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SEMI-TOPOLOGICAL CLASSIFICATION OF LINE PATTERNS IN THE PLANE

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A b s t r a c t. A large class of patterns, consisted of lines being situated in the plane, is classified into equivalence classes, each of which is a conveyor of meaning of a shape. Invariants of this classification are developed and, in particular, for a more regular subclass of these patterns, invariant matrices are defined being unique arithmetic codes of these shapes. Then, an algorithm is established as the way of transformation of so called associated matrices, formed as a result of local inspection of patterns, into invariant ones which express the global properties of these patterns. Using the language of psychology, we could say that this investigation is the study of percepts and the establishment of their meaning.

AMS Mathematics Subject Classification (2000): 68T10

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1. *Introduction*

In a number of papers (first of all in M. Marjanović, R. Tomović, S. Stanković, A Topological Approach to Recognition of Line Figures, Bull. T.CVII, No 19, de l'Académie Serbe des Sciences et des Arts, pp. 43–64, 1994), a large class of line patterns in the plane have been defined and their semi-topological classification established. In this paper we redefine this class of objects and the way of their classification, avoiding all topological terms. In fact, following this approach, we consider patterns to be families of arcs rather than their union as it was the case in our previous papers and when such unions were called forms (or figures). These families of the arcs are of the two kinds—some are graphs of continuous function in the plane supplied with a coordinate system (called stretching), the others are vertical intervals in that plane. Their classification is semi-topological in the sense that the size of these arcs and the shape of stretching ones may vary but their type and orientation stay preserved.

Recognition of patterns is based on their invariant properties of which, particularly discriminating, are invariant matrices defined in Section 4 of this paper and which are considered here for the first time.

To grasp the intuitive idea of this morphology, the way how a look is cast at an object of observation has to be fixed. In this paper, the post of observation could be imagined to be the point in the plane down, at infinity and a look is cast along the directions going straight upwards. Then, a pattern is seen to be split into layers overlying one above the other. This also explains the essential role of the coordinate system that we suppose to be fixed throughout all our considerations.

2. *Definition of line patterns and their decomposition into layers*

Let E^2 be the Euclidean plane supplied with a fixed rectangular coordinate system Σ. Then, to each point of E^2 a unique pair of real numbers is assigned, being its coordinates with respect to Σ.

Let $[a, b]$ be a closed interval belonging to x-axis of Σ. If f is a continuous function defined on $[a, b]$, then its graph will be called a *stretching arc* in E^2. The *left* and *right end points* of this arc are the points $(a, f(a))$ and $(b, f(b))$, respectively. The point $(x, f(x))$ for $x \in (a, b)$ is the *interior point* of this arc and the set $\{(x, f(x)) \mid x \in (a, b)\}$ is its *interior*.

For a point a belonging to x-axis of Σ and a closed interval $[c, d]$ belonging to y-axis of Σ, the set $\{a\} \times [c, d]$ will be called a *vertical arc* in E^2. The points (a, c) and (a, d) are the *lower* and *upper end points* of this arc, the point (a, y) for $y \in (c, d)$ its *interior point* and the set $\{(a, y) \mid y \in (c, d)\}$ is its *interior.*

Let Φ be a finite family of stretching and vertical arcs in E^2 satisfying the following conditions: (i) *Two stretching arcs can intersect only at their end points*, (ii) *Each vertical arc intersects at least one stretching arc. This intersection is one of the end points of the stretching arc*, (iii) *No two vertical arcs intersect.*

Then, the family Φ is called a *line pattern.* Let us notice that the condition (ii) excludes the existence of isolated vertical arcs.

For a stretching arc $\alpha \in \Phi$, its left (right) end point will be denoted by $end_-(\alpha)$, $(end_+(\alpha))$. A vertical arc $\omega \in \Phi$ which intersects α at $end_-(\alpha)$ $(end_+(\alpha))$ will be called *left (right) end component* of α and denoted by $C_-(\alpha)$ $(C_+(\alpha))$. For the sake of simplicity, when α is a stretching arc in E^2, we also denote by α the corresponding continuous function and for $x \in \operatorname{dom}(\alpha)$, $\alpha(x)$ denotes the value of this function at the point x.

For two stretching arcs α and β in E^2 for which $\operatorname{dom}(\alpha) \cap \operatorname{dom}(\beta) \neq \emptyset$, we say that α *is lower than* β if there exist a point $x \in \operatorname{dom}(\alpha) \cap \operatorname{dom}(\beta)$ such that $\alpha(x) < \beta(x)$. In this case, $\alpha(x) \leq \beta(x)$ for each $x \in \operatorname{dom}(\alpha) \cap \operatorname{dom}(\beta)$, with equality being possible only at the abscissas of end points of these arcs. Two pairs of such arcs are illustrated in Fig. 1.

Fig. 1

(When no coordinate system is indicated, we suppose that the lower and left edges of the page are x-axis and y-axis respectively).

The relationship "to be lower than" defines a relation on the set of all stretching arcs in E^2. We will somewhat modify it when it is considered to be applied to the set of stretching arcs of a line pattern Φ. Namely, the components of stretching arcs will be viewed as they were "big" end points. Thereby, we modify the above relation in the way that two stretching arcs α and β having the same component, being right for one of them and left for the other one, are not considered to be related even if $\alpha(x) > \beta(x)$ or

$\alpha(x) < \beta(x)$ at the point $x \in \mathrm{dom}\,(\alpha) \cap \mathrm{dom}\,(\beta)$. For example, the pairs of arcs illustrated in Fig. 2 are not related in Φ.

Fig. 2

As being defined on the set of stretching arcs in Φ, this relation extends to a unique order relation "$<$" on this set. Namely, if α and β are stretching arcs in Φ, then we write $\alpha < \beta$ if there exists a sequence of stretching arcs in Φ: $\alpha_0 = \alpha, \alpha_1, \ldots, \alpha_k = \beta$, $(k \geq 1)$ such that α_i is lower than α_{i+1} in Φ, $(i = 0, \ldots, k-1)$. We will call the order relation "$<$" the *vertical ordering* of stretching arcs in Φ. In Fig. 3 a line pattern is given

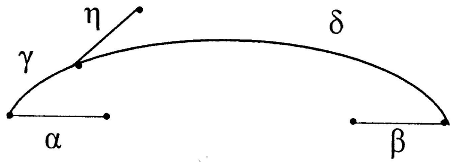

Fig. 3

where: $\alpha < \gamma$, $\beta < \delta$, $\delta < \eta$, $\beta < \eta$. Minimal elements of this ordered set are: α and β.

A line pattern is simple if it contains no two stretching arcs which are related. For instance the following pattern

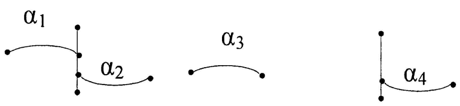

Fig. 4

is simple. The set of all stretching arcs of a *simple* pattern Φ can be ordered in the sequence α_1, α_2, $\ldots$, α_n according to the rule that the right end point of $\mathrm{dom}\,(\alpha_i)$ is equal or less than the left end point of $\mathrm{dom}\,(\alpha_{i+1})$, $(i = 1, \ldots, n-1)$. The number n is called the *length* of Φ.

We will call an end point of a stretching arc in Φ a *node*, when it does note belong to a vertical arc. For example in the case of the following pattern

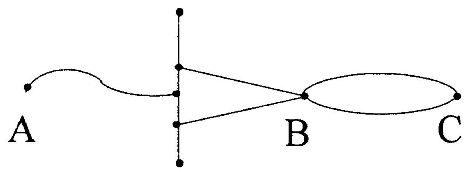

Fig. 5

the points A, B and C are its nodes.

Let Φ be a line pattern. As we have seen it, the set of all stretching arcs in Φ is an ordered set and let Λ_1 be those of these arcs which are minimal with respect to this ordering, taken together with all their end components. Then Λ_1 is a simple line pattern which we call the *first layer* of Φ. By removing from Φ all stretching arcs in Λ_1 together with those of its vertical arcs which are not components of some of remaining arcs, a subfamily Φ_1 of Φ is obtained. The subfamily Φ_1 is again a line pattern. Let Λ_2 be the first layer of Φ_1, then Λ_2 is also simple and we call it the *second layer* of Φ. Again, by removing from Φ_1 all stretching arcs in Λ_2 together with their end components not being end components of some of remaining arcs, a subfamily Φ_2 of Φ_1 is obtained. The same procedure is applied until a family Φ_{m-1} is obtained and which is simple itself. Then, we call Φ_{m-1}, the *m-th layer* of Φ and we write $\Lambda_m = \Phi_{m-1}$. We call the sequence

$$\Lambda_1, \Lambda_2, \ldots, \Lambda_m$$

the *decomposition* of Φ *into layers* and the number m, the *height* of Φ. For instance, in Fig. 6 the decomposition of a line pattern into layers is illustrated and the height of the pattern is 3.

Let m be the height of a line pattern Φ and let $n(i)$ be the length of its layer Λ_i, $i = 1, \ldots, m$. Then, there exists a biunivoque correspondence between the set of all pairs (i, j), $(i = 1, \ldots, m)$, $(j = 1, \ldots, n(i))$ and the

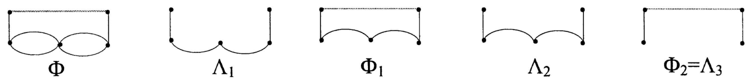

Fig. 6

set of all stretching arcs in Φ. Let α_{ij} be such an arc corresponding to (i, j). Let Φ and Φ' be two line patterns having the same height m and let for each i, $(i = 1, \ldots, m)$, the layers Λ_i and Λ'_i have the same length $n(i)$. Then, to each stretching arc α_{ij} in Φ, the stretching arc α'_{ij} in Φ' is corresponded and this correspondence is also biunivoque. We say that such two arcs are *analogous* and that the two patterns Φ and Φ' are *similar*. Two pairs of similar patterns are given in Fig. 7:

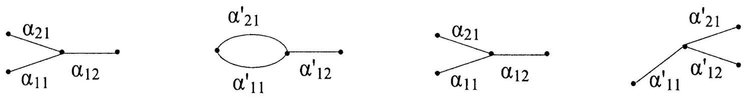

Fig. 7

3. *Classification of line patterns*

In what follows we restrict the class of line patterns to *those of them which have no vertical arc.* Classification in general case is somewhat more complicated and we omit it here.

Let α and β be two arcs of a pattern Φ. If α and β intersect at the point A, then A is the end of each of these arcs, being of one of the following types: left of α and left of β – (l, l), left of α and right of β – (l, r), right of α and left of β – (r, l), right of α and right of β – (r, r). Thus, for a pair of intersecting arcs, one of these four possibilities determines the *type of their intersection.* Let now Φ and Φ' be two similar line patterns and let α, β and α', β' be pairs of analogous arcs in Φ and Φ', respectively. Then Φ is *semi-topologically equivalent to* Φ' if the following condition is satisfied: *For* *any two arcs α and β in Φ, α and β intersect at the point A if and only if α' and β' intersect at the point A' and these intersections are of the same type.*

Examples of pairs of similar patterns which are not equivalent are illustrated in Fig. 7: $end_-(\alpha_{11}) \neq end_-(\alpha_{21})$ and $end_-(\alpha'_{11}) = end_-(\alpha'_{21})$ in the first case, $end_+(\alpha_{11}) = end_+(\alpha_{21})$ and $end_+(\alpha'_{11}) \neq end_+(\alpha'_{21})$ in the second.

Let us notice that when two patterns are equivalent then there exists a biunivoque correspondence between their sets of nodes. Moreover, the numbers of stretching arcs related to analogous nodes are equal. Namely, let the point A be a node in Φ and let $\Psi_- = \{\alpha \mid end_-(\alpha) = A\}$, $\Psi_+ = \{\alpha \mid end_+(\alpha) = A\}$. It may happen that one of these two sets is empty. Let us suppose that $\Psi_- \neq \emptyset$ and let $\alpha_0 \in \Psi_-$. Then, the point $A' = end_-(\alpha'_0)$ is a node in Φ' and the sets Ψ_- and $\Psi'_- = \{\alpha' \mid end_-(\alpha') = A'\}$ as well as the sets Ψ_+ and $\Psi'_+ = \{\alpha' \mid end_+(\alpha') = A'\}$ have the same number of elements.

Now we consider the way how a matrix with entries 1's and 0's is attached to a pattern Φ as a procedure of its arithmetic codification. Let Φ be a line pattern and let m be its height and $\Lambda_1, \ldots, \Lambda_m$ its decomposition into layers. By projecting orthogonally onto x-axis all nodes of Φ, an increasing sequence of points $a_1, a_2, \ldots, a_s$ is obtained. The matrix M_Φ that is attached to Φ will be of the type $(2s-1) \times m$ and to each layer Λ_i, i-th row of M_Φ will be attached. For $i = 1, \ldots, m$ and $k = 1, \ldots, s$, let ξ be $(i, 2k-1)$-entry of M_Φ.

If Λ_i has a node projecting onto a_k, then ξ is 1 if that node does not belong to any Λ_j, $j < i$ and is 0 if it belongs to some Λ_j, $j < i$.

If an interior point of a stretching arc of Λ_i projects on a_k, then ξ is 1, and ξ is 0 when no point of $|\Lambda_i|$ projects onto a_k, ($|\Lambda_i|$ is the union of all arcs of Λ_i).

For $i = 1, \ldots, m$ and $k = 1, \ldots, s-1$, let η be $(i, 2k)$-entry of M_Φ. The open interval (a_k, a_{k+1}) is either contained into the projection of $|\Lambda_i|$, when $\eta = 1$, or is disjoint with that projection, when $\eta = 0$. We call M_Φ the *matrix associated with the pattern* Φ and the entries of M_Φ belonging to even columns will be called "running".

For example, the matrix associated with the pattern in Fig. 8:
will be

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

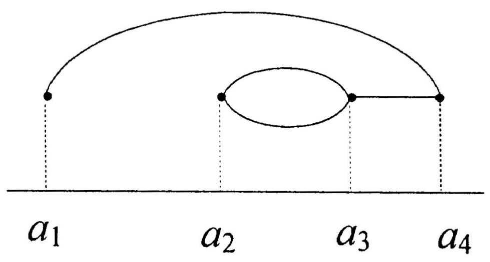

Fig. 8

The three patterns in Fig. 9:

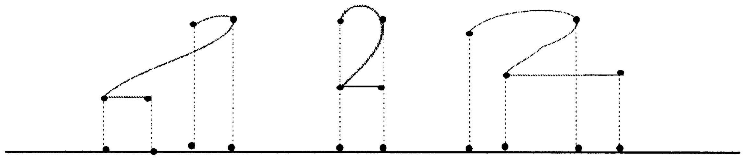

Fig. 9

are equivalent but the matrices associated with them are different:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

For the second pattern in Fig. 9, we could say that it is well-balanced while the first and the third are unbalanced. This possible disorder of parts (arcs) affects the form of the associated matrices.

4. *Invariant matrices*

For a more efficient study, we have to confine our considerations to a still smaller class of patterns, requiring properties which make their structure more regular. Loosely speaking, now we aim to define a subclass of line patterns whose layers, when properly extended, would stretch over the same domain.

Let Φ be a pattern and $\Lambda_1, \ldots, \Lambda_m$ its decomposition into layers. A layer Λ_i is called *connected* if each two of its successive arcs have a common end. In other words, the set $|\Lambda_i|$ is connected and it represents a stretching arc itself. Our first restricting condition will be

(a) *All layers of* Φ *are connected.*

As a consequence of this condition, it follows that *the domain of* Φ (i.e. the projection of $|\Phi|$ onto x-axis, where $|\Phi|$ is the union of all arcs in Φ) is a *connected set* (i.e. an interval). Indeed, when $S = \mathrm{dom}\,(\Phi)$ is disconnected and $S = [a,b] \cup (S \setminus [a,b])$ is a disconnection, let Φ_1 be all arcs in Φ projecting onto $[a,b]$ and Φ_2 those of them projecting onto $S \setminus [a,b]$. The first layer Λ_1' of Φ_1 and the first layer Λ_1'' of Φ_2 make together the first layer Λ_1 of Φ. Being $|\Lambda_1| = |\Lambda_1'| \cup |\Lambda_1''|$ a disconnection, the pattern Φ does not satisfy (a).

Let Λ be a layer of Φ and α be its first (last) stretching arc. The point $End_-(\Lambda) = end_-(\alpha)$ ($End_+(\Lambda) = end_+(\alpha)$) will be called *left (right) end point* of Λ and the set

$$Int(\Lambda) = |\Lambda| \setminus \{End_-(\Lambda), End_+(\Lambda)\}$$

will be called the *interior* of Λ. Our second restricting condition will be:

(b) *For each* i *and each* j, ($i \neq j$) $End_-(\Lambda_i)$ *and* $End_+(\Lambda_i)$ *do not belong to* $Int(\Lambda_j)$.

Our third restricting condition will be:

(c) *For each* i, j, k *in* $\{1, 2, \ldots, m\}$ *such that* $i < j < k$ *if a node* A *belongs to* Λ_i *and* Λ_k, *then it also belongs to* Λ_j.

As an immediate consequence of (c), it follows that when $\Lambda_{i(1)}, \ldots, \Lambda_{i(k)}$, $i(1) < \cdots < i(k)$ are all layers to which a node A belongs, then $i(1), \ldots, i(k)$ are successive integers. A line pattern Φ (without vertical arcs) satisfying the conditions (a), (b) and (c) will be called *even*.

Let us notice that the conditions (a), (b) and (c) express invariant properties of patterns, what means that as soon as a pattern Φ satisfies one of these conditions each equivalent to it pattern Φ' does as well. To be an even pattern is also an invariant property.

For the class of even patterns, together with the concept of height, an invariant meaning of the length can also be given. To do it we need to fix a number of related technical details.

A sequence $\alpha_1, \alpha_2, \ldots, \alpha_s$ of stretching arcs of a pattern Φ, which satisfies the condition $end_+(\alpha_1) = end_-(\alpha_2), \ldots, end_+(\alpha_{s-1}) = end_-(\alpha_s)$

will be called a *stretching chain ending at* $end_+(\alpha_s)$. The number s will be called the length of that chain. If A is a node of Φ, then the maximal length of all chains ending at A will be called the *order of the node* A and will be denoted by $ord(A)$. Now, the *length of a pattern* Φ is the number

$$length(\Phi) = \max\{\, ord(A) \mid A \text{ is a node of } \Phi\,\}.$$

Remarks: (a) *The number of stretching arcs of a layer* Λ *of* Φ *is less or equal to* $length(\Phi)$. (b) *When the nodes of a layer* Λ *of* Φ *are arranged according to the increasing order of their projections to* $dom(\Phi)$*, then their orders form an increasing sequence.* (c) *The concepts* $ord(A)$ *and* $length(\Phi)$ *are invariant, what means that for each two equivalent patterns* Φ *and* Φ'*:* $ord(A) = ord(A')$ *and* $length(\Phi) = length(\Phi')$, (where A and A' are analogous nodes).

For the pattern Φ in Fig. 10, $length(\Phi) = 4$ and the lengths of all layers are equal to 3, while the numbers assigned to the nodes are their orders.

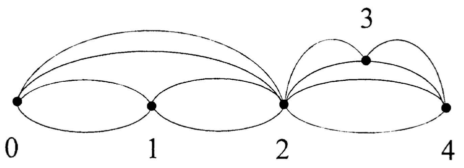

Fig. 10

This example also shows that the orders of nodes of a layer need not be successive integers. Now we describe the way how a matrix is assigned to each equivalence class of even patterns.

Let Φ be an even pattern of height m and length t. Let us define a matrix of the type $m \times (2t+1)$ by corresponding to each layer Λ_i of Φ, the row of the following form

$$a_1^i \quad 1 \quad a_3^i \quad 1 \quad \dots \quad 1 \quad a_{2t+1}^i \ ,$$

where a^i's are defined as it follows:

$$a_1^i = \begin{cases} 1, & \text{when the node } End_-(\Lambda_i) \text{ does not belong to } \Lambda_{i-1} \\ 0, & \text{when the node } End_-(\Lambda_i) \text{ belongs to } \Lambda_{i-1}, \end{cases}$$

$$a^i_{2t+1} = \begin{cases} 1, & \text{when the node } End_+(\Lambda_i) \text{ does not belong to } \Lambda_{i-1} \\ 0, & \text{when the node } End_+(\Lambda_i) \text{ belongs to } \Lambda_{i-1}. \end{cases}$$

Let now $0 < j < t$. When Λ_i has a node A of order j, then

$$a^i_{2j+1} = \begin{cases} 1, & \text{when } A \text{ does not belong to } \Lambda_{i-1} \\ 0, & \text{when } A \text{ belongs to } \Lambda_{i-1}, \end{cases}$$

and when Λ_i does not have a node of order j, then

$$a^i_{2j+1} = 1.$$

It is worth noticing that this definition is given in invariant terms and that two equivalent patterns Φ and Φ' will have the same invariant matrix assigned. Since this assignment is unique for all patterns belonging to the same equivalence class $[\Phi]$, we call this matrix the *invariant matrix* of Φ and we denote it by $M_{[\Phi]}$:

$$M_{[\Phi]} = \begin{bmatrix} a^m_1 & 1 & \dots & 1 & a^m_{2j+1} & 1 & \dots & 1 & a^m_{2t+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a^1_1 & 1 & \dots & 1 & a^1_{2j+1} & 1 & \dots & 1 & a^1_{2t+1} \end{bmatrix}.$$

For example, the invariant matrix assigned to the pattern in Fig. 10 will be

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

As we have seen it, the associated matrices of the three equivalent patterns in Fig. 9 are different. Their height is 3 and length 1 and their invariant matrix is

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The second of those patterns is "good" (well balanced) and its associated matrix coincides with the invariant one. Let us note that a pattern Φ is "poor" when unbalanced with stretching arcs unnecessarily uneven in size and shape. A "good" pattern Φ' is equivalent to Φ, balanced and even in size and shape at least to the degree its structure permits such regularity. The "best" is the arithmetical code of Φ in the form of its invariant matrix.

5. *Invariant matrices are unique arithmetic codifications of equivalent patterns*

Let us notice that both invariant matrices of the patterns in Fig. 11

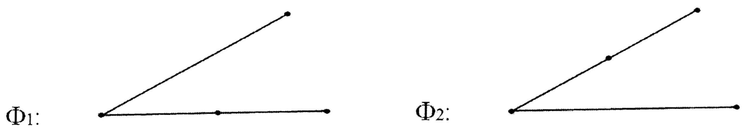

Fig. 11

coincide with

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Both patterns Φ_1 and Φ_2 have a node where only two arcs intersect at right end point of one of them and left end point of the other. When only two arcs α and β intersect in a node of a pattern being right end point of one of them and left end point of the other, then such a node will be called *superfluous*. By replacing α and β with their union $\alpha \cup \beta$, which is again a stretching arc, this superfluous node is removed. A pattern having no superfluous node will be called *canonical*. By removing successfully all superfluous nodes of a pattern, a canonical pattern is obtained. Proceeding further, *we assume that all patterns under consideration are canonical*.

(Let us notice that a pattern may be canonical but when its layers are removed, the remaining subpatterns need not be. This is the reason why this assumption refers to a pattern as a whole, not to its subpatterns).

Now we are going to state some characteristic properties of invariant matrices. Let Φ be a pattern of length t and $\alpha_1, \alpha_2, \ldots, \alpha_t$ the maximal chain in Φ. Then, the order of the node $end_+(\alpha_k) = A$, $(0 < k < t)$ is k. Indeed, when there would exist a chain of length m, $(m > k)$ ending at A, then that chain together with the arcs $\alpha_{k+1}, \ldots, \alpha_t$ would be a chain in Φ of length larger than t, what contradicts our assumption. Hence, *for each* $k \in \{1, \ldots, t-1\}$ *there exists a node of* Φ *of order* k. If A is a node of order k, $(0 < k < t)$ then A is left and right end point of at least two pairs of arcs in Φ. Hence, a subcolumn of the $(2k+1)$-column of the invariant matrix which is corresponded to A has one of the following forms

$$\begin{array}{ccc} & 0 & \\ 0 & 0 & \\ 1, & 1, & \ldots \end{array}$$

dependently on the number of pairs of arcs which intersect at A. Such subcolumns will be called *nodal.* In view of this all, we state:

(a) *Each* $(2k+1)$*-column, (*$0<k<t$*) of the invariant matrix* $M_{[\Phi]}$ *contains at least one nodal subcolumn.*

(b) *As for the first and last columns of an invariant matrix, they uniquely split into nodal subcolumns of the form*

$$\begin{array}{cccc} & & 0 & \\ & 0 & 0 & \\ 1, & 1, & 1, & \ldots \end{array}$$

and its last row consists of 1's.

When A is a node of order k, $(0<k\leq t)$, then there exists a chain of arcs in Φ having the length k and ending at A. In view of this remark, we state

(c) *For each nodal subcolumn of the* $(2k+1)$*-column, (*$0<k\leq t$*), there exists a sequence of elements of* $M_{[\Phi]}$

$$a_1\;1\;a_3,\; a'_3\;1\;a_5,\; \ldots,\; a'_{2k+1}\;1\;a_{2k+1}$$

where a *and* a'*, with the same subscripts, belong to the same nodal subcolumn.*

From (c), it follows that there exists no array of an invariant matrix of the form

$$\begin{array}{cccc} 1 & 1 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{array}$$

where last column of the array is a nodal subcolumn and where no of 1's from the first three columns belongs to a nodal subcolumn of $M_{[\Phi]}$.

As an example, now we consider the Pythagorean pentagram Π, represented in Fig. 12.

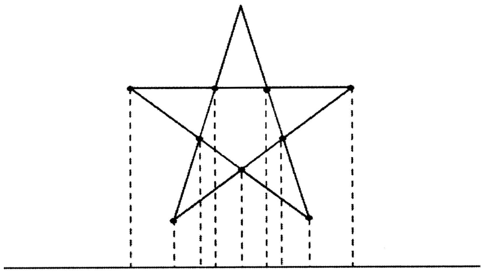

Fig. 12

Its associated matrix is

$$M_\Pi = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

The length of the pattern Π is 5 and its invariant matrix is

$$M_{[\Pi]} = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Up to the equivalence, the invariant matrix determines a pattern Π', with respect to which it will also be its associated matrix. Indeed, let us start with a blue print which consists of six vertical (dotted) lines and one horizontal line being x-axis (Fig. 13). According to the last row of $M_{[\Pi]}$, the first layer Λ_1 of Π' is constructed to be a segment stretching from the first vertical line to the last and its nodal points are marked on vertical lines. Then, two circular arcs are constructed to correspond to the sequences of entries of the third row: 0 1 1 1 0 and 0 1 1 1 1 1 0 and nodal points are marked on each of them, (Fig. 13). Thus, the layer Λ_2 is constructed. Similarly Λ_3 and Λ_4 are constructed, (Fig. 13).

The variant of Pythagorean pentagram represented in Fig. 13 is well-balanced and hence, a "good" one as far as the way of looking from a post at the point in infinity is concerned. Its associated matrix coincides with the invariant one, what we take as a criterion for a pattern to be well-balanced. It is

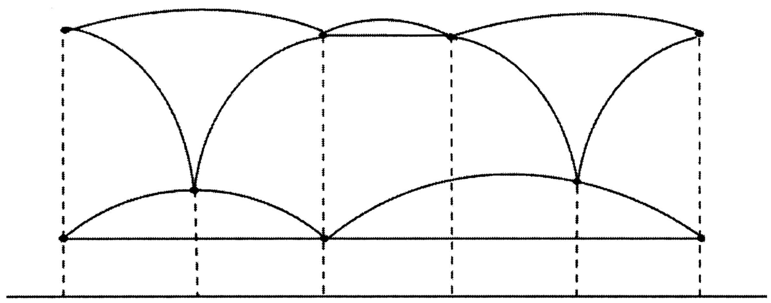

Fig. 13

interesting to notice that the star-shaped variant of the pentagram shown in Fig. 14,

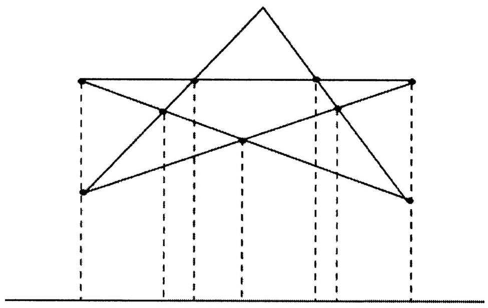

Fig. 14

(though more symmetric than that in Fig. 13), is not again well-balanced. Its associated matrix is the following one

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and evidently different from its invariant matrix.

The usual regular variant of the Pythagorean pentagram (Fig. 12) is even less balanced than the star-shaped pattern in Fig. 14. The regularity of that variant of pentagram is related to the way how evenly its vertices are situated along the circumscribed circle, what has no significance in the case of semi-topological classification. And, let us also add that when we speak of variants of the Pythagorean pentagram, we mean that such patterns are semi-topologically equivalent.

From the fact that invariant matrices are defined in invariant terms, it follows that two equivalent patterns have the same invariant matrix, but the converse of this implication is also true.

Let us notice that sequences of successive entries $a_s^k\,1\,a_{s+1}^k\,\ldots a_r^k$ of rows of an invariant matrix $M_{[\Phi]}$, where a_s^k and a_r^k are the only terms belonging to nodal subcolumns, determine all arcs of Φ. For two such sequences, two corresponding arcs intersect when these sequences have terms at their ends belonging to the same nodal subcolumns. Thereby, an invariant matrix reveals the whole structure of a pattern, determining its arcs and the way of their intersection. Thus, we end this section with the following statement:

Two canonic patterns are semi-topologically equivalent if and only if their invariant matrices are equal.

6. *Transforming an associated matrix into the invariant one*

The associated matrix of a pattern Φ is formed by casting a look at it along vertical lines. The sight of Φ changes only passing through those lines containing nodes, that is in a finite number of cases. Thus, this matrix is formed according to the way a pattern stands, without using any of its global properties. But, as we have seen it, two equivalent patterns may have quite different associated matrices, thereby they are not a very efficient tool for discrimination of patterns. Let us also remark that, when all 1's in each column of an associated matrix are summed up, a sequence of numbers is obtained, called *crossing numbers.* In a time, more than twenty years ago, these numbers were used in pattern recognition but abandoned latter because of their inefficiency. Although the associated matrices are a refinement of crossing numbers, being not invariant, they cannot be efficiently used for recognition, either. This is the reason why we turn our attention to the way how an associated matrix is transformed into the invariant one.

Proceeding further we maintain the assumption that the patterns under consideration are even and canonical. First we prove the following

Proposition 1. *Let A be a left (right) node of a pattern Φ and let Λ_i, $\dots$, Λ_{i+k-1} be all layers which contain A. Let $[0,a]$ be the domain of Φ and let $prEnd_-(\Lambda_i) = t_s > 0$, $(prEnd_+(\Lambda_i) = t_s < a)$. Then, there exists a pattern Φ' equivalent to Φ and having all layers not included in Λ_i, $\dots$, Λ_{i+k-1} identical with the corresponding ones of Φ and $prEnd_-(\Lambda'_i) = 0$, $(prEnd_+(\Lambda'_i) = a)$.*

P r o o f. We will prove this proposition when $prEnd_-(\Lambda_i) = t_s > 0$ and the case $prEnd_+(\Lambda_i) = t_s < a$ is proved analogously. Let $\delta > 0$ be such a number that $[t_s, t_s + \delta]$ contains no projection of nodes of Φ apart from t_s. Let c and d be lower and upper bounds of the set $\{y \mid (x,y) \in |\Phi|\}$ respectively and let m be the height of Φ. Let $y = \alpha(x)$ be the function whose graph is $|\Lambda_{i-1}|$, $(i > 1)$ and $y = \beta(x)$ the function whose graph is $|\Lambda_{i+k}|$, $(i + k \leq m)$ and both of them possibly extended on $[0, t_s + \delta]$ so that the relation $\alpha(x) < \beta(x)$ is preserved for each $x \in [0, t_s + \delta]$. (This is feasible, according to the properties (a), (b) and (c) in Section 4). For $i = 1$ let $\alpha(x) = c$ and for $n + k = m + 1$, let $\beta(x) = d$. Let $\eta = [\alpha(0) + \beta(0)]/2$ and $A' = (0, \eta)$.

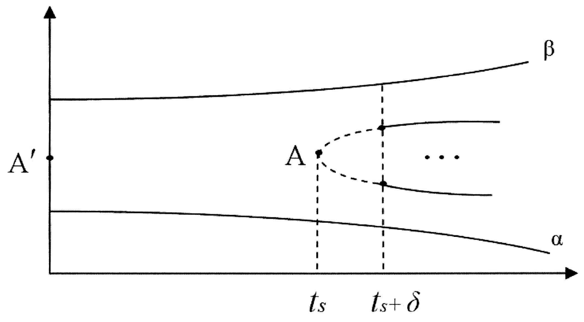

Fig. 15

Let us remove $([0, t_s + \delta] \times \mathbb{R}) \cap (|\Lambda_i| \cup \cdots \cup |\Lambda_{i+k-1}|)$. Then the parts of arcs meeting at A, which are in $[t_s + \delta, a] \times \mathbb{R}$, can be continuously extended through the strip between $y = \alpha(x)$ and $y = \beta(x)$ so that they do not intersect and that they meet at A'. In that way the pattern Φ' is obtained which is equivalent to Φ.

Two matrices M_Φ and $M_{\Phi'}$ have all their entries equal which are out of the following arrays

$$\begin{array}{cccccccccc} 0 & \ldots & 0 & 0 & -(i+k-1)\text{-th row}- & 0 & \ldots & 1 & 1 \\ & \ldots & & & & & \ldots & & \\ 0 & \ldots & 0 & 0 & & 0 & \ldots & 1 & 1 \\ 0 & \ldots & 0 & 1 & -i\text{-th row}- & 1 & \ldots & 1 & 1 \end{array}$$

Let us add that an end nodal column may also be: 1, thereby the arrays of the form

$$0 \quad \ldots \quad 0 \quad 1 \quad -i\text{-th row}- \quad 1 \quad \ldots 1 \quad 1$$

are also included. □

Based on Proposition 1, we define *elementary transformations of the first type*, being the replacements of one array by another one as it is indicated just below

$$\begin{bmatrix} & \ldots & & & \\ 0 & \ldots & 0 & 0 & \\ & \ldots & & & \ldots \\ 0 & \ldots & 0 & 0 & \\ 0 & \ldots & 0 & 1 & \\ & \ldots & & & \end{bmatrix} \xrightarrow{e_1} \begin{bmatrix} & \ldots & & & \\ 0 & \ldots & 1 & 1 & \\ & \ldots & & & \ldots \\ 0 & \ldots & 1 & 1 & \\ 1 & \ldots & 1 & 1 & \\ & \ldots & & & \end{bmatrix}$$

$$\begin{bmatrix} & & & \ldots & \\ & 0 & 0 & \ldots & 0 \\ \ldots & & & \ldots & \\ & 0 & 0 & \ldots & 0 \\ & 1 & 0 & \ldots & 0 \\ & & & \ldots & \end{bmatrix} \xrightarrow{e_1} \begin{bmatrix} & & & \ldots & \\ & 1 & 1 & \ldots & 0 \\ \ldots & & & \ldots & \\ & 1 & 1 & \ldots & 0 \\ & 1 & 1 & \ldots & 1 \\ & & & \ldots & \end{bmatrix}$$

All other entries out of the indicated arrays are equal, (and the case of subcolumns of the form:1, is considered to be also included). A transformation of this type will be called *pulling of the end nodal subcolumns to the end position.*

The next proposition will be the basis for definition of elementary transformations of the second type.

Proposition 2. *Let* $[0, a]$ *be the domain of a pattern* Φ *and let all its left end nodes project onto* 0 *and right ones onto* a*. Let* (t_j) *be the increasing sequence of projections of its other nodes and* t_{j-1}, t_j *two of its terms. Let* A *be a node of* Φ *projecting onto* t_j *and* $\Lambda_i, \ldots, \Lambda_{i+k-1}$ *all layers of* Φ *containing* A*. If none of the layers* $\Lambda_i, \ldots, \Lambda_{i+k-1}$ *has a node projecting onto* t_{j-1}*, then there exists a pattern* Φ' *equivalent to* Φ *and having all layers* *not included in* $\Lambda_i, \ldots, \Lambda_{i+k-1}$ *identical with the corresponding ones of* Φ *and a node* A' *projecting onto* t_{j-1}.

P r o o f. Let $y = \alpha(x)$ and $y = \beta(x)$ be the functions as they have been defined in the proof of the previous proposition. Let $\delta > 0$ be such a number that the interval $[t_{j-1} - \delta, t_j + \delta]$ contains no other point apart from t_{j-1}, t_j. Let $\eta = [\alpha(t_{j-1}) + \beta(t_{j-1})]/2$ and let $A' = (t_{j-1}, \eta)$.

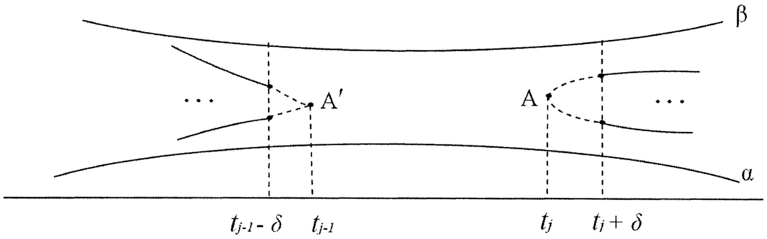

Fig. 16

Removing the set $([t_{j-1} - \delta, t_j + \delta] \times \mathbb{R}) \cap (|\Lambda_i| \cup \cdots \cup |\Lambda_{i+k-1}|)$, the parts of arcs of layers $\Lambda_i, \ldots, \Lambda_{i+k-1}$ in $[t_j + \delta, a] \times \mathbb{R}$, having A as their left end point, can be extended through the strip between $y = \alpha(x)$ and $y = \beta(x)$ so that they do not intersect and meet at the point A'. Similarly the parts of arcs of $\Lambda_i, \ldots, \Lambda_{i+k-1}$ in $[0, t_{j-1} - \delta] \times \mathbb{R}$, having A as their right end point, can be extended through the same strip to meet at the point A', without intersecting each other. As a result of this construction, a pattern Φ' is obtained being equivalent to Φ. $\square$

On the basis of Proposition 2, we define *elementary transformations of the second type* to be the replacing of an array by another one as it is indicated just below

$$\begin{bmatrix} & & \cdots & & \\ & 1 & 1 & 0 & \\ \cdots & & \cdots & & \cdots \\ & 1 & 1 & 0 & \\ & 1 & 1 & 1 & \\ & & \cdots & & \end{bmatrix} \xrightarrow{e_2} \begin{bmatrix} & & \cdots & & \\ & 0 & 1 & 1 & \\ \cdots & & \cdots & & \cdots \\ & 0 & 1 & 1 & \\ & 1 & 1 & 1 & \\ & & \cdots & & \end{bmatrix}$$

We call this type of transformations *pulling of a nodal subcolumn to the left.*

When all possible transformations of the first type are performed, a matrix is obtained having 1's in each of its even columns. Continuing with transformations of the second type, when they all are performed a matrix is obtained having possibly arrays consisted of three successive columns of 1's. Such arrays are superfluous and their replacing by single columns of 1's will be called *contraction*, what is the *elementary operation of the third type.*

$$\begin{bmatrix} & 1 & 1 & 1 & \\ \dots & & \dots & & \dots \\ & 1 & 1 & 1 & \\ & 1 & 1 & 1 & \end{bmatrix} \xrightarrow{e_3} \begin{bmatrix} & 1 & \\ \dots & \dots & \dots \\ & 1 & \\ & 1 & \end{bmatrix}$$

Starting with the associated matrix M_Φ of a pattern Φ, by pulling the end subcolumns to the end position, then by pulling nodal subcolumns to the left and finally, by performing all possible contractions, a matrix M'_Φ is obtained which will be called the *transformed matrix* of Φ. The matrix M_Φ has as its stable, invariant arrays the nodal subcolumns which, under these transformations, change their position but not their form and meaning of being recordings of nodes. The transformed matrix is just a nice arrangement of nodal subcolumns.

Now the content of this section culminates with the following

Theorem 3. *Let Φ be an even pattern and let M'_Φ be its transformed matrix. Then, the matrix M'_Φ coincides with the invariant matrix $M_{[\Phi]}$ of Φ.*

P r o o f. As a result of performed elementary transformations, the matrix M'_Φ has a nodal subcolumn in each of its columns of odd order. Let $1 < j < length(\Phi)$ and let

$$N_j = \begin{matrix} 0 \\ \dots \\ 1 \end{matrix}$$

be a nodal subcolumn of $(2j+1)$-column of M'_Φ which corresponds to the node A. (A possible nodal subcolumn:1 is necessarily in one of the end columns of M'_Φ). Since N_j cannot be pulled to the left, there exists a nodal subcolumn N_{j-1} in $(2j-1)$-column of M'_Φ and thereby, M'_Φ has an array: $a_{j-1}\,1\,a_j$ corresponding to a stretching arc and where $a_{j-1} \in N_{j-1}$, $a_j \in N_j$. Continuing in this way, a sequence

$$a_1\ 1\ a_3\ \dots\ a_{j-1}\ 1\ a_j$$

is obtained showing that Φ has a chain ending at A and that the order of A is j. Comparing the procedures how rows of M'_Φ and $M_{[\Phi]}$ are formed, we see that the corresponding rows of the two matrices are, entry by entry, equal. Hence, $M'_\Phi = M_{[\Phi]}$. □

As an example, let us consider the numeral "8" as it is represented in Fig. 17.

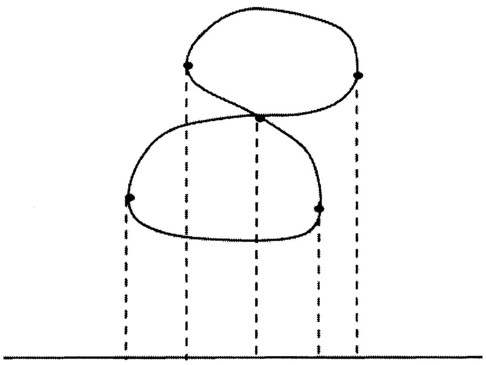

Fig. 17

Its associated matrix is

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

After performing two elementary operations of the first type, it becomes

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Now, performing one elementary operation of the second type, we get the matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

which, after two contractions, becomes

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

On one hand, the last of the above matrices is the invariant matrix of the pattern represented in Fig. 17 and on the other, the associated matrix of the regular numeral "8".

As our next example, we consider an unbalanced copy of the Solomon's seal, represented in Fig. 18.

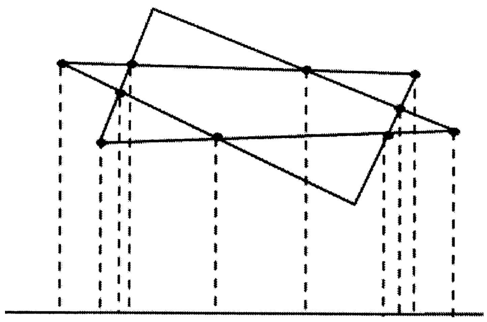

Fig. 18

The associated matrix of this pattern is

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Performing two operations of the first type the matrix becomes

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

After the performance of all elementary operations of the second type, then the following matrix is obtained

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Simplifying this matrix by performing four contractions, the transformed matrix of the pattern in Fig. 18 is obtained

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

which is also the invariant matrix of this pattern. On the other hand, this matrix is also the associated matrix of the well-balanced form of the Solomon's seal (Fig. 19).

As our third example of transforming a associated matrix, let us take the matrix M_Π which corresponds to the regular Pythagorean pentagram Π (Fig. 12). Performing the transformations of the first type, we obtain the matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and then, all transformations of the second type, the matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

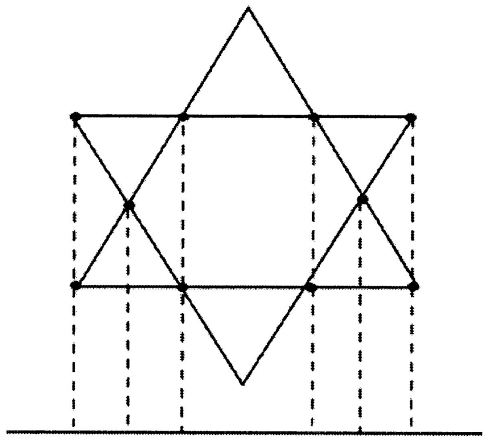

Fig. 19

After three contractions, we get the transformed matrix of Π

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

which coincides with the matrix $M_{[\Pi]}$.

7. *Additional comments*

As it has been already said, classification of line patterns is restricted here to those of them having no vertical arcs. The rationale behind it is the possibility to obtain unique arithmetic codifications in the form of invariant matrices for a subclass of patterns. A definition of equivalent patterns in general case is somewhat more complicated, but since it would be of no effect in the frame of this paper, it was omitted.

The purpose of the consideration of equivalent line patterns is the possibility to establish for them a logically well founded concept of shape and to study its invariant properties. Being ideal representations of shapes, line patterns are well suitable for theoretical considerations as well as they are useful signposts showing how a more realistic recognition process should be directed. Particularly delicate is the recognition of vertical arcs as extremely unstable elements of a shape. To be more specific, let us suppose that a shape has been perceived as consisting of arcs. Then a number as a threshold should be fixed so that each arc having its projection on x-axis less than that number should be taken to be vertical. Such a situation is illustrated in Fig. 20, by two "poor" forms of line patterns.

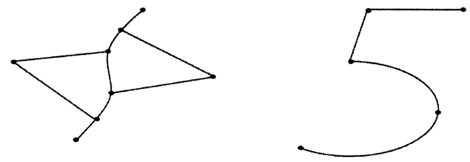

Fig. 20

Then, following this way of perception, these two "poor" forms are recognized as it is represented in Fig. 21.

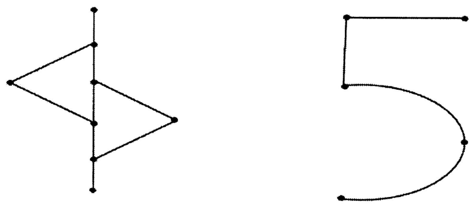

Fig. 21

A dilemma also exists: What is more efficient, to consider patterns having vertical arcs or first to "detect" vertical arcs and then, to consider quotient patterns obtained by collapsing such arcs to a point. For example, the quotient patterns of those in Fig. 21, are illustrated in Fig. 22.

Fig. 22

The second possibility expressed in the above dilemma was also a reason why we have paid more attention to the family of patterns without vertical arcs. But a more decisive factor to resolve this dilemma could only be found in the further study of pattern recognition.

And again, when all 1's in each column of an invariant matrix are summed up, an invariant sequence of crossing numbers is obtained. For example, in the case of patterns in Fig. 23

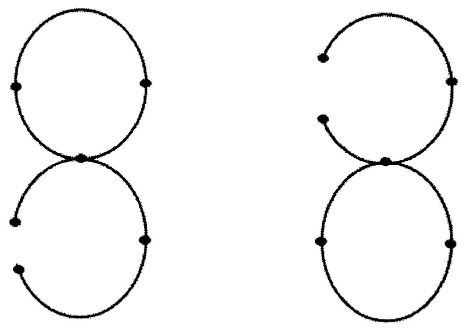

Fig. 23

two sequences coincide with 34342 and still the two patterns are not equivalent. Of course, the invariant matrices of these patterns differ.

Finally, let us remark that the "view" of a pattern depends on the way how a coordinate system is fixed. Rotating the system another "view" of the same pattern is obtained, whereby new characteristics of that shape follow depending on the angle of rotation.

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