Coloring digraphs by iterated antichains

Svatopluk Poljak

Abstract. We show that the minimum chromatic number of a product of two \( n \)-chromatic graphs is either bounded by 9, or tends to infinity. The result is obtained by the study of coloring iterated adjoints of a digraph by iterated antichains of a poset.

Keywords: graph product, chromatic number, antichain
Classification: 05C15, 06A10

This note is motivated by a conjecture by Hedetniemi on the chromatic number of the product of two graphs. (The product \( G \times H \) of two unoriented graphs \( G \) and \( H \) is the graph on the vertex set \( V(G) \times V(H) \) and with the edges \((u_1, u_2), (v_1, v_2)\) for \( u_1v_1 \in E(G_1) \) and \( u_2v_2 \in E(G_2) \).) Hedetniemi [H] conjectured that

\[ \chi(G \times H) = \min(\chi_G, \chi_H) \]

for any pair \( G \) and \( H \) of graphs. The conjecture is also sometimes called the Lovász–Hedetniemi conjecture. The inequality ‘\( \leq \)’ in the conjecture is obvious, and it is also easy to see that the conjecture is valid for 1-, 2-, and 3-chromatic graphs. The validity for 4-chromatic graphs has been proved in [ES]. On the other hand, no lower bound on \( \chi(G \times H) \) is known. It is even not known whether the function \( f(n) \) defined by \( f(n) = \min\{\chi(G \times H) \mid \chi_G = \chi_H = n\} \) tends to infinity for \( n \to \infty \). However, it has been proved in [PR] that if the function is bounded, then \( f(n) \leq 16 \) for all \( n \). The purpose of this note is to decrease the bound from 16 to 9.

A survey of other known results on Hedetniemi’s conjecture can be found in [DSW], and some further related results have been published in [HHMN]. A special case was proved also in [T].

The result here is obtained by extending the technique of coloring digraphs by antichains (see [HE] and [PR]) to coloring iterated adjoints of digraphs by iterated antichains.

Let \( L \) be a poset and let \( A(L) \) be the set of all (not necessarily maximal) antichains of \( L \). We introduce a partial order on \( A(L) \) as follows. For \( a, a' \in A(L) \), we write \( a < a' \), if for every \( x \in a \) there is some \( y \in a' \) such that \( x < y \). It is easy to check that if \( a < a' \) and \( a' < a \), then \( a = a' \), and \( a < a' \) and \( a' < a'' \) give \( a < a'' \). For \( i > 0 \), we define \( A^i(L) = A(A^{i-1}(L)) \). (Note that our construction of a poset on antichains slightly differs from that of Dilworth [D], where only maximum sized antichains were considered.)

Let \( G = (V, E) \) be a digraph. We say that a mapping \( f \) from \( V \) to a poset \( L \) is a homomorphism, if \( f(u) < f(v) \) for every edge \( uv \in E \).
The adjoint $\delta G$ of a digraph $G$ is the digraph whose vertex set is $E(G)$, and edges of $\delta G$ are the pairs of consecutive edges of $G$, i.e. $E(\delta G) = \{ (uv, vw) \mid uv, vw \in E(G) \}$. For $i > 0$, we define the $i$-th adjoint $\delta^i G = \delta(\delta^{i-1} G)$.

**Lemma 1.** Let $G$ be a digraph and $L$ be a poset. Then there is a homomorphism $f$ from $G$ to $A(L)$, if and only if there is a homomorphism $\phi$ from $\delta G$ to $L$.

**Proof:** Let $f$ be a homomorphism from $G$ to $A(L)$. We define $\phi$ as follows. Given $e = uv \in V(\delta G)$, where $e$ is an edge of $G$, choose an arbitrary $x \in f(u)$ for which $\{ x \} < f(v)$, and set $\phi(e) = x$. (A suitable $x$ must exist since $f(u) < f(v)$.) We check that the mapping $\phi$ is a homomorphism from $\delta G$ to $L$. Let $ee' \in E(\delta G)$, where $e = uv$ and $e' = vw$ are edges of $G$. We have $\phi(e) < \phi(e')$ since $\phi(e) \in f(u)$, $\phi(e') \in f(v)$ and $\{ \phi(e) \} < f(v)$.

Conversely, let $\phi$ be a homomorphism from $\delta G$ to $L$. We define a homomorphism $f$ as follows. Given $u \in V(G)$, let $S(u) = \{ \phi(uw) \mid vw \in E(G) \}$. Since $S(u)$ is not necessarily an antichain, we define $f(u)$ as the set of the maximal elements of $S(u)$. It is straightforward to check that $f(u) < f(v)$ for $uv \in E(G)$. $\square$

The chromatic number $\chi G$ of a digraph $G$ is the chromatic number of the graph obtained from $G$ after forgetting the orientation of the edges. Equivalently, it is the minimum $k$ for which there is a homomorphism from $G$ to $D_k$, where $D_k$ denotes the discrete poset on $k$ elements. A digraph $G = (V, E)$ is said to be symmetric, if for every edge $uv$ it contains also the reversed edge $vu$. For a poset $L$, $\alpha(L)$ denotes the size of the maximum antichain in $L$.

**Theorem 2.** Let $G$ be a symmetric digraph, and $i$ a nonnegative positive integer. Then $\chi(\delta^i G)$ is equal to the minimum $k$ for which $\chi G$ is less or equals $\alpha(A^i(D_k))$.

**Proof:** Let $\chi(\delta^i G) = k$. Then there is a homomorphism $f$ from $\delta^i G$ to $D_k$, and hence also a homomorphism $\phi$ from $G$ to $A^i(D_k)$ by the repeated use of Lemma 1. Since $G$ is symmetric, $\phi(u)$ and $\phi(v)$ are incomparable elements of $A^i(D_k)$ for every edge $uv$ of $G$. Let $H$ be the complement of the comparability graph of $A^i(D_k)$. The existence of $\phi$ implies that $\chi(\delta^i G) \leq \chi H$. Since $H$ is a perfect graph, $\chi H$ equals the size of the maximum clique of $H$, which is the size of the maximum antichain in $A^i(D_k)$. Hence the inequality $\chi G \leq \alpha(A^i(D_k))$ is established.

Conversely, let $\chi G \leq \alpha(A^i(D_k))$. Then there is a homomorphism $\phi$ from $\delta^i G$ to $A^i(D_k)$. By a repeated use of Lemma 1, there is a homomorphism $f$ from $\delta^i G$ to $D_k$. Clearly, $f$ is a coloring of $\delta^i G$ since $D_k$ is discrete. Hence $\chi(\delta^i G) \leq k$. $\square$

We recall that $D_k$ is a discrete poset. Then $A(D_k)$ is the set of all subsets of $\{1, 2, \ldots, k\}$ ordered by inclusion, and $A^2(D_k)$ is the set of the Sperner systems on the underlying $k$-element set.

**Lemma 3.** We have $\alpha(A^2(D_3)) = 4$.

**Proof:** The following four sets $\{\{1, 2\}\}, \{\{2, 3\}\}, \{\{1, 3\}\}$ and $\{\{1\}, \{2\}, \{3\}\}$ form an antichain in $A^2(D_3)$. It is easy to check that it is an antichain of the maximum size. $\square$
Lemma 4 ([HE]). We have \( \chi(\delta G) \geq \log_2 \chi G \).

The product \( G_1 \times G_2 \) of two digraph \( G_1 \) and \( G_2 \) is the digraph with the vertex set \( V(G_1) \times V(G_2) \) and the edges \((u_1, u_2), (v_1, v_2)\) for \( u_1v_1 \in E(G_1) \) and \( u_2v_2 \in E(G_2) \).

We define \( g(n) \) as the minimum chromatic number of the product of two \( n \)-chromatic digraphs. It has been proved in [PR] that the function \( g \) is either bounded by 4, or tends to infinity. Here we present an improvement of that result.

**Theorem 5.** The function \( g(n) \) is either bounded by 3, or tends to infinity.

**Proof:** Assume that the function \( g \) is bounded by a constant \( c \), i.e. for all \( n \) sufficiently large, say \( n > n_0 \), we have \( g(n) = c \). It has been proved in [PR] that \( c \leq 4 \). For a contradiction, assume that \( c = 4 \). Let \( n > 2^{2^{n_0}} \), and \( G_1 \) and \( G_2 \) be a pair of \( n \)-chromatic digraphs for which \( \chi(G_1 \times G_2) = \chi H = 4 \), where \( H = G_1 \times G_2 \).

Since \( \alpha A^2(D_3) = 4 \) by Lemma 3, we have \( \chi(\delta^2 H) \leq 3 \) by Theorem 2.

On the other hand, we have \( \chi(\delta^2 G_1), \chi(\delta^2 G_2) > n_0 \) by Lemma 4, and hence \( \chi(\delta^2 G_1 \times \delta^2 G_2) \geq 4 \) by our assumption on \( g \). Since \( \delta^2 H = \delta^2(G_1 \times G_2) = \delta^2 G_1 \times \delta^2 G_2 \) (the latter equality is easy to see, cf. Proposition 2.2 of [PR]), we get \( \chi(\delta^2 H) = 4 \), which is a contradiction.

Let \( h(n) = \min\{\max(\chi(G_1 \times G_2), \chi(G_1 \times G_2^{-1})) \mid G_1 \text{ and } G_2 \text{ are digraphs with } \chi G_1 = \chi G_2 = n\} \), where \( G_2^{-1} \) denotes the digraph obtained from \( G_2 \) by reversing the edges. Quite analogously as above, it is possible to show that \( h(n) \) is either bounded by 3 or tends to infinity. However, it is not yet excluded that \( g(n) \) is bounded, while \( h(n) \) is not.

**Theorem 6.** The minimum chromatic number of a product of two \( n \)-chromatic graphs is either bounded by 9, or tends to infinity.

**Proof:** Let \( f(n) \) be the minimum chromatic number of a product of two (undirected) \( n \)-chromatic graphs. The statement follows from the inequality \( h(n) \leq f(n) \leq h^2(n) \) established in the proof of Theorem 3.6 of [PR].

I have been recently informed by V. Rödl that the possibility of improving the construction of [PR] was also observed by J. Schmerl.

**References**


Faculty of Mathematics and Physics, Charles University, Malostranské nám. 25, 118 00 Prague 1, Czechoslovakia

(Received August 23, 1990)