \dot{t} \subset s\}$. This makes sense because by (b) above we have indeed that $(p_n{\restriction}\gamma)\langle\sigma{\restriction}B_n^\gamma\rangle\Vdash S^\gamma_{n,\sigma} \in p_n(\gamma)$.

Similarly, $p_n\langle\sigma\rangle = (p_n{\restriction}\gamma)\langle\sigma{\restriction}B_n^\gamma\rangle^\frown(p_n(\gamma)_{S^\gamma_{n,\sigma}})^\frown p_n{\restriction}[\gamma+1,\alpha)$, where

$$\gamma = max\{\alpha_{F(i)} : i < n\}.$$

Note that in this case $A_n^\gamma = n$. The key point is that

$$\{(p_n{\restriction}\gamma)\langle\sigma\rangle \mid \sigma \in 2^{B_n^\gamma}\}$$

forms a maximal antichain below $p_n{\restriction}\gamma$, and the same holds for $\{p_n\langle\sigma\rangle \mid \sigma \in 2^n\}$ and p_n. Further, if $\gamma < \delta$ and $\sigma \in 2^{B_n^\gamma}$ is a subfunction of $\tau \in 2^{B_n^\delta}$ (or 2^n), then $(p_n{\restriction}\delta)\langle\tau\rangle$ (or $p_n\langle\tau\rangle$) $\le (p_n{\restriction}\gamma)\langle\sigma\rangle$.

Now for the details of the construction. In the case $n = 0$, put $S_{n,\sigma} = \sigma = \langle\rangle$, and let all $(p_0{\restriction}\gamma)\langle\rangle = p_0{\restriction}\gamma$, $p_0\langle\rangle = p_0$. Then all the conditions are satisfied trivially. Given the construction for n, construct for $n+1$ as follows. Let $\delta = \alpha_{F(n)}$; since for all $i < n$ $\alpha_i \in supp(p_i)$ this is defined (not all α_i have been). Choose, for each $\sigma \in 2^{B^\delta_{n+1}}$, $p^{**}_\sigma \le (p_n{\restriction}\delta)\langle\sigma\rangle$ and $S^\delta_{n+1,\tau}$ for $\tau \supset \sigma, \tau \in 2^{A^\delta_{n+1}}$ such that $p^{**}_\sigma\Vdash S^\delta_{n+1,\tau} \in p_n(\delta)$, and such that

- for distinct τ, the $S^\delta_{n+1,\tau}$ are incompatible,

- in case $\delta \in \{\alpha_{F(i)} : i < n\}$, $p^{**}_\sigma\Vdash S^\delta_{n,\tau{\restriction}A_n^\delta} \subset S^\delta_{n+1,\tau}$.

Now recursively for each

$$\gamma \in \{\alpha_{F(i)} > \delta \mid i < n\},$$

we produce

$$S^\gamma_{n+1,\sigma}$$

for $\sigma \in 2^{A^\gamma_{n+1}}$, as follows.

Say that we have chosen for γ, and want to choose for

$$\gamma' = min(\{\alpha_{F(i)} : i < n\} \setminus \gamma + 1).$$

For $j \in \{0,1\}$ fix $G^j_{\gamma'}$ generic such that $(p_n \restriction \gamma')\langle \vartheta ^\frown \langle j \rangle\rangle \in G^j_{\gamma'}$, where $dom(\vartheta) = B^{\gamma'}_n = A^\gamma_n$, $\tau \subset \vartheta$. Then look at $S^{\gamma'}_{n,\chi}$, $\chi \supset \vartheta$, $dom(\chi) = A^{\gamma'}_n$. These form a finite maximal antichain in $p_n(\gamma')$. So, in the models $V[G^0_{\gamma'}]$ and $V[G^1_{\gamma'}]$ we may find

$$S^{\gamma'}_{n+1,\chi^\frown\langle 0\rangle},\ S^{\gamma'}_{n+1,\chi^\frown\langle 1\rangle}$$

which are incompatible such that $S^{\gamma'}_{n,\chi} \subset S^{\gamma'}_{n+1,\chi^\frown\langle j\rangle} \in p_n(\gamma')$ holds in $V[G^j_{\gamma'}]$.

Having chosen the

$$S^\gamma_{n+1,\vartheta\restriction A^\gamma_{n+1}}$$

we now choose p_{n+1}. This choice induces our choices of the $(p_{n+1}\restriction\gamma)\langle\sigma\rangle$ and $p_{n+1}\langle\sigma\rangle$. For $\sigma \in 2^{<\omega}$, and $q \in \mathbb{S}_\alpha$, let $\phi(\sigma, q)$ be the statement that for each $\gamma \in \{\alpha_{F(i)} : i < |\sigma|\}$, $q\restriction\gamma$ is consistent with

$$S^\gamma_{|\sigma|,\sigma\restriction A^\gamma_{|\sigma|}}$$

being an initial segment of the stem of $q(\gamma)$. The fusion requirement is maintained by requiring that $\phi(\sigma, p_{n+1})$ holds for each $\sigma \in 2^{n+1}$. We have chosen the

$$S^\gamma_{n+1,\sigma\restriction A^\gamma_{n+1}}$$

so that $\phi(\sigma, p_n)$ always holds. Further, these statements are mutually incompatible. Let $q_\sigma \leq p_n$ force that

$$S^\gamma_{n+1,\sigma\restriction A^\gamma_{n+1}}$$

is an initial segment of the stem of $p_n(\gamma)$, and furthermore extend the q_σ so that they decide incompatible initial segments $S_{n+1,\sigma}$ of $\dot{f}$. Then by the same recursive procedure as for choosing the

$$S^\gamma_{n+1,\sigma\restriction A^\gamma_{n+1}}$$

we can choose p_{n+1} so that the q_σ $(= p_{n+1}\langle\sigma\rangle)$ form a maximal antichain below p_{n+1} as desired. This completes the standard argument and the construction for $n + 1$.

This completes Step 1.

For Step 2, we first give a definition of $S = \dot{S}_{G_\alpha}$ in the generic extension. That is, suppose G_α is $\mathbb{S}_\alpha$-generic, with $p \in G_\alpha$. Then $f = \dot{f}_{G_\alpha} \in T$, and there is a unique $y \in 2^\omega$ such that $f = \dot{f}_{G_\alpha} \supset S_{n,y\restriction n}$ for all n. Let $\{l_n : n < \omega\}$ be such that $\{\alpha_{l_n} : n \in \omega\}$ is strictly increasing and converges to α. Choose the l_n's so that $\alpha_{l_n} > \alpha_i$ for all $i < l_n$ - this will be useful in Step 3. Let Z be the set of $z \in 2^\omega$ such that

- $\forall n \forall j \in F^{-1}(\{l_n\})\ z(min(F^{-1}(\{l_n\}) \setminus (j+1))) = y(j)$,
- $\forall j \notin F^{-1}(\{l_n : n \in \omega\})\ z(j) = y(j)$.

Note that for each n, z can take any value at $min\{i \mid F(i) = l_n\}$. Let

$$S = \{g \mid \exists z \in Z \forall n\ S_{n,z\restriction n} \subset g\}.$$

S is a perfect subset of T, so there a name $\dot{S}$ such that $\dot{S}_{G_\alpha} = S$. Also note the following (still in $V[G_\alpha]$): if $g \in S$ then in $V[g]$ we can reconstruct $z \in Z$ such that $\forall n S_{n,z\restriction n} \subset g$. From z we can reconstruct y, and from y we can reconstruct f. So $f \in V[g]$. This means that all branches of S are reals which arise only in $V[G_\alpha]$.

On the other hand, by [23] we have that exactly one new constructibility degree, above all the previous degrees, arises in $V[G_\alpha]$. So f, S and all the branches of S have the same constructibility degree. This completes Step 2.

Step 3: First, fix a name $\dot{g}$ and a condition $q_0 \leq p$ such that $q_0 \Vdash \dot{g} \in [\dot{S}]$. As above, we have a name $\dot{y}$ such that $p \Vdash \dot{f} = \bigcup_{n\in\omega} S_{n,\dot{y}\restriction n}$. This gives us a name $\dot{z}$ such that $q_0 \Vdash \dot{g} = \bigcup_{n\in\omega} S_{n,\dot{z}\restriction n}$, where $\dot{z}$ is forced to be in the set Z defined from $\dot{y}$ as above.

We construct a fusion sequence $\langle q_n : n < \omega \rangle$, where q_0 is the given condition. The result of the fusion will be q. The construction of the tree U will be induced by the construction, for each $\sigma \in 2^{<\omega}$, of a sequence $T_{|\sigma|,\sigma} \in 2^{<\omega}$ and a condition $q_{|\sigma|}\langle\sigma\rangle \leq q_{|\sigma|}$ such that

- $\{T_{|\sigma|,\sigma} \mid \sigma \in 2^{<\omega}\}$ forms a perfect tree, with extension and incompatibility in accordance with the corresponding σ's,
- each $q_{|\sigma|}\langle\sigma\rangle \Vdash T_{|\sigma|,\sigma} \subset \dot{z}$,
- for each n, $\{q_n\langle\sigma\rangle \mid \sigma \in 2^n\}$ forms a maximal antichain below q_n.
- the length of $T_{|\sigma|,\sigma}$ depends only on n (and will denoted by k_n).

Then $U = \{t \in 2^{<\omega} \mid \exists n \in \omega, \sigma \in 2^n (t \subset S_{k_n, T_{n,\sigma}})\}$ will be as desired. Along the way we also fix ordinals β_n ($n \in \omega$) such that $supp(q) = \{\beta_n : n \in \omega\}$. Note that $\{\alpha_n : n \in \omega\} \subset \{\beta_n : n \in \omega\}$. We stipulate that $\beta_n \leq \alpha_n$ for all n, and also that if $\beta_n \in \{\alpha_i : i \in \omega\}$ then there exists $i \leq n$ such that $\beta_n = \alpha_i$. We will use the following sets, where $\gamma < \alpha$ and $n \in \omega$:

$$C_n^\gamma = \{i < n \mid \beta_{F(i)} \leq \gamma\}, D_n^\gamma = \{i < n \mid \beta_{F(i)} < \gamma\},$$

(so $C_n^\gamma \setminus D_n^\gamma = \{i < n \mid \beta_{F(i)} = \gamma\}$).

Additionally, we produce

i. sequences $T^\gamma_{n,\sigma\restriction C_n^\gamma} \in 2^{<\omega}$ (of length $\geq |C_n^\gamma|$), where $n \in \omega$, $\sigma \in 2^n$, and $\gamma \in \{\beta_{F(i)} : i < n\}$

ii. $U^\gamma_{n,\sigma\restriction C_n^\gamma} \in 2^{<\omega}$, where $n \in \omega$, $\sigma \in 2^n$, and $\gamma \in \{\beta_{F(i)} : i < n\}$

such that

a. for all $n \in \omega$, $\gamma \in \{\beta_{F(i)} : i < n\}$,

- for all distinct $\sigma \in 2^{C_n^\gamma}$, the $T^\gamma_{n,\sigma}$ are all pairwise incompatible,
- for all $\sigma \in 2^{C^\gamma_{n+1}}$, $T^\gamma_{n,\sigma\restriction C_n^\gamma} \subset T^\gamma_{n+1,\sigma}$,

b. for all $n \in \omega$ and $\gamma \in \{\beta_{F(i)} : i < n\}$,

- for all distinct $\sigma \in 2^{C_n^\gamma}$, the $U^\gamma_{n,\sigma}$ are all pairwise incompatible,
- for all $\sigma \in 2^{C^\gamma_{n+1}}$, $U^\gamma_{n,\sigma\restriction C_n^\gamma} \subset U^\gamma_{n+1,\sigma}$,

c. for all $n \in \omega$, $\sigma \in 2^n$ and $\gamma \in \{\beta_{F(i)} : i < n\} \cap \{\alpha_i : i \in \omega\}$,

$$U^\gamma_{n,\sigma\restriction C_n^\gamma} = S^\gamma_{k_n, T_{n,\sigma}\restriction A^\gamma_{k_n}}.$$

d. for all $n \in \omega$, $\gamma \in \{\beta_{F(i)} : i < n\}$ and $\sigma \in 2^{D_n^\gamma}$, there exists $(q_n\restriction\gamma)\langle\sigma\rangle \leq q_n$ such that $(q_n\restriction\gamma)\langle\sigma\rangle$ forces

- $\{T^\gamma_{n,\tau} \mid \sigma \subset \tau \wedge dom(\tau) = C_n^\gamma\} \subset q_n(\gamma)$,
- $\forall T \in q_n(\gamma) \exists \tau \in 2^{C_n^\gamma} (\sigma \subset \tau \wedge (T \subset T^\gamma_{n,\tau} \vee T^\gamma_{n,\tau} \subset T))$,

i.e., each $T^\gamma_{n,\tau}$ canonically defines the $|C^\gamma_n \setminus D^\gamma_n|$-th splitting level of $q_n(\gamma)$, so together they define a finite maximal antichain in $q_n(\gamma)$. In particular, if $\beta_{F(n)} > \gamma$, then for all $\sigma \in 2^{n+1}$

$$T^\gamma_{n+1,\sigma\restriction C^\gamma_{n+1}} = T^\gamma_{n,\sigma\restriction C^\gamma_n}.$$

As specified by (d), the sets $q_n\restriction\gamma\langle\sigma\rangle$ and $q_n\langle\sigma\rangle$ are built up from the $T^\gamma_{n,\sigma}$'s in the same way that the sets $p_n\restriction\gamma\langle\sigma\rangle$ and $p_n\langle\sigma\rangle$ were built from the $S^\gamma_{n,\sigma}$'s in Step 1, using $\{\beta_i : i < \omega\}$ in place of $\{\alpha_i : i < \omega\}$. Furthermore, the condition r is constructed in the same way from sets $r_n\restriction\gamma\langle\sigma\rangle$ and $r_n\langle\sigma\rangle$ built in the same way from the $U^\gamma_{n,\sigma}$'s. Conditions (a) and (b) then induce the isomorphism π between $\mathbb{S}_\alpha\restriction q$ and $\mathbb{S}_\alpha\restriction r$. The remaining point, that $\dot{g}_{G_\alpha} = \dot{f}_{G'_\alpha}$ whenever $G_\alpha \subset \mathbb{S}_\alpha$ is generic with $q \in G_\alpha$, follows from condition (c) and the fact that each $q_n\langle\sigma\rangle\Vdash T_{|\sigma|,\sigma} \subset \dot{z}$.

The construction requires one more condition:

e. if $n \in \omega$, $\sigma, \bar\sigma \in 2^n$, $\gamma \geq \beta_{F(i)}$ is the minimal ordinal in

$$\{\beta_{F(i)} : i \leq n\} \cap \{\alpha_j : j < \omega\},$$

and $\sigma\restriction C^\gamma_n = \bar\sigma\restriction C^\gamma_n$, then there is a $j \in A^\gamma_{k_{n+1}} \setminus (A^\gamma_{k_n} \cup B^\gamma_{k_{n+1}})$ such that $T_{n+1,\sigma^\frown\langle 0\rangle}(j) \neq T_{n+1,\bar\sigma^\frown\langle 1\rangle}(j)$.

Now for the details. By the argument for Step 1, we can construct the $T^\gamma_{n,\sigma\restriction C^\gamma_n}$'s to satisfy (a) and (d). We may further assume by augmenting the previous argument that the sequences $T_{n,\sigma}$ ($\sigma \in 2^n$) all have the same length k_n. To satisfy (e), we add another condition to the step $n \to n+1$ of the construction. Fix $n \in \omega, \sigma, \bar\sigma \in 2^n$. Let $\delta = \beta_{F(n)}$ and let $\gamma \geq \beta_{F(i)}$ be the least ordinal in $\{\beta_{F(i)} : i \leq n\} \cap \{\alpha_j : j < \omega\}$, if such an ordinal exists. Let G_γ be generic, with

$$(q_n\restriction\gamma)\langle\sigma\restriction D^\gamma_n\rangle = (q_n\restriction\gamma)\langle\bar\sigma\restriction D^\gamma_n\rangle \in G_\gamma.$$

Now consider

$$(q_n(\gamma))_{T^\gamma_{n,\sigma\restriction C^\gamma_n}}$$

in $V[G_\gamma]$. Since $q_n \leq p$,

$$(q_n(\gamma))_{T^\gamma_{n,\sigma\restriction C^\gamma_n}} \leq p(\gamma).$$

By the construction of p, $p(\gamma)$ is the set of s for which there exists k, τ such that

- $dom(\tau) = A^\gamma_k$,
- $s \subset S^\gamma_{k,\tau}$,
- $\forall j \in \omega(\alpha_{F(j)} < \gamma \Rightarrow \tau(j) = H(j))$,

where the function $H : \{i \in \omega \mid \alpha_{F(i)} < \gamma\} \to 2$ is canonically given by the generic G_γ as in Step 1, i.e.,

$$H = \bigcup\{\rho: B^\gamma_m \to 2 \mid m \in \omega, (p_m{\restriction}\gamma)\langle\rho\rangle \in G_\gamma\}.$$

Fix $\bar{\imath} \in \omega$ such that $\gamma = \alpha_{F(\bar{\imath})}$. Still arguing in $V[G_\gamma]$, we can find extensions $\bar{q}, \bar{\bar{q}}$ of

$$(q_n(\gamma))_{T^\gamma_{n,\sigma\restriction C^\gamma_n}}$$

such that for some $j \in F^{-1}(\{F(\bar{\imath})\}) \setminus k_n$ and some $\bar{\tau}, \bar{\bar{\tau}} \in 2^{A^\gamma_{k_{n+1}}}$ (assuming that we have chosen k_{n+1} to be large enough) we have $\bar{\tau}(j) \neq \bar{\bar{\tau}}(j)$ and $stem(\bar{q}) = S^\gamma_{k_{n+1},\bar{\tau}}$, $stem(\bar{\bar{q}}) = S^\gamma_{k_{n+1},\bar{\bar{\tau}}}$. By the definitions of $\dot{y}$ in Step 2 and (b) in Step 1, this means $\bar{q}$ and $\bar{\bar{q}}$ force different values to $\dot{y}(j)$, so they force different values to $\dot{z}(j)$ or to $\dot{z}(min(F^{-1}(\{F(\bar{\imath})\}) \setminus (j+1)))$. In either case, by letting

$$T^\gamma_{n+1,\sigma^\frown\langle 0\rangle\restriction D^\gamma_{n+1}}, T^\gamma_{n+1,\bar{\sigma}^\frown\langle 1\rangle\restriction D^\gamma_{n+1}}$$

extend the sequences $S^\gamma_{k_{n+1},\bar{\tau}}$ and $S^\gamma_{k_{n+1},\bar{\bar{\tau}}}$ respectively, we ensure that $T_{n+1,\sigma^\frown\langle 0\rangle}$ and $T_{n+1,\bar{\sigma}^\frown\langle 1\rangle}$ are distinct at some

$$j \in A^\gamma_{k_{n+1}} \setminus (A^\gamma_{k_n} \cup B^\gamma_{k_{n+1}}).$$

The construction for Step 3 is essentially finished. We just need to see that conditions (b) and (c) worked out. First note that in the case where $\gamma \notin \{\alpha_i : i \in \omega\}$ we can arbitrarily choose $U^\gamma_{n,\sigma\restriction C^\gamma_n}$ to satisfy (b) because clause (c) is void. So assume $\gamma \in \{\alpha_i : i \in \omega\}$. We first check for each n that the requirement in (c) is well-defined. That is, we show that for all $\gamma \in \{\beta_{F(i)} : i < n\} \cap \{\alpha_i : i \in \omega\}$, if $\sigma, \bar{\sigma} \in 2^n$ satisfy $\sigma{\restriction}C^\gamma_n = \bar{\sigma}{\restriction}C^\gamma_n$, then $T_{n,\sigma}{\restriction}A^\gamma_{k_n} = T_{n,\bar{\sigma}}{\restriction}A^\gamma_{k_n}$.

For $n = 0$ there is nothing to show. For $n+1 \in \omega$, fix

$$\gamma \in \{\beta_{F(i)} : i < n+1\} \cap \{\alpha_i : i \in \omega\}.$$

Say $\gamma = \beta_{F(i_0)}$. Assume that $\sigma, \bar{\sigma} \in 2^{n+1}$ are such that $\sigma{\restriction}C^\gamma_{n+1} = \bar{\sigma}{\restriction}C^\gamma_{n+1}$. Let $j \in A^\gamma_{k_{n+1}}$. Then $\alpha_{F(j)} \leq \gamma$. Also note (trivially) that

$$(q_{n+1}{\restriction}\gamma+1)\langle\sigma{\restriction}C^\gamma_{n+1}\rangle = (q_{n+1}{\restriction}\gamma+1)\langle\bar{\sigma}{\restriction}C^\gamma_{n+1}\rangle$$

and that

$$q_{n+1}\langle\sigma\rangle\Vdash T_{n+1,\sigma}\subset\dot z,\ q_{n+1}\langle\bar\sigma\rangle\Vdash T_{n+1,\bar\sigma}\subset\dot z,$$

so they decide $\dot z(j)$. However, by Steps 1 and 2, if two conditions below p force different values to $\dot z(j)$, then

- in case $j\in F^{-1}(\{m\})$, where $m\notin\{l_{\bar n}:\bar n\in\omega\}$, they force different values to $\dot y(j)$, which means the conditions are forced by $1_{\mathbb{S}_{\alpha_{F(j)}}}$ to take different values at the $\alpha_{F(j)}$th stage of the iteration, which is not the case for $q_{n+1}\langle\sigma\rangle$ and $q_{n+1}\langle\bar\sigma\rangle$
- in case $j\in F^{-1}(\{l_{\bar n}\})$ for some $\bar n$, if $j\neq min(F^{-1}(\{l_{\bar n}\}))$, they still force different values to some $\dot y(\bar j)$, where $\bar j<j$ and $F(\bar j)=l_{\bar n}=F(j)$, so the argument reduces to the previous case
- in case $j=min(F^{-1}(\{l_{\bar n}\}))$ for some $\bar n$, we have $\beta_{F(j)}\le\alpha_{F(j)}\le\gamma=\beta_{F(i_0)}$, so $j\le i_0$ by (**) in Step 1, and the choice of the l_n's in Step 2. Let $i_1\le i_0$ be minimal such that $\beta_{F(i_1)}\ge\alpha_{F(j)}$. We still have $j\le i_1$, for the same reason, and also that $C^{\beta_{F(i_1)}}_{i_1+1}=i_1+1$. Also, $C^{\beta_{F(i_1)}}_{i_1+1}\subset C^\gamma_{n+1}$ because $\beta_{F(i_1)}\le\gamma$. Let $\tau=\sigma\restriction(i_1+1)=\bar\sigma\restriction(i_1+1)$. Then
$$T_{i_1+1,\tau}=T_{n+1,\sigma}\restriction k_{i_1+1}=T_{n+1,\bar\sigma}\restriction k_{i_1+1},$$
and since $j\le i_1\le k_{i_1}<k_{i_1+1}$, they agree at j.

This verifies (c). Finally, we check (b). The inclusion relation is immediate. We need to check incompatibility, that is, that if $\sigma\restriction C^\gamma_n=\bar\sigma\restriction C^\gamma_n$ and $n\in C^\gamma_{n+1}$ (so that $\sigma^\frown\langle0\rangle\restriction C^\gamma_{n+1}\neq\bar\sigma^\frown\langle1\rangle\restriction C^\gamma_{n+1}$), then $U^\gamma_{n+1,\sigma^\frown\langle0\rangle\restriction C^\gamma_{n+1}}$ and $U^\gamma_{n+1,\bar\sigma^\frown\langle1\rangle\restriction C^\gamma_{n+1}}$ are incompatible. By (c) it suffices to show that

$$T_{n+1,\sigma^\frown\langle0\rangle}\restriction A^\gamma_{k_{n+1}}\neq T_{n+1,\bar\sigma^\frown\langle1\rangle}\restriction A^\gamma_{k_{n+1}}.$$

This follows by induction.

For $T_{0,\langle\rangle}\restriction A^\gamma_{k_0}$ (the case $n=-1$) there is nothing to show. Given that this holds for n, we argue for $n+1$. Now $\beta_{F(n)}\le\gamma$, since $n\in C^\gamma_{n+1}$. Also, by the assumption in (b), $\gamma=\beta_{F(i)}$, for some $i\le n$. Let γ' be the least member of $\{\beta_{F(i)}:i\le n\}\cap\{\alpha_j:j\in\omega\}$ (recall that γ is in this intersection). By (e) there is $j\in A^{\gamma'}_{k_{n+1}}\setminus A^{\gamma'}_{k_n}$ such that

$$T_{n+1,\sigma^\frown\langle0\rangle}(j)\neq T_{n+1,\bar\sigma^\frown\langle1\rangle}(j),$$

so we are done.

This finishes Step 3, and the proof. □

6. *Axiom 3*

It is easy to see that Axiom 3 holds if $\mathfrak{d} = Cov(\mathcal{SN}) = \aleph_1$. We show here that Axiom 3 implies $\mathfrak{d} = \aleph_1$. Axiom 3 as stated trivially implies $Cov(\mathcal{N}) = \aleph_1$, and in the second subsection we show that this would still hold even if we restricted Axiom 3 to sets of measure zero.

6.1 Remark. As pointed out in [30], Axiom 2 + $\mathfrak{d} = \aleph_1$ + $Cov(\mathcal{SN}) \leq \aleph_2$ trivially implies that $\mathfrak{c} \leq \aleph_2$. This follows from the fact that each strong measure zero set is contained in a $G_{\mathfrak{d}}$ set of strong measure zero, and so by Axiom 2 must have cardinality less than or equal to $\aleph_2$ since it can't contain a perfect set. By the same reasoning, $\mathfrak{d} = \aleph_1$, $Cov(\mathcal{SN}) = \aleph_1$ and $\mathrm{PSP}(\aleph_2, G_{\aleph_1})$ together imply CH.

Axiom 3 implies that there exists an ω_1-sequence of functions from ω to $\mathbb{R}^+$ such that no $G_{\aleph_1}$ set which is f-coverable for every f in the sequence contains a perfect set. This follows from the fact that H has no decreasing ω_2 sequences, and so there is an ω_1 sequence of elements of H with no lower bound in H. Any perfect set coverable by every function in the sequence would be a counterexample to Axiom 3. Since perfect sets cannot have strong measure zero, the existence of such a sequence follows from $\mathfrak{d} = \aleph_1$. The converse also holds.

Theorem 6.2. *If there exists a κ-sequence of functions from ω to $\mathbb{R}^+$ such that no G_κ set which is f-coverable for every f in the sequence contains a perfect set then $\mathfrak{d} \leq \kappa$.*

P r o o f. We prove the contrapositive. Assume that $\mathfrak{d} > \kappa$. Given a sequence $\langle f_\alpha : \alpha < \kappa \rangle$ of functions from ω to $\mathbb{R}^+$, for each $\alpha < \kappa$ define $f'_\alpha : \omega \to \mathbb{R}^+$ by letting

$$f'_\alpha(n) = min\{f_\alpha(m) : m \in [2^{n+1} - 2, 2^{n+2} - 3]\}.$$

Using $\mathfrak{d} > \kappa$, let $g : \omega \to \mathbb{R}^+$ be such that for all $\alpha < \kappa$ $\{n \in \omega \mid g(n) < f'_\alpha(n)\}$ is infinite. Now build a binary tree of intervals such that the members of the nth level (not counting the root as a node these are the $(2^{n+1} - 2)$th to $(2^{n+2} - 3)$th nodes of the tree) are disjoint and of diameter less than $g(n)$. Let P be the set of reals arising from paths through this tree, and note that

P is a perfect set. For each α, then, we have infinitely many n such that P can be covered by a sequence of intervals of diameters as prescribed by $f_\alpha\restriction[2^{n+1}-2, 2^{n+2}-3]$, so P is f_α-coverable. $\square$

In certain cirumstances the requirement that we decompose into an increasing sequence of smaller sets is not restrictive. These circumstances do not always hold, though, see Theorem 6.37.

Lemma 6.3. *Assume $\mathfrak{d} = \aleph_1$ and let A be a set of reals such that for any countable set G of functions $g\colon\omega \to \mathbb{R}^+$ for which A is not g-coverable there is an $h\colon\omega \to \mathbb{R}^+$ such that A is not h-coverable and $h \geq g$ mod-finite for all $g \in G$. If $A = \bigcup_{\alpha<\omega_1} B_\alpha$, where each $B_\alpha \lhd A$, then there is an increasing sequence $\langle D_\alpha : \alpha < \omega_1\rangle$ such that $A = \bigcup_{\alpha<\omega_1} D_\alpha$ and each $D_\alpha \lhd A$.*

P r o o f. We want to write the union of the B_α's as an increasing union of sets smaller than A. We may assume that each B_α is a G_δ-set, and so by Lemma 5.5 each B_α is a union of compact sets $\langle C_{\alpha\zeta} : \zeta < \omega_1\rangle$. We show that each $D_\beta \lhd A$, where $D_\beta = \bigcup\{C_{\alpha\zeta} : \alpha, \zeta < \beta\}$. Since the $C_{\alpha\zeta}$'s are compact, it suffices to find for each $\beta < \omega_1$ a function $h_\beta\colon\omega \to \omega$ such that A is not h_β-coverable but each $C_{\alpha\zeta}$, $\alpha, \zeta < \beta$, is. Our assumption on A gives us such an h_β. $\square$

6.1. *Cardinal invariants and Axiom 3*

For $f : \omega \to \mathbb{R}^+$, $Cov(f)$ is the least κ such that there is a κ-sequence of f-sets whose union is the reals. In this context we say that f is *nontrivial* if $\sum_{n\in\omega} f(n)$ is finite, since otherwise $Cov(f) = 1$. We define two more cardinal invariants.

6.4 Definition. The least cardinal κ such that for some nontrivial $f \in (\mathbb{R}^+)^\omega$, $Cov(f) = \kappa$ is denoted $\mathfrak{mc}$ (minCov). The least cardinal κ such that for all nontrivial $f \in (\mathbb{R}^+)^\omega$, $Cov(f) \leq \kappa$ is denoted $\mathfrak{sc}$ (supCov).

Note that $\mathfrak{sc} \geq \mathfrak{mc} \geq Cov(\mathcal{N})$ trivially.

6.5 Remark. If $\mathfrak{sc} = \aleph_1$, the ω_1-many f-coverable sets can be taken to be an increasing sequence (even if $\mathfrak{d} > \aleph_1$). This follows from the fact that for any $f : \omega \to \mathbb{R}^+$ there is a $g : \omega \to \mathbb{R}^+$ such that any countable union of g-coverable sets is f-coverable. Assuming that f is strictly decreasing, any g such that

$$g(n) \leq f\left(\frac{(\mathrm{n}+1)(\mathrm{n}+2)}{2}\right)$$

for all n suffices.

Axiom 3 then follows from $\mathfrak{d} = \mathfrak{sc} = \aleph_1$.

6.6 Remark. Unlike the case for $\mathfrak{d} = Cov(\mathcal{SN}) = \aleph_1$, we know of no argument other than our version of Gödel's argument which proves $\mathfrak{c} \leq \aleph_2$ from our first two axioms plus $\mathfrak{d} = \mathfrak{sc} = \aleph_1$. However, if we replace Axiom 3 by $\mathfrak{d} = \mathfrak{sc} = \aleph_1$ in Theorem 3.1, then we can replace Axiom 1 with $\mathfrak{d}_1 = \aleph_2$, which unlike Axiom 1 does follow from G1+G2. The point is that using $\mathfrak{d} = Cov(\mathcal{SN}) = \aleph_1$ we can carry out Gödel's construction so that each descending sequence of B_α's becomes strong measure zero in at most ω_1 many steps, so each real is in a strong measure zero set corresponding to some member of a given witness to the value of $\mathfrak{d}_1$.

In this section, we show the following, among other things.

- $\mathfrak{mc}$ and $\mathfrak{sc}$ can be characterized in terms of covering numbers for trees.
- $Cov(\mathcal{N}) = \aleph_1$ does not imply $\mathfrak{sc} = \aleph_1$.
- $\mathfrak{b} > Cov(\mathcal{N})$ implies $\mathfrak{mc} = Cov(\mathcal{N})$.
- $\mathfrak{sc} = \aleph_1$ does not imply $(Non(\mathcal{M}) = \aleph_1 \vee Cov(\mathcal{SN}) = \aleph_1)$.
- $\mathfrak{d} = \mathfrak{sc} = \aleph_1$ does not imply $Cov(\mathcal{SN}) \leq \aleph_2$.

We also leave several questions open.

Many arguments about $\mathfrak{mc}$ and $\mathfrak{sc}$ are easier to carry out in terms of covering numbers for trees. First, we need to relate the two types of covering. Given $s \in 2^{<\omega}$, let $x_s = \Sigma_{i<|s|} \frac{s(i)}{2^i}$, and let $I_s = (x_{s^\frown\langle 0\rangle}, x_{s^\frown\langle 1\rangle})$. Then, minus the countable dense set

$$D = \{x_s : s \in 2^{<\omega}\} = \{\frac{m}{2^n} : m < 2^n \in \omega\},$$

we can identify the unit interval with 2^ω by identifying each real a with the unique $f \in 2^\omega$ such that $a \in I_{f\restriction n}$ for all $n < \omega$, and thus $[s]$ with I_s. The key point is the following.

Lemma 6.7. *Let I be an interval contained in $(0,1)$ of width r. Let n be the largest integer such that $2^{-n} \geq r$. Then there exist $s_0, s_1 \in 2^n$ such that $I \subset [s_0] \cup [s_1]$.*

P r o o f. Given I, let $n_0 < \omega$ be least such that there exists $d = \frac{m}{2^{n_0}} \in D \cap I$. Note that $2^{-n_0} \geq r$ and that d is unique. Let $s \in 2^{<\omega}$ be such that

$d = x_s$. If $n_0 = n$, then we can let $s_0 = s_1 = s$. Otherwise, d splits I into two pieces. Let s_0 be the extension of $s^\frown\langle 0\rangle$ of length n which takes value 1 on every integer in the extension of the domain, and let s_1 be the extension of $s^\frown\langle 1\rangle$ of length n which takes value 0 on every integer in the extension of the domain. Then $I \subset [s_0] \cup [s_1]$. □

6.8 Definition. Given a function $g\colon\omega \to \omega$, say that $A \subset 2^\omega$ is a *g-selection* if A is of the form $\bigcap_{m<\omega} \bigcup\{[s_n] : m < n < \omega\}$ where each $s_n \in 2^{g(n)}$. We let $TCov(g)$ be the least κ such that there is a set of g-selections of size κ with union 2^ω.

In this context, g is *nontrivial* if $\sum_{n\in\omega} \frac{g(n)}{2^n}$ is finite.

Lemma 6.9. *For every $g\colon\omega \to \omega$ there is an $f\colon\omega \to \mathbb{R}^+$ such that every g-selection is coverable by two f-sets and every f-set is coverable by a g-selection.*

For every $f\colon\omega \to \mathbb{R}^+$ there is a $g\colon\omega \to \omega$ such that every f-set is coverable by two g-selections and every g-selection is coverable by two f-sets.

P r o o f. For the first part, we may assume that g is nondecreasing. Then we define f by letting $f(n) = 2^{-g(2n+1)}$. Then given any g-selection, for each n we can cover the $(2n+1)$th interval by one of size $f(n)$, and the $(2n)$th (for $n > 0$) interval by one of size $f(n-1)$. Since every point of the g-selection appears infinitely often in either the odd intevervals or the even ones, these two f-sets cover the given g-selection. That every f-set is coverable by a g-selection follows from Lemma 6.7.

For the second part, given $f\colon\omega \to \mathbb{R}^+$, define g by letting $g(n)$ be the greatest m such that $2^{-m} \geq f(n)$. Then every f-set can be covered by two g-selections by Lemma 6.7, and every g-selection can be covered by two f-sets since $g(n) \leq 2f(n)$. □

Corollary 6.10. *Let $Q \subset P \subset (0,1)$. Then there is a $f : \omega \to \mathbb{R}^+$ such that Q is f-coverable but P is not contained in a countable union of f-sets if and only if there is a $g\colon\omega \to \omega$ such that Q is contained in a g-selection but P is not contained in a countable union of f-selections.*

This shows that the spectrum of values $Cov(f)$ for nontrivial $f\colon\omega \to \mathbb{R}^+$ is the same as the spectrum of values $TCov(g)$ for $g\colon\omega \to \omega$. In particular, we have

Corollary 6.11.

$$\mathfrak{mc} = inf\{TCov(g) \mid g \in \omega^\omega \;\; nontrivial\}.$$

$$\mathfrak{sc} = sup\{TCov(g) \mid g \in \omega^\omega \ nontrivial\}.$$

It is a standard fact that every continuous map between metric compacta is uniformly continuous. This leads to the following observations.

Lemma 6.12. *If $P \subset 2^\omega$ is a perfect set, then for any continuous 1-1, onto function $g : 2^\omega \to P$ there is a nondecreasing function $h : \mathbb{R} \to \mathbb{R}$ such that for all $f : \omega \to \mathbb{R}$ and $A \subset P$, if A is $(h \circ f)$-coverable then $g^{-1}[A]$ is f-coverable.*

Corollary 6.13. *If there exists a perfect set P such that for every $f : \omega \to \omega$ P is a union of $\aleph_1$ many f-coverable sets, then $\mathfrak{sc} = \aleph_1$.*

To see that $Cov(\mathcal{N}) = \aleph_1$ does not imply $\mathfrak{sc} = \aleph_1$ we use the following characterization of $Non(\mathcal{M})$.

Theorem 6.14. *[3], Theorem 2.4.7) $Non(\mathcal{M})$ is equal to the least κ such that there is a set $F \subset \omega^\omega$ of cardinality κ such that for every $g \in \omega^\omega$ there is an $f \in F$ such that $f(n) = g(n)$ for infinitely many n.*

Theorem 6.15. $\mathfrak{sc} \leq Non(\mathcal{M})$.

P r o o f. Applying Theorem 6.14, let $F \subset \omega^\omega$ of cardinality $Non(\mathcal{M})$ be such that for all $g \in \omega^\omega$ there is an $f \in F$ such that g and f agree on an infinite set. Let $h : \omega \to \mathbb{Q}^+$, and let $\{A_i : i < \omega\}$ be a basis of intervals for $\mathbb{R}$. For $f \in F$, let X_f be the h-set defined by letting O^f_n be $A_{f(n)}$ if the diameter of $A_{f(n)}$ is less than $h(n)$, and $\emptyset$ otherwise.

Given a real x, define a function $g_x \in \omega^\omega$ by letting $g_x(n)$ be the least m such that the diameter of A_m is less than $h(n)$ and $x \in A_m$. Then letting $f \in F$ agree with g_x on an infinite set, we have that $x \in X_f$. □

Theorem 6.16. *If $\mathfrak{d} < Non(\mathcal{M})$ then $Non(\mathcal{M}) = \mathfrak{sc}$.*

P r o o f. By Theorem 6.15 we have that $\mathfrak{sc} \leq Non(\mathcal{M})$.

Let μ be the Lebesgue measure on ω^ω. Assume that $\mathfrak{d}, \mathfrak{sc} < Non(\mathcal{M})$. Let M be an elementary substructure of a sufficiently large $H(\theta)$ of cardinality $max\{\mathfrak{d}, \mathfrak{sc}\}$. By Theorem 6.14 there is a function $g \in \omega^\omega$ such that for all $f \in \omega^\omega \cap M$, $f(n) = g(n)$ for only finitely many n. By the elementarity and cardinality of M, there is an $f \in \omega^\omega \cap M$ such that for all n $f(n) > g(n)$. For each $n \in \omega$, let $h(n) \in \mathbb{Q}^+$ be smaller than

$$min\{\mu(\sigma^* f, \tau^* 0) \mid \sigma \neq \tau \in \omega^n \wedge \forall i < n \ \sigma(i) \leq \tau(i) < f(i)\},$$

where $\sigma^* f$ and $\tau^* 0$ are the members of ω^ω extending σ and τ such that for $i \geq n$, $\sigma^* f(i) = f(i)$ and $\tau^* 0(i) = 0$. Then since $|M| \geq \mathfrak{sc}$, there is an h-set defined by a sequence of intervals $\langle O_i : i < \omega \rangle$ in M such that $g \in \bigcap_{n<\omega} \bigcup_{i \geq n} O_i$. By the definition of h, though, each O_i can intersect at most one set of the form

$$\{r \in \omega^\omega \mid r \restriction i = \sigma \wedge \forall j < \omega \; r(j) < \sigma^* f(j)\}$$

for some $\sigma \in \omega^i$. Then letting σ_i be the unique such $\sigma \in \omega^i$ if it exists, and letting $t(i) = \sigma_{i+1}(i)$, we have that t and g agree on an infinite set, which is a contradiction since $t \in M$. □

The following theorem is an improvement of a result in [4].

Theorem 6.17. *([9]) If GCH holds, then for any regular cardinal κ, there is a forcing which preserves cardinals and makes the following hold.*

1. $\mathfrak{d} = Cov(\mathcal{N}) = \aleph_1$.
2. $Non(\mathcal{M}) = \kappa$.

Theorems 6.16 and 6.17 give us the following.

Corollary 6.18. $Cov(\mathcal{N}) = \omega_1$ *does not imply* $\mathfrak{sc} = \aleph_1$.

Theorem 6.19. *If* $\mathfrak{b} > Cov(\mathcal{N})$*, then* $\mathfrak{mc} = Cov(\mathcal{N})$.

Theorem 6.19 follows from Lemmas 6.23 and 6.24 below. We will use the following definitions from [3], relating $\mathfrak{mc}$ to another type of covering number for trees. Assume that $H \in \omega^\omega$ is such that $\sum_{n<\omega} \frac{1}{\mathrm{H(n)}}$ is finite - we will call such H *nontrivial* as well. Let

$$\mathcal{C}_H = \{s \in ([\omega]^{<\omega})^\omega \mid \sum_{n<\omega} \frac{|\mathrm{s(n)}|}{\mathrm{H(n)}} < \infty\}$$

and

$$\mathcal{X}_H = \prod_{n<\omega} H(n) = \{x \in \omega^\omega \mid \forall n < \omega \; x(n) < H(n)\}.$$

Then

$$Cov(\mathcal{C}_H) = min\{|A| \mid A \subset \mathcal{C}_H \wedge \; \forall x \in \mathcal{X}_H \exists s \in A \exists^\infty n (x(n) \in s(n))\}.$$

Note that we can identity $\mathcal{X}_H$ with the interval $[0, 1]$ (minus countably many points) by first diving $[0, 1]$ into $H(0)$ many equal intervals, then dividing each of these into $H(1)$ many, and so on. As a consequence, $Cov(\mathcal{C}_H) \geq Cov(\mathcal{N})$. Bartoszyński has shown the following.

Theorem 6.20. *([3], Theorem 2.5.12) If $Cov(\mathcal{N}) < \mathfrak{b}$ then*

$$Cov(\mathcal{N}) = min\{Cov(\mathcal{C}_H) : H \in \omega^\omega \textit{ nontrivial } \}.$$

6.21 Question. Is it consistent that $Cov(\mathcal{N}) < Cov(\mathcal{C}_H)$ for all nontrivial $H \in \omega^\omega$?

6.22 Conjecture. It is consistent to have $Cov(\mathcal{N}) = \mathfrak{d} = \aleph_1$ and $\mathfrak{mc} = \mathfrak{sc} = \aleph_2$.

The proof of the following is essentially the same as the proof of Theorem 2.5.12 in [3]

Lemma 6.23. *For every nontrivial $f \in (\mathbb{R}^+)^\omega$ there is a nontrivial $H \in \omega^\omega$ such that $Cov(\mathcal{C}_H) \leq Cov(f)$.*

Lemma 6.24. *Assume that $\mathfrak{b} > Cov(\mathcal{C}_H)$. Then there is a nontrivial $f \in (\mathbb{R}^+)^\omega$ such that $Cov(\mathcal{C}_H) \geq Cov(f)$.*

P r o o f. Let $A \subset \mathcal{C}_H$ witness $\mathfrak{b} > Cov(\mathcal{C}_H)$, and canonically identify $\mathcal{X}_H$ with $[0,1]$. For $s \in A$, let $g_s \in \omega^\omega$ be such that $g_s(k)$ is least such that

$$\sum_{n \geq g_s(k)} \frac{|\mathrm{s(n)}|}{\mathrm{H(n)}} < \frac{1}{2^{\mathrm{k}}}.$$

Fix g such that for all $s \in A$ $\{n \mid g_s(n) \geq g(n)\}$ is finite. For each $k < \omega$, let

$$l_k = \frac{1}{\prod_{\mathrm{i<g(k+1)}} \mathrm{H(i)}}$$

and let

$$t_k = \frac{\prod_{\mathrm{i<g(k+1)}} \mathrm{H(i)}}{2^{\mathrm{k}}}.$$

Now define $f \in (\mathbb{Q}^+)^\omega$ so that for each k, f takes the value l_k exactly t_k times. Then f is nontrivial. It remains to see that each of the sets represented by a member of A is f-coverable.

For each $s \in A$, for cofinitely many $k \in \omega$,

$$(**) \quad \sum_{n=g(k)}^{g(k+1)-1} \frac{|\mathrm{s(n)}|}{\mathrm{H(n)}} < \frac{1}{2^{\mathrm{k}}}.$$

For such s and k, we can think of $\bigcup_{n=g(k)}^{g(k+1)-1} s(n)$ as representing

$$\sum_{n=g(k)}^{g(k+1)-1} \left(|s(n)| \cdot \prod_{i \in (g(k+1) \setminus \{n\})} H(i) \right)$$

many pairwise disjoint (or identical) intervals, all of which have length l_k. By $(**)$, the collection of these intervals must have size less than t_k. $\square$

A set of reals X is *strongly meager* [2] if for each null set Y,

$$\{x + y \mid \exists x \in X, y \in Y\} \neq \mathbb{R}.$$

We let $\mathcal{SM}$ denote the class of strongly meager sets. These sets are in fact meager [3].

Theorem 6.25. *There exists a nontrivial $f : \omega \to \mathbb{Q}^+$ such that $Cov(f) \leq Non(\mathcal{SM})$.*

P r o o f. Let X be not strongly meager, and Y a null set such that $X + Y = \mathbb{R}$. Then there is a nontrivial $f \in (\mathbb{Q}^+)^\omega$ such that Y is f-coverable. For any real x, $x + Y = \{x + y \mid y \in Y\}$ is also f-coverable. $\square$

The following chart, where larger ($\geq$, since if CH holds they are all equal) invariants point to smaller ones, summarizes the relationships between some of the invariants mentioned in this section.

$$\begin{tikzcd}
\mathfrak{mc} \arrow[d] & Non(\mathcal{SM}) \arrow[l] & Non(\mathcal{M}) \arrow[l] \arrow[dl] \\
Cov(\mathcal{N}) & \mathfrak{sc} \arrow[ul] & Cov(\mathcal{SN}) \arrow[l]
\end{tikzcd}$$

6.26 Question. Is it consistent to have, with $\mathfrak{c}$ arbitrarily large, for each cardinal $\kappa \leq \mathfrak{c}$ a nontrivial f such that $Cov(f) = \kappa$?

Although $\mathfrak{sc} = \aleph_1$ follows from each of $Non(\mathcal{M}) = \aleph_1$ and $Cov(\mathcal{SN}) = \aleph_1$, the following shows that it implies neither, as $\mathfrak{b} = \aleph_2$ implies $Non(\mathcal{M}) = \aleph_2$ and the Borel Conjecture implies that $Cov(\mathcal{SN}) = \mathfrak{c}$.

Theorem 6.27. *After adding ω_2 many Mathias (or Laver) reals to a model of CH, the following hold.*

1. $\mathfrak{b} = \aleph_2$.

2. The Borel Conjecture.

3. $\mathfrak{sc} = \aleph_1$,

P r o o f. The first two consequences are well known ([7], [22]). For the third, the key property is the following standard fact. Let M be an intermediate model in the iteration, and $\tau \in M$ a name in the rest of the iteration for a function from ω to ω. Let $\langle x_n : n \in \omega \rangle \in M$ be a sequence of finite subsets of ω and p a condition forcing that each $\tau(n)$ will be in x_n. Then there exists $q \leq p$ and a sequence of finite sets $\langle y_n \in [x_n]^{2^{n^2}} : n < \omega \rangle$ such that for all n $q \Vdash \tau(n) \in y_n$. Now, given a name σ for a subset of ω, and a decreasing function $g : \omega \to \mathbb{R}^+$ in an intermediate model M, for each $n \in \omega$ let m_n be such that

$$\frac{1}{2^{\mathrm{m_n}}} < g\left(\sum_{i=0}^{n} 2^{i^2}\right),$$

and let τ be a name for a function from ω to ω such that $\tau(n)$ is a code for $\sigma \restriction m_n$. Let x_n be the set of codes for the members of $\mathcal{P}(m_n)$. Then applying the key fact, we can shrink to a condition allowing just 2^{n^2} possibilities for each $\sigma \restriction m_n$. Listing the decoded versions of these possibilities, we get a description of a g-set which has the realization of σ as a member. Since CH holds in each intermediate model, then, there is an ω_1-sequence of g-sets covering the reals in the final model, and so $\mathfrak{sc} = \aleph_1$ holds there. $\square$

Gödel's proof shows that Axioms 1 and 3 together imply that $Cov(\mathcal{SN}) \leq \aleph_2$. By the theorem below, $Cov(\mathcal{SN}) \leq \aleph_2$ does not follow from $\mathfrak{sc} = \aleph_1$ plus $\mathfrak{d} = \aleph_1$.

Theorem 6.28. *Let $\kappa \geq \aleph_2$ have uncountable cofinality. After forcing to add κ many simultaneous Sacks reals to a model of CH with countable support, $\mathfrak{sc} = \mathfrak{d} = \aleph_1$ and $Cov(\mathcal{SN}) = \kappa$.*

P r o o f. We give a proof that $Cov(\mathcal{SN}) = \kappa$. The other parts are standard. For $A \subset \kappa$, let $\mathbb{S}_A$ be the countable support product of the copies of Sacks forcing ($\mathbb{S}$) with index in A. Since $\mathbb{S}_\kappa$ is ω^ω-bounding, $\mathfrak{d} = \aleph_1$, and so every strong measure zero set is contained in a $G_{\aleph_1}$ strong measure zero set. So it suffices to consider such sets C. Every such C is coded in some extension via $\mathbb{S}_{A_C}$, where $A_C \subset \kappa$, $|A_C| \leq \aleph_1$. Hence it suffices to prove that if $\alpha \notin A_C$, then s_α, the Sacks real added by $\mathbb{S}_{\{\alpha\}}$, does not belong to C.

This, however, is easy. Working in the ground model, let $p \in \mathbb{S}_\kappa$ be any condition. Let p_α be its αth coordinate. Let $f \in (\mathbb{R}^+)^\omega$ be a function such that whenever $\{I_n^i : n \in \omega, i < n\}$ is a set of intervals in 2^ω such that each I_n^i has length $\leq f(n)$, then $[p_\alpha] \setminus \{y \in 2^\omega \mid \exists^\infty \langle n, i\rangle (y \in I_n^i)\}$ contains a perfect set.

Let $\dot{C}$ be the $\mathbb{S}_{A_C}$-name for C. It is forced to be contained in an f-coverable G_δ set, say $\dot{D}$. By the Sacks property of $\mathbb{S}_{A_C}$, there are $q \leq p \restriction A_C$ and a set of intervals $\{I_n^i : n \in \omega, i < n\}$ such that I_n^i has length less than or equal $f(n)$, and such that

$$q \Vdash \dot{D} \subset \{y \in 2^\omega \mid \exists^\infty \langle n, i\rangle (y \in I_n^i)\}.$$

Now find $q_\alpha \leq p_\alpha$ such that $[q_\alpha] \cap \{y \in 2^\omega \mid \exists^\infty \langle n, i\rangle (y \in I_n^i)\} = \emptyset$. Then the condition r defined by

- $r_\alpha = q_\alpha$
- $r \restriction A_C = q$
- $r \restriction \kappa \setminus (\{\alpha\} \cup A_C) = p \restriction \kappa \setminus (\{\alpha\} \cup A_C)$

clearly forces that $\dot{s}_\alpha \notin \dot{D}$. Hence $r \Vdash \dot{s}_\alpha \notin \dot{C}$, as required. □

6.29 Question. After adding $\kappa \geq \omega_2$ many simultaneous Sacks reals to a model of CH, does PSP$(\aleph_2, G_{\aleph_1})$ hold? Is every strong measure zero set of cardinality $\aleph_1$ or less?

6.30 Remark. For many ω^ω-bounding forcings, every strong measure zero set has cardinality $\aleph_1$ or less after an iteration of length ω_2. This holds for the 'infinitely often equal reals' forcing, but fails after the corresponding product forcing (see [3]).

6.31 Remark. It is also shown in [2] that after adding an arbitrary number of random reals to a model of CH, $\mathfrak{d} = \aleph_1$ but all strong measure zero sets have cardinality $\aleph_1$ or less. In this model, $\mathfrak{sc} > \aleph_1$.

6.32 Remark. Adding uncountably many Cohen reals preserves Axioms 1 and 2 and forces $\mathfrak{sc} = \aleph_1$, so together they do not imply a bound on the continuum.

6.2. *Axiom 3 and decompositions*

Axiom 3 clearly follows from $\mathfrak{d} = \mathfrak{sc} = \aleph_1$, but we would like to see whether it is in fact weaker. As we have defined it, Axiom 3 implies $Cov(\mathcal{N}) = \aleph_1$. Theorem 6.36 below shows that for a certain class of perfect sets, if some P in this class can be written as a union of $\mathfrak{d}$ many sets $Q \lhd P$, then $Cov(\mathcal{N}) \leq \mathfrak{d}$. Therefore, $Cov(\mathcal{N}) = \aleph_1$ is a consequence of any decomposition scheme along the lines of Gödel's orignal proof.

6.33 Definition. A perfect set P is *uniformly perfect* if P is the set of paths through a tree $T \subset 2^{<\omega}$ such that on each level of T either all or none of the nodes split.

6.34 Remark. If T is the subset of $2^{<\omega}$ whose members take the value 0 on every even member of their domains, then T represents a uniformly perfect set P of measure 0.

For a finite branching tree T, a subtree S of T has measure 0 in T if $\lim_{n\to\infty} \frac{|\mathrm{A_n \cap S}|}{2^{\mathrm{n}}} = 0$, where A_n is the set of nth splitting nodes of T. The idea behind this definition is that if a tree $T \subset 2^{<\omega}$ represents a perfect set P, then P has measure zero if and only if $lim_{n\to\infty} \frac{|\mathrm{2^n \cap T}|}{2^{\mathrm{n}}} = 0$.

Lemma 6.35. *Let $Q \lhd P$ be perfect sets, with P uniformly perfect. Then the image of Q under the canonical bijection between P and 2^ω has measure 0.*

P r o o f. Citing Corollary 6.10, we work in terms of coverings of the trees representing Q and P. Let $S, T \subset 2^{<\omega}$ represent Q and P respectively, and let $g \in \omega^\omega$ be such that some g-selection covers T, but S cannot be covered by inifinitely many g-selections. We want to see that $lim_{n\to\infty} \frac{|2^n \cap S|}{|\mathrm{2^n \cap T}|} = 0$. Notice that the limit exists since the values are nonincreasing.

Say that there is some m such that for all n $\frac{|2^n \cap S|}{|\mathrm{2^n \cap T}|} \geq 2^{-m}$. For each $n < \omega$, let A_n indicate the first level of T to have size 2^n. Fix $\bar{t} \in A_m$. We will show that for any set of finite sequences $\langle s_i : i < \omega \rangle$ defining a g-selection covering S, there is a g-selection $\langle t_i : i < \omega \rangle$ covering $T \cap [\bar{t}]$, where $[t]$ denotes the set of all extensions in T of the sequence $[t]$.

Since P is compact, we can choose integers $\{n_i, j_i : i < \omega\}$ such that for all i

$$2^{n_i} \cap S \subset \bigcup \{[s_j] : j_i \leq j < j_{i+1}\}.$$

For each $i < \omega$ we will pick t_i of the same length as s_i, in such a way that for all $i < \omega$,

$$[\bar{t}] \cap 2^{n_i} \cap T \subset \bigcup\{[t_j] : j_i \leq j < j_{i+1}\}.$$

We have the following by induction on n: if $\langle p_i : i < k\rangle$ is a set of finite sequences such that

$$\sum_{i<k} |\{t \in A_n \mid p_i \subset t\}| \geq 2^r$$

for some integer $r \leq n$, then for any $t^* \in A_{n-r}$ there is a set $\langle q_i : i < k\rangle$ of sequences such that $length(p_i) = length(q_i)$ for all $i < k$, and such that

$$\{t \in A_n \mid \exists i < k \; q_i \subset t\} = \{t \in A_n \mid t^* \subset t\}.$$

For the induction step to n, note that either the number of p_i's in A_n is even, or

$$\sum_{i<k} |\{t \in A_n \mid p_i \subset t\}| \geq 2^r$$

holds even if we remove one such p_i. Then by pairing off the p_i's in A_{n-1} we can replace them with shorter sequences and apply the induction hypothesis.

So by the fact that $\frac{|2^n \cap S|}{|2^n \cap T|} \geq 2^{-m}$ for all n, we can choose the t_i's as desired. □

Lemmas 5.5, 6.12 and 6.35 give the following.

Theorem 6.36. *(* $\mathfrak{d} = \aleph_1$*) If there exists a uniformly perfect set P contained in a set $\bigcup_{\alpha<\omega_1} Q_\alpha$, where each $Q_\alpha \lhd P$ then $Cov(\mathcal{N}) = \aleph_1$.*

P r o o f. Say that P and Q_α ($\alpha < \omega_1$) are as in the statement of the theorem. Since each $Q_\alpha \lhd P$ as witnessed by a G_δ set covering Q_α, we may assume that each Q_α is G_δ. By Lemma 5.5, each Q_α is a union of ω_1-many perfect sets, so by Lemmas 6.35 and 6.12 $Cov(\mathcal{N}) = \aleph_1$. □

We would like to know whether the condition in Lemma 6.35 that P is uniformly perfect is necessary.

Certain strenghtenings of Axiom 3 do imply that $\mathfrak{mc} = \aleph_1$.

Theorem 6.37. *If the set of functions H in Axiom 3 is required to be linearly ordered under mod-finite domination, then this strengthened version of the axiom implies $\mathfrak{mc} = \aleph_1$.*

P r o o f. Fix H as in the statement of Axiom 3. If $\mathfrak{mc} > \aleph_1$, then there is a sequence $\langle (A_i, f_i) \in \mathcal{P}(\mathbb{R}) \times H : i < \omega\rangle$ such that each A_i is f_i-coverable but not f_j-coverable for any $j < i$. To construct such a sequence, using

Theorem 6.36, let $\langle (B_\alpha, g_\alpha) \in \mathcal{P}(\mathbb{R}) \times H : \alpha < \omega_1 \rangle$ be such that each g_α is nontrivial, each B_α is a g_α-set and $\bigcup_{\alpha<\omega_1} B_\alpha = \mathbb{R}$. Let $A_0 = B_0$ and $f_0 = g_0$. Then given $(A_j, f_j) \in \mathcal{P}(\mathbb{R})$ $(j < i)$, define $h\colon \omega \to \mathbb{R}^+$ by letting $h(ai+b) = f_b(a)$ whenever $a, b \in \omega$ with $b < i$. Using $\mathfrak{mc} > \aleph_1$, some B_α is not h-coverable, so we can let $A_i = B_\alpha$ and $f_i = g_\alpha$.

Now by Lemma 5.5, by shrinking the A_i's if necessary, we can assume that they are all compact, and that there are disjoint intervals I_i $(i \in \omega)$ such that $A_i \subset I_i$.

Let $D = \bigcup_{i<\omega} A_i$. Now if H is linearly ordered by mod-finite domination, then D is h-coverable for all $h \in H$ not dominated mod-finite by some f_i. But if D can be written as an increasing union of ω_1 many sets which are each f_i for some integer i, then there is a fixed integer i such that D can be covered by ω_1-many f_i-coverable sets. But D was constructed to make this impossible. $\square$

There are many other questions one could ask in this area, especially : does Axiom 3 imply $\mathfrak{sc} = \omega_1$?

7. *Appendix : Chart of Models*

Model:	1	2	3	4	5	6	7
Axiom 1	t	t	t	t	t	t	F
Axiom 2	t	T	F	F	T	T	?
Axiom 3	t	F	F	T	F	T	T
PSP$(\aleph_1, G_{\aleph_1})$	F	T	f	F	T	F	F
PSP$(\aleph_2, G_{\aleph_1})$	t	T	F	F	T	T	?
$\mathfrak{d} = \aleph_1$	t	F	T	T	F	T	T
$Cov(\mathcal{SN}) = \aleph_1$	t	T	F	T	F	F	F
$\mathfrak{sc} = \aleph_1$	t	T	F	T	T	T	T
$\mathcal{SN} \subset [\mathbb{R}]^{\aleph_1}$	t	F	T	F	T	T	T?

T and F correspond to true and false, t and f to trivially true and trivially false. Question marks indicate open questions or, if accompanied by T or F, conjectures. The models listed are as follows, where each forcing is conducted over a model of GCH, and 'many' means $\geq \aleph_3$, so models 2, 3, 4 and 7 do not satisfy $\mathfrak{c} \leq \aleph_2$.

1. Ground model.

2. Adding many Cohen reals.

3. Adding many random reals.
4. Adding many reals by c.c.c. forcing, followed by ω_1 Hechler reals.
5. Adding ω_2 Mathias reals.
6. Adding ω_2 Sacks reals.
7. Adding many Sacks reals simultaneously.

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