Divulgaciones Matemáticas Vol. 15 No. 1(2007), pp. 1–9

On the Controllability of a Type of Large Scale CNN with Delays

Sobre la Controlabilidad de un Tipo de RNC a Gran Escala con Retardos

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Abstract

In this paper we study the approximate controllability of a particular type of large scale CNN (Cellular Neural Network) with delays given by: $\begin{cases} \dot{x} & = A_0x(t) + \sum_{j=1}^{N} A^{(j)}x(t-h_j) + B_0u, \ t \geq 0, \\ x(0) & = r, \ r \in \mathbf{R}^n, \\ x(\theta) & = f(\theta), \ \ \theta \in [-h, 0), \end{cases}$
where $0 < h_1 < h_2 < \cdots < h_N$ represent the point delays, $h = h_N$, the matrices $B_0, A_0, A^{(j)} \in \mathcal{L}(\mathbf{R}^n)$, $i = 1, 2, \dots N$, the control u belong to $L_2([0, \tau], \mathbf{R}^n)$ and $f \in L^2([-h, 0]; \mathbf{R}^n)$. Moreover, $A_0 = \mathrm{diag}(A_1, \cdots, A_N)$, $B_0 = \mathrm{diag}(B_1, \cdots, B_N)$ and $A^{(j)}$, $j = 1, \cdots, N$ is an $n \times n$ block matrix

$$A^{(j)} = \begin{pmatrix} 0 & \cdots & A_{1j} & \cdots & 0 \\ 0 & \cdots & A_{2j} & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & A_{Nj} & \cdots & 0 \end{pmatrix},$$

with $A_i, \ A_{ij}, \ B_i \in \mathcal{L}(\mathbf{R}^{n_0}), \forall \tau > 0$.
Key words and phrases: Large Scale System, Cellular Neural Network, Exact and Approximate Controllability.

Received 2006/10/05. Revised 2007/09/03. Accepted 2007/09/10.
MSC (2000): Primary 93B05; Secondary 93C25.

Resumen

En este trabajo se estudia la controlabilidad aproximada de un tipo particular de RNC (Red Neural Celular) a gran escala con retardos dados por:

$$\begin{cases} \dot{x} & = A_0 x(t) + \sum_{j=1}^{N} A^{(j)} x(t-h_j) + B_0 u, \ t \geq 0, \\ x(0) & = r, \ r \in \mathbf{R}^n, \\ x(\theta) & = f(\theta), \ \ \theta \in [-h, 0), \end{cases}$$

donde $0 < h_1 < h_2 < \cdots < h_N$ representan los puntos de retardo, $h = h_N$, las matrices $B_0, A_0, A^{(j)} \in \mathcal{L}(\mathbf{R}^n)$, $i = 1, 2, \ldots N$, el control u pertenece a $L_2([0,\tau], \mathbf{R}^n)$ y $f \in L^2([-h,0]; \mathbf{R}^n)$. Más aún, $A_0 = \mathrm{diag}(A_1, \cdots, A_N)$, $B_0 = \mathrm{diag}(B_1, \cdots, B_N)$ y $A^{(j)}$, $j = 1, \cdots, N$ es la matriz de bloques $n \times n$

$$A^{(j)} = \begin{pmatrix} 0 & \cdots & A_{1j} & \cdots & 0 \\ 0 & \cdots & A_{2j} & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & A_{Nj} & \cdots & 0 \end{pmatrix},$$

con $A_i, \ A_{ij}, \ B_i \in \mathcal{L}(\mathbf{R}^{n_0}), \forall \tau > 0$.
Palabras y frases clave: Sistema a gran escala, Red Neural Celular, Controlabilidad exacta y aproximada.

1 Introduction

In recent years, there have been a considerable attention to the study of stability and designs for large scale time delayed CNN's. Razumikhin-Type Theorems ([8], [15]) and M-matrix properties are some of the tools used to prove stability ([1], [10]). In [16] for instance, they give a stability criterion for this type of system by means of the comparison method and the M-matrix properties already mentioned; moreover, in [11] and [13] they obtain some stability results by using techniques of quasidiagonal dominance. In [16] a unified analysis method for stability of large scale systems with and without time delays is established.

In this paper we consider a delayed large scale cellular neural network similar to the one considered in [16]; that is to say, the uncontrolled system with time delays in state given by

$$\begin{cases} \dot{x}_i & = A_i x_i(t) + A_{ii} x_i(t - h_{ii}(t)) + \sum\limits_{j=1, j\neq i}^{N} A_{ij} x_j(t - h_{ij}(t)), \\ & \qquad\qquad\qquad\qquad i = 1, \cdots, N, \\ x_i(0) & = r_i,\ r_i \in \mathbf{R}^{n_i}, \\ x_i(\theta) & = f_i(\theta), \quad \theta \in [-h, 0), \end{cases} \tag{1}$$

where $x_i \in \mathbf{R}^{n_i},\ \sum\limits_{i=1}^{n} n_i = N,\ A_i,\ A_{ii},\ A_{ij}$ are constant matrices with appropriate dimensions, $0 < h_{ij} < h,\ i, j = 1, \cdots, N$ are time dependent bounded and continuous delays.

The controlled system under our consideration is

$$\begin{cases} \dot{x}_i & = A_i x_i(t) + A_{ii} x_i(t - h_{ii}(t)) + \sum\limits_{j=1, j\neq i}^{N} A_{ij} x_j(t - h_{ij}(t)) + B_i u_i, \\ & \qquad\qquad\qquad\qquad i = 1, \cdots, N, \\ x_i(0) & = r_i,\ r_i \in \mathbf{R}^{n_i}, \\ x_i(\theta) & = f_i(\theta), \quad \theta \in [-h, 0), \end{cases} \tag{2}$$

with $x_i \in \mathbf{R}^{n_0};\ A_i,\ A_{ij},\ B_i \in \mathcal{L}(\mathbf{R}^{n_0}),\ u_i \in L_2([0,\tau], \mathbf{R}^{n_0}), \forall \tau \geq 0;\ 0 < h_i < h$ are continuous and bounded delays; h is constant.
The system (2) is equivalent to

$$\begin{cases} \dot{x} & = A_0 x(t) + \sum_{j=1}^{N} A^{(j)} x(t - h_j(t)) + B_0 u,\ t \geq 0, \\ x(0) & = r,\ r \in \mathbf{R}^n, \\ x(\theta) & = f(\theta), \quad \theta \in [-h, 0), \end{cases} \tag{3}$$

in $\mathbf{R}^n$, for certain $A_0,\ A^{(j)},\ B_0$. In fact, for $n = n_0 N$ and for any $i = 1, \cdots, N$ we set

$$x_i = (x_i^{(1)}, x_i^{(2)}, \cdots, x_i^{(n_0)})^T, \quad u_i = (u_i^{(1)}, u_i^{(2)}, \cdots, u_i^{(n_0)})^T.$$

With this in mind (2) becomes

$$(\dot{x}_i^{(1)}, \cdots, \dot{x}_i^{(n_0)})^T = A_i (x_i^{(1)}(t), \cdots, x_i^{(n_0)}(t)) + \sum_{j=1}^{N} A_{ij} (x_j^{(1)}(t - h_j(t)), \cdots \\ \cdots, x_j^{(n_0)}(t - h_j(t)))^T + B_i (u_i^{(1)}, \cdots, u_i^{(n_0)})^T. \tag{4}$$

Now we rewrite (4) in a compact form, that is, as one set of equations.

$$(\dot{x}_1,\cdots,\dot{x}_N)^T = (A_1x_1(t),\cdots,A_Nx_N(t))^T + (\sum_{j=1}^{N} A_{1j}x_j(t-h_j(t)),\cdots,$$
$$\cdots,\sum_{j=1}^{N} A_{Nj}x_j(t-h_j(t)))^T + (B_1u_1,\cdots,B_Nu_N)^T.$$

Then, if we put $A_0 = \operatorname{diag}(A_1,\cdots,A_N)$, $B_0 = \operatorname{diag}(B_1,\cdots,B_N)$, $x = (x_1,\cdots,x_N)^T$and $u = (u_1,\cdots,u_N)^T$, we get that A_0, B_0 are $n \times n$ block matrices $(n = n_0N)$; $x \in \mathbf{R}^n$, $u \in L_2([0,\tau],\mathbf{R}^n)$ and (4) takes the required form looks

$$\dot{x} = A_0x(t) + \sum_{j=i}^{N} A^{(j)}x(t-h_j(t)) + B_0u, t \geq 0 \tag{5}$$

with $A^{(j)}$, $j = 1,\cdots,N$ an $n \times n$ block matrix given as

$$A^{(j)} = \begin{pmatrix} 0 & \cdots & A_{1j} & \cdots & 0 \\ 0 & \cdots & A_{2j} & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & A_{Nj} & \cdots & 0 \end{pmatrix}.$$

In order to apply Theorem 4.2.10 from [6] we have to assume that the delay functions h_j are constants, otherwise, we have to prove an analogous theorem first. So, we shall prove the controllability of system (3) when the functions $h_j(t) = h_j \equiv \mathit{constant}$ are constants:

$$\begin{cases} \dot{x} & = A_0x(t) + \sum_{j=1}^{N} A^{(j)}x(t-h_j) + B_0u, \ t \geq 0, \\ x(0) & = r, \ r \in \mathbf{R}^n, \\ x(\theta) & = f(\theta), \ \ \theta \in [-h,0), \end{cases} \tag{6}$$

To this purpose: First, we rewrite this delay system as an ordinary differential in an appropriate Hilbert product space using Semigroups Theory. Second, we use the variation of constant formula or mild solution of this ordinary differential equation in order to define controllability. Then, we use the well-known result on the rank condition for the approximate controllability of delay system from [6] to derive our main result. Finally, in the conclusion section we consider the possibility to study system (3) with time-dependent delays and diffusion coefficients as a future reacher.

2 Abstract Formulation of the Problem

In this section we shall choose the space where this problem will be set up as an abstract control system governed by an ordinary differential equation an appropriate Hilbert space. In fact, we consider the Hilbert space $\mathcal{M}_2([-h,0];\mathbf{R}^n) = \mathbf{R}^n \oplus L_2([-h,0];\mathbf{R}^n)$ with the usual innerproduct given by:

$$\left\langle \begin{pmatrix} r_1 \\ f_1 \end{pmatrix}, \begin{pmatrix} r_2 \\ f_2 \end{pmatrix} \right\rangle = \langle r_1, r_2\rangle_{\mathbf{R}^n} + \langle f_1, f_2\rangle_{L_2}.$$

Define the following operator in the space $\mathcal{M}_2$ for $t \geq 0$ by

$$T(t)\begin{pmatrix} r \\ f(.) \end{pmatrix} = \begin{pmatrix} z(t) \\ z(t+\cdot) \end{pmatrix} \tag{7}$$

where $z(\cdot)$ is the only solution of the system (6). The following Theorem can be find in [6].

Theorem 2.1. *The family of operators $\{T(t)\}_{t\geq 0}$ defined by (7) is a strongly continuous semigroup on $\mathcal{M}_2$.*

Then, the system (6) is equivalent to the following system of ordinary differential equations in $\mathcal{M}_2$:

$$\begin{cases} \dfrac{dz(t)}{dt} = \Lambda z(t) + Bu(t), \quad t > 0, \\ z(0) \ = z_0 = (r, f(\cdot))^T, \end{cases} \tag{8}$$

where Λ is the infinitesimal generator of the semigroup $\{T(t)\}_{t\geq 0}$ and $Bu = (B_0 u, 0)^T$. Moreover, in [6] they prove the following lemma:

Lemma 2.2. *Let Λ be the infinitesimal generator of the semi-group $\{T(t)\}_{t\geq 0}$. Then*

$$\Lambda\begin{pmatrix} r \\ f(.) \end{pmatrix} = \begin{pmatrix} A_0 r + \sum_{j=1}^{N} A^{(j)} f(-h_j) \\ \\ \dfrac{\partial f}{\partial \theta} \end{pmatrix} \ ; \ -h \leq \theta \leq 0,$$

$$D(\Lambda) = \{\begin{pmatrix} r \\ f(\cdot) \end{pmatrix} \in \mathcal{M}_2 : f \ \ is \ \ a.c., \frac{\partial f}{\partial \theta} \in L^2([-h,0];\mathbf{R}) \ \ and \ \ f(0) = r\},$$

Hence, the solution of system (6) is given by the variation of constants formula or mild solution:

$$z(t) = T(t)z_0 + \int_0^t T(t-s)Bu(s)ds. \tag{9}$$

This formula has been extended in [4],[7],[3] and [5] to parabolic differential equations with delay. Particularly in [5], where they express the associated semigroup as a series of strongly continuous semigroups and orthogonal projections related with the eigenvalues of the Laplacian operator ($A = -\frac{\partial}{\partial^2}$); this representation allows them to reduce the controllability of this partial differential equation with delay to a family of ordinary delay equations and prove the main result of this work.

3 Proof of the Main Theorems

In this section we shall prove the results announced in the introduction and abstract of this work. To this end, we will give the definition of exact and approximate controllability in terms of the control system governed by the abstract ordinary differential equation (8).

Definition 3.1. ([6], [9], [12], [14]) System (8) is approximately controllable on $[0,\tau]$, $0 < \tau < \infty$ (in finite time) if for any z_0, z_1 in $M_2([-h,0])$ and $\epsilon > 0$ there exists $u \in L_1([0,\tau], \mathbf{R}^n)$ such that $||z(\tau, z_0, u) - z_1|| \leq \epsilon$, where $z(\cdot,\cdot,\cdot)$ is the solution (mild solution) of (8).

Definition 3.2. ([6], [9], [12], [14]) System (8) is exactly controllable on $[0,\tau]$, $0 < \tau < \infty$ (in finite time) in case $\epsilon = 0$ in the foregoing definition.

Remark 3.3. The following result was proved in [2]: If the semigroup $\{T(t)\}$ is compact, then the general system $z' = Az + Bu(t)$ can never be exactly controllable for any $\tau > 0$. It is well known that the heat equation and the delay differential equation generate a compact semigroup, therefore system (8) is not exactly controllable. Also, since B is a compact operator, applying Theorems 1.1 and 1.2 of [14] and Theorem 4.1.5 of [6] we obtain that the system (8) is not exactly controllable.

In order to apply Theorem 4.2.10 from [6], we have to consider the following matrices:

$$\Delta(\lambda) = \lambda I_{n\times n} - A_0 - \sum_{j=1}^{N} A^{(j)} e^{-\lambda h_j}, \quad \forall \lambda \in \mathbf{C}, \tag{10}$$

$$B_0 = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & & B_N \end{pmatrix}, \quad A^{(N)} := \begin{pmatrix} 0 & \cdots & A_{1N} \\ 0 & \cdots & A_{2N} \\ \vdots & & \vdots \\ 0 & \cdots & A_{NN} \end{pmatrix}, \tag{11}$$

$$(\Delta(\lambda) : B_0) = \begin{pmatrix} B_0 & \vdots & \Delta(\lambda)B_0 & \vdots & \Delta(\lambda)^2 B_0 & \vdots & \cdots & \vdots & \Delta(\lambda)^{n-1} B_0 \end{pmatrix} \tag{12}$$

and

$$(A^{(N)} : B_0) = \begin{pmatrix} B_0 & \vdots & A^{(N)} B_0 & \vdots & {A^{(N)}}^2 B_0 & \vdots & \cdots & \vdots & {A^{(N)}}^{n-1} B_0 \end{pmatrix} \tag{13}$$

where $A_{ij},\ B_k;\ i,j,k = 1, \cdots, N$ are previously defined.

Now we are ready to formulate our main results:

Theorem 3.4. *If* $\mathrm{rank}(B_i) = n_0,\ \forall i = 1, \cdots, N$*, then system (8) is approximately controllable.*

Proof. If $\mathrm{rank}(B_i) = n_0\ \forall i = 1, \cdots, N$, then $\mathrm{rank}(B_0) = n = n_0 N$, therefore looking at (12) and (13) it is straightforward that

$$\mathrm{rank}(A^{(N)} : B_0) = \mathrm{rank}(\Delta(\lambda) : B_0) = n, \quad \forall \lambda \in \mathbf{C}, \tag{14}$$

which guarantee the approximate controllability of (8) according to Theorem 4.2.10 from [6]. □

Remark 3.5. In general a system like (8) will be approximately controllable if the condition (14) is satisfied, and it will depend on a particular Cellular Neural Network with delays.

4 Conclusion and Future Works

As one can see, the main result of this paper is based on Theorem 4.2.10 from [6], which assumes that the delays h_j are constants: so, in order to consider the case when the $h_j = h_j(t)$ depends on time t one has to extend Theorem 4.2.10 from [6] to this case, which does not seem to be straightforward. In other word, the following problem is open:

$$\begin{cases} \dot{x} & = A_0 x(t) + \sum_{j=1}^{N} A^{(j)} x(t - h_j(t)) + B_0 u,\ t \geq 0, \\ x(0) & = r,\ r \in \mathbf{R}^n, \\ x(\theta) & = f(\theta), \quad \theta \in [-h, 0), \end{cases} \tag{15}$$

where $0 < h_i(t) < h$ are continuous and bounded delays; h is constant.

Using the variation of constant formula for parabolic equation with delay found in [5], one can intent to investigate the approximate controllability of the following CNN with diffusion coefficients and delays:

$$\begin{cases} \dfrac{\partial z(t,x)}{\partial t} & = & D\Delta z + \displaystyle\sum_{j=1}^{N} A^j z(t-h_j,x) + B_0 u(t), \quad t \in (0,\tau], \\ \dfrac{\partial z}{\partial \eta} & = & 0, \quad x \in \partial\Omega, \quad t \in (0,\tau], \\ z(0,x) & = & \phi_0(x), \quad x \in \Omega, \\ z(s,x) & = & \phi(s,x), \quad s \in [-h,0), \quad x \in \Omega \end{cases} \tag{16}$$

where $0 < h_1 < h_2 < \cdots < h_N$ represent the point delays, $h = h_N$, $B, A^j \in \mathcal{L}(\mathbf{R}^n)$, $j = 1, 2, \dots N$, u belong to $L^2([0,\tau];U)$ $(U = L^2(\Omega, \mathbf{R}^n))$, D is a $n \times n$ non diagonal matrix whose eigenvalues are semi-simple with non negative real part, and $\phi_0 \in Z$, $\phi \in L^2([-h,0];Z)$ with $Z = U$.

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