Divulgaciones Matemáticas Vol. 10 No. 1(2002), pp. 43–49

Some Results on Semigroups

Algunos Resultados sobre Semigrupos

Mohamed Akkouchi (makkouchi@caramail.com)
Département de Mathématiques
Faculté des Sciences-Semlalia
Université Cadi Ayyad
Avenue du Prince My. Abdellah. B.P. 2390
40.000, Marrakech. MAROC (MOROCCO)

Abstract

In this note we state some results concerning the perturbation problem of strongly continuous semigroups on a Banach space X and provide some new characterizations of the uniformly continuous semigroups on X.
Key words and phrases: Strongly continuous semigroups on Banach spaces. Perturbations of semigroups. Characterizations of uniformly continuous semigroups.

Resumen

En esta nota establecemos algunos resultados concernientes al problema de perturbación de semigrupos fuertemente continuos sobre un espacio de Banach X y proporcionamos algunas nuevas caracterizaciones de los semigrupos uniformemente continuos sobre X.
Palabras y frases clave: semigrupos fuertemente continuos sobre un espacio de Banach. Perturbaciones de semigrupos. Caracterizaciones de semigrupos uniformemente continuos.

1 Introduction

1.1 In semigroup theory, we have the following perturbation problem. Let $(A, D(A))$ be the generator of a strongly continuous semigroup on a Banach space X. Consider another operator $(B, D(B))$ on X, and let c be a complex constant. Find the conditions such that the sum $A + cB$ generates a

Recibido 2000/10/16. Aceptado 2002/04/10.
MSC (2000): 47D06.

strongly continuous semigroup on X. The purpose of this paper is twofold. The first purpose is to provide some answers to this problem. The results obtained in are established in Section 2. The second purpose is to provide some new characterizations of uniformly continuous semigroups on X. These characterizations are established in Section 3. The section four contains some observations and remarks in connection with the results obtained in the sections 2 and 3. For the sequel, we need to make some recalls and notations.

1.2 Let $(X, \|.\|)$ be a (complex) Banach space. Let $(A, D(A))$ be a (possibly unbounded) operator on X. Consider a second operator $(B, D(B))$ on X. We say (see [1]) that B is (relatively) A-bounded if $D(A) \subset D(B)$ and if there exist positive constants a, b such that

$$\|Bx\| \leq a \|Ax\| + b \|x\|, \tag{1.1}$$

for all x in the domain $D(A)$ of A. The space $D(A)$ will be endowed with the norm $\|x\|_A := \|Ax\| + \|x\|$. A subset D of $D(A)$ will be called a core of A if it is dense in $D(A)$ equipped with the norm $\|.\|_A$.

Suppose now, that $(A, D(A))$ is the generator of a bounded strongly continuous semigroup $(T(t))_{t\geq 0}$ on X. Then (see [2], p. 7) A verifies the Landau-Kolmogorov inequality, that is

$$\|Ax\|^2 \leq 4M^2 \left\|A^2x\right\| \|x\|, \ \forall x \in D(A^2), \tag{1.2}$$

where M is a positive constant such that $\sup_{t\geq 0} \|T(t)\| \leq M$. We set $D(A^\infty) := \cap_{n\geq 1} D(A^n)$. We recall (see [1], p. 53) that $D(A^\infty)$ is a core of A and that $D(A^n)$ is a core of A for every integer $n \geq 1$.

2 Some results on perturbations of semigroups

The purpose of this section is to provide some results in connection with the perturbation problems of semigroups on Banach spaces. Our main result in this area of investigations is the following.

2.1 Theorem: *Let $(A, D(A))$ be the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on the Banach space X. Let $(B, D(B))$ be a second operator on X verifying the following assumptions*
(H_1) $R(B) \subset D(A)$, where $R(B)$ is the range of B.
(H_2) The set $D := D(B^\infty)$ is a core of A such that

$$\|Bx\|^2 \leq \left\|B^2x\right\| \|x\|, \ \forall x \in D. \tag{2.1}$$

(H_3) *B is A-bounded.*
(H_4) *$A \circ B$ is B-bounded.*
Then B and $A \circ B$ are bounded on X. Consequently, the operators $(A + cB, D(A))$ and $(A + cA \circ B, D(A))$ are generators of strongly continuous semigroups on X for every complex number c.

Proof: a) replacing x by $B^{n-1}x$ in (2.1), we deduce

$$\|B^n x\|^2 \leq \left\|B^{n+1}x\right\| \left\|B^{n-1}x\right\|, \ \forall x \in D, \text{ and } \forall n \in \mathbb{N}^*, \tag{2.2}$$

where $\mathbb{N}^*$ is the set of all strictly positive integers. By induction (2.2) yields to

$$\|B^n x\|^2 \leq \left\|B^{2n}x\right\| \|x\|, \ \forall x \in D, \text{ and } \forall n \in \mathbb{N}^*. \tag{2.3}$$

The assumptions (H_3) and (H_4) will ensure the existence of four positive constants a_1, a_2, b_1, b_2 such that

$$\begin{aligned} \|Bx\| &\leq a_1 \|Ax\| + b_1 \|x\| \ \forall x \in D, \\ \|A \circ Bx\| &\leq a_2 \|Ax\| + b_2 \|x\| \ \forall x \in D. \end{aligned} \tag{2.4}$$

From the last inequalities, we obtain

$$\|Bx\|_A \leq a \|Ax\| + b \|x\| \ \forall x \in D, \tag{2.5}$$

where $a = a_1(1 + a_2)$ and $b = (1 + a_2)b_1 + b_2$. Since D is a core of A, we deduce from (2.5) that the operator $B : (D(A), \|\|_A) \longrightarrow (D(A), \|\|_A)$ is bounded. Thus we can find a positive constant β such that

$$\|Bx\|_A \leq \beta \|x\|_A, \ \forall x \in D(A). \tag{2.6}$$

b) By using (2.3) we get, for every integer n and every $x \in D$, the following inequalities

$$\begin{array}{rcl} \left\|B^{2^n}x\right\|^2 & \leq & \left\|B^{2^{n+1}}x\right\| \|x\|, \\ \left\|B^{2^{n-1}}x\right\|^{2^2} & \leq & \left\|B^{2^n}x\right\|^2 \|x\|^2, \\ \cdots\cdots\cdots\cdots & \cdots & \cdots\cdots\cdots\cdots\cdots \\ \|Bx\|^{2^{n+1}} & \leq & \left\|B^2 x\right\|^{2^n} \|x\|^{2^n}. \end{array}$$

These inequalities allow us to deduce

$$\|Bx\|^{2^{n+1}} \leq \left\|B^{2^{n+1}}x\right\| \|x\|^{2^{n+1}-1}, \ \forall x \in D \text{ and } \forall n \in \mathbb{N}. \tag{2.7}$$

For every $x \in D \setminus \{0\}$, we get from (2.6) and (2.7) the following inequality

$$\frac{\|Bx\|}{\|x\|} \leq \beta \left[\frac{\|x\|_A}{\|x\|} \right]^{\frac{1}{2^{n+1}}}, \ \forall n \in \mathbb{N}. \tag{2.8}$$

Letting $n \to \infty$ in (2.8), we conclude that

$$\|Bx\| \leq \beta \|x\|, \ \forall x \in D. \tag{2.9}$$

Since D is a core of A then it is dense in $D(A)$. Hence, (2.9) says that the operator B extends in a unique manner to the whole space X in a bounded operator. From (2.4), we deduce that $A \circ B$ extends in a unique manner to X in a bounded operator. The remainder is a consequence from the general theory of semigroups (see Theorem 1.3, p. 158 in [1]). □

2.2 Corollary: *Let H be a Hilbert space endowed with the inner product $<,>$, and let $(A, D(A))$ be the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on H. Let $(B, D(B))$ be a second operator on H verifying (H_1), (H_3), (H_4) together with the following assumption*
$(H_2)'$ $< Bx, y > = < x, By >$, for all $x, y \in D$, where D is a core of A contained in the set $D(B^\infty)$.
Then B and $A \circ B$ are bounded on X. Consequently, the operators $(A + cB, D(A))$ and $(A + cA \circ B, D(A))$ are generators of strongly continuous semigroups on X for every complex number c.

Proof: It is sufficient to see that $(H_2)'$ implies (H_2). But this is an easy consequence of the Cauchy-Schwarz inequality applied to the hermitian form $F : D \times D \longrightarrow \mathbb{C}$ given by $F(x, y) := < Bx, By >$. □

2.3 Remark: The previous results are valid if we replace the assumptions (H_3) and (H_4) by the following assumptions
$(H_3)'$ $D(A) \subset D(B)$, and
$(H_4)'$ the operator $B : (D(A), \|.\|_A) \longrightarrow (D(A), \|.\|_A)$ is bounded.

3 Characterizations of uniformly continuous semigroups

The purpose of this section is to establish some new characterizations for the uniformly continuous semigroups on Banach spaces. Our first result is the following.

3.1 Theorem: *Let $(A, D(A))$ be the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on the Banach space X. Then the following assertions are equivalent.*
(1) The semigroup $(T(t))_{t\geq 0}$ is uniformly continuous.
(2) The operator $A : (D(A^2), \|.\|_A) \longrightarrow (D(A), \|.\|_A)$ is bounded.
(3) The operator $A^2 : (D(A^2), \|.\|_A) \longrightarrow (X, \|.\|)$ is bounded.

Proof: a) We know that there exist constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that

$$\|T(t)\| \leq Me^{\omega t} \tag{3.1}$$

for all $t \geq 0$. Then replacing A by $A-\omega I$, we may suppose that $\sup_{t\geq 0} \|T(t)\| \leq M$.
b) Suppose that (1) holds true. Then $D(A) = D(A^\infty) = X$ and $A \in \mathcal{L}(X)$ the Banach space of all bounded operators on X. Then for every $x \in X$, we have $\|Ax\|_A \leq [1 + \|A\|]\|x\|_A$. Thus (2) is true.
c) Suppose that (2) holds true. Then there exists a positive constant $C > 0$ such that $\|Ax\|_A = \left\|A^2x\right\| + \|Ax\| \leq C[\|Ax\| + \|x\|]$ for all $x \in D(A^2)$. This inequality implies (3).
d) Suppose that (3) holds true. Then there exists a positive constant $C > 0$ such that $\left\|A^2x\right\| \leq C[\|Ax\| + \|x\|]$ for all $x \in D(A^2)$. By using the Landau-Kolmogorov inequality we deduce that there exists another positive constant designated again by C such that $\|Ax\|^2 \leq C[\|Ax\|+\|x\|]\,\|x\|$ for all $x \in D(A^2)$. Let us denote $S_{D(A^2)}$ the set of unit vectors in $D(A^2)$. Then we have

$$\|Au\|^2 - C\,\|Au\| - C \leq 0,$$

for all $u \in S_{D(A^2)}$. Since the set S of numbers $s \in [0, \infty[$ verifying $s^2 - Cs - C \leq 0$ is bounded by the constant $\frac{C+\sqrt{C[C+4]}}{2}$ we deduce that $\sup_{u\in S_{D(A^2)}} \|Au\|$ is finite, which means that A is bounded, hence (1) is true. □

By induction and similar arguments to those used in the proof of Theorem 3.1, one can establish the following

3.2 Theorem: *Let $(A, D(A))$ be the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on the Banach space X. Then the following assertions are equivalent.*
(1) The semigroup $(T(t))_{t\geq 0}$ is uniformly continuous.
(2) For all integer $n \in \mathbb{N}^*$, the operator $A^n : (D(A^{n+1}), \|.\|_A) \longrightarrow (D(A), \|.\|_A)$ is bounded.

(3) There exists at least an integer $n \in \mathbb{N}^*$, such that the operator $A^n : (D(A^{n+1}), \|.\|_A) \longrightarrow (D(A), \|.\|_A)$ is bounded.
(4) For all integer $n \in \mathbb{N}^*$, the operator $A^n : (D(A^n), \|.\|_A) \longrightarrow (X, \|.\|)$ is bounded.
(5) There exists at least an integer $n \in \mathbb{N}^*$, such that the operator $A^n : (D(A^n), \|.\|_A) \longrightarrow (X, \|.\|)$ is bounded.

4 Remarks

Let X be a Banach space and let let $(A, D(A))$ be the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X verifying $\|T(t)\| \leq M$ for all $t \geq 0$, where M is a positive constant. Then we have the following observations.

4.1 The operator A is A^2-bounded. Indeed, one has $D(A^2) \subset D(A)$, and by using Landau-Kolmogorov inequality and Young inequality, we get

$$\|Ax\| \leq 2M[\left\|A^2x\right\| \|x\|]^{\frac{1}{2}} \leq M[\left\|A^2x\right\| + \|x\|]$$

for all $x \in D(A^2)$.

4.2 Suppose that $D(A^2) = D(A)$, then the following assertions are equivalent:
(i) The semigroup $(T(t))_{t\geq 0}$ is uniformly continuous.
(ii) The operator A^2 is A-bounded.

4.3 Suppose that $X = H$ is a Hilbert space endowed with an inner product $<,>$ and that A verifies the following property:

$$< Ax, y >=< x, Ay >, \ \forall\, x, y \in D(A). \tag{4.1}$$

Then the Landau-Kolomogorov inequality is improved to the following inequality

$$\|Ax\|^2 \leq \left\|A^2x\right\| \|x\| \quad \forall\, x, y \in D(A^2), \tag{4.2}$$

which is a consequence of Cauchy-Schwarz inequality.

We set $\mathcal{E}_A$ the collection of all linear subspaces E of H verifying
(i) $D(A) \subset E$, and
(ii) There exists a norm $\|.\|_E$ for which E is a Banach space and the identity map $i_E : (E, \|.\|_E) \longrightarrow H$ is continuous.
Then, the semigroup $(T(t))_{t\geq 0}$ is uniformly continuous if one of the following assertions is satisfied:
(1) For all $E \in \mathcal{E}_A$, the operator $A : (D(A^2), \|.\|_E) \longrightarrow (D(A), \|.\|_E)$ is bounded.

(2) There exists at least an $E \in \mathcal{E}_A$ such that the operator $A : (D(A^2), \|.\|_E) \longrightarrow (D(A), \|.\|_E)$ is bounded.

References

[1] Engel, K-J., Nagel, R. *Semigroups for Linear Evolution Equations*, Graduate Texts in Math., vol. 194, Springer-Verlag, 2000.

[2] Pazy, A. *Semigroups for Linear Operators and Applications to Partial Differential Equations*, Appl. Math. Sci., vol. 44, Springer-Verlag, 1983.