Duality theory of spaces of vector-valued continuous functions
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Abstract. Let $X$ be a completely regular Hausdorff space, $E$ a real normed space, and let $C_b(X, E)$ be the space of all bounded continuous $E$-valued functions on $X$. We develop the general duality theory of the space $C_b(X, E)$ endowed with locally solid topologies; in particular with the strict topologies $\beta_z(X, E)$ for $z = \sigma, \tau, t$. As an application, we consider criteria for relative weak-star compactness in the spaces of vector measures $M_z(X, E')$ for $z = \sigma, \tau, t$. It is shown that if a subset $H$ of $M_z(X, E')$ is relatively $\sigma(M_z(X, E'), C_b(X, E))$-compact, then the set $\text{conv}(S(H))$ is still relatively $\sigma(M_z(X, E'), C_b(X, E))$-compact ($S(H)$ = the solid hull of $H$ in $M_z(X, E')$). A Mackey-Arens type theorem for locally convex-solid topologies on $C_b(X, E)$ is obtained.

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1. Introduction and preliminaries

Let $X$ be a completely regular Hausdorff space and let $(E, \| \cdot \|_E)$ be a real normed space. Let $B_E$ and $S_E$ stand for the closed unit ball and the unit sphere in $E$, and let $E'$ stand for the topological dual of $(E, \| \cdot \|_E)$. Let $C_b(X, E)$ be the space of all bounded continuous functions $f : X \to E$. We will write $C_b(X)$ instead of $C_b(X, \mathbb{R})$, where $\mathbb{R}$ is the field of real numbers. For a function $f \in C_b(X, E)$ we will write $\|f\|(x) = \|f(x)\|_E$ for $x \in X$. Then $\|f\| \in C_b(X)$ and the space $C_b(X, E)$ can be equipped with the norm $\|f\|_\infty = \sup_{x \in X} \|f\|(x) = \|\|f\||_\infty$, where $\|u\|_\infty = \sup_{x \in X} |u(x)|$ for $u \in C_b(X)$.

It turns out that the notion of solidness in the Riesz space (= vector lattice) $C_b(X)$ can be lifted in a natural way to $C_b(X, E)$ (see [NR]). Recall that a subset $H$ of $C_b(X, E)$ is said to be solid whenever $\|f_1\| \leq \|f_2\|$ (i.e., $\|f_1(x)\|_E \leq \|f_2(x)\|_E$ for all $x \in X$) and $f_1 \in C_b(X, E)$, $f_2 \in H$ imply $f_1 \in H$. A linear topology $\tau$ on $C_b(X, E)$ is said to be locally solid if it has a local base at 0 consisting of solid sets. A linear topology $\tau$ on $C_b(X, E)$ that is at the same time locally convex and locally solid will be called a locally convex-solid topology.

In [NR] we examine the general properties of locally solid topologies on the space $C_b(X, E)$. In particular, we consider the mutual relationship between locally solid topologies on $C_b(X, E)$ and $C_b(X)$. It is well known that the so-called
strict topologies $\beta_z(X, E)$ on $C_b(X, E)$ ($z = t, \tau, \sigma, g, p$) are locally convex-solid topologies (see [Kh, Theorem 8.1], [KhO2, Theorem 6], [KhV1, Theorem 5]).

For a linear topological space $(L, \xi)$, by $(L, \xi)'$ (or $L'_\xi$) we will denote its topological dual. We will write $C_b(X, E)'$ and $C_b(X)'$ instead of $(C_b(X, E), \| \cdot \|_\infty)'$ and $(C_b(X), \| \cdot \|_\infty)'$ respectively. By $\sigma(L, M)$ and $\tau(L, M)$ we will denote the weak topology and the Mackey topology with respect to a dual pair $\langle L, M \rangle$. For terminology concerning locally solid Riesz spaces we refer to [AB1], [AB2].

In the present paper, we develop the duality theory of the space $C_b(X, E)$ endowed with locally solid topologies (in particular, the strict topologies $\beta_z(X, E)$), where $z = \sigma, \tau, t$.

In Section 2 we examine the topological dual of $C_b(X, E)$ endowed with a locally solid topology $\tau$. We obtain that $(C_b(X, E), \tau)'$ is an ideal of $C_b(X, E)'$. We consider a mutual relationship between topological duals of the spaces $C_b(X)$ and $C_b(X, E)$, which allows us to examine in a unified manner continuous linear functionals on $C_b(X, E)$ by means of continuous linear functionals on $C_b(X)$.

In Section 3 we consider criteria for relative weak-star compactness in spaces of vector measures $M_z(X, E')$ for $z = \sigma, \tau, t$. In particular, we show that if a subset $H$ of $M_z(X, E')$ is relatively $\sigma(M_z(X, E'), C_b(X, E))$-compact, then $\text{conv}(S(H))$ is still relatively $\sigma(M_z(X, E'), C_b(X, E))$-compact (here $S(H)$ stand for the solid hull of $H$ in $M_z(X, E')$; see Definition 3.1 below).

Section 4 deals with the absolute weak and the absolute Mackey topologies on $C_b(X, E)$. A Mackey–Arens type theorem for locally convex-solid topologies on $C_b(X, E)$ is obtained.

Now we recall some properties of locally solid topologies on $C_b(X, E)$ as set out in [NR]. A seminorm $\rho$ on $C_b(X, E)$ is said to be solid whenever $\rho(f_1) \leq \rho(f_2)$ if $f_1, f_2 \in C_b(X, E)$ and $\|f_1\| \leq \|f_2\|$. Note that a solid seminorm on the vector lattice $C_b(X)$ is usually called a Riesz seminorm (see [AB1]).

**Theorem 1.1** (see [NR, Theorem 2.2]). For a locally convex topology $\tau$ on $C_b(X, E)$ the following statements are equivalent:

(i) $\tau$ is generated by some family of solid seminorms;

(ii) $\tau$ is a locally convex-solid topology.

From Theorem 1.1 it follows that any locally convex-solid topology $\tau$ on $C_b(X, E)$ admits a local base at 0 formed by sets which are simultaneously absolutely convex and solid.

Recall that the algebraic tensor product $C_b(X) \otimes E$ is the subspace of $C_b(X, E)$ spanned by the functions of the form $u \otimes e$, $(u \otimes e)(x) = u(x)e$, where $u \in C_b(X)$ and $e \in E$.

Now we briefly explain the general relationship between locally convex-solid topologies on $C_b(X)$ and $C_b(X, E)$ (see [NR]). Given a Riesz seminorm $p$ on
$C_b(X)$ let us set

$$p^\vee(f) := p(\|f\|) \quad \text{for all} \quad f \in C_b(X, E).$$

It is seen that $p^\vee$ is a solid seminorm on $C_b(X, E)$. From now on let $e_0 \in S_E$ be fixed. Given a solid seminorm $\rho$ on $C_b(X, E)$ one can define a Riesz seminorm $\rho^\wedge$ on $C_b(X)$ by:

$$\rho^\wedge(u) := \rho(u \otimes e_0) \quad \text{for all} \quad u \in C_b(X).$$

One can easily show:

**Lemma 1.2** (see [NR, Lemma 3.1]). (i) If $\rho$ is a solid seminorm on $C_b(X, E)$, then $(\rho^\wedge)^\vee(f) = \rho(f)$ for all $f \in C_b(X, E)$.

(ii) If $p$ is a Riesz seminorm on $C_b(X)$, then $(p^\vee)^\wedge(u) = p(u)$ for all $u \in C_b(X)$.

Let $\tau$ be a locally convex-solid topology on $C_b(X, E)$ and let $\{\rho_\alpha : \alpha \in A\}$ be a family of solid seminorms on $C_b(X, E)$ that generates $\tau$. By $\tau^\wedge$ we will denote the locally convex-solid topology on $C_b(X)$ generated by the family $\{\rho_\alpha^\wedge : \alpha \in A\}$.

Next, let $\xi$ be a locally convex-solid topology on $C_b(X)$ and let $\{p_\alpha : \alpha \in A\}$ be a family of solid seminorms on $C_b(X)$ that generates $\xi$. By $\xi^\vee$ we will denote the locally convex-solid topology on $C_b(X, E)$ generated by the family $\{p_\alpha^\vee : \alpha \in A\}$.

As an immediate consequence of Lemma 1.2 we have:

**Theorem 1.3** (see [NR, Theorem 3.2]). For a locally convex-solid topology $\tau$ on $C_b(X, E)$ (resp. $\xi$ on $C_b(X)$) we have:

$$(\tau^\wedge)^\vee = \tau \quad (\text{resp.} \ (\xi^\vee)^\wedge = \xi).$$

The strict topologies $\beta_z(X, E)$ on $C_b(X, E)$, where $z = t, \tau, \sigma, g, p$ have been examined in [F], [KhC], [Kh], [KhO1], [KhO2], [KhO3], [KhV1], [KhV2]. In this paper we will consider the strict topologies $\beta_z(X, E)$, where $z = t, \tau, \sigma$. We will write $\beta_z(X)$ instead of $\beta_z(X, \mathbb{R})$.

Now we recall the concept of a strict topology on $C_b(X, E)$. Let $\beta X$ stand for the Stone-Čech compactification of $X$. For $v \in C_b(X)$, $\overline{v}$ denotes its unique continuous extension to $\beta X$. For a compact subset $Q$ of $\beta X \setminus X$ let $C_Q(X) = \{v \in C_b(X) : \overline{v} \upharpoonright Q \equiv 0\}$. Let $\beta_Q(X, E)$ be the locally convex topology on $C_b(X, E)$ defined by the family of solid seminorms $\{q_v : v \in C_Q(X)\}$, where $q_v(f) = \sup_{x \in X} |v(x)| \|f\|(x)$ for $f \in C_b(X, E)$.

Now let $\mathcal{C}$ be some family of compact subsets of $\beta X \setminus X$. The strict topology $\beta_\mathcal{C}(X, E)$ on $C_b(X, E)$ determined by $\mathcal{C}$ is the greatest lower bound (in the class of locally convex topologies) of the topologies $\beta_Q(X, E)$, as $Q$ runs over $\mathcal{C}$ (see [NR] for more details). In particular, it is known that $\beta_\mathcal{C}(X, E)$ is locally solid (see [NR, Theorem 4.1]).
The strict topologies \( \beta_\tau(X, E) \) and \( \beta_\sigma(X, E) \) on \( C_b(X, E) \) are obtained by choosing the family \( C_\tau \) of all compact subsets of \( \beta X \setminus X \) and the family \( C_\sigma \) of all zero subsets of \( \beta X \setminus X \) as \( C \), resp. In view of [NR, Corollary 4.4] for \( z = \tau, \sigma \) we have
\[
\beta_z(X)^\vee = \beta_z(X, E) \quad \text{and} \quad \beta_z(X, E)^\wedge = \beta_z(X).
\]

The strict topology \( \beta_t(X, E) \) on \( C_b(X, E) \) is generated by the family \( \{ g_v : v \in C_0(X) \} \), where \( C_0(X) \) denotes the space of scalar-valued continuous functions on \( X \), vanishing at infinity. It is easy to show that
\[
\beta_t(X)^\vee = \beta_t(X, E) \quad \text{and} \quad \beta_t(X, E)^\wedge = \beta_t(X).
\]

2. Topological dual of \( C_b(X, E) \) with locally solid topologies

For a linear functional \( \Phi \) on \( C_b(X, E) \) let us put
\[
|\Phi|(f) = \sup \{ |\Phi(h)| : h \in C_b(X, E), \|h\| \leq \|f\| \}.
\]

The next theorem gives a characterization of the space \( C_b(X, E)^\prime \).

**Theorem 2.1.** We have
\[
C_b(X, E)^\prime = \{ \Phi \in C_b(X, E)^\# : |\Phi|(f) < \infty \text{ for all } f \in C_b(X, E) \},
\]
where \( C_b(X, E)^\# \) denotes the algebraic dual of \( C_b(X, E) \).

**Proof:** Indeed, by the way of contradiction, assume that for some \( \Phi_0 \in C_b(X, E)^\prime \) we have \( |\Phi_0|(f_0) = \infty \) for some \( f_0 \in C_b(X, E) \). Hence there exists a sequence \( (h_n) \) in \( C_b(X, E) \) such that \( \|h_n\| \leq \|f_0\| \) and \( |\Phi_0(h_n)| \geq n \) for all \( n \in \mathbb{N} \). Since \( \|n^{-1}h_n\|_\infty \to 0 \), we get \( n^{-1}\Phi_0(h_n) \to 0 \), which is in contradiction with \( |\Phi_0(h_n)| \geq n \).

Next, assume by the way of contradiction that there exists a linear functional \( \Phi_0 \) on \( C_b(X, E) \) such that \( |\Phi_0|(f) < \infty \) for all \( f \in C_b(X, E) \) and \( \Phi_0 \notin C_b(X, E)^\prime \). Then there exists a sequence \( (f_n) \) in \( C_b(X, E) \) such that \( \|f_n\|_\infty = 1 \) and \( |\Phi_0(f_n)| > n^3 \) for all \( n \in \mathbb{N} \). Since \( \sum_{n=1}^\infty \frac{1}{n^2}\|f_n\|_\infty < \infty \) and the space \( (C_b(X), \|\cdot\|_\infty) \) is complete, there exists \( u_0 \in C_b(X)^+ \) such that \( \sum_{n=1}^\infty \frac{1}{n^2}\|f_n\| = u_0 \). Let \( f_0 = u_0 \otimes e_0 \) for some fixed \( e_0 \in S_E \). Then \( \frac{1}{n^2}\|f_n\| \leq \|f_0\| = u_0 \). Hence for all \( n \in \mathbb{N} \), \( n < |\Phi_0(f_n/n^2)| \leq |\Phi_0|(f_n/n^2) \leq |\Phi_0|(f_0) < \infty \), which is impossible. Thus the proof is complete. \( \square \)

Now we consider the concept of solidness in \( C_b(X, E)^\prime \).

**Definition 2.1.** For \( \Phi_1, \Phi_2 \in C_b(X, E)^\prime \) we will write \( |\Phi_1| \leq |\Phi_2| \) whenever \( |\Phi_1|(f) \leq |\Phi_2|(f) \) for all \( f \in C_b(X, E) \). A subset \( A \) of \( C_b(X, E)^\prime \) is said to be *solid* whenever \( |\Phi_1| \leq |\Phi_2| \) with \( \Phi_1 \in C_b(X, E)^\prime \) and \( \Phi_2 \in A \) implies \( \Phi_1 \in A \). A linear subspace \( I \) of \( C_b(X, E)^\prime \) will be called an *ideal* whenever \( I \) is solid.
Since the intersection of any family of solid subsets of $C_b(X, E)'$ is solid, every subset $A$ of $C_b(X, E)'$ is contained in the smallest (with respect to the inclusion) solid set called the solid hull of $A$ and denoted by $S(A)$. Note that

$$S(A) = \{ \Phi \in C_b(X, E)' : |\Phi| \leq |\Psi| \text{ for some } \Psi \in A \}.$$  

**Lemma 2.2.** Let $\Phi \in C_b(X, E)'$. Then for $f \in C_b(X, E)$,

$$|\Phi|(f) = \sup \{|\Psi(f)| : \Psi \in C_b(X, E)', |\Psi| \leq |\Phi| \}.$$  

Moreover, if $A$ is a subset of $C_b(X, E)'$ then for $f \in C_b(X, E)$ we have

$$\sup \{|\Phi|(f) : \Phi \in A\} = \sup \{|\Psi(f)| : \Psi \in S(A)\}$$

$$= \sup \{|\Psi(f)| : \Psi \in \text{conv} (S(A))\}.$$  

**Proof:** Note first that $|\Phi|$ is a seminorm on $C_b(X, E)$. To see that $|\Phi|(f_1 + f_2) \leq |\Phi|(f_1) + |\Phi|(f_2)$ holds for $f_1, f_2 \in C_b(X, E)$ with $f_1, f_2 \neq 0$, assume that $h \in C_b(X, E)$ and $\|h\| \leq \|f_1 + f_2\|$. Then for $h_i = (\|f_1\|/(\|f_1\| + \|f_2\|))h$ for $i = 1, 2$ we have $h = h_1 + h_2$ and $\|h_i\| \leq \|f_i\|$ for $i = 1, 2$. Thus $|\Phi(h_i)| \leq |\Phi(h_1)| + |\Phi(h_2)| \leq |\Phi|(h_1) + |\Phi|(h_2) \leq |\Phi|(f_1) + |\Phi|(f_2)$. Hence $|\Phi|(f_1 + f_2) \leq |\Phi|(f_1) + |\Phi|(f_2)$, as desired. Moreover, one can easily show that $|\Phi|(\lambda f) = |\lambda| |\Phi|(f)$ for all $\lambda \in \mathbb{R}$.  

For a fixed $f_0 \in C_b(X, E)$ we define a functional $\Psi_0$ on the linear subspace $L_{f_0} = \{ \lambda f_0 : \lambda \in \mathbb{R} \}$ of $C_b(X, E)$ by putting $\Psi_0(\lambda f_0) = |\lambda| |\Phi|(f_0)$ for $\lambda \in \mathbb{R}$. It is clear that $\Psi_0$ is a linear functional on $L_{f_0}$ and $|\Psi_0(\lambda f_0)| = |\Phi|(f_0)$ for $\lambda \in \mathbb{R}$. Then by the Hahn–Banach extension theorem there exists a linear functional $\Psi$ on $C_b(X, E)$ such that $|\Psi|(f) \leq |\Phi|(f)$ for all $f \in C_b(X, E)$ and $\Psi(\lambda f_0) = \Psi_0(\lambda f_0)$ for all $\lambda \in \mathbb{R}$. Since $\Psi$ is linear and $|\Phi|(f) = |\Phi|(-f)$ we get $|\Psi(f)| \leq |\Phi|(f)$ for all $f \in C_b(X, E)$. To see that $|\Psi| \leq |\Phi|$ let $f \in C_b(X, E)$ and take $h \in C_b(X, E)$ with $\|h\| \leq \|f\|$. Then $|\Psi(h)| \leq |\Phi|(|h|) \leq |\Phi|(f)$, so $|\Psi(f)| \leq |\Phi|(f)$. Thus $|\Psi| \leq |\Phi|$. Moreover, $\Psi(f_0) = \Psi_0(f_0) = |\Phi|(f_0)$, so

$$|\Phi|(f_0) = \sup \{|\Psi(f_0)| : \Psi \in C_b(X, E)', |\Psi| \leq |\Phi|\}.$$  

Thus $(*)$ is shown. As a consequence of $(*)$ we easily obtain that $(**)$ holds. \qed

We now introduce the concept of a solid dual system. Let $I$ be an ideal of $C_b(X, E)'$ separating the points of $C_b(X, E)$. Then the pair $(C_b(X, E), I)$, under its natural duality

$$\langle f, \Phi \rangle = \Phi(f) \quad \text{for} \quad f \in C_b(X, E), \quad \Phi \in I$$

will be referred to as a solid dual system.

For a subset $A$ of $C_b(X, E)$ and a subset $B$ of $I$ let us set

$$A^0 = \{ \Phi \in I : |\langle f, \Phi \rangle| \leq 1 \text{ for all } f \in A \},$$

$$^0B = \{ f \in C_b(X, E) : |\langle f, \Phi \rangle| \leq 1 \text{ for all } \Phi \in B \}.$$  

By making use of Lemma 2.2 we can get the following result.
Theorem 2.3. Let $(C_b(X, E), I)$ be a solid dual system.

(i) If a subset $A$ of $C_b(X, E)$ is solid, then $A^0$ is a solid subset of $I$.

(ii) If a subset $B$ of $I$ is solid, then $0B$ is a solid subset of $C_b(X, E)$.

Proof: (i) Let $|\Phi_1| \leq |\Phi_2|$ with $\Phi_1 \in I$ and $\Phi_2 \in A^0$. Assume that $f \in A$ and let $h \in C_b(X, E)$ with $\|h\| \leq \|f\|$. Then $h \in A$, because $A$ is solid, so $|\Phi_2(h)| \leq 1$. Hence $|\Phi_2|(f) \leq 1$. Thus $|\Phi_1(f)| \leq |\Phi_1|(f) \leq 1$, so $\Phi_1 \in A^0$. This means that $A^0$ is a solid subset of $I$.

(ii) Let $\|f_1\| \leq \|f_2\|$ with $f_1 \in C_b(X, E)$ and $f_2 \in 0B$. To see that $f_1 \in 0B$ assume that $\Phi \in B$. Since $B$ is a solid subset of $I$, by Lemma 2.2 the identity $|\Phi|(f_2) = \sup \{|\Psi(f_2)| : \Psi \in B, |\Psi| \leq |\Phi|\}$ holds. Thus for every $\Psi \in B$ with $|\Psi| \leq |\Phi|$ we have $|\Psi(f_2)| \leq 1$, so $|\Phi|(f_2) \leq 1$. Since $|\Phi(f_1)| \leq |\Phi|(f_1) \leq |\Phi|(f_2) \leq 1$, we get $f_1 \in 0B$, as desired. \qed

Theorem 2.4. Let $\tau$ be a locally solid topology on $C_b(X, E)$. Then $(C_b(X, E), \tau)'$ is an ideal of $C_b(X, E)'$.

Proof: To show that $(C_b(X, E), \tau)' \subset C_b(X, E)'$, by the way of contradiction assume that for some $\Phi_0 \in (C_b(X, E), \tau)'$ we have $\Phi_0 \notin C_b(X, E)'$, so in view of Theorem 2.1 we get $|\Phi_0|(f_0) = \infty$ for some $f_0 \in C_b(X, E)$. Hence there exists a sequence $(h_n)$ in $C_b(X, E)$ such that $\|h_n\| \leq \|f_0\|$ and $|\Phi_0(h_n)| \geq n$ for $n \in \mathbb{N}$. Since $n^{-1}f_0 \to 0$ for $\tau$, and $\tau$ is locally solid, we get $n^{-1}h_n \to 0$ for $\tau$. Hence $\Phi_0(n^{-1}h_n) \to 0$, which is in contradiction with $|\Phi_0(h_n)| \geq n$.

To see that $(C_b(X, E), \tau)'$ is an ideal of $C_b(X, E)'$ assume that $|\Phi_1| \leq |\Phi_2|$ with $\Phi_1 \in C_b(X, E)'$ and $\Phi_2 \in (C_b(X, E), \tau)'$. Let $f_\alpha \xrightarrow{\tau} 0$ and $\varepsilon > 0$ be given. Then there exists a net $(h_\alpha)$ in $C_b(X, E)$ such that $\|h_\alpha\| \leq \|f_\alpha\|$ for each $\alpha$ and $|\Phi_2|(f_\alpha) \leq |\Phi_2(h_\alpha)| + \varepsilon$. Clearly $h_\alpha \xrightarrow{\tau} 0$, because $\tau$ is locally solid, so $\Phi_2(h_\alpha) \to 0$. Since $|\Phi_1(f_\alpha)| \leq |\Phi_1(f_\alpha)| \leq |\Phi_2|(f_\alpha) \leq |\Phi_2(f_\alpha)| + \varepsilon$, we get $\Phi_1(f_\alpha) \to 0$, so $\Phi_1 \in (C_b(X, E), \tau)'$, as desired. \qed

Theorem 2.5. For a Hausdorff locally convex topology $\tau$ on $C_b(X, E)$ the following statements are equivalent:

(i) $\tau$ is locally solid;

(ii) $(C_b(X, E), \tau)'$ is an ideal of $C_b(X, E)'$ and for every $\tau$-equicontinuous subset $A$ of $(C_b(X, E), \tau)'$ its solid hull $S(A)$ is also $\tau$-equicontinuous.

Proof: (i) $\implies$ (ii) By Theorem 2.4 $(C_b(X, E), \tau)'$ is an ideal of $C_b(X, E)'$, and thus we have the solid dual system $(C_b(X, E), (C_b(X, E), \tau)')$. Assume that a subset $A$ of $(C_b(X, E), \tau)'$ is equicontinuous. Hence $A \subset V^0$ for some solid $\tau$-neighbourhood $V$ of zero. Hence $S(A) \subset S(V^0) = V^0$ (see Theorem 2.3). This means that $S(A)$ is a $\tau$-equicontinuous subset of $(C_b(X, E), \tau)'$.

(ii) $\implies$ (i) Let $B_\tau$ be a local base at zero for $\tau$ consisting of absolutely convex, $\tau$-closed sets. Assume that $V$ is $\tau$-neighbourhood of zero. Then there exists $U \in B_\tau$
such that $U \subset V$. Moreover, the polar set $U^0$ is a $\tau$-equicontinuous subset of $(C_b(X, E), \tau)'$. By our assumption $S(U^0)$ is also $\tau$-equicontinuous. Hence there exists $W \in B_\tau$ such that $W \subset 0S(U^0)$. Since the set $0S(U^0)$ is solid in $C_b(X, E)$, $S(W) \subset 0S(U^0) \subset 0(U^0) = \overline{\text{absconv} U^\tau} = U \subset V$. This shows that $\tau$ is locally solid, as desired. \hfill \Box

For each $\Phi \in C_b(X, E)'$ let

$$\varphi_\Phi(u) = \sup \{ |\Phi(h)| : h \in C_b(X, E), \|h\| \leq u \} \text{ for } u \in C_b(X)^+.$$ 

One can easily show that $\varphi_\Phi : C_b(X)^+ \to \mathbb{R}^+$ is an additive and positively homogeneous mapping (see [KhO1, Lemma 1]), so $\varphi_\Phi$ has a unique positive extension to a linear mapping from $C_b(X)$ to $\mathbb{R}$ (denoted by $\varphi_\Phi$ again) and given by

$$\varphi_\Phi(u) = \varphi_\Phi(u^+) - \varphi_\Phi(u^-) \text{ for all } u \in C_b(X)$$

(see [AB, Lemma 3.1]). Hence $\varphi_\Phi = |\varphi_\Phi|$ holds on $C_b(X)^+$. Since $C_b(X)' = C_b(X)\sim$ (the order dual of $C_b(X)$) (see [AB2, Corollary 12.5]), we get $\varphi_\Phi \in C_b(X)'$. Moreover, we have:

$$\varphi_\Phi(\|f\|) = |\Phi|(f) \text{ for } f \in C_b(X, E)$$

and

$$\varphi_\Phi(u) = |\Phi|(u \otimes e_0) \text{ for } u \in C_b(X)^+.$$ 

The following lemma will be useful.

**Lemma 2.6.** (i) Assume that $L$ is an ideal of $C_b(X)'$. Then the set

$$C_b(X, E)'_L := \{ \Phi \in C_b(X, E)' : \varphi_\Phi \in L \}$$

is an ideal of $C_b(X, E)'$.

(ii) Assume that $I$ is an ideal of $C_b(X, E)'$. Then the set

$$C_b(X)'_I := \{ \varphi \in C_b(X)' : |\varphi| \leq \varphi_\Phi \text{ for some } \Phi \in I \}$$

is an ideal of $C_b(X)'$ and $C_b(X, E)'_{C_b(X)'_I} = I$.

**Proof:** (i) We first show that $C_b(X, E)'_L$ is a linear subspace of $C_b(X, E)'$. Assume that $\Phi_1, \Phi_2 \in C_b(X, E)'_L$, i.e., $\varphi_{\Phi_1}, \varphi_{\Phi_2} \in L$. It is easy to show that $\varphi_{\Phi_1+\Phi_2}(u) \leq (\varphi_{\Phi_1} + \varphi_{\Phi_2})(u)$ for $u \in C_b(X)^+$, so $\varphi_{\Phi_1+\Phi_2} \in L$, i.e., $\Phi_1 + \Phi_2 \in C_b(X, E)'_L$. Next, let $\Phi \in C_b(X, E)'_I$ and $\lambda \in \mathbb{R}$. Then $\varphi_\Phi \in L$ and since $\varphi_{\lambda\Phi} = \lambda\varphi_\Phi$, we get $\lambda\Phi \in C_b(X, E)'_L$. 

To show that $C_b(X, E)^\prime_L$ is solid in $C_b(X, E)^\prime$, assume that $|\Phi_1| \leq |\Phi_2|$ with $\Phi_1 \in C_b(X, E)^\prime$ and $\Phi_2 \in C_b(X, E)^\prime_L$, i.e., $\varphi_{\Phi_2} \in L$. Then for $u \in C_b(X)^+$ we have $\varphi_{\Phi_1}(u) = |\Phi_1|(u \otimes e_0) \leq |\Phi_2|(u \otimes e_0) = \varphi_{\Phi_2}(u)$. Hence $\varphi_{\Phi_1} \in L$, because $L$ is an ideal of $C_b(X)^\prime$. Thus $\Phi_1 \in C_b(X, E)^\prime_L$, as desired.

(ii) To prove that $C_b(X)^\prime_I$ is an ideal of $C_b(X)^\prime$ assume that $|\varphi_1| \leq |\varphi_2|$, where $\varphi_1 \in C_b(X)^\prime$ and $\varphi_2 \in C_b(X)^\prime_I$. Then $|\varphi_2| \leq \varphi_{\Phi}$ for some $\Phi \in I$, so $|\varphi_1| \leq \varphi_{\Phi}$, and this means that $\varphi_1 \in C_b(X)^\prime_I$.

To show that $I \subset C_b(X, E)^\prime_{C_b(X)^\prime_I}$, assume that $\Phi \in I$. Then $\varphi_{\Phi} \in C_b(X)^\prime_I$, so $\Phi \in C_b(X, E)^\prime_{C_b(X)^\prime_I}$.

Now, we assume that $\Phi \in C_b(X, E)^\prime_{C_b(X)^\prime_I}$, i.e., $\Phi \in C_b(X, E)^\prime$ and $\varphi_{\Phi} \in C_b(X)^\prime_I$. It follows that there exists $\Phi_0 \in I$ such that $\varphi_{\Phi} \leq \varphi_{\Phi_0}$. Hence for every $f \in C_b(X, E)$ we have $|\Phi|(f) = \varphi_{\Phi}(\|f\|) \leq \varphi_{\Phi_0}(\|f\|) = |\Phi_0|(f)$. Thus $\Phi \in I$, because $I$ is an ideal of $C_b(X, E)^\prime$.

Let $A$ be a subset of $C_b(X, E)^\prime_{\tau}$. Then $S(A) \subset C_b(X, E)^\prime_{\tau}$ as $C_b(X, E)^\prime_{\tau}$ is solid (by Theorem 2.4). Hence

$$S(A) = \{ \Phi \in C_b(X, E)^\prime_{\tau} : |\Phi| \leq |\Psi| \text{ for some } \Psi \in A \}.$$

In view of Lemma 2.2 for a subset $A$ of $C_b(X, E)^\prime$ and $f \in C_b(X, E)$ we have:

$$\sup \{ |\Phi|(f) : \Phi \in A \} = \sup \{ \varphi_{\Phi}(\|f\|) : \Phi \in A \} = \sup \{ |\Psi|(f) : \Psi \in S(A) \}.$$

**Theorem 2.7.** Let $\tau$ be a locally convex-solid Hausdorff topology on $C_b(X, E)$. Then for a subset $A$ of $C_b(X, E)^\prime$ the following statements are equivalent:

(i) $A$ is $\tau$-equicontinuous;

(ii) conv $(S(A))$ is $\tau$-equicontinuous;

(iii) $S(A)$ is $\tau$-equicontinuous;

(iv) the subset $\{ \varphi_{\Phi} : \Phi \in A \}$ of $C_b(X)^\prime$ is $\tau^\wedge$-equicontinuous.

**Proof:**

(i) $\implies$ (ii) In view of Theorem 2.4 we have a solid dual system $(C_b(X, E), C_b(X, E)^\prime_{\tau})$. Let $A$ be $\tau$-equicontinuous. Then by Theorem 1.1 there is a convex solid $\tau$-neighbourhood $V$ of zero such that $A \subset V^0$. Hence conv $(S(A)) \subset$ conv $(S(V^0)) = V^0$ (see Theorem 2.3), and this means that conv $(S(A))$ is still $\tau$-equicontinuous.

(ii) $\implies$ (iii) It is obvious.

(iii) $\implies$ (iv) Assume that the subset $S(A)$ of $C_b(X, E)^\prime$ is $\tau$-equicontinuous. Let $\{ \rho_{\alpha} : \alpha \in \mathcal{A} \}$ be a family of solid seminorms on $C_b(X, E)$ that generates $\tau$. Given $\varepsilon > 0$ there exist $\alpha_1, \ldots, \alpha_n \in \mathcal{A}$ and $\eta > 0$ such that sup $\{ |\Psi(f)| : \Psi \in S(A) \} \leq \varepsilon$
whenever \( \rho_{\alpha_i}(f) \leq \eta \) for \( i = 1, 2, \ldots, n \). To show that \( \{ \varphi_{\Phi} : \Phi \in A \} \) is \( \tau^\wedge \)-equicontinuous, it is enough to show that sup \( \{ |\varphi_{\Phi}(u)| : \Phi \in A \} \leq \varepsilon \) whenever \( \rho_{\alpha_i}^\wedge(u) \leq \eta \) for \( i = 1, 2, \ldots, n \). Indeed, let \( u \in C_b(X) \) and \( \rho_{\alpha_i}^\wedge(u) \leq \eta \) for \( i = 1, 2, \ldots, n \). Then \( \rho_{\alpha_i}(u \otimes e_0) \leq \eta \) (\( i = 1, 2, \ldots, n \)), so sup \( \{ |\Psi(u \otimes e_0)| : \Psi \in S(A) \} \leq \varepsilon \). Hence, in view of (\( + \)) we obtain that sup \( \{ \varphi_{\Phi}(u) : \Phi \in A \} \leq \varepsilon \), because \( \|u \otimes e_0\| = |u| \). But \( |\varphi_{\Phi}(u)| \leq \varphi_{\Phi}(\|u\|) \), and the proof is complete.

(iv) \( \implies \) (i) Assume that the set \( \{ \varphi_{\Phi} : \Phi \in A \} \) is \( \tau^\wedge \)-equicontinuous. Let \( \{ \rho_\alpha : \alpha \in A \} \) be a family of solid seminorms on \( C_b(X, E) \) that generates \( \tau \). Given \( \varepsilon > 0 \) there exist \( \alpha_1, \ldots, \alpha_n \in A \) and \( \eta > 0 \) such that sup \( \{ \varphi_{\Phi}(u) : \Phi \in A \} \leq \varepsilon \) whenever \( u \in C_b(X) \) and \( \rho_{\alpha_i}(u) \leq \eta \) for \( i = 1, 2, \ldots, n \). Let \( f \in C_b(X, E) \) with \( \rho_{\alpha_i}(f) \leq \eta \) for \( i = 1, 2, \ldots, n \). Since \( \rho_{\alpha_i}(\|f\|) = \rho_{\alpha_i}^\wedge(\|f\| \otimes e_0) = \rho_{\alpha_i}(f) \) \( (i = 1, 2, \ldots, n) \), sup \( \{ |\varphi_{\Phi}(\|f\|)| : \Phi \in A \} \leq \varepsilon \). But \( |\Phi(f)| \leq |\Phi(f) - \varphi_{\Phi}(\|f\|)| \), so sup \( \{ |\Phi(f)| : \Phi \in A \} \leq \varepsilon \). This means that \( A \) is \( \tau \)-equicontinuous.

**Corollary 2.8.** Let \( \tau \) be a locally convex-solid topology on \( C_b(X, E) \). Then for \( \Phi \in C_b(X, E)' \) the following statements are equivalent:

(i) \( \Phi \) is \( \tau \)-continuous;

(ii) \( \varphi_{\Phi} \) is \( \tau^\wedge \)-continuous.

**Corollary 2.9.** Let \( \xi \) be a locally convex-solid topology on \( C_b(X) \). Then for \( \Phi \in C_b(X, E)' \) the following statements are equivalent:

(i) \( \Phi \) is \( \xi^\vee \)-continuous;

(ii) \( \varphi_{\Phi} \) is \( \xi \)-continuous.

**Remark.** For the equivalence (i) \( \iff \) (iv) of Theorem 2.7 for the strict topologies \( \beta_z(X, E) \) \( (z = \sigma, \tau, t, \infty, g) \) see [KhO3, Lemma 2].

**Corollary 2.10.** (i) Let \( \xi \) be a locally convex-solid topology on \( C_b(X) \). Then

\[
(C_b(X), \xi)' = \left\{ \varphi \in C_b(X)' : |\varphi| \leq \varphi_{\Phi} \text{ for some } \Phi \in (C_b(X, E), \xi^\vee)' \right\}.
\]

(ii) Let \( \tau \) be a locally convex-solid topology on \( C_b(X, E) \). Then

\[
(C_b(X), \tau^\wedge)' = \left\{ \varphi \in C_b(X)' : |\varphi| \leq \varphi_{\Phi} \text{ for some } \Phi \in (C_b(X, E), \tau)' \right\}.
\]

**Proof:** (i) Let \( \varphi \in (C_b(X), \xi)' \). Define a linear functional \( \Phi_0 \) on the subspace \( C_b(X)(e_0) (= \{ u \otimes e_0 : u \in C_b(X) \}) \) of \( C_b(X, E) \) by putting \( \Phi_0(u \otimes e_0) = \varphi(u) \) for \( u \in C_b(X) \). Let \( \{ p_\alpha : \alpha \in A \} \) be a family of Riesz seminorms generating \( \xi \). Since \( \varphi \in (C_b(X), \xi)' \), there exist \( c > 0 \) and \( \alpha_1, \ldots, \alpha_n \in A \) such that for \( u \in C_b(X) \)

\[
|\Phi_0(u \otimes e_0)| = |\varphi(u)| \leq c \max_{1 \leq i \leq n} p_{\alpha_i}(u) = c \max_{1 \leq i \leq n} p_{\alpha_i}^\vee(u \otimes e_0).
\]
This means that $\Phi_0 \in (C_b(X)(e_0), \xi^\vee | C_b(X)(e_0))^\prime$, so by the Hahn-Banach extension theorem there is $\Phi \in (C_b(X,E), \xi^\vee)^\prime$ such that $\Phi(u \otimes e_0) = \varphi(u)$ for all $u \in C_b(X)$. We shall now show that $|\varphi| \leq \varphi_\Phi$, i.e., $|\varphi|(u) \leq \varphi_\Phi(u)$ for all $u \in C_b(X)^+$. Indeed, let $u \in C_b(X)^+$ be given and let $v \in C_b(X)$ with $|v| \leq u$. Then we have $|\varphi(v)| = |\Phi(v \otimes e_0)| \leq \varphi_\Phi(u)$, so $|\varphi| \leq \varphi_\Phi$, as desired.

Next, assume that $\varphi \in C_b(X)^\prime$ with $|\varphi| \leq \varphi_\Phi$ for some $\Phi \in (C_b(X,E), \xi^\vee)^\prime$. In view of Corollary 2.9, $\varphi_\Phi \in (C_b(X), \xi)^\prime$ and since $(C_b(X), \xi)^\prime$ is an ideal of $C_b(X)^\prime$, we conclude that $\varphi \in (C_b(X), \xi)^\prime$.

(ii) It follows from (i), because $(\tau^\vee)^\vee = \tau$. \hfill $\square$

It is well known that if $L$ is a $\sigma$-Dedekind complete vector-lattice and if $H$ is a relatively $\sigma(L\sim, L)$-compact subset of $L\sim$ (resp. a relatively $\sigma(L\sim_c, L)$-compact subset of $L\sim_c$), then the set $\text{conv}(S(H))$ is still relatively $\sigma(L\sim, L)$-compact (resp. relatively $\sigma(L\sim_c, L)$-compact) (see [AB, Corollary 20.12, Corollary 20.10]) (here $L\sim$ and $L\sim_c$ stand for the order continuous dual and the $\sigma$-order continuous dual of $L$ resp.).

Now, we shall show that this property holds in $(C_b(X,E)^\prime_{\beta_z}, \sigma(C_b(X,E)^\prime_{\beta_z}, C_b(X,E)))$ for $z = \sigma, \tau, t$.

Recall that a completely regular Hausdorff space $X$ is called a $P$-space if every $G_\delta$ set in $X$ is open (see [GJ, p. 63]).

The following result will be of importance.

**Theorem 2.11.** Let $H$ be a norm-bounded and $\sigma(C_b(X,E)^\prime_{\beta_z}, C_b(X,E))$-compact subset of $C_b(X,E)^\prime_{\beta_z}$, where $z = \sigma$ (resp. $z = \tau$ and $X$ is a paracompact space; resp. $z = \tau$ and $X$ is a $P$-space). Then $H$ is $\beta_z(X,E)$-equicontinuous.

**Proof:** See [KhO1, Theorem 5] for $z = \sigma$; [Kh, Theorem 6.1] for $z = \tau$ and [KhC, Lemma 3] for $z = t$. \hfill $\square$

Now we are ready to state our main result.

**Theorem 2.12.** Let $H$ be a norm bounded subset of $C_b(X,E)^\prime_{\beta_z}$, where $z = \sigma$ (resp. $z = \tau$ and $X$ is a paracompact space; resp. $z = t$ and $X$ is a $P$-space). Then the following statements are equivalent:

(i) $H$ is relatively countably $\sigma(C_b(X,E)^\prime_{\beta_z}, C_b(X,E))$-compact;

(ii) $H$ is $\beta_z(X,E)$-equicontinuous;

(iii) $\text{conv}(S(H))$ is relatively $\sigma(C_b(X,E)^\prime_{\beta_z}, C_b(X,E))$-compact;

(iv) $S(H)$ is relatively $\sigma(C_b(X,E)^\prime_{\beta_z}, C_b(X,E))$-compact;

(v) $H$ is relatively $\sigma(C_b(X,E)^\prime_{\beta_z}, C_b(X,E))$-compact.

**Proof:** (i) $\implies$ (ii) See Theorem 2.11.

(ii) $\implies$ (iii) In view of Theorem 2.7 the set $\text{conv}(S(H))$ is $\beta_z(X,E)$-equicontinuous, i.e., there is a neighbourhood of 0 for $\beta_z(X,E)$ such that $\text{conv}(S(H)) \subset V^0$
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(= the polar set with respect to the dual pair \( \langle C_b(X, E), C_b(X, E)'_{\beta_z} \rangle \)). Then by the Banach-Alaoglu's theorem the set \( V^0 \) is \( \sigma(C_b(X, E)'_{\beta_z}, C_b(X, E)) \)-compact, so the set \( \text{conv}(S(H)) \) is relatively \( \sigma(C_b(X, E)'_{\beta_z}, C_b(X, E)) \)-compact.

(iii) \( \implies \) (iv) \( \implies \) (v) \( \implies \) (i) It is obvious. \( \square \)

3. Weak-star compactness in some spaces of vector measures

In this section we consider criteria for relative weak-star compactness in some spaces of vector measures \( M_z(X, E') \) for \( z = \sigma, \tau, t \). In particular, by making use of Theorem 2.11 we show that if a subset \( H \) of \( M_z(X, E') \) is relatively \( \sigma(M_z(X, E'), C_b(X, E)) \)-compact, then the set \( \text{conv}(S(H)) \) is still relatively \( \sigma(M_z(X, E'), C_b(X, E)) \)-compact (here \( S(H) \) stand for the solid hull of \( H \) is \( M_z(X, E') \)). We start by recalling some notions and results concerning the topological measure theory (see [V], [S], [Wh]).

Let \( B(X) \) be the algebra of subsets of \( X \) generated by the zero sets. Let \( M(X) \) be the space of all bounded finitely additive regular (with respect to the zero sets) measures on \( B(X) \). The spaces of all \( \sigma \)-additive, \( \tau \)-additive and tight members of \( M(X) \) will be denoted by \( M_\sigma(X) \), \( M_\tau(X) \) and \( M_t(X) \) respectively (see [V], [Wh]). It is well known that \( M_z(X) \) for \( z = \sigma, \tau, t \) are ideals of \( M(X) \) (see [Wh, Theorem 7.2]).

**Theorem 3.1** (A.D. Alexandroff; [Wh, Theorem 5.1]). For a linear functional \( \varphi : C_b(X) \to \mathbb{R} \) the following statements are equivalent.

(i) \( \varphi \in C_b(X)' \).

(ii) There exists a unique \( \mu \in M(X) \) such that

\[
\varphi(u) = \varphi_\mu(u) = \int_X u \, d\mu \quad \text{for all} \quad u \in C_b(X).
\]

Moreover, \( \mu \geq 0 \) if and only if \( \varphi_\mu(u) \geq 0 \) for all \( u \in C_b(X)^+ \).

By \( M(X, E') \) we denote the set of all finitely additive measures \( m : B(X) \to E' \) with the following properties:

(i) For every \( e \in E \), the function \( m_e : B(X) \to \mathbb{R} \) defined by \( m_e(A) = m(A)(e) \), belongs to \( M(X) \).

(ii) \( |m|(X) < \infty \), where for \( A \in B(X) \)

\[
|m|(A) = \sup \left\{ \left| \sum_{i=1}^n m(B_i)(e_i) \right| : \bigcup_{i=1}^n B_i = A, \ B_i \in B(X), \ B_i \cap B_j = \emptyset \right. \right. \\
\left. \left. \quad \text{for } i \neq j, \ e_i \in B_E, \ n \in \mathbb{N} \right\}.
\]
For \( z = \sigma, \tau, t \) let

\[
M_z(X, E') = \{ m \in M(X, E') : m_e \in M_z(X) \text{ for every } e \in E \}.
\]

It is well known that \( |m| \in M(X) \) (resp. \(|m| \in M_z(X)\) for \( z = \sigma, \tau, t \)) whenever \( m \in M(X, E') \) (resp. \( m \in M_z(X, E') \) for \( z = \sigma, \tau, t \)) (see [F, Proposition 3.9]).

Now we are ready to define the notion of solidness in \( M(X, E') \).

**Definition 3.1.** For \( m_1, m_2 \in M(X, E') \) we will write \(|m_1| \leq |m_2|\) whenever \(|m_1|(B) \leq |m_2|(B)\) for every \( B \in B(X) \). A subset \( H \) of \( M(X, E') \) is said to be solid whenever \(|m_1| \leq |m_2|\) with \( m_1 \in M(X, E') \) and \( m_2 \in H \) imply \( m_1 \in H \). A linear subspace \( I \) of \( M(X, E') \) will be called an ideal of \( M(X, E') \) whenever \( I \) is a solid subset of \( M(X, E') \).

**Proposition 3.2.** \( M_z(X, E') \) \((z = \sigma, \tau, t)\) is an ideal of \( M(X, E') \).

**Proof:** Let \(|m_1| \leq |m_2|\), where \( m_1 \in M(X, E') \) and \( m_2 \in M_z(X, E') \). Then \(|m_1| \in M(X) \) and \(|m_2| \in M_z(X) \), and since \( M_z(X) \) is an ideal of \( M(X) \) we conclude that \(|m_1| \in M_z(X) \). For each \( e \in E \) we have \(|(m_1)_e|(B) \leq \|e\|_E |m_1|(B)| \) for \( B \in B(X) \), so \((m_1)_e \in M_z(X) \), i.e., \( m_1 \in M_z(X, E') \). \( \square \)

Since the intersection of any family of solid subsets of \( M(X, E') \) is solid, every subset \( H \) of \( M(X, E') \) is contained in the smallest (with respect to inclusion) solid set called the solid hull of \( H \) and denoted by \( S(H) \). Note that

\[
S(H) = \{ m \in M(X, E') : |m| \leq |m'| \text{ for some } m' \in H \}.
\]

Now we recall some results concerning a characterization of the topological duals of \((C_b(X, E), \beta_z(X, E))\) in terms of the spaces \( M_z(X, E') \) \((z = \sigma, \tau, t)\).

**Theorem 3.3.** Assume that \( \beta_z(X, E) \) is the strict topology on \( C_b(X, E) \), where \( z = \sigma \) and \( C_b(X) \otimes E \) is dense in \((C_b(X, E), \beta_\sigma(X, E))\) (resp. \( z = \tau \); resp. \( z = t \)). Then for a linear functional \( \Phi \) on \( C_b(X, E) \) the following statements are equivalent.

(i) \( \Phi \) is \( \beta_z(X, E) \)-continuous.

(ii) There exists a unique \( m \in M_z(X, E') \) such that

\[
\Phi(f) = \Phi_m(f) = \int_X f \, dm \text{ for every } f \in C_b(X, E).
\]

(iii) The functional \( \varphi_\Phi \) is \( \beta_z(X) \)-continuous.

Moreover, \( \|\Phi_m\| = |m|(X) \) for \( m \in M_z(X, E') \).

**Proof:** (i) \( \iff \) (ii) See [Kh, Theorem 5.3] for \( z = \sigma \); [Kh, Corollary 3.9] for \( z = \tau \); [F1, Theorem 3.13] for \( z = t \).

(ii) \( \iff \) (iii) It follows from Corollary 2.8, because \( \beta_z(X, E)^\wedge = \beta_z(X) \). \( \square \)
**Lemma 3.4.** Assume that \( m \in M_z(X, E') \), where \( z = \sigma \) and \( C_b(X) \otimes E \) is dense in \((C_b(X, E), \beta_\sigma(X, E))\) (resp. \( z = \tau \); resp. \( z = t \)). Then

\[
\varphi_{\Phi_m}(u) = \int_X u \, d|m| = \varphi_{|m|}(u) \quad \text{for all} \quad u \in C_b(X).
\]

**Proof:** Let \( u \in C_b(X)^+ \) and \( m \in M_z(X, E') \). Then for \( h \in C_b(X, E) \) with \( \|h\| \leq u \) by \([F_2, \text{Lemma 3.11}]\) we have

\[
|\Phi_m(h)| = \left| \int_X h \, d|m| \right| \leq \int_X \|h\| \, d|m| \leq \int_X u \, d|m| = \varphi_{|m|}(u).
\]

Hence

\[
\varphi_{\Phi_m}(u) = |\Phi_m|(u \otimes e_0) = \sup \{ |\Phi_m(h)| : h \in C_b(X, E), \|h\| \leq u \} \leq \varphi_{|m|}(u).
\]

On the other hand, in view of \([Kh, \text{Theorem 2.1}]\) we have

\[
\varphi_{|m|}(u) = \int_X u \, d|m| = \sup \{ |\Phi_m(g)| : g \in C_b(X) \otimes E, \|g\| \leq u \},
\]

so \( \varphi_{|m|}(u) \leq \varphi_{\Phi_m}(u) \). Thus \( \varphi_{|m|}(u) = \varphi_{\Phi_m}(u) \) for all \( u \in C_b(X) \).

**Lemma 3.5.** Assume that \( m_1, m_2 \in M_z(X, E') \), where \( z = \sigma \) and \( C_b(X) \otimes E \) is dense in \((C_b(X, E), \beta_\sigma(X, E))\) (resp. \( z = \tau \); resp. \( z = t \)). Then the following statements are equivalent:

(i) \( |m_1| \leq |m_2| \), i.e., \( |m_1|(B) \leq |m_2|(B) \) for every \( B \in B(X) \);

(ii) \( \varphi_{|m_1|}(u) \leq \varphi_{|m_2|}(u) \) for every \( u \in C_b(X)^+ \);

(iii) \( |\Phi_{m_1}|(f) \leq |\Phi_{m_2}|(f) \) for every \( f \in C_b(X, E) \).

**Proof:** (i) \( \iff \) (ii) It easily follows from Theorem 3.1.

(ii) \( \implies \) (iii) In view of Lemma 3.4 we get

\[
|\Phi_{m_1}|(f) = \varphi_{\Phi_{m_1}}(\|f\|) = \varphi_{|m_1|}(\|f\|) \\
\leq \varphi_{|m_2|}(\|f\|) = \varphi_{\Phi_{m_2}}(\|f\|) = |\Phi_{m_2}|(f).
\]

(iii) \( \implies \) (ii) By Lemma 3.3 for \( u \in C_b(X)^+ \) and \( e_0 \in S_E \) we have

\[
\varphi_{|m_1|}(u) = \varphi_{\Phi_{m_1}}(u) = |\Phi_{m_1}|(u \otimes e_0) \\
\leq |\Phi_{m_2}|(u \otimes e_0) = \varphi_{\Phi_{m_2}}(u) = \varphi_{|m_2|}(u).
\]
Lemma 3.6. Assume that $H \subset M_z(X, E')$, where $z = \sigma$ and $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_\sigma(X, E))$ (resp. $z = \tau$; resp. $z = t$) and let $\Phi_H = \{\Phi_m : m \in H\}$. Then $\text{conv} (S(\Phi_H)) = \Phi_{\text{conv} (S(H))}$.

Proof: Assume that $\Phi \in \text{conv} (S(\Phi_H))$. Then $\Phi = \sum_{i=1}^n \alpha_i \Phi_{m_i} = \Phi \sum_{i=1}^n \alpha_i m_i$, where $m_i \in M_z(X, E')$ and $\alpha_i \geq 0$ for $i = 1, 2, \ldots, n$ with $\sum_{i=1}^n \alpha_i = 1$, and $|\Phi_{m_i}| \leq |\Phi_{m_i'}|$ for some $m_i \in H$ and $i = 1, 2, \ldots, n$. In view of Lemma 3.5 $|m_i| \leq |m_i'|$, i.e., $m_i \in S(H)$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^n \alpha_i m_i \in \text{conv} (S(H))$. This means that $\Phi \in \Phi_{\text{conv} (S(H))}$.

Assume that $\Phi \in \Phi_{\text{conv} (S(H))}$. Then $\Phi = \Phi \sum_{i=1}^n \alpha_i m_i = \sum_{i=1}^n \alpha_i \Phi_{m_i}$, where $m_i \in M_z(X, E')$ and $\alpha_i \geq 0$ for $i = 1, 2, \ldots, n$ with $\sum_{i=1}^n \alpha_i = 1$, and $|m_i| \leq |m_i'|$ for some $m_i \in H$ and $i = 1, 2, \ldots, n$. By Lemma 3.5 $|\Phi_{m_i}| \leq |\Phi_{m_i'}|$ for $i = 1, 2, \ldots, n$, so $\Phi \in \text{conv} (S(\Phi_H))$.

Corollary 3.7. Assume that $m_0 \in M_z(X, E')$, where $z = \sigma$ and $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_\sigma(X, E))$ (resp. $z = \tau$; resp. $z = t$) and let $e \in S_E$. Then for every $u \in C_b(X)^+$ we have:

$$\int_X u \, d|m_0| = \sup \left\{ \left| \int_X u \, dm_e \right| : m \in M_z(X, E'), \, |m| \leq |m_0| \right\}.$$ 

Proof: Let $m_0 \in M_z(X, E')$ and $e \in S_E$. Assume that $\Phi \in C_b(X, E)'$ and $|\Phi| \leq |\Phi_{m_0}|$. Since $\Phi_{m_0} \in C_b(X, E)'_{\beta_z}$ (see Theorem 3.3), by making use of Theorem 2.4 we get $\Phi \in C_b(X, E)'_{\beta_z}$. Hence in view of Theorem 3.3 and Lemma 3.5 we see that $\Phi = \Phi_m$ for some $m \in M_z(X, E')$ with $|m| \leq |m_0|$.

Moreover, it is easy to observe that for every $m \in M(X, E')$ and $u \in C_b(X)$ we have:

$$\int_X (u \otimes e) \, dm = \int_X u \, dm_e.$$

Thus in view of Lemma 3.4, Lemma 2.2 and Lemma 3.5 we get:

$$\int_X u \, dm_0 = \varphi_{\Phi_{m_0}}(u) = |\Phi_{m_0}| (u \otimes e)$$

$$= \sup \left\{ |\Phi(u \otimes e)| : \Phi \in C_b(X, E)', \, |\Phi| \leq |\Phi_{m_0}| \right\}$$

$$= \sup \left\{ |\Phi_m(u \otimes e)| : m \in M_z(X, E'), \, |m| \leq |m_0| \right\}$$

$$= \sup \left\{ \left| \int_X (u \otimes e) \, dm \right| : m \in M_z(X, E'), \, |m| \leq |m_0| \right\}$$

$$= \sup \left\{ \left| \int_X u \, dm_e \right| : m \in M_z(X, E'), \, |m| \leq |m_0| \right\}.$$

To state our main result we recall some definitions (see [Wh, Definition 11.13, Definition 11.23, Theorem 10.3]).
A subset $A$ of $M_\sigma(X)$ (resp. $M_\tau(X)$) is said to be uniformly $\sigma$-additive (resp. uniformly $\tau$-additive) if whenever $u_n(x) \downarrow 0$ for every $x \in X$, $u_n \in C_b(X)^+$ (resp. $u_\alpha \downarrow 0$ for every $x \in X$, $u_\alpha \in C_b(X)^+$), then sup $\{|\int_X u_n \, d\mu| : \mu \in A\} \to 0$ (resp. sup $\{|\int_X u_\alpha \, d\mu| : \mu \in A\} \to 0$).

A subset $A$ of $M_t(X)$ is said to be uniformly tight if given $\varepsilon > 0$ there exists a compact subset $K$ of $X$ such that sup $\{|\mu|(X \setminus K) : \mu \in A\} \leq \varepsilon$.

Now we are in position to prove our desired result.

**Theorem 3.8.** For a subset $H$ of $M_z(X, E')$, where $z = \sigma$ and $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_\sigma(X, E))$ (resp. $z = \tau$ and $X$ is paracompact; resp. $z = t$ and $X$ is a $P$-space) the following statements are equivalent.

(i) $H$ is relatively $\sigma(M_z(X, E'), C_b(X, E))$-compact.

(ii) conv$(S(H))$ is relatively $\sigma(M_z(X, E'), C_b(X, E))$-compact.

(iii) The set $\{ |m| : m \in H \}$ in $M_z(X)^+$ is uniformly $\sigma$-additive for $z = \sigma$, (resp. uniformly $\tau$-additive for $z = \tau$; resp. uniformly tight for $z = t$).

**Proof:** (i) $\implies$ (ii) It is seen that $H$ is relatively $\sigma(M_z(X, E'), C_b(X, E))$-compact if and only if $\Phi_H$ is relatively $\sigma(C_b(X, E)', \beta_\sigma(C_b(X, E))$-compact. Hence by Theorem 2.12 and Lemma 3.6 the set $\Phi_{\text{conv}}(S(H))$ is still relatively $\sigma(C_b(X, E)', C_b(X, E))$-compact. This means that conv$(S(H))$ is relatively $\sigma(M_z(X, E'), C_b(X, E))$-compact.

(ii) $\iff$ (iii) In view of Theorem 2.12 $H$ is relatively $\sigma(M_z(X, E'), C_b(X, E))$-compact if and only if $\Phi_H$ is $\beta_\sigma(C_b(X, E))$-equicontinuous; hence in view of Theorem 2.7 and Lemma 3.4 the subset $\{ \varphi_{|m|} : m \in H \}$ of $(C_b(X), \beta_\sigma(X))'$ is $\beta_\sigma(X)$-equicontinuous. It is known that the subset $\{ \varphi_{|m|} : m \in H \}$ of $(C_b(X), \beta_\sigma(X))'$ is $\beta_\sigma(X)$-equicontinuous if and only if the set $\{|m| : m \in H\}$ in $M_z(X)^+$ is uniformly $\sigma$-additive for $z = \sigma$ (see [Wh, Theorem 11.14]) (resp. uniformly $\tau$-additive for $z = \tau$ (see [Wh, Theorem 11.24]); resp. uniformly tight for $z = t$ (see [Wh, Theorem 10.7])).

\[\square\]

4. A Mackey-Arens type theorem for locally convex-solid topologies on $C_b(X, E)$

Let $I$ be an ideal of $C_b(X, E)'$ separating points of $C_b(X, E)$. For each $\Phi \in I$ let us put

$$\rho_\Phi(f) = |\Phi|(f) \quad \text{for} \quad f \in C_b(X, E).$$

One can show that $\rho_\Phi$ is a solid seminorm on $C_b(X, E)$ (see the proof of Lemma 2.2). We define the absolute weak topology $|\sigma|(C_b(X, E), I)$ on $C_b(X, E)$ as
the locally convex-solid topology generated by the family \( \{ \rho_\Phi : \Phi \in I \} \). In view of Lemma 2.2 we have
\[
\rho_\Phi(f) = |\Phi|(f) = \sup \{ |\Psi(f)| : \Psi \in I, \; |\Psi| \leq |\Phi| \}.
\]
This means that \( |\sigma|(C_b(X, E), I) \) is the topology of uniform convergence on sets of the form \( \{ \Psi \in I : |\Psi| \leq |\Phi| \} = S(\{ \Phi \}) \), where \( \Phi \in I \).

Assume that \( L \) is an ideal of \( C_b(X) \)' separating the points of \( C_b(X) \). For each \( \varphi \in L \) the function \( p_\varphi(u) = |\varphi|(|u|) \) for \( u \in C_b(X) \) defines a Riesz semi-norm on \( C_b(X) \). The family \( \{ p_\varphi : \varphi \in I \} \) defines a locally convex-solid topology \( |\sigma|(C_b(X), L) \) on \( C_b(X) \), called the absolute weak topology generated by \( L \) (see [AB]).

Recall that \( |\sigma|(C_b(X), L)^\vee \) is the locally convex-solid topology on \( C_b(X, E) \) generated by the family \( \{ p_\varphi^\vee : \varphi \in L \} \), where \( p_\varphi^\vee(f) = p_\varphi(|f|) \) for \( f \in C_b(X, E) \).

We shall need the following result.

**Lemma 4.1.** Let \( I \) be an ideal of \( C_b(X, E) \)' separating the points of \( C_b(X, E) \). Then
\[
|\sigma|(C_b(X, E), I) = |\sigma|(C_b(X), C_b(X)'_I)^\vee
\]
where \( C_b(X)'_I = \{ \varphi \in C_b(X)' : |\varphi| \leq \varphi_\Phi \; \text{ for some } \Phi \in I \} \).

**Proof:** Let \( \varphi \in C_b(X)' \), i.e., \( |\varphi| \leq \varphi_\Phi \) for some \( \Phi \in I \). Then for \( f \in C_b(X, E) \) we have
\[
p_\varphi^\vee(f) = p_\varphi(|f|) = |\varphi|(|f|) \leq \varphi_\Phi(|f|) = |\Phi|(f) = \rho_\Phi(f).
\]
This means that \( |\sigma|(C_b(X), C_b(X)'_I)^\vee \subset |\sigma|(C_b(X, E), I) \).

Next, let \( \Phi \in I \). Then for \( f \in C_b(X, E) \) we have
\[
\rho_\Phi(f) = |\Phi|(f) = \varphi_\Phi(|f|) = p_\varphi^\vee(|f|) = p_\varphi^\vee(f).
\]
This shows that \( |\sigma|(C_b(X, E), I) \subset |\sigma|(C_b(X), C_b(X)'_I)^\vee \), and the proof is complete. \( \square \)

Now we are ready to state the main result of this section.

**Theorem 4.2.** Let \( I \) be an ideal of \( C_b(X, E) \)' separating the points of \( C_b(X, E) \). Then
\[
(C_b(X, E), |\sigma|(C_b(X, E), I))' = I.
\]

**Proof:** To see that \( (C_b(X, E), |\sigma|(C_b(X, E), I))' \subset I \) assume that \( \Phi \in (C_b(X, E), |\sigma|(C_b(X, E), I))' \). In view of Lemma 2.6 we have to show that \( \Phi \in C_b(X, E)'_{C_b(X)'_I} \), that is \( \Phi \in C_b(X, E)' \) and \( \varphi_\Phi \in C_b(X)'_I \). In fact, we know
that \((C_b(X), |\sigma|(C_b(X), C_b(X)'_I))' = C_b(X)'_I\) (see [AB1, Theorem 6.6]). Assume that \(u_\alpha \to 0\) for \(|\sigma|(C_b(X), C_b(X)'_I)\). It is enough to show that \(\varphi_\Phi(u_\alpha) \to 0\).

Indeed, \(u_\alpha \otimes e_0 \to 0\) for \(|\sigma|(C_b(X), C_b(X)'_I)\), because for each \(\varphi \in C_b(X)'_I\), \(p_{\varphi}^\vee(u_\alpha \otimes e_0) = p_{\varphi}(u_\alpha)\). Hence by Theorem 4.1 \(u_\alpha \otimes e_0 \to 0\) for \(|\sigma|(C_b(X, E), I)\).

Since \(|\varphi_\Phi(u_\alpha)| \leq |\varphi_\Phi(u_\alpha| = |\Phi|(u_\alpha \otimes e_0) = \rho_\Phi(u_\alpha \otimes e_0)\), we obtain that \(\varphi_\Phi(u_\alpha) \to 0\).

Now let \(\Phi \in I\). Then for \(f \in C_b(X, E)\), \(|\Phi(f)| \leq |\Phi(f) = \rho_\Phi(f)\), so \(\Phi\) is \(|\sigma|(C_b(X, E), I)\)-continuous, i.e., \(\Phi \in (C_b(X, E), |\sigma|(C_b(X, E), I))'\), as desired.

\[\square\]

As an application of Theorem 4.2 we have:

**Corollary 4.3.** Let \(I\) be an ideal of \(C_b(X, E)'\) separating the points of \(C_b(X, E)\). Then for a subset \(H\) of \(C_b(X, E)\) the following statements are equivalent:

(i) \(H\) is bounded for \(|\sigma|(C_b(X, E), I)\);

(ii) \(S(H)\) is bounded for \(|\sigma|(C_b(X, E), I)\).

**Proof:** (i) \(\Rightarrow\) (ii) By Theorem 4.2 and the Mackey theorem (see [Wi, Theorem 8.4.1]) \(H\) is bounded for \(|\sigma|(C_b(X, E), I)\). Since the topology \(|\sigma|(C_b(X, E), I)\) is locally solid, \(S(H)\) is bounded for \(|\sigma|(C_b(X, E), I)\). Hence \(S(H)\) is bounded for \(|\sigma|(C_b(X, E), I)\).

(ii) \(\Rightarrow\) (i) It is obvious. \(\square\)

**Lemma 4.4.** Let \(I_z = \{\Phi_m : m \in M_z(X, E')\}\), where \(z = \sigma\) and \(C_b(X) \otimes E\) is dense in \((C_b(X, E), \beta_\sigma(X, E))\) (resp. \(z = \tau\); resp. \(z = t\)). Then

\[C_b(X)'_{I_z} = \{\varphi_\mu : \mu \in M_z(X)\}\].

**Proof:** Assume that \(\varphi \in C_b(X)'_I\), i.e., \(\varphi \in C_b(X)'\) and \(|\varphi| \leq \varphi_{\Phi_m}\) for some \(m \in M_z(X, E')\). Then \(\varphi = \varphi_\mu\) for some \(\mu \in M(X)\), and \(|\varphi_\mu| = |\varphi_\mu| \leq \varphi_\Phi_m = \varphi_m|\) (see Lemma 3.4). It follows that \(|\mu| \leq |m|\), where \(|m| \in M_\sigma(X)^+\). Since \(M_z(X)\) is an ideal of \(M(X)\), we get \(\mu \in M_z(X)\).

Conversely, assume that \(\mu \in M_z(X)\) and \(e_0 \in S_E\) and let \(e^* \in E'\) be such that \(e^*(e_0) = 1\) and \(\|e^*\|_{E'} = 1\). Let us set \(m(B) = \mu(B)e^*\) for all \(B \in B(X)\). Then \(m : B(X) \to E'\) is finitely additive, and for each \(e \in E\) we have \(m_e(B) = m(B)(e) = (e^*(e))\mu(B)\) for all \(B \in B(X)\). Hence \(m_e \in M_z(X)\) for each \(e \in E\). It is easy to show that \(|m(B)| = |\mu(B)|\) for all \(B \in B(X)\), so \(|m| \in M_z(X)\). Hence \(m \in M_z(X, E')\), and \(|\varphi_\mu| = |\varphi_\mu| = |\varphi_m| = \varphi_{\Phi_m}\), so \(\varphi_\mu \in C_b(X)'_{I_z}\), as desired. \(\square\)

As an application of Lemma 4.1 and Lemma 4.4 we get:
Corollary 4.5. For $z = \sigma$ and $C_b(X) \otimes E$ dense in $(C_b(E), \beta_\sigma(X, E))$ (resp. $z = \tau$; resp. $z = t$) we have:

$$|\sigma|(C_b(X, E), M_z(X, E')) = |\sigma|(C_b(X), M_z(X))$$

and

$$|\sigma|(C_b(X, E), M_z(X, E'))^\wedge = |\sigma|(C_b(X), M_z(X)).$$

We now define the absolute Mackey topology $|\tau|(C_b(X, E), I)$ on $C_b(X, E)$ as the topology on uniform convergence on the family of all solid absolutely convex $\sigma(I, C_b(X, E))$-compact subsets of $I$. In view of Theorem 2.3 $|\tau|(C_b(X, E), I)$ is a locally convex-solid topology.

The following theorem strengthens the classical Mackey-Arens theorem for the class of locally convex-solid topologies on $C_b(X, E)$.

Theorem 4.6. Let $\tau$ be a locally convex-solid topology on $C_b(X, E)$ and let $(C_b(X, E), \tau)' = I_\tau$. Then

$$|\sigma|(C_b(X, E), I_\tau) \subset \tau \subset |\tau|(C_b(X, E), I_\tau).$$

Proof: To show that $|\sigma|(C_b(X, E), I_\tau) \subset \tau$, assume that $(f_\alpha)$ is a sequence in $C_b(X, E)$ and $f_\alpha \xrightarrow{\tau} 0$. Let $\Phi \in I_\tau$ and $\varepsilon > 0$ be given. Then there exists a net $(h_\alpha)$ in $C_b(X, E)$ such that $\|h_\alpha\| \leq \|f_\alpha\|$ and $\rho_\Phi(f_\alpha) = |\Phi(f_\alpha)| \leq |\Phi(h_\alpha)| + \varepsilon$. Since $\tau$ is locally solid, $h_\alpha \xrightarrow{\tau} 0$. Hence $h_\alpha \rightarrow 0$ for $\sigma(C_b(X, E), I_\tau)$, so $\Phi(h_\alpha) \rightarrow 0$, because $\sigma(C_b(X, E), I_\tau) \subset \tau$. Thus $\rho_\Phi(f_\alpha) \rightarrow 0$, and this means that $f_\alpha \rightarrow 0$ for $|\sigma|(C_b(X, E), I_\tau)$.

Now we show that $\tau \subset |\tau|(C_b(X, E), I_\tau)$. Indeed, let $B_\tau$ be a local base at zero for $\tau$ consisting of solid absolutely convex and $\tau$-closed sets and let $V \in B_\tau$. Then by Theorem 2.3 and the Banach-Alaoglu’s theorem, $V^0$ is a solid absolutely convex and $\sigma(I_\tau, C_b(X, E))$-compact subset of $I_\tau$. Hence

$$0(V^0) = \text{abs conv } V^\sigma = \text{abs conv } V^\tau = V,$$

so $\tau$ is the topology of uniform convergence on the family $\{V^0 : V \in B_\tau\}$. It follows that $\tau \subset |\tau|(C_b(X, E), I_\tau).$ \hfill $\square$

Corollary 4.7. Let $I_z = \{\Phi_m : m \in M_z(X, E')\}$, where $z = \sigma$ and $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_z(X, E))$ (resp. $z = \tau$ and $X$ is paracompact; resp. $z = t$ and $X$ is a $P$-space). Then

$$\beta_z(X, E) = |\tau|(C_b(X, E), M_z(X, E')) = \tau(C_b(X, E), M_z(X, E')),$$

and for a locally convex-solid topology $\tau$ on $C_b(X, E)$ with $C_b(X, E)' = I_z$ we have:

$$|\sigma|(C_b(X, E), M_z(X, E')) \subset \tau \subset \beta_z(X, E).$$
Proof: It is known that under our assumptions $\beta_z(X,E)$ is a Mackey topology (see [KhO$_1$, Corollary 6] for $z = \sigma$, [Kh, Theorem 6.2] for $z = \tau$ and [Kh, Theorem 5] for $z = t$). Hence $\tau(C_b(X,E), M_z(X,E')) = \beta_z(X,E)$. On the other hand, since $\beta_z(X,E)$ is a locally convex-solid topology and $(C_b(X,E), \beta_z(X,E))' = I_z$, by Corollary 4.6 we get $\beta_z(X,E) \subset |\tau|(C_b(X,E), M_z(X,E'))$. □

References


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