Rings of continuous functions vanishing at infinity

A.R. Aliabad, F. Azarpanah, M. Namdari

Abstract. We prove that a Hausdorff space $X$ is locally compact if and only if its topology coincides with the weak topology induced by $C_\infty(X)$. It is shown that for a Hausdorff space $X$, there exists a locally compact Hausdorff space $Y$ such that $C_\infty(X) \cong C_\infty(Y)$. It is also shown that for locally compact spaces $X$ and $Y$, $C_\infty(X) \cong C_\infty(Y)$ if and only if $X \cong Y$. Prime ideals in $C_\infty(X)$ are uniquely represented by a class of prime ideals in $C^*(X)$. $\infty$-compact spaces are introduced and it turns out that a locally compact space $X$ is $\infty$-compact if and only if every prime ideal in $C_\infty(X)$ is fixed. The existence of the smallest $\infty$-compact space in $\beta X$ containing a given space $X$ is proved. Finally some relations between topological properties of the space $X$ and algebraic properties of the ring $C_\infty(X)$ are investigated. For example we have shown that $C_\infty(X)$ is a regular ring if and only if $X$ is an $\infty$-compact $P_\infty$-space.

Keywords: $\sigma$-compact, pseudocompact, $\infty$-compact, $\infty$-compactification, $P_\infty$-space, $P$-point, regular ring, fixed and free ideals

Classification: 54C40

1. Introduction

Throughout this article, the space $X$ stands for a nonempty completely regular Hausdorff space. We denote by $C(X)$ ($C^*(X)$) the ring of all (bounded) real valued continuous functions on the space $X$, ideals are assumed to be proper ideals and the reader is referred to [7] for undefined terms and notations. Kohls in [9] has proved that the intersection of all free maximal ideals in $C^*(X)$ is precisely the set $C_\infty(X)$ consisting of all continuous functions $f$ in $C(X)$ which vanish at infinity, in the sense that $\{ x \in X : |f(x)| \geq \frac{1}{n} \}$ is compact for each $n \in \mathbb{N}$. Kohls has also shown that the set $C_K(X)$ of all functions in $C(X)$ with compact support is the intersection of all the free ideals in $C(X)$ and of all the free ideals in $C^*(X)$. $C_K(X)$ is an ideal of $C(X)$ and it is easy to see that $C_\infty(X)$ is an ideal in $C^*(X)$ but not in $C(X)$, see also [4], [9] and 7D in [7]. In fact $C_\infty(X)$ is a subring of $C(X)$ and topological spaces $X$ for which $C_\infty(X)$ is an ideal of $C(X)$ are characterized in [4]. Our main purpose in this article is the study of the ring structure of $C_\infty(X)$ and of the relations between topological properties of the space $X$ and algebraic properties of the ring $C_\infty(X)$.
This article consists of four sections. In Section 2, we will characterize locally compact spaces \( X \) by the structure of the ring \( C_\infty(X) \). We will see that for studying the ring \( C_\infty(X) \), it suffices to consider the topological space \( X \) to be a locally compact space. It is shown that whenever \( X \) and \( Y \) are locally compact, then \( C_\infty(X) \cong C_\infty(Y) \) if and only if \( X \cong Y \). This part of article is also presented in ICM 2002, see [11]. Section 3 is devoted to the ideal structure of the ring \( C_\infty(X) \) and to a new compactness concept, namely the \( \infty \)-compactness. In this section prime ideals of \( C_\infty(X) \) are investigated and using a special class of prime ideals in \( C^*(X) \), a unique representation for prime ideals of \( C_\infty(X) \) is given. \( \infty \)-compact spaces are those spaces \( X \) for which \( C_K(X) = C_\infty(X) \). We show that for a locally compact space \( X \), every prime ideal in \( C_\infty(X) \) is fixed if and only if \( X \) is an \( \infty \)-compact space. The existence of the smallest \( \infty \)-compact space in \( \beta X \) containing \( X \) is also proved in this section. We denote this smallest \( \infty \)-compact space by \( \infty X \) and we call it the \( \infty \)-compactification of the space \( X \). In the last results of the Section 3, we have characterized the type of points in \( \infty X \setminus X \). We have shown that every point in \( \infty X \setminus X \) is a non-P-point in \( \beta X \). In Section 4, the relations between algebraic properties of \( C_\infty(X) \) and topological properties of the space \( X \) are studied. We have shown that the ring \( C_\infty(X) \) is regular if and only if \( X \) is an \( \infty \)-compact \( P_\infty \)-space (a space \( X \) for which \( Z(f) \) is open for every \( f \in C_\infty(X) \)). We will also observe that the ring \( C_\infty(X) \) has a finite Goldie dimension if an only if the only open locally compact subsets of \( X \) are finite sets. Finally, locally compact spaces \( X \) are characterized for which the ring \( C_\infty(X) \) is a Baer ring or a p.p. ring.

The following proposition and its corollary are proved in [4]. They will be used in the next sections.

**Proposition 1.1.** \( C_\infty(X) \) is an ideal in \( C(X) \) if and only if every open locally compact subset of \( X \) is relatively pseudocompact. (A subset \( U \) of \( X \) is called relatively pseudocompact if \( f(U) \) is bounded for all \( f \in C(X) \)).

**Corollary 1.2.** Let \( X \) be a locally compact Hausdorff space. Then \( C_\infty(X) \) is an ideal in \( C(X) \) if and only if \( X \) is a pseudocompact space.

We also need the following lemma.

**Lemma 1.3.** No point of \( A \subseteq X \) has a compact neighborhood in \( X \) if and only if \( f(A) = \{0\} \) for all \( f \in C_\infty(X) \).

**Proof:** If \( a \in A \) and \( f(a) \neq 0 \) for some \( f \in C_\infty(X) \), then there exists \( n \in \mathbb{N} \) such that \( \frac{1}{n} < |f(a)| \) and hence \( H = \{ x \in X : |f(x)| \geq \frac{1}{n+1} \} \) is a compact neighborhood of \( a \), a contradiction. Now suppose that the point \( a \) has a compact neighborhood \( H \). Then there exists \( f \in C(X) \) such that \( f(a) = 1 \) and \( f(X \setminus \text{int} H) = \{0\} \). Since for every \( n \in \mathbb{N} \) we have \( \{ x \in X : |f(x)| \geq \frac{1}{n} \} \subseteq H \), the closed set \( \{ x \in X : |f(x)| \geq \frac{1}{n} \} \) is compact and hence \( f \in C_\infty(X) \). This proves the converse. \( \square \)
For proof of the following proposition, see Corollary 3.6 in [12].

**Proposition 1.4.** Let $A$ be a commutative algebra over the rationals with unity. Let $I$ be an ideal of $A$. Then an ideal $D$ of $I$ is a maximal ideal of $I$ if and only if $D = M \cap I$ for some maximal ideal $M$ in $A$.

### 2. Characterization of locally compact spaces $X$ by the ring $C_\infty(X)$

We recall that for any topological space $X$, the set of all continuous real valued functions which vanish at infinity is a ring, which is denoted by $C_\infty(X)$. In fact for every $f, g \in C_\infty(X)$, we have \( \{ x \in X : |f(x) + g(x)| \geq \frac{1}{n} \} \subseteq \{ x \in X : |f(x)| \geq \frac{1}{2n} \} \cup \{ x \in X : |g(x)| \geq \frac{1}{2n} \} \) and \( \{ x \in X : |f(x)g(x)| \geq \frac{1}{n} \} \subseteq \{ x \in X : |f(x)| \geq \frac{1}{\sqrt{n}} \} \cup \{ x \in X : |g(x)| \geq \frac{1}{\sqrt{n}} \} \). By the following propositions and corollaries, for studying the ring $C_\infty(X)$, we may consider the space $X$ to be a locally compact space.

**Proposition 2.1.** For a Hausdorff space $X$, the following statements are equivalent:

1. $X$ is locally compact;
2. $\mathfrak{B} = \{ X \setminus Z(f) : f \in C_\infty(X) \}$ is a base for open sets in $X$;
3. the collection $C_\infty(X)$ separates points from closed sets (i.e., whenever $F$ is a closed set in $X$ and $x_0 \notin F$, then there exists $f \in C_\infty(X)$ such that $f(x_0) = 1$ and $f(F) = \{0\}$).

**Proof:** (1)$\rightarrow$(2). Let $G$ be an open set in $X$ and $x_0 \in G$. Then there exists a compact set $H$ such that $x_0 \in \text{int} \, H \subseteq H \subseteq G$. Now define $f \in C(X)$ with $f(x_0) = 1$ and $f(X \setminus \text{int} \, H) = \{0\}$. Since $\{ x \in X : |f(x)| \geq \frac{1}{n} \} \subseteq X \setminus Z(f) \subseteq H$, $\{ x \in X : |f(x)| \geq \frac{1}{n} \}$ is compact, $\forall n \in \mathbb{N}$, i.e., $f \in C_\infty(X)$ and clearly $x_0 \in X \setminus Z(f) \subseteq G$, i.e., $\mathfrak{B}$ is a base for open sets in $X$.

(2)$\rightarrow$(3). Is clear.

(3)$\rightarrow$(1). For every open set $G$ and $x_0 \in G$, there exists $f \in C_\infty(X)$ such that $f(G) = \{0\}$ and $f(x_0) = 1$. Therefore $x_0 \in \{ x \in X : |f(x)| \geq \frac{1}{n} \} \subseteq G$ and by letting $H = \{ x \in X : |f(x)| \geq \frac{1}{2} \}$, $H$ is compact and $x_0 \in \text{int} \, H \subseteq H \subseteq G$ which means that $X$ is locally compact.

**Corollary 2.2.** If $X$ is a Hausdorff space, then $X$ is locally compact if and only if its topology coincides with the weak topology induced by $C_\infty(X)$.

**Proposition 2.3.** For every Hausdorff space $X$, whenever $C_\infty(X) \neq \{0\}$, then there exists a locally compact space $Y$ such that $C_\infty(X) \cong C_\infty(Y)$. In fact the space $Y$ may be considered as a nonempty open locally compact subspace of $X$.

**Proof:** Let $Y$ be the set of all points in $X$ which have a compact neighborhood. Clearly $Y$ is a locally compact open subspace of $X$ and since $C_\infty(X) \neq \{0\}$,
We may also assume that $Y \neq \emptyset$. Then, for otherwise $X$ itself would be a locally compact space. Define $\sigma : C_\infty(X) \to C_\infty(Y)$ by $\sigma(f) = f|_Y$, $\forall f \in C_\infty(X)$. Since by Lemma 1.3, $f(X \setminus Y) = 0$, evidently $\sigma$ is a one to one function. $\sigma$ is also onto, for if $g \in C_\infty(Y)$, then we define $g^* : X \to \mathbb{R}$ such that $g^*(x) = g(x)$, $\forall x \in Y$ and $g^*(x) = 0$, $\forall x \in X \setminus Y$. To see the continuity of $g^*$, it is enough to show that $g^*$ is continuous on the nonempty set $X \setminus Y$. Given $x \in X \setminus Y$ and $\epsilon > 0$, the set $\{x \in Y : |g(x)| \geq \epsilon\}$ is compact in $Y$ and hence in $X$. Therefore $G = X \setminus \{x \in Y : |g(x)| \geq \epsilon\} = \{x \in X : |g^*(x)| < \epsilon\}$ is an open set in $X$ and $g^*(G) \subseteq (-\epsilon, \epsilon)$, i.e., $g^*$ is continuous at $x \in X \setminus Y$. On the other hand, $\{x \in X : |g^*(x)| \geq \frac{1}{n}\} = \{x \in Y : |g(x)| \geq \frac{1}{n}\}$ implies that $g^* \in C_\infty(X)$. Now $\sigma(g^*) = g$, i.e., $\sigma$ is onto. Finally, for every $f, g \in C_\infty(X)$ it is easy to see that $\sigma(f + g) = \sigma(f) + \sigma(g)$ and $\sigma(fg) = \sigma(f)\sigma(g)$, i.e., $C_\infty(X) \cong C_\infty(Y)$.

**Proposition 2.4.** If $X$ is a completely regular Hausdorff space, then every maximal ideal of $C_\infty(X)$ is fixed. In fact every maximal ideal of $C_\infty(X)$ is of the form $M_x \cap C_\infty(X)$, where $M_x$ is a fixed maximal ideal in $C(X)$ and the point $x$ has a compact neighborhood.

**Proof:** Since $C_\infty(X)$ is the intersection of all free maximal ideals in $C^*(X)$, by Proposition 1.4, every maximal ideal in $C_\infty(X)$ is of the form $M^*_p \cap C_\infty(X)$, where $p \in X$ and $C_\infty(X) \nsubseteq M^*_p$. But if $C_\infty(X) \subseteq M^*_p$ for some $p \in X$, then $f(p) = 0$ for all $f \in C_\infty(X)$ and by Lemma 1.3, the point $p$ has no compact neighborhood. Hence if we consider $A$ to be the set of all points of $X$ which have no any compact neighborhood, then the collection of all maximal ideals of $C_\infty(X)$ is $\{M^*_x \cap C_\infty(X) : x \in X \setminus A\}$. On the other hand, $M^*_x = C^*(X) \cap M_x$, for all $x \in X$, see 4.7 in [7]. This implies that every maximal ideal of $C_\infty(X)$ is of the form $M_x \cap C_\infty(X)$, where $x \in X \setminus A$.

By the above proposition, whenever $X$ is locally compact, the only maximal ideals of $C_\infty(X)$ are of the form $M_p \cap C_\infty(X)$, where $p \in X$, i.e., we have a one-to-one correspondence between $X$ and the set $\mathcal{M}$ of all maximal ideals of $C_\infty(X)$. If $\mathcal{M}$ is equipped with the hull-kernel topology, then using this topological space, as in [7, Theorem 4.9], we have the following theorem.

**Theorem 2.5.** Two locally compact spaces $X$ and $Y$ are homeomorphic if and only if $C_\infty(X)$ and $C_\infty(Y)$ are isomorphic.

We conclude this section by the following proposition which is evident by Corollary 2.2 and the fact that every idempotent of $C_\infty(X)$ is in $C_K(X)$. We recall that a topological space $X$ is said to be zero-dimensional if it has a base consisting of open-closed sets. We refer the reader to [6] for more facts about the zero-dimensional spaces.

**Proposition 2.6.** A Hausdorff space $X$ is a locally compact zero-dimensional space if and only if its topology coincides with the weak topology induced by the set of idempotents of $C_\infty(X)$. 
3. Prime ideals of $C_\infty(X)$ and $\infty$-compact spaces

We devote this section to some important ideals related to $C_\infty(X)$. Prime ideals in $C_\infty(X)$, the $z$-ideal $C_\ell(X)$, the ideal $C_K(X)$ and the ideal $C_R(X) = \bigcap_{p \in C(X) \setminus \mathcal{X}} M_p$ are important ideals related to $C_\infty(X)$. First of all we show that $C_\ell(X)$ is the smallest $z$-ideal in $C(X)$ containing $C_\infty(X)$. Next we will characterize topological spaces $X$ for which $C_\infty(X) = C_K(X)$ or $C_\infty(X) = C_R(X)$. Studying the prime ideals of $C_\infty(X)$ and characterization of the type of points in the remainder $\infty X \setminus X$ are the final parts of this section.

We need the following useful lemma which is also proved in [4].

**Lemma 3.1.** Let $A$ be an open subset of $X$. Then $A = X \setminus Z(f)$ for some $f \in C_\infty(X)$ if and only if $A$ is a $\sigma$-compact locally compact subset of $X$.

**Proof:** Let $A = X \setminus Z(f)$ for some $f \in C_\infty(X)$. Then $A = \bigcup_{n=1}^\infty A_n$, where $A_n = \{x \in X : |f(x)| \geq \frac{1}{n}\}$. Since each $A_n$ is compact, $A$ is $\sigma$-compact. If $x \in A$, there exists $n_0 \in \mathbb{N}$ such that $x \in \{y \in X : |f(y)| > \frac{1}{n_0}\} \subseteq A_{n_0}$. Thus we get $A$ is a locally compact subset of $X$ and this proves the necessity. For sufficiency, let $A$ be a $\sigma$-compact locally compact subset of $X$. Then $A = \bigcup_{n=1}^\infty A_n$, where $A_n$ is compact and $A_n \subseteq \text{int} \ A_{n+1}$ for all $n \in \mathbb{N}$, see [6, p.250]. Now for each $n \in \mathbb{N}$, there exists $f_n \in C(X)$ such that $f(X) \subseteq [0,1]$, $f_n(A_n) = \{1\}$ and $f_n(X \setminus \text{int} \ A_{n+1}) = \{0\}$. Then $f = \sum_{n=1}^\infty f_n/2^n$ is an element of $C(X)$ by the Weierstrass M-test. Clearly $A = X \setminus Z(f)$. We claim that $f \in C_\infty(X)$. Let $x_0 \notin A_{n+1}$. Then $f_1(x_0) = \cdots = f_n(x_0) = 0$ and so $f(x_0) \leq \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^n} < \frac{1}{n}$. Therefore $x_0 \notin \{x \in X : |f(x)| \geq \frac{1}{n}\}$, and hence $\{x \in X : |f(x)| \geq \frac{1}{n}\} \subseteq A_{n+1}$ and so we get $f \in C_\infty(X)$. \qed

In fact the collection of all the complement of $\sigma$-compact locally compact subsets of $X$ is a $z$-filter $\mathcal{F}$ in $X$ containing $Z[C_\infty(X)]$. By the next proposition, $Z^{-1}[\mathcal{F}]$ is the smallest $z$-ideal in $C(X)$ containing $C_\infty(X)$.

**Proposition 3.2.** Let

$$C_\ell(X) = \{f \in C(X) : X \setminus Z(f) \text{ is locally compact } \sigma\text{-compact}\}.$$ 

Then $C_\ell(X)$ is the smallest $z$-ideal in $C(X)$ containing $C_\infty(X)$ or $C_\ell(X)$ is all of $C(X)$.

**Proof:** If $g \in C(X)$ and $f \in C_\ell(X)$, then $X \setminus Z(fg) \subseteq X \setminus Z(f)$ and clearly $X \setminus Z(fg)$ is also locally compact $\sigma$-compact, i.e., $fg \in C_\ell(X)$. Since $X \setminus Z(f+g) \subseteq (X \setminus Z(f)) \cup (X \setminus Z(g))$, we have $f+g \in C_\ell(X)$ for every $f, g \in C_\ell(X)$. Hence $C_\ell(X)$ is an ideal in $C(X)$ and it is evident that $C_\ell(X)$ is a $z$-ideal containing $C_\infty(X)$. Now suppose that $I$ is a $z$-ideal in $C(X)$ such that $C_\infty(X) \subseteq I$. If $f \in C_\ell(X)$, then $X \setminus Z(f)$ is locally compact $\sigma$-compact and hence by Lemma 3.1, there exists $g \in C_\infty(X)$ such that $Z(f) = Z(g)$. But
\( g \in C_\infty(X) \subseteq I \) and \( I \) is a \( z \)-ideal, hence \( f \in I \), i.e., \( C_{l\sigma}(X) \subseteq I \). We note that \( C_{l\sigma}(X) = C(X) \) if and only if \( X \) is a locally compact \( \sigma \)-compact space. \( \square \)

We recall that \( C_K(X) = \bigcap_{p \in \beta X \setminus X} O^{*p} = \bigcap_{p \in \beta X \setminus X} O^p \) and \( C_\infty(X) = \bigcap_{p \in \beta X \setminus X} M^{*p} \), see 7E and 7F in [7]. Obviously \( C_K(X) \subseteq C_\infty(X) \) and \( C_K(X) = C_{\infty}(X) \) if and only if every open locally compact \( \sigma \)-compact subset of \( X \) is contained in a compact set in \( X \), see [4, Proposition 2.1]. For convenience, whenever \( C_K(X) = C_\infty(X) \) we call \( X \) an \( \infty \)-compact space. For example, \( \mathbb{N} \) and \( \mathbb{Q} \) are \( \infty \)-compact spaces. Moreover, if we denote \( C_{R}(X) = \bigcap_{p \in \upsilon X \setminus X} M^p \), where \( \upsilon X \) is the realcompactification of \( X \), then \( C_\infty(X) \subseteq C_{l\sigma}(X) \subseteq C_R(X) \). To show the second inclusion, \( C_\infty(X) = \bigcap_{p \in \beta X \setminus X} M^{*p} \) implies that

\[
C_\infty(X)C(X) = \bigcap_{p \in \beta X \setminus X} M^{*p}C(X) \subseteq \bigcap_{p \in \beta X \setminus X} M^{*p}C(X).
\]

Now by parts b and c of 7.9 in [7], \( M^{*p}C(X) = C(X) \), \( \forall p \in \beta X \setminus \upsilon X \) and \( M^{*p}C(X) = M^p \), \( \forall p \in \upsilon X \); hence \( C_\infty(X)C(X) \subseteq \bigcap_{p \in \upsilon X \setminus X} M^p = C_R(X) \). Since \( C_{l\sigma}(X) \) is the smallest \( z \)-ideal containing \( C_\infty(X) \) and \( C_R(X) \) is also a \( z \)-ideal containing \( C_\infty(X) \), we have \( C_{l\sigma}(X) \subseteq C_R(X) \).

The following proposition shows that for a locally compact space \( X \), the equality \( C_\infty(X) = C_R(X) \) is equivalent to pseudocompactness of the space \( X \).

**Proposition 3.3.** For a locally compact space \( X \), \( C_\infty(X) = C_R(X) \) if and only if \( X \) is a pseudocompact space.

**Proof:** If \( X \) is pseudocompact, then \( \upsilon X = \beta X \), see 8A in [7]. Hence

\[
C_\infty(X) = \bigcap_{p \in \beta X \setminus X} M^{*p} = \bigcap_{p \in \upsilon X \setminus X} M^p = \bigcap_{p \in \upsilon X \setminus X} M^p = C_K(X).
\]

Conversely, suppose that \( C_\infty(X) = \bigcap_{p \in \upsilon X \setminus X} M^p \); then \( C_\infty(X) \) is an ideal in \( C(X) \) and hence \( X \) should be a pseudocompact space by Corollary 1.2. \( \square \)

**Proposition 3.4.** Every locally compact \( \infty \)-compact space is a pseudocompact space.

**Proof:** Let \( X \) be a locally compact \( \infty \)-compact space. Then \( C_\infty(X) = C_K(X) \), i.e., \( C_\infty(X) \) is an ideal in \( C(X) \). Now by Corollary 1.2, \( X \) is a pseudocompact space. \( \square \)

**Corollary 3.5.** Every locally compact \( \infty \)-compact and realcompact space is compact.

The converse of the Proposition 3.4 is not true, i.e., not every locally compact pseudocompact space has to be an \( \infty \)-compact space.
Example 3.6. Consider the Tychonoff plank space $T$. $T$ is a locally compact pseudocompact space and the ring $C(T)$ has only one free maximal ideal $M^t$, where $t = (\omega_1, \omega)$ and $M^t \neq O^t$, see 8.20 in [7]. Now since $T$ is pseudocompact, $M^{*t} = M^t$ and $C_\infty(X) = M^{*t} \neq O^t = C_K(X)$, i.e., $T$ is not $\infty$-compact.

Next we are going to characterize prime ideals of the subring $C_\infty(X)$ via prime ideals of $C^*(X)$. By Spec$(C_\infty(X))$, we mean the set of all prime ideals of the ring $C_\infty(X)$. For details of spectrum for general rings, see [8]. The spectrum of $C_\infty(X)$ might be empty only whenever $C_\infty(X) = (0)$.

Proposition 3.7. For every completely regular Hausdorff space $X$, we have

$$\text{Spec}(C_\infty(X)) = \{ P^* \cap C_\infty(X) : P^* \text{ is a prime ideal in } C^*(X) \}
\text{ and } C_\infty(X) \notin P^* \}.$$

We have $C_\infty(X) \neq (0)$ if and only if Spec$(C_\infty(X)) \neq \emptyset$.

Proof: For every prime ideal $P^*$ in $C^*(X)$ with $C_\infty(X) \notin P^*$, clearly $P^* \cap C_\infty(X)$ is a prime ideal in $C_\infty(X)$. Conversely, let $P_\infty$ be a prime ideal in $C_\infty(X)$. Then $P_\infty$ is an ideal in $C^*(X)$, for if $f \in P_\infty$ and $g \in C^*(X)$, then $fg = f^{1/3}f^{2/3}g$ and $f^{2/3}g \in C_\infty(X)$, $f^{1/3} \in P_\infty$ imply that $fg \in P_\infty$. Now suppose that $P^*$ is a prime ideal in $C^*(X)$ minimal over $P_\infty$ and disjoint from the multiplicatively closed set $C_\infty(X) - P_\infty$. It goes without saying that $P_\infty = P^* \cap C_\infty(X)$. To prove the second part of the proposition, suppose that $C_\infty(X) \neq (0)$. Then by Proposition 2.3, there exists a nonempty locally compact space $Y$ such that $C_\infty(X) \cong C_\infty(Y)$. Hence it is enough to show that Spec$(C_\infty(Y)) \neq \emptyset$. If $Y$ is compact, then $C_\infty(X) = C^*(X)$ and clearly Spec$(C_\infty(X)) \neq \emptyset$. Thus suppose that $Y$ is not compact. Since $Y$ is locally compact and noncompact, then by 4D in [7], $C_K(Y)$ is free and hence no fixed prime ideal of $C^*(Y)$ contains $C_\infty(Y)$. On the other hand, since $C_\infty(Y)$ is a free ideal of $C^*(X)$, by Theorem 3.1 in [2], $C_\infty(Y)$ intersects every nonzero ideal in $C^*(X)$ nontrivially. Therefore if $P^*$ is a fixed prime ideal in $C^*(Y)$, we have $C_\infty(Y) \notin P^*$ and $P^* \cap C_\infty(Y) \neq (0)$ which means that Spec$(C_\infty(Y))$ contains at least a nonzero prime ideal. The converse is evident, for $C_\infty(X) = (0)$ implies that Spec$(C_\infty(X)) = \emptyset$.

To establish a one-to-one correspondence between prime ideals of $C_\infty(X)$ and a subclass of prime ideals of $C^*(X)$, we need the following lemma which will also be used in Section 4.

Lemma 3.8. Let $I$ be an ideal in a commutative ring $R$. Suppose that $Q$ and $P$ are ideals in $R$ and $P$ is prime. If $P$ does not contain $I$ and $Q \cap I \subseteq P \cap I$, then $Q \subseteq P$. In particular, if $Q$ is also a prime ideal and $Q \cap I = P \cap I$, then $P = Q$.

Proof: $Q \cap I \subseteq P \cap I$ implies that $Q \cap I \subseteq P$. Since $P$ is prime and $I \notin P$, we have $Q \subseteq P$. □
The following proposition shows that every prime ideal $P_\infty$ of $C_\infty(X)$ has a unique representation of the form $P_\infty = P^* \cap C_\infty(X)$, where $P^*$ is a prime ideal in $C^*(X)$.

**Proposition 3.9.** Let $D$ be the collection of all prime ideals of $C^*(X)$ which do not contain $C_\infty(X)$. Then $\Phi : D \rightarrow \text{Spec}(C_\infty(X))$ defined by $\Phi(P^*) = P^* \cap C_\infty(X)$ is a one-to-one correspondence.

**Proof:** Using Proposition 3.7 and Lemma 3.8 the proof is evident. \qed

If $X$ has no point with compact neighborhood, then $C_\infty(X) = \{0\}$ is contained in every ideal of $C^*(X)$. Even if the space $X$ is locally compact, many prime ideals of $C^*(X)$ may contain $C_\infty(X)$. In the following proposition, we show that whenever $X$ is a locally compact $\infty$-compact space, then all free prime ideals of $C^*(X)$ contain $C_\infty(X)$.

**Proposition 3.10.** A locally compact Hausdorff space $X$ is $\infty$-compact if and only if every prime ideal in $C_\infty(X)$ is fixed.

**Proof:** Let $X$ be an $\infty$-compact space and $P_\infty$ be a prime ideal in $C_\infty(X)$. By Proposition 3.7, there exists a prime ideal $P^*$ in $C^*(X)$ such that $P_\infty = P^* \cap C_\infty(X)$, where $C_\infty(X) \not\subseteq P^*$. $P^*$ is not free, for otherwise $C_\infty(X) = C_K(X) \subseteq P^*$, by $\infty$-compactness of $X$ and 4D in [7], a contradiction. Hence $P^*$ is fixed and therefore $P_\infty$ is fixed too. Conversely suppose that every prime ideal in $C_\infty(X)$ is fixed but $X$ is not $\infty$-compact, i.e., $C_\infty(X) \not= C_K(X)$. Hence there exists $f \in C_\infty(X)$ such that $f \notin C_K(X)$. Now consider the prime ideal $P^*$ in $C^*(X)$ containing $C_K(X)$ but not $f$. Since $X$ is locally compact, then by 4D in [7], $C_K(X)$ is free, so $P^*$ is free. Since $C_\infty(X) \not\subseteq P^*$, $P_\infty = P^* \cap C_\infty(X)$ is a prime ideal in $C_\infty(X)$ by Proposition 3.7. Now $C_K(X) \subseteq P^* \cap C_\infty(X) = P_\infty$ implies that $P_\infty$ is also free which contradicts our hypothesis. \qed

**Remark 3.11.** $C_\infty(X)$ may be contained in no prime ideal of $C(X)$. In fact this happens if and only if $X$ is a locally compact $\sigma$-compact space. To see this, let $P$ be a prime ideal in $C(X)$ such that $C_\infty(X) \subseteq P$. Thus there exists a maximal ideal $M$ in $C(X)$ such that $C_\infty(X) \subseteq M$. Since $C_\sigma(X)$ is the smallest $z$-ideal containing $C_\infty(X)$, $C_\sigma(X) \subseteq M$ by Proposition 3.2, which implies that $C_\sigma(X)$ is an ideal in $C(X)$. By definition of the ideal $C_\sigma(X)$, this shows that $X$ is not locally compact or $X$ is not $\sigma$-compact. Conversely, suppose that $X$ is either not locally compact or not $\sigma$-compact. Then $C_\sigma(X)$ is an ideal of $C(X)$. Now $C_\sigma(X)$ is contained in a maximal ideal of $C(X)$. Clearly, that maximal ideal which is also a prime ideal in $C(X)$ contains $C_\infty(X)$.

$C_\infty(X)$ may contain a prime ideal of $C^*(X)$. If $P^*$ is a prime ideal in $C^*(X)$ and $P^* \subseteq C_\infty(X)$, then $P^* \subseteq \bigcap_{x \in \beta X \setminus X} M^{*x}$ and since every prime ideal in $C^*(X)$ is contained in a unique maximal ideal in $C^*(X)$, $C_\infty(X) = M^{*x}$, where $\beta X \setminus X = \{x\}$. This shows that $C_\infty(X)$ contains a prime ideal of $C^*(X)$ if and
only if the cardinal number of the remainder $\beta X \setminus X$ is 1. In this case $C_\infty(X)$ itself is a maximal ideal in $C^*(X)$.

It is time to show the existence of the smallest $\infty$-compact space in $\beta X$ containing the space $X$. To avoid the confusion, we denote the ideals $M^p$ and $O^p$ in $C(X)$ by $M^p(X)$ and $O^p(X)$, respectively. The corresponding ideals in $C^*(X)$ are also denoted by $M^{*p}(X)$ and $O^{*p}(X)$.

**Theorem 3.12.** Let $\{Y_\alpha\}_{\alpha \in S}$ be a collection of $\infty$-compact spaces such that $X \subseteq Y_\alpha \subseteq \beta X$, $\forall \alpha \in S$. Then $Y = \bigcap_{\alpha \in S} Y_\alpha$ is also an $\infty$-compact space.

**Proof:** First suppose that $X \subseteq T \subseteq \beta X$ and define the map $\varphi : C^*(X) \to C^*(T)$ by $\varphi(f) = f^\beta|_T$ (denote $f^\beta|_T$ by $f^T$). It is clear that $\varphi$ is an isomorphism. Moreover, for every $p \in \beta X$, we have $\varphi(O^p(X)) = O^p(T)$ and $\varphi(M^{*p}(X)) = M^{*p}(T)$. To see this let $\varphi(f) \in \varphi(O^p(X))$, where $f \in O^p(X)$. Then $p \in \text{int}_{\beta X} Z(f^\beta) = \text{int}_{\beta X} Z(f^T)$ and hence $f^T \in O^p(T)$ implies that $\varphi(O^p(X)) \subseteq O^p(T)$. Since $\varphi$ is an isomorphism, similarly $\varphi^{-1}(O^p(T)) \subseteq O^p(X)$ and hence $\varphi(O^p(X)) = O^p(T)$. The proof of $\varphi(M^{*p}(X)) = M^{*p}(T)$ is similar. More generally, whenever $A \subseteq \beta X$ we have also $\varphi(O^{*A}(X)) = O^{*A}(T)$ and $\varphi(M^{*A}(X)) = M^{*A}(T)$.

Now for every $\alpha \in S$, let $\varphi_\alpha : C^*(X) \to C^*(Y_\alpha)$ be an isomorphism defined by $\varphi_\alpha(f) = f^{Y_\alpha}$, $\forall f \in C^*(Y)$. By the above argument we have

$$C_K(Y) = O^{*\beta Y \setminus Y}(Y) = O^{*\beta Y \setminus Y_\alpha}(Y) = O^*\bigcup_{\alpha \in S} (\beta Y \setminus Y_\alpha)(Y) = \bigcap_{\alpha \in S} O^{*\beta Y \setminus Y_\alpha}(Y)$$

$$= \bigcap_{\alpha \in S} \varphi_\alpha^{-1}(O^{*\beta Y \setminus Y_\alpha}(Y_\alpha)) = \bigcap_{\alpha \in S} \varphi_\alpha^{-1}(C_K(Y_\alpha)) = \bigcap_{\alpha \in S} \varphi_\alpha^{-1}(C_\infty(Y_\alpha))$$

$$= \bigcap_{\alpha \in S} \varphi_\alpha^{-1}(M^{*\beta Y \setminus Y_\alpha}(Y_\alpha)) = \bigcap_{\alpha \in S} M^{*\beta Y \setminus Y_\alpha}(Y_\alpha) = M^{*\bigcup_{\beta Y \setminus Y_\alpha}(Y_\alpha)}(Y)$$

$$= M^{*\beta Y \setminus Y_\alpha}(Y) = C_\infty(Y).$$

**Corollary 3.13.** For every completely regular Hausdorff space $X$, there is an smallest $\infty$-compact space in $\beta X$ containing $X$.

**Proof:** By Theorem 3.12, this smallest $\infty$-compact space is the intersection of all $\infty$-compact spaces in $\beta X$ containing $X$. 

We conclude this section by the following lemmas and proposition which characterize the type of points in $\infty X \setminus X$. First we note that, if $X \subseteq Y \subseteq \beta X$, then a point $p \in \beta X$ is said to be a $P$-point with respect to $Y$ if $O^p(Y) = M^p(Y)$. In case $Y = X$, we apply $O^p = M^p$ instead of $O^p(X) = M^p(X)$ and briefly we say that $p$ is a $P$-point.
Lemma 3.14. Suppose that $p \in \beta X$ and $X \subseteq Y \subseteq \beta X$. Then for every $f \in C^*(X)$, $f \in Op(X)$ if and only if $f^Y \in Op(Y)$.

PROOF: We consider $\varphi_Y : C^*(X) \rightarrow C^*(Y)$ defined by $\varphi_Y(f) = f^Y$, $\forall f \in C^*(X)$. As was pointed out in the proof of Theorem 3.12, $\varphi_Y(M^{*p}(X)) = M^{*p}(Y)$ and $\varphi_Y(O^{*p}(X)) = O^{*p}(Y)$. Hence for every $f \in C^*(X)$, $\varphi_Y(f) = f^Y \in Op(Y) \cap C^*(Y) = O^{*p}(Y)$ if and only if $f \in \varphi_Y^{-1}(O^{*p}(Y)) = O^{*p}(X)$ which is equivalent to $f \in Op(X)$. □

Lemma 3.15. Suppose that $p \in \beta X$ and $X \subseteq Y \subseteq \beta X$. If $p$ is a P-point with respect to $Y$, then it is also a P-point with respect to $X$.

PROOF: We suppose that $f \in M^p(X)$ and consider $g = \frac{f^2}{1+f^2}$. Hence $Z(f) = Z(g)$ and therefore $g \in M^p(X) \cap C^*(X)$. Thus $p \in cl_{\beta X} Z(f) = cl_{\beta X} Z(g) \subseteq cl_{\beta X}(Z(g^Y))$ implies that $g^Y \in M^p(Y) = Op(Y)$ and by Lemma 3.14, $g \in Op(X)$. Hence $f \in Op(X)$, i.e., $p$ is a P-point with respect to $X$. □

Proposition 3.16. If $p_0 \in \infty X \setminus X$, then $p_0$ is a non-P-point with respect to $\infty X$ and hence it is a non-P-point with respect to $\beta X$.

PROOF: We put $Y = \infty X$ and $T = Y \setminus \{p_0\}$. Thus $T$ is not $\infty$-compact and therefore there exists $f \in C_\infty(T) - C_K(T)$. For every $p \in \beta Y \setminus Y = \beta X \setminus \infty X \subseteq \beta X \setminus T = \beta T \setminus T$ we have $f^\beta(p) = 0$. However, if we let $g = f^Y$, then $g^\beta(p) = f^\beta(p) = 0$, $\forall p \in \beta Y \setminus Y$ and hence $g \in C_\infty(Y)$ implies that $g \in C_K(Y)$. Therefore $p \in int_{\beta X} Z(g^\beta) = int_{\beta X} Z(f^\beta)$, $\forall p \in \beta Y \setminus Y$ and hence $f \in O^{*p}(T)$, $\forall p \in (\beta T \setminus T) \setminus \{p_0\}$. Now $f \notin O^{*p_0}(T)$ since $f \notin C_K(T)$, and by Lemma 3.14, $g = f^Y \notin Op_0(Y)$. But $g(p_0) = f^\beta(p_0) = 0$ and hence $g \in M^{p_0}(Y)$, i.e., $p_0$ is not a P-point with respect to $Y$. Finally, by Lemma 3.15, $p_0$ is not also a P-point with respect to $\beta X$. □

Corollary 3.17. If for a topological space $X$, we put

$$\Pi = \{p \in \beta X \setminus X : p \; \text{is a P-point in} \; \beta X\}$$

then $\infty X \subseteq \beta X \setminus \Pi$. Moreover if $\beta X \setminus \Pi \subseteq Y \subseteq \beta X$, then $Y$ is an $\infty$-compact space containing $\infty X$.

4. Relations between algebraic properties of $C_\infty(X)$ and topological properties of $X$

In this section we present topological characterizations of some algebraic properties of the ring $C_\infty(X)$. We will characterize topological spaces $X$ for which the ring $C_\infty(X)$ is a regular ring, has a finite Goldie dimension, a p.p. ring and a Baer ring. First of all we consider $C_\infty(X)$ to be a regular ring. A ring $R$ is called regular if for every $a \in R$, there exists $b \in R$ with $a = a^2b$. A completely
regular Hausdorff space $X$ is said to be a P-space if every $G_\delta$-set (zero-set) in $X$ is an open set. It is well-known that $C(X)$ is a regular ring if and only if $X$ is a P-space, see Theorem 14.29 and 4J in [7]. Whenever $Z(f)$ is open for every $f \in C_\infty(X)$, we call $X$ a P$_\infty$-space. The following theorem shows that $C_\infty(X)$ is a regular ring if and only if $X$ is an $\infty$-compact P$_\infty$-space.

**Theorem 4.1.** The following statements are equivalent:

1. $C_\infty(X)$ is a regular ring;
2. every open locally compact $\sigma$-compact set in $X$ is compact;
3. $\forall f \in C_\infty(X), X \setminus Z(f)$ is compact;
4. $X$ is an $\infty$-compact P$_\infty$-space;
5. $\forall p \in X, M_p \cap C_\infty(X) = O_p \cap C_K(X)$.

**Proof:** (1)→(2). By Lemma 3.1, every open locally compact $\sigma$-compact set is of the form $X \setminus Z(f)$ for some $f \in C_\infty(X)$. Since $C_\infty(X)$ is regular, there exists $g \in C_\infty(X)$ such that $f^2g = f$. Now $f(fg - 1) = 0$ implies that $\{x : (fg)(x) \neq 1\} = Z(f)$, i.e., $Z(f)$ is open. On the other hand, $g(x) = \frac{1}{f(x)}$ for every $x \in X \setminus Z(f)$ and hence $g(x) \geq \frac{1}{N}$, where $N$ is an upper bound for $|f|$ (note that every member of $C_\infty(X)$ is bounded). Therefore

$$X \setminus Z(f) \subseteq \{x \in X : |g(x)| \geq \frac{1}{N}\} = A_N.$$

Since $X \setminus Z(f)$ is closed and $A_N$ is compact, $X \setminus Z(f)$ is also compact.

(2)→(3)→(4)→(5). Evident.

(5)→(1). (5) implies that for every $f \in C_\infty(X)$, $Z(f)$ is open and $X \setminus Z(f)$ is compact. Now for every $f \in C_\infty(X)$, we define $g(x) = 0$ for $x \in Z(f)$ and $g(x) = \frac{1}{f(x)}$ for $x \in X \setminus Z(f)$. By pasting lemma, $g \in C(X)$ and $\{x \in X : |g(x)| \geq \frac{1}{N}\} \subseteq X \setminus Z(f)$ implies that $\{x \in X : |g(x)| \geq \frac{1}{N}\}$ is compact, i.e., $g \in C_\infty(X)$ and $f^2g = f$ means that $C_\infty(X)$ is regular. □

**Remark 4.2.** Clearly every P-space is a P$_\infty$-space but every P$_\infty$-space is not necessarily a P-space. For example let $S$ be a P-space and consider the space $X$, the free union of spaces $S$ and $\mathbb{Q}$ (Q with usual topology). By Lemma 1.3, for every $f \in C_\infty(X)$, we have $f(\mathbb{Q}) = 0$ and since $S$ is a P-space, $Z(f)$ is open $\forall f \in C_\infty(X)$, i.e., $X$ is a P$_\infty$-space. But $\mathbb{Q}$ is not a P-space and hence $X$ is not a P-space either.

**Proposition 4.3.** Let $X$ be a locally compact Hausdorff space. If $X$ is a P$_\infty$-space, then it is also a P-space.

**Proof:** If $X$ is a P$_\infty$-space, then $M^*_x \cap C_\infty(X) = O^*_x \cap C_\infty(X)$, $\forall x \in X$. Since $M^*_x$ is prime in $C^*(X)$, then by Lemma 3.8, either $M^*_x = O^*_x$ or $C_\infty(X) \subseteq O^*_x$. But $C_\infty(X) \subseteq O^*_x$ does not happen, for if $K$ and $H$ are compact neighborhoods of
A.R. Aliabad, F. Azarpanah, M. Namdari

Proof: If it is a P-space by Proposition 4.3. Now according to Proposition 3.4, \( X \) is a regular ring, then by Theorem 4.1, \( X \) is pseudocompact P-space which should be finite by 4K in [7]. □

Corollary 4.4. Let \( X \) be a locally compact Hausdorff space. Then \( C_\infty(X) \) is a regular ring if and only if \( X \) is finite.

Proof: If \( X \) is finite, then clearly \( C_\infty(X) \) is a regular ring. Conversely, if \( C_\infty(X) \) is a regular ring, then by Theorem 4.1, \( X \) is an \( \infty \)-compact \( P_\infty \)-space and hence it is a P-space by Proposition 4.3. Now according to Proposition 3.4, \( X \) is a pseudocompact P-space which should be finite by 4K in [7]. □

Next we characterize spaces \( X \) for which the ring \( C_\infty(X) \) has a finite Goldie dimension. Before doing this, we need to characterize uniform ideals and essential ideals in \( C_\infty(X) \). A nonzero ideal \( I \) in a commutative ring \( R \) is called essential if it intersects every nonzero ideal nontrivially, and it is called uniform if any two nonzero ideals contained in \( I \) intersect nontrivially. In [2, Proposition 1.1], it is shown that the ideal \( I \) in \( C(X) \) is uniform if and only if it is minimal, i.e., \( I \) is generated by an idempotent \( e \in C(X) \) such that \( X \setminus Z(e) \) is singleton. In [2, Proposition 3.1], it is also shown that an ideal \( E \) in \( C(X) \) is essential if and only if \( \text{int}_X \cap Z[E] = \emptyset \), i.e., \( \cap Z[E] \) is nowhere dense. By the following proposition, analogous criteria hold for essential ideals and uniform ideals in \( C_\infty(X) \). First we need the following lemma.

Lemma 4.5. Let \( f, g \in C_\infty(X) \).

(a) If there exists \( n_0 \in \mathbb{N} \) such that \( \{ x \in X : |g(x)| < \frac{1}{n_0} \} \subseteq Z(f) \), then \( f \) is a multiple of \( g \) in \( C_\infty(X) \).

(b) If \( |f| \leq |g|^r \) for some \( r > 1 \), then \( f \) is a multiple of \( g \) in \( C_\infty(X) \).

Proof: (a) We define \( h(x) = f(x)/g(x) \) for \( |g(x)| \geq \frac{1}{2n_0} \) and \( h(x) = 0 \) for \( |g(x)| \leq \frac{1}{2n_0} \). Clearly \( h \in C(X) \) and \( f \equiv gh \). But for every \( n \in \mathbb{N} \), we have

\[
\{ x \in X : |h(x)| \geq \frac{1}{n} \} \subseteq \{ x \in X : |f(x)| \geq \frac{1}{2n_0n} \}
\]

which implies that \( \{ x \in X : |h(x)| \geq \frac{1}{n} \} \) is compact for any \( n \in \mathbb{N} \), i.e., \( h \in C_\infty(X) \).

(b) By problem 1D in [7], there exists \( h \in C(X) \) such that \( f = gh \). Now \( |gh| \leq |g|^r \) implies that \( \{ x \in X : |h(x)| \geq \frac{1}{n} \} \subseteq \{ x \in X : |g(x)|^{r-1} \geq \frac{1}{n} \} \) and hence \( h \in C_\infty(X) \). □

Proposition 4.6. (a) An ideal \( E \) in \( C_\infty(X) \) is essential if and only if \( \cap Z[E] \) is nowhere compact (i.e., \( \cap Z[E] \) does not contain any nonempty compact neighborhood).

(b) An ideal \( I \) in \( C_\infty(X) \) is uniform if and only if \( I = (f) \) for some \( f \in C_\infty(X) \), where \( X \setminus Z(f) \) is a singleton.
Proof: (a) Suppose $E$ is an essential ideal in $C_\infty(X)$ and $B = \bigcap Z[E]$ is not nowhere compact. Then there exists a compact set $A$ with $A \subseteq B$ and $\text{int } A \neq \emptyset$. Let $a \in \text{int } A$ and define $f \in C(X)$ such that $f(X \setminus \text{int } A) = \{0\}$ and $f(a) = 1$. Hence $\{x \in X : |f(x)| \geq \frac{1}{n}\} \subseteq A$ implies that $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is compact, i.e., $f \in C_\infty(X)$. Now if there exists $g \in C_\infty(X)$ such that $g \in (f) \cap E$, then $Z(f) \subseteq Z(g)$ implies that $X \setminus Z(g) \subseteq X \setminus Z(f) \subseteq A \subseteq B \subseteq Z(g)$ and hence $g = 0$ which contradicts the essentiality of $E$ in $C_\infty(X)$. Conversely, let $\bigcap Z[E]$ be nowhere compact, $0 \neq f \in C_\infty(X)$ and $a \in X \setminus Z(f)$. Then there exists $n \in \mathbb{N}$ such that $|f(a)| \geq \frac{1}{n}$ and hence $a$ is in the compact set $\{x \in X : |f(x)| \geq \frac{1}{n}\}$. Since $\bigcap Z[E]$ is nowhere compact, there exists $b \in \{x \in X : |f(x)| \geq \frac{1}{n}\} \setminus \bigcap Z[E]$ which implies that there is $g \in E$, such that $g(b) \neq 0$ and hence $0 \neq fg \in (f) \cap E$, i.e., $E$ is essential in $C_\infty(X)$.

(b) Let $I$ be a uniform ideal in $C_\infty(X)$ and $f \in I$. First we show that $X \setminus Z(f)$ is a singleton. Suppose that $x_0, y_0 \in X \setminus Z(f)$ and $x_0 \neq y_0$. By Lemma 3.1, $X \setminus Z(f)$ is a locally compact subspace of $X$ and hence there exist two disjoint compact neighborhoods $G$ and $H$ in $X \setminus Z(f)$ of points $x_0$ and $y_0$ respectively. Since $X \setminus Z(f)$ is open in $X$, $G$ and $H$ are also compact neighborhoods in $X$. Now we define two functions $g, h \in C(X)$ such that $g(x_0) = 1 = h(y_0)$ and $g(X \setminus \text{int } G) = \{0\} = h(X \setminus \text{int } H)$. Since $\{x \in X : |g(x)| \geq \frac{1}{n}\} \subseteq G$ and $G$ is compact, $\{x \in X : |g(x)| \geq \frac{1}{n}\}$ is also compact, i.e., $g \in C_\infty(X)$. Similarly, $h \in C_\infty(X)$. Now consider the principal subideals $(fg)$ and $(fh)$ of $I$. Since $I$ is a uniform ideal, there exists $0 \neq k \in (fg) \cap (fh)$ and hence there exists $z \in X \setminus Z(g)$ with $k(z) \neq 0$. Now $kg = 0$ contradicts $k(z)g(z) \neq 0$ and therefore $X \setminus Z(f)$ is a singleton, say $X \setminus Z(f) = \{x_0\}$. Next we show that for every $g \in I$, we have also $X \setminus Z(g) = \{x_0\}$. Let $X \setminus Z(g) = \{y_0\}$ and $y_0 \neq x_0$. For the principal subideals $(f)$ and $(g)$ of $I$, we have $(f) \cap (g) = \{0\}$, for if $h \in (f) \cap (g)$, then $Z(f) \cup Z(g) = X \setminus Z(h)$ implies that $h = 0$. This contradicts the uniformity of $I$ and hence $X \setminus Z(g) = \{x_0\}$. Therefore we have shown that there exists an isolated point $x_0 \in X$ such that $X \setminus Z(f) = \{x_0\}$, $\forall f \in I$. Finally, suppose that $f, g \in I$ and $f'(0) = \alpha$. Then there exists $n \in \mathbb{N}$ such that $|\alpha| \geq \frac{1}{n}$ and hence $\{x \in X : |f(x)| < \frac{1}{n}\} \subseteq Z(g)$ which implies that $g$ is a multiple of $f$ by Lemma 4.5. This shows that $I = (f)$. The converse is evident. \qed

It is well-known that if a ring $R$ has a finite Goldie dimension, then there exists an integer $n > 0$ such that any direct sum of nonzero ideals in $R$ has always $m$ terms, where $m \leq n$ and there is a direct sum of uniform ideals with $n$ terms which is essential in $R$, see [8] and [10].

Proposition 4.7. $C_\infty(X)$ has a finite Goldie dimension if and only if every open locally compact set in $X$ is finite.

Proof: If $C_\infty(X) = (0)$, then every locally compact set in $X$ is empty. Now suppose that $C_\infty(X) \neq (0)$ has a finite Goldie dimension and let $G$ be a locally
compact open set in $X$. Hence there exists $n > 0$ such that the direct sum of $n$ uniform ideals $I_1, I_2, \ldots, I_n$ in $C_\infty(X)$ is an essential ideal $E$ in $C_\infty(X)$. By Proposition 4.6, there is an isolated point $x_i \in X$ and $f_i \in I_i$ such that $I_i = (f_i)$, where $X \setminus Z(f_i) = \{x_i\}$, for $i = 1, 2, \ldots, n$. This implies that $\bigcap Z[I] = X \setminus \{x_1, x_2, \ldots, x_n\}$ and again by Proposition 4.6, $X \setminus \{x_1, x_2, \ldots, x_n\}$ does not contain any nonempty compact neighborhood. Thus $G \cap (X \setminus \{x_1, x_2, \ldots, x_n\}) = \emptyset$ and hence $G \subseteq \{x_1, x_2, \ldots, x_n\}$, i.e., $G$ is finite. The converse is obvious. □

**Corollary 4.8.** If $X$ is a locally compact Hausdorff space, then $C_\infty(X)$ has a finite Goldie dimension if and only if $X$ is finite.

Finally we characterize the locally compact spaces $X$ for which $C_\infty(X)$ is a p.p. ring or a Baer ring. A topological space $X$ is called **extremally (basically) disconnected** if each open (cozero) set in $X$ has an open closure. A commutative ring $R$ is a p.p. (Baer) ring if for any $a \in R \ (S \subseteq R)$, Ann$(a)$ (Ann$(S)$) is the principal ideal generated by an idempotent. In [1] and [3], it is shown that $X$ is basically (extremally) disconnected if and only if $C(X)$ is a p.p. (Baer) ring.

**Theorem 4.9.** Let $X$ be a locally compact space.

(a) $C_\infty(X)$ is a p.p. ring if and only if $X$ is a basically disconnected compact space.

(b) $C_\infty(X)$ is a Baer ring if and only if $X$ is an extremally disconnected compact space.

**Proof:** (a) Let $C_\infty(X)$ be a p.p. ring. Then for every $0 \neq f \in C_\infty(X)$, there exists an idempotent $e \in C_\infty(X)$ such that Ann$(f) = (e)$. Therefore $X \setminus Z(e) \subseteq \text{int } Z(f)$. We show that $X \setminus Z(e) = \text{int } Z(f)$. Let $x \in \text{int } Z(f)$ but $x \notin X \setminus Z(e)$ and define $g \in C(X)$ such that $g(X \setminus \text{int } K) = \{0\}$ and $g(x) = 1$, where $K$ is a compact neighborhood of $x$ contained in $\text{int } Z(f) \cap Z(e)$. Hence $g \in C_\infty(X)$ and $gf = 0$ but $g \notin (e)$, for $Z(e) \nsubseteq Z(g)$ (as $g(x) = 1$, $e(x) = 0$), a contradiction. This implies that $X \setminus Z(e) = \text{int } Z(f)$ and hence $Z(e) = \text{cl}_X (X \setminus Z(f))$. Now if we take $f \in C_K(X)$, then $Z(e)$ and $X \setminus Z(e)$ are compact, i.e., $X$ is compact. We have also shown that for every $f \in C_\infty(X)$, $\text{int } Z(f)$ is closed. Since $X$ is compact, $C_\infty(X) = C(X)$ and hence for every $f \in C(X)$, $\text{int } Z(f)$ is closed, i.e., $X$ is basically disconnected. Conversely, if $X$ is a compact space, then $C_\infty(X) = C(X)$ and since $X$ is basically disconnected, $C_\infty(X)$ is a p.p. ring by [1, Lemma 3].

(b) If $C_\infty(X)$ is a Baer ring, then it is p.p. ring and hence by part (a), $X$ is compact, i.e., $C_\infty(X) = C(X)$. Now part (b) is well-known for compact spaces, see [5]. □

**Corollary 4.10.** Let $X$ be a locally compact non-compact space. Then $C_\infty(X)$ is never a p.p. (Baer) ring.
References


A.R. Aliabady:
DEPARTMENT OF MATHEMATICS, CHAMRAN UNIVERSITY, AHVAZ, IRAN

F. Azarpanah, M. Namdari:
INSTITUTE FOR STUDIES IN THEORETICAL PHYSICS AND MATHEMATICS (IPM), TEHRAN, IRAN
E-mail: aliabady_r@cua.ac.ir
azarpanah@ipm.ir
namdari@ipm.ir

(Received July 23, 2003, revised January 15, 2004)