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The Origin of Tauberian Operators

El Origen de los Operadores Tauberianos

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Abstract

We describe the steps that led to introduce tauberian operators in Banach space theory in order to apply abstract methods to solve a problem in summability theory.
Key words and phrases: tauberian operator, tauberian matrix, weakly compact operator, reflexive Banach space.

Resumen

Describimos los pasos que llevaron a introducir los operadores tauberianos en la teoría de espacios de Banach, con el fin de aplicar métodos abstractos al estudio de un problema de sumabilidad.
Palabras y frases clave: operador tauberiano, matriz tauberiana, operador débilmente compacto, espacio de Banach reflexivo.

1 Introduction

In 1976, Kalton and Wilansky [19] coined the term *tauberian* to designate those (bounded linear) operators $T\colon X \longrightarrow Y$ acting between Banach spaces that satisfy

$$T^{**}(X^{**} \setminus X) \subset Y^{**} \setminus Y\,. \tag{1}$$

Since then, tauberian operators have found many successful applications in Banach space theory. For instance, they have been used in the celebrated

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factorization of Davis, Figiel, Johnson and Pełczyński [9], in the equivalence between the Radon-Nikodym property and the Krein-Milman property [25], and in the study of norm-attainment of linear functionals on subspaces [23] and measures of non-weakly compactness [3]. Of course, tauberian operators have been extensively studied ([16], [18], [14]), and also, have been generalized, localized, dualized, etc. giving rise to the appearance of new related classes of operators (co-tauberian operators [27], [2], [12], semi-tauberian operators [5], supertauberian operators [26], [13], strongly tauberian operators [24], and other classes that are quoted in [1]).

While a significative part of the properties and applications of tauberian operators in Banach space theory is summarized in [11], and a detailed exposition of these results will appear in [15], this paper is mainly concerned with the origin and circumstances under which tauberian operators appeared in Banach space theory. In particular, we intend to find answers for the following pair of questions:

Question 1. Why are they called *tauberian*?

Question 2. When and why did those operators come into sight?

Section 2 is devoted to Question 1. The remaining sections tell us how tauberian operators, –a typical notion of Banach space theory–, come from a classic problem of summability: the identification of all those matrices that sum no divergent sequence. That problem is explained with more detail in Section 3, where we include a brief account of the first attempts in solving it. In particular, we show how Crawford used some techniques of duality, which led to the application of functional analysis in summability theory (indeed, one of the goals of [31], according to his author, was to popularize Crawford's results). In section 4, we explain how a further abstraction on Crawford's result gave birth to the notion of tauberian operator such at is has been defined in formula (1).

2 Tauberian conditions in summability

In order to answer Question 1, we need to go back in time to 1897, when Tauber proved that if

$$\lim_{x\to 1^-} \sum_{n=0}^{\infty} a_n x^n = \lambda \tag{2}$$

and

(3) $$\lim_n n^{-1}a_n = 0$$

then

(4) $$\sum_{n=0}^{\infty} a_n = \lambda.$$

This is a conditioned converse of Abel's theorem which states that (2) is a consequence of (4) without the mediation of any hypothesis such as (3). Since then, it has been customary to classify certain types of theorems into abelian or tauberian according to the following: an *abelian (*or direct*) theorem* is a theorem whose converse fails but becomes true if certain additional hypothesis –usually named *tauberian condition*– is considered, in which case, that modified converse is called *tauberian theorem.* Indeed, Hardy [17] described this classification with the following words:

> *"A tauberian theorem may be defined as the corrected form of the false converse of an abelian theorem. An abelian theorem asserts that, if a sequence or function behaves regularly, then some average of it behaves regularly."*

It is not simple at all to provide a more precise definition of tauberian theorem in regard to the large list of fields where tauberian theorems occur. We refer to the recent monograph [20] for an authorized description of this topic.

Let us fix now an operator $T\colon X \longrightarrow Y$ and consider the following statement:

(5) (x_n) *contains a weakly convergent subsequence if* (Tx_n) *is convergent and the tauberian condition of boundedness of* (x_n) *holds.*

The main result in [19] establishes that statement (1) is satisfied by T if and only if (5) is so. This fact evidences the tauberian character of those operators satisfying (1), which answers Question 1.

In some sense, the corresponding *abelian* version of the tauberian operators are the *weakly compact operators*, which are defined as those operators $T\colon X \longrightarrow Y$ for which $T^{**}(X^{**}) \subset Y$, which turns out to be equivalent to say that for every bounded sequence (x_n) in X, (Tx_n) contains a weakly convergent subsequence.

3 Tauberian matrices

About Question 2, we will see that the concept of tauberian operator deeps its roots into summability theory, a branch of mathematics whose original purpose is assigning limits to sequences that are not convergent in the usual sense. One of the typical techniques in summability theory is the matrix method: consider an infinite matrix $A = (a_{ij})_{i=1}^{\infty}{}_{j=1}^{\infty}$. A sequence of real numbers $x = (x_i)_i$ is said to be A-summable (or A-limitable) if the sequence $Ax := (\sum_{j=1}^{\infty} a_{ij}x_j)_i$ is well defined and convergent. In that case, $\lim_i Ax$ is denoted $\lim_A x_i$ and assigned to the sequence x. Thus, denoting by c *the space of all convergent sequences* of scalar numbers, answers to the following questions are needed:

How is the set ω_A formed by all the sequences x for which Ax does exist?

How is the set c_A formed by all the A-summable sequences?

Does c_A contain c?

If $c \subset c_A$, does A preserve limits?

When $c \subset c_A$, A is called *conservative*. Moreover if $\lim_i x_i = \lim_A x_i$ for all $(x_i) \in c$ then A is called *normal*. A genuine example of the interest in normal matrices that sum bounded divergent sequences is provided by Féjer's theorem, which uses the Cesàro matrix to recover any function $f \in L_p(0, 2\pi)$ from its Fourier series.

General study of matrix methods was only affordable after the discovery in 1911 of the classical Toeplitz-Silverman conditions which assert that a matrix $A = (a_{ij})_{i=1}^{\infty}{}_{j=1}^{\infty}$ is conservative if and only if

(i) $\|A\| := \sup_i \sum_j |a_{ij}| < \infty$;

(ii) *there exists* $s := \lim_i s_i$, *where* $s_i := \sum_j a_{ij}$;

(iii) *there exists* $a_j := \lim_i a_{ij}$ *for each* j.

Indeed, Toeplitz-Silverman conditions allow us to identify every conservative matrix A with the operator $S_A : c \longrightarrow c$ and also with $T_A : \ell_\infty \longrightarrow \ell_\infty$, being both operators defined by the expression Ax when x belongs respectively to the domains c or ℓ_∞, so $\|S_A\| = \|T_A\| = \|A\|$.

Searching for criteria to decide whether or not a conservative matrix sums a bounded divergent sequence became an engaging activity during the fifties: [22], [28], [32], etc. The next decade brought new characterizations with an undoubtedly algebraic taste. Thus, Copping [7] obtained the following result:

(6) *Let A be a conservative matrix such that T_A is injective. Then A sums no bounded divergent sequence if and only if there is a conservative matrix B which is a left inverse of A.*

In 1964, Wilansky [30] improved Copping's result replacing the injectivity of T_A by the weaker condition of injectivity of S_A. For the same matrices that same year, Berg [4] obtained the following characterization:

(7) *Let A be a conservative matrix such that S_A is injective. Then A sums no bounded divergent sequence if and only if A is not a left-topological divisor of zero, that is, there exists $\varepsilon > 0$ such that for every norm one element $x \in c$, $\|Ax\| \geq \varepsilon$.*

Obviously, if S_A is injective then A is a left-topological divisor of zero if and only if the range of S_A is not closed. A definitive improvement dropped the hypothesis of injectivity of S_A in (7):

(8) *A conservative matrix A sums no bounded divergent sequence if and only if the operator $S_A : c \longrightarrow c$ has closed range and finite-dimensional null-space.*

A conservative matrix that sums no bounded divergent sequence is called *tauberian* [31].

Statement (8) was obtained with different proofs by Crawford in 1966 [8], Whitley in 1967 [29], and Garling and Wilansky in 1972 [10]. Each of the above mentioned papers meant a new stage in the increasing presence of functional analysis in summability theory, which paved the way for tauberian operators appearance. Crawford's main contribution to the proof of (8) is the introduction of duality techniques by means of the following result:

(9) *Given a conservative matrix A, we have that ${T_A}^{-1}(c) \subset c$ if and only if $S_A^{**\,-1}(c) \subset c$.*

Note that, in general, the operators T_A and S_A^{**} are not equal. Indeed, T_A is represented by the matrix A, but since the canonical embedding of c into its bidual space, ℓ_∞, maps every sequence (x_i) to $(\lim_i x_i, x_1, x_2, \ldots)$, the operator S_A^{**} is represented by the matrix

$$P = \begin{pmatrix} s & a_1 & a_2 & \cdots \\ s_1 - s & a_{11} - a_1 & a_{12} - a_2 & \cdots \\ s_2 - s & a_{21} - a_1 & a_{22} - a_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

The difficulty caused by the fact that $A \neq P$ was solved by Crawford by substitution of c for an isomorphic space, c_0, and taking advantage of the fact that for every operator $L\colon c_0 \longrightarrow c_0$, both L and L^{**} are representable by the same matrix. Thus he considered the surjective isomorphism $U\colon c_0 \longrightarrow c$ that maps e_1 to the constant sequence $(1, 1, \ldots)$ and e_i to e_{i-1} for $i > 1$, and takes the operator $R := U^{-1} S_A U$ which is matrix representable by P. Next, by means of classical techniques of matrix summability, Crawford obtains the following:

(10) $\quad {T_A}^{-1}(c) \subset c$ *if and only if* $(R^{**})^{-1}(c_0) \subset c_0$*;*

and since R is an isomorphism, statement (10) yields trivially (9).

4 Tauberian operators

Garling and Wilansky's innovation with respect to Crawford's proof is that they study an operator $T\colon X \longrightarrow Y$ satisfying $T^{**-1}(Y) \subset X$ prior to consideration of the case $X = Y = c$. So they deduce the following results:

(11) *Let $T\colon X \longrightarrow Y$ be an operator. Consider the conditions*

(i) $T^{**-1}(Y) \subset X$,

(ii) $N(T^{**}) \subset X$,

(iii) $N(T)$ *is reflexive.*

Then (i)⇒(ii)⇒(iii) *and neither implication can be reversed;*

(12) *If $R(T)$ is closed, then* (i), (ii) *and* (iii) *are equivalent.*

So Garling and Wilansky obtain (8) with the next argument: if A is a conservative matrix that sums no bounded divergent sequence then Crawford's result (9) yields $S_A^{**-1}(c) \subset c$, and by condition (i) in (11), it follows that $N(S_A^{**})$ is reflexive, and therefore finite-dimensional, because c contains no infinite-dimensional reflexive subspace. They offer no new proof of the fact that $R(S_A)$ is closed. Conversely, if $R(S_A)$ is closed and $N(S_A)$ is finite-dimensional then $N(S_A)$ is trivially reflexive, so (12) shows that $S_A^{**-1}(c) \subset c$, hence (9) yields that A sums no bounded divergent sequence.

As far as we know, Crawford's statement (9) contains the first application of tauberian operators, but condition (i) in (11) is the first appearance of tauberian operators with the same level of generality given in (1). Garling and Wilansky stimulated interest in tauberian operators posing the following questions:

Question 3. *For which pairs of non-reflexive Banach spaces X and Y can the assumption "closed range" be dropped in (12)?*

Question 4. *For which non-reflexive Banach spaces X and Y does condition (i) in (11) imply $R(T)$ closed?*

Kalton and Wilansky [19] found the following two results that include sufficient and necessary conditions for the equivalence between the three statements of (11).

Theorem 1. *For every $T \in \mathcal{L}(X, Y)$, the following statements are equivalent:*

(a) *T is tauberian;*

(b) *$N(T^{**}) = N(T)$ and $T(B_X)$ is closed;*

(c) *$N(T^{**}) = N(T)$ and $\overline{T(B_X)}$ is contained in $R(T)$.*

The notation B_X represents the unit closed ball of X centered at the origin.

Theorem 2. *For every $T \in \mathcal{L}(X, Y)$ the following statements are equivalent:*

(a) *$N(T^{**}) = N(T)$;*

(b) *if (x_n) is a bounded sequence in X and (Tx_n) is weakly null then (x_n) contains a weakly convergent subsequence;*

(c) *if (x_n) is a bounded sequence in X and (Tx_n) is null then (x_n) contains a weakly convergent subsequence.*

Theorems 1 and 2 became essential in any further study of tauberian operators, and were proved using only functional analysis techniques. The paper [19], where these theorems appeared, popularized the term *tauberian* for the operators defined in (1).

Full answers to Questions 3 and 4 are still unknown. However, the following sufficient condition was shown in [19]:

(13) *If X contains no reflexive infinite-dimensional subspace and $T : X \longrightarrow Y$ is tauberian then T is upper semi-Fredholm.*

Let us recall that an operator $T\colon X \longrightarrow Y$ is said to be *upper semi-Fredholm* if it has closed range and finite dimensional kernel. The reader will realize that (13), combined with Crawford's result (9), yields an immediate proof of (8). This observation was made by Wilansky in [31, section 17.6]. But the most important fact concerning [19] is that it roused research focused on tauberian operators. In fact, Kalton and Wilansky suggested that statement (13) could be extended to more Banach spaces X other than those with no reflexive infinite-dimensional subspaces. In particular, as c_0 is isomorphic to a space of continuous functions, they posed the following question:

Question 5. Given a pair of spaces of continuous functions, $C(K)$ and $C(L)$, is a tauberian map $T\colon C(K) \longrightarrow C(L)$ an isomorphism in some sense?

Question 5 was partially solved by Lotz, Peck and Porta [21], who proved that a compact space K is scattered if and only if every injective tauberian operator $T\colon C(K) \longrightarrow Y$ is an isomorphism.

It is not difficult to prove that an operator T is tauberian if T^{**} is so. Kalton and Wilansky asked [19]:

Question 6. When is it true that an operator $T\colon X \longrightarrow Y$ is tauberian if and only if T^{**} is so?

Question 6 was suggested by the fact that its answer is positive when T has closed range. Nevertheless, it was shown in [2] that the answer is negative, in general.

It is quite simple to prove that an operator $T\colon X \longrightarrow Y$ is tauberian if and only if the induced operator $T^{co}\colon X^{**}/X \longrightarrow Y^{**}/Y$, given by

$$T^{co}(x^{**} + X) := T^{**}(x^{**}) + Y,$$

is injective. Kalton and Wilansky asked:

Question 7. Given an operator $T \in \mathcal{L}(X, Y)$, when is T^{co} an isomorphism?

The operators T for which T^{co} is an isomorphism have been studied by Rosenthal, who called them *strongly tauberian* [24]. He proved that if an operator T is strongly tauberian then T^{**} is also strongly tauberian.

5 Final remarks

Such as we have accounted, the first work entirely devoted to tauberian operators is [19], which came to light in 1976. But there are two papers more concerning tauberian operators, [9] and [33], respectively published in 1974 and 1976. The authors of [19] and [33], prior submission, were acquainted with the contents of the three mentioned papers, but a closer look to them reveals that actually [9], [19] and [33] are mathematically independent and pursue different ends. Indeed, in [33], Yang extends Fredholm's theory to tauberian operators with closed range. His results lead to a homological description of reflexivity in Banach spaces. In [9], Davis, Figiel, Johnson and Pełczyński obtained their famous factorization for operators. Finally, [19] can be regarded as the continuation of the work of Garling and Wilansky [10], putting an endpoint to a longstanding problem in summability theory: the characterization of the tauberian matrices. These arguments have led us to consider [10] and [19] as the seminal papers in the study of tauberian operators.

Let us notice that the role played by the tauberian operators in the solution of the aforementioned problem has been recognized by the summability theorists [20, p. 262]. As accounts of summability theory, we recommend [6], [20] or [31]. The first two ones are exhaustive monographs, while the third one is concise and contains most of the results of Crawford's Ph.D. thesis.

The opposite character of tauberian operators and weakly compact operators is also observable in other pairs of classes of operators. For instance, upper semi-Fredholm operators and compact operators. In [1], these pairs of classes of operators has been studied from a homological point of view.

Of course, more questions, results and applications concerning tauberian operators appear in the papers we have cited. The reader interested in the subject can find an exhaustive reference list in [1].

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