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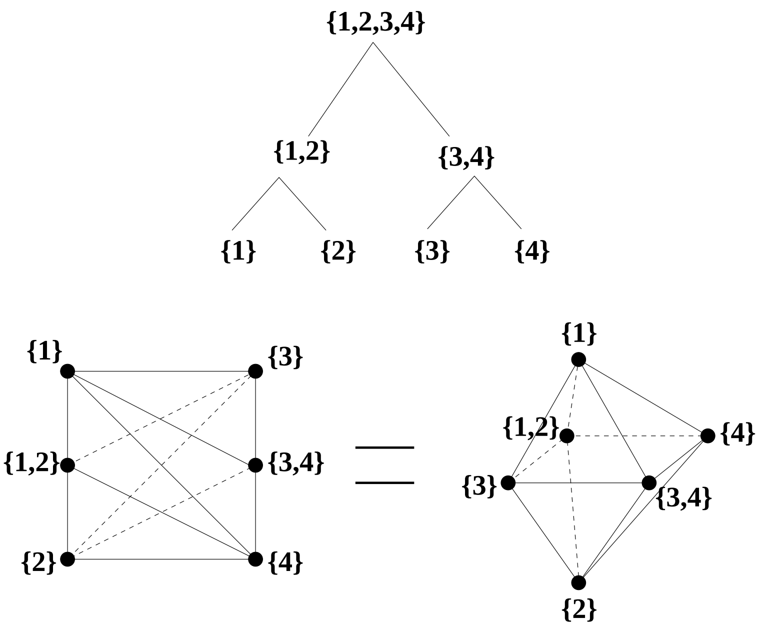

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Hirzebruch-Mumford Proportionality and Locally Symmetric Varieties of Orthogonal Type

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Abstract. For many classical moduli spaces of orthogonal type there are results about the Kodaira dimension. But nothing is known in the case of dimension greater than 19. In this paper we obtain the first results in this direction. In particular the modular variety defined by the orthogonal group of the even unimodular lattice of signature $(2, 8m + 2)$ is of general type if $m \geq 5$.

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1 Modular varieties of orthogonal type

Let L be an integral indefinite lattice of signature $(2, n)$ and $(\ , \ )$ the associated bilinear form. By $\mathcal{D}_L$ we denote a connected component of the homogeneous type IV complex domain of dimension n

$$\mathcal{D}_L = \{[w] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (w, w) = 0,\ (w, \overline{w}) > 0\}^+.$$

$\mathrm{O}^+(L)$ is the index 2 subgroup of the integral orthogonal group $\mathrm{O}(L)$ that leaves $\mathcal{D}_L$ invariant. Any subgroup Γ of $\mathrm{O}^+(L)$ of finite index determines a modular variety

$$\mathcal{F}_L(\Gamma) = \Gamma \setminus \mathcal{D}_L.$$

By [BB] this is a quasi-projective variety.

For some special lattices L and subgroups $\Gamma < \mathrm{O}^+(L)$ one obtains in this way the moduli spaces of polarised abelian or Kummer surfaces ($n = 3$, see [GH]), the moduli space of Enriques surfaces ($n = 10$, see [BHPV]), and the moduli spaces of polarised or lattice-polarised K3 surfaces ($0 < n \leq 19$, see [Nik1, Dol]). Other interesting modular varieties of orthogonal type include the period domains of irreducible symplectic manifolds: see [GHS3].

It is natural to ask about the birational type of $\mathcal{F}_L(\Gamma)$. For many classical moduli spaces of orthogonal type there are results about the Kodaira dimension, but nothing is known in the case of dimension greater than 19. In this paper we obtain the first results in this direction. We determine the Kodaira dimension of many quasi-projective varieties associated with two series of even lattices.

To explain what these varieties are, we first introduce the *stable orthogonal group* $\widetilde{\mathrm{O}}(L)$ of a nondegenerate even lattice L. This is defined (see [Nik2] for more details) to be the subgroup of $\mathrm{O}(L)$ which acts trivially on the discriminant group $A_L = L^\vee/L$, where $L^\vee$ is the dual lattice. If $\Gamma < \mathrm{O}(L)$ then we write $\widetilde{\Gamma} = \Gamma \cap \widetilde{\mathrm{O}}(L)$. Note that if L is unimodular then $\widetilde{\mathrm{O}}(L) = \mathrm{O}(L)$.

The first series of varieties we want to study, which we call the modular varieties of *unimodular type*, is

$$\mathcal{F}_{II}^{(m)} = \mathrm{O}^+(II_{2,8m+2})\backslash\mathcal{D}_{II_{2,8m+2}}. \tag{1}$$

$\mathcal{F}_{II}^{(m)}$ is of dimension $8m + 2$ and arises from the even unimodular lattice of signature $(2, 8m + 2)$

$$II_{2,8m+2} = 2U \oplus mE_8(-1),$$

where U denotes the hyperbolic plane and $E_8(-1)$ is the negative definite lattice associated to the root system E_8. The variety $\mathcal{F}_{II}^{(2)}$ is the moduli space of elliptically fibred K3 surfaces with a section (see e.g. [CM, Section 2]). The case $m = 3$ is of particular interest: it arises in the context of the fake Monster Lie algebra [B1].

The second series, which we call the modular varieties of *K3 type*, is

$$\mathcal{F}_{2d}^{(m)} = \widetilde{\mathrm{O}}^+(L_{2d}^{(m)})\backslash\mathcal{D}_{L_{2d}^{(m)}}. \tag{2}$$

$\mathcal{F}_{2d}^{(m)}$ is of dimension $8m + 3$ and arises from the lattice

$$L_{2d}^{(m)} = 2U \oplus mE_8(-1) \oplus \langle -2d\rangle,$$

where $\langle -2d\rangle$ denotes a lattice generated by a vector of square $-2d$.

The first three members of the series $\mathcal{F}_{2d}^{(m)}$ have interpretations as moduli spaces. $\mathcal{F}_{2d}^{(2)}$ is the moduli space of polarised $K3$ surfaces of degree $2d$. For $m = 1$ the 11-dimensional variety $\mathcal{F}_{2d}^{(1)}$ is the moduli space of lattice-polarised K3 surfaces, where the polarisation is defined by the hyperbolic lattice $\langle 2d\rangle \oplus E_8(-1)$ (see [Nik1, Dol]). For $m = 0$ and d prime the 3-fold $\mathcal{F}_{2d}^{(0)}$ is the moduli space of polarised Kummer surfaces (see [GH]).

Theorem 1.1 *The modular varieties of unimodular and* K3 *type are varieties of general type if* m *and* d *are sufficiently large. More precisely:*

(i) If $m \geq 5$ *then the modular varieties* $\mathcal{F}_{II}^{(m)}$ *and* $\mathcal{F}_{2d}^{(m)}$ *(for any* $d \geq 1$*) are of general type.*

(ii) For $m = 4$ *the varieties* $\mathcal{F}_{2d}^{(4)}$ *are of general type if* $d \geq 3$ *and* $d \neq 4$.

(iii) For $m = 3$ *the varieties* $\mathcal{F}_{2d}^{(3)}$ *are of general type if* $d \geq 1346$.

(iv) For $m = 1$ *the varieties* $\mathcal{F}_{2d}^{(1)}$ *are of general type if* $d \geq 1537488$.

Remark. The methods of this paper are also applicable if $m = 2$. Using them, one can show that the moduli space $\mathcal{F}_{2d}^{(2)}$ of polarised K3 surfaces of degree $2d$ is of general type if $d \geq 231000$. This case was studied in [GHS2], where, using a different method involving special pull-backs of the Borcherds automorphic form Φ_{12} on the domain $\mathcal{D}_{II_{2,26}}$, we proved that $\mathcal{F}_{2d}^{(2)}$ is of general type if $d > 61$ or $d = 46$, 50, 54, 57, 58, 60.
The methods of [GHS2] do not appear to be applicable in the other cases studied here. Instead, the proof of Theorem 1.1 depends on the existence of a good toroidal compactification of $\mathcal{F}_L(\Gamma)$, which was proved in [GHS2], and on the exact formula for the Hirzebruch-Mumford volume of the orthogonal group found in [GHS1].
We shall construct pluricanonical forms on a suitable compactification of the modular variety $\mathcal{F}_L(\Gamma)$ by means of modular forms. Let $\Gamma < \mathrm{O}^+(L)$ be a subgroup of finite index, which naturally acts on the affine cone $\mathcal{D}_L^\bullet$ over $\mathcal{D}_L$. In what follows we assume that $\dim \mathcal{D}_L \geq 3$.

Definition 1.2 *A modular form of weight* k *and character* $\chi\colon \Gamma \to \mathbb{C}^*$ *with respect to the group* Γ *is a holomorphic function*

$$F : \mathcal{D}_L^\bullet \to \mathbb{C}$$

which has the two properties

$$\begin{aligned} F(tz) &= t^{-k}F(z) \quad \forall\, t \in \mathbb{C}^*, \\ F(g(z)) &= \chi(g)F(z) \quad \forall\, g \in \Gamma. \end{aligned}$$

The space of modular forms is denoted by $M_k(\Gamma, \chi)$. The space of cusp forms, i.e. modular forms vanishing on the boundary of the Baily–Borel compactification of $\Gamma\backslash\mathcal{D}_L$, is denoted by $S_k(\Gamma, \chi)$. We can reformulate the definition of modular forms in geometric terms. Let $F \in M_{kn}(\Gamma, \det^k)$ be a modular form, where n is the dimension of $\mathcal{D}_L$. Then

$$F(dZ)^k \in H^0(\mathcal{F}_L(\Gamma)^\circ, \Omega^{\otimes k}),$$

where dZ is a holomorphic volume form on $\mathcal{D}_L$, Ω is the sheaf of germs of canonical n-forms on $\mathcal{F}_L(\Gamma)$ and $\mathcal{F}_L(\Gamma)^\circ$ is the open smooth part of $\mathcal{F}_L(\Gamma)$ such that the projection $\pi : \mathcal{D}_L \to \Gamma\backslash\mathcal{D}_L$ is unramified over $\mathcal{F}_L(\Gamma)^\circ$.

The main question in the proof of Theorem 1.1 is how to extend the form $F(dZ)^k$ to $\mathcal{F}_L(\Gamma)$ and to a suitable toroidal compactification $\mathcal{F}_L(\Gamma)^{\mathrm{tor}}$. There are three possible kinds of obstruction to this, which we call (as in [GHS2]) elliptic, reflective and cusp obstructions. Elliptic obstructions arise if $\mathcal{F}_L(\Gamma)^{\mathrm{tor}}$ has non-canonical singularities arising from fixed loci of the action of the group Γ. Reflective obstructions arise because the projection π is branched along divisors whose general point is smooth in $\mathcal{F}_L(\Gamma)$. Cusp obstructions arise when we extend the form from $\mathcal{F}_L(\Gamma)$ to $\mathcal{F}_L(\Gamma)^{\mathrm{tor}}$.
The problem of elliptic obstructions was solved for $n \geq 9$ in [GHS2].

Theorem 1.3 *([GHS2, Theorem 2.1]) Let L be a lattice of signature $(2,n)$ with $n \geq 9$, and let $\Gamma < \mathrm{O}^+(L)$ be a subgroup of finite index. Then there exists a toroidal compactification $\mathcal{F}_L(\Gamma)^{\mathrm{tor}}$ of $\mathcal{F}_L(\Gamma) = \Gamma\backslash\mathcal{D}_L$ such that $\mathcal{F}_L(\Gamma)^{\mathrm{tor}}$ has canonical singularities and there are no branch divisors in the boundary. The branch divisors in $\mathcal{F}_L(\Gamma)$ arise from the fixed divisors of reflections.*

Reflective obstructions, that is branch divisors, are a special problem related to the orthogonal group. They do not appear in the case of moduli spaces of polarised abelian varieties of dimension greater than 2, where the modular group is the symplectic group. There are no quasi-reflections in the symplectic group even for $g = 3$.
The branch divisor is defined by special reflective vectors in the lattice L. This description is given in §2. To estimate the reflective obstructions we use the Hirzebruch-Mumford proportionality principle and the exact formula for the Hirzebruch-Mumford volume of the orthogonal group found in [GHS1]. We do the numerical estimation in §4.
We treat the cusp obstructions in §3, using special cusp forms of low weight (the lifting of Jacobi forms) constructed in [G2] and the low-weight cusp form trick (see [G2] and [GHS2]).

2 The branch divisors

To estimate the obstruction to extending pluricanonical forms to a smooth projective model of $\mathcal{F}_L(\widetilde{\mathrm{O}}^+(L))$ we have to determine the branch divisors of the projection

$$\pi\colon \mathcal{D}_L \to \mathcal{F}_L(\widetilde{\mathrm{O}}^+(L)) = \widetilde{\mathrm{O}}^+(L) \setminus \mathcal{D}_L. \tag{3}$$

According to [GHS2, Corollary 2.13] these divisors are defined by reflections $\sigma_r \in \mathrm{O}^+(L)$, where

$$\sigma_r(l) = l - \frac{2(l,r)}{(r,r)}r,$$

coming from vectors $r \in L$ with $r^2 < 0$ that are *stably reflective*: by this we mean that r is primitive and σ_r or $-\sigma_r$ is in $\widetilde{\mathrm{O}}^+(L)$. By a (k)-*vector* for $k \in \mathbb{Z}$ we mean a primitive vector r with $r^2 = k$.

Let D be the exponent of the finite abelian group A_L and let the *divisor* $\operatorname{div}(r)$ of $r \in L$ be the positive generator of the ideal (l, L). We note that $r^* = r/\operatorname{div}(r)$ is a primitive vector in $L^\vee$. In [GHS2, Propositions 3.1–3.2] we proved the following.

Lemma 2.1 *Let L be an even integral lattice of signature $(2, n)$. If $\sigma_r \in \widetilde{\mathrm{O}}^+(L)$ then $r^2 = -2$. If $-\sigma_r \in \widetilde{\mathrm{O}}^+(L)$, then $r^2 = -2D$ and $\operatorname{div}(r) = D \equiv 1$ mod 2 or $r^2 = -D$ and $\operatorname{div}(r) = D$ or $D/2$.*

We need also the following well-known property of the stable orthogonal group.

Lemma 2.2 *For any sublattice M of an even lattice L the group $\widetilde{\mathrm{O}}(M)$ can be considered as a subgroup of $\widetilde{\mathrm{O}}(L)$.*

Proof. Let $M^\perp$ be the orthogonal complement of M in L. We have as usual

$$M \oplus M^\perp \subset L \subset L^\vee \subset M^\vee \oplus (M^\perp)^\vee.$$

We can extend $g \in \widetilde{\mathrm{O}}(M)$ to $M \oplus M^\perp$ by putting $g|_{M^\perp} \equiv \mathrm{id}$. It is clear that $g \in \widetilde{\mathrm{O}}(M \oplus M^\perp)$. For any $l^\vee \in L^\vee$ we have $g(l^\vee) \in l^\vee + (M \oplus M^\perp)$. In particular, $g(l) \in L$ for any $l \in L$ and $g \in \widetilde{\mathrm{O}}(L)$. $\square$

We can describe the components of the branch locus in terms of homogeneous domains. For r a stably reflective vector in L we put

$$H_r = \{[w] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (w, r) = 0\},$$

and let $\mathcal{N}$ be the union of all hyperplane sections $H_r \cap \mathcal{D}_L$ over all stably reflective vectors r.

Proposition 2.3 *Let $r \in L$ be a stably reflective vector: suppose that r and L do not satisfy $D = 4$, $r^2 = -4$, $\operatorname{div}(r) = 2$. Let K_r be the orthogonal complement of r in L. Then the associated component $\pi(H_r \cap \mathcal{D}_L)$ of the branch locus $\mathcal{N}$ is of the form $\widetilde{\mathrm{O}}^+(K_r)\backslash\mathcal{D}_{K_r}$.*

Proof. We have $H_r \cap \mathcal{D}_L = \mathbb{P}(K_r) \cap \mathcal{D}_L = \mathcal{D}_{K_r}$. Let

$$\Gamma_{K_r} = \{\varphi \in \widetilde{\mathrm{O}}^+(L) \mid \varphi(K_r) = K_r\}. \tag{4}$$

Γ_{K_r} maps to a subgroup of $\mathrm{O}^+(K_r)$. The inclusion of $\widetilde{\mathrm{O}}(K_r)$ in $\widetilde{\mathrm{O}}(L)$ (Lemma 2.2) preserves the spinor norm (see [GHS1, §3.1]), because K_r has signature $(2, n-1)$ and so $\widetilde{\mathrm{O}}^+(K_r)$ becomes a subgroup of $\widetilde{\mathrm{O}}^+(L)$.
Therefore the image of Γ_{K_r} contains $\widetilde{\mathrm{O}}^+(K_r)$ for any r. Now we prove that this image coincides with $\widetilde{\mathrm{O}}^+(K_r)$ for all r, except perhaps if $D = 4$, $r^2 = -4$ and $\operatorname{div}(r) = 2$.

Let us consider the inclusions

$$\langle r\rangle \oplus K_r \subset L \subset L^\vee \subset \langle r\rangle^\vee \oplus K_r^\vee.$$

By standard arguments (see [GHS2, Proposition 3.6]) we see that

$$|\det K_r| = \frac{|\det L|\cdot |r^2|}{\operatorname{div}(r)^2} \quad \text{and} \quad [L:\langle r\rangle \oplus K_r] = \frac{|r^2|}{\operatorname{div}(r)} = 1 \ \text{ or } \ 2.$$

If the index is 1, then it is clear that the image of Γ_{K_r} is $\widetilde{\mathrm{O}}^+(K_r)$. Let us assume that the index is equal to 2. In this case the lattice $\langle r\rangle^\vee$ is generated by $r^\vee = -r/(r,r) = r^*/2$, where $r^* = r/\operatorname{div}(r)$ is a primitive vector in $L^\vee$. In particular $r^\vee$ represents a non-trivial class in $\langle r\rangle^\vee \oplus K_r^\vee$ modulo $L^\vee$. Let us take $k^\vee \in K_r^\vee$ such that $k^\vee \notin L^\vee$. Then $k^\vee + r^\vee \in L^\vee$ and

$$\varphi(k^\vee) - k^\vee \equiv r^\vee - \varphi(r^\vee) \mod L.$$

We note that if $\varphi \in \Gamma_{K_r}$ then $\varphi(r) = \pm r$. Hence

$$\varphi(k^\vee) - k^\vee \equiv \begin{cases} 0 \mod L & \text{if } \varphi(r) = r \\ r^* \mod L & \text{if } \varphi(r) = -r \end{cases}$$

Since $\varphi(r^*) \equiv r^* \mod L$, we cannot have $\varphi(r) = -r$ unless $\operatorname{div}(r) = 1$ or 2. Therefore we have proved that $\varphi(k^\vee) \equiv k^\vee \mod K_r$ ($K_r = K_r^\vee \cap L$), except possibly if $D = 4$, $r^2 = -4$, $\operatorname{div}(r) = 2$. □

The group $\widetilde{\mathrm{O}}^+(L)$ acts on $\mathcal{N}$. We need to estimate the number of components of $\widetilde{\mathrm{O}}^+(L) \setminus \mathcal{N}$. This will enable us to estimate the reflective obstructions to extending pluricanonical forms which arise from these branch loci.
For the even unimodular lattice $II_{2,8m+2}$ any primitive vector r has $\operatorname{div}(r) = 1$. Consequently r is stably reflective if and only if $r^2 = -2$.
For $L_{2d}^{(m)}$ the reflections and the corresponding branch divisors arise in two different ways, according to Lemma 2.1. We shall classify the orbits of such vectors.

Proposition 2.4 *Suppose d is a positive integer.*

(i) *Any two (-2)-vectors in the lattice $II_{2,8m+2}$ are equivalent modulo $\mathrm{O}^+(II_{2,8m+2})$, and the orthogonal complement of a (-2)-vector r is isometric to*
$$K_{II}^{(m)} = U \oplus mE_8(-1) \oplus \langle 2\rangle.$$

(ii) *There is one $\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})$-orbit of (-2)-vectors r in $L_{2d}^{(m)}$ with $\operatorname{div}(r) = 1$. If $d \equiv 1 \bmod 4$ then there is a second orbit of (-2)-vectors, with $\operatorname{div}(r) = 2$. The orthogonal complement of a (-2)-vector r in $L_{2d}^{(m)}$ is isometric to*
$$K_{2d}^{(m)} = U \oplus mE_8(-1) \oplus \langle 2\rangle \oplus \langle -2d\rangle,$$

if $\operatorname{div}(r) = 1$, *and to*

$$N_{2d}^{(m)} = U \oplus mE_8(-1) \oplus \begin{pmatrix} 1 & 2 \\ \frac{1-d}{2} & 1 \end{pmatrix},$$

if $\operatorname{div}(r) = 2$.

(iii) *The orthogonal complement of a* $(-2d)$*-vector* r *in* $L_{2d}^{(m)}$ *is isometric to*

$$II_{2,8m+2} = 2U \oplus mE_8(-1)$$

if $\operatorname{div}(r) = 2d$, *and to*

$$K_2^{(m)} = U \oplus mE_8(-1) \oplus \langle 2 \rangle \oplus \langle -2 \rangle \quad \textit{or} \quad T_{2,8m+2} = U \oplus U(2) \oplus mE_8(-1)$$

if $\operatorname{div}(r) = d$.

(iv) *Suppose* $d > 1$. *The number of* $\widetilde{\mathrm{O}}(L_{2d}^{(m)})$*-orbits of* $(-2d)$*-vectors with* $\operatorname{div}(r) = 2d$ *is* $2^{\rho(d)}$. *The number of* $\widetilde{\mathrm{O}}(L_{2d}^{(m)})$*-orbits of* $(-2d)$*-vectors with* $\operatorname{div}(r) = d$ *is*

$$\begin{cases} 2^{\rho(d)} & \textit{if } d \textit{ is odd or } d \equiv 4 \mod 8; \\ 2^{\rho(d)+1} & \textit{if } d \equiv 0 \mod 8; \\ 2^{\rho(d)-1} & \textit{if } d \equiv 2 \mod 4. \end{cases}$$

Here $\rho(d)$ *is the number of prime divisors of* d.

Proof. If the lattice L contains two hyperbolic planes then according to the well-known result of Eichler (see [E, §10]) the $\widetilde{\mathrm{O}}^+(L)$-orbit of a primitive vector $l \in L$ is completely defined by two invariants: by its length (l, l) and by its image $l^* + L$ in the discriminant group A_L, where $l^* = l/\operatorname{div}(l)$.
i) If u is a primitive vector of an even unimodular lattice $II_{2,8m+2}$ then $\operatorname{div}(u) = 1$ and there is only one $\mathrm{O}(II_{2,8m+2})$-orbit of (-2)-vectors. Therefore we can take r to be a (-2)-vector in U, and the form of the orthogonal complement is obvious.
ii) In the lattice $L_{2d}^{(m)}$ we fix a generator h of its $\langle -2d \rangle$-part. Then for any $r \in L_{2d}^{(m)}$ we can write $r = u + xh$, where $u \in II_{2,8m+2}$ and $x \in \mathbb{Z}$. It is clear that $\operatorname{div}(r)$ divides r^2. If $f | \operatorname{div}(r)$, where $f = 2$, d or $2d$, then the vector u is also divisible by f. Therefore the (-2)-vectors form two possible orbits of vectors with divisor equal to 1 or 2. If $r^2 = -2$ and $\operatorname{div}(r) = 2$ then $u = 2u_0$ with $u_0 \in 2U \oplus mE_8(-1)$ and we see that in this case $d \equiv 1 \mod 4$. This gives us two different orbits for such d. In both cases we can find a (-2)-vector r in the sublattice $U \oplus \langle -2d \rangle$. Elementary calculation gives us the orthogonal complement of r.
iii) This was proved in [GHS2, Proposition 3.6] for $m = 2$. For general m the proof is the same.

iv) To find the number of orbits of $(-2d)$-vectors we have to consider two cases.
a) Let $\operatorname{div}(r) = 2d$. Then $r = 2du + xh$ and $r^* \equiv (x/2d)h \mod L$, where $u \in II_{2,8m+2}$ and x is modulo $2d$. Moreover $(r,r) = 4d^2(u,u) - x^2 2d = -2d$. Thus $x^2 \equiv 1 \mod 4d$. This congruence has $2^{\rho(d)}$ solutions modulo $2d$. For any such $x \mod 2d$ we can find a vector u in $2U \oplus mE_8(-1)$ with $(u,u) = (x^2-1)/2d$. Then $r = 2du + xh$ is primitive (because u is not divisible by any divisor of x) and $(r,r) = -2d$.
b) Let $\operatorname{div}(r) = d$. Then $r = du + xh$, where u is primitive, $r^* \equiv (x/d)h \mod L$ and x is modulo d. We have $(r^*, r^*) \equiv -2x^2/d \mod 2\mathbb{Z}$ and $x^2 \equiv 1 \mod d$. For any solution modulo d we can find as above $u \in 2U \oplus mE_8(-1)$ such that $r = du + xh$ is primitive and $(r,r) = -2d$. It is easy to see that the number of solutions $\{x \mod d \,|\, x^2 \equiv 1 \mod d\}$ is as stated. □

Remark. To calculate the number of the branch divisors arising from vectors r with $r^2 = -2d$ one has to divide the corresponding number of orbits found in Proposition 2.4(iv) by 2 if $d > 2$. This is because $\pm r$ determine different orbits but the same branch divisor. For $d = 2$ the proof shows that there is one divisor for each orbit given in Proposition 2.4(iv).

3 Modular forms of low weight

In this section we let $L = 2U \oplus L_0$ be an even lattice of signature $(2,n)$ with two hyperbolic planes. We choose a primitive isotropic vector c_1 in L. This vector determines a 0-dimensional cusp and a tube realisation of the domain $\mathcal{D}_L$. The tube domain (see the definition of $\mathcal{H}(L_1)$ below) is a complexification of the positive cone of the hyperbolic lattice $L_1 = c_1^\perp/c_1$. If $\operatorname{div}(c_1) = 1$ we call this cusp *standard* (as above, by [E] there is only one standard cusp). In this case $L_1 = U \oplus L_0$. In [GHS2, §4] we proved that any 1-dimensional boundary component of $\widetilde{\mathrm{O}}^+(L) \setminus \mathcal{D}_L$ contains the standard 0-dimensional cusp if every isotropic (with respect to the discriminant form: see [Nik2, §1.3]) subgroup of A_L is cyclic.
Let us fix a 1-dimensional cusp by choosing two copies of U in L. (One has to add to c_1 a primitive isotropic vector $c_2 \in L_1$ with $\operatorname{div}(c_2) = 1$). Then $L = U \oplus L_1 = U \oplus (U \oplus L_0)$ and the construction of the tube domain may be written down simply in coordinates. We have

$$\mathcal{H}(L_1) = \mathcal{H}_n = \{Z = (z_n, \ldots, z_1) \in \mathbb{H}_1 \times \mathbb{C}^{n-2} \times \mathbb{H}_1;\ (\operatorname{Im} Z, \operatorname{Im} Z)_{L_1} > 0\},$$

where $Z \in L_1 \otimes \mathbb{C}$ and $(z_{n-1}, \ldots, z_2) \in L_0 \otimes \mathbb{C}$. (We represent Z as a column vector.) An isomorphism between $\mathcal{H}_n$ and $\mathcal{D}_L$ is given by

$$\begin{aligned} p\colon \mathcal{H}_n &\longrightarrow \mathcal{D}_L \\ Z = (z_n, \ldots, z_1) &\longmapsto \big(-\tfrac{1}{2}(Z,Z)_{L_1} : z_n : \cdots : z_1 : 1\big). \end{aligned} \tag{5}$$

The action of $\mathrm{O}^+(L \otimes \mathbb{R})$ on $\mathcal{H}_n$ is given by the usual fractional linear transformations. A calculation shows that the Jacobian of the transformation of $\mathcal{H}_n$

defined by $g \in \mathrm{O}^+(L\otimes\mathbb{R})$ is equal to $\det(g)j(g,Z)^{-n}$, where $j(g,Z)$ is the last ($(n+2)$-nd) coordinate of $g\big(p(Z)\big) \in \mathcal{D}_L$. Using this we define the automorphic factor

$$\begin{array}{rrcl} J\colon & \mathrm{O}^+(L\otimes\mathbb{R})\times\mathcal{H}_{n+2} & \to & \mathbb{C}^* \\ & (g,Z) & \mapsto & (\det g)^{-1}\cdot j(g,Z)^n. \end{array}$$

The connection with pluricanonical forms is the following. Consider the form

$$dZ = dz_1\wedge\cdots\wedge dz_n \in \Omega^n(\mathcal{H}_n).$$

$F(dZ)^k$ is a Γ-invariant k-fold pluricanonical form on $\mathcal{H}_n$, for Γ a subgroup of finite index of $\mathrm{O}^+(L)$, if $F\big(g(Z)\big) = J(g,Z)^kF(Z)$ for any $g\in\Gamma$; in other words if $F\in M_{nk}(\Gamma,\det^k)$ (see Definition 1.2). To prove Theorem 1.1 we need cusp forms of weight smaller than the dimension of the corresponding modular variety.

Proposition 3.1 *For unimodular type, cusp forms of weight* $12+4m$ *exist: that is*

$$\dim S_{12+4m}(\mathrm{O}^+(II_{2,8m+2})) > 0.$$

For K3 *type we have the bounds*

$$\begin{array}{rcl} \dim S_{11+4m}(\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})) & > & 0 \ \textit{ if }\ d>1; \\ \dim S_{10+4m}(\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})) & > & 0 \ \textit{ if }\ d\ge 1; \\ \dim S_{7+4m}(\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})) & > & 0 \ \textit{ if }\ d\ge 4; \\ \dim S_{6+4m}(\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})) & > & 0 \ \textit{ if }\ d=3 \ \textit{ or }\ d\ge 5; \\ \dim S_{5+4m}(\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})) & > & 0 \ \textit{ if }\ d=5 \ \textit{ or }\ d\ge 7; \\ \dim S_{2+4m}(\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})) & > & 0 \ \textit{ if }\ d>180. \end{array}$$

Proof. For any $F(Z)\in M_k(\widetilde{\mathrm{O}}^+(L))$ we can consider its Fourier-Jacobi expansion at the 1-dimensional cusp fixed above

$$F(Z) = f_0(z_1) + \sum_{m\ge 1} f_m(z_1;z_2,\dots z_{n-1})\exp(2\pi i m z_n).$$

A lifting construction of modular forms $F(Z)\in M_k(\widetilde{\mathrm{O}}^+(L))$ with trivial character by means of the first Fourier–Jacobi coefficient is given in [G1], [G2]. We note that $f_1(z_1;z_2,\dots,z_{n-1})\in J_{k,1}(L_0)$, where $J_{k,1}(L_0)$ is the space of the Jacobi forms of weight k and index 1. A more general construction of the additive lifting was given in [B2] but for our purpose the construction of [G2] is sufficient.

The dimension of $J_{k,1}(L_0)$ depends only on the discriminant form and the rank of L_0 (see [G2, Lemma 2.4]). In particular, for the special cases of $L = II_{2,8m+2}$ and $L = L_{2d}^{(m)}$ we have

$$J_{k+4m,1}^{\text{cusp}}(mE_8(-1)) \cong S_k(\mathrm{SL}_2(\mathbb{Z}))$$

and

$$J_{k+4m,1}^{\text{cusp}}(mE_8(-1) \oplus \langle -2d \rangle) \cong J_{k,d}^{\text{cusp}},$$

where $J_{k,d}^{\text{cusp}}$ is the space of the usual Jacobi cusp forms in two variables of weight k and index d (see [EZ]) and $S_k(\mathrm{SL}_2(\mathbb{Z}))$ is the space of weight k cusp forms for $\mathrm{SL}_2(\mathbb{Z})$.

The lifting of a Jacobi cusp form of index one is a cusp form of the same weight with respect to $\mathrm{O}^+(II_{2,8m+2})$ or $\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})$ with trivial character. The fact that we get a cusp form was proved in [G2] for maximal lattices, i.e., if d is square-free. In [GHS2, §4] we extended this to all lattices L for which the isotropic subgroups of the discriminant A_L are all cyclic, which is true in all cases considered here.

To prove the unimodular type case of Proposition 3.1 we can take the Jacobi form corresponding to the cusp form $\Delta_{12}(\tau)$. Using the Jacobi lifting construction we obtain a cusp form of weight $12 + 4m$ with respect to $\mathrm{O}^+(II_{2,8m+2})$.

For the K3 type case we need the dimension formula for the space of Jacobi cusp forms $J_{k,d}^{\text{cusp}}$ (see [EZ]). For a positive integer l one sets

$$\{l\}_{12} = \begin{cases} \lfloor \frac{l}{12} \rfloor & \text{if } l \not\equiv 2 \bmod 12 \\ \lfloor \frac{l}{12} \rfloor - 1 & \text{if } l \equiv 2 \bmod 12. \end{cases}$$

Then if $k > 2$ is even

$$\dim J_{k,d}^{\text{cusp}} = \sum_{j=0}^{d} \left(\{k + 2j\}_{12} - \left\lfloor \frac{j^2}{4d} \right\rfloor \right),$$

and if k is odd

$$\dim J_{k,d}^{\text{cusp}} = \sum_{j=1}^{d-1} \left(\{k - 1 + 2j\}_{12} - \left\lfloor \frac{j^2}{4d} \right\rfloor \right).$$

This gives the bounds claimed. For $k = 2$, using the results of [SZ] one can also calculate $\dim J_{2,d}^{\text{cusp}}$: there is an extra term, $\lceil \sigma_0(d)/2 \rceil$, where $\sigma_0(d)$ denotes the number of divisors of d. This gives $\dim J_{2,d}^{\text{cusp}} > 0$ if $d > 180$ and for some smaller values of d. □

4 Kodaira dimension results

In this section we prove Theorem 1.1. We first explain the geometric background. Let $\mathcal{F}_L(\Gamma)^{\text{tor}}$ be a toroidal compactification as in Theorem 1.3. In particular all singularities are canonical and there is no ramification divisor which is contained in the boundary. Then the canonical divisor (as a $\mathbb{Q}$-divisor) is given by $K_{\mathcal{F}_L(\Gamma)^{\mathrm{tor}}} = nM - V - D$ where M is the line bundle of modular forms of weight 1, n is the dimension of $\mathcal{F}_L(\Gamma)$, V is the branch locus (which is given by reflections) and D is the boundary. Hence in order to construct k-fold pluricanonical forms we must find modular forms of weight kn which vanish of order k along the branch divisor and the boundary. This also suffices since $\mathcal{F}_L(\Gamma)^{\mathrm{tor}}$ has canonical singularities.

Our strategy is the following. For $\Gamma \subseteq \widetilde{\mathrm{O}}^+(L)$ we choose a cusp form $F_a \in S_a(\Gamma)$ of low weight a, i.e. a strictly less than the dimension. Then we consider elements $F \in F_a^k M_{k(n-a)}(\Gamma, \det^k)$: for simplicity we assume that k is even. Such an F vanishes to order at least k on the boundary of any toroidal compactification. Hence if dZ is the volume element on $\mathcal{D}_L$ defined in §3 it follows that $F(dZ)^k$ extends as a k-fold pluricanonical form to the general point of every boundary component of $\mathcal{F}_L(\Gamma)^{\mathrm{tor}}$. Since we have chosen the toroidal compactification so that all singularities are canonical and that there is no ramification divisor which is contained in the boundary the only obstructions to extending $F(dZ)^k$ to a smooth projective model are the reflective obstructions, coming from the ramification divisor of the quotient map $\pi\colon \mathcal{D}_L \to \mathcal{F}_L(\Gamma)$ studied in §2.

Let $\mathcal{D}_K$ be an irreducible component of this ramification divisor. Recall from Proposition 2.3 that $\mathcal{D}_K = \mathbb{P}(K \otimes \mathbb{C}) \cap \mathcal{D}_L$ where $K = K_r$ is the orthogonal complement of a stably reflective vector r. For the lattices chosen in Theorem 1.1 all irreducible components of the ramification divisor are given in Proposition 2.4.

Proposition 4.1 *We assume that k is even and that the dimension $n \geq 9$. For $\Gamma \subseteq \widetilde{\mathrm{O}}^+(L)$, the obstruction to extending forms $F(dZ)^k$ where $F \in F_a^k M_{k(n-a)}(\Gamma)$ to $\mathcal{F}_L(\Gamma)^{\mathrm{tor}}$ lies in the space*

$$B = \bigoplus_K B(K) = \bigoplus_K \bigoplus_{\nu=0}^{k/2-1} M_{k(n-a)+2\nu}(\Gamma \cap \widetilde{\mathrm{O}}^+(K)),$$

where the direct sum is taken over all irreducible components $\mathcal{D}_K$ of the ramification divisor of the quotient map $\pi\colon \mathcal{D}_L \to \mathcal{F}(\Gamma)$.

Proof. Let $\sigma \in \Gamma$ be plus or minus a reflection whose fixed point locus is $\mathcal{D}_K$. We can extend the differential form provided that F vanishes of order k along every irreducible component $\mathcal{D}_K$ of the ramification divisor.

If F_a vanishes along $\mathcal{D}_K$ then K gives no restriction on the second factor of the modular form F.

Now let $\{w = 0\}$ be a local equation for $\mathcal{D}_K$. Then $\sigma^*(w) = -w$ (this is independent of whether σ or $-\sigma$ is the reflection). For every modular form $F \in M_k(\Gamma)$ of even weight we have $F\big(\sigma(z)\big) = F(z)$. This implies that if $F(z) \equiv 0$ on $\mathcal{D}_K$, then F vanishes to even order on $\mathcal{D}_K$.

We denote by $M_{2b}(\Gamma)(-\nu\mathcal{D}_K)$ the space of modular forms of weight $2b$ which vanish of order at least ν along $\mathcal{D}_K$. Since the weight is even we have $M_{2b}(\Gamma)(-\mathcal{D}_K) = M_{2b}(\Gamma)(-2\mathcal{D}_K)$. For $F \in M_{2b}(\Gamma)(-2\nu\mathcal{D}_K)$ we consider $(F/w^{2\nu})$ as a function on $\mathcal{D}_K$. From the definition of modular form (Definition 1.2) it follows that this function is holomorphic, $\Gamma \cap \Gamma_K$-invariant (see equation (4)) and homogeneous of degree $2b + 2\nu$. Thus $(F/w^{2\nu})|_{\mathcal{D}_K} \in M_{2(b+\nu)}(\Gamma \cap \Gamma_K)$. In Proposition 2.3 we saw that, Γ_K contains $\widetilde{\mathrm{O}}^+(K)$ as subgroup of $\widetilde{\mathrm{O}}^+(L)$(with equality in almost all cases), so we may replace $\Gamma \cap \Gamma_K$ by $\Gamma \cap \widetilde{\mathrm{O}}^+(K)$. In this way we obtain an exact sequence

$$0 \to M_{2b}(\Gamma)(-(2+2\nu)\mathcal{D}_K) \to M_{2b}(\Gamma)(-2\nu\mathcal{D}_K) \to M_{2(b+\nu)}(\Gamma \cap \widetilde{\mathrm{O}}^+(K)),$$

where the last map is given by $F \mapsto F/w^{2\nu}$. This gives the result. □

Now we proceed with the proof of Theorem 1.1.

Let L be a lattice of signature $(2, n)$ and $\Gamma < \widetilde{\mathrm{O}}^+(L)$: recall that k is even. According to Proposition 4.1 we can find pluricanonical differential forms on $\mathcal{F}_L(\Gamma)^{\mathrm{tor}}$ if

$$C_B(\Gamma) = \dim M_{k(n-a)}(\Gamma) - \sum_K \dim B(K) > 0, \tag{6}$$

where summation is taken over all irreducible components of the ramification divisor (see the remark at the end of §2). It now remains to estimate the dimension of $B(K)$ for each of the finitely many components of the ramification locus in the cases we are interested in, namely $\Gamma = \mathrm{O}^+(II_{2,8m+2})$ and $\Gamma = \widetilde{\mathrm{O}}^+(L_{2d}^{(m)})$.

According to the Hirzebruch-Mumford proportionality principle

$$\dim M_k(\Gamma) = \frac{2}{n!}\,\mathrm{vol}_{HM}(\Gamma)k^n + O(k^{n-1}).$$

The exact formula for the Hirzebruch-Mumford volume vol_{HM} for any indefinite orthogonal group was obtained in [GHS1]. It depends mainly on the determinant and on the local densities of the lattice L. Here we simply quote the estimates of the dimensions of certain spaces of cusp forms.

The case of $II_{2,8m+2}$ is easier because the branch divisor has only one irreducible component defined by any (-2)-vector r. According to Proposition 2.4 the orthogonal complement K_r is $K_{II}^{(m)}$. This lattice differs from the lattice $L_2^{(m)}$, whose Hirzebruch-Mumford volume was calculated in [GHS1, §3.5], only by one copy of the hyperbolic plane. Therefore

$$\mathrm{vol}_{HM}\,\widetilde{\mathrm{O}}^+(L_2^{(m)}) = (B_{8m+4}/(8m+4))\,\mathrm{vol}_{HM}\,\widetilde{\mathrm{O}}^+(K_{II}^{(m)}),$$

and hence, for even k,

$$\dim M_k(\widetilde{\mathrm{O}}^+(K_{II}^{(m)})) = \frac{2^{1-4m}}{(8m+1)!}\cdot\frac{B_2\ldots B_{8m+2}}{(8m+2)!!}\,k^{8m+1} + O(k^{8m}),$$

where the B_i are Bernoulli numbers. Assume that $m \geq 3$. Let us take a cusp form

$$F \in S_{4m+12}(\mathrm{O}^+(II_{2,8m+2}))$$

from Proposition 3.1. In this case the dimension of the obstruction space B of Proposition 4.1 for the pluricanonical forms of order $k = 2k_1$ is given by

$$\sum_{\nu=0}^{k_1-1} \dim M_{(4m-10)k+2\nu}(\widetilde{\mathrm{O}}^+(K_{II}^{(m)})) =$$
$$\frac{2^{4m+2}}{(8m+2)!} \cdot \frac{B_2 \dots B_{8m+2}}{(8m+2)!!}\big((1+\frac{1}{4m-10}\big)^{8m+2} - 1\big)((4m-10)k_1)^{8m+2}$$
$$+O(k^{8m+1})$$

In [GHS1, §3.3] we computed the leading term of the dimension of the space of modular forms for $\mathrm{O}^+(II_{2,8m+2})$. Comparing these two we see that the constant $C_B(\mathrm{O}^+(II_{2,8m+2}))$ in the obstruction inequality (6) is positive if and only if

$$\frac{B_{4m+2}}{4m+2} > \big(1+\frac{1}{4m-10}\big)^{8m+2} - 1. \tag{7}$$

Moreover $\mathcal{F}_{II}^{(m)}$ is of general type if $C_B(\mathrm{O}^+(II_{2,8m+2})) > 0$. From Stirling's formula

$$5\sqrt{\pi n}\big(\frac{n}{\pi e}\big)^{2n} > |B_{2n}| > 4\sqrt{\pi n}\big(\frac{n}{\pi e}\big)^{2n}. \tag{8}$$

Using this estimate we easily obtain that (7) holds if $m \geq 5$. Therefore we have proved Theorem 1.1 for the lattice $II_{2,8m+2}$.

Next we consider the lattice $L_{2d}^{(m)}$ of K3 type. For this lattice the branch divisor of $\mathcal{F}_{2d}^{(m)}$ is calculated in Proposition 2.4. It contains one or two (if $d \equiv 1 \bmod 4$) components defined by (-2)-vectors and some number of components defined by $(-2d)$-vectors. To estimate the obstruction constant $C_B(\Gamma)$ in (6) we use the dimension formulae for the space of modular forms with respect to the group $\widetilde{\mathrm{O}}^+(M)$, where M is one of the following lattices from Proposition 2.4: $L_{2d}^{(m)}$ (the main group); $K_{2d}^{(m)}$ and $N_{2d}^{(m)}$ (the (-2)-obstruction); $M_{2,8m+2}$, $K_2^{(m)}$ and $T_{2,8m+2}$ (the $(-2d)$-obstruction). The corresponding dimension formulae were found in [GHS1] (see §§3.5, 3.6.1–3.6.2, 3.3 and 3.4). The branch divisor of $(-2d)$-type appears only if $d > 1$. We note that

$$\mathrm{vol}_{HM}(\widetilde{\mathrm{O}}^+(T_{2,8m+2})) > \mathrm{vol}_{HM}(\widetilde{\mathrm{O}}^+(K_2^{(m)})). \tag{9}$$

Therefore in order to estimate $C_B(\Gamma)$ we can assume that all $(-2d)$-divisors defined by stably reflective $(-2d)$-vectors r with $\mathrm{div}(r) = d$ (see Proposition 2.4) are of the type $T_{2,8m+2}$.

We put $k = 2k_1$, $w = n - a$ and $n = 8m + 3$. For the obstruction constant in (6) we obtain

$$C_B(\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})) > \dim M_{2k_1 w}(\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})) - B_{(-2)} - B_{(-2d)} \tag{10}$$

where

$$B_{(-2)} = \dim B(K_{2d}^{(m)}) + \dim B(N_{2d}^{(m)}),$$

$$B_{(-2d)} = 2^{\rho(d)-1}(\dim B(M_{2,8m+2}) + 2^{h_d} \dim B(T_{2,8m+2}))$$

and $B(K)$ is the obstruction space from Proposition 4.1. By h_d we denote the sum $\delta_{0,d(8)} - \delta_{2,d(4)}$, where $d(n)$ is $d \bmod n$ and δ is the Kronecker delta (see Proposition 2.4 and the remark following it).
For any lattice considered above

$$\begin{aligned} B(K) &= \sum_{\nu=0}^{k_1-1} \dim M_{2(k_1 w+\nu)}(\widetilde{\mathrm{O}}^+(K)) \\ &= \frac{2^{8m+3}}{(8m+3)!} E_w(8m+3) \mathrm{vol}_{HM}(\widetilde{\mathrm{O}}^+(K))(k_1 w)^{8m+3} + O(k_1^{8m+2}) \end{aligned}$$

where $E_w(8m+3) = (1+\frac{1}{w})^{8m+3}$.
All terms in (10) contain a common factor. First

$$\dim M_{2k_1 w}(\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})) = C_{m,d}^{k_1,w} \left| \frac{B_{8m+4}}{B_{4m+2}} \right| \sqrt{d} + O(k_1^{8m+2}), \tag{11}$$

where

$$C_{m,d}^{k_1,w} = \frac{2^{4m+1+\delta_{1,d}}}{(8m+3)!} \frac{|B_2 \dots B_{8m+2}|}{(8m+2)!!} \frac{|B_{4m+2}|}{4m+2} d^{4m+\frac{3}{2}} \prod_{p|d} (1-p^{-(4m+2)})(k_1 w)^{8m+3}.$$

We note that $2^{4m+1}\frac{B_{4m+2}}{4m+2} = \pi^{-(4m+2)}\Gamma(4m+2)\zeta(4m+2)$.
From [GHS1, (16)] it follows that

$$\mathrm{vol}_{HM}(\widetilde{\mathrm{O}}^+(K_{2d}^{(m)})) = 2^{\delta_{1,d}-\delta_{4,d(8)}} \frac{B_2 \dots B_{8m+2}}{(8m+2)!!} d^{4m+\frac{3}{2}} \pi^{-(4m+2)} \Gamma(4m+2) L(4m+2, \left(\frac{4d}{*}\right)).$$

We can use the formula for the volume of $N_{2d}^{(m)}$ in the following form:

$$\mathrm{vol}_{HM}(\widetilde{\mathrm{O}}^+(N_{2d}^{(m)})) = 2^{1+\delta_{1,d}-(8m+4)} d^{4m+\frac{3}{2}} \frac{B_2 \dots B_{8m+2}}{(8m+2)!!} \pi^{-(4m+2)} \Gamma(4m+2) L(4m+2, \left(\frac{d}{*}\right))$$

(see [GHS1, 3.6.2]). It follows that

$$B_{(-2)} = C_{m,d}^{k_1,w} E_w(8m+3)(2^{8m+3-\delta_{4,d(8)}} P_K(4m+2) + P_N(4m+2)) + O(k_1^{8m+2})$$

where

$$P_K(n) = (1-2^{-n})^{\delta_{0,d(2)}} \frac{L(n, (\frac{4d}{*}))}{L(n, \chi_{0,4d})} \prod_{p|d} \frac{1-p^{-n}}{1+p^{-n}}$$

and

$$P_N(n) = \frac{L(n, (\frac{d}{*}))}{L(n, \chi_{0,d})} \prod_{p|d} \frac{1-p^{-n}}{1+p^{-n}}.$$

Here $\chi_{0,f}$ denotes the principal Dirichlet character modulo f.
We note that $|P_K(n)| < 1$ and $|P_N(n)| < 1$ for any d. We conclude that

$$B_{(-2)} < C^{k_1,w}_{m,d} E_w(8m+3) b_{(-2)}$$

where $b_{(-2)} = 2^{8m+3} - 1$.
The $(-2d)$-contribution is calculated according to [GHS1, 3.3–3.4]. We note that $\widetilde{\mathrm{O}}^+(T_{2,8m+2})$ is a subgroup of $\widetilde{\mathrm{O}}^+(M_{2,8m+2})$. We obtain

$$B_{(-2d)} \le C^{k_1,w}_{m,d} E_w(8m+3) b_{(-2d)}$$

where for $d > 2$

$$b_{(-2d)} = \frac{2^{\rho(d)}}{d} \left(\frac{4}{d}\right)^{4m+\frac{1}{2}} 4(2^{h_d}(1+2^{-(4m+2)} - 2^{-(8m+3)}) + 2^{-(8m+3)}).$$

As a result we see that that the obstruction constant $C_B(\widetilde{\mathrm{O}}^+(L^{(m)}_{2d}))$ is positive if

$$\beta^{(w)}_{m,d} = \left|\frac{B_{4m+2}}{B_{8m+4}}\right| E_w(8m+3)(b_{(-2)} + b_{(-2d)}) < \sqrt{d}.$$

Using (8) we get

$$\left|\frac{B_{4m+2}}{B_{8m+4}}\right| < \frac{5}{4\sqrt{2}} \left(\frac{\pi e}{2m+1}\right)^{4m+2} \frac{1}{2^{8m+4}}.$$

For $m \ge 5$ we choose a cusp form F_a of weight $a = 4m+10$, i.e. we take $w = 4m-7$ in Proposition 4.1. Such a cusp form exists for all $d \ge 1$ by Proposition 3.1. Using the fact that $\beta_{(-2d)} \le \beta_{(-4)}$ for any $d \ge 2$ and the value $b_{(-4)} = 2^{4m+\frac{5}{2}}$, we see that

$$\beta^{(4m-7)}_{m,d} < (1+\frac{1}{4m-7})^{8m+3} \frac{5}{8\sqrt{2}} \left(\frac{\pi e}{2m+1}\right)^{4m+2} \frac{2^{8m+3} + 2^{4m+\frac{5}{2}} + 1}{2^{8m+3}},$$

which is smaller than 1 if $m \ge 5$. This proves Theorem 1.1 for $m \ge 5$.
For $m = 4$ there exists a cusp form F_a of weight $4m+6$ if $d \ne 1, 2, 4$, i.e. we take $w = 4m-3$. To see that $\beta^{(13)}_{4,d} < \sqrt{d}$ we need check this only for $d = 3$ because $b_{(-2d)} < b_{(-6)}$ for $d > 3$. One can do it by direct calculation.

For $m \leq 3$ we choose F_a of weight $4m+2$, i.e. we take $w = 4m+1$. Such a cusp form exists if $d > 180$ according to Proposition 3.1. For such d we see that $\beta_{(-2d)} < 1$. Then the obstruction constant $C_B(\widetilde{\mathrm{O}}^+(L_{2d}^{(m)}))$ is positive if

$$\left|\frac{B_{4m+2}}{B_{8m+2}}\right|(1+\frac{1}{4m+1})^{8m+3}(2^{8m+3}+2) < \sqrt{d}.$$

This inequality gives us the bound on d in Theorem 1.1.
This completes the proof of Theorem 1.1.

In the proof of Theorem 1.1 above we have seen that the (-2)-part of the branch divisor forms the most important reflective obstruction to the extension of the $\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})$-invariant differential forms to a smooth compact model of $\mathcal{F}_{2d}^{(m)}$. Let us consider the double covering $\mathcal{SF}_{2d}^{(m)}$ of $\mathcal{F}_{2d}^{(m)}$ for $d > 1$ determined by the special orthogonal group:

$$\mathcal{SF}_{2d}^{(m)} = \widetilde{\mathrm{SO}}^+(L_{2d}^{(m)}) \setminus \mathcal{D}_{L_{2d}^{(m)}} \to \mathcal{F}_{2d}^{(m)}.$$

Here the branch divisor does not contain the (-2)-part. Theorem 4.2 below shows that there are only five exceptional varieties $\mathcal{SF}_{2d}^{(m)}$ with $m > 0$ and $d > 1$ that are possibly not of general type.
The variety $\mathcal{SF}_{2d}^{(2)}$ can be interpreted as the moduli space of K3 surfaces of degree $2d$ with spin structure: see [GHS2, §5]. The three-fold $\mathcal{SF}_{2d}^{(0)}$ is the moduli space of $(1,t)$-polarised abelian surfaces.

Theorem 4.2 *The variety $\mathcal{SF}_{2d}^{(m)}$ is of general type for any $d > 1$ if $m \geq 3$. If $m = 2$ then $\mathcal{SF}_{2d}^{(2)}$ is of general type if $d \geq 3$. If $m = 1$ then $\mathcal{SF}_{2d}^{(1)}$ is of general type if $d = 5$ or $d \geq 7$.*

Proof. The case $m = 2$ is [GHS2, Theorem 5.1], and the result for $m \geq 5$ is immediate from Theorem 1.1. For $m = 1$, 3 and 4 we can prove more than what follows from Theorem 1.1.
The branch divisor of $\mathcal{SF}_{2d}^{(m)}$ is defined by the reflections in vectors $r \in L_{2d}^{(m)}$ such that $-\sigma_r \in \widetilde{\mathrm{SO}}^+(L_{2d}^{(m)})$, because the rank of $L_{2d}^{(m)}$ is odd. Therefore $r^2 = -2d$, by Proposition 2.1.
If $F \in M_{2k+1}(\widetilde{\mathrm{SO}}^+(L_{2d}^{(m)}))$ is a modular form (note that the character det is trivial), $d > 1$ and $z \in \mathcal{D}^\bullet_{L_{2d}^{(m)}}$ is such that $(z,r) = 0$, then

$$F(z) = F(-\sigma_r(z)) = F(-z) = (-1)^{2k+1}F(z).$$

Therefore any modular form of odd weight for $\widetilde{\mathrm{SO}}^+(L_{2d}^{(m)})$ vanishes on the branch divisor.

To apply the low-weight cusp form trick used in the proof of Theorem 1.1 one needs a cusp form of weight smaller than $\dim \mathcal{SF}_{2d}^{(m)} = 8m+3$. By Proposition 3.1 there exists a cusp form $F_{11+4m} \in S_{11+4m}(\widetilde{\mathrm{SO}}^+(L_{2d}^{(m)}))$. For $m \geq 3$ we have that $11+4m < 8m+3$. Therefore the differential forms $F_{11+4m}^k F_{(4m-8)k}(dZ)^k$, for arbitrary $F_{(4m-8)k} \in M_{(4m-8)k}(\widetilde{\mathrm{SO}}^+(L_{2d}^{(m)}))$, extend to the toroidal compactification of $\mathcal{SF}_{2d}^{(m)}$ constructed in Theorem 1.3. This proves the cases $m \geq 3$ of the theorem.

For the case $m=1$ we use a cusp form of weight 9 with respect to $\widetilde{\mathrm{SO}}^+(L_{2d}^{(1)})$ constructed in Proposition 3.1. □

We can obtain some information also for some of the remaining cases.

Proposition 4.3 *The spaces $\mathcal{SF}_8^{(1)}$ and $\mathcal{SF}_{12}^{(1)}$ have non-negative Kodaira dimension.*

Proof. By Proposition 3.1 there are cusp forms of weight 11 for $\widetilde{\mathrm{SO}}^+(L_8^{(1)})$ and $\widetilde{\mathrm{SO}}^+(L_{12}^{(1)})$. The weight of these forms is equal to the dimension. By the well-known observation of Freitag [F, Hilfssatz 2.1, Kap. III] these cusp forms determine canonical differential forms on the 11-dimensional varieties $\mathcal{SF}_8^{(1)}$ and $\mathcal{SF}_{12}^{(1)}$. □

These varieties may perhaps have intermediate Kodaira dimension as it seems possible that a reflective modular form of canonical weight exists for $L_8^{(1)}$ and $L_{12}^{(1)}$.

In [GHS2] we used pull-backs of the Borcherds modular form Φ_{12} on $\mathcal{D}_{II_{2,26}}$ to show that many moduli spaces of K3 surfaces are of general type. We can also use Borcherds products to prove results in the opposite direction.

Theorem 4.4 *The Kodaira dimension of $\mathcal{F}_{II}^{(m)}$ is $-\infty$ for $m = 0$, 1 and 2.*

Proof. For $m=0$ we can see immediately that the quotient is rational: a straightforward calculation gives that $\mathcal{F}_{II}^{(0)} = \Gamma\backslash\mathbb{H}_1 \times \mathbb{H}_1$ where $\mathbb{H}_1$ is the usual upper half plane and Γ is the degree 2 extension of $\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$ by the involution which interchanges the two factors. Compactifying this, we obtain the projective plane $\mathbb{P}_2$.

For $m = 1$, 2 we argue differently. There are modular forms similar to Φ_{12} for the even unimodular lattices $II_{2,10}$ and $II_{2,18}$. They are Borcherds products Φ_{252} and Φ_{127} of weights 252 and 127 respectively, defined by the automorphic functions

$$\Delta(\tau)^{-1}(\tau)E_4(\tau)^2 = q^{-1} + 504 + q(\dots)$$

and

$$\Delta(\tau)^{-1}(\tau)E_4(\tau) = q^{-1} + 254 + q(\dots),$$

where $q = \exp(2\pi i\tau)$ and $\Delta(\tau)$ and $E_4(\tau)$ are the Ramanujan delta function and the Eisenstein series of weight 4 (see [B1]). The divisors of Φ_{252} and Φ_{127} coincide with the branch divisors of $\mathcal{F}_{II}^{(1)}$ and $\mathcal{F}_{II}^{(2)}$ defined by the (-2)-vectors. Moreover Φ_{252} and Φ_{127} each vanishes with order one along the respective divisor. Therefore if $F_{10k}(dZ)^k$ (or $F_{18k}(dZ)^k$) defines a pluricanonical differential form on a smooth model of a toroidal compactification of $\mathcal{F}_{II}^{(1)}$ or $\mathcal{F}_{II}^{(2)}$, then F_{10k} (or F_{18k}) is divisible by Φ_{252}^k (or Φ_{127}^k), since F_{10k} or F_{18k} must vanish to order at least k along the branch divisor. This is not possible, because the quotient would be a holomorphic modular form of negative weight. □

We have already remarked that the space $\mathcal{F}_{II}^{(2)}$ is the moduli space of K3 surfaces with an elliptic fibration with a section. Using the Weierstrass equations it is then clear that this moduli space is unirational (such K3 surfaces are parametrised by a linear system in the weighted projective space $\mathbb{P}(4,6,1,1)$). In fact it is even rational: see [Le].

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The Euler Characteristic of a Category

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Abstract. The Euler characteristic of a finite category is defined and shown to be compatible with Euler characteristics of other types of object, including orbifolds. A formula is proved for the cardinality of a colimit of sets, generalizing the classical inclusion-exclusion formula. Both rest on a generalization of Rota's Möbius inversion from posets to categories.

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Introduction

We first learn of Euler characteristic as 'vertices minus edges plus faces', and later as an alternating sum of ranks of homology groups. But Euler characteristic is much more fundamental than these definitions make apparent, as has been made increasingly explicit over the last fifty years; it is something akin to cardinality or measure. More precisely, it is the fundamental dimensionless quantity associated with an object.

The very simplest context for Euler characteristic is that of finite sets, and of course the fundamental way to assign a quantity to a finite set is to count its elements. Euler characteristic of topological spaces can usefully be thought of as a generalization of cardinality; for instance, it obeys the same laws with respect to unions and products.

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In a more sophisticated context, integral geometry, Euler characteristic also emerges clearly as the fundamental dimensionless invariant. A subset of $\mathbb{R}^n$ is *polyconvex* if it is a finite union of compact convex subsets. Let V_n be the vector space of finitely additive measures, invariant under Euclidean transformations, defined on the polyconvex subsets of $\mathbb{R}^n$. Hadwiger's Theorem [KR] states that $\dim V_n = n + 1$. (See also [Sc2], and [MS] for an important application to materials science.) A natural basis consists of one d-dimensional measure for each $d \in \{0, \ldots, n\}$: for instance, $\{\text{Euler characteristic, perimeter, area}\}$ when $n = 2$. Thus, up to scalar multiplication, Euler characteristic is the unique dimensionless measure on polyconvex sets.

Schanuel [Sc1] showed that for a certain category of polyhedra, Euler characteristic is determined by a simple universal property, making its fundamental nature transparent.

All of the above makes clear the importance of defining and understanding Euler characteristic in new contexts. Here we do this for finite categories.

Categories are often viewed as large structures whose main purpose is organizational. However, some different viewpoints will be useful here. A combinatorial point of view is that a category is a directed graph (objects and arrows) equipped with some extra structure (composition and identities). We will concentrate on finite categories (those with only finitely many objects and arrows), which also suits the combinatorial viewpoint, and the composition and identities will play a surprisingly minor role.

A topological point of view is that a category can be understood through its classifying space. This is formed by starting with one 0-cell for each object, then gluing in one 1-cell for each arrow, one 2-cell for each commutative triangle, and so on.

Both of these points of view will be helpful in what follows. The topological perspective is heavily used in the sequel [BL] to this paper.

With topology in mind, one might imagine simply transporting the definition of Euler characteristic from spaces to categories via the classifying space functor, as with other topological invariants: given a category $\mathbb{A}$, define $\chi(\mathbb{A})$ as the Euler characteristic of the classifying space $B\mathbb{A}$. The trouble with this is that the Euler characteristic of $B\mathbb{A}$ is not always defined. Below we give a definition of the Euler characteristic of a category that agrees with the topological Euler characteristic when the latter exists, but is also valid in a range of situations when it does not. It is a rational number, not necessarily an integer.

A version of the definition can be given very succinctly. Let $\mathbb{A}$ be a finite category; totally order its objects as $a_1, \ldots, a_n$. Let Z be the matrix whose (i, j)-entry is the number of arrows from a_i to a_j. Let $M = Z^{-1}$, assuming that Z is invertible. Then $\chi(\mathbb{A})$ is the sum of the entries of M. Of course, the reader remains to be convinced that this definition is the right one.

The foundation on which this work rests is a generalization of Möbius–Rota inversion (§1). Rota developed Möbius inversion for posets [R]; we develop it for categories. (A poset is viewed throughout as a category in which each hom-set has at most one element: the objects are the elements of the poset, and there is an arrow $a \longrightarrow b$ if and only if $a \leq b$.) This leads, among other things, to a 'representation formula': given any functor known to be a sum of representables, the formula tells us the representation explicitly. This in turn can be used to solve enumeration problems, in the spirit of Rota's paper.

However, the main application of this generalized Möbius inversion is to the theory of the Euler characteristic of a category (§2). We actually use a different definition than the one just given, equivalent to it when Z is invertible, but valid for a wider class of categories. It depends on the idea of the 'weight' of an object of a category. The definition is justified in two ways: by showing that it enjoys the properties that the name would lead one to expect (behaviour with respect to products, fibrations, etc.), and by demonstrating its compatibility with Euler characteristics of other types of structure (groupoids, graphs, topological spaces, orbifolds). There is an accompanying theory of Lefschetz number.

The technology of Möbius inversion and weights also solves another problem: what is the cardinality of a colimit? For example, the union of a family of sets and the quotient of a set by a free action of a group are both examples of colimits of set-valued functors, and there are simple formulas for their cardinalities. (In the first case it is the inclusion-exclusion formula.) We generalize, giving a formula valid for any shape of colimit (§3).

Rota and his school proved a large number of results on Möbius inversion for posets. As we will see repeatedly, many are not truly order-theoretic: they are facts about categories in general. In particular, important theorems in Rota's original work [R] generalize from posets to categories (§4).

(The body of work on Möbius inversion in finite lattices is not, however, so ripe for generalization: a poset is a lattice just when the corresponding category has products, but a finite category cannot have products unless it is, in fact, a lattice.)

Other authors have considered different notions of Möbius inversion for categories; notably, there is that developed by Content, Lemay and Leroux [CLL] and independently by Haigh [H]. This generalizes both Rota's notion for posets and Cartier and Foata's for monoids [CF]. (Here a monoid is viewed as a one-object category.) The relation between their approach and ours is discussed in §4. Further approaches, not discussed here, were taken by Dür [D] and Lück [Lü].

In the case of groupoids, our Euler characteristic of categories agrees with Baez and Dolan's groupoid cardinality [BD]. The cardinality of the groupoid of finite sets and bijections is $e = 2.718\ldots$, and there are connections to exponential generating functions and the species of Joyal [J, BLL]. Paré has a definition of the cardinality of an endofunctor of the category of finite sets [Pa]; I do not know whether this can be related to the definition here of the Lefschetz number of an endofunctor.

The view of Euler characteristic as generalized cardinality is promoted in [Sc1], [BD] and [Pr1]. The appearance of a non-integral Euler characteristic is nothing new: see for instance Wall [Wl], Bass [Ba] and Cohen [Co], and the discussion of orbifolds in §2.

Ultimately it would be desirable to have the Euler characteristic of categories described by a universal property, as Schanuel did for polyhedra [Sc1]. For this, it may be necessary to relax the constraints of the present work, where for simplicity our categories are required to be finite and the coefficients are required to lie in the ring of rational numbers. Rather than asking, as below, 'does this category have Euler characteristic (in $\mathbb{Q}$)?', we should perhaps ask 'in what rig (semiring) does the Euler characteristic of this category lie?' However, this is not pursued here.

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1 Möbius inversion

We consider a finite category $\mathbb{A}$, writing $\mathrm{ob}\,\mathbb{A}$ for its set of objects and, when a and b are objects, $\mathbb{A}(a,b)$ for the set of maps from a to b.

Definition 1.1 We denote by $R(\mathbb{A})$ the $\mathbb{Q}$-algebra of functions $\mathrm{ob}\,\mathbb{A} \times \mathrm{ob}\,\mathbb{A} \longrightarrow \mathbb{Q}$, with pointwise addition and scalar multiplication, multiplication defined by

$$(\theta\phi)(a,c) = \sum_{b \in \mathbb{A}} \theta(a,b)\phi(b,c)$$

($\theta, \phi \in R(\mathbb{A})$, $a, c \in \mathbb{A}$), and the Kronecker δ as unit.

The *zeta function* $\zeta_{\mathbb{A}} = \zeta \in R(\mathbb{A})$ is defined by $\zeta(a,b) = |\mathbb{A}(a,b)|$. If ζ is invertible in $R(\mathbb{A})$ then $\mathbb{A}$ is said to *have Möbius inversion*; its inverse $\mu_{\mathbb{A}} = \mu = \zeta^{-1}$ is the *Möbius function* of $\mathbb{A}$.

If a total ordering is chosen on the n objects of $\mathbb{A}$ then $R(\mathbb{A})$ can be regarded as the algebra of $n \times n$ matrices over $\mathbb{Q}$. The defining equations of the Möbius function are

$$\sum_b \mu(a,b)\zeta(b,c) = \delta(a,c) = \sum_b \zeta(a,b)\mu(b,c)$$

for all $a, c \in \mathbb{A}$. By finite-dimensionality, $\mu\zeta = \delta$ if and only if $\zeta\mu = \delta$.

The definitions above could be made for directed graphs rather than categories, since they do not refer to composition. However, this generality seems to be inappropriate. For example, the definition of Möbius inversion will lead to a definition of Euler characteristic, and if we use graphs rather than categories then we obtain something other than 'vertices minus edges'. Proposition 2.10 clarifies this point.

A different notion of Möbius inversion for categories has been considered; see §4.

Examples 1.2 a. Any finite poset A has Möbius inversion; this special case was investigated by Rota [R] and others. We may compute $\mu(a,c)$ by induction on the number of elements between a and c:

$$\mu(a,c) = \delta(a,c) - \sum_{b:a\leq b<c} \mu(a,b).$$

In particular, $\mu(a,c) = 0$ unless $a \leq c$, and $\mu(a,a) = 1$ for all a. See also Theorem 1.4 and Corollary 1.5.

b. Let M be a finite monoid, regarded as a category with unique object $\star$. (The arrows of the category are the elements of the monoid, and composition in the category is multiplication in the monoid.) Then $\zeta(\star,\star) = |M|$, so $\mu(\star,\star) = 1/|M|$.

c. Let $N \geq 0$. Write $\mathbb{D}_N^{\mathrm{inj}}$ for the category with objects $0,\ldots,N$ whose maps $a \longrightarrow b$ are the order-preserving injections $\{1,\ldots,a\} \longrightarrow \{1,\ldots,b\}$. Then $\zeta(a,b) = \binom{b}{a}$, and it is easily checked that $\mu(a,b) = (-1)^{b-a}\binom{b}{a}$. If we use surjections instead of injections then $\zeta(a,b) = \binom{a-1}{b-1}$ and $\mu(a,b) = (-1)^{a-b}\binom{a-1}{b-1}$.

A category with Möbius inversion must be *skeletal* (isomorphic objects must be equal), for otherwise the matrix of ζ would have two identical rows. The property of having Möbius inversion is not, therefore, invariant under equivalence of categories.

In general we cannot hope to just spot the Möbius function of a category. In 1.3–1.7 we make tools for computing Möbius functions. These cover large classes of categories, although not every finite skeletal category has Möbius inversion (1.11(d), (e)).

Let $n \geq 0$, let $\mathbb{A}$ be a category or a directed graph, and let $a, b \in \mathbb{A}$. An *n-path* from a to b is a diagram

$$a = a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} a_n = b \tag{1}$$

in $\mathbb{A}$. It is a *circuit* if $a = b$, and (when $\mathbb{A}$ is a category) *nondegenerate* if no f_i is an identity.

Lemma 1.3 *The following conditions on a finite category $\mathbb{A}$ are equivalent:*

a. every idempotent in $\mathbb{A}$ is an identity

b. every endomorphism in $\mathbb{A}$ is an automorphism

c. every circuit in $\mathbb{A}$ consists entirely of isomorphisms.

Proof (a) $\Rightarrow$ (b) follows from the fact that if f is an element of a finite monoid then some positive power of f is idempotent. The other implications are straightforward. □

Theorem 1.4 *Let $\mathbb{A}$ be a finite skeletal category in which the only idempotents are identities. Then $\mathbb{A}$ has Möbius inversion given by*

$$\mu(a,b) = \sum (-1)^n / |\mathrm{Aut}(a_0)| \cdots |\mathrm{Aut}(a_n)|$$

where $\mathrm{Aut}(a)$ *is the automorphism group of* $a \in \mathbb{A}$ *and the sum runs over all* $n \geq 0$ *and paths (1) for which* $a_0, \ldots, a_n$ *are all distinct.*

Proof First observe that for a path (1) in $\mathbb{A}$, if $a_0 \neq a_1 \neq \cdots \neq a_n$ then the a_is are all distinct. Indeed, if $0 \leq i < j \leq n$ and $a_i = a_j$ then the sub-path running from a_i to a_j is a circuit, so by Lemma 1.3, f_{i+1} is an isomorphism, and by skeletality, $a_i = a_{i+1}$.

Now let $a, c \in \mathbb{A}$ and define μ by the formula above. We have

$$\begin{aligned}
\sum_{b \in \mathbb{A}} \mu(a,b)\zeta(b,c) &= \mu(a,c)\zeta(c,c) + \sum_{b: b \neq c} \mu(a,b)\zeta(b,c) \\
&= |\mathrm{Aut}(c)| \left\{ \mu(a,c) + \sum_{b: b\neq c,\ g \in \mathbb{A}(b,c)} \mu(a,b)/|\mathrm{Aut}(c)| \right\} \\
&= |\mathrm{Aut}(c)| \left\{ \mu(a,c) + \sum (-1)^n / |\mathrm{Aut}(a_0)| \cdots |\mathrm{Aut}(a_n)||\mathrm{Aut}(c)| \right\},
\end{aligned}$$

where the last sum is over all $n \geq 0$ and paths

$$a = a_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} a_n = b \xrightarrow{g} c$$

such that $a_0 \neq \cdots \neq a_n \neq c$. By definition of μ, the term in braces collapses to 0 if $a \neq c$ and to $1/|\mathrm{Aut}(a)|$ if $a = c$. Hence $\sum_b \mu(a,b)\zeta(b,c) = \delta(a,c)$, as required. $\square$

Corollary 1.5 *Let $\mathbb{A}$ be a finite skeletal category in which the only endomorphisms are identities. Then $\mathbb{A}$ has Möbius inversion given by*

$$\mu(a,b) = \sum_{n \geq 0} (-1)^n |\{\textit{nondegenerate } n\textit{-paths from } a \textit{ to } b\}| \in \mathbb{Z}.$$

$\square$

When $\mathbb{A}$ is a poset, this is Philip Hall's theorem (Proposition 3.8.5 of [St] and Proposition 6 of [R]).

An *epi-mono factorization system* $(\mathcal{E}, \mathcal{M})$ on a category $\mathbb{A}$ consists of a class $\mathcal{E}$ of epimorphisms in $\mathbb{A}$ and a class $\mathcal{M}$ of monomorphisms in $\mathbb{A}$, satisfying axioms [FK]. The axioms imply that every map f in $\mathbb{A}$ can be expressed as me for some $e \in \mathcal{E}$ and $m \in \mathcal{M}$, and that this factorization is essentially unique: the other pairs $(e', m') \in \mathcal{E} \times \mathcal{M}$ satisfying $m'e' = f$ are those of the form (ie, mi^{-1}) where i is an isomorphism. Typical examples are the categories of sets, groups and rings, with $\mathcal{E}$ as all surjections and $\mathcal{M}$ as all injections.

Theorem 1.6 *Let $\mathbb{A}$ be a finite skeletal category with an epi-mono factorization system $(\mathcal{E}, \mathcal{M})$. Then $\mathbb{A}$ has Möbius inversion given by*

$$\mu(a,b) = \sum (-1)^n / |\mathrm{Aut}(a_0)| \cdots |\mathrm{Aut}(a_n)|$$

where the sum is over all $n \geq r \geq 0$ and paths (1) such that $a_0, \ldots, a_r$ are distinct, $a_r, \ldots, a_n$ are distinct, $f_1, \ldots, f_r \in \mathcal{M}$, and $f_{r+1}, \ldots, f_n \in \mathcal{E}$.

Proof The objects of $\mathbb{A}$ and the arrows in $\mathcal{E}$ determine a subcategory of $\mathbb{A}$, also denoted $\mathcal{E}$; it satisfies the hypotheses of Theorem 1.4 and therefore has Möbius inversion. The same is true of $\mathcal{M}$.

Any element $\alpha \in \mathbb{Q}^{\mathrm{ob}\,\mathbb{A}} = \prod_{a \in \mathbb{A}} \mathbb{Q}$ gives rise to an element of $R(\mathbb{A})$, also denoted α and defined by $\alpha(a,b) = \delta(a,b)\alpha(b)$. This defines a multiplication-preserving map from $\mathbb{Q}^{\mathrm{ob}\,\mathbb{A}}$ to $R(\mathbb{A})$, where the multiplication on $\mathbb{Q}^{\mathrm{ob}\,\mathbb{A}}$ is coordinatewise. We have elements $|\mathrm{Aut}|$ and $1/|\mathrm{Aut}|$ of $\mathbb{Q}^{\mathrm{ob}\,\mathbb{A}}$, where, for instance, $|\mathrm{Aut}|(a) = |\mathrm{Aut}(a)|$.

By the essentially unique factorization property, $\zeta_{\mathbb{A}} = \zeta_{\mathcal{E}} \cdot \frac{1}{|\mathrm{Aut}|} \cdot \zeta_{\mathcal{M}}$. Hence $\mathbb{A}$ has Möbius function $\mu_{\mathbb{A}} = \mu_{\mathcal{M}} \cdot |\mathrm{Aut}| \cdot \mu_{\mathcal{E}}$. Theorem 1.4 applied to $\mu_{\mathcal{M}}$ and $\mu_{\mathcal{E}}$ then gives the formula claimed. □

Example 1.7 Let $N \geq 0$ and write $\mathbb{F}_N$ for the full subcategory of **Set** with objects $1, \ldots, N$, where n denotes a (chosen) n-element set. Let $\mathcal{E}$ be the set of surjections in $\mathbb{F}_N$ and $\mathcal{M}$ the set of injections; then $(\mathcal{E}, \mathcal{M})$ is an epi-mono factorization system. Theorem 1.6 gives a formula for the inverse of the matrix $(i^j)_{i,j}$. For instance, take $N = 3$; then $\mu(1,2)$ may be computed as follows:

Paths	Contribution to sum
$1 \xrightarrow{2} 2$	$-2/1!2! = -1$
$1 \xrightarrow{3} 3 \xrightarrow{6} 2$	$3 \cdot 6/1!3!2! = 3/2$
$1 \xrightarrow{2} 2 \xrightarrow{6} 3 \xrightarrow{6} 2$	$-2 \cdot 6 \cdot 6/1!2!3!2! = -3$

Here '$1 \xrightarrow{2} 2$' means that there are 2 monomorphisms from 1 to 2, '$3 \xrightarrow{6} 2$' that there are 6 epimorphisms from 3 to 2, etc. Hence $\mu(1,2) = -1 + 3/2 - 3 = -5/2$.

One of the uses of the Möbius function is to calculate Euler characteristic (§2). Another is to calculate representations. Specifically, suppose that we have a **Set**-valued functor known to be *familially representable*, that is, a coproduct of representables. The Yoneda Lemma tells us that the family of representing objects is unique (up to isomorphism). But if we have Möbius inversion, there is actually a formula for it:

Proposition 1.8 *Let $\mathbb{A}$ be a finite category with Möbius inversion and let $X : \mathbb{A} \longrightarrow$ **Set** be a functor satisfying*

$$X \cong \sum_a r(a) \mathbb{A}(a, -)$$

for some natural numbers $r(a)$ $(a \in \mathbb{A})$. Then

$$r(a) = \sum_b |X(b)| \mu(b, a)$$

for all $a \in \mathbb{A}$.

In the first formula, $\sum$ denotes coproduct of **Set**-valued functors.

Proof Follows from the definition of Möbius function. $\square$

In the spirit of Rota's programme, this can be applied to solve counting problems, as illustrated by the following standard example.

Example 1.9 A *derangement* is a permutation without fixed points. We calculate d_n, the number of derangements of n letters.

Fix $N \geq 0$. Take the category $\mathbb{D}_N^{\mathrm{inj}}$ of Example 1.2(c) and the functor $S : \mathbb{D}_N^{\mathrm{inj}} \longrightarrow \mathbf{Set}$ defined as follows: $S(n)$ is S_n, the underlying set of the nth symmetric group, and if $f \in \mathbb{D}_N^{\mathrm{inj}}(m, n)$ and $\tau \in S_m$, the induced permutation $S_f(\tau) \in S_n$ acts as τ on the image of f and fixes all other points. Any permutation consists of a derangement together with some fixed points, so there is an isomorphism of sets

$$S_n \cong \sum_m d_m \mathbb{D}_N^{\mathrm{inj}}(m, n)$$

where $\sum$ denotes disjoint union. Then by Proposition 1.8 and Example 1.2(c),

$$d_n = \sum_m |S_m| \mu(m, n) = \sum_m m! (-1)^{n-m} \binom{n}{m} = n! \left(\frac{1}{0!} - \frac{1}{1!} + \cdots + \frac{(-1)^n}{n!} \right).$$

To set up the theory of Euler characteristic we will not need the full strength of Möbius invertibility; the following suffices.

Definition 1.10 Let $\mathbb{A}$ be a finite category. A *weighting* on $\mathbb{A}$ is a function $k^\bullet : \mathrm{ob}\, \mathbb{A} \longrightarrow \mathbb{Q}$ such that for all $a \in \mathbb{A}$,

$$\sum_b \zeta(a, b) k^b = 1.$$

A *coweighting* $k_\bullet$ on $\mathbb{A}$ is a weighting on $\mathbb{A}^{\mathrm{op}}$.

Note that $\mathbb{A}$ has Möbius inversion if and only if it has a unique weighting, if and only if it has a unique coweighting; they are given by

$$k^a = \sum_b \mu(a, b), \qquad k_b = \sum_a \mu(a, b).$$

Examples 1.11 a. Let $\mathbb{L}$ be the category

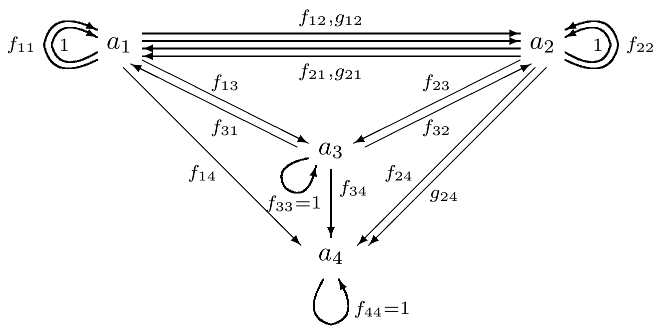
$$\begin{tikzcd}[row sep=small] & b_1 \\ a \arrow[ur] \arrow[dr] & \\ & b_2. \end{tikzcd}$$

Then the unique weighting $k^\bullet$ on $\mathbb{L}$ is $(k^a, k^{b_1}, k^{b_2}) = (-1, 1, 1)$.

b. Let M be a finite monoid, regarded as a category with unique object $\star$. Again there is a unique weighting $k^\bullet$, with $k^\star = 1/|M|$.

c. If $\mathbb{A}$ has a terminal object 1 then $\delta(-, 1)$ is a weighting on $\mathbb{A}$.

d. A finite category may admit no weighting at all. (This can happen even when the category is Cauchy-complete, in the sense defined in the Appendix.) An example is the category $\mathbb{A}$ with objects and arrows



where if $a_i \xrightarrow{p} a_j \xrightarrow{q} a_k$ and neither p nor q is an identity then $q \circ p = f_{ik}$.

e. A category may certainly have more than one weighting: for instance, if $\mathbb{A}$ is the category consisting of two objects and a single isomorphism between them, a weighting on $\mathbb{A}$ is any pair of rational numbers whose sum is 1. But even a skeletal category may admit more than one weighting. Indeed, the full subcategories $\mathbb{B} = \{a_1, a_2\}$ and $\mathbb{C} = \{a_1, a_2, a_3\}$ of the category $\mathbb{A}$ of the previous example both have a 1-dimensional space of weightings.

In contrast to Möbius invertibility, the property of admitting at least one weighting is invariant under equivalence:

Lemma 1.12 *Let $\mathbb{A}$ and $\mathbb{B}$ be equivalent finite categories. Then $\mathbb{A}$ admits a weighting if and only if $\mathbb{B}$ does.*

Proof Let $F : \mathbb{A} \longrightarrow \mathbb{B}$ be an equivalence. Given $a \in \mathbb{A}$, write C_a for the number of objects in the isomorphism class of a. Take a weighting $l^\bullet$ on $\mathbb{B}$ and put $k^a = (\sum_{b: b \cong F(a)} l^b)/C_a$. I claim that $k^\bullet$ is a weighting on $\mathbb{A}$.

To prove this, choose representatives $a_1, \ldots, a_m$ of the isomorphism classes of objects of $\mathbb{A}$; then $F(a_1), \ldots, F(a_m)$ are representatives of the isomorphism classes of objects of $\mathbb{B}$. Let $a' \in \mathbb{A}$. For any $a \in \mathbb{A}$, the numbers $\zeta(a', a)$ and k^a depend only on the isomorphism class of a. Hence

$$\begin{aligned}
\sum_{a \in \mathbb{A}} \zeta(a', a) k^a &= \sum_{i=1}^{m} \sum_{a: a \cong a_i} \zeta(a', a) k^a \\
&= \sum_{i=1}^{m} C_{a_i} \zeta(a', a_i) k^{a_i} \\
&= \sum_{i=1}^{m} \sum_{b: b \cong F(a_i)} \zeta(a', a_i) l^b \\
&= \sum_{b \in \mathbb{B}} \zeta(F(a'), b) l^b \\
&= 1,
\end{aligned}$$

as required. □

Weightings and Möbius functions are compatible with sums and products of categories. We write $\sum_{i \in I} \mathbb{A}_i$ for the sum of a family $(\mathbb{A}_i)_{i \in I}$ of categories, also called the coproduct or disjoint union and written $\coprod_{i \in I} \mathbb{A}_i$. The following lemma is easily verified.

Lemma 1.13 *Let $n \geq 0$ and let $\mathbb{A}_1, \ldots, \mathbb{A}_n$ be finite categories.*

a. *If each $\mathbb{A}_i$ has a weighting $k_i^\bullet$ then $\sum_i \mathbb{A}_i$ has a weighting $l^\bullet$ given by $l^a = k_i^a$ whenever $a \in \mathbb{A}_i$. If each $\mathbb{A}_i$ has Möbius inversion then so does $\sum_i \mathbb{A}_i$, where for $a \in \mathbb{A}_i$ and $b \in \mathbb{A}_j$,*

$$\mu_{\sum \mathbb{A}_k}(a, b) = \begin{cases} \mu_{\mathbb{A}_i}(a, b) & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

b. *If each $\mathbb{A}_i$ has a weighting $k_i^\bullet$ then $\prod_i \mathbb{A}_i$ has a weighting $l^\bullet$ given by $l^{(a_1, \ldots, a_n)} = k_1^{a_1} \cdots k_n^{a_n}$. If each $\mathbb{A}_i$ has Möbius inversion then so does $\prod_i \mathbb{A}_i$, with*

$$\mu_{\prod \mathbb{A}_i}((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = \mu_{\mathbb{A}_1}(a_1, b_1) \cdots \mu_{\mathbb{A}_n}(a_n, b_n).$$

□

Thinking of $R(\mathbb{A})$ as a matrix algebra (as described after Definition 1.1), the part of (a) concerning Möbius inversion merely says that the inverse of a block sum of matrices is the block sum of the inverses.

To every **Set**-valued functor X there is assigned a 'category of elements' $\mathbb{E}(X)$. (See the Appendix for a review of definitions.) This is also true of functors X taking values in **Cat**, the category of small categories and functors, even when X is only a weak or 'pseudo' functor. We say that a **Set**- or **Cat**-valued functor X is *finite* if $\mathbb{E}(X)$ is finite. When the domain category $\mathbb{A}$ is finite, this just means that each set or category $X(a)$ is finite.

LEMMA 1.14 *Let $\mathbb{A}$ be a finite category and $X : \mathbb{A} \longrightarrow$ **Cat** a finite weak functor. Suppose that we have weightings on $\mathbb{A}$ and on each $X(a)$, all written $k^\bullet$. Then there is a weighting on $\mathbb{E}(X)$ defined by $k^{(a,x)} = k^a k^x$ ($a \in \mathbb{A}$, $x \in X(a)$).*

PROOF Let $a \in \mathbb{A}$ and $x \in X(a)$. Then

$$\begin{aligned}\sum_{(b,y)\in\mathbb{E}(X)} \zeta((a,x),(b,y))k^b k^y &= \sum_b \sum_{f\in\mathbb{A}(a,b)} \left(\sum_{y\in X(b)} \zeta((X(f))x, y)k^y \right) k^b \\ &= \sum_b \zeta(a,b)k^b = 1.\end{aligned}$$

□

This result will be used to show how Euler characteristic behaves with respect to fibrations.

2 EULER CHARACTERISTIC

In this section, the Euler characteristic of a category is defined and its basic properties are established. The definition is justified by a series of propositions showing its compatibility with the Euler characteristics of other types of object: graphs, topological spaces, and orbifolds. There follows a brief discussion of the Lefschetz number of an endofunctor.

LEMMA 2.1 *Let $\mathbb{A}$ be a finite category, $k^\bullet$ a weighting on $\mathbb{A}$, and $k_\bullet$ a coweighting on $\mathbb{A}$. Then $\sum_a k^a = \sum_a k_a$.*

PROOF

$$\sum_b k^b = \sum_b \left(\sum_a k_a \zeta(a,b) \right) k^b = \sum_a k_a \left(\sum_b \zeta(a,b)k^b \right) = \sum_a k_a.$$

□

If $\mathbb{A}$ admits a weighting but no coweighting then $\sum_a k^a$ may depend on the weighting $k^\bullet$ chosen: see Example 4.8 of [BL].

Definition 2.2 A finite category $\mathbb{A}$ *has Euler characteristic* if it admits both a weighting and a coweighting. Its *Euler characteristic* is then

$$\chi(\mathbb{A}) = \sum_a k^a = \sum_a k_a \in \mathbb{Q}$$

for any weighting $k^\bullet$ and coweighting $k_\bullet$.

With the Gauss–Bonnet Theorem in mind, one might think of weight as an analogue of curvature: summed over the whole structure, it yields the Euler characteristic.

Any category $\mathbb{A}$ with Möbius inversion has Euler characteristic, $\chi(\mathbb{A}) = \sum_{a,b} \mu(a,b)$, as in the Introduction.

Examples 2.3

a. If $\mathbb{A}$ is a finite discrete category then $\chi(\mathbb{A}) = |\mathrm{ob}\,\mathbb{A}|$.

b. If M is a finite monoid then $\chi(M) = 1/|M|$. (We continue to view monoids as one-object categories.) When M is a group, this can be understood as follows: M acts freely on the contractible space EM, which has Euler characteristic 1; one would therefore expect the quotient space BM to have Euler characteristic $1/|M|$. (Compare [Wl] and [Co].)

c. By Corollary 1.5, a finite poset A has Euler characteristic $\sum_{n\geq 0}(-1)^n c_n \in \mathbb{Z}$, where c_n is the number of chains in A of length n. (See [Pu], [Fo], [R] and [Fa] for connections with poset homology, and §4 for further comparisons with the Rota theory.) More generally, the results of §1 lead to formulas for the Euler characteristic of any finite category that either has no non-trivial idempotents or admits an epi-mono factorization system.

   For example, let $\mathbb{A}$ be a category with no non-trivial idempotents. Let $\mathbb{B}$ be a skeleton of $\mathbb{A}$, that is, a full subcategory containing exactly one object from each isomorphism class of $\mathbb{A}$. Theorem 1.4 tells us that $\mathbb{B}$ has Möbius inversion and gives us a formula for its Möbius function, hence for its Euler characteristic. Proposition 2.4(b) below then implies that $\mathbb{A}$ has Euler characteristic, equal to that of $\mathbb{B}$.

d. By 1.11(c) and its dual, if $\mathbb{A}$ has Euler characteristic and either an initial or a terminal object then $\chi(\mathbb{A}) = 1$; moreover, if $\mathbb{A}$ has both an initial and a terminal object then it does have Euler characteristic. This applies, for instance, to the category $\mathbb{C}$ of 1.11(e). Hence having Möbius inversion is a strictly stronger property than having Euler characteristic, even for skeletal categories.

e. Euler characteristic is not invariant under Morita equivalence. Recall that categories $\mathbb{A}$ and $\mathbb{B}$ are *Morita equivalent* if their presheaf categories $[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$ and $[\mathbb{B}^{\mathrm{op}}, \mathbf{Set}]$ are equivalent; see [Bo], for instance. Equivalent categories are Morita equivalent, but not conversely. For instance, take $\mathbb{A}$ to be the two-element monoid consisting of the identity and an idempotent, and $\mathbb{B}$ to be the category generated by objects and arrows

$$b_1 \underset{s}{\overset{i}{\rightleftarrows}} b_2$$

subject to $si = 1$. Then $\mathbb{A}$ and $\mathbb{B}$ are Morita equivalent but not equivalent. Moreover, their Euler characteristics are different: $\chi(\mathbb{A}) = 1/2$ by (b), but $\chi(\mathbb{B}) = 1$ by (d).

Clearly $\chi(\mathbb{A}^{\mathrm{op}}) = \chi(\mathbb{A})$, one side being defined when the other is. The next few propositions set out further basic properties of Euler characteristic.

PROPOSITION 2.4 *Let $\mathbb{A}$ and $\mathbb{B}$ be finite categories.*

a. *If there is an adjunction $\mathbb{A} \rightleftarrows \mathbb{B}$ and both $\mathbb{A}$ and $\mathbb{B}$ have Euler characteristic then $\chi(\mathbb{A}) = \chi(\mathbb{B})$.*

b. *If $\mathbb{A} \simeq \mathbb{B}$ then $\mathbb{A}$ has Euler characteristic if and only if $\mathbb{B}$ does, and in that case $\chi(\mathbb{A}) = \chi(\mathbb{B})$.*

In (a), it may be that one category has Euler characteristic but the other does not: consider, for instance, the unique functor from the category $\mathbb{A}$ of 1.11(d) to the terminal category.

PROOF

a. Suppose that $\mathbb{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{B}$ with $F \dashv G$. Then $\zeta(F(a), b) = \zeta(a, G(b))$ for all $a \in \mathbb{A}$, $b \in \mathbb{B}$; write $\zeta(a, b)$ for their common value. Take a coweighting $k_\bullet$ on $\mathbb{A}$ and a weighting $k^\bullet$ on $\mathbb{B}$. Then $\sum_a k_a = \sum_b k^b$ by the same proof as that of Lemma 2.1.

b. The first statement follows from Lemma 1.12 and its dual, and the second from (a). $\square$

EXAMPLE 2.5 If $\mathbb{B}$ is a category with an initial or a terminal object then $\chi(\mathbb{A}^{\mathbb{B}}) = \chi(\mathbb{A})$ for all $\mathbb{A}$, provided that both Euler characteristics exist. Indeed, if 0 is initial in $\mathbb{B}$ then evaluation at 0 is right adjoint to the diagonal functor $\mathbb{A} \longrightarrow \mathbb{A}^{\mathbb{B}}$.

PROPOSITION 2.6 *Let $n \geq 0$ and let $\mathbb{A}_1, \ldots, \mathbb{A}_n$ be finite categories that all have Euler characteristic. Then $\sum_i \mathbb{A}_i$ and $\prod_i \mathbb{A}_i$ have Euler characteristic, with*

$$\chi\left(\sum_i \mathbb{A}_i\right) = \sum_i \chi(\mathbb{A}_i), \qquad \chi\left(\prod_i \mathbb{A}_i\right) = \prod_i \chi(\mathbb{A}_i).$$

PROOF Follows from Lemma 1.13. $\square$

Example 2.7 Let $\mathbb{A}$ be a finite groupoid. Choose one object a_i from each connected-component of $\mathbb{A}$, and write G_i for the automorphism group of a_i. Then $\mathbb{A} \simeq \sum_i G_i$, so by 2.3(b), 2.4(b) and 2.6, we have $\chi(\mathbb{A}) = \sum_i 1/|G_i|$. This is what Baez and Dolan call the cardinality of the groupoid $\mathbb{A}$ [BD].

One might also ask whether $\chi(\mathbb{A}^{\mathbb{B}}) = \chi(\mathbb{A})^{\chi(\mathbb{B})}$. By 2.3(d), 2.5 and 2.6, the answer is yes if every connected-component of $\mathbb{B}$ has an initial or a terminal object (and all the Euler characteristics exist). But in general the answer is no: for instance, take $\mathbb{A}$ to be the 2-object discrete category and $\mathbb{B}$ to be the category of 3.4(b). See also Propp [Pr2], Speed [Sp], and §5, 6 of Rota [R].

An important property of topological Euler characteristic is its behaviour with respect to fibre bundles (or more generally, fibrations). Take a space A with connected-components $A_1, \ldots, A_n$, take a fibre bundle E over A, and write X_i for the fibre in the ith component. Then under suitable hypotheses, $\chi(E) = \sum_i \chi(A_i)\chi(X_i)$.

There is an analogy between topological fibrations and categorical fibrations, which are functors satisfying a certain condition. (In this discussion I will use 'fibration' to mean what is usually called an opfibration; the difference is inessential.) The crucial property of fibrations of categories is that for any category $\mathbb{A}$, the fibrations with codomain $\mathbb{A}$ correspond naturally to the weak functors $\mathbb{A} \longrightarrow \mathbf{Cat}$. Given a fibration $P : \mathbb{E} \longrightarrow \mathbb{A}$, define a functor $X : \mathbb{A} \longrightarrow \mathbf{Cat}$ by taking $X(a)$, for each $a \in \mathbb{A}$, to be the *fibre* over a: the subcategory of $\mathbb{E}$ whose objects e are those satisfying $P(e) = a$ and whose arrows f are those satisfying $P(f) = 1_a$. Conversely, given a weak functor $X : \mathbb{A} \longrightarrow \mathbf{Cat}$, the corresponding fibration is the category of elements $\mathbb{E}(X)$ together with the projection functor to $\mathbb{A}$. For details, see [Bo], for instance.

The formula for the Euler characteristic of a fibre bundle has a categorical analogue. Since in general the fibres of a fibration over $\mathbb{A}$ vary within each connected-component of $\mathbb{A}$, the formula for categories is more complicated. We state the result in terms of $\mathbf{Cat}$-valued functors rather than fibrations; the proof follows from Lemma 1.14.

Proposition 2.8 *Let $\mathbb{A}$ be a finite category and $X : \mathbb{A} \longrightarrow \mathbf{Cat}$ a finite weak functor. Let $k^\bullet$ be a weighting on $\mathbb{A}$ and suppose that $\mathbb{E}(X)$ and each $X(a)$ have Euler characteristic. Then*

$$\chi(\mathbb{E}(X)) = \sum_a k^a \chi(X(a)).$$

□

Examples 2.9 a. When X is a finite $\mathbf{Set}$-valued functor, $\chi(\mathbb{E}(X)) = \sum_a k^a |X(a)|$. For example, let M be a finite monoid. A finite functor $X : M \longrightarrow \mathbf{Set}$ is a finite set S with a left M-action. Following [BD], we write $\mathbb{E}(X)$ as $S//M$, the *lax quotient* of S by M. (Its objects are the elements of S, and the arrows $s \longrightarrow s'$ are the elements $m \in M$ satisfying $ms = s'$.) Then $\chi(S//M) = |S|/|M|$.

b. Define a sequence $(\mathbb{S}^n)_{n\geq -1}$ of categories inductively as follows. $\mathbb{S}^{-1}$ is empty. Let $\mathbb{L}$ be the category of 1.11(a); define $X : \mathbb{L} \longrightarrow \mathbf{Cat}$ by $X(a) = \mathbb{S}^{n-1}$ and $X(b_1) = X(b_2) = \mathbf{1}$ (the terminal category); put $\mathbb{S}^n = \mathbb{E}(X)$. Then explicitly, $\mathbb{S}^n$ is the poset

$$\begin{tikzcd} c_0 \arrow[r] \arrow[dr] & c_1 \arrow[r] \arrow[dr] & \cdots \arrow[r] \arrow[dr] & c_n \\ d_0 \arrow[r] \arrow[ur] & d_1 \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & d_n. \end{tikzcd}$$

(If we take the usual expression of the topological n-sphere S^n as a CW-complex with two cells in each dimension $\leq n$ then $\mathbb{S}^n$ is the set of cells ordered by inclusion; S^n is the classifying space of $\mathbb{S}^n$.)

Each $\mathbb{S}^n$ is a poset, so has Euler characteristic. By Proposition 2.8,

$$\chi(\mathbb{S}^n) = -\chi(\mathbb{S}^{n-1}) + 2\chi(\mathbf{1}) = 2 - \chi(\mathbb{S}^{n-1})$$

for all $n \geq 0$; also $\chi(\mathbb{S}^{-1}) = 0$. Hence $\chi(\mathbb{S}^n) = 1 + (-1)^n$.

The next three propositions show how the Euler characteristics of various types of structure are compatible with that of categories.

First, Euler characteristic of categories extends Euler characteristic of graphs. More precisely, let $G = (G_1 \rightrightarrows G_0)$ be a directed graph, where G_1 is the set of edges and G_0 the set of vertices. We will show that if $F(G)$ is the free category on G then $\chi(F(G)) = |G_0| - |G_1|$. This only makes sense if $F(G)$ is finite, which is the case if and only if G is finite and circuit-free; then $F(G)$ is also circuit-free. (A directed graph is *circuit-free* if it contains no circuits of non-zero length, and a category is *circuit-free* if every circuit consists entirely of identities.)

Proposition 2.10 *Let G be a finite circuit-free directed graph. Then $\chi(F(G))$ is defined and equal to $|G_0| - |G_1|$.*

Proof Given $a, b \in G_0$, write $\zeta_G(a, b)$ for the number of edges from a to b in G. Then $\zeta_{F(G)} = \sum_{n\geq 0} \zeta_G^n$ in $R(F(G))$, the sum being finite since G is circuit-free. Hence $\mu_{F(G)} = \delta - \zeta_G$, and the result follows. □

This suggests that in the present context, it is more fruitful to view a graph as a special category (via F) than a category as a graph with structure. Compare the comments after Definition 1.1.

The second result compares the Euler characteristics of categories and topological spaces. We show that under suitable hypotheses, $\chi(B\mathbb{A}) = \chi(\mathbb{A})$, where $B\mathbb{A}$ is the classifying space of a category $\mathbb{A}$ (that is, the geometric realization of its nerve $N\mathbb{A}$). To ensure that $B\mathbb{A}$ has Euler characteristic, we assume that $N\mathbb{A}$ contains only finitely many nondegenerate simplices; then

$$\chi(B\mathbb{A}) = \sum_{n\geq 0} (-1)^n |\{\text{nondegenerate } n\text{-simplices in } N\mathbb{A}\}|.$$

An n-simplex in $N\mathbb{A}$ is just an n-path in $\mathbb{A}$, and is nondegenerate in the sense of simplicial sets if and only if it is nondegenerate as a path, so $\mathbb{A}$ must contain only finitely many nondegenerate paths. This is the case if and only if $\mathbb{A}$ is circuit-free, if and only if $\mathbb{A}$ is skeletal and contains no endomorphisms except identities. So by Corollary 1.5, we have:

Proposition 2.11 *Let $\mathbb{A}$ be a finite skeletal category containing no endomorphisms except identities. Then $\chi(B\mathbb{A})$ is defined and equal to $\chi(\mathbb{A})$.* □

For the final compatibility result, consider the following schematic diagrams:

$$\begin{tikzcd} \{\text{triangulated manifolds}\} \arrow[d] \arrow[dr, "\chi"] & \\ \{\text{posets}\} \arrow[r, "\chi"'] & \mathbb{Z} \end{tikzcd} \qquad \begin{tikzcd} \{\text{triangulated orbifolds}\} \arrow[d] \arrow[dr, "\chi"] & \\ \{\text{categories}\} \arrow[r, "\chi"'] & \mathbb{Q}. \end{tikzcd}$$

On the left, we start with a compact manifold M equipped with a finite triangulation. As shown in §3.8 of [St], the topological Euler characteristic of M is equal to the Euler characteristic of the poset of simplices in the triangulation, ordered by inclusion. We generalize this result from manifolds to orbifolds, which entails replacing posets by categories and $\mathbb{Z}$ by $\mathbb{Q}$.

Let M be a compact orbifold equipped with a finite triangulation. (See [MP] for definitions.) The simplices in the triangulation form a poset P, and if $p \in P$ is a d-dimensional simplex then $\downarrow p = \{q \in P \mid q \leq p\}$ is isomorphic to the poset $\mathbb{P}_{d+1}$ of nonempty subsets of $\{1, \ldots, d+1\}$, with $p \in \downarrow p$ corresponding to $\{1, \ldots, d+1\} \in \mathbb{P}_{d+1}$. Every $p \in P$ has a stabilizer group $G(p)$, and

$$\chi(M) = \sum_{p \in P} (-1)^{\dim p} / |G(p)|.$$

On the other hand, the groups $G(p)$ fit together to form a *complex of finite groups* on P^{op}, that is, a weak functor $G : P^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ taking values in finite groups (regarded as one-object categories) and injective homomorphisms; see §3 of [M]. This gives a finite category $\mathbb{E}\,(G)$. For example, when M is a manifold, each group $G(p)$ is trivial and $\mathbb{E}\,(G) \cong P$.

The following result is joint with Ieke Moerdijk.

Proposition 2.12 *Let M be a compact orbifold equipped with a finite triangulation. Let G be the resulting complex of groups. Then $\chi(\mathbb{E}\,(G))$ is defined and equal to $\chi(M)$.*

Proof Every arrow in $\mathbb{E}\,(G)$ is monic, so by Theorem 1.4, $\mathbb{E}\,(G)$ has Euler characteristic. Moreover, P is a finite poset, so has a unique coweighting $k_\bullet$, and $\chi(\mathbb{E}\,(G)) = \sum_p k_p / |G(p)|$ by the dual of Proposition 2.8.

The coweight of p in P is equal to the coweight of p in $\downarrow p \cong \mathbb{P}_{d+1}$, where $d = \dim p$. The unique coweighting $k_\bullet$ on $\mathbb{P}_{d+1}$ is given by $k_J = (-1)^{|J|-1}$, so $k_p = (-1)^{(d+1)-1} = (-1)^{\dim p}$. The result follows. □

We now turn to the theory of Lefschetz number. Let $F : \mathbb{A} \longrightarrow \mathbb{A}$ be an endofunctor of a category $\mathbb{A}$. The category $\mathbf{Fix}\, F$ has as objects the (strict) fixed points of F, that is, the objects $a \in \mathbb{A}$ such that $F(a) = a$; a map $a \longrightarrow b$ in $\mathbf{Fix}\, F$ is a map $f : a \longrightarrow b$ in $\mathbb{A}$ such that $F(f) = f$.

Definition 2.13 Let F be an endofunctor of a finite category. Its *Lefschetz number* $L(F)$ is $\chi(\mathbf{Fix}\, F)$, when this exists.

The Lefschetz number is, then, the sum of the (co)weights of the fixed points. This is analogous to the standard Lefschetz fixed point formula, (co)weight playing the role of index. The following results further justify the definition.

Proposition 2.14 *Let $\mathbb{A}$ be a finite category.*

a. *$L(1_{\mathbb{A}}) = \chi(\mathbb{A})$, one side being defined if and only if the other is.*

b. *If $\mathbb{B}$ is another finite category and $\mathbb{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{B}$ are functors then $L(GF) = L(FG)$, one side being defined if and only if the other is.*

c. *Let $F : \mathbb{A} \longrightarrow \mathbb{A}$ and write $BF : B\mathbb{A} \longrightarrow B\mathbb{A}$ for the induced map on the classifying space of $\mathbb{A}$. If $\mathbb{A}$ is skeletal and contains no endomorphisms except identities then $L(F) = L(BF)$, with both sides defined.*

In the special case that $\mathbb{A}$ is a poset, part (c) is Theorem 1.1 of [BB].

Proof For (a) and (b), just note that $\mathbf{Fix}\, 1_{\mathbb{A}} \cong \mathbb{A}$ and $\mathbf{Fix}\, GF \cong \mathbf{Fix}\, FG$. For (c), recall from the proof of Proposition 2.11 that $N\mathbb{A}$ has only finitely many nondegenerate simplices; then

$$\begin{aligned} L(BF) &= \sum_{n \geq 0} (-1)^n |\{\text{nondegenerate } n\text{-simplices in } N\mathbb{A} \text{ fixed by } NF\}| \\ &= \sum_{n \geq 0} (-1)^n |\{\text{nondegenerate } n\text{-paths in } \mathbf{Fix}\, F\}| \\ &= L(F), \end{aligned}$$

using Corollary 1.5 in the last step. □

An *algebra* for an endofunctor F of $\mathbb{A}$ is an object $a \in \mathbb{A}$ equipped with a map $h : F(a) \longrightarrow a$. With the evident structure-preserving morphisms, algebras for F form a category $\mathbf{Alg}\, F$. There is a dual notion of *coalgebra* (where now $h : a \longrightarrow F(a)$), giving a category $\mathbf{Coalg}\, F$.

Proposition 2.15 *Let F be an endofunctor of a finite skeletal category $\mathbb{A}$ containing no endomorphisms except identities. Then $\chi(\mathbf{Alg}\, F) = L(F) = \chi(\mathbf{Coalg}\, F)$, with all three terms defined.*

PROOF First observe that $\mathbb{A}$ is circuit-free. Now, the inclusion $\mathbf{Fix}\,F \longrightarrow \mathbf{Alg}\,F$ has a right adjoint R: given an algebra (a,h), circuit-freeness implies that $F^N(a)$ is a fixed point for all sufficiently large N, and $R(a,h) = F^N(a)$. The Euler characteristics of $\mathbf{Alg}\,F$ and $\mathbf{Fix}\,F$ exist, by Corollary 1.5, and are equal, by Proposition 2.4(a). The statement on coalgebras follows by duality. $\square$

For example, if f is an endomorphism of a finite poset A then the subposets

$$\{a \in A \mid f(a) \leq a\}, \qquad \{a \in A \mid f(a) = a\}, \qquad \{a \in A \mid f(a) \geq a\}$$

all have the same Euler characteristic.

The theory of Euler characteristic presented here can be extended in at least two directions.

First, we can relax the finiteness assumption. For instance, the category of finite sets and bijections should have Euler characteristic $\sum_{n=0}^{\infty} 1/|S_n| = e$, as observed in [BD]. See the remarks after Corollary 4.3.

Second, the Euler characteristic of categories is defined in terms of the cardinality of finite sets, and the theory can be generalized to $\mathcal{V}$-enriched categories whenever there is a suitable notion of cardinality or Euler characteristic of objects of $\mathcal{V}$. For example, $\mathcal{V}$ might be the category of finite-dimensional vector spaces, with dimension playing the role of cardinality, and this leads to an Euler characteristic for finite linear categories. For another example, a *0-category* is a set and an *n-category* is a category enriched in $(n-1)$-categories; iterating, we obtain an Euler characteristic for finite n-categories. In particular, if $\mathbf{S}^n$ is the n-category consisting of two parallel n-cells then $\chi(\mathbf{S}^n) = 1+(-1)^n$.

3 THE CARDINALITY OF A COLIMIT

The main theorem of this section generalizes the formulas

$$|X \cup Y| = |X| + |Y| - |X \cap Y|, \qquad\qquad |S/G| = |S|/|G|$$

where X and Y are finite subsets of some larger set and S is a finite set acted on freely by a finite group G.

Take a finite functor $X : \mathbb{A} \longrightarrow \mathbf{Set}$. The colimit (or direct limit, or inductive limit) $\varinjlim X$ can be viewed as the gluing-together of the sets $X(a)$. Its cardinality depends on the way in which these sets are glued together, which in turn is determined by the action of X on arrows, so in general there is no formula for $|\varinjlim X|$ purely in terms of the cardinalities $|X(a)|$ $(a \in \mathbb{A})$.

Suppose, however, that we are in the extreme case that there are no unforced equations of the type $(X(f))(x) = (X(f'))(x')$, where f and f' are arrows in $\mathbb{A}$. For pushouts, this means that the two functions along which we are pushing out are injective; when $\mathbb{A}$ is a group G, so that X is a set with a

G-action, it means that the action is free. In this extreme case, $|\varinjlim X|$ can be calculated as a weighted sum of the cardinalities $|X(a)|$.

We now make this precise. Recall from §1 that a **Set**-valued functor is said to be familially representable if it is a sum of representables.

Proposition 3.1 *Let $\mathbb{A}$ be a finite category and $k^\bullet$ a weighting on $\mathbb{A}$. If $X : \mathbb{A} \longrightarrow \mathbf{Set}$ is finite and familially representable then $|\varinjlim X| = \sum_a k^a |X(a)|$.*

Proof The result holds if X is representable, since then $|\varinjlim X| = 1$. On the other hand, the class of functors X for which the conclusion holds is clearly closed under finite sums. □

To make use of this, we need a way of recognizing familially representable functors. Carboni and Johnstone [CJ1, CJ2] show that when $\mathcal{A}$ satisfies certain hypotheses, including having all limits, a functor $\mathcal{A} \longrightarrow \mathbf{Set}$ is familially representable if and only if it preserves connected limits. This does not help directly, because our categories $\mathbb{A}$ are finite, and a finite category does not have even all finite limits unless it is a lattice.

However, a standard philosophy applies: when $\mathbb{A}$ fails to have all limits of a certain type, it is rarely useful to consider the functors $\mathbb{A} \longrightarrow \mathbf{Set}$ preserving limits of that type; the correct substitute is the class of functors that are suitably 'flat'. The notion of flatness appropriate here will be called nondegeneracy. (This is unrelated to the usage of 'nondegenerate' in §1.)

Definition 3.2 Let $\mathbb{A}$ be a small category. A functor $X : \mathbb{A} \longrightarrow \mathbf{Set}$ is *nondegenerate* if $\mathbb{E}(X)$ has the following diagram-completion properties:

$$\begin{tikzcd} & \cdot \arrow[dr] & \\ \cdot \arrow[ur, dotted] \arrow[dr, dotted] & & \cdot \\ & \cdot \arrow[ur] & \end{tikzcd} \qquad\qquad \begin{tikzcd} \cdot \arrow[r, dotted] & \cdot \arrow[r, shift left] \arrow[r, shift right] & \cdot \end{tikzcd}$$

Explicitly, this means that

a. given arrows $a \xrightarrow{f} b \xleftarrow{f'} a'$ in $\mathbb{A}$ and $x \in X(a)$, $x' \in X(a')$ satisfying $(X(f))(x) = (X(f'))(x')$, there exist arrows $a \xleftarrow{g} c \xrightarrow{g'} a'$ and $z \in X(c)$ satisfying $fg = f'g'$, $(X(g))(z) = x$, and $(X(g'))(z) = x'$, and

b. given arrows $a \overset{f}{\underset{f'}{\rightrightarrows}} b$ in $\mathbb{A}$ and $x \in X(a)$ satisfying $(X(f))(x) = (X(f'))(x)$, there exist $c \xrightarrow{g} a$ and $z \in X(c)$ satisfying $fg = f'g$ and $(X(g))(z) = x$.

This is the most concrete form of the definition. For further explanation, see the Appendix; for references, see [Ln]. In the Appendix (Lemma 5.2) it is shown that under suitable hypotheses, nondegeneracy is equivalent to familial representability, and from this we deduce a more applicable form of Proposition 3.1:

Theorem 3.3 *Let $\mathbb{A}$ be a finite Cauchy-complete category and $k^\bullet$ a weighting on $\mathbb{A}$. If $X : \mathbb{A} \longrightarrow \mathbf{Set}$ is finite and nondegenerate then $|\varinjlim X| = \sum_a k^a |X(a)|$.* $\square$

Using the fact that $\varinjlim X$ is the set of connected-components of $\mathbb{E}(X)$, this may be rephrased as $|\pi_0(\mathbb{E}(X))| = \sum k^a |X(a)|$. On the other hand, Proposition 2.8 implies that $\chi(\mathbb{E}(X)) = \sum k^a |X(a)|$. Indeed, under the hypotheses of the Theorem, X is familially representable, so each connected-component of $\mathbb{E}(X)$ has an initial object, so $\chi(\mathbb{E}(X)) = |\pi_0(\mathbb{E}(X))|$.

Examples 3.4

a. Let $\mathbb{L}$ be the category of 1.11(a). A functor $X : \mathbb{L} \longrightarrow \mathbf{Set}$ is nondegenerate if and only if both functions $X(a) \longrightarrow X(b_i)$ are injective. In that case, Theorem 3.3 says that

$$|X(b_1) +_{X(a)} X(b_2)| = |X(b_1)| + |X(b_2)| - |X(a)|$$

where the set on the left-hand side is a pushout.

b. Let $\mathbb{B}$ be the category $\left(a \underset{g}{\overset{f}{\rightrightarrows}} b \right)$. A functor $X : \mathbb{B} \longrightarrow \mathbf{Set}$ is nondegenerate if and only if the two functions $X(f)$, $X(g)$ are injective and have disjoint images. The unique weighting $k^\bullet$ on $\mathbb{B}$ is $(k^a, k^b) = (-1, 1)$, and

$$|(X(b))/\sim| = |X(b)| - |X(a)|$$

where $\sim$ is the equivalence relation generated by $(X(f))(x) \sim (X(g))(x)$ for all $x \in X(a)$.

c. Let G be a group. A functor $X : G \longrightarrow \mathbf{Set}$ is a set S equipped with a left G-action; the functor is nondegenerate if and only if the action is free. Theorem 3.3 then says that the number of orbits is $|S|/|G|$.

d. The Theorem can be viewed as a generalized inclusion-exclusion principle. (Compare [R].) Let $n \geq 0$ and let $\mathbb{P}_n$ be the poset of nonempty subsets of $\{1, \ldots, n\}$, ordered by inclusion. (So $\mathbb{P}_2^{\mathrm{op}}$ is the category $\mathbb{L}$ of (a).) Its unique coweighting $k_\bullet$ is defined by $k_J = (-1)^{|J|-1}$. Given subsets $S_1, \ldots, S_n$ of some set, there is a nondegenerate functor $X : \mathbb{P}_n^{\mathrm{op}} \longrightarrow \mathbf{Set}$ defined on objects by $X(J) = \bigcap_{j \in J} S_j$ and on maps by inclusion. Theorem 3.3 gives the inclusion-exclusion formula,

$$|S_1 \cup \cdots \cup S_n| = \sum_{\emptyset \neq J \subseteq \{1,\ldots,n\}} (-1)^{|J|-1} \left| \bigcap_{j \in J} S_j \right|.$$

Corollary 3.5 *Let $\mathbb{A}$ be a finite Cauchy-complete category admitting a weighting. Let $X, Y : \mathbb{A} \longrightarrow \mathbf{Set}$ be finite nondegenerate functors satisfying $|X(a)| = |Y(a)|$ for all $a \in \mathbb{A}$. Then $|\varinjlim X| = |\varinjlim Y|$.* □

The condition that $\mathbb{A}$ admits a weighting cannot be dropped: consider the category $\mathbb{A}$ of Example 1.11(d) and the functors $X = \mathbb{A}(a_1, -) + \mathbb{A}(a_4, -)$, $Y = \mathbb{A}(a_2, -)$.

If $\mathbb{A}$ not only has a weighting but admits Möbius inversion then a sharper statement can be made (Proposition 1.8).

4 Relations with Rota's theory

In 1964, Gian-Carlo Rota published his seminal paper [R] on Möbius inversion in posets. The name is motivated as follows: in the poset of positive integers ordered by divisibility, $\mu(a, b) = \mu(b/a)$ whenever a divides b, where the μ on the right-hand side is the classical Möbius function. He was not the first to define Möbius inversion in posets—Weisner, Hall, and Ward preceded him—but Rota's contribution was the decisive one; in particular, he realized the power of the method in enumerative combinatorics. The history of Möbius inversion is well described in [R], [G] and [St].

In this section we discover that some of the principal results in Rota's theory are the order-theoretic shadows of more general categorical facts. We also examine briefly a different generalization of Möbius–Rota inversion, proposed by other authors.

Given a poset A, Rota considered its *incidence algebra* $I(A)$, which is the subring of $R(A)$ consisting of the integer-valued $\theta \in R(A)$ such that $\theta(a, b) = 0$ whenever $a \not\leq b$. By Example 1.2(a) or Corollary 1.5, $\mu \in I(A)$.

In posets, then, $\zeta(a, b) = 0 \Rightarrow \mu(a, b) = 0$. More generally:

Theorem 4.1 *If $\mathbb{A}$ is a finite category with Möbius inversion then, for $a, b \in \mathbb{A}$,*

$$\zeta(a, b) = 0 \Rightarrow \mu(a, b) = 0.$$

The proof uses a combinatorial lemma.

Lemma 4.2 *Let $n \geq 2$ and $\sigma \in S_{n-1}$. Then there exist $k \geq 1$ and $p_0, \ldots, p_k$ such that*

$$p_0 = 1, \quad p_1, \ldots, p_{k-1} \in \{1, \ldots, n-1\}, \quad p_k = n,$$

and $p_r = \sigma(p_{r-1}) + 1$ for each $r \in \{1, \ldots, k\}$.

Proof Suppose not; then there is an infinite sequence $(p_r)_{r \geq 0}$ of elements of $\{1, \ldots, n-1\}$ satisfying $p_0 = 1$ and $p_r = \sigma(p_{r-1}) + 1$ for all $r \geq 1$. Let ε be the endomorphism of the finite set $\{p_r \mid r \geq 0\}$ defined by $\varepsilon(p) = \sigma(p) + 1$. Then ε is injective but not surjective (since 1 is not in its image), contradicting finiteness. □

PROOF OF THEOREM 4.1 Write the objects of $\mathbb{A}$ as $a_1, \ldots, a_n$. There is an $n \times n$ matrix Z defined by $Z_{ij} = \zeta(a_i, a_j)$, and Z is invertible over $\mathbb{Q}$ with $(Z^{-1})_{ij} = \mu(a_i, a_j)$. Suppose that $i, j \in \{1, \ldots, n\}$ and $Z_{ij} = 0$. Certainly $i \neq j$, so $n \geq 2$ and we may assume that $(i, j) = (1, n)$. By the standard formula for the inverse of a matrix, our task is to prove that the $(n, 1)$-minor of Z is 0.

The $(n, 1)$-minor of Z is

$$\sum_{\sigma \in S_{n-1}} \pm Z_{1,\sigma(1)+1} \cdots Z_{n-1,\sigma(n-1)+1},$$

and in fact we will prove that each summand is 0. Indeed, let $\sigma \in S_{n-1}$. Take $p_0, \ldots, p_k$ as in the Lemma. By hypothesis, there is no map $a_1 \longrightarrow a_n$ in $\mathbb{A}$. Categories have composition, so there is no diagram

$$a_1 = a_{p_0} \longrightarrow a_{p_1} \longrightarrow \cdots \longrightarrow a_{p_k} = a_n$$

in $\mathbb{A}$. Hence $\zeta(a_{p_{r-1}}, a_{p_r}) = 0$ for some $r \in \{1, \ldots, k\}$, giving $Z_{p_{r-1},\sigma(p_{r-1})+1} = 0$, as required. $\square$

Given objects a, c of a category $\mathbb{A}$, let $\mathbb{A}_{a,c}$ be the full subcategory consisting of those $b \in \mathbb{A}$ for which there exist arrows $a \longrightarrow b \longrightarrow c$. Theorem 4.1 easily implies:

COROLLARY 4.3 *Let $\mathbb{A}$ be a finite category. Then $\mathbb{A}$ has Möbius inversion if and only if $\mathbb{A}_{a,c}$ has Möbius inversion for all $a, c \in \mathbb{A}$, and in that case the Möbius function of $\mathbb{A}_{a,c}$ is the restriction of that of $\mathbb{A}$.* $\square$

These results suggest a way of relaxing the finiteness assumption on our categories. It extends to categories the local finiteness condition on posets used in the Rota theory. Let $\mathbb{A}$ be a category for which each subcategory $\mathbb{A}_{a,c}$ is finite. Then each hom-set $\mathbb{A}(a, b)$ has finite cardinality, $\zeta(a, b)$, and there is a $\mathbb{Q}$-algebra

$$\hat{R}(\mathbb{A}) = \{\theta : \mathrm{ob}\,\mathbb{A} \times \mathrm{ob}\,\mathbb{A} \longrightarrow \mathbb{Q} \mid \text{for } a, b \in \mathbb{A},\ \zeta(a, b) = 0 \Rightarrow \theta(a, b) = 0\}$$

with operations defined as for $R(\mathbb{A})$. Evidently $\zeta \in \hat{R}(\mathbb{A})$, and $\mathbb{A}$ may be said to have Möbius inversion if ζ has an inverse μ in $\hat{R}(\mathbb{A})$. By Theorem 4.1, this extends the definition for finite categories. For example, the skeletal category $\mathbb{D}^{\mathrm{inj}}$ of finite totally ordered sets and order-preserving injections has Möbius inversion; compare Example 1.2(c).

The main theorem in Rota's paper [R] relates the Möbius functions of two posets linked by a Galois connection. Viewing a poset as a special category, a Galois connection is nothing but a (contravariant) adjunction, and Rota's theorem is a special case of the following result.

Proposition 4.4 *Let $\mathbb{A}$ and $\mathbb{B}$ be finite categories with Möbius inversion. Let $\mathbb{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{B}$ be an adjunction, $F \dashv G$. Then for all $a \in \mathbb{A}$, $b \in \mathbb{B}$,*

$$\sum_{a' : F(a') = b} \mu(a, a') = \sum_{b' : G(b') = a} \mu(b', b).$$

Proof Write $\zeta(a, b) = \zeta(F(a), b) = \zeta(a, G(b))$. Then for all $a \in \mathbb{A}$, $b \in \mathbb{B}$,

$$\sum_{a' : Fa' = b} \mu(a, a') = \sum_{a' \in \mathbb{A}} \mu(a, a')\delta(F(a'), b) = \sum_{a' \in \mathbb{A},\, b' \in \mathbb{B}} \mu(a, a')\zeta(a', b')\mu(b', b).$$

The result follows by symmetry. □

For example, when l is an element of a finite lattice L, the inclusion of the sub-poset $\{x \in L \mid x \leq l\}$ into L has right adjoint $(- \wedge l)$, giving Weisner's Theorem (p.351 of [R]).

The Euler characteristic of posets has been studied extensively; see [St] for references. Given a finite poset A, the classifying space BA always has Euler characteristic, which by Proposition 2.11 is equal to the Euler characteristic of the category A. On the other hand, we may form a new poset $\widetilde{A}$ by adjoining to A a least element 0 and a greatest element 1, and then $\chi(A) = \mu_{\widetilde{A}}(0, 1) + 1$: see [R] or §3.8 of [St]. This result can be extended from posets to categories:

Proposition 4.5 *Let $\mathbb{A}$ be a finite category. Write $\widetilde{\mathbb{A}}$ for the category obtained from $\mathbb{A}$ by freely adjoining an initial object 0 and a terminal object 1. If $\mathbb{A}$ has Möbius inversion then $\widetilde{\mathbb{A}}$ does too, and $\chi(\mathbb{A}) = \mu_{\widetilde{\mathbb{A}}}(0, 1) + 1$.*

Proof Suppose that $\mathbb{A}$ has Möbius inversion. Let $\mathbb{A}_0$ be the category obtained from $\mathbb{A}$ by freely adjoining an initial object 0. Extend $\mu \in R(\mathbb{A})$ to a function $\mu \in R(\mathbb{A}_0)$ by defining

$$\mu(0, b) = -\sum_{a \in \mathbb{A}} \mu(a, b), \qquad \mu(a, 0) = 0, \qquad \mu(0, 0) = 1$$

$(b, a \in \mathbb{A})$. It is easily checked that this is the Möbius function of $\mathbb{A}_0$.

Dually, if $\mathbb{B}$ is a finite category with Möbius inversion then the category $\mathbb{B}_1$ obtained from $\mathbb{B}$ by freely adjoining a terminal object 1 also has Möbius inversion, with $\mu(c, 1) = -\sum_{b \in \mathbb{B}} \mu(c, b)$ for all $c \in \mathbb{B}$. Take $\mathbb{B} = \mathbb{A}_0$: then $\mathbb{A}_{01} = \widetilde{\mathbb{A}}$ has Möbius inversion, and

$$\mu(0, 1) = -\sum_{b \in \mathbb{A}_0} \mu(0, b) = -\sum_{b \in \mathbb{A}} \mu(0, b) - \mu(0, 0) = \sum_{a, b \in \mathbb{A}} \mu(a, b) - 1 = \chi(\mathbb{A}) - 1.$$

□

Remark Given categories $\mathbb{B}, \mathbb{A}$ and a functor $M : \mathbb{B}^{\mathrm{op}} \times \mathbb{A} \longrightarrow \mathbf{Set}$, the *collage* of M is the category $\mathbb{C}$ formed by taking the disjoint union of $\mathbb{B}$ and $\mathbb{A}$ and adjoining one arrow $b \longrightarrow a$ for each $b \in \mathbb{B}$, $a \in \mathbb{A}$ and $m \in M(b,a)$, with composition defined using M [CKW]. Assuming finiteness, if $\mathbb{B}$ and $\mathbb{A}$ have Möbius inversion then so does $\mathbb{C}$:

$$\mu_{\mathbb{C}}(b,b') = \mu_{\mathbb{B}}(b,b'), \qquad \mu_{\mathbb{C}}(a,a') = \mu_{\mathbb{A}}(a,a'), \qquad \mu_{\mathbb{C}}(a,b) = 0,$$
$$\mu_{\mathbb{C}}(b,a) = -\sum_{b',a'} \mu_{\mathbb{B}}(b,b')\,|M(b',a')|\,\mu_{\mathbb{A}}(a',a)$$

($b, b' \in \mathbb{B}$, $a, a' \in \mathbb{A}$). In the proof above, the calculation of the Möbius function of $\mathbb{A}_0$ is the special case where $\mathbb{B}$ is the terminal category and M has constant value 1. The ordinal sum of posets is another special case. Moreover, one easily deduces a formula for the Euler characteristic of a collage, which in the special case of posets is essentially Theorem 3.1 of Walker [Wk].

Let us now look at the different generalization of Rota's Möbius inversion proposed, independently, by Content, Lemay and Leroux [CLL] and by Haigh [H]. (See also [Lr] and §4 of [La]. Haigh briefly considered the same generalization as here, too; see 3.5 of [H].) Given a sufficiently finite category $\mathbb{A}$, they take the algebra $I(\mathbb{A})$ of functions from {arrows of $\mathbb{A}$} to $\mathbb{Q}$ (or more generally, to some base commutative ring), with a convolution product:

$$(\theta\phi)(f) = \sum_{hg=f} \theta(g)\phi(h).$$

Taking $\zeta \in I(\mathbb{A})$ to have constant value 1, they call the *Möbius function* of $\mathbb{A}$ the inverse $\mu = \zeta^{-1}$ in $I(\mathbb{A})$, if it exists. When $\mathbb{A}$ is a poset, this agrees with Rota; when $\mathbb{A}$ is a monoid, it agrees with Cartier and Foata [CF].

They seek to solve a harder problem than we do: if a finite category $\mathbb{A}$ has Möbius inversion in their sense then it does in ours (with $\mu(a,b) = \sum_{f\in\mathbb{A}(a,b)} \mu(f)$), but not conversely. For instance, a non-trivial finite group never has Möbius inversion in their sense, but always does in ours.

5 Appendix: category theory

Here follows a skeletal account of some standard notions: category of elements, flat functors, and Cauchy-completeness. Details can be found in texts such as [Bo]. Throughout, $\mathbb{A}$ denotes a small category.

Let $X : \mathbb{A} \longrightarrow \mathbf{Set}$. The *category of elements* $\mathbb{E}(X)$ of X has as objects all pairs (a,x) where $a \in \mathbb{A}$ and $x \in X(a)$, and as maps $(a,x) \longrightarrow (a',x')$ all maps $f : a \longrightarrow a'$ in $\mathbb{A}$ such that $(X(f))(x) = x'$.

Similarly, let $X : \mathbb{A} \longrightarrow \mathbf{Cat}$, where $\mathbf{Cat}$ is the category of small categories and functors. Then X has a *category of elements* $\mathbb{E}(X)$; its objects are pairs (a,x) where $a \in \mathbb{A}$ and $x \in X(a)$, and its maps $(a,x) \longrightarrow (a',x')$ are pairs (f, ξ) where $f : a \longrightarrow a'$ in $\mathbb{A}$ and $\xi : (X(f))(x) \longrightarrow x'$ in $X(a')$. This definition can be made even when X is a *weak functor* or *pseudofunctor*, that is, only preserves composition and identities up to coherent isomorphism. The weak functors $\mathbb{A} \longrightarrow \mathbf{Cat}$ correspond to the fibrations over $\mathbb{A}^{\mathrm{op}}$; see [Bo].

A set can be viewed as a discrete category (one in which the only maps are the identities). From this point of view, **Set**-valued functors are special **Cat**-valued functors, and the second definition of the category of elements extends the first.

Any two functors $Y : \mathbb{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ and $X : \mathbb{A} \longrightarrow \mathbf{Set}$ have a tensor product $Y \otimes X$, a set, defined by

$$Y \otimes X = \left(\coprod_{a \in \mathbb{A}} Y(a) \times X(a) \right) / \sim$$

where $\sim$ is the equivalence relation generated by $(y, (X(f))(x)) \sim ((Y(f))(y), x)$ whenever $f : a \longrightarrow b$, $x \in X(a)$ and $y \in Y(b)$. (It may be helpful to think of X and Y as left and right $\mathbb{A}$-modules.) A functor $X : \mathbb{A} \longrightarrow \mathbf{Set}$ is *flat* if

$$- \otimes X : [\mathbb{A}^{\mathrm{op}}, \mathbf{Set}] \longrightarrow \mathbf{Set}$$

preserves finite limits. An equivalent condition is that $\mathbb{E}(X)$ is *cofiltered*, that is, every finite diagram in $\mathbb{E}(X)$ admits at least one cone.

Proposition 5.1 *The following conditions on a functor $X : \mathbb{A} \longrightarrow \mathbf{Set}$ are equivalent:*

a. X is nondegenerate (in the sense of 3.2)

b. every connected-component of $\mathbb{E}(X)$ is cofiltered

c. X is a sum of flat functors.

d. $- \otimes X : [\mathbb{A}^{\mathrm{op}}, \mathbf{Set}] \longrightarrow \mathbf{Set}$ preserves finite connected limits

Proof See [Ln] or [ABLR]. □

An idempotent $e : a \longrightarrow a$ in $\mathbb{A}$ *splits* if there exist $a \underset{i}{\overset{s}{\rightleftarrows}} b$ such that $si = 1$ and $is = e$. The category $\mathbb{A}$ is *Cauchy-complete* if every idempotent in $\mathbb{A}$ splits. (This is a very weak form of completeness. Let $\mathbb{I}$ be the category consisting of one object, the identity on it, and an idempotent u. Then a splitting of e is precisely a limit of the functor $\mathbb{I} \longrightarrow \mathbb{A}$ defined by $u \longmapsto e$.) All of the examples of categories in this paper are Cauchy-complete, except that a finite monoid is Cauchy-complete if and only if it is a group.

Lemma 5.2 *Let $\mathbb{A}$ be a Cauchy-complete category and $X : \mathbb{A} \longrightarrow \mathbf{Set}$ a finite functor. Then X is familially representable if and only if X is nondegenerate.*

As in §1, 'finite' means that $\mathbb{E}(X)$ is a finite category.

Proof By Proposition 5.1, it is enough to prove that a finite functor X is representable if and only if it is flat. 'Only if' is immediate.

For 'if', suppose that X is flat. Then $\mathbb{E}(X)$ is cofiltered and finite, so the identity functor $1_{\mathbb{E}(X)}$ admits a cone. Also, $\mathbb{E}(X)$ is Cauchy-complete since $\mathbb{A}$ is. Now, if $\mathbb{C}$ is a Cauchy-complete category and $(j \xrightarrow{p_c} c)_{c\in\mathbb{C}}$ is a cone on $1_{\mathbb{C}}$ then p_j is idempotent, and the object through which it splits is initial. Hence $\mathbb{E}(X)$ has an initial object; equivalently, X is representable. $\square$

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Essential Spectrum of Multiparticle Brown–Ravenhall Operators in External Field

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Abstract. The essential spectrum of multiparticle Brown–Ravenhall operators is characterized in terms of two–cluster decompositions for a wide class of external fields and interparticle interactions and for the systems with prescribed symmetries.

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1 Introduction

It is well known that the eigenvalues of the one–particle Dirac operator are in much better accordance with the spectroscopic data then the eigenvalues of the Schrödinger operator. However, due to the presence of the negative continuum of positronic states the multiparticle Coulomb–Dirac operator has no eigenvalues and its essential spectrum is the whole real line. Coupling with the quantized electromagnetic field does not correct this situation. However, there are ways to construct a *semibounded* operator which will take the relativistic effects into account. Such models, although nonlocal, find their applications in numerical studies of heavy elements and cosmology, where the relativistic effects cannot be ignored.

The most obvious choice of the kinetic energy (sometimes called Chandrasekhar or Herbst operator) given by $\sqrt{\mathbf{p}^2c^2 + m^2c^4}$, $\mathbf{p}$ and m being the momentum and mass of the particle, suffers from the lack of semiboundedness for nuclear charges exceeding 87, as shown in [9]. Most other operators considered in the literature are obtained by reducing the (multiparticle) Dirac operator onto some subspace on which it becomes semibounded. One of such models, extensively studied recently, is by Brown and Ravenhall [4], see also Bethe and Salpeter [3], Sucher [18, 19]. In this model every particle is confined the positive spectral subspace of the *free* Dirac operator. Since the multiplication by interaction potentials does not leave this subspace invariant, the potential energy terms are projected back by the corresponding projector.

The mathematical study of the Brown–Ravenhall operator started from the one–particle case in the article of Evans, Perry, and Siedentop [7]. The authors have proved that the atomic Hamiltonian is semibounded from below for nuclear charges not exceeding 124. This makes the Brown–Ravenhall model applicable to all existing elements. It was also proved in [7] that the essential spectrum of the one–particle atomic Brown–Ravenhall operator is $[mc^2, \infty)$ with m being the mass of the particle, and that the singular continuous spectrum is empty.

Further studies of the Brown–Ravenhall operator include the improved lower bounds by Tix [21, 22] (see also Burenkov and Evans [5]) in the atomic case, the proof that the eigenvalues of Brown–Ravenhall operator are strictly bigger than those of the one–particle Dirac operator by Griesemer et al. [8], proofs of stability of one-electron molecule by Balinsky and Evans [2], the proof of stability of matter by Hoever and Siedentop [10], and the asymptotic result on the ground state energy for large atomic charges Z (with Z/c fixed) by Cassanas and Siedentop [6].

The essential spectrum of the multiparticle operator was characterized by Jakubaßa–Amundsen [12, 13], and in our joint work with S. Vugalter [16] in terms of two–cluster decompositions. This is usually referred to as HVZ theorem after the well known result for the multiparticle Schrödinger operator. In [11] an analogous result is proved in presence of the constant magnetic field. It is also proved in [16] that the neutral atoms or positively charged atomic ions have infinitely many bound states.

In all these previous studies the nuclei were considered as fixed sources of the external field, the particles were assumed to be identical, and the interaction potentials were purely Coulombic.

In this paper we generalize the HVZ theorem of [12, 13, 16] as follows: We allow any number of (massive) particles of the system to be identical. We allow quite general matrix interaction potentials. In particular, our result applies in the presence of the magnetic fields if the vector potential decays at infinity in some weak sense. Another problem we address is the reduction to any irreducible representations of the groups of rotation–reflection symmetry and permutations of identical particles. Note that such a reduction allows to analyze the eigenvalues of some irreducible representations even if they are embedded into the continuous spectrum of some other representations. For some particular models (including atoms and molecules in the Born–Oppenheimer approximation) the existence of such eigenvalues can be shown along the same lines as in [16].

From the technical point of view, the nonlocality of the model due to the presence of the spectral projections of the free Dirac operator is overcome with the same ideas as in [16]. One more complication should be stressed: for the Brown–Ravenhall operator the center of mass motion cannot be separated in the same way as it is usually done for Schrödinger operators, where the complete

Hamiltonian without external field can be represented in suitable coordinates as

$$\mathcal{H} = A \otimes I + I \otimes B,$$

where A describes the free motion of the center of mass and B is the internal Hamiltonian of the system (see [14]). Such a decomposition appears to be especially fruitful in the presence of rotation symmetries. Since it cannot be obtained for pseudorelativistic operators due to the form of kinetic energy, we have used a completely different approach based on the commutation of the Hamiltonian with the *absolute value* of the total momentum of the system.

Note that the proof of the HVZ theorem for a system of particles described by the Chandrasekhar operator was till now not known for operators reduced to irreducible representations of the rotation–reflection symmetry group (see the article of Lewis, Siedentop and Vugalter [15] for the case without such reductions). Such a proof can now be obtained as a simplified modification of the proof given in this paper.

In Section 2 we introduce the model and make the necessary assumptions. At the end of this section we formulate the main result in Theorem 6. The rest of the article contains the proof of this theorem.

2 Setup and Main Result

$[A, B] = AB - BA$ is the commutator of two operators. $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ stand for the inner product and the norm in $L_2(\mathbb{R}^{3d}, \mathbb{C}^{4^d})$, where d is the dimension of the underlying configuration space. Irrelevant constants are denoted by C. I_Ω is the indicator function of the set Ω. For a selfadjoint operator A we denote its spectrum and the corresponding sesquilinear form by $\sigma(A)$ and $\langle A \cdot, \cdot \rangle = \langle \cdot, A \cdot \rangle$, respectively. We use the conventional units $\hbar = c = 1$. Sometimes we denote the unitary Fourier transform by $\widehat{\cdot}$.

In the Hilbert space $L_2(\mathbb{R}^3, \mathbb{C}^4)$ the Dirac operator describing a particle of mass $m > 0$ is given by

$$D_m = -i\boldsymbol{\alpha} \cdot \nabla + \beta m,$$

where $\boldsymbol{\alpha} := (\alpha_1, \alpha_2, \alpha_3)$ and β are the 4×4 Dirac matrices [20]. The form domain of D_m is the Sobolev space $H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$ and the spectrum is $(-\infty, -m] \cup [m, +\infty)$. Let Λ_m be the orthogonal projector onto the positive spectral subspace of D_m:

$$\Lambda_m := \frac{1}{2} + \frac{-i\boldsymbol{\alpha} \cdot \nabla + \beta m}{2\sqrt{-\Delta + m^2}}. \tag{2.1}$$

We consider a finite system of N particles with positive masses m_n, $n = 1, \ldots, N$. To simplify the notation we write D_n and Λ_n for D_{m_n} and Λ_{m_n}, respectively. Let $\mathfrak{H}_N := \bigotimes_{n=1}^{N} \Lambda_n L_2(\mathbb{R}^3, \mathbb{C}^4)$ be the Hilbert space with the inner product induced by those of $\bigotimes_{n=1}^{N} L_2(\mathbb{R}^3, \mathbb{C}^4) \cong L_2(\mathbb{R}^{3N}, \mathbb{C}^{4^N})$. In this space the

N–particle Brown–Ravenhall operator is formally given by

$$\mathcal{H}_N = \Lambda^N \biggl(\sum_{n=1}^{N} (D_n + V_n) + \sum_{n<j}^{N} U_{nj} \biggr) \Lambda^N, \tag{2.2}$$

with

$$\Lambda^N := \prod_{n=1}^{N} \Lambda_n = \bigotimes_{n=1}^{N} \Lambda_n. \tag{2.3}$$

Here and below the indices n and j indicate the particle, on whose coordinates the corresponding operator acts. In (2.2) V_n is the external field potential for the n^{th} particle, i.e., the operator of multiplication by a hermitian 4×4 matrix–function $V_n(\mathbf{x}_n)$, $n = 1, \dots, N$, and U_{nj} is the potential energy of the interaction between the n^{th} and j^{th} particles, given by the operator of multiplication by a hermitian 16×16 matrix–function $U_{nj}(\mathbf{x}_n - \mathbf{x}_j)$, $n < j = 1, \dots, N$. More explicitly, if we let $s_j \in \{1, 2, 3, 4\}$ be the spinor index of the j^{th} particle, then

$$\begin{aligned}
&(V_n \psi)(\mathbf{x}_1, s_1; \dots; \mathbf{x}_n, s_n; \dots; \mathbf{x}_N, s_N) \\
&:= \sum_{\widetilde{s}_n} V_n^{s_n, \widetilde{s}_n}(\mathbf{x}_n) \psi(\mathbf{x}_1, s_1; \dots; \mathbf{x}_n, \widetilde{s}_n; \dots; \mathbf{x}_N, s_N),
\end{aligned}$$

and

$$\begin{aligned}
&(U_{nj} \psi)(\mathbf{x}_1, s_1; \dots; \mathbf{x}_n, s_n; \dots; \mathbf{x}_j, s_j; \dots; \mathbf{x}_N, s_N) \\
&:= \sum_{\widetilde{s}_n, \widetilde{s}_j} U_{nj}^{s_n s_j, \widetilde{s}_n \widetilde{s}_j}(\mathbf{x}_n - \mathbf{x}_j) \psi(\mathbf{x}_1, s_1; \dots; \mathbf{x}_n, \widetilde{s}_n; \dots; \mathbf{x}_j, \widetilde{s}_j; \dots; \mathbf{x}_N, s_N).
\end{aligned}$$

Before we make other assumptions on the interaction potentials, let us consider possible decompositions of the system into two clusters. Let $Z = (Z_1, Z_2)$ be a decomposition of the index set $I := \{1, \dots, N\}$ into two disjoint subsets:

$$I = Z_1 \cup Z_2, \quad Z_1 \cap Z_2 = \varnothing.$$

Let

$$N_j := \# Z_j, \quad j = 1, 2 \tag{2.4}$$

be the number of particles in each cluster. We will write $n \# j$ if n and j belong to different clusters. Let

$$\mathcal{H}_{Z,1} := \sum_{n \in Z_1} (D_n + V_n) + \sum_{\substack{n,j \in Z_1 \\ n<j}} U_{nj}, \tag{2.5}$$

$$\mathcal{H}_{Z,2} := \sum_{n \in Z_2} D_n + \sum_{\substack{n,j \in Z_2 \\ n<j}} U_{nj}. \tag{2.6}$$

We omit $\mathcal{H}_{Z,j}$ if $Z_j = \varnothing$, $j = 1, 2$. Let us introduce the operators corresponding to noninteracting clusters, with the second cluster transferred far away from the sources of the external field:

$$\widetilde{\mathcal{H}}_{Z,j} := \Lambda_{Z,j}\mathcal{H}_{Z,j}\Lambda_{Z,j}, \quad \text{in} \quad \mathfrak{H}_{Z,j} := \underset{n\in Z_j}{\otimes} \Lambda_n L_2(\mathbb{R}^3, \mathbb{C}^4), \quad j = 1, 2, \tag{2.7}$$

where

$$\Lambda_{Z,j} := \prod_{n\in Z_j} \Lambda_n.$$

We make the following assumptions:

Assumption 1 *There exists $C > 0$ such that for any Z and $j = 1, 2$*

$$\big|\langle \mathcal{H}_{Z,j}\varphi, \psi\rangle\big| \leqslant C\|\varphi\|_{H^{1/2}}\|\psi\|_{H^{1/2}}, \quad \textit{for any} \quad \varphi, \psi \in \underset{n\in Z_j}{\otimes} H^{1/2}(\mathbb{R}^3, \mathbb{C}^4). \tag{2.8}$$

For Coulomb interaction potentials (2.8) follows from Kato's inequality.

Assumption 2 *There exist $C_1 > 0$ and $C_2 \in \mathbb{R}$ such that for any Z*

$$\begin{gathered}\langle \widetilde{\mathcal{H}}_{Z,j}\psi, \psi\rangle \geqslant C_1\langle \sum_{n\in Z_j} D_n\psi, \psi\rangle - C_2\|\psi\|^2, \\ \textit{for any} \quad \psi \in \underset{n\in Z_j}{\otimes} \Lambda_n H^{1/2}(\mathbb{R}^3, \mathbb{C}^4), \quad j = 1, 2.\end{gathered} \tag{2.9}$$

Remark 3 *Note that for $\psi \in \underset{n\in Z_j}{\otimes} \Lambda_n H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$ the metric*

$$\langle \sum_{n\in Z_j} D_n\psi, \psi\rangle^{1/2} = \Big\| \sum_{n\in Z_j} |D_n|^{1/2}\psi\Big\|$$

is equivalent to the norm of ψ in $\underset{n\in Z_j}{\otimes} H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$, since

$$\Lambda_n D_n \Lambda_n = \Lambda_n |D_n| \Lambda_n = \Lambda_n \sqrt{-\Delta + m_n^2}\Lambda_n. \tag{2.10}$$

An equivalent formulation of Assumption 2 is that the operator $\widetilde{\mathcal{H}}_{Z,j}$ is semibounded from below even if we multiply all the interaction potentials by $1 + \varepsilon$ with $\varepsilon > 0$ small enough. This is only slightly more restrictive than the semiboundedness of $\widetilde{\mathcal{H}}_{Z,j}$.

Assumption 4 *For any $R > 0$ there exists a finite constant $C_R \geqslant 0$ such that*

$$\sum_{n=1}^{N} \Big(\int_{|\mathbf{x}|\leqslant R} \big|V_n(\mathbf{x})\big|^2 d\mathbf{x}\Big)^{1/2} + \sum_{n<j}^{N} \Big(\int_{|\mathbf{x}|\leqslant R} \big|U_{nj}(\mathbf{x})\big|^2 d\mathbf{x}\Big)^{1/2} \leqslant C_R. \tag{2.11}$$

This means that the interaction potentials are locally square integrable.

ASSUMPTION 5 *For any $\varepsilon > 0$ there exists $R > 0$ big enough such that for all $n = 1, \ldots, N$*

$$\|V_n I_{\{|\mathbf{x}_n|>R\}}\psi\| \leqslant \varepsilon \big\||D_n|^{1/2}\psi\big\|, \quad \textit{for all} \quad \psi \in H^{1/2}(\mathbb{R}^3, \mathbb{C}^4), \tag{2.12}$$

and for all $n < j = 1, \ldots, N$

$$\begin{aligned} \|U_{nj} I_{\{|\mathbf{x}_n - \mathbf{x}_j|>R\}}\varphi\| \leqslant \varepsilon \min\Big\{\big\||D_n|^{1/2}\varphi\big\|, \big\||D_j|^{1/2}\varphi\big\|\Big\}, \\ \textit{for all} \quad \varphi \in H^{1/2}(\mathbb{R}^6, \mathbb{C}^{16}). \end{aligned} \tag{2.13}$$

By Remark 3 this assumption is *weaker* then the decay of L_∞ norms of the interaction potentials at infinity.
It follows from (2.9) and Remark 3 that for any Z there exists a constant $C > 0$ such that for any $\psi \in \bigotimes_{n \in Z_j} \Lambda_n H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$

$$\|\psi\|^2_{H^{1/2}} \leqslant C\big(\langle \widetilde{\mathcal{H}}_{Z,j}\psi, \psi\rangle + \|\psi\|^2\big), \quad j = 1, 2. \tag{2.14}$$

Hence by Assumptions 1 and 2, the quadratic forms of operators (2.7) (and, in particular, $\mathcal{H}_N$) are semibounded from below and closed on $\bigotimes_{n \in Z_j} \Lambda_n H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$. Thus these operators are well–defined in the form sense.
Some particles of the system (say, k^{th} and l^{th}) can be identical (in which case $m_k = m_l$, $V_k = V_l$, and $U_{kj} = U_{lj}$ for all j). Then the operator $\mathcal{H}_N$ can be reduced to the subspace of functions which transform in a certain way under permutations of identical particles. The most physically motivated assumption is that any transposition of two identical particles should change the sign of the wave function $\psi \in \mathfrak{H}_N$ describing the system. This is the Pauli principle applied to the identical fermions (the model describes spin 1/2 particles, thus fermions).
Let Π be the subgroup of the symmetric group $\mathcal{S}_N$ generated by transpositions of identical particles. We denote the number of elements of Π by h_Π. Let E be some irreducible representation of Π with dimension d_E and character ξ_E. For $\psi \in \mathfrak{H}_N$ let

$$P^E\psi := \frac{d_E}{h_\Pi} \sum_{\pi \in \Pi} \overline{\xi_E(\pi)}\pi\psi, \tag{2.15}$$

where π is the operator of permutation:

$$(\pi\psi)(\mathbf{x}_1, s_1; \ldots; \mathbf{x}_N, s_N) = \psi(\mathbf{x}_{\pi^{-1}(1)}, s_{\pi^{-1}(1)}; \ldots; \mathbf{x}_{\pi^{-1}(N)}, s_{\pi^{-1}(N)}).$$

Here $s_1, \ldots, s_N$ are the spinor coordinates of the particles. The operator P^E defined in (2.15) is the projector to the subspace of functions in $\mathfrak{H}_N$ which transform according to the representation E of Π. Since any $\pi \in \Pi$ commutes with $\mathcal{H}_N$, P^E reduces $\mathcal{H}_N$. Let $\mathcal{H}_N^E$ be the corresponding reduced selfadjoint operator in

$$\mathfrak{H}_N^E := P^E \mathfrak{H}_N.$$

For a decomposition $Z = (Z_1, Z_2)$ let Π_j^Z be the group generated by transpositions of identical particles inside Z_j, $j = 1, 2$. For any irreducible representation E_j of Π_j^Z with dimension d_{E_j} and character ξ_{E_j} the projection to the space of functions in $\mathfrak{H}_{Z,j}$ transforming according to E_j under the action of Π_j^Z is given by

$$P^{E_j}\psi := \frac{d_{E_j}}{h_{\Pi_j^Z}} \sum_{\pi \in \Pi_j^Z} \overline{\xi_{E_j}(\pi)} \pi\psi, \quad \psi \in \mathfrak{H}_{Z,j},$$

where $h_{\Pi_j^Z}$ is the cardinality of Π_j^Z. Projectors P^{E_j} reduce operators $\widetilde{\mathcal{H}}_{Z,j}$. We introduce the reduced operators $\widetilde{\mathcal{H}}_{Z,j}^{E_j}$ in

$$\mathfrak{H}_{Z,j}^{E_j} := P^{E_j} \mathfrak{H}_{Z,j}, \quad j = 1, 2.$$

Given an irreducible representation E of Π and a decomposition $Z = (Z_1, Z_2)$, we have

$$\mathfrak{H}_N^E \subset \bigoplus_{(E_1,E_2)} \left(\mathfrak{H}_{Z,1}^{E_1} \otimes \mathfrak{H}_{Z,2}^{E_2}\right), \tag{2.16}$$

where $E_{1,2}$ are some irreducible representations of $\Pi_{1,2}^Z$. We write $(E_1, E_2) \underset{Z}{\prec} E$ if the corresponding term cannot be omitted on the r. h. s. of (2.16) without violation of the inclusion.

Apart from permutations of identical particles the operator $\mathcal{H}_N^E$ can have some rotation–reflection symmetries. Let γ be an orthogonal transform in $\mathbb{R}^3$: the rotation around the axis directed along a unit vector $\mathbf{n}_\gamma$ through an angle φ_γ, possibly combined with the reflection $\mathbf{x} \mapsto -\mathbf{x}$. The corresponding unitary operator O_γ acts on the functions $\psi \in \mathfrak{H}^N$ as (see [20], Chapter 2)

$$(O_\gamma\psi)(\mathbf{x}_1, \dots, \mathbf{x}_N) = \prod_{n=1}^N e^{-i\varphi_\gamma \mathbf{n}_\gamma \cdot \mathbf{S}_n} \psi(\gamma^{-1}\mathbf{x}_1, \dots, \gamma^{-1}\mathbf{x}_N).$$

Here $\mathbf{S}_n = -\frac{i}{4}\alpha_n \wedge \alpha_n$ is the spin operator acting on the spinor coordinates of the n^{th} particle. The compact group of orthogonal transformations γ such that O_γ commutes with V_n and U_{nj} for all n, $j = 1, \dots, N$ (and thus with $\mathcal{H}_N^E$) we denote by Γ. Further, we decompose $\mathfrak{H}_N^E$ into the orthogonal sum

$$\mathfrak{H}_N^E = \bigoplus_{\alpha \in A} \mathfrak{H}_N^{D_\alpha, E}, \tag{2.17}$$

where $\mathfrak{H}_N^{D_\alpha,E}$ consists of functions which transform under O_γ according to some irreducible representation D_α of Γ, and A is the set indexing all such irreducible representations. The decomposition (2.17) reduces $\mathcal{H}_N^E$. We denote the selfadjoint restrictions of $\mathcal{H}_N^E$ to $\mathfrak{H}_N^{D_\alpha,E}$ by $\mathcal{H}_N^{D_\alpha,E}$. For any fixed irreducible representation D with dimension d_D and character ζ_D the orthogonal projector in $\mathfrak{H}_N$ onto the subspace of functions which transform according to D is

$$P^D := d_D \int_\Gamma \overline{\zeta_D(\gamma)} O_\gamma d\mu(\gamma),$$

where μ is the invariant probability measure on Γ.
For $j = 1, 2$ let D_j be some irreducible representations of Γ with dimensions d_{D_j} and characters ζ_{D_j}. The corresponding projectors in $\mathfrak{H}_{Z,j}$ are given by

$$P^{D_j} = d_{D_j} \int_\Gamma \overline{\zeta_{D_j}(\gamma)} O_{\gamma,j} d\mu(\gamma),$$

where $O_{\gamma,j}$ is the restriction of O_γ to $\mathfrak{H}_{Z,j}$:

$$(O_{\gamma,j}\psi)(\mathbf{x}_{n_1}, \dots, \mathbf{x}_{n_{N_j}}) = \prod_{n \in Z_j} e^{-i\varphi_\gamma \mathbf{n}_\gamma \cdot \mathbf{S}_n} \psi(\gamma^{-1}\mathbf{x}_{n_1}, \dots, \gamma^{-1}\mathbf{x}_{n_{N_j}}).$$

Given representations D_j and E_j, projector $P^{D_j} P^{E_j} = P^{E_j} P^{D_j}$ reduces $\widetilde{\mathcal{H}}_{Z,j}$. We denote the reduced operators in

$$\mathfrak{H}_{Z,j}^{D_j,E_j} := P^{D_j} P^{E_j} \mathfrak{H}_{Z,j}$$

by $\widetilde{\mathcal{H}}_{Z,j}^{D_j,E_j}$. Let

$$\varkappa_j(Z, D_j, E_j) := \inf \sigma(\widetilde{\mathcal{H}}_{Z,j}^{D_j,E_j}). \tag{2.18}$$

We write $(D_1, E_1; D_2, E_2) \underset{Z}{\prec} (D, E)$ if the corresponding term cannot be omitted on the r. h. s. of

$$\mathfrak{H}_N^{D,E} \subset \bigoplus_{\substack{(D_1,E_1)\\(D_2,E_2)}} \big(\mathfrak{H}_{Z,1}^{D_1,E_1} \otimes \mathfrak{H}_{Z,2}^{D_2,E_2}\big)$$

without violation of the inclusion. For $Z_2 \neq \varnothing$ let

$$\begin{aligned}&\varkappa(Z, D, E)\\ &:= \begin{cases} \inf\big\{\varkappa_1(Z, D_1, E_1) + \varkappa_2(Z, D_2, E_2) : (D_1, E_1; D_2, E_2) \underset{Z}{\prec} (D, E)\big\}, & Z_1 \neq \varnothing, \\ \varkappa_2(Z, D, E), & Z_1 = \varnothing. \end{cases}\end{aligned} \tag{2.19}$$

The main result of the article is

Theorem 6 *Suppose Assumptions 1, 2, 4, and 5 hold true. For $N \in \mathbb{N}$ let D be some irreducible representation of Γ, and E some irreducible representation of Π, such that $P^D P^E \neq 0$. Then*

$$\sigma_{\mathrm{ess}}(\mathcal{H}_N^{D,E}) = \big[\varkappa(D, E), \infty\big),$$

where

$$\varkappa(D, E) = \min\big\{\varkappa(Z, D, E) : Z = (Z_1, Z_2),\ Z_2 \neq \varnothing\big\}. \tag{2.20}$$

Remark 7 *We only need Assumption 2 for the operators* $\widetilde{\mathcal{H}}_{Z,j}^{D_j,E_j}$ *which appear in* (2.18)*,* (2.19)*.*

3 Commutator Estimates

3.1 One Particle Commutator Estimate

Lemma 8 *Let* $\chi \in C_B^2(\mathbb{R}^3)$ *(i. e. a bounded twice–differentiable function with bounded derivatives). Then for* $m_n > 0$ *the commutator* $[\chi, \Lambda_n]$ *is a bounded operator from* $L_2(\mathbb{R}^3, \mathbb{C}^4)$ *to* $H^1(\mathbb{R}^3, \mathbb{C}^4)$*. There exists* $C(m) > 0$ *such that*

$$\big\|[\chi, \Lambda_n]\big\|_{L_2(\mathbb{R}^3,\mathbb{C}^4)\to H^1(\mathbb{R}^3,\mathbb{C}^4)} \leqslant C(m_n)\big(\|\nabla\chi\|_{L_\infty} + \|\partial^2\chi\|_{L_\infty}\big). \tag{3.1}$$

Here $\|\partial^2\chi\|_{L_\infty} = \max\limits_{\substack{\mathbf{z}\in\mathbb{R}^3\\ k,l\in\{1,2,3\}}} \big|\partial_{kl}^2\chi(\mathbf{z})\big|$.

Proof. In the coordinate representation for $f \in C_0^1(\mathbb{R}^3, \mathbb{C}^4)$ the operator Λ_n acts as

$$(\Lambda_n f)(\mathbf{x}) = \frac{f(\mathbf{x})}{2} + \frac{im_n}{2\pi^2}\lim_{\varepsilon\to+0}\int_{|\mathbf{y}-\mathbf{x}|\geqslant\varepsilon} \frac{\boldsymbol{\alpha}\cdot(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3} K_1\big(m_n|\mathbf{x}-\mathbf{y}|\big) f(\mathbf{y})d\mathbf{y}$$
$$+ \frac{m_n^2}{4\pi^2}\int_{\mathbb{R}^3}\left(\beta\frac{K_1\big(m_n|\mathbf{x}-\mathbf{y}|\big)}{|\mathbf{x}-\mathbf{y}|} + \frac{i\boldsymbol{\alpha}\cdot(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}K_0\big(m_n|\mathbf{x}-\mathbf{y}|\big)\right)f(\mathbf{y})d\mathbf{y},$$

where the limit on the r. h. s. is the limit in $L_2(\mathbb{R}^3, \mathbb{C}^4)$ (see Appendix B of [16], where this formula is derived in the case $m_n = 1$). The rest of the proof is an obvious modification of the proof of Lemma 1 of [16], where the case $m_n = 1$ is considered. •

Remark 9 *Since we only deal with a finite number of particles with positive masses, we will not trace the m-dependence of the constant in* (3.1) *any longer.*

3.2 Multiparticle Commutator Estimate

Lemma 10 *For any* $d, k \in \mathbb{N}$ *there exists* $C > 0$ *such that for any* $\chi \in C_B^1(\mathbb{R}^d)$ *and* $u \in H^{1/2}(\mathbb{R}^d, \mathbb{C}^k)$

$$\|\chi u\|_{H^{1/2}(\mathbb{R}^d,\mathbb{C}^k)} \leqslant C\big(\|\chi\|_{L_\infty(\mathbb{R}^d)} + \|\nabla\chi\|_{L_\infty(\mathbb{R}^d)}\big)\|u\|_{H^{1/2}(\mathbb{R}^d,\mathbb{C}^k)}. \tag{3.2}$$

Proof of Lemma 10. We can choose the norm in $H^{1/2}(\mathbb{R}^d, \mathbb{C}^k)$ as (see [1], Theorem 7.48)

$$\|u\|^2_{H^{1/2}(\mathbb{R}^d,\mathbb{C}^k)} := \|u\|^2_{L_2(\mathbb{R}^d,\mathbb{C}^k)} + \iint \frac{\big|u(\mathbf{x}) - u(\mathbf{y})\big|^2}{|\mathbf{x}-\mathbf{y}|^{d+1}} d\mathbf{x}d\mathbf{y}.$$

Then

$$\begin{aligned}\|\chi u\|^2_{H^{1/2}(\mathbb{R}^d,\mathbb{C}^k)} &= \|\chi u\|^2_{L_2(\mathbb{R}^d,\mathbb{C}^k)} + \iint \frac{\big|\chi(\mathbf{x})u(\mathbf{x})-\chi(\mathbf{y})u(\mathbf{y})\big|^2}{|\mathbf{x}-\mathbf{y}|^{d+1}}d\mathbf{x}d\mathbf{y}\\ \leqslant \|\chi\|^2_{L_\infty}\|u\|^2_{L_2} &+ \iint\Big(\frac{\big|\chi(\mathbf{x})\big|^2\big|u(\mathbf{x})-u(\mathbf{y})\big|^2}{|\mathbf{x}-\mathbf{y}|^{d+1}} + \frac{\big|\chi(\mathbf{x})-\chi(\mathbf{y})\big|^2\big|u(\mathbf{y})\big|^2}{|\mathbf{x}-\mathbf{y}|^{d+1}}\Big)d\mathbf{x}d\mathbf{y}\\ &\leqslant \|\chi\|^2_{L_\infty}\|u\|^2_{H^{1/2}} + \sup_{\mathbf{y}\in\mathbb{R}^d}\int\frac{\big|\chi(\mathbf{x})-\chi(\mathbf{y})\big|^2}{|\mathbf{x}-\mathbf{y}|^{d+1}}d\mathbf{x}\|u\|^2_{L_2}.\end{aligned} \tag{3.3}$$

The supremum on the r. h. s. of (3.3) can be estimated as

$$\begin{aligned}\sup_{\mathbf{y}\in\mathbb{R}^d}\int\frac{\big|\chi(\mathbf{x})-\chi(\mathbf{y})\big|^2}{|\mathbf{x}-\mathbf{y}|^{d+1}}d\mathbf{x} &\leqslant \sup_{\mathbf{y}\in\mathbb{R}^d}\int_{|\mathbf{x}-\mathbf{y}|\leqslant 1}\frac{\big|\chi(\mathbf{x})-\chi(\mathbf{y})\big|^2}{|\mathbf{x}-\mathbf{y}|^{d+1}}d\mathbf{x}\\ + \sup_{\mathbf{y}\in\mathbb{R}^d}\int_{|\mathbf{x}-\mathbf{y}|>1}\frac{\big|\chi(\mathbf{x})-\chi(\mathbf{y})\big|^2}{|\mathbf{x}-\mathbf{y}|^{d+1}}d\mathbf{x} &\leqslant |\mathbb{S}^{d-1}|\big(\|\nabla\chi\|^2_{L_\infty}+4\|\chi\|^2_{L_\infty}\big),\end{aligned} \tag{3.4}$$

where $|\mathbb{S}^{d-1}|$ is the area of $(d-1)$–dimensional unit sphere. Substituting (3.4) into (3.3) we obtain (3.2). •

LEMMA 11 *For any $\chi \in C_B^2(\mathbb{R}^{3N})$ the operator $[\chi, \Lambda^N]$ is bounded in $H^{1/2}(\mathbb{R}^{3N}, \mathbb{C}^{4^N})$, and for any $\psi \in H^{1/2}(\mathbb{R}^{3N}, \mathbb{C}^{4^N})$ we have*

$$\big\|[\chi,\Lambda^N]\psi\big\|_{H^{1/2}} \leqslant C\big(\|\nabla\chi\|_{L_\infty}+\|\partial^2\chi\|_{L_\infty}\big)\big(\|\chi\|_{L_\infty}+\|\nabla\chi\|_{L_\infty}\big)\|\psi\|_{H^{1/2}} \tag{3.5}$$

with C depending only on N and the masses of the particles.

PROOF. Successively commuting χ with Λ_n, $n = 1, \dots, N$ (see (2.3)) we obtain

$$[\chi,\Lambda^N] = \sum_{n=1}^{N}\prod_{k=1}^{n-1}\Lambda_k[\chi,\Lambda_n]\prod_{l=n+1}^{N}\Lambda_l, \tag{3.6}$$

where the empty products should be replaced by identity operators. By (2.1) the operators Λ_n are bounded in $H^{1/2}$ for any $n = 1, \dots, N$. This, together with (3.6) and Lemmata 8 and 10, implies (3.5). •

4 LOWER BOUND OF THE ESSENTIAL SPECTRUM

In this section we prove that

$$\inf\sigma_{\mathrm{ess}}(\mathcal{H}_N^{D,E}) \geqslant \varkappa(D,E). \tag{4.1}$$

4.1 Partition of Unity

Lemma 12 *There exists a set of nonnegative functions $\{\chi_Z\}$ indexed by possible 2–cluster decompositions $Z = (Z_1, Z_2)$ satisfying*

1. $\chi_Z \in C^\infty(\mathbb{R}^{3N})$ *for all* Z;
2. $\chi_Z(\kappa \mathbf{X}) = \chi_Z(\mathbf{X})$ *for all* $|\mathbf{X}| = 1,\ \kappa > 1,\ Z_2 \neq \varnothing$;
3. $$\sum_Z \chi_Z^2(\mathbf{X}) = 1, \text{ for all } \mathbf{X} \in \mathbb{R}^{3N}; \tag{4.2}$$
4. *There exists* $C > 0$ *such that for any* $\mathbf{X} \in \operatorname{supp} \chi_Z$ $$\min\left\{|\mathbf{x}_j - \mathbf{x}_n| : \mathbf{x}_j \in Z_1,\ \mathbf{x}_n \in Z_2;\ |\mathbf{x}_n| : \mathbf{x}_n \in Z_2\right\} \geqslant C|\mathbf{X}|; \tag{4.3}$$
5. $\chi_Z(\gamma\mathbf{x}_1, \ldots, \gamma\mathbf{x}_N) = \chi_Z(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ *for any orthogonal transformation* γ;
6. χ_Z *is invariant under permutations of variables preserving* $Z_{1,2}$.

Proof. The proof is essentially based on the modification of the argument given in [17], Lemma 2.4.

1. We first prove that for any $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_N) \in \mathbb{R}^{3N}$ with $|\mathbf{X}| = 1$ there exists a 2–cluster decomposition $Z = (Z_1, Z_2)$ such that

$$\min\left\{|\mathbf{x}_j - \mathbf{x}_n| : \mathbf{x}_j \in Z_1,\ \mathbf{x}_n \in Z_2;\ |\mathbf{x}_n| : \mathbf{x}_n \in Z_2\right\} > N^{-3/2}. \tag{4.4}$$

Indeed, let k be such that $|\mathbf{x}_k| \geqslant |\mathbf{x}_j|$ for all $j = 1, \ldots, N$. Then, since $|\mathbf{X}| = 1$,

$$|\mathbf{x}_k| \geqslant N^{-\frac{1}{2}}. \tag{4.5}$$

Choose Cartesian coordinates in $\mathbb{R}^3$ with the first axis passing through the origin and $\mathbf{x}_k$, so that $\mathbf{x}_k = \big(|\mathbf{x}_k|, 0, 0\big)$. Consider N regions

$$R_1 := \left\{\mathbf{x} \in \mathbb{R}^3 : x^1 \leqslant |\mathbf{x}_k|/N\right\},$$
$$R_l := \Big\{\mathbf{x} \in \mathbb{R}^3 : x^1 \in \big((l-1)|\mathbf{x}_k|/N,\ l|\mathbf{x}_k|/N\big]\Big\}, \quad l = 2, \ldots, N.$$

At least one of these regions does not contain $\mathbf{x}_j$ with $j \neq k$. Let l_0 be the maximal index of such regions. Let Z_2 be the set of indices n such that $\mathbf{x}_n \in \bigcup_{l > l_0} R_l$. Z_2 is nonempty since $\mathbf{x}_k \in Z_2$. Setting $Z_1 := I \setminus Z_2$ we observe that

$$\min\left\{|\mathbf{x}_j - \mathbf{x}_n| : \mathbf{x}_j \in Z_1,\ \mathbf{x}_n \in Z_2;\ |\mathbf{x}_n| : \mathbf{x}_n \in Z_2\right\} > |\mathbf{x}_k|/N,$$

which together with (4.5) implies (4.4).

2. Choose $\eta \in C^\infty\big(\mathbb{R}_+, [0,1]\big)$ so that

$$\eta(t) \equiv \begin{cases} 0, & t \in [0,1], \\ 1, & t \in [2, \infty). \end{cases}$$

Let

$$\zeta_Z(\mathbf{X}) := \begin{cases} \eta\big(2|\mathbf{X}|\big) \prod_{n\in Z_2} \eta\Big(\frac{2|\mathbf{x}_n|}{|\mathbf{X}|N^{-3/2}}\Big) \prod_{j\in Z_1} \eta\Big(\frac{2|\mathbf{x}_j - \mathbf{x}_n|}{|\mathbf{X}|N^{-3/2}}\Big), & Z_2 \neq \varnothing, \\ 1 - \eta\big(2|\mathbf{X}|\big), & Z_2 = \varnothing. \end{cases} \tag{4.6}$$

Functions (4.6) satisfy conditions 1, 2, 4 (with $C = N^{-3/2}/2$), 5, and 6 of Lemma 12. Moreover, by the first part of the proof

$$\sum_Z \zeta_Z(\mathbf{X}) \geqslant 1, \quad \text{for all} \quad \mathbf{X} \in \mathbb{R}^{3N}.$$

Hence all the conditions are satisfied by the functions

$$\chi_Z := \zeta_Z^{1/2}\Big(\sum_Z \zeta_Z\Big)^{-1/2}.$$

•

Let

$$\chi_Z^R(\mathbf{X}) := \chi_Z(\mathbf{X}/R), \tag{4.7}$$

where the functions χ_Z are defined in Lemma 12. The derivatives of χ_Z^R decay as R tends to infinity:

$$\|\nabla\chi_Z^R\|_\infty \leqslant CR^{-1}, \quad \|\partial^2\chi_Z^R\|_\infty \leqslant CR^{-2}. \tag{4.8}$$

To simplify the notation we omit the superscript R further on.

4.2 Cluster Decomposition and Lower Bound

We now estimate from below the quadratic form of $\mathcal{H}_N^{D,E}$ on a function ψ from $\mathfrak{H}_N^{D,E} \cap \Lambda^N \bigotimes_{n=1}^{N} H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$, which is the form domain of $\mathcal{H}_N^{D,E}$.

$$\begin{aligned} \langle \mathcal{H}_N^{D,E}\psi, \psi\rangle &= \langle \Big(\sum_{n=1}^N (D_n + V_n) + \sum_{n<j}^N U_{nj}\Big) \sum_Z \chi_Z^2 \psi, \psi\rangle \\ &= \sum_Z \langle \Big(\sum_{n=1}^N (D_n + V_n) + \sum_{n<j}^N U_{nj}\Big) \chi_Z\psi, \chi_Z\psi\rangle. \end{aligned}$$

Here we have used (4.2) and the relation

$$\sum_Z \langle f, \sum_{n=1}^N \nabla_n(\chi_Z^2 g)\rangle = \sum_Z \langle \chi_Z f, \sum_{n=1}^N \nabla_n(\chi_Z g)\rangle + \sum_Z \langle f, \sum_{n=1}^N \nabla_n\Big(\frac{\chi_Z^2}{2}\Big) g\rangle \tag{4.9}$$

which holds for any $f, g \in \overset{N}{\underset{n=1}{\otimes}} H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$. The last term on the r. h. s. of (4.9) is equal to zero due to (4.2). Thus

$$\begin{aligned}\langle \mathcal{H}_N^{D,E}\psi, \psi\rangle = \sum_{Z=(Z_1,Z_2)} \Big(&\langle (\mathcal{H}_{Z,1} + \mathcal{H}_{Z,2})\Lambda^N \chi_Z \psi, \Lambda^N \chi_Z \psi\rangle \\ &+ \langle (\mathcal{H}_{Z,1} + \mathcal{H}_{Z,2})[\chi_Z, \Lambda^N]\psi, \Lambda^N \chi_Z \psi\rangle \\ &+ \langle (\mathcal{H}_{Z,1} + \mathcal{H}_{Z,2})\chi_Z \psi, [\chi_Z, \Lambda^N]\psi\rangle \\ &+ \langle \sum_{n \in Z_2} V_n \chi_Z^2 \psi, \psi\rangle + \langle \sum_{\substack{n<j \\ n \# j}} U_{nj} \chi_Z^2 \psi, \psi\rangle \Big).\end{aligned} \tag{4.10}$$

By (2.12), (2.13), (4.3), (4.7), and (2.14) the terms at the last line of (4.10) can be estimated as

$$\langle \sum_{n \in Z_2} V_n \chi_Z^2 \psi, \psi\rangle + \langle \sum_{\substack{n<j \\ n \# j}} U_{nj} \chi_Z^2 \psi, \psi\rangle \geqslant -\varepsilon_1(R)\big(\langle \mathcal{H}_N^{D,E}\psi, \psi\rangle + \|\psi\|^2\big) \tag{4.11}$$

with $\varepsilon_1(R) \to 0$ as $R \to \infty$. The terms at the second and third lines of (4.10) can also be estimated as

$$\begin{aligned}&\langle (\mathcal{H}_{Z,1} + \mathcal{H}_{Z,2})[\chi_Z, \Lambda^N]\psi, \Lambda^N \chi_Z \psi\rangle + \langle (\mathcal{H}_{Z,1} + \mathcal{H}_{Z,2})\chi_Z \psi, [\chi_Z, \Lambda^N]\psi\rangle \\ &\geqslant -\varepsilon_2(R)\big(\langle \mathcal{H}_N^{D,E}\psi, \psi\rangle + \|\psi\|^2\big), \quad \varepsilon_2(R) \underset{R\to\infty}{\longrightarrow} 0,\end{aligned}$$

according to (2.8), (3.2), (3.5), (4.8), and (2.14). For $Z_2 \neq \varnothing$ we estimate the terms at the first line of (4.10) in the following way (recall the definitions (2.18), (2.19) and (2.20)):

$$\begin{aligned}&\langle (\mathcal{H}_{Z,1} + \mathcal{H}_{Z,2})\Lambda^N \chi_Z \psi, \Lambda^N \chi_Z \psi\rangle \\ &= \sum_{(D_1,E_1;D_2,E_2) \underset{Z}{\prec} (D,E)} \langle (\mathcal{H}_{Z,1} P^{D_1} P^{E_1} + \mathcal{H}_{Z,2} P^{D_2} P^{E_2})\Lambda^N \chi_Z \psi, \Lambda^N \chi_Z \psi\rangle \\ &\geqslant \sum_{(D_1,E_1;D_2,E_2) \underset{Z}{\prec} (D,E)} \langle \big(\varkappa_1(Z, D_1, E_1) P^{D_1} P^{E_1} \\ &+ \varkappa_2(Z, D_2, E_2) P^{D_2} P^{E_2}\big)\Lambda^N \chi_Z \psi, \Lambda^N \chi_Z \psi\rangle \\ &\geqslant \varkappa(D, E)\langle \Lambda^N \chi_Z \psi, \Lambda^N \chi_Z \psi\rangle \\ &= \varkappa(D, E)\langle \chi_Z^2 \psi, \psi\rangle + \varkappa(D, E)\langle [\Lambda^N, \chi_Z]\psi, \chi_Z \psi\rangle.\end{aligned} \tag{4.12}$$

By (3.2), (3.5), (4.8), and (2.14) the last term on the r. h. s. of (4.12) can be estimated as

$$\varkappa(D, E)\langle [\Lambda^N, \chi_Z]\psi, \chi_Z \psi\rangle \geqslant -\varepsilon_3(R)\big(\langle \mathcal{H}_N^{D,E}\psi, \psi\rangle + \|\psi\|^2\big), \quad \varepsilon_3(R) \underset{R\to\infty}{\longrightarrow} 0. \tag{4.13}$$

Substituting the estimates (4.11) — (4.13) into (4.10) we obtain

$$\langle \mathcal{H}_N^{D,E}\psi,\psi\rangle \geqslant \varkappa(D,E)\langle \sum_{\substack{Z=(Z_1,Z_2)\\ Z_2\neq\varnothing}} \chi_Z^2\psi,\psi\rangle + \langle \mathcal{H}_N^{D,E}\Lambda^N\chi_{(I,\varnothing)}\psi, \Lambda^N\chi_{(I,\varnothing)}\psi\rangle$$
$$- \varepsilon_4(R)\big(\langle \mathcal{H}_N^{D,E}\psi,\psi\rangle + \|\psi\|^2\big), \quad \varepsilon_4(R) \underset{R\to\infty}{\longrightarrow} 0. \tag{4.14}$$

4.3 Estimate Inside of the Compact Region

It remains to estimate from below the quadratic form of $\mathcal{H}_N^{D,E}$ on $\Lambda^N\chi_{(I,\varnothing)}\psi$. Note that according to Lemma 12 and (4.7) $\operatorname{supp}\chi_{(I,\varnothing)} \subset [-2R,2R]^{3N}$. To simplify the notation let

$$\chi_0 := \chi_{(I,\varnothing)}.$$

Lemma 13 *For $M>0$ let*

$$W_M := \big\{\mathbf{p}\in\mathbb{R}^{3N} : |p_i| \leqslant M, i=1,\dots,3N\big\},\ \widetilde{W}_M := \mathbb{R}^{3N}\setminus W_M.$$

There exists a finite set $Q_M \subset L_2(\mathbb{R}^{3N})$ such that for any $f\in L_2(\mathbb{R}^{3N})$ with $\operatorname{supp} f \subset [-2R,2R]^{3N}$, $f \perp Q_M$ *holds*

$$\|\hat f\|_{L_2(\widetilde{W}_M)} \geqslant \frac{1}{2}\|\hat f\|_{L_2(\mathbb{R}^{3N})}.$$

The proof of Lemma 13 is analogous to the proof of Theorem 7 of [23] and is given in Appendix C of [16].

It follows from (2.9) that for any $M>0$

$$\langle \mathcal{H}_N^{D,E}\Lambda^N\chi_0\psi, \Lambda^N\chi_0\psi\rangle \geqslant C_1\langle \sum_{n=1}^N D_n I_{\widetilde{W}_M}\Lambda^N\chi_0\psi, \Lambda^N\chi_0\psi\rangle - C_2\|\chi_0\psi\|^2. \tag{4.15}$$

Here $I_{\widetilde{W}_M}$ is the operator of multiplication by the characteristic function of $\widetilde{W}_M$ in momentum space.
We choose

$$M := 8\big(\varkappa(D,E) + C_2\big)C_1^{-1} \tag{4.16}$$

and assume henceforth that $f := \chi_0\psi$ is orthogonal to the set Q_M defined in Lemma 13. Since in momentum space the operator D_n acts on functions from $\Lambda_n L_2(\mathbb{R}^3,\mathbb{C}^4)$ as multiplication by $\sqrt{|\mathbf{p}|^2+m_n^2}$, by construction of $\widetilde{W}_M$ we have

$$\langle \sum_{n=1}^N D_n I_{\widetilde{W}_M}\Lambda^N\chi_0\psi, \Lambda^N\chi_0\psi\rangle \geqslant M\|I_{\widetilde{W}_M}\Lambda^N\chi_0\psi\|^2. \tag{4.17}$$

Inequalities (4.15) and (4.17) imply

$$\begin{aligned}
&\langle \mathcal{H}_N^{D,E}\Lambda^N\chi_0\psi, \Lambda^N\chi_0\psi\rangle \geqslant C_1M\|I_{\widetilde{W}_M}\Lambda^N\chi_0\psi\|^2 - C_2\|\chi_0\psi\|^2 \\
&\geqslant C_1M\Big(\|I_{\widetilde{W}_M}\chi_0\psi\| - \big\|I_{\widetilde{W}_M}[\Lambda^N,\chi_0]\psi\big\|\Big)^2 - C_2\|\chi_0\psi\|^2 \\
&\geqslant C_1M\Big(\frac{1}{2}\|I_{\widetilde{W}_M}\chi_0\psi\|^2 - \big\|I_{\widetilde{W}_M}[\Lambda^N,\chi_0]\psi\big\|^2\Big) - C_2\|\chi_0\psi\|^2 \\
&\geqslant 4\big(\varkappa(D,E)+C_2\big)\|I_{\widetilde{W}_M}\chi_0\psi\|^2 \\
&- 8\big(\varkappa(D,E)+C_2\big)\big\|[\Lambda^N,\chi_0]\psi\big\|^2 - C_2\|\chi_0\psi\|^2.
\end{aligned} \tag{4.18}$$

At the last step we have used (4.16). The second term on the r. h. s. of (4.18) can be estimated analogously to (4.13) as

$$-8\big(\varkappa(D,E)+C_2\big)\big\|[\Lambda^N,\chi_0]\psi\big\|^2 \geqslant -\varepsilon_5(R)\big(\langle \mathcal{H}_N^{D,E}\psi,\psi\rangle + \|\psi\|^2\big),\ \varepsilon_5(R) \underset{R\to\infty}{\longrightarrow} 0.$$

For the first term on the r. h. s. of (4.18) Lemma 13 implies

$$4\|I_{\widetilde{W}_M}\chi_0\psi\|^2 \geqslant \|\chi_0\psi\|^2. \tag{4.19}$$

As a consequence of (4.18) — (4.19), we have

$$\begin{gathered}
\langle \mathcal{H}_N^{D,E}\Lambda^N\chi_0\psi, \Lambda^N\chi_0\psi\rangle \geqslant \varkappa(D,E)\|\chi_0\psi\|^2 - \varepsilon_5(R)\big(\langle \mathcal{H}_N^{D,E}\psi,\psi\rangle + \|\psi\|^2\big), \\
\varepsilon_5(R) \underset{R\to\infty}{\longrightarrow} 0.
\end{gathered} \tag{4.20}$$

4.4 Completion of the Proof

By (4.14), (4.20), and (4.2)

$$\langle \mathcal{H}_N^{D,E}\psi,\psi\rangle \geqslant \varkappa(D,E)\|\psi\|^2 - \varepsilon_6(R)\big(\langle \mathcal{H}_N^{D,E}\psi,\psi\rangle + \|\psi\|^2\big), \quad \varepsilon_6(R) \underset{R\to\infty}{\longrightarrow} 0.$$

for any ψ in the form domain of $\mathcal{H}_N^{D,E}$ orthogonal to the finite set of functions (cardinality of this set depends on R). This implies the discreteness of the spectrum of $\mathcal{H}_N^{D,E}$ below $\varkappa(D,E)$ and thus (4.1).

5 Spectrum of the Free Cluster

In this section we characterize the spectrum of the cluster Z_2 which does not interact with the external field.

Proposition 14 *For any irreducible representations D_2, E_2 of rotation–reflection and permutation groups the spectrum of $\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2}$ is*

$$\sigma(\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2}) = \sigma_{\mathrm{ess}}(\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2}) = \big[\varkappa_2(Z,D_2,E_2),\infty\big),$$

with some $\varkappa_2(Z,D_2,E_2) \in \mathbb{R}$.

PROOF. Let us introduce the new coordinates in the configuration space $\mathbb{R}^{3N_2}$ of the cluster $Z_2 = \{n_1, \dots, n_{N_2}\}$, in the same manner as it is done in [15]. Let $M := \sum_{n\in Z_2} m_n$ be the total mass of the particles constituting the cluster. We introduce

$$\begin{aligned} \mathbf{y}_0 &:= \frac{1}{M}\sum_{n\in Z_2} m_n \mathbf{x}_n, \\ \mathbf{y}_k &:= \mathbf{x}_{n_{k+1}} - \mathbf{x}_{n_1}, \quad k = 1, \dots, N_2 - 1. \end{aligned} \tag{5.1}$$

The Jacobian of this variable change is one. Here $\mathbf{y}_0$ is the coordinate of the center of mass, whereby $\mathbf{y}_k$, $k = 1, \dots, N_2 - 1$ are the internal coordinates of the cluster. Accordingly,

$$\begin{aligned} \mathbf{x}_{n_1} &= \mathbf{y}_0 - \frac{1}{M}\sum_{k=1}^{N_2-1} m_{n_{k+1}}\mathbf{y}_k, \\ \mathbf{x}_{n_{l+1}} &= \mathbf{y}_0 + \mathbf{y}_l - \frac{1}{M}\sum_{k=1}^{N_2-1} m_{n_{k+1}}\mathbf{y}_k, \quad l = 1, \dots, N_2 - 1. \end{aligned} \tag{5.2}$$

The momentum operators in the new coordinates are

$$\begin{aligned} \mathbf{p}_{n_1} &:= -i\nabla_{\mathbf{x}_{n_1}} = \frac{m_{n_1}}{M}\mathbf{P} - \sum_{k=1}^{N_2-1}(-i\nabla_{\mathbf{y}_k}), \\ \mathbf{p}_{n_k} &:= -i\nabla_{\mathbf{x}_{n_k}} = \frac{m_{n_k}}{M}\mathbf{P} + (-i\nabla_{\mathbf{y}_{k-1}}), \quad k = 2, \dots, N_2, \end{aligned} \tag{5.3}$$

where $\mathbf{P}$ is the total momentum of the cluster:

$$\mathbf{P} := \sum_{n\in Z_2} -i\nabla_{\mathbf{x}_n} = -i\nabla_{\mathbf{y}_0}.$$

Let $\mathcal{F}_0$ be the partial Forurier transform on $\mathfrak{H}_{Z,2}^{D_2,E_2}$ defined by

$$(\mathcal{F}_0 f)(\mathbf{P}, \mathbf{y}_1, \dots, \mathbf{y}_{N_2-1}) := \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} f(\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_{N_2-1}) e^{-i\mathbf{P}\mathbf{y}_0} d\mathbf{y}_0.$$

By (2.6) and (2.7) we have

$$\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2} = \mathcal{F}_0^{-1}\widehat{\Lambda}_{Z,2}\widehat{\mathcal{H}}_{Z,2}^{D_2,E_2}\widehat{\Lambda}_{Z,2}\mathcal{F}_0,$$

where in the new coordinates

$$\widehat{\mathcal{H}}_{Z,2}^{D_2,E_2} := \sum_{n\in Z_2}(\boldsymbol{\alpha}_n\cdot\mathbf{p}_n + \beta_n m_n) + \sum_{k=2}^{N_2-1} U_{n_1 n_k}(\mathbf{y}_k) + \sum_{1<k<l\leqslant N_2-1} U_{n_k n_l}(\mathbf{y}_k - \mathbf{y}_l), \tag{5.4}$$

$$\widehat{\Lambda}_{Z,2} := \prod_{n\in Z_2} \widehat{\Lambda}_n, \tag{5.5}$$

$$\widehat{\Lambda}_n := \frac{1}{2} + \frac{\boldsymbol{\alpha}_n \cdot \mathbf{p}_n + \beta_n m_n}{2\sqrt{\mathbf{p}_n^2 + m_n^2}},$$

operators $\mathbf{p}_n$ are given by (5.3), and $\mathbf{P}$ should now be interpreted as multiplication by the vector–function. The operators (5.4) and (5.5) obviously commute with $\mathfrak{P} := |\mathbf{P}|$. The operator $\mathcal{F}_0^{-1}\mathfrak{P}\mathcal{F}_0$ (unlike $\mathcal{F}_0^{-1}\mathbf{P}\mathcal{F}_0$) is well–defined in $\mathfrak{H}_{Z,2}^{D_2,E_2}$, since it commutes with P^{D_2} an P^{E_2} in $\mathfrak{H}_{Z,2}$. This implies that $\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2}$ commutes with $\mathcal{F}_0^{-1}\mathfrak{P}\mathcal{F}_0$.

Let $\omega := \mathbf{P}/\mathfrak{P} \in S^2$. We decompose the Hilbert space $\mathfrak{H}_{Z,2}^{D_2,E_2}$ into the direct integral

$$\mathfrak{H}_{Z,2}^{D_2,E_2} = \int_0^\infty \oplus \mathfrak{H}_{Z,2}^{D_2,E_2,\mathfrak{P}} \mathfrak{P}^2 d\mathfrak{P}. \tag{5.6}$$

The fibre space $\mathfrak{H}_{Z,2}^{D_2,E_2,\mathfrak{P}}$ can be considered as a subspace of $L_2(\mathbb{R}^{3N_2-3} \times S^2, \mathbb{C}^{4^{N_2}})$ with the inner product

$$\langle f, g\rangle_* := \int_{\mathbb{R}^{3(N_2-1)}\times S^2} \langle f, g\rangle_{\mathbb{C}^{4^{N_2}}} d\mathbf{y}_1 \cdots d\mathbf{y}_{N_2-1} d\omega.$$

For $f \in \mathfrak{H}_{Z,2}^{D_2,E_2}$ the corresponding element of $\mathfrak{H}_{Z,2}^{D_2,E_2,\mathfrak{P}}$ is given by

$$f_{\mathfrak{P}} := \mathcal{F}_0 f|_{|\mathbf{P}|=\mathfrak{P}}.$$

We have

$$\|f\|^2 = \int_0^\infty \|f_{\mathfrak{P}}\|_*^2 \mathfrak{P}^2 d\mathfrak{P} \tag{5.7}$$

in compliance with (5.6). The form domain of $\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}}$ is

$$\mathfrak{D}^{\mathfrak{P}} := \Lambda_{Z,2}^{\mathfrak{P}} P^{D_2} P^{E_2} H^{1/2}(\mathbb{R}^{3(N_2-1)} \times S^2, \mathbb{C}^{4^{N_2}}),$$

where $\Lambda_{Z,2}^{\mathfrak{P}}$ is given by (5.5) with the only difference that we should replace $\mathbf{P}$ by $\omega\mathfrak{P}$ in (5.3). The operators on fibres of the direct integral (5.6) are

$$\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}} := \Lambda_{Z,2}^{\mathfrak{P}} \mathcal{H}_{Z,2}^{D_2,E_2,\mathfrak{P}} \Lambda_{Z,2}^{\mathfrak{P}},$$

where $\mathcal{H}_{Z,2}^{D_2,E_2,\mathfrak{P}}$ is given by the r. h. s. of (5.4) with $\mathbf{P}$ replaced by $\omega\mathfrak{P}$ in (5.3). We thus have

$$\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2} = \int_0^\infty \oplus \widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}} \mathfrak{P}^2 d\mathfrak{P}. \tag{5.8}$$

The spectrum of $\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2}$ can be represented as

$$\sigma(\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2}) = \overline{\operatorname{ess} \bigcup_{\mathfrak{P}\in\mathbb{R}_+} \sigma(\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}})}, \tag{5.9}$$

where the essential union is taken with respect to the Lebesgue measure in $\mathbb{R}_+$. The bottom of the spectrum of $\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}}$ is given by

$$\mu(\mathfrak{P}) := \inf_{\psi\in\mathfrak{D}^{\mathfrak{P}}} \frac{\langle \widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}}\psi,\psi\rangle_*}{\|\psi\|_*^2}. \tag{5.10}$$

Lemma 15 *Function* (5.10) *is continuous on* $\mathbb{R}_+$.

Proof of Lemma 15. Let us fix $\mathfrak{P}\in\mathbb{R}_+$ and $\varepsilon>0$. We will prove that $\big|\mu(\mathfrak{P}+\mathfrak{p})-\mu(\mathfrak{P})\big|<\varepsilon$ if $|\mathfrak{p}|$ is small enough. Choose $\psi\in\mathfrak{D}^{\mathfrak{P}}$ such that

$$\left|\frac{\langle \widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}}\psi,\psi\rangle_*}{\|\psi\|_*^2}-\mu(\mathfrak{P})\right| \leqslant \frac{\varepsilon}{2}. \tag{5.11}$$

Let

$$\phi := \Lambda_{Z,2}^{\mathfrak{P}+\mathfrak{p}}\psi \in \mathfrak{D}^{\mathfrak{P}+\mathfrak{p}}.$$

We have

$$\phi-\psi = (\Lambda_{Z,2}^{\mathfrak{P}+\mathfrak{p}}-\Lambda_{Z,2}^{\mathfrak{P}})\psi = \sum_{k=1}^{N_2}\prod_{i<k}\Lambda_{n_i}^{\mathfrak{P}+\mathfrak{p}}(\Lambda_{n_k}^{\mathfrak{P}+\mathfrak{p}}-\Lambda_{n_k}^{\mathfrak{P}})\prod_{j>k}\Lambda_{n_j}^{\mathfrak{P}}\psi. \tag{5.12}$$

Let $\mathcal{F}$ be the unitary Fourier transform in $L_2(\mathbb{R}^{3(N_2-1)}\times S^2,\mathbb{C}^{4^{N_2}})$ defined by

$$\begin{aligned}&(\mathcal{F}\xi)(\omega,\mathbf{q}_1,\dots,\mathbf{q}_{N_2-1})\\ &:= (2\pi)^{3(1-N_2)/2}\int\limits_{\mathbb{R}^{3(N_2-1)}}\xi(\omega,\mathbf{y}_1,\dots,\mathbf{y}_{N_2-1})e^{-i\sum\limits_{k=1}^{N_2-1}\mathbf{q}_k\cdot\mathbf{y}_k}d\mathbf{y}_1\cdots d\mathbf{y}_{N_2-1}.\end{aligned}$$

We can rewrite (5.12) as

$$\phi-\psi = \mathcal{F}^{-1}\sum_{k=1}^{N_2}\prod_{i<k}\widehat{\Lambda}_{n_i}^{\mathfrak{P}+\mathfrak{p}}(\widehat{\Lambda}_{n_k}^{\mathfrak{P}+\mathfrak{p}}-\widehat{\Lambda}_{n_k}^{\mathfrak{P}})\prod_{j>k}\widehat{\Lambda}_{n_j}^{\mathfrak{P}}\mathcal{F}\psi, \tag{5.13}$$

where $\widehat{\Lambda}_n^{\mathfrak{P}}$, $n\in Z_2$ are the operators of multiplication by the symbols

$$\widehat{\Lambda}_n^{\mathfrak{P}} := \frac{1}{2}+\frac{\boldsymbol{\alpha}_n\cdot\widehat{\mathbf{p}}_n+\beta_n m_n}{2\sqrt{\widehat{\mathbf{p}}_n^2+m_n^2}}, \tag{5.14}$$

$$\begin{aligned}\widehat{\mathbf{p}}_{n_1} &:= \frac{m_{n_1}}{M}\omega\mathfrak{P}-\sum_{k=1}^{N_2-1}\mathbf{q}_k,\\ \widehat{\mathbf{p}}_{n_k} &:= \frac{m_{n_k}}{M}\omega\mathfrak{P}+\mathbf{q}_{k-1},\quad k=2,\dots,N_2.\end{aligned} \tag{5.15}$$

The matrix–functions (5.14) are uniformly continuous in $\mathfrak{P}$. Thus by (5.13)

$$\|\phi-\psi\|_{H^{1/2}(\mathbb{R}^{3(N_2-1)}\times S^2,\mathbb{C}^{4^{N_2}})} \leqslant C\sum_{k=1}^{N_2}\|\widehat{\Lambda}_{n_k}^{\mathfrak{P}+\mathfrak{p}}-\widehat{\Lambda}_{n_k}^{\mathfrak{P}}\|_{L_\infty}\|\psi\|_{H^{1/2}} \underset{|\mathfrak{p}|\to 0}{\longrightarrow} 0. \quad (5.16)$$

We write

$$\begin{aligned}\langle\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}+\mathfrak{p}}\phi,\phi\rangle_* &= \langle\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}}\psi,\psi\rangle_* + \langle\mathcal{H}_{Z,2}^{D_2,E_2,\mathfrak{P}}(\phi-\psi),\psi\rangle_* \\ &+ \langle\mathcal{H}_{Z,2}^{D_2,E_2,\mathfrak{P}}\phi,(\phi-\psi)\rangle_* + \langle(\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}+\mathfrak{p}}-\mathcal{H}_{Z,2}^{D_2,E_2,\mathfrak{P}})\phi,\phi\rangle_*.\end{aligned} \quad (5.17)$$

The second and third terms on the r. h. s. of (5.17) tend to zero as $|\mathfrak{p}|\to 0$ according to (5.16) and (2.8). The last term also tends to zero for small $|\mathfrak{p}|$, since the symbol of the difference is

$$\mathcal{F}(\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}+\mathfrak{p}}-\mathcal{H}_{Z,2}^{D_2,E_2,\mathfrak{P}})\mathcal{F}^{-1} = \sum_{n\in Z_2}\frac{m_n}{M}\boldsymbol{\alpha}_n\cdot\omega\mathfrak{p}.$$

From (5.16) and (5.17) follows that

$$\left|\frac{\langle\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}}\psi,\psi\rangle_*}{\|\psi\|_*^2}-\frac{\langle\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}+\mathfrak{p}}\phi,\phi\rangle_*}{\|\phi\|_*^2}\right| \leqslant \frac{\varepsilon}{2}, \quad (5.18)$$

if $|\mathfrak{p}|$ is small enough. Hence by (5.11) and (5.18) for any $\varepsilon>0$

$$\left|\mu(\mathfrak{P}+\mathfrak{p})-\mu(\mathfrak{P})\right|<\varepsilon$$

for $|\mathfrak{p}|$ small enough. •

Now we prove that μ is semibounded from below and tends to infinity as $|\mathfrak{P}|\to\infty$. This, together with (5.9) and Lemma 15, implies that the spectrum of $\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2}$ is purely essential and is concentrated on a semi–axis. Proposition 14 will be thus proved.

According to (2.9) for $j=2$ and (2.10) we have

$$\begin{aligned}\langle\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2}\psi,\psi\rangle &\geqslant C_1\langle\sum_{n\in Z_2}\sqrt{-\Delta_n+m_n^2}\psi,\psi\rangle - C_2\|\psi\|^2, \\ &\text{for any}\quad \psi\in P^DP^E\bigotimes_{n\in Z_2}\Lambda_nH^{1/2}(\mathbb{R}^3,\mathbb{C}^4).\end{aligned} \quad (5.19)$$

Since all the operators corresponding to the quadratic forms involved in (5.19) commute with $\mathcal{F}_0^{-1}\mathfrak{P}\mathcal{F}_0$, it follows from (5.8) that for almost all $\mathfrak{P}$ the inequality

$$\langle\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}}\psi,\psi\rangle_* \geqslant C_1\langle\sum_{n\in Z_2}\sqrt{\widehat{\mathbf{p}}_n^2+m_n^2}\mathcal{F}\psi,\mathcal{F}\psi\rangle_* - C_2\|\psi\|_*^2 \quad (5.20)$$

holds for every $\psi\in\mathfrak{D}^{\mathfrak{P}}$, where $\widehat{\mathbf{p}}_n$ are defined in (5.15). Thus μ is semibounded from below. Since by (5.15)

$$\mathfrak{P} = \Big|\sum_{n\in Z_2}\widehat{\mathbf{p}}_n\Big|,$$

there exists $n \in Z_2$ such that

$$|\widehat{\mathbf{p}}_n| \geqslant \frac{\mathfrak{P}}{N_2}$$

and hence

$$\sum_{n \in Z_2} \sqrt{\widehat{\mathbf{p}}_n^2 + m_n^2} \geqslant \frac{\mathfrak{P}}{N_2}.$$

Thus the r. h. s. of (5.20) tends to infinity as $\mathfrak{P} \to \infty$. •

6 Absence of Gaps

We are now ready to finish the proof of Theorem 6 by proving that

$$\big[\varkappa(D,E), \infty\big) \subseteq \sigma(\mathcal{H}_N^{D,E}). \tag{6.1}$$

Let us first fix a decomposition Z on which the minimum is attained in (2.20). Following the general strategy of [14], we will prove that for any irreducible representations $(D_1, E_1; D_2, E_2) \underset{Z}{\prec} (D, E)$ any

$$\lambda \geqslant \varkappa_1(Z, D_1, E_1) + \varkappa_2(Z, D_2, E_2)$$

belongs to $\sigma(\mathcal{H}_N^{D,E})$. This will imply (6.1) according to the definition (2.19). Let

$$\lambda_1 := \lambda - \varkappa_1(Z, D_1, E_1) \geqslant \varkappa_2(Z, D_2, E_2). \tag{6.2}$$

We will use the notation and results of Section 5. The following lemma is a slight modification of Theorem 8.11 of [14] and is proved along the same lines:

Lemma 16 *Let A be a selfadjoint operator in a Hilbert space $\mathfrak{H}$ and $U(\gamma)$ be a continuous representation of a compact group Γ by unitary operators in $\mathfrak{H}$ such that $U(\gamma)\operatorname{Dom} A \subset \operatorname{Dom} A$ and $U(\gamma)A = AU(\gamma)$ for any $\gamma \in \Gamma$. Then for any irreducible (matrix) representation D of Γ the corresponding subspace $P^D\mathfrak{H}$ reduces A. For every $\lambda \in \sigma(A^D)$ where A^D is the reduced operator and every $\varepsilon > 0$ there exists a D–generating subspace G of* Dom A *such that*

$$\|Au - \lambda u\| \leqslant \varepsilon \|u\|, \ \textit{for all } u \in G.$$

Remark 17 *Recall that a subspace G of $\mathfrak{H}$ is called D*–generating *if the operator $U(\gamma)|G$ is unitary in G for all $\gamma \in \Gamma$ and there exists an orthonormal base in G such that for every $\gamma \in \Gamma$ the operator $U(\gamma)|G$ is represented by the matrix $D(\gamma)$.*

Proof of Lemma 16. Let r be the dimension of the representation $D : \gamma \mapsto \big(D_{lk}(\gamma)\big)_{l,k=1}^r$. Let us introduce in $\mathfrak{H}$ the bounded operators P_{lk} by

$$P_{lk} := r \int_\Gamma \overline{D_{lk}(\gamma)} U(\gamma) d\mu(\gamma), \quad l, k = 1, \ldots, r,$$

where μ is the invariant probability measure on Γ. It is shown in the proof of Theorem 8.11 of [14] that P_{ll} are orthogonal projections onto mutually orthogonal subspaces of $\mathfrak{H}$ and that

$$P^D = \sum_{l=1}^{r} P_{ll}. \tag{6.3}$$

In fact, P_{ll} is the projection on the subspace of function which belong to the l^{th} row of the representation D. Moreover, P_{lk} is a partial isometry between $P_{kk}\mathfrak{H}$ and $P_{ll}\mathfrak{H}$. Since $\lambda \in \sigma(A^D)$, there exists a vector $u_0 \in \operatorname{Dom} A^D$ such that

$$\|A^D u_0 - \lambda u_0\| \leqslant \varepsilon \|u_0\|.$$

It follows from (6.3) that there exists $l \in \{1, \dots, r\}$ such that $\|P_{ll}u_0\| \geqslant r^{-1}$. We can thus define $u_l := P_{ll}u_0/\|P_{ll}u_0\|$ and then $u_k := P_{kl}u_l$ for $k = 1, \dots, r$. The subspace G spanned by $\{u_k\}_{k=1}^r$ satisfies the statement of the lemma. •

Let

$$r_j := \dim(D_j \otimes E_j), \quad j = 1, 2. \tag{6.4}$$

Since $\varkappa_1(Z, D_1, E_1)$ belongs to the spectrum of $\widetilde{\mathcal{H}}_{Z,1}^{D_1,E_1}$ (see definition (2.18)), by Lemma 16 we can choose a sequence of $(D_1 \otimes E_1)$–generating subspaces $\{G_q\}_{q=1}^\infty$ of $\operatorname{Dom}(\widetilde{\mathcal{H}}_{Z,1}^{D_1,E_1})$ such that for all $q \in \mathbb{N}$

$$\left\|\widetilde{\mathcal{H}}_{Z,1}^{D_1,E_1}\phi_q - \varkappa_1(Z, D_1, E_1)\phi_q\right\|_{\mathfrak{H}_{Z,1}} \leqslant q^{-1}\|\phi_q\|_{\mathfrak{H}_{Z,1}}, \text{ for all } \phi_q \in G_q. \tag{6.5}$$

Analogously, for any $\mathfrak{P} \geqslant 0$ we can find a sequence $\{G_q^{\mathfrak{P}}\}_{q=1}^\infty$ of $(D_2 \otimes E_2)$–generating subspaces of $\operatorname{Dom}\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}}$ such that

$$\left\|\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2,\mathfrak{P}}\psi_q^{\mathfrak{P}} - \mu(\mathfrak{P})\psi_q^{\mathfrak{P}}\right\|_* \leqslant q^{-1}\left\|\psi_q^{\mathfrak{P}}\right\|_*, \text{ for all } \psi_q^{\mathfrak{P}} \in G_q^{\mathfrak{P}}. \tag{6.6}$$

Moreover, we can choose an orthonormal basis $\{\psi_{q,l}^{\mathfrak{P}}\}_{l=1}^{r_2}$ in $G_q^{\mathfrak{P}}$ in such a way that for every $q \in \mathbb{N}$ and $l = 1, \dots, r_2$ $\psi_{q,l}$ belongs to the l^{th} row of the representation $(D_2 \otimes E_2)$ and satisfies (6.6). By Proposition 14, Lemma 15, and (6.2) we can choose $\mathfrak{P}_0$ in such a way that

$$\mu(\mathfrak{P}_0) = \lambda_1. \tag{6.7}$$

We choose $R_q > q$ so that (2.12) and (2.13) hold true for all $n, j = 1, \dots, N$, $n < j$ with

$$\varepsilon := q^{-1}(N_1 + 1)^{-1} N_2^{-1/2} C_1^{1/2} \big(C_2 + |\lambda_1| + 2\big)^{-1/2}, \tag{6.8}$$

where $N_{1,2}$ are the numbers of elements in $Z_{1,2}$, and $C_{1,2}$ are the constants in (2.9) for $j = 2$, and so that for some orthonormal base $\{\phi_{q,k}\}_{k=1}^{r_1}$ of G_q

$$\left\|\Big(1 - \prod_{j \in Z_1} I_{\{|\mathbf{x}_j| < R_q\}}\Big)\phi_{q,k}\right\|_{L_2(\mathbb{R}^{3N_1}, \mathbb{C}^{4^{N_1}})} \leqslant \frac{\nu_0}{4 d_E^2 r_1 r_2}, \tag{6.9}$$

where d_E is the dimension of E, $r_{1,2}$ are defined in (6.4), and the constant $\nu_0 > 0$ depending only on E, E_1, E_2 will be specified later in the proof of Lemma 21.

By Assumption 4 and Lemma 15, we can choose a sequence of positive numbers $\{\delta_q\}_{q=1}^{\infty}$ tending to zero in such a way that

$$\left|\mu(\mathfrak{P}) - \lambda_1\right| \leqslant q^{-1} \quad \text{for all} \quad \mathfrak{P} \in [\mathfrak{P}_0, \mathfrak{P}_0 + \delta_q], \tag{6.10}$$

$$\frac{1}{2\pi^2}(\mathfrak{P}_0 + \delta_q)^2 \delta_q C_{R_q} < q^{-2}, \tag{6.11}$$

where C_{R_q} is the constant in (2.11), and

$$\frac{1}{2\pi^2}(\mathfrak{P}_0 + \delta_q)^2 \delta_q \cdot \frac{4}{3}\pi R_q^3 < \frac{\nu_0^2}{16 d_E^4 r_1^2 r_2^2}. \tag{6.12}$$

Let us choose a function $f_q \in L_2(\mathbb{R}_+)$ with $\operatorname{supp} f_q \subset [\mathfrak{P}_0, \mathfrak{P}_0 + \delta_q]$ so that

$$\int_{\mathfrak{P}_0}^{\mathfrak{P}_0+\delta_q} \left|f_q(\mathfrak{P})\right|^2 \mathfrak{P}^2 d\mathfrak{P} = 1. \tag{6.13}$$

Let

$$\begin{aligned} &\psi_{q,l}(\mathbf{y}_0, \dots, \mathbf{y}_{N_2-1}) \\ &:= \frac{1}{(2\pi)^{\frac{3}{2}}} \int\limits_{\mathfrak{P}_0}^{\mathfrak{P}_0+\delta_q} \int\limits_{S^2} e^{i\mathfrak{P}\omega\mathbf{y}_0} f_q(\mathfrak{P}) \psi_{q,l}^{\mathfrak{P}}(\omega, \mathbf{y}_1, \dots, \mathbf{y}_{N_2-1}) \mathfrak{P}^2 d\omega d\mathfrak{P}, \end{aligned} \tag{6.14}$$

where $\{\mathbf{y}_0, \dots, \mathbf{y}_{N_2-1}\}$ and $\{\mathbf{x}_n\}_{n\in Z_2}$ are related by (5.1) and (5.2). It follows from (6.13) and the choice of $\psi_{q,l}^{\mathfrak{P}}$ that

$$\|\psi_{q,l}\|_{\mathfrak{H}_{Z,2}} = 1, \quad l = 1, \dots, N_2, \tag{6.15}$$

and that $\psi_{q,l}$ belongs to the l^{th} row of $(D_2 \otimes E_2)$. Clearly the linear subspace $\widetilde{G}_q$ spanned by $\{\psi_{q,l}\}_{l=1}^{r_2}$ is a $(D_2 \otimes E_2)$–generating subspace of $\operatorname{Dom} \widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2}$.

Lemma 18 *For any* $n \in Z_2$ *and* $\psi \in \widetilde{G}_q$ *with* $\|\psi\| = 1$ *the one–particle density*

$$\rho_{\psi,n}(\mathbf{x}_n) := \int_{\mathbb{R}^{3N_2-3}} \left|\psi(\mathbf{x}_{n_1}, \dots, \mathbf{x}_{n_{N_2}})\right|^2 (d\mathbf{x}_{n_1} \cdots d\mathbf{x}_{n_{N_2}})/d\mathbf{x}_n$$

satisfies

$$\|\rho_{\psi,n}\|_{L_\infty(\mathbb{R}^3)} \leqslant \frac{1}{2\pi^2}(\mathfrak{P}_0 + \delta_q)^2 \delta_q.$$

Proof. By (6.14)

$$
\begin{aligned}
&\|\rho_{\psi,n}\|_{L_\infty(\mathbb{R}^3)} \leqslant (2\pi)^{-3/2}\|\widehat{\rho}_{\psi,n}\|_{L_1(\mathbb{R}^3)} \\
&= \frac{1}{(2\pi)^6}\int_{\mathbb{R}^3}\bigg|\int_{\mathbb{R}^{3N_2}}\int_{\mathfrak{P}_0}^{\mathfrak{P}_0+\delta_q}\int_{S^2}\int_{\mathfrak{P}_0}^{\mathfrak{P}_0+\delta_q}\int_{S^2} e^{-i\mathbf{p}(\mathbf{y}_0+\mathbf{r}_n)}e^{-i\mathfrak{P}\omega\mathbf{y}_0}\overline{f_q(\mathfrak{P})} \\
&\times \psi_q^{\mathfrak{P}*}(\omega,\mathbf{y}_1,\dots,\mathbf{y}_{N_2-1})e^{i\widetilde{\mathfrak{P}}\widetilde{\omega}\mathbf{y}_0}f_q(\widetilde{\mathfrak{P}})\psi_q^{\widetilde{\mathfrak{P}}}(\widetilde{\omega},\mathbf{y}_1,\dots,\mathbf{y}_{N_2-1})\mathfrak{P}^2\widetilde{\mathfrak{P}}^2 \\
&\times d\widetilde{\omega}\,d\widetilde{\mathfrak{P}}\,d\omega\,d\mathfrak{P}\,d\mathbf{y}_0\,d\mathbf{y}_1\cdots d\mathbf{y}_{N_2-1}\bigg|d\mathbf{p},
\end{aligned}
\tag{6.16}
$$

where $\mathbf{r}_n := \mathbf{x}_n - \mathbf{y}_0$, see (5.2). Integrating the r. h. s. of (6.16) in $\mathbf{y}_0$ we obtain $(2\pi)^3\delta(\mathbf{p}+\mathfrak{P}\omega-\widetilde{\mathfrak{P}}\widetilde{\omega})$ from all the factors involving $\mathbf{y}_0$. Estimating the absolute value of the integral by the integral of absolute value and taking into account that $\int\delta(\mathbf{p}+\dots)d\mathbf{p} = 1$ we get

$$
\begin{aligned}
&\|\rho_{\psi,n}\|_{L_\infty(\mathbb{R}^3)} \leqslant \frac{1}{(2\pi)^3}\int_{\mathbb{R}^{3N_2-3}}\int_{\mathfrak{P}_0}^{\mathfrak{P}_0+\delta_q}\int_{S^2}\int_{\mathfrak{P}_0}^{\mathfrak{P}_0+\delta_q}\int_{S^2}\big|f_q(\mathfrak{P})\big|\big|f_q(\widetilde{\mathfrak{P}})\big| \\
&\times \big|\psi_q^{\mathfrak{P}}(\omega,\mathbf{y}_1,\dots,\mathbf{y}_{N_2-1})\big|\big|\psi_q^{\widetilde{\mathfrak{P}}}(\widetilde{\omega},\mathbf{y}_1,\dots,\mathbf{y}_{N_2-1})\big|\mathfrak{P}^2\widetilde{\mathfrak{P}}^2 \\
&\times d\widetilde{\omega}\,d\widetilde{\mathfrak{P}}\,d\omega\,d\mathfrak{P}\,d\mathbf{y}_1\cdots d\mathbf{y}_{N_2-1} \leqslant \frac{1}{(2\pi)^3}4\pi(\mathfrak{P}_0+\delta_q)^2\delta_q,
\end{aligned}
\tag{6.17}
$$

where at the last step we have used Schwarz inequality and $\|\psi\| = 1$. The formal calculation (6.16) — (6.17) is justified by the fact that the integral over $\mathbb{R}^{3N_2}$ can be considered as a limit of integrals over expanding finite volumes, since $\psi \in L_2(\mathbb{R}^{3N_2})$. •

Corollary 19 *For any $W \in L_2(\mathbb{R}^3)$, $n \in Z_2$, and $\psi \in \widetilde{G}_q$ with $\|\psi\| = 1$ we have*

$$
\int_{\mathbb{R}^{3N_2}}\big|W(\mathbf{x}_n)\psi(\mathbf{x}_{n_1},\dots,\mathbf{x}_{n_{N_2}})\big|^2 d\mathbf{x}_{n_1}\cdots d\mathbf{x}_{n_{N_2}} \leqslant \frac{1}{2\pi^2}(\mathfrak{P}_0+\delta_q)^2\delta_q\|W\|^2.
$$

Let F_q be the subspace of $\mathfrak{H}_N$ spanned by the functions

$$
\begin{aligned}
\varphi_{q,k,l}(\mathbf{x}_1,\dots,\mathbf{x}_N) := \phi_{q,k}(\mathbf{x}_j : j \in Z_1)\otimes\psi_{q,l}(\mathbf{x}_n : n \in Z_2),& \\
k = 1,\dots,r_1,\quad l = 1,\dots,r_2,&
\end{aligned}
\tag{6.18}
$$

where $\{\phi_{q,k}\}_{k=1}^{r_1}$ and $\{\psi_{q,l}\}_{l=1}^{r_2}$ are orthonormal bases of G_q and $\widetilde{G}_q$, respectively. We obviously have $\|\varphi_{q,k,l}\|_{L_2(\mathbb{R}^{3N},\mathbb{C}^{4^N})} = 1$.

Lemma 20 *For any $q \in \mathbb{N}$ $F_q \subset \operatorname{Dom}\mathcal{H}_N$. For any $\varphi \in F_q$*

$$
\big\|(\mathcal{H}_N-\lambda)\varphi\big\| \leqslant 5q^{-1}r_1^{1/2}r_2^{1/2}\|\varphi\|.
$$

Proof. It is enough to show that the functions (6.18) belong to $\operatorname{Dom}\mathcal{H}_N$ and satisfy

$$\big\|(\mathcal{H}_N-\lambda)\varphi_{q,k,l}\big\|_{L_2(\mathbb{R}^{3N},\mathbb{C}^{4^N})}\leqslant 5q^{-1}. \tag{6.19}$$

Indeed, by triangle and Cauchy inequalities for

$$\varphi=\sum_{k=1}^{r_1}\sum_{l=1}^{r_2}c_{kl}\varphi_{q,k,l} \tag{6.20}$$

we have

$$\begin{aligned}\big\|(\mathcal{H}_N-\lambda)\varphi\big\|&\leqslant\sum_{k=1}^{r_1}\sum_{l=1}^{r_2}|c_{kl}|\big\|(\mathcal{H}_N-\lambda)\varphi_{q,k,l}\big\|\\&\leqslant\sup_{k,l}\big\|(\mathcal{H}_N-\lambda)\varphi_{q,k,l}\big\|r_1^{1/2}r_2^{1/2}\|\varphi\|.\end{aligned}$$

The operator domain of $\mathcal{H}_N$ can be characterized as the set of functions ξ from the form domain $\overset{N}{\underset{n=1}{\otimes}}\Lambda_nH^{1/2}(\mathbb{R}^3,\mathbb{C}^4)$ on which the sesquilinear form $\langle\mathcal{H}_N\xi,\cdot\rangle$ is a bounded linear functional in $\mathfrak{H}_N$. Functions (6.18) belong to $\overset{N}{\underset{n=1}{\otimes}}\Lambda_nH^{1/2}(\mathbb{R}^3,\mathbb{C}^4)$ by construction. By (2.2), (2.5), and (2.6) we have

$$\mathcal{H}_N=\mathcal{H}_{Z,1}+\mathcal{H}_{Z,2}+\Lambda^N\bigg(\sum_{n\in Z_2}V_n+\sum_{\substack{n<j\\ n\# j}}U_{nj}\bigg)\Lambda^N. \tag{6.21}$$

The sesquilinear forms $\langle(\mathcal{H}_{Z,1}+\mathcal{H}_{Z,2})\varphi_{q,k,l},\cdot\rangle$ are bounded linear functionals over $L_2(\mathbb{R}^{3N},\mathbb{C}^{4^N})$, since $\phi_{q,k}\in\operatorname{Dom}(\widetilde{\mathcal{H}}^{D_1,E_1}_{Z,1})$ and $\psi_{q,l}\in\operatorname{Dom}\widetilde{H}^{D_2,E_2}_{Z,2}$. Moreover, by (6.5)

$$\Big\|\big(\mathcal{H}_{Z,1}-\varkappa_1(Z,D_1,E_1)\big)\varphi_{q,k,l}\Big\|=\Big\|\big(\widetilde{\mathcal{H}}^{D_1,E_1}_{Z,1}-\varkappa_1(Z,D_1,E_1)\big)\phi_{q,k}\Big\|\leqslant q^{-1},$$

and by (6.6), (6.7), (6.10), (6.14), and (6.15)

$$\big\|(\mathcal{H}_{Z,2}-\lambda_1)\varphi_{q,k,l}\big\|=\big\|(\widetilde{\mathcal{H}}^{D_2,E_2}_{Z,2}-\lambda_1)\psi_{q,l}\big\|\leqslant 2q^{-1}. \tag{6.22}$$

In view of (6.21)—(6.22) and (6.2), to prove that $\varphi_{q,k,l}\in\operatorname{Dom}\mathcal{H}_N$ and that (6.19) holds true it is enough to obtain that

$$\bigg\|\bigg(\sum_{n\in Z_2}V_n+\sum_{\substack{n<j\\ n\# j}}U_{nj}\bigg)\varphi_{q,k,l}\bigg\|\leqslant 2q^{-1}. \tag{6.23}$$

To do this, we first note that by (2.12), (2.13), and Cauchy inequality

$$\Big\|\Big(\sum_{n\in Z_2} V_n I_{\{|\mathbf{x}_n|>R_q\}}+\sum_{\substack{n<j\\ n\#j}} U_{nj} I_{\{|\mathbf{x}_n-\mathbf{x}_j|>R_q\}}\Big)\varphi_{q,k,l}\Big\|$$

$$\leqslant \varepsilon(N_1+1)\sum_{n\in Z_2}\big\||D_n|^{\frac12}\psi_{q,l}\big\| \leqslant \varepsilon(N_1+1)N_2^{\frac12}\Big(\sum_{n\in Z_2}\big\||D_n|^{\frac12}\psi_{q,l}\big\|^2\Big)^{\frac12}. \tag{6.24}$$

By (2.9), (6.15), and (6.22),

$$\begin{aligned}\sum_{n\in Z_2}\|D_n^{1/2}\psi_{q,l}\|^2 &\leqslant C_1^{-1}\Big(\big\|(\widetilde{\mathcal{H}}_{Z,2}^{D_2,E_2}-\lambda_1)\psi_{q,l}\big\|+C_2+|\lambda_1|\Big)\\ &\leqslant C_1^{-1}\big(C_2+|\lambda_1|+2q^{-1}\big).\end{aligned} \tag{6.25}$$

Thus by (6.24), (6.25) and (6.8) for $q\geqslant 1$ we obtain

$$\Big\|\Big(\sum_{n\in Z_2} V_n I_{\{|\mathbf{x}_n|>R_q\}}+\sum_{\substack{n<j\\ n\#j}} U_{nj} I_{\{|\mathbf{x}_n-\mathbf{x}_j|>R_q\}}\Big)\varphi_{q,k,l}\Big\| \leqslant q^{-1}. \tag{6.26}$$

Now the scalar functions

$$V_{n,q}(\mathbf{x}) := \big|V_n(\mathbf{x})\big| I_{\{|\mathbf{x}|\leqslant R_q\}}(\mathbf{x}) \quad\text{and}\quad U_{nj,q}(\mathbf{x}) := \big|U_{nj}(\mathbf{x})\big| I_{\{|\mathbf{x}|\leqslant R_q\}}(\mathbf{x}) \tag{6.27}$$

are square integrable by (2.11). By Corollary 19, for $n\in Z_2$

$$\|V_{n,q}\varphi_{q,k,l}\|^2 = \|V_{n,q}\psi_{q,l}\|^2 \leqslant \frac{1}{2\pi^2}\delta_q(\mathfrak{P}_0+\delta_q)^2\|V_{n,q}\|^2_{L_2(\mathbb{R}^3)} \tag{6.28}$$

and for $n<j$, $n\#j$

$$\|U_{nj,q}\varphi_{q,k,l}\|^2 \leqslant \sup_{\mathbf{z}\in\mathbb{R}^3}\big\|U_{nj,q}(\cdot-\mathbf{z})\psi_{q,l}\big\|^2 \leqslant \frac{1}{2\pi^2}\delta_q(\mathfrak{P}_0+\delta_q)^2\|U_{nj,q}\|^2_{L_2(\mathbb{R}^3)}. \tag{6.29}$$

Hence by (6.27), (6.28), (6.29), (2.11), and (6.11)

$$\Big\|\Big(\sum_{n\in Z_2} V_n I_{\{|\mathbf{x}_n|\leqslant R_q\}}+\sum_{\substack{n<j\\ n\#j}} U_{nj} I_{\{|\mathbf{x}_n-\mathbf{x}_j|\leqslant R_q\}}\Big)\varphi_{q,k,l}\Big\| \leqslant q^{-1}. \tag{6.30}$$

It remains to add (6.26) and (6.30) to obtain (6.23), finishing the proof of the lemma. •

The subspace F_q spanned by the functions (6.18) is $D_1\otimes E_1\otimes D_2\otimes E_2$–generating. Since $(D_1,E_1;D_2,E_2)\underset{Z}{\prec}(D,E)$, F_q contains some nontrivial D–generating subspace. Hence the subspace $K_q := P^D F_q$ is not equal to $\{0\}$ and is contained in F_q.

LEMMA 21 *There exists a constant $C_E > 0$ such that for every $q \in \mathbb{N}$*

$$\|P^E\varphi\| \geqslant C_E\|\varphi\|, \quad \textit{for all} \quad \varphi \in F_q. \tag{6.31}$$

PROOF. Projector (2.15) can be written as

$$P^E = \frac{d_E}{h_\Pi} \sum_{\pi \in \Pi_1^Z \times \Pi_2^Z} \overline{\xi_E(\pi)}\pi + \frac{d_E}{h_\Pi} \sum_{\pi \in \Pi \setminus (\Pi_1^Z \times \Pi_2^Z)} \overline{\xi_E(\pi)}\pi. \tag{6.32}$$

We will denote the first term in (6.32) by Q^E, and the second by R^E. Then

$$\|P^E\varphi\|^2 = \langle \varphi, P^E\varphi\rangle = \langle \varphi, Q^E\varphi\rangle + \langle \varphi, R^E\varphi\rangle. \tag{6.33}$$

Relation $(D_1, E_1; D_2, E_2) \underset{Z}{\prec} (D, E)$ implies that the representation $E|\Pi_1^Z \times \Pi_2^Z$ is unitarily equivalent to a sum $\overset{k}{\underset{i=0}{\oplus}} n_i E^{(i)}$, where $n_i > 0$ are multiplicities of the irreducible representations $E^{(i)}$ of the group $\Pi_1^Z \times \Pi_2^Z$ with $E^{(0)} = E_1 \otimes E_2$. For the corresponding characters this gives

$$\xi_E(\pi) = \sum_{i=0}^k n_i \xi^{(i)}(\pi), \quad \text{for all} \quad \pi \in \Pi_1^Z \times \Pi_2^Z.$$

Hence

$$Q^E = \sum_{i=0}^k \nu_i P_i,$$

where $\nu_i > 0$ and P_i is the projector corresponding to the representation $E^{(i)}$. By construction, $P_0\varphi = \varphi$ for any $\varphi \in F_q$, hence $P_i\varphi = 0$ for $i = 1, \ldots, k$. Thus for any $\varphi \in F_q$

$$\langle \varphi, Q^E\varphi\rangle = \nu_0\|\varphi\|^2, \quad \nu_0 > 0. \tag{6.34}$$

We will now estimate the second term on the r. h. s. of (6.33). For any $n \in Z_2$ and any $\psi \in \widetilde{G}_q$ with $\|\psi\| = 1$ by Corollary 19 and (6.12) we have

$$\|I_{\{|\mathbf{x}_j| < R_q\}}\psi\|^2 \leqslant \frac{\nu_0^2}{16 d_E^4 r_1^2 r_2^2}. \tag{6.35}$$

For any functions (6.18) and any $\pi \in \Pi$ inequality (6.9) implies that

$$\big|\langle \varphi_{q,k,l}, \pi\varphi_{q,\widetilde{k},\widetilde{l}}\rangle\big| \leqslant \langle \prod_{j \in Z_1} I_{\{|\mathbf{x}_j| < R_q\}} |\varphi_{q,k,l}|, \pi|\varphi_{q,\widetilde{k},\widetilde{l}}|\rangle_{L_2(\mathbb{R}^{3N})} + \frac{\nu_0}{4 d_E^2 r_1 r_2}.$$

Now if $\pi \in \Pi \setminus (\Pi_1^Z \times \Pi_2^Z)$, then there exists $j_0 \in Z_1$ such that $\pi j_0 \in Z_2$. Hence by (6.35)

$$\langle \prod_{j \in Z_1} I_{\{|\mathbf{x}_j| < R_q\}} |\varphi_{q,k,l}|, \pi|\varphi_{q,\widetilde{k},\widetilde{l}}|\rangle \leqslant \langle |\varphi_{q,k,l}|, I_{\{|\mathbf{x}_{j_0}| < R_q\}} \pi|\varphi_{q,\widetilde{k},\widetilde{l}}|\rangle \leqslant \frac{\nu_0}{4 d_E^2 r_1 r_2}.$$

Thus

$$\big|\langle\varphi_{q,k,l},\pi\varphi_{q,\widetilde{k},\widetilde{l}}\rangle\big| \leqslant \frac{\nu_0}{2d_E^2 r_1 r_2}, \quad \pi\in\Pi\setminus(\Pi_1^Z\times\Pi_2^Z). \tag{6.36}$$

Any $\varphi\in F_q$ can be written as (6.20). By (6.36) and Cauchy inequality for any $\pi\in\Pi\setminus(\Pi_1^Z\times\Pi_2^Z)$

$$\big|\langle\varphi,\pi\varphi\rangle\big| \leqslant \sum_{k,l,\widetilde{k},\widetilde{l}} |c_{kl}||c_{\widetilde{k}\widetilde{l}}|\big|\langle\varphi_{q,k,l},\pi\varphi_{q,\widetilde{k},\widetilde{l}}\rangle\big| \leqslant \frac{\nu_0}{2d_E^2}\|\varphi\|^2. \tag{6.37}$$

Since the number of elements of $\Pi\setminus(\Pi_1^Z\times\Pi_2^Z)$ does not exceed d_Π and for any π $\big|\xi_E(\pi)\big|\leqslant d_E$ as a trace of unitary matrix of dimension d_E, (6.37) implies that

$$\big|\langle\varphi,R^E\varphi\rangle\big|\leqslant\nu_0\|\varphi\|^2/2.$$

By (6.33) and (6.34) we conclude that (6.31) holds with $C_E=\sqrt{\nu_0/2}$. •
Lemmata 20 and 21 imply that $L_q := P^E K_q$ is a nontrivial subspace of $\operatorname{Dom}\mathcal{H}_N^{D,E}$ and for every $f=P^E\varphi\in L_q$

$$\big\|(\mathcal{H}_N^{D,E}-\lambda)f\big\|\leqslant\big\|(\mathcal{H}_N-\lambda)\varphi\big\|\leqslant 5q^{-1}r_1^{\frac12}r_2^{\frac12}\|\varphi\|\leqslant 5q^{-1}r_1^{\frac12}r_2^{\frac12}C_E^{-1}\|f\|, \quad q\in\mathbb{N}.$$

This implies that $\lambda\in\sigma(\mathcal{H}_N^{D,E})$, and thus finishes the proof of Theorem 6.

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Motivic Splitting Lemma

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Abstract. Let M be a Chow motive over a field F. Let X be a smooth projective variety over F and N be a direct summand of the motive of X. Assume that over the generic point of X the motives M and N become isomorphic to a direct sum of twisted Tate motives. The main result of the paper says that if a morphism $f : M \to N$ splits over the generic point of X then it splits over F, i.e., N is a direct summand of M. We apply this result to various examples of motives of projective homogeneous varieties.

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1 Introduction

By a variety X over a field F we always mean a reduced and irreducible scheme of finite type over F. By $F(X)$ we denote the function field of X.

Definition 1.1. Let M be a Chow motive over F. We say M is *split* over F if it is a direct sum of twisted Tate motives over F. We say a motive M is *generically split* if there exists a smooth projective variety X over F and an integer l such that M is a direct summand of the twisted motive $M(X)\{l\}$ of X and M is split over $F(X)$. In particular, a smooth projective variety X is called *generically split* if its Chow motive $M(X)$ is split over $F(X)$.

The classical examples of such varieties are Severi-Brauer varieties, Pfister quadrics and maximal orthogonal Grassmannians. In the present paper we provide useful technical tool to study motivic decompositions of generically split varieties (motives). Namely, we prove the following

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Theorem 1.2. *Let M be a Chow motive over a field F. Let X be a smooth projective variety over F and N be a direct summand of the motive of X. Assume that M and N are split over $F(X)$. Then a morphism $M \to N$ splits, i.e. N is a direct summand of M, if it splits over $F(X)$.*

The paper is organized as follows. In section 2 we introduce the category of Chow motives over a relative base. In section 3 we provide the version of the Rost nilpotence theorem for generically split varieties. In section 4 we prove the main result of this paper (see the above theorem). The last section is devoted to various applications and examples.

2 Chow motives over a relative base

Let X be a variety over a field F. We say X is *essentially smooth* over F if it is an inverse limit of smooth varieties X_i over F taken with respect to open embeddings. Let $\mathrm{CH}^m(X;\Lambda) = \mathrm{CH}^m(X) \otimes_{\mathbb{Z}} \Lambda$ denote the Chow group of codimension m cycles on X with coefficients in a commutative ring Λ. If X is essentially smooth, then $\mathrm{CH}^m(X;\Lambda) = \varinjlim \mathrm{CH}^m(X_i;\Lambda)$, where the limit is taken with respect to the pull-backs induced by open embeddings.
In the present section we introduce the category of *Chow motives* over an essentially smooth variety X with Λ-coefficients. Our arguments follow the paper [9].

I. First, we define the *category of correspondences* $\mathcal{C}(X;\Lambda)$. The objects of $\mathcal{C}(X;\Lambda)$ are smooth projective maps $Y \to X$. The morphisms are given by

$$\mathrm{Hom}([Y \to X],[Z \to X]) = \oplus_i \mathrm{CH}^{\dim(Z_i/X)}(Y \times_X Z_i;\Lambda),$$

where the sum is taken over all irreducible components Z_i of Z of relative dimensions $\dim(Z_i/X)$. The composition of two morphisms is given by the usual correspondence product

$$\psi \circ \phi = (p_{Y,T})_*\big((p_{Y,Z})^*(\phi) \cdot (p_{Z,T})^*(\psi)\big),$$

where $\phi \in \mathrm{Hom}([Y \to X],[Z \to X])$, $\psi \in \mathrm{Hom}([Z \to X],[T \to X])$ and $p_{Y,T}$, $p_{Y,Z}$, $p_{Z,T}$ are projections $Y \times_X Z \times_X T \to Y \times_X T$, $Y \times_X Z$, $Z \times_X T$. The category $\mathcal{C}(X;\Lambda)$ is a tensor additive category, where the direct sum is given by $[Y \to X] \oplus [Z \to X] := [Y \coprod Z \to X]$ and the tensor product by $[Y \to X] \otimes [Z \to X] := [Y \times_X Z \to X]$ (cf. [9, §2-4]). As usual we denote by $\phi^t \in \mathrm{CH}(Z \times_X Y)$ the transposition of a cycle $\phi \in \mathrm{CH}(Y \times_X Z)$.
The category of *effective* Chow motives $Chow^{\mathrm{eff}}(X;\Lambda)$ can be defined as the pseudo-abelian completion of $\mathcal{C}(X;\Lambda)$. Namely, the objects are pairs (U,ρ), where U is an object of $\mathcal{C}(X;\Lambda)$ and $\rho \in \mathrm{End}_{\mathcal{C}(X;\Lambda)}(U)$ is a projector, i.e. $\rho \circ \rho = \rho$. The morphisms between (U_1,ρ_1) and (U_2,ρ_2) are given by the group $\rho_2 \circ \mathrm{Hom}_{\mathcal{C}(X;\Lambda)}(U_1,U_2) \circ \rho_1$. The composition of morphisms is induced by the correspondence product. In the case $X = \mathrm{Spec}(F)$ and $\Lambda = \mathbb{Z}$ we obtain

the usual category $Chow^{\mathrm{eff}}(F)$ of effective Chow motives over F with integral coefficients (cf. [9, §5]).
Consider the projective line $\mathbb{P}^1$ over F. The projector $\rho = [\mathrm{Spec}(F) \times \mathbb{P}^1] \in \mathrm{CH}^1(\mathbb{P}^1 \times \mathbb{P}^1)$ defines an object $(\mathbb{P}^1, \rho)$ in $Chow^{\mathrm{eff}}(F)$ called the *Tate motive* over F and denoted by $\mathbb{Z}\{1\}$ (cf. [9, §6]).

II. We have two types of restriction functors.

1) For any morphism $f\colon X_1 \to X_2$ of essentially smooth varieties we have a tensor additive functor

$$res_{X_2/X_1}\colon\ \mathcal{C}(X_2;\Lambda) \to \mathcal{C}(X_1;\Lambda)$$

given on the objects by $[Y_2 \to X_2] \mapsto [Y_2 \times_{X_2} X_1 \to X_1]$ and on the morphisms by $\phi \mapsto (\mathrm{id} \times f)^*(\phi)$, where $\mathrm{id} \times f : (Y_2 \times_{X_2} Z_2) \times_{X_2} X_1 \to Y_2 \times_{X_2} Z_2$ is the natural map. It induces a functor on pseudo-abelian completions

$$res_{X_2/X_1}\colon\ Chow^{\mathrm{eff}}(X_2;\Lambda) \to Chow^{\mathrm{eff}}(X_1;\Lambda).$$

2) For any homomorphism of commutative rings $h\colon \Lambda \to \Lambda'$ we have a tensor additive functor

$$res_{\Lambda'/\Lambda}\colon\ \mathcal{C}(X,\Lambda) \to \mathcal{C}(X;\Lambda')$$

which is identical on objects and is given by $\mathrm{id} \otimes h\colon\ \mathrm{CH}(Y \times_X Z;\Lambda) \to \mathrm{CH}(Y \times_X Z;\Lambda')$ on morphisms. Again, it induces a functor on pseudo-abelian completions

$$res_{\Lambda'/\Lambda}\colon\ Chow^{\mathrm{eff}}(X;\Lambda) \to Chow^{\mathrm{eff}}(X;\Lambda').$$

Observe that the functor $res_{\Lambda'/\Lambda}$ commutes with res_{X_2/X_1}. We denote by $res_{X_2/X_1,\Lambda'/\Lambda}$ the composite $res_{X_2/X_1} \circ res_{\Lambda'/\Lambda}$. To simplify the notation we omit X_2 (resp. Λ), if $X_2 = \mathrm{Spec}\, F$ (resp. $\Lambda = \mathbb{Z}$).
Let $f\colon X \to \mathrm{Spec}\, F$ and $h\colon \mathbb{Z} \to \Lambda$ be the structure maps. Then $res_{X,\Lambda}\colon\ Chow^{\mathrm{eff}}(F) \to Chow^{\mathrm{eff}}(X;\Lambda)$. Given a motive N over F we denote by $N_{X,\Lambda}$ its image $res_{X,\Lambda}(N)$ in $Chow^{\mathrm{eff}}(X;\Lambda)$. The image $\mathbb{Z}\{1\}_{X,\Lambda}$ of the Tate motive is denoted by T and is called the *Tate motive over* X. Let M be a motive from $Chow^{\mathrm{eff}}(X;\Lambda)$ and $l \geq 0$ be an integer. The tensor product $M \otimes T^{\otimes l}$ is denoted by $M\{l\}$ and is called the *twist* of M. The *trivial Tate motive* $T^{\otimes 0}$ will be denoted Λ (thus, $T^{\otimes l} = \Lambda\{l\}$).
The same arguments as in the proof of [9, Lemma of §8] show that for any motives U and V from $Chow^{\mathrm{eff}}(X;\Lambda)$ and $l \geq 0$ the natural map

$$\mathrm{Hom}_{Chow^{\mathrm{eff}}(X;\Lambda)}(U,V) \to \mathrm{Hom}_{Chow^{\mathrm{eff}}(X;\Lambda)}(U\{l\},V\{l\}) \tag{1}$$

given by $\phi \mapsto \phi \otimes \mathrm{id}_T$ is an isomorphism.

III. We define the category $Chow(X;\Lambda)$ of Chow motives over X with Λ-coefficients as follows. The objects are pairs (U,l), where U is an object of $Chow^{\mathrm{eff}}(X;\Lambda)$ and l is an integer. The morphisms are given by

$$\mathrm{Hom}((U,l),(V,m)) := \lim_{N\to+\infty} \mathrm{Hom}_{Chow^{eff}(X;A)}(U\{N+l\},V\{N+m\}).$$

This is again a tensor additive category, where the sum and the product are given by

$$(U,l)\oplus(V,m) := (U\{l-n\}\oplus V\{m-n\},n),\ \text{where } n=\min(l,m),$$

$$(U,l)\otimes(V,m) := (U\otimes V, l+m).$$

Observe that the Tate motive T is isomorphic to $([\mathrm{id}: X\to X],1)$ and, hence, it is invertible in $(Chow(X;\Lambda),\otimes)$. Moreover, we can say that $Chow(X;\Lambda)$ is obtained from $Chow^{\mathrm{eff}}(X;\Lambda)$ by inverting T (cf. [9, §8]).
According to (1) the natural functor $Chow^{\mathrm{eff}}(X;\Lambda)\to Chow(X;\Lambda)$ given by $U\mapsto (U,0)$ is fully faithful and the restriction $res_{X,\Lambda}$ descend to the respective functor $res_{X,\Lambda}\colon Chow(F)\to Chow(X;\Lambda)$.
For a smooth projective morphism $Y\to X$ we denote by $M(Y\to X)$ its effective motive $([Y\to X],\mathrm{id})$ considered as an object of $Chow(X;\Lambda)$. If $X=\mathrm{Spec}\,F$ and $\Lambda=\mathbb{Z}$, then we denote the motive $M(Y\to X)$ simply by $M(Y)$. By definition there is a natural identification

$$\mathrm{Hom}_{Chow(X;\Lambda)}(M(Y\to X)\{i\},M(Z\to X)\{j\}) = \mathrm{CH}^{\dim(Z/X)+j-i}(Y\times_X Z;\Lambda).$$

IV. Let M be an object of $Chow(X;\Lambda)$. We define the Chow group with low index $\mathrm{CH}_m(M;\Lambda)$ of M as

$$\mathrm{CH}_m(M;\Lambda) := \mathrm{Hom}_{Chow(X;\Lambda)}(\Lambda\{m\},M)$$

and the Chow group with upper index $\mathrm{CH}^m(M;\Lambda)$ as

$$\mathrm{CH}^m(M;\Lambda) := \mathrm{Hom}_{Chow(X;\Lambda)}(M,\Lambda\{m\}).$$

Observe that if $M=M(Y\to X)$, then we obtain the usual Chow groups $\mathrm{CH}^{\dim(Y/X)-m}(Y;\Lambda)$ and $\mathrm{CH}^m(Y;\Lambda)$ of a variety Y. A composite with a morphism $f: M\to N$ induces a homomorphism between the Chow groups $R_m(f)\colon \mathrm{CH}_m(M)\to\mathrm{CH}_m(N)$ and $R^m(f)\colon \mathrm{CH}^m(N)\to\mathrm{CH}^m(M)$ called the *realization map.*

3 The Rost nilpotence

We will extensively use the following version of the Rost nilpotence (cf. [14, Proposition 9])

Proposition 3.1. *Let N be a generically split motive over a field F. Then for any field extension E/F and any coefficient ring Λ the kernel of the restriction*

$$res_{E/F} : \mathrm{End}_F(N) \to \mathrm{End}_E(N_E)$$

consists of nilpotents.

To simplify the notation we denote by $\mathrm{End}_X(M)$ the endomorphism group $\mathrm{Hom}_{Chow(X;\Lambda)}(M, M)$, where M is a motive over a variety X.

Proof. Recall that (see Definition 1.1) a motive N over F is generically split if there exists a smooth projective variety X and $l \in \mathbb{Z}$ such that N is a direct summand of $M(X)\{l\}$ and $N_K = res_{K/F}(N)$ is split, where $K = F(X)$ denotes the function field of X.
We may assume that N is a direct summand of $M(X)$ (that is, $l = 0$). Since for a split motive M and a field extension E/L, the map $\mathrm{End}_L(M_L) \to \mathrm{End}_E(M_E)$ is an isomorphism, we may assume that $E = K$.
Consider the composite of ring homomorphisms

$$res_{K/F} \colon \mathrm{End}_F(N) \xrightarrow{res_{X/F}} \mathrm{End}_X(N_X) \xrightarrow{res_{K/X}} \mathrm{End}_K(N_K),$$

where the last map is induced by passing to the generic point $\mathrm{Spec}\, K \to X$.
Observe that $\mathrm{End}_K(N_K) = \varinjlim \mathrm{End}_U(N_U)$, where the limit is taken over all open subvarieties $U \subset X$. Then $\ker(res_{K/X}) = \cup_U \ker(res_{U/X})$ and by Lemma 3.2 the kernel of $res_{K/X}$ consists of nilpotents.
On the other hand, the map $res_{X/F}$ is injective. Indeed, since N is a direct summand of $M(X)$, $\mathrm{End}_F(N)$ is a subring of $\mathrm{End}_F(M(X))$ and $\mathrm{End}_X(N_X)$ is a subring of $\mathrm{End}_X(M(X)_X)$. So, it is sufficient to prove the injectivity for the case $N = M(X)$. The restriction $res_{X/F} \colon \mathrm{End}_F(M(X)) \to \mathrm{End}_X(M(X)_X)$ coincides with the pull-back $\pi_{1,2}^* \colon \mathrm{CH}(X \times X; \Lambda) \to \mathrm{CH}(X \times X \times X; \Lambda)$ induced by the projection on the first two coordinates. And $\pi_{1,2}^*$ splits by $(\mathrm{id}_X \times \Delta_X)^* \colon \mathrm{CH}(X \times X \times X; \Lambda) \to \mathrm{CH}(X \times X; \Lambda)$, where $\Delta_X \colon X \to X \times X$ is the diagonal. The proposition is proven. □

Lemma 3.2. *Let X be a smooth projective variety over F and Λ be a commutative ring. Let $U \subset X$ be an open embedding. Then for any motive M from $Chow(X;\Lambda)$ the kernel of the restriction map*

$$res_{U/X} : \mathrm{End}_X(M) \to \mathrm{End}_U(M_U)$$

consists of nilpotents.

Proof. If M is a direct summand of $[Y \to X]\{i\}$, then $\mathrm{End}_X(M)$ is a subring of $\mathrm{End}_X(M(Y \to X))$ and it is sufficient to study the case $M = M(Y \to X)$.
Recall that $\mathrm{End}_X(M(Y \to X)) = \mathrm{CH}^{\dim(Y)-\dim(X)}(Y \times_X Y; \Lambda)$.
Let ϕ be an element from the kernel of $res_{U/X}$. Let $j \colon Z \to X$ be the reduced closed complement to U in X. Then by the localization sequence for Chow groups the cycle ϕ belongs to the image of the induced push-forward

$$(\mathrm{id}_{(Y\times_X Y)} \times j)_* \colon \mathrm{CH}((Y \times_X Y) \times_X Z; \Lambda) \to \mathrm{CH}(Y \times_X Y; \Lambda).$$

Let $\operatorname{codim}(Z)$ be the minimum of codimensions of irreducible components of Z, and $d := [\frac{\dim(X)}{\operatorname{codim}(Z)}]+1$. We claim that the d-th power $\phi^{\circ d}$ of ϕ taken with respect to the correspondence product is trivial. Indeed, $\phi^{\circ d} = (\pi_{1,d+1})_*(\phi_1 \cdot \phi_2 \cdot \ldots \cdot \phi_d)$, where $\phi_i = \pi^*_{i,i+1}(\phi)$ and the map $\pi_{i,i'} \colon Y^{\times(d+1)} \to Y \times_X Y$ is the projection on the i-th and i'-th components. Since $\pi^*_{i,i'} \circ (\mathrm{id}_{(Y\times_X Y)} \times j)_*$ coincides with $(\mathrm{id}_{Y^{\times(d+1)}} \times j)_* \circ (\pi_{i,i'} \times \mathrm{id}_Z)^*$, all cycles ϕ_i belong to the image of the push-forward

$$(\mathrm{id}_{Y^{\times(d+1)}} \times j)_* \colon \mathrm{CH}(Y^{\times(d+1)} \times_X Z) \to \mathrm{CH}(Y^{\times(d+1)}).$$

By Proposition 6.1 applied to the projection $Y^{\times(d+1)} \to X$ and the closed embedding $j \colon Z \hookrightarrow X$ we obtain that the product

$$\phi_1 \cdot \ldots \cdot \phi_d \in \big((\mathrm{id}_{Y^{\times(d+1)}} \times j)_* \,\mathrm{CH}(Y^{\times(d+1)} \times_X Z)\big)^d$$

is trivial. Therefore, $\phi^{\circ d}$ is trivial as well. □

We finish this section with the following

Definition 3.3. Given motive M over a field F and a field extension L/F we say a cycle in $\mathrm{CH}(M_L)$ is *rational* if it is in the image of the restriction map $res_{L/F}$.

Observe that the rationality of cycles is preserved by push-forward and pull-back maps. It also respects addition, intersection and correspondence product of cycles.

4 Motivic splitting lemma

In the present section we prove the main result of this paper

Theorem 4.1. *Let M be a Chow motive over a field F. Let X be a smooth projective variety over F and N be a direct summand of the motive of X. Assume that M and N are split over the function field $K = F(X)$. Then a morphism $f \colon M \to N$ splits, i.e. N is a direct summand of M, if it splits over K.*

Proof. To construct a section of f we apply recursively the following procedure starting from $g = 0$ and such m that $\mathrm{CH}^i(N_K) = 0$, for $i < m$.

> For a morphism $g : N \to M$ such that the realization morphism $R^i(f_K \circ g_K)$ is the identity on $\mathrm{CH}^i(N_K)$ for $i < m$, we construct a new morphism $g' : N \to M$ such that $R^i(f_K \circ g'_K)$ is the identity on $\mathrm{CH}^i(N_K)$ for $i \le m$.

Since the motive N_K splits, for the corresponding projector ρ_N over K we may write $(\rho_N)_K = \sum_l \omega_l \times \omega_l^\vee$ for certain $\omega_l \in \mathrm{CH}^*(X_K)$ and $\omega_l^\vee \in \mathrm{CH}_*(X_K)$ such that $\deg(\omega_l \cdot \omega_m^\vee) = \delta_{l,m}$. Elements ω_l form a basis of $\mathrm{CH}^*(N_K) = (\rho_N)_K \circ \mathrm{CH}^*(X_K) \subset \mathrm{CH}^*(X_K)$.

Consider the surjection $\mathrm{CH}^m(X \times X) \twoheadrightarrow \mathrm{CH}^m(K \times_F X) = \mathrm{CH}^m(X_K)$. Let Ω_l be a preimage of an element ω_l of $\mathrm{CH}^m(X_K)$.

Consider the difference $\mathrm{id} - f \circ g$ and denote it by h. Assume that over K it sends a basis element ω_j to a cycle α_j. Since $R^i(h_K)$ is trivial for all $i < m$, the cycle $h_K = h_K \circ (\rho_N)_K$ can be written as

$$h_K = \sum_{\operatorname{codim} \alpha_l = m} \alpha_l \times \omega_l^\vee + \sum_{\operatorname{codim} \alpha_j > m} \alpha_j \times \omega_j^\vee \in \mathrm{CH}^{\dim X}(X_K \times X_K). \quad (2)$$

From (2) we immediately see that

$$\alpha_l = \mathrm{pr}_{1*}(\Omega_{l,K} \cdot h_K) \in \mathrm{CH}^m(X_K) \text{ is rational.} \quad (3)$$

Also, $\alpha_l \circ (\rho_N)_K = \alpha_l$.

The realization $R^m(f_K)$ is a $\mathbb{Z}$-linear map $\mathrm{CH}^m(N_K) \to \mathrm{CH}^m(M_K)$. Let $C = (c_{ij})$ be the respective matrix of coefficients, i.e.,

$$R^m(f_K) : \omega_i \mapsto \sum_j c_{ji}\theta_j,$$

where $\{\theta_i\}$ is a $\mathbb{Z}$-basis of $\mathrm{CH}^m(M_K)$. Let $s : N_K \to M_K$ be a section of f_K. The realization map $R^m(s)$ is a left inverse to $R^m(f_K)$. Hence, for the respective matrix of coefficients $D = (d_{ij})$ we have

$$R^m(s) : \theta_i \mapsto \sum_j d_{ji}\omega_j$$

and $D \cdot C = \mathrm{id}$, i.e., $\sum_j d_{ij}c_{jk} = \delta_{ik}$. For each α_l define the morphism $u_l : N \to M$ as

$$u_l = \sum_i d_{li}\Theta_i^\vee \circ (\mathrm{pr}_1^*(\alpha_l) \cdot \Delta_X) \circ p_N,$$

where $\Theta_i^\vee$ is a preimage of an element $\theta_i^\vee$ of $\mathrm{CH}_m(M_K)$ by means of the canonical surjection $\mathrm{Hom}_F(M(X)(m)[2m], M) \to \mathrm{CH}_m(M_K)$ and $p_N : N \to M(X)$ be the morphism presenting N as a direct summand of $M(X)$. By definition, u_l is a rational morphism and the realization $R^m(u_l)$ is given by

$$\theta_i \mapsto d_{li}\alpha_l$$

Hence, the composite $R^m(f \circ u_l) = R^m(u_l) \circ R^m(f)$ maps ω_i to $\delta_{il}\alpha_l$.

Set $\tilde{g} = g + \sum_l u_l$. By construction, the realization $R(f \circ \tilde{g})$ is the identity on $\mathrm{CH}^i(N_K)$ for $i \leq m$. Consider the endomorphism $\mathrm{id} - f \circ \tilde{g}$ of N. Over K its realization $R^i(\mathrm{id} - f \circ \tilde{g})$ is trivial for each $i \leq m$.

Recursion step is proven and we obtain map $g' : N \to M$ such that $(f \circ g')_K = id_{N_K}$. Let $q = id - f \circ g'$. By the Proposition 3.1, $q^r = 0$, for some r. Set $g = g' \circ (id + q + q^{\circ 2} + \ldots + q^{\circ(r-1)})$. Then $f \circ g = id_N$ and N is a direct summand of M. □

5 Examples and Applications

Geometric construction of a generalized Rost motive. Let p be a prime and F be a field of characteristic different from p. Let n be a positive integer. To each nonzero cyclic subgroup $\langle\alpha\rangle$ in $K_n^M(F)/p$ consisting of pure symbols one can assign some motive M_α in the category $Chow(F;\mathbb{Z}/p\mathbb{Z})$, which satisfies the following property
For an arbitrary field extension E/F

$\alpha|_E \neq 0 \iff (M_\alpha)_E = res_{E/F}(M_\alpha)$ is indecomposable;
$\alpha|_E = 0 \iff (M_\alpha)_E$ is split.

It follows from the results of V. Voevodsky and M. Rost that for a given subgroup such motive always exists and is unique (see [17, § 5] and [15, Prop. 5.9]). Moreover, when split it is isomorphic to

$$\bigoplus_{i=0}^{p-1}\mathbb{Z}/p\mathbb{Z}\{i\cdot \tfrac{p^{n-1}-1}{p-1}\}.$$

Such a motive is called a *generalized Rost motive with $\mathbb{Z}/p\mathbb{Z}$-coefficients.*

Definition 5.1. A motive with integral coefficients which specializes modulo p into a generalized Rost motive and splits modulo q for every prime q different from p will be called an *integral generalized Rost motive* and denoted by $\mathcal{R}_{n,p}$.

Integral generalized Rost motives, hypothetically, should be parameterized not by the pure cyclic subgroups of $\mathrm{K}_n^M(F)/p$, but by the pure symbols of $\mathrm{K}_n^M(F)/p$ up to a sign. The existence of integral generalized Rost motives is known for $n=2$ and arbitrary p, for $p=2$ and arbitrary n, and for the pair $n=3, p=3$. All these examples are essentially due to M. Rost.
As the first application of Theorem 4.1 we obtain the construction of the classical integral Rost motive corresponding to a Pfister form.

Corollary 5.2. *(cf. [14, Theorem 17.(9) and Proposition 19]) Let X be a hyperplane section of a n-fold Pfister quadric Y over a field F. Then $M(Y)\simeq M(X)\{1\}\oplus\mathcal{R}_{n,2}$, where $\mathcal{R}_{n,2}$ is an integral Rost motive.*

Proof. In the proof we use several auxiliary facts concerning quadrics and their motives which can be found in [5].
Let ϕ_X and ϕ_Y be the quadratic forms which define X and Y. By definition ϕ_X is a subform of codimension 1 of the Pfister form ϕ_Y. According to [5, Def.5.1.2 and Thm.5.3.4.(a)] Y becomes isotropic over $K=F(X)$. This fact together with [5, Prop.4.2.1] implies that both ϕ_X and ϕ_Y become totally split (hyperbolic) over K. Then by [5, E.10.8] the motives $M(X)_K$ and $M(Y)_K$ are split over K.
Let Γ_e be the graph of the closed embedding $e\colon X\hookrightarrow Y$. The respective correspondence cycle $[\Gamma_e]\in \mathrm{CH}_{\dim X}(X\times Y)$ induces the realization map $R^*(\Gamma_e)$

which coincides with the pull-back $e^*\colon \mathrm{CH}^*(Y) \to \mathrm{CH}^*(X)$ (see §2.IV). It is known that the Chow ring of a hyperbolic quadric is generated by two elements $\langle h, l\rangle$, where h is the class of a hyperplane section and l is the class of a maximal totally isotropic subspace. In this notation the pull-back e_K^* maps $h_Y \mapsto h_X$ and $l_Y \mapsto l_X$, i.e. maps the ring $\mathrm{CH}(Y_K)$ *onto* the ring $\mathrm{CH}(X_K)$. The latter means that $R^*(\Gamma_e)$ and, therefore, the transposed correspondence cycle $[\Gamma_e]^t \in \mathrm{CH}_{\dim Y-1}(Y \times X)$ have a section over K.
Take $f = [\Gamma_e]^t\colon M(Y) \to M(X)\{1\}$ and apply Theorem 1.2. We obtain the decomposition $M(Y) = M(X)\{1\} \oplus N$, where N is such that

$$N_K = \mathbb{Z} \oplus \mathbb{Z}\{2^{n-1} - 1\}.$$

Let E/F be a field extension. The Pfister quadric Y corresponds to some pure symbol $\alpha \in \mathrm{K}_n^M(F)/2$ (see [5, §9.4]) with the property that $\alpha|_E = 0$ if and only if Y_E has a rational point. Consider the specialization $N_{E,\mathbb{Z}/2}$ with $\mathbb{Z}/2$-coefficients. We have the following chain of equivalences: $N_{E,\mathbb{Z}/2}$ is decomposable $\Leftrightarrow$ $N_{E,\mathbb{Z}/2}$ contains $\mathbb{Z}/2$ as a direct summand $\Leftrightarrow$ $M(Y;\mathbb{Z}/2)_E$ contains $\mathbb{Z}/2$ as a direct summand $\Leftrightarrow$ (see [14, §1.4]) Y_E has a zero-cycle of odd degree $\Leftrightarrow$ (Springer Theorem) Y_E has a rational point. At the same time, the specialization $N_{\mathbb{Z}/p}$ is split for any odd prime p, since $M(Y;\mathbb{Z}/p)$ is split. Hence, N is an integral generalized Rost motive corresponding to the symbol α. ☐

To provide the next application we use several auxiliary facts concerning Albert algebras and Cayley planes which can be found in [4], [8], [11], [12]. We use the notation of [12, §3].
Consider an Albert algebra J defined by means of the first Tits construction. Let $\mathrm{F}_4(J)$ and $\mathrm{E}_6(J)$ denote the respective simple groups of types F_4 and E_6. Let X be the variety of maximal parabolic subgroups of $\mathrm{F}_4(J)$ of type P_4. Let Y be the variety of maximal parabolic subgroups of $\mathrm{E}_6(J)$ of type P_1 Here P_i corresponds to a standard parabolic subgroup generated by the Borel subgroup and all unipotent subgroups corresponding to linear spans of all simple roots with no i-th terms (our enumeration of roots follows Bourbaki). The variety Y is called a *(twisted) Cayley plane*.
Observe that there is a closed embedding $e\colon X \hookrightarrow Y$ such that over the splitting field K of J the class $[X_K] \in \operatorname{Pic} Y_K$ generates the Picard group of Y_K. In other words, X_K is a hyperplane section of Y_K (see [8, 6.3]).

Corollary 5.3. *Let X and Y be as above. Then $M(Y) \simeq M(X)\{1\} \oplus \mathcal{R}_{3,3}$, where $\mathcal{R}_{3,3}$ is an integral generalized Rost motive corresponding to the Serre-Rost invariant $g_3(J)$ in $K_3^M(F)/3$.*

Proof. We follow the previous proof step by step.
Let K denote the function field of X. Analyzing the Tits indices of $\mathrm{F}_4(J)$ we conclude that J becomes reduced over K. Moreover, since J is defined by means of the *first* Tits construction, J becomes split over K. By definition it implies that both groups and varieties become split over K.

Consider now the graph Γ_e of the closed embedding $e\colon X \hookrightarrow Y$. As before, the respective correspondence cycle $[\Gamma_e]$ induces the realization map $R^*(\Gamma_e)$ which coincides with the pull-back e^*. The Chow rings $\mathrm{CH}(X_K)$ and $\mathrm{CH}(Y_K)$ are generated by $\langle h, g_1^4\rangle$ (see [10, 4.10]) and $\langle H, \sigma_4', \sigma_8\rangle$ (see [4, 5.1]). By the Lefschetz hyperplane theorem the pull-back e^* has to be an isomorphisms on all graded components of codimensions ≤ 7. This immediately implies that e^* maps $H \mapsto h$ and $\sigma_4' \mapsto g_1^4$, i.e. maps the ring $\mathrm{CH}(Y_K)$ *onto* the ring $\mathrm{CH}(X_K)$. So $R^*(\Gamma_e)$ and, therefore, the transposed cycle $[\Gamma_e]^t$ have a section over K. Take $f = [\Gamma_e]^t\colon M(Y) \to M(X)\{1\}$ and apply Theorem 1.2. We obtain the decomposition $M(Y) = M(X)\{1\} \oplus N$, where the motive N is such that

$$N_K = \mathbb{Z} \oplus \mathbb{Z}\{4\} \oplus \mathbb{Z}\{8\}.$$

Let E/F be a field extension. Let $\alpha = g_3(J) \in K_3^M(F)/3$ be the Serre-Rost invariant of the Jordan algebra J (see [13]). Analyzing the Tits indices of $\mathrm{E}_6(J)$ we see that $\alpha|_E = 0$ if and only if Y_E has a zero-cycle of degree coprime to 3. Consider the specialization $N_{E,\mathbb{Z}/3}$ with $\mathbb{Z}/3$-coefficients. Similar to the quadric case there is a chain of equivalences which says that $N_{E,\mathbb{Z}/3}$ is decomposable $\Leftrightarrow$ Y_E has a zero-cycle of degree coprime to 3. At the same time, the specialization $N_{\mathbb{Z}/p}$ is split for any prime $p \neq 3$, since $M(Y;\mathbb{Z}/p)$ is split. Therefore, N is an integral generalized Rost motive corresponding to the symbol α. □

Remark 1. Observe that in view of the main result of [10] we obtain the following decomposition

$$M(Y) \simeq \bigoplus_{i=0}^{8} \mathcal{R}_{3,3}\{i\}.$$

So from the motivic point of view the variety Y is a 3-analog of a Pfister quadric.

Projective homogeneous varieties of type F_4. As before let J be an Albert algebra defined by means of the first Tits construction. Let $\mathrm{F}_4(J)$ be the respective group of type F_4. Let X be the same as before, i.e. the variety of maximal parabolic subgroups of type P_4 of $\mathrm{F}_4(J)$. Let Y be the variety of maximal parabolic subgroups of type P_3 of $\mathrm{F}_4(J)$. Observe that Y has dimension 20.

Corollary 5.4. *Let X and Y be as above. Then the motive $M(X;\mathbb{Z})$ is isomorphic to a direct summand of the motive $M(Y;\mathbb{Z})$.*

Proof. Since the Albert algebra J splits over the function field K of X, the motives $M(X)$ and $M(Y)$ become split over K as well. By the main result of [10] $M(X)$ splits as

$$M(X) \simeq \bigoplus_{i=0}^{7} \mathcal{R}_{3,3}\{i\},$$

where $\mathcal{R}_{3,3}$ is the integral generalized Rost motive corresponding to $g_3(J)$.
Let Z be the variety of parabolic subgroups of type $P_{3,4}$ of $\mathrm{F}_4(J)$. Observe that Z has dimension 21 and there is a map $\mathrm{pr}_{XY} = (\mathrm{pr}_X, \mathrm{pr}_X)\colon Z \to X \times Y$, where pr_X, pr_Y are the quotient maps. For each $i = 0 \dots 7$ consider the composite

$$f_{\alpha_i} : M(Y) \xrightarrow{\mathrm{pr}_{XY*}(\alpha_i)} M(X) \to \mathcal{R}_{3,3}\{i\}, \text{ where } \alpha_i \in \operatorname{Pic} Z.$$

Set $f = \bigoplus_{i=0}^{7} f_{\alpha_i}\colon M(Y) \to M(X)$. Assume that we can choose $\alpha_i \in \operatorname{Pic} Z$ in such a way that the realization map $R^*(f)$ becomes split injective over K. Then by Theorem 1.2 applied f, the motive $M(X)$ is isomorphic to a direct summand of $M(Y)$.
So to prove the corollary it is enough to find $\alpha_i \in \operatorname{Pic} Z$, $i = 0 \dots 7$, such that $R^*(f)$ is split injective over K.
Observe that the restriction map $res_{K/F}\colon \operatorname{Pic} Z \to \operatorname{Pic} Z_K$ is an isomorphism (see [10, Lemma 4.3]). Therefore, we may assume that $\alpha_i \in \operatorname{Pic} Z_K$. Observe also that the ring structures of $\mathrm{CH}(X_K)$, $\mathrm{CH}(Y_K)$ and $\mathrm{CH}(Z_K)$ are known. We have $R^*(f_{\alpha_i})_K = R^*(\alpha_i)_K \circ R^*(\rho_i)_K$, where ρ_i is an idempotent defining $\mathcal{R}_{3,3}\{i\}$. Both realizations $R^*(\rho_i)_K$ and $R^*(\alpha_i)_K$ can be described explicitly on generators. Indeed, the realization $R^*(\alpha_i)_K$ is given by the composite $\mathrm{CH}(X_K) \xrightarrow{\mathrm{pr}_X^*} \mathrm{CH}(Z_K) \xrightarrow{\cdot\alpha_i} \mathrm{CH}(Z_K) \xrightarrow{\mathrm{pr}_{Y*}} \mathrm{CH}(Y_K)$, where the maps pr_X^* and pr_{Y*} can be described using [10, §3]. The explicit description of the cycles $(\rho_i)_K$ is provided in [10, 5.5].
Let $\{\alpha_i = c_{1i}g_1 + c_{2i}g_2\}_{i=0\dots7}$, $c_{1i}, c_{2i} \in \mathbb{Z}$, be the presentation of the cycles α_i in terms of a fixed $\mathbb{Z}$-basis $\langle g_1, g_2\rangle$ of $\operatorname{Pic} Z_K$. Since all realization maps $R^*(\alpha_i)_K$, $R^*(\rho_i)_K$ are $\mathbb{Z}$-linear, the question of split injectivity of $R^*(f)_K$ translates into the problem of solving certain system of $\mathbb{Z}$-linear equations in 16 variables $\{c_{1i}, c_{2i}\}_{i=0\dots7}$. Direct computations show that this system has a solution. This finishes the proof of the corollary. □

Twisted forms of Grassmannians. Consider a Grassmannian $\mathbb{G}(d, n)$ of d-dimensional planes in a n-dimensional affine space. Its twisted form is called a generalized Severi-Brauer variety and denoted by $\mathrm{SB}_d(A)$, where A is the respective central simple algebra of degree n (see [7, §1.C]). The next corollary relates the motive of a generalized Severi-Brauer variety with the motive of usual Severi-Brauer variety.

Corollary 5.5. *Let A and B be two central division algebras of degree n with $[A] = \pm d[B]$ in the Brauer group $\mathrm{Br}(F)$, where d and n are coprime. Then the motive of the Severi-Brauer variety $\mathrm{SB}(A)$ is a direct summand in the motive of the generalized Severi-Brauer variety $\mathrm{SB}_d(B)$.*

Proof. We construct the morphism $f : M(\mathrm{SB}_d(B)) \to M(\mathrm{SB}(A))$ as follows. Consider the Plücker embedding $pl : \mathrm{SB}_d(B) \to \mathrm{SB}(\Lambda^d B)$. It induces the morphism $M(\mathrm{SB}_d(B)) \to M(\mathrm{SB}(\Lambda^d B))$, where $\Lambda^d B$ is the d-th lambda power of B (see [7, II.10.A]). By [6, Cor. 1.3.2] the motive $M(\mathrm{SB}(\Lambda^d B))$ splits as a direct sum of shifted copies of $M(\mathrm{SB}(A))$, where $[A] = d[B]$ in $\mathrm{Br}(F)$. Take f to be the composite of the Plücker embedding and the projection $M(\mathrm{SB}(\Lambda^d B)) \to M(\mathrm{SB}(A))$.

We claim that f has a section (splits) over the generic point of $\mathrm{SB}(A)$. Indeed, it is equivalent to the fact that for each $m = 0, \ldots, n-1$

$$g.c.d._i(c_i^{(m)}) = 1$$

where $c_i^{(m)}$ are degrees of the Schubert varieties generating $\mathrm{CH}^m(\mathbb{G}(d,n))$. The latter can be computed using explicit formulas for degrees of Schubert varieties provided for instance in [3, Ch. 14, Ex. 14.7.11.(ii)].

Then by Theorem 1.2 the motive $M(\mathrm{SB}(A))$ is a direct summand in $M(\mathrm{SB}_d(B))$. Observe that the motives $M(\mathrm{SB}(A))$ and $M(\mathrm{SB}(A^{\mathrm{op}}))$ are isomorphic. So replacing $[A]$ by $[A^{op}] = -[A]$ doesn't change anything. □

Compactifications of a Merkurjev-Suslin variety. Here we follow definitions and notation of [16]. Let A be a cubic division algebra over F. Recall that a smooth compactification D of a Merkurjev-Suslin variety $\mathcal{MS}(A,c)$ can be identified with the smooth hyperplane section of the twisted form $X = \mathrm{SB}_3(M_2(A))$ of Grassmannian $\mathbb{G}(3,6)$. Using Theorem 1.2 one obtains a shortened proof of the main result of [16]

Corollary 5.6. *Let D be the smooth projective variety introduced above. Then*

$$M(D) \simeq \bigoplus_{i=1}^{5} M(\mathrm{SB}(A))\{i\} \oplus \mathcal{R}_{3,3},$$

where $\mathcal{R}_{3,3}$ is an integral generalized Rost motive. In other words, from the motivic point of view the variety D can be viewed as a 3-analog of a Norm quadric.

Proof. Let $i : D \hookrightarrow X$ denote the closed embedding. It induces the map $\Gamma_i \colon M(D) \to M(X)$. The variety X is a projective homogeneous PGL_6-variety corresponding to a maximal parabolic subgroup of type P_3. According to the Tits indices for the group $\mathrm{PGL}_{M_2(A)}$ the parabolic subgroup P_3 is defined over F and, hence, X is isotropic. By [2, Thm. 7.5] the motive of X splits as

$$M(X) = \mathbb{Z} \oplus Q\{1\} \oplus Q\{4\} \oplus \mathbb{Z}\{9\},$$

where $Q = M(\mathrm{SB}(A) \times \mathrm{SB}(A^{\mathrm{op}})) = \bigoplus_{i=0}^{2} M(\mathrm{SB}(A))\{i\}$ by the projective bundle theorem. Hence, we obtain

$$M(X) = \mathbb{Z} \oplus \bigoplus_{i=1}^{6} M(\mathrm{SB}(A))\{i\} \oplus \mathbb{Z}\{9\}. \tag{4}$$

We define f to be the composite of Γ_i and the canonical projection from $M(X)$ to the direct summand $\bigoplus_{i=1}^{5} M(\mathrm{SB}(A))\{i\}$ of (4). Observe that the motive

$M(D)$ splits over the generic point of $\mathrm{SB}(A)$. The direct computations (using multiplication tables provided in [16]) show that f has a section over $F(\mathrm{SB}(A))$. By Theorem 1.2 we conclude that

$$M(D) \simeq \bigoplus_{i=1}^{5} M(\mathrm{SB}(A))\{i\} \oplus N$$

for some motive N which splits over $F(\mathrm{SB}(A))$ as $\mathbb{Z} \oplus \mathbb{Z}\{4\} \oplus \mathbb{Z}\{8\}$.
Note that both D and the twisted form of F_4/P_4 (given by the first Tits construction) split the same symbol $\mathfrak{g}_3$ in $K_3^M(F)/3$. This implies that there is a morphism $f : N_{\mathbb{Z}/3} \to \mathcal{R}_{3,3}$ of motives with $\mathbb{Z}/3$-coefficients which becomes an isomorphism over the separable closure of F, where $\mathcal{R}_{3,3}$ is a generalized Rost motive corresponding to $\mathfrak{g}_3$. Since N is split over the generic point of the twisted form of F_4/P_4, $\mathcal{R}_{3,3}$ is a direct summand of $N_{\mathbb{Z}/3}$ which implies that $\mathcal{R}_{3,3} \simeq N_{\mathbb{Z}/3}$. Finally observe that $N_{\mathbb{Z}/p}$ splits if $p \neq 3$. ◻

6 Appendix

Proposition 6.1. *Let X be a smooth quasi-projective variety, $\pi\colon Y \to X$ a smooth morphism and $i\colon Z \hookrightarrow X$ a closed embedding. Consider the Cartesian square*

$$\begin{array}{ccc} Y_Z & \xrightarrow{i'} & Y \\ {\scriptstyle \pi_Z}\downarrow & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{i} & X \end{array}$$

Then $(\operatorname{im} i'_*)^d = 0$ for $d = [\frac{\dim(X)}{\operatorname{codim}_X(Z)}] + 1$, where $\operatorname{codim}_X(Z)$ is the minimum of codimensions of irreducible components of Z.

It is sufficient to prove the following:

Lemma 6.2. *Let $\pi\colon Y \to X$ be a smooth morphism, with X smooth quasi-projective, and $i_1\colon Z_1 \hookrightarrow X$, $i_2\colon Z_2 \hookrightarrow X$ closed embeddings.*
Then there exists a closed embedding $i_3\colon Z_3 \hookrightarrow X$ such that

$$\operatorname{codim}(Z_3) \geq \operatorname{codim}(Z_1) + \operatorname{codim}(Z_2) \quad \textit{and} \quad \operatorname{im}(i'_1)_* \cdot \operatorname{im}(i'_2)_* \subset \operatorname{im}(i'_3)_*,$$

where $i'_l\colon Y_{Z_l} \hookrightarrow Y$, $l = 1, 2, 3$ is obtained from the respective Cartesian square.

Proof. We have $(i'_1)_*(a) \cdot (i'_2)_*(b) = \Delta_X^*((i'_1 \times i'_2)_*(a \times b))$. The diagonal map $\Delta_Y\colon Y \to Y \times Y$ is the composition $Y \xrightarrow{\phi} Y \times_X Y \xrightarrow{f_W} Y \times Y$, where ϕ is the relative diagonal and the second map is the natural embedding. By Lemma 6.3 applied to $B = X \times X$, $V = X$, $f = \Delta_X$, $T = Z_1 \times Z_2$ and $W = Y \times Y$ we obtain a closed embedding $h\colon Z \hookrightarrow X$ such that

$$\operatorname{codim}(Z) \geq \operatorname{codim}(Z_1) + \operatorname{codim}(Z_2) \text{ and } \operatorname{im}(f_W^* \circ (i'_1 \times i'_2)_*) \subset \operatorname{im}(h_{W*}).$$

Consider the Cartesian square

$$\begin{tikzcd} Y \arrow[r, "\phi"] & Y \times_X Y \\ Y_Z \arrow[u, "h'"] \arrow[r, "\phi_Z"] & (Y \times_X Y)_Z \arrow[u, "h_W"'] \end{tikzcd}$$

By [3, Theorem 6.2(a)], $\phi^* \circ h_{W*} = h'_* \circ \phi^!$. Hence, $\mathrm{im}(\Delta_X^* \circ (i_1 \times i_2)_*) \subset \mathrm{im}(h'_*)$ and the lemma is proven. $\square$

Lemma 6.3. *Let $V \xrightarrow{f} B \xleftarrow{g} T$ be closed embeddings with regular f, and smooth quasi-projective B. Let $\varepsilon\colon W \to B$ be a smooth morphism. Consider two Cartesian diagrams:*

$$\begin{tikzcd} W_V \arrow[r, "f_W"] \arrow[d, "\varepsilon_V"'] & W \arrow[d, "\varepsilon"] & W_T \arrow[l, "g_W"'] \arrow[d, "\varepsilon_T"] \\ V \arrow[r, "f"] & B & T \arrow[l, "g"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T \arrow[r, "g"] & B \\ \tilde{T} \arrow[u, "\tilde{f}"] \arrow[r, "\tilde{g}"] & V \arrow[u, "f"'] \end{tikzcd}$$

There exists a closed embedding $h\colon Z \hookrightarrow V$ such that $\mathrm{codim}(h) \geq \mathrm{codim}(g)$ *and* $\mathrm{im}(f_W^* \circ g_{W*}) \subset \mathrm{im}(h_{W*})$.

Proof. Consider the Cartesian square

$$\begin{tikzcd} W_T \arrow[r, "g_W"] & W \\ W_{\tilde{T}} \arrow[u, "\tilde{f}_W"] \arrow[r, "\tilde{g}_W"] & W_V \arrow[u, "f_W"'] \end{tikzcd}$$

By [3, Theorem 6.2(a)], $f_W^* \circ g_{W*} = \tilde{g}_{W*} \circ f_W^!$. The morphism $f_W^!\colon \mathrm{CH}^*(W_T) \to \mathrm{CH}^*(W_{\tilde{T}})$ is given by the composition:

$$\mathrm{CH}^*(W_T) \xrightarrow{\sigma} \mathrm{CH}^*(\mathcal{C}_W) \xrightarrow{\rho_{W*}} \mathrm{CH}^*(\mathcal{N}_W) \xrightarrow{s^*} \mathrm{CH}^*(W_{\tilde{T}}),$$

where σ is the specialization map from [3, §5.2], $\mathcal{C}_W = C_{W_T}(W_{\tilde{T}}) = C_T(\tilde{T}) \times_B W$ is the normal cone of the morphism $\tilde{f}_W$ and $\mathcal{N}_W = W_{\tilde{T}} \times_{W_V} \mathcal{N}_{f_W} = (\tilde{T} \times_V \mathcal{N}_f) \times_B W$ is the total space of the vector bundle $\tilde{g}_W^*(\mathcal{N}_{f_W}) = (\varepsilon_{\tilde{T}} \circ \tilde{g})^*(\mathcal{N}_f)$ over $W_{\tilde{T}}$, $\rho_W\colon \mathcal{C}_W \hookrightarrow \mathcal{N}_W$ is the closed embedding and $s\colon W_{\tilde{T}} \to \mathcal{N}_W$ is the zero section.

Consider the Cartesian square of projective completions of $\mathcal{C}_W$ and $\mathcal{N}_W$

$$\begin{tikzcd} \mathbb{P}(\mathcal{C}_W \oplus \mathcal{O}) \arrow[r, "\overline{\rho}_W"] & \mathbb{P}(\mathcal{N}_W \oplus \mathcal{O}) \\ \mathcal{C}_W \arrow[u, "e_{\mathcal{C}}"] \arrow[r, "\rho_W"] & \mathcal{N}_W \arrow[u, "e_{\mathcal{N}}"'] \end{tikzcd}$$

By [3, Proposition 3.3] the morphism $s^* \circ \rho_{W_*} \colon \mathrm{CH}^*(\mathcal{C}_W) \to \mathrm{CH}^*(W_{\tilde{T}})$ is given by $s^* \circ \rho_{W_*}(x) = \pi_{W_*}(c_d(\tilde{g}_W^* \mathcal{N}_{f_W} \otimes \mathcal{O}(1)) \cdot \overline{\rho}_{W_*}(y))$, where $e_{\mathcal{C}}^*(y) = x$, $\pi_W \colon \mathbb{P}(\mathcal{N}_W \oplus \mathcal{O}) \to W_{\tilde{T}}$ is the projection and $d = \mathrm{codim}(f_W) = \mathrm{codim}(f)$. By Lemma 6.4, we can choose a cycle α representing $c_d(\tilde{g}^* \mathcal{N}_f \otimes \mathcal{O}(1))$ in such a way that $|\alpha| \cap \mathbb{P}(\mathcal{C} \oplus \mathcal{O})$ has codimension d in $\mathbb{P}(\mathcal{C} \oplus \mathcal{O})$. Consider $Z := \pi(|\alpha| \cap \mathbb{P}(\mathcal{C} \oplus \mathcal{O}))$ and the closed embedding $j \colon Z \hookrightarrow \tilde{T}$. Then for arbitrary $x \in \mathrm{CH}^*(\mathbb{P}(\mathcal{C}_W \oplus \mathcal{O}))$ we have $|\pi_{W_*}(\varepsilon_{\tilde{T}}^*(\alpha) \cdot \overline{\rho}_{W_*}(x))| \subset \varepsilon^{-1}(Z)$. This implies that $\mathrm{im}(f_W^!) \subset \mathrm{im}(j_{W_*})$ and $\mathrm{im}(\tilde{g}_{W_*} \circ f_W^!) \subset \mathrm{im}(h_{W_*})$, where $h = \tilde{g} \circ j$. At the same time, $\mathrm{codim}(h) \geq \mathrm{codim}(g)$, and the lemma is proven. □

Lemma 6.4. *Let X be a quasi-projective variety, and Z_l, $l = 1, \dots, n$ be closed irreducible subvarieties of dimensions d_l. Let $\mathcal{V}$ be a vector bundle over X. Then there exists a representative α_d of $c_d(\mathcal{V})$ such that $|\alpha_d| \cap Z_l$ has dimension $\leq d_l - d$.*

Proof. The total Chern class $c_\bullet(\mathcal{V})$ is the inverse of the total Segre class $s_\bullet(\mathcal{V})$, and $s_i(\mathcal{V}) = \pi_*(c_1(\mathcal{O}(1))^{n-1+i})$, where $\pi : \mathbb{P}_X(\mathcal{V}) \to X$ is the projection, and $n = \dim(\mathcal{V})$. Thus, the general case of our statement follows by the inductive application of the one with $d = 1$, and $\mathcal{V}$ - linear bundle. Indeed, since $c_d([X]) = -\sum_{j=1}^{d} \pi_*(c_1(\mathcal{O}(1))^{n-1+j}(\pi^{-1}(c_{d-j}([X]))))$, and α_{d-j} can be chosen with the needed property, it is sufficient to apply the above particular case to the set of irreducible components of $\pi^{-1}(Z_l \cap |\alpha_{d-j}|)$, $l = 1, \dots, n; j = 1, \dots, d$ inside $\mathbb{P}_X(\mathcal{V})$. Finally, the case $d = 1$ and linear $\mathcal{V}$ follows from the presentation $\mathcal{V} = \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$, where $\mathcal{L}_i$ have "sufficiently many sections", which is possible, since X is quasi-projective. □

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On Tameness and Growth Conditions

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Abstract. We study discrete subsets of $\mathbb{C}^d$, relating "tameness" with growth conditions.

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1. Results

A discrete subset D in $\mathbb{C}^n$ ($n \geq 2$) is called "tame" if there exists a holomorphic automorphism ϕ of $\mathbb{C}^n$ such that $\phi(D) = \mathbb{Z} \times \{0\}^{n-1}$ (see [3]). If there exists a linear projection π of $\mathbb{C}^n$ onto some $\mathbb{C}^k$ $(0 < k < n)$ for which the image $\pi(D)$ is discrete, then D is tame ([3]). If D is a discrete subgroup (e.g. a lattice) of the additive group $(\mathbb{C}^n+)$, then D must be tame ([1], lemma 4.4 in combination with corollary 2.6). On the other hand there do exist discrete subsets which are not tame (see [3], theorem 3.9).
Here we will investigate how "tameness" is related to growth conditions for D. Slow growth implies tameness as we well see. On the other hand, rapid growth can not imply non-tameness, since every discrete subset of $\mathbb{C}^{n-1}$ is tame regarded as subset of $\mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C}$.
The key method is to show that sufficiently slow growth implies that a generic linear projection will have discrete image for D.
The main result is:

Theorem 1. *Let n be a natural number and let v_k be a sequence of elements in $V = \mathbb{C}^n$.*
Assume that
$$\sum_k \frac{1}{||v_k||^{2n-2}} < \infty$$
Then $D = \{v_k : k \in \mathbb{N}\}$ is tame, *i.e., there exists a biholomorphic map $\phi : \mathbb{C}^n \to \mathbb{C}^n$ such that*
$$\phi(D) = \mathbb{Z} \times \{0\}^{n-1}.$$

This growth condition is fulfilled for discrete subgroups of rank at most $2n-3$, implying the following well-known fact:

Corollary 1. *Let Γ be a discrete subgroup of $\mathbb{Z}$-rank at most $2n-3$ of the additive group $(\mathbb{C}^n,+)$.*
Then Γ is a tame discrete subset of $\mathbb{C}^n$.

While this is well-known (even with no condition on the $\mathbb{Z}$-rank of Γ), our approach yields the additional information that these discrete subsets remain tame after a small deformation:

Corollary 2. *Let Γ be a discrete subgroup of $\mathbb{Z}$-rank at most $2n-3$ of the additive group $(\mathbb{C}^n,+)$, $0<\lambda<1$ and $K>0$. Let D be a subset of $\mathbb{C}^n$ for which there exists a bijective map $\zeta:\Gamma\to D$ with*

$$||\zeta(v)-v||\le\lambda||v||+K$$

for all $v\in\Gamma$.
Then D is a tame discrete subset of $\mathbb{C}^n$.

This confirms the idea that tame sets should be stable under deformation. Similarily one would hope that the category of non-tame sets is also stable under deformation. Here, however, one has to be careful not to be too optimistic, because in fact the following is true:

Proposition. *For every non-tame discrete subset $D\subset\mathbb{C}^n$ $(n>1)$ there is a tame discrete subset D' and a bijection $\alpha:D\to D'$ such that*

$$||\alpha(v)-v||\le\frac{1}{\sqrt{2}}||v|| \quad \forall v\in D$$

and

$$||w-\alpha^{-1}(w)||\le||w|| \quad \forall w\in D'.$$

In particular, if D is a tame discrete subset and $\zeta:D\to\mathbb{C}^n$ is a bijective map with $||\zeta(v)-v||\le||v||$ for all $v\in D$, it is possible that $\zeta(D)$ is not tame.
Still, one might hope for a positive answer to the following question:

Question. *Let $n\in\mathbb{N}$, $n\ge2$, let $1>\lambda>0$, $K>0$, let D be a tame discrete subset of $\mathbb{C}^n$ and let $\zeta:D\to\mathbb{C}^n$ be a map such that*

$$||\zeta(v)-v||\le\lambda||v||+K$$

for all $v\in D$. Does this imply that $\zeta(D)$ is a tame *discrete subset of $\mathbb{C}^n$?*

Technically, the following is the key point for the proof of our main result (theorem 1):

Theorem 2. *Let $n>d>0$. Let V be a complex vector space of dimension n and let v_k be a sequence of elements in V.*
Assume that

$$\sum_k\frac{1}{||v_k||^{2d}}<\infty$$

Then there exists a complex linear map $\pi:V\to\mathbb{C}^d$ such that the set of all $\pi(v_k)$ is discrete in $\mathbb{C}^d$.

In a similar way on can prove such a result for real vector spaces:

Theorem 3. *Let $n > d > 0$. Let V be a real vector space of dimension n and let v_k be a sequence of elements in V.*
Assume that

$$\sum_k \frac{1}{||v_k||^d} < \infty$$

Then there exists a real linear map $\pi : V \to \mathbb{R}^d$ such that the set of all $\pi(v_k)$ is discrete in $\mathbb{R}^d$.

For the proof of the existence of a linear projection π with $\pi(D)$ discrete we proceed by regarding randomly chosen linear projections and verifying that the image of D under a random projection has discrete image with probability 1 if the above stated series converges.

2. Proofs

First we deduce an auxiliary lemma.

Lemma 1. *Let $k, m > 0$, $n = k + m$ and let S denote the unit sphere in $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^m$. Furthermore let*

$$M_\epsilon = \{(v, w) \in \mathbb{R}^k \times \mathbb{R}^m : ||v|| \leq \epsilon, (v, w) \in S\}.$$

Then there are constants $\delta > 0$, $C_1 > C_2 > 0$ such that for all $\epsilon < \delta$ we have

$$C_1 \epsilon^k \geq \lambda(M_\epsilon) \geq C_2 \epsilon^k$$

where λ denotes the rotationally invariant probability measure on S.

Proof. For each $\epsilon \in]0, 1[$ there is a bijection

$$\phi_\epsilon : B \times S' \to M_\epsilon$$

where

$$B = \{v \in \mathbb{R}^k : ||v|| \leq 1\}, \quad S' = \{w \in \mathbb{R}^m : ||w|| = 1\}$$

and

$$\phi_\epsilon(v, w) = \left(\epsilon v; \sqrt{1 - ||\epsilon v||^2} w\right).$$

The functional determinant for ϕ_ϵ equals

$$\epsilon^k \left(\sqrt{1 - ||\epsilon v||^2}\right)^m.$$

It follows that

$$\epsilon^k \left(\sqrt{1 - \epsilon^2}\right)^m volume(S' \times B) \leq volume(M_\epsilon) \leq \epsilon^k volume(S' \times B),$$

which in turn implies

$$\lim_{\epsilon \to 0} \epsilon^{-k} \frac{volume(M_\epsilon)}{volume(S' \times B)} = 1.$$

Hence the assertion. □

Lemma 2. *Let* Γ *be a discrete subgroup of* $\mathbb{Z}$*-rank* d *in* $V = \mathbb{R}^n$.
Then

$$\sum_{\gamma\in\Gamma} ||\gamma||^{-d-\epsilon} < \infty$$

for all $\epsilon > 0$.

Proof. Since all norms on a finite-dimensional vector space are equivalent, there is no loss in generality if we assume that the norm is the maximum norm and $\Gamma = \mathbb{Z}^d \times \{0\}^{n-d}$. Then the assertion is an easy consequence of the fact that $\sum_{n\in\mathbb{N}} n^{-s} < \infty$ if and only if $s > 1$. □

Now we proceed with the proof of theorem 2:

Proof. We fix a surjective linear map $L : V \to W = \mathbb{C}^d$. Let K denote $U(n)$ (the group of unitary complex linear transformations of V). For each $g \in K$ we define a linear map $\pi_g : V \to W$ as follows:

$$\pi_g : v \mapsto L(g \cdot v).$$

For $k \in \mathbb{N}$ and $r \in \mathbb{R}^+$ define

$$S_{k,r} = \{g \in K : ||\pi_g(v_k)|| \leq r\},$$

$$M_{N,r} = \{g \in K : \#\{k \in \mathbb{N} : g \in S_{k,r}\} \geq N\}$$

and

$$M_r = \cap_N M_{N,r}.$$

Now for each $g \in K$ the set $\{\pi_g(v_k) : k \in \mathbb{N}\}$ is discrete unless there is a number $r > 0$ such that infinitely many distinct image points are contained in a ball of radius r. By the definition of the sets M_r it follows that $\{\pi_g(v_k) : k \in \mathbb{N}\}$ is discrete unless $g \in M = \cup M_r$.
Let us now assume that there is no linear map $L' : V \to W$ with $L'(D)$ discrete. Then $K = M$. In particular $\mu(M) > 0$, where μ denotes the Haar measure on the compact topological group K. Since the sets M_r are increasing in r, we have

$$M = \cup_{r\in\mathbb{R}^+} M_r = \cup_{r\in\mathbb{N}} M_r$$

and may thus deduce that $\mu(M_r) > 0$ for some number r. Fix such a number $r > 0$ and define $c = \mu(M_r) > 0$. Then $\mu(M_{N,r}) \geq c$ for all N, since $M_r = \cap M_{N,r}$. However, for fixed N and r we have

$$N\mu(M_{N,r}) \leq \sum_k \mu(S_{k,r}).$$

Hence

$$\sum_{k\in\mathbb{N}} \mu(S_{k,r}) \geq N\mu(M_{N,r}) \geq Nc$$

for all $N \in \mathbb{N}$. Since $c > 0$, it follows that $\sum_k \mu(S_{k,r}) = +\infty$.

Let us now embedd $\mathbb{C}^d$ into $\mathbb{C}^n$ as the orthogonal complement of $\ker L$. In this way we may assume that L is simply the map which projects a vector onto its first d coordinates, i.e.,

$$L(w_1, \dots, w_n) = (w_1, \dots, w_d; 0, \dots, 0).$$

Now $g \in S_{k,r}$ is equivalent to the condition that $g(v_k)$ is a real multiple of an element in M_ϵ where M_ϵ is defined as in lemma 1 with $\epsilon = r/||v_k||$. Using lemma 1 we may deduce that $\sum_k \mu(S_{k,r})$ converges if and only if $\sum_k ||v_k||^{-2d}$ converges. □

Proof of theorem 1. The growth condition allows us to employ theorem 2 in order to deduce that there is a linear projection onto a space of complex dimension $d-1$ which maps D onto a discrete image. By the results of Rosay and Rudin it follows that D is tame. □

Proof of the proposition. We fix a decomposition $\mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1}$ and write D as the union of all $(a_k, b_k) \in \mathbb{C} \times \mathbb{C}^{n-1}$ $(k \in \mathbb{N})$. We define

$$\alpha(a_k, b_k) = \begin{cases} (a_k, 0) & \text{if } ||a_k|| > ||b_k|| \\ (0, b_k) & \text{if } ||a_k|| \leq ||b_k|| \end{cases}$$

Then $D' = \alpha(D)$ is tame because each of the projections to one of the two factors $\mathbb{C}$ and $\mathbb{C}^{n-1}$ maps D' onto a discrete subset.
The other assertions follow from the triangle inequality. □

The proof of thm. 3 works in the same way as the proof of thm. 2, simply using the group of all orthogonal transformations instead of the group of unitary transformations.

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Dynamical Symmetries in Supersymmetric Matrix[1]

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Abstract. We reveal a dynamical SU(2) symmetry in the asymptotic description of supersymmetric matrix models. We also consider a recursive approach for determining the ground state, and point out some additional properties of the model(s).

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1 Introduction

Supersymmetric matrix models derive from superstring theory which ultimately aims at a quantum field theoretic model of all known forces, including gravity. Some of the basic mathematical properties of supersymmetric matrix models are still open and pose a challenge to mathematics.

One of the key properties of supersymmetric matrix models - often assumed for granted in physics, but difficult to prove mathematically - is the existence of a ground state. I.e., the self-adjoint and nonnegative Hamiltonian operator $H = H^* \geq 0$ specifying the supersymmetric matrix model under consideration is assumed to have an eigenvalue at 0, the bottom of its spectrum. Since its spectrum is purely essential and covers the entire positive half axis, $\sigma(H) = [0, \infty)$ (see [3, 11]), the existence of zero-energy eigenstates, i.e., the non-triviality $\mathrm{Ker}(H) \neq 0$ of the zero-energy subspace, is not a consequence of standard methods of regular perturbation theory.

The Hamiltonian H acts on (an appropriate dense domain in) the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^{d(N^2-1)}) \otimes \mathcal{F}$ of square-integrable functions of coordinate variables

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$x \equiv (x_{tA})_{t\in\mathcal{D},A\in\mathcal{N}}$ with values in the fermion Fock space $\mathcal{F}$ spanned by vectors of the form $\lambda^\dagger_{\alpha_1,A_1}\cdots\lambda^\dagger_{\alpha_k,A_k}|0\rangle$, where $\{\lambda^\dagger_{\alpha A}\}_{\alpha\in\mathcal{S},A\in\mathcal{N}}$ are standard fermion creation operators, and $|0\rangle$ is the vacuum vector. The index ranges are denoted by $\mathcal{D} := \{1,2,\ldots,d\}$, $\mathcal{N} := \{1,2,\ldots,N^2-1\}$, and $\mathcal{S} := \{1,2,\ldots,d-1\}$. Note that $\dim_{\mathbb{C}}\mathcal{F} = 2^{(d-1)(N^2-1)} < \infty$.
On $\mathcal{H}$, the Hamiltonian assumes the form

$$H \;=\; -\Delta_x \otimes \mathbf{1} + V(x)\otimes\mathbf{1} + H_F, \tag{1}$$

where Δ_x is the Laplacian on $\mathbb{R}^{d(N^2-1)}$. The potential V is a homogenous, quartic polynomial in the coordinates x,

$$V(x) \;=\; \frac{1}{2} f_{ABC}\, f_{AB'C'}\, x_{sB}\, x_{tC}\, x_{sB'}\, x_{tC'}, \tag{2}$$

with $(f_{ABC})_{A,B,C\in\mathcal{N}}$ being real, antisymmetric structure constants of SU(N) and using Einstein's summation convention (i.e., repeated indices are summed over). The operator H_F, the fermionic part of the Hamiltonian, is linear in x, but quadratic in the fermion creation operators $\lambda^\dagger_{\alpha A}$ and their adjoints $\lambda_{\alpha A} = (\lambda^\dagger_{\alpha A})^*$, the fermion annihilation operators,

$$\begin{aligned} H_F \;=\; f_{ABC}\Big\{ & -2i\Big(\sum_{t=1}^{d-2} x_{t,C}\,\Gamma^t_{\alpha,\beta}\Big)\,\lambda^\dagger_{\alpha A}\,\lambda_{\beta B} \\ & + (x_{d-1,C} + i x_{d,C})\,\lambda^\dagger_{\alpha A}\,\lambda^\dagger_{\alpha B} \;-\; (x_{d-1,C} - i x_{d,C})\,\lambda_{\alpha A}\,\lambda_{\alpha B}\Big\}, \end{aligned} \tag{3}$$

where $(\Gamma^t)_{t=1}^{d-2}$ are purely imaginary, antisymmetric $(d-1)\times(d-1)$ matrices that represent the Clifford algebra $\{\Gamma^s,\Gamma^t\} = 2\delta_{st}\cdot\mathbf{1}_{(d-1)\times(d-1)}$, with $s,t\in\{1,2,\ldots,d-2\}$ and $d\in\{2,3,5,9\}$.
The Hamiltonian H commutes with the generators $\{J_A\}_{A\in\mathcal{N}}$ of su(N), where $J_A = \frac{-i}{2} f_{ABC}(2x_{sB}\partial_{sC} + \lambda^\dagger_{\alpha B}\lambda_{\alpha C} + \lambda_{\alpha C}\lambda^\dagger_{\alpha B})$ and $\partial_{sC} := \frac{\partial}{\partial x_{sC}}$, and the ground state sought for is required to be SU(N)-invariant. That is, the spectral analysis of H is carried out on the subspace $\mathcal{H}_0 = \bigcap_{A\in\mathcal{N}}\mathrm{Ker}(J_A) \subseteq \mathcal{H}$. On $\mathcal{H}_0$, the Hamiltonian H arises as the restriction of the square of supercharges $(Q_\beta)_{\beta\in\mathcal{S}}$. These supercharges are selfadjoint (matrix-valued, first-order partial differential) operators on $\mathcal{H}$, which we don't describe here in detail, but we note that the Hamiltonian $H\restriction_{\mathcal{H}_0} = Q_\beta^2\restriction_{\mathcal{H}_0} \geq 0$, is manifestly nonnegative on $\mathcal{H}_0$.
Two main difficulties arise in the analysis of H:
(a) The quartic potential V has many vanishing directions. E.g., given $\vec{e} = (e_A)_{A\in\mathcal{N}} \in \mathbb{R}^{N^2-1}$ and denoting $\vec{x}_t = (x_{tA})_{A\in\mathcal{N}}$, the potential $V(x)$ vanishes on all hyperplanes $M(\vec{e}) = \{x \mid \forall t\in\mathcal{D} :\ \vec{x}_t \in \mathbb{R}\vec{e}\}$. So, even though the potential V grows to $\lim_{\eta\to\infty} V(\eta x) = \infty$, for almost all $x\in\mathbb{R}^{d(N^2-1)}$, this growth at infinity is not confining enough for H to have purely discrete spectrum, as shown in [3].

(b) The fermionic part H_F of the Hamiltonian is indefinite, so it doesn't contribute an obviously confining term to $-\Delta + V(x)$. Yet, their sum H restricted to $\mathcal{H}_0$ is nonnegative and is expected to yield a zero eigenvalue at the bottom of its spectrum, for $d = 9$. In contrast, if $d = 2$ and $N = 2$ then zero is not an eigenvalue of H, as was shown in [5].

A lot of effort was put into the question of existence of zero-energy states in these SU(N)-invariant supersymmetric matrix models given by $H \upharpoonright_{\mathcal{H}_0}$. The original formulation uses Clifford variables $\{\Theta_{\hat\alpha,A}\}_{\hat\alpha\in\widehat{\mathcal{S}},A\in\mathcal{N}}$, $\{\Theta_{\hat\alpha,A},\Theta_{\hat\beta,B}\} = \delta_{\hat\alpha\hat\beta}\delta_{AB}$ rather than fermion creation and annihilation operators employed here, where $\widehat{\mathcal{S}} := \{1,2,\ldots,2(d-1)\}$ and the relation between Clifford variables and the fermion creation and annihilation operators is the standard one, $\Theta_{\alpha,A} := \frac{1}{\sqrt{2}}\big(\lambda^\dagger_{\alpha A} + \lambda_{\alpha A}\big)$ and $\Theta_{\alpha+d-1,A} := \frac{-i}{\sqrt{2}}\big(\lambda^\dagger_{\alpha A} - \lambda_{\alpha A}\big)$, for all $\alpha \in \mathcal{S}$ and $A \in \mathcal{N}$. In terms of these Clifford variables, the Hamiltonian reads $\widetilde{H} = [-\Delta_x + V(x)] \otimes \mathbf{1} + \widetilde{H}_F$, where

$$\widetilde{H}_F \;=\; i\, f_{ABC}\, x_{tC}\, \gamma^t_{\hat\alpha,\hat\beta}\, \Theta_{\hat\alpha,A}\, \Theta_{\hat\beta,B}, \tag{4}$$

and $(\gamma^t)_{t\in\mathcal{D}}$ are real, symmetric $2(d-1)\times 2(d-1)$ matrices given by

$$\gamma^t \;:=\; \begin{pmatrix} 0 & i\Gamma^t \\ -i\Gamma^t & 0 \end{pmatrix}, \qquad \gamma^8 \;:=\; \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad \gamma^9 \;:=\; \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \tag{5}$$

with $t = 1,2,\ldots,d-2$ [6]. The matrices $(\gamma^t)_{t\in\mathcal{D}}$ form a real representation of the Clifford algebra $\{\gamma^s,\gamma^t\} = 2\delta_{st}\cdot\mathbf{1}$, with $s,t \in \mathcal{D}$, of minimal dimension, provided $d = 2,3,5,9$.

The reason for recalling the form $\widetilde{H}$ of the Hamiltonian is that $\widetilde{H}$ is manifestly Spin(d)-invariant. The fermion creation and annihilation operators leading to (3) correspond to the particular choice (5) of $(\gamma^t)_{t\in\mathcal{D}}$. In attempts to construct a ground state explicitly [9], fermion creation and annihilation operators are used rather than Clifford variables. This is so, mainly because they provide the Hilbert space on which the Hamiltonian acts irreducibly from the very beginning. Namely, the creation and annihilation operators, $\lambda^\dagger_{\alpha A}, \lambda_{\alpha A}$, with $\alpha \in \mathcal{S}$ and $a \in \mathcal{N}$, form a representation of the canonical anticommutation relations (CAR): $\{\lambda^\dagger_{\alpha A},\lambda^\dagger_{\beta B}\} = \{\lambda_{\alpha A},\lambda_{\beta B}\} = 0$ and $\{\lambda_{\alpha A},\lambda^\dagger_{\beta B}\} = \delta_{\alpha\beta}\delta_{AB}$, where the anticommutator is $\{a,b\} := ab + ba$. The CAR have an explicit representation as linear operators on the fermion Fock space

$$\mathcal{F} \;=\; \bigoplus_{k=0}^{(d-1)(N^2-1)} \operatorname{span}\big\{\lambda^\dagger_{\alpha_1,A_1}\cdots\lambda^\dagger_{\alpha_k,A_k}|0\rangle \;\big|\; \alpha_j \in \mathcal{S},\ A_j \in \mathcal{N}\big\}, \tag{6}$$

which is a complex Hilbert space of dimension $2^{(d-1)(N^2-1)}$. The vectors $\big\{\lambda^\dagger_{\alpha_1,A_1}\cdots\lambda^\dagger_{\alpha_k,A_k}|0\rangle \;\big|\; \alpha_j \in \mathcal{S},\ A_j \in \mathcal{N}\big\} \subseteq \mathcal{F}$ form an orthonormal basis; $|0\rangle$ is called vacuum vector. The Hilbert space $\mathcal{H}$ can be viewed as a direct integral

$$\mathcal{H} \;=\; \int^{\oplus}_{\mathbb{R}^{d(N^2-1)}} \mathcal{F}\, dx \;=\; L^2\big(\mathbb{R}^{d(N^2-1)};\, \mathcal{F}\big), \tag{7}$$

whose elements are linear combinations of the form

$$\Psi(x) = \sum_{k=0}^{(d-1)(N^2-1)} \Psi_k(x), \tag{8}$$

$$\Psi_k(x) = \sum_{\alpha_1,\ldots,\alpha_k\in\mathcal{S}} \sum_{A_1,\ldots,A_k\in\mathcal{N}} \psi^{(k)}_{\alpha_1,A_1\ldots,\alpha_k,A_k}(x)\, \lambda^\dagger_{\alpha_1,A_1}\cdots\lambda^\dagger_{\alpha_k,A_k}|0\rangle. \tag{9}$$

While the fermion creation and annihilation operators above (and in attempts to explicitly construct a ground state as in [9]) are chosen *independently of* x, the asymptotic form of the ground state wave function was determined with the help of *space-dependent* fermions [7, 6, 5]. The analysis of the asymptotic form $H^\infty\Psi = 0$ of the solutions of $H\Psi = 0$ leads for $N = 2$ and $d = 2, 3, 5$ to absence of a zero-energy states, as proved in [5], since these solutions are not square-integrable at $r \to \infty$, where $r > 0$ is introduced in (12), below. On the other hand, for $d = 9$, this asymptotic form of the wave function is square-integrable at infinity, and it is believed that for $d = 9$, the supersymmetric matrix models do possess zero enery eigenstates, for all $N \in \mathbb{N}$. This belief is supported by the recent existence proof for a related model [4].
Main Results: In this paper we study the asymptotic Hamiltonian H^∞ described in detail in (16)–(18). The asympotic Hamiltonian $H^\infty = H^\infty_B + H^\infty_F$ splits into a bosonic part H^∞_B and a fermionic part H^∞_F, similar to the full Hamiltonian H.
The bosonic Hamiltonian H^∞_B is a sum of harmonic oscillators and we first focus our attention on the ground state subspace of H^∞_B with corresponding ground state energy $2(d-1)$. This leads us to study the spectral properties of $2(d-1) + H^\infty_F$. We derive a dynamical SU(2) symmetry in (39) and observe the formation of 'Cooper pairs' [e.g., in the ground state of $2(d-1) + H^\infty_F$ computed in (46) and (48)] that arise in the SO(d)-breaking formulation when diagonalizing certain ingredients of the fermionic part of the Hamiltonian.
Thereafter, we transform the zero energy equation on Fock space into a system of graded equations (52) obtained by its natural grading derived from the fermion number. We show that this system of equations can be solved by a recursive insertion (58) of solutions, provided a certain invertibility condition on the graded Hamiltonians hold, which is known to hold true for the first recursion step (54). We finally observe a sum rule for the graded equations and apply this to the asymptotic ground state of $s_d + H^\infty_F$ (62)–(64).
To ease the reading, we carry out our analysis first in the case $N = 2$, i.e., for the asymptotic SU(2) theory. In the last section we note that several features extend to the non-asymptotic case and/or to general $N \geq 2$. We mostly restrict the dimension d to the most interesting case $d = 9$.

2 Asymptotic form of the Hamiltonian

The bosonic configuration space is a set of $d = 2, 3, 5$, or 9 traceless hermitian matrices $\{X_s\}_{s=1}^d$ corresponding to the Lie algebra su(N) of the gauge group

SU(N). Given selfadjoint generators $\vec{T} \equiv (T_A)_{A\in\mathcal{N}}$ of su(N) with $[T_A, T_B] = i f_{ABC} T_C$, the coordinates x derive from expanding $X_t = x_{tA} T_A = \vec{x}_t \cdot \vec{T}$ in these generators.

For simplicity, we start by taking $N = 2$ and $2\vec{T}$ to be the Pauli matrices $\vec{\sigma} \equiv (\sigma_1, \sigma_2, \sigma_3)$, so $f_{ABC} = \epsilon_{ABC}$. The potential, which for general $N \geq 2$ is given by

$$V(x) \;:=\; -\frac{1}{2} \sum_{s,t\in\mathcal{D}} \operatorname{Tr}\big([X_s, X_t]^2\big), \tag{10}$$

assumes the form

$$V(x) \;=\; \frac{1}{2} \sum_{s,t\in\mathcal{D}} \big(\vec{x}_s \wedge \vec{x}_t\big)^2, \tag{11}$$

where $(\vec{x} \wedge \vec{y})_A = \epsilon_{ABC}\, x_B\, y_C$, as usual. Observe from (11) that $V(x)$ vanishes if $x \in M := \bigcup_{\vec{e}\in\mathbb{R}^3} M(\vec{e})$, recalling that $M(\vec{e})$ are the hyperplanes $M(\vec{e}) = \{x \mid \forall t \in \mathcal{D} :\ \vec{x}_t \in \mathbb{R}\vec{e}\}$. We remark that, for $N = 2$, the condition $x \in M$ is even equivalent to $V(x) = 0$, while a necessary condition for the vanishing of the potential is more complicated for $N > 2$. We coordinatize $x \in \mathbb{R}^{3d}$ by (see, e.g., [6, 5])

$$\vec{x}_t \;=\; r\, E_t\, \vec{e} \;+\; r^{-1/2}\, \vec{y}_t, \tag{12}$$

for $t = 1, \ldots, d$, where $\vec{e} \in \mathbb{R}^3$ and $E = (E_1, \ldots, E_d) \in \mathbb{R}^d$ are unit vectors, $r > 0$, and $\vec{y}_t \in \mathbb{R}^3$ are perpendicular to both E and $\vec{e}$ in the sense that $E_s \vec{y}_s = \vec{e} \cdot \vec{y}_t = 0$, for all $t = 1, \ldots, d$. They derive from $x \in \mathbb{R}^{3d} \setminus \{0\}$ by the requirement that the euclidean length of the projection $x^{\parallel}$ of x along $\mathbb{R} \cdot E \otimes \vec{e}$ be maximal. Indeed, $\vec{e}$ and E are normalized eigenvectors of $(x_{tA} x_{tB})_{A,B=1}^3$ and $(x_{sA} x_{tA})_{s,t=1}^d$, respectively, corresponding to the largest eigenvalue $r^2 > 0$ which, in turn, is the square of the length $r = |x^{\parallel}| = \langle E \otimes \vec{e}, x\rangle = E_t x_{tA} e_A$ of $x^{\parallel}$. The component $x^{\perp} = x - x^{\parallel}$ perpendicular to $E \otimes \vec{e}$ then yields $\vec{y}_t = r^{1/2} \vec{x}_t^{\perp}$. Writing E as $E(\widetilde{E}, \theta, \varphi) = \big(\cos\theta\, \widetilde{E},\ \sin\theta\, \cos\varphi,\ \sin\theta\, \sin\varphi\big)$ in spherical coordinates, the coordinate vectors $\vec{x}_t$ assume the form

$$\vec{x}_t \;=\; r\, \cos\theta\, \widetilde{E}_t\, \vec{e} \;+\; r^{-1/2}\, \vec{y}_t, \tag{13}$$

for $t = 1, \ldots, d-2$, and

$$\vec{x}_{d-1} + i\vec{x}_d \;=\; r\, \sin\theta\, e^{i\varphi}\, \vec{e} \;+\; r^{-1/2}\, (\vec{y}_{d-1} + i\vec{y}_d), \tag{14}$$

where $\vec{e} \in \mathbb{R}^3$ and $\widetilde{E} = (\widetilde{E}_1, \widetilde{E}_2, \ldots, \widetilde{E}_{d-2}) \in \mathbb{R}^{d-2}$ are unit vectors, $\theta \in (-\pi/2, \pi/2)$, $\varphi \in (0, 2\pi)$, and $r > 0$.

To derive the asymptotic form of the Hamiltonian (cf. [5, 9]), we substitute (13)–(14) (and the corresponding differentials) into H, divide by r, and obtain

$$\frac{1}{r} H \quad \to \quad H^{\infty} \tag{15}$$

as the resulting limit, as $r \to \infty$. Note that, while the difference of H/r and H^{∞} is of lower order in r, the limit $r \to \infty$ is formal, as this difference is an unbounded (differential) operator. Moreover, it ignores the question whether the coordinate change (13)–(14) defines a global diffeomorphism. The asymptotic Hamiltonian H^∞ in (15) is of the form

$$H^\infty \;=\; H_B^\infty \;+\; H_F^\infty\,, \tag{16}$$

where the bosonic part

$$H_B^\infty \;:=\; -(1 - E_s E_t)\,(1 - e_A e_B)\,\frac{\partial^2}{\partial y_{sA}\,\partial y_{tB}} \;+\; y_{sA}\,y_{sA} \tag{17}$$

is a sum of harmonic oscillators in the variables $\vec{y}_t$ in all $s_d = 2(d-1)$ spatial directions perpendicular to both E and $\vec{e}_t$, with ground state energy equal to $s_d = 2(d-1) = 2, 4, 8, 16$, for $d = 2, 3, 5, 9$, respectively, and ground state eigenvector $\Phi_0(\vec{y}_1, \ldots, \vec{y}_d) = \exp[-(1 - E_s E_t)\,(1 - e_A e_B)\,y_{sA}\,y_{tB}]$.
The fermionic part H_F^∞ of the asymptotic Hamiltonian H^∞ in (16) results from (3) by inserting (13)–(14), with $\vec{y}_t = 0$ and $r = 1$,

$$\begin{aligned} H_F^\infty \;&=\; 2\cos\theta\,(-ie_C\epsilon_{ABC})\,\Gamma_{\alpha\beta}\,\lambda_{\alpha A}^\dagger\,\lambda_{\beta B} \\ &\quad + \sin\theta\,e^{i\varphi}\,(e_C\epsilon_{ABC})\,\lambda_{\alpha A}^\dagger\lambda_{\alpha B}^\dagger + \sin\theta\,e^{-i\varphi}\,(e_C\epsilon_{ABC})\,\lambda_{\alpha B}\lambda_{\alpha A}, \end{aligned} \tag{18}$$

with $\Gamma_{\alpha\beta} := \sum_{t=1}^{d-2} \widetilde{E}_t \Gamma_{\alpha\beta}^t$.
We henceforth assume the unit vectors $\widetilde{E} \in \mathbb{R}^{d-2}$, $\vec{e} \in \mathbb{R}^3$ and $\theta \in (-\pi/2, \pi/2)$, $\varphi \in (0, 2\pi)$ to be fixed. Our goal is to find an explicit unitary transformation on Fock space $\mathcal{F}$ which brings H_F^∞ into normal (i.e., particle number preserving) form. It is a well-known fact (see, e.g., [2, 1]) that, since H_F^∞ is quadratic in the creation and annihilation operators, such a unitary exists and is of the form $\lambda_k^\dagger \mapsto u_{k,\ell}\lambda_\ell^\dagger + v_{k,\ell}\lambda_\ell^\dagger$, i.e., linear. While the existence of such a unitary follows from the diagonalizability of self-adjoint matrices, the determination of that unitary in an *explicit and managable form* can be difficult and depends on the special properties of the model (here: H_F^∞) under consideration.
Note that both $(M_{AB} := -ie_C\epsilon_{ABC})_{A,B=1,2,3}$ and $(\Gamma_{\alpha\beta})_{\alpha,\beta=1}^{d-1}$ are imaginary, antisymmetric, and thus self-adjoint matrices.
Since $M\vec{v} = i\vec{e} \wedge \vec{v}$, an orthonormal basis $\{\vec{e}, \vec{n}_+, \vec{n}_-\} \subseteq \mathbb{C}^3$ of eigenvectors, $M\vec{e} = 0$,

$$M\vec{n}_\pm \;=\; i\vec{e} \wedge \vec{n}_\pm \;=\; \pm\vec{n}_\pm \tag{19}$$

is given by the usual orthonormal *dreibein*: $\vec{e} \perp \vec{n}_+ \perp \vec{n}_-$. We choose $\vec{n}_\pm = \vec{n}_\pm(\vec{e})$ to depend continuously on $\vec{e}$ and to obey $\overline{\vec{n}_\pm} = \vec{n}_\mp$. Hence

$$-ie_C\epsilon_{ABC} \;=\; (\vec{n}_+)_A\,\overline{(\vec{n}_+)_B} - (\vec{n}_-)_A\,\overline{(\vec{n}_-)_B} \;=\; (\vec{n}_+)_A\,(\vec{n}_-)_B - (\vec{n}_-)_A\,(\vec{n}_+)_B. \tag{20}$$

Similary, for $d = 3, 5, 9$, we observe that, due to $\Gamma^2 = \widetilde{E}^2 \cdot \mathbf{1} = \mathbf{1}$ and $\operatorname{Tr}\Gamma = 0$, there is an orthonormal basis $\{\tilde{e}_{\sigma j}|\ \sigma = \pm, j = 1, \ldots, (d-1)/2\} \subseteq \mathbb{C}^{d-1}$ of eigenvectors of Γ such that

$$\Gamma\tilde{e}_{\pm j} \;=\; \pm\tilde{e}_{\pm j}, \tag{21}$$

for all $j = 1, \ldots, (d-1)/2$. Again we choose $\widetilde{E} \mapsto \tilde{e}_{\pm j}$ continuous and $\overline{\tilde{e}_{\pm j}} = \tilde{e}_{\mp j}$. So,

$$\Gamma_{\alpha\beta} \;=\; \sum_j \Big[(\tilde{e}_{+j})_\alpha \, (\tilde{e}_{-j})_\beta \;-\; (\tilde{e}_{-j})_\alpha \, (\tilde{e}_{+j})_\beta \Big], \tag{22}$$

where the summation ranges over $j = 1, \ldots, (d-1)/2$.
Using the orthonormal (eigen)vectors $\tilde{e}_{\pm j}$, $\vec{n}_\pm$, and $\vec{n}_0 := \vec{e}$, we define space-dependent fermion creation operators (for $d = 3,\, 5,\, 9$)

$$\lambda^\dagger_{\sigma j \tau} \;:=\; (\tilde{e}_{\sigma j})_\alpha \, (\vec{n}_\tau)_A \, \lambda^\dagger_{\alpha A}, \tag{23}$$

where $\sigma \in \{+,-\}$, $j = 1, \ldots, (d-1)/2$, and $\tau \in \{+,-,0\}$. Note that the matrix U defined by $U_{\sigma j \tau, \alpha A} := (\tilde{e}_{\sigma j})_\alpha \, (\vec{n}_\tau)_A$ is unitary and, hence, $\lambda^\dagger_{\alpha A} \mapsto \lambda^\dagger_{\sigma j \tau}$ is implemented by a unitary conjugation on (the operators on) Fock space $\mathcal{F}$. I.e., $\lambda_{\sigma j \tau} |0\rangle = 0$ and $\lambda^\dagger_{\sigma j \tau}, \lambda_{\sigma j \tau}$ fulfill the CAR.
Using the new creation operators $\lambda^\dagger_{\sigma j \tau}$, we introduce

$$A^\dagger_j \;:=\; i e^{i\varphi} \, \lambda^\dagger_{+j+} \, \lambda^\dagger_{-j-}, \tag{24}$$

$$B^\dagger_j \;:=\; i e^{-i\varphi} \, \lambda^\dagger_{-j+} \, \lambda^\dagger_{+j-}, \tag{25}$$

and $A_j := (A^\dagger_j)^*$ and $B_j := (B^\dagger_j)^*$, for $j = 1, \ldots, (d-1)/2$, which may be considered *(Cooper) pair creation and annihilation operators.* Note that these operators obey commutation relations somewhat reminiscent to the canonical ones, namely

$$[A^\dagger_j \,,\, A^\dagger_k] \;=\; [B^\dagger_j \,,\, B^\dagger_k] \;=\; [A^\dagger_j \,,\, B^\dagger_k] \;=\; [A_j \,,\, B^\dagger_k] \;=\; 0, \tag{26}$$

$$[A_j \,,\, A^\dagger_k] \;=\; \delta_{kj}\big(N^{(A)}_j - 1\big) \;:=\; \delta_{kj}\big(\lambda^\dagger_{+j+}\,\lambda_{+j+} \;+\; \lambda^\dagger_{-j-}\,\lambda_{-j-} \;-\; 1\big), \tag{27}$$

$$[B_j \,,\, B^\dagger_k] \;=\; \delta_{kj}\big(N^{(B)}_j - 1\big) \;:=\; \delta_{kj}\big(\lambda^\dagger_{-j+}\,\lambda_{-j+} \;+\; \lambda^\dagger_{+j-}\,\lambda_{+j-} \;-\; 1\big). \tag{28}$$

The asymptotic Hamiltonian H^∞, when acting on the ground state of H^∞_B, can be written as

$$s_d \;+\; H^\infty_F \;=\; H^\infty_0 \;+\; H^\infty_+ \;+\; H^\infty_-, \tag{29}$$

where

$$H^\infty_0 \;:=\; s_d \;+\; 2\cos\theta \sum_j (N^{(A)}_j - N^{(B)}_j), \tag{30}$$

$$H^\infty_+ \;:=\; 2\sin\theta \sum_j (A^\dagger_j + B^\dagger_j), \tag{31}$$

$$H^\infty_- \;:=\; 2\sin\theta \sum_j (A_j + B_j). \tag{32}$$

We remark that the degrees of freedom defined by the parallel fermions $\lambda^\dagger_{\pm j 0} = (\tilde{e}_{\pm j})_\alpha e_A \lambda_{\alpha A}$ do not appear in H^∞_F and can be dropped, henceforth.

For the d=2 case we instead of (23) define $\lambda_\pm^\dagger := (\vec{n}_\pm)_A \lambda_A^\dagger$, and the corresponding expressions for the asymptotic Hamiltonian in (29) are simply

$$H_0^\infty = 2, \quad H_+^\infty = 2C^\dagger, \quad H_-^\infty = 2C, \quad C^\dagger := ie^{i\varphi}\,\lambda_+^\dagger\,\lambda_-^\dagger. \tag{33}$$

3 Dynamical Symmetry

For definiteness, we restrict our attention in this and the following sections to the most interesting case: $d = 9$. Denoting

$$J_+ \otimes \mathbf{1} := A^\dagger := \textstyle\sum_j A_j^\dagger, \quad \mathbf{1} \otimes J_+ := B^\dagger := \textstyle\sum_j B_j^\dagger, \tag{34}$$
$$J_- \otimes \mathbf{1} := A := \textstyle\sum_j A_j, \quad \mathbf{1} \otimes J_- := B := \textstyle\sum_j B_j, \tag{35}$$
$$J_3 \otimes \mathbf{1} := \tfrac{1}{2}(N^{(A)} - 4) := \tfrac{1}{2}\big(\textstyle\sum_j N_j^{(A)} - 4\big), \tag{36}$$
$$\mathbf{1} \otimes J_3 := \tfrac{1}{2}(N^{(B)} - 4) := \tfrac{1}{2}\big(\textstyle\sum_j N_j^{(B)} - 4\big), \tag{37}$$

with

$$[J_+, J_-] = 2J_3, \quad [J_3, J_\pm] = \pm J_\pm, \quad J_\pm = J_1 \pm iJ_2, \tag{38}$$

Eqs. (29)–(32) can be written as

$$4 + \frac{1}{4}H_F^\infty = (2 + \cos\theta\, J_3 + \sin\theta\, J_1) \otimes \mathbf{1} + \mathbf{1} \otimes (2 - \cos\theta\, J_3 + \sin\theta\, J_1), \tag{39}$$

thus exhibiting the dynamical symmetry mentioned above. (We recall that a dynamical symmetry refers to the situation that the Hamiltonian, being one of the generators of a symmetry Lie group, has nontrivial commutation relations with the other symmetry generators rather than commuting with them.)

The relevant SU(2) representations are the tensor product of four spin $\frac{1}{2}$ representations, i.e., direct sums of two singlets [note that both $(A_1A_3 + A_2A_4 - A_1A_4 - A_2A_3)|0\rangle$ and $(A_1A_2 + A_3A_4 - A_1A_4 - A_2A_3)|0\rangle$ are annihilated by $A^\dagger$, A , and $\frac{1}{2}(N^{(A)} - 4)$], three spin 1 representations, and (most importantly, as providing the zero-energy state of H) one spin 2 representation acting irreducibly on the space spanned by the orthonormal states

$$|0\rangle, \quad \frac{1}{2}\,A^\dagger|0\rangle, \quad \frac{1}{\sqrt{24}}\,(A^\dagger)^2|0\rangle, \quad \frac{1}{12}\,(A^\dagger)^3|0\rangle, \quad \frac{1}{4!}\,(A^\dagger)^4|0\rangle. \tag{40}$$

Restricting to that space (correspondingly for the $B^\dagger$'s), we can write

$$J_+ \;=\; \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}, \quad J_- \;=\; \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{41}$$

$$J_3 \;=\; \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}. \tag{42}$$

Since the spectrum of $\sin\theta J_1 \pm \cos\theta J_3$ is the same as that of J_3, the spectrum of $4+\frac{1}{4}H_F^\infty$ clearly consists of all integers between zero and eight,

$$\sigma\Big(4+\frac{1}{4}H_F^\infty\Big) \;=\; \{0,1,2,\ldots,8\}. \tag{43}$$

Its unique zero-energy state Ψ is most easily obtained by solving individually, for each $A_j^\dagger$ resp. $B_j^\dagger$ degree of freedom,

$$\Big(1\pm\cos\theta\;\sigma_3^{(j)}+\sin\theta\;\sigma_1^{(j)}\Big)\,\Psi \;=\; e^{\mp\frac{1}{2}\theta i\sigma_2^{(j)}}\big(1\pm\sigma_3^{(j)}\big)e^{\pm\frac{1}{2}\theta i\sigma_2^{(j)}}\,\Psi \;\stackrel{!}{=}\; 0, \tag{44}$$

where we identify

$$2\,J_k \;=\; \sigma_k\otimes 1\otimes 1\otimes 1 \;+\ldots+\; 1\otimes 1\otimes 1\otimes\sigma_k \;\equiv\; \sum_{j=1}^{4}\sigma_k^{(j)}. \tag{45}$$

In our notation $\sigma_3^{(j)}|0\rangle=-|0\rangle$ and $\sigma_3^{(j)}A_j^\dagger|0\rangle=+A_j^\dagger|0\rangle$, and we easily find the solution to (44) as

$$\Psi \;=\; \Big(\prod_j e^{-\frac{\theta}{2}i\sigma_2^{(j)}}\Big)\Big(\prod_j e^{\frac{\theta}{2}i\sigma_2^{(j)}}B_j^\dagger\Big)|0\rangle \;=\; \frac{1}{4!}\,e^{-\theta i(J_2\otimes 1-1\otimes J_2)}\,(B^\dagger)^4|0\rangle. \tag{46}$$

Using the nilpotency of $A_j^\dagger$ and $B_j^\dagger$ for

$$e^{\alpha(A_j^\dagger-A_j)}|0\rangle=\cos\alpha\,e^{\tan\alpha\;A_j^\dagger}|0\rangle \text{ and } e^{-\alpha(B_j^\dagger-B_j)}B_j^\dagger|0\rangle=\sin\alpha\,e^{\cot\alpha\;B_j^\dagger}|0\rangle, \tag{47}$$

the ground state can also be written as

$$\begin{aligned}\Psi \;&=\; \qquad\qquad (48)\\ &=\; \frac{1}{16}\,e^{-4i\varphi}\,(\sin\theta)^{-4}\prod_j\big\{\big(\sin\theta-(1-\cos\theta)A_j^\dagger\big)\big(\sin\theta-(1+\cos\theta)B_j^\dagger\big)\big\}|0\rangle\\ &=\; \frac{1}{16}\,e^{-4i\varphi}\,(\sin\theta)^4\;\exp\big[-\frac{1-\cos\theta}{\sin\theta}A^\dagger\;-\;\frac{1+\cos\theta}{\sin\theta}B\big]|0\rangle \;\sim\; e^{-C_\theta}|0\rangle,\end{aligned}$$

with $C_\theta := \frac{1-\cos\theta}{\sin\theta}(J_+ \otimes 1) + \frac{1+\cos\theta}{\sin\theta}(1 \otimes J_+)$. Alternatively, one can solve the 2×2 matrix eigenvector equations resulting from (44),

$$\big(1 + \cos\theta\,(N_j^{(A)} - 1) + \sin\theta\,(A_j^\dagger + A_j)\big)\Psi = 0, \tag{49}$$

$$\big(1 - \cos\theta\,(N_j^{(B)} - 1) + \sin\theta\,(B_j^\dagger + B_j)\big)\Psi = 0, \tag{50}$$

to obtain (48).
For d=2 the asymptotic ground state is easily found from (33),

$$\Psi = \tfrac{1}{\sqrt{2}} e^{-C^\dagger}|0\rangle = \tfrac{1}{\sqrt{2}}(1 - C^\dagger)|0\rangle. \tag{51}$$

An interesting feature of the form (46) for the ground state is that it expresses it as a spin-rotation by an angle θ applied to some reference state $(B^\dagger)^4|0\rangle$ (which itself also varies in the first $d-2$ directions in space according to (13), (23), (21)).

4 Graded chain of Hamiltonians

We henceforth drop the superscript "∞" and write $H_0 = H_0^\infty$, $H_+ = H_+^\infty$, and $H_- = H_-^\infty$. Consider the grade- resp. fermion number-ordered equations

$$\begin{array}{ccccccl} & & H_0\Psi_0 & + & H_-\Psi_2 & = & 0, \\ H_+\Psi_0 & + & H_0\Psi_2 & + & H_-\Psi_4 & = & 0, \\ H_+\Psi_2 & + & H_0\Psi_4 & + & H_-\Psi_8 & = & 0, \\ & & \vdots & & & & \\ H_+\Psi_{12} & + & H_0\Psi_{14} & + & H_-\Psi_{16} & = & 0, \\ H_+\Psi_{14} & + & H_0\Psi_{16} & & & = & 0, \end{array} \tag{52}$$

which we obtain by writing $\Psi = \sum_{n=0}^{16} \Psi_n$, requiring $(N^{(A)} + N^{(B)})\Psi_n = n\,\Psi_n$ and the ground state equation

$$(16 + H_F^\infty)\Psi = (H_0 + H_+ + H_-)(\Psi_0 + \Psi_2 + \ldots + \Psi_{16}) \stackrel{!}{=} 0. \tag{53}$$

(Recall that we have dropped the eight non-dynamical parallel fermions $\lambda_{\pm j0}^\dagger = (\tilde{e}_{\pm j})_\alpha e_A \lambda_{\alpha A}$.)
The following method to construct the ground state we believe to be relevant also for the fully interacting, non-asymptotic theory. We use the first equation in (52) to express Ψ_0 in terms of Ψ_2,

$$\Psi_0 = -H_0^{-1} H_- \Psi_2. \tag{54}$$

H_0 is certainly invertible on the zero-fermion subspace, even in the full theory (cf. [9]). Inserting (54), the second equation in (52) can be written as

$$H_2\Psi_2 + H_-\Psi_4 = 0, \quad \text{with} \quad H_2 := H_0 - H_+ H_0^{-1} H_-, \tag{55}$$

which yields

$$\Psi_2 \; = \; -H_2^{-1} H_- \Psi_4, \tag{56}$$

provided H_2 is invertible on $H_-\Psi_4$, resp. the two-fermion sector of Fock space. Continuing in this manner, denoting

$$\widehat{\mathscr{H}_{2k}} \; := \; \operatorname{span}\left\{(A^\dagger)^m (B^\dagger)^n |0\rangle\right\}_{m,n=0,1,2,3,4,\; m+n=k} \tag{57}$$

for the considered $2k$-fermion subspace, we find that if we assume invertibility of H_{2k} on $\widehat{\mathscr{H}_{2k}}$ we can form

$$H_{2(k+1)} \; := \; H_0 - H_+ H_{2k}^{-1} H_- \tag{58}$$

on $\widehat{\mathscr{H}_{2(k+1)}}$ and solve for Ψ_{2k} in terms of $\Psi_{2(k+1)}$. The final equation for Ψ_{16} is $H_{16}\Psi_{16} = 0$.
For concreteness, denote an orthonormal basis of $\widehat{\mathscr{H}} = \oplus_k \widehat{\mathscr{H}_{2k}}$ by $|k,l\rangle := |k\rangle \otimes |l\rangle$, where, as in (40),

$$|k\rangle := \frac{1}{k!\sqrt{\binom{4}{k}}} J_+^k |0\rangle. \tag{59}$$

Then, e.g., $H_+ H_0^{-1} H_-$ acts on $\widehat{\mathscr{H}}$ 'tridiagonally' according to

$$\begin{aligned}
\frac{1}{\sin^2\theta} H_+ H_0^{-1} H_- |k,l\rangle \; = \;& \\
& \left(\frac{k(5-k)}{4+(k-l-1)\cos\theta} + \frac{l(5-l)}{4+(k-l+1)\cos\theta}\right)|k,l\rangle \\
& + \frac{\sqrt{l(5-l)(k+1)(4-k)}}{4+(k-l+1)\cos\theta}|k+1,l-1\rangle \\
& + \frac{\sqrt{k(5-k)(l+1)(4-l)}}{4+(k-l-1)\cos\theta}|k-1,l+1\rangle.
\end{aligned} \tag{60}$$

Calculating the spectra of H_{2k} on $\widehat{\mathscr{H}_{2k}}$ (e.g., with the help of a computer) one can verify the invertibility of all H_{2k} on $\widehat{\mathscr{H}_{2k}}$ for $k < 8$, while H_{16} is identically zero on $\widehat{\mathscr{H}_{16}}$. Hence, one can also start with the state $\Psi_{16} \sim A^4 B^4 |0\rangle$ (with correct normalization in θ) and generate the lower grade parts of the full asymptotic ground state Ψ using the relations (54), (56), etc.
We finish this section by noting a simple consequence of the graded form (52) of the ground state equation $H\Psi = 0$ (for general d and N). Taking the inner product of the grade $2k$-equation with Ψ_{2k} yields

$$\begin{aligned}
\langle \Psi_{2k}, H_- \Psi_{2(k+1)} \rangle \;&=\; -\langle H_0 \rangle_{2k} \,-\, \langle \Psi_{2k}, H_+ \Psi_{2(k-1)} \rangle \\
&=\; -\langle H_0 \rangle_{2k} \,-\, \overline{\langle \Psi_{2(k-1)}, H_- \Psi_{2k} \rangle},
\end{aligned} \tag{61}$$

where $\langle H_0\rangle_{2k} := \langle\Psi_{2k}, H_0\Psi_{2k}\rangle$. The first equation reads $\langle\Psi_0, H_-\Psi_2\rangle = -\langle H_0\rangle_0$ which is real. The second then becomes $\langle\Psi_2, H_-\Psi_4\rangle = -\langle H_0\rangle_2 + \langle H_0\rangle_0$, and so on, so that in the last step one obtains

$$\sum_{k=0}^{\Lambda}(-1)^k\langle H_0\rangle_{2k} \;=\; 0, \tag{62}$$

where Λ is the total number of fermions in the relevant Fock space.
It is instructive to verify (62) for the asymptotic $N = 2$ case studied above, since there all relevant terms can be calculated explicitly. Using the basis (59) and the notation $\alpha := 1 - \cos\theta$, $\beta := 1 + \cos\theta$, we find

$$\Psi \sim e^{-C_\theta}|0\rangle \;=\; \sum_k \frac{(-1)^k\sqrt{\binom{4}{k}}}{(\sin\theta)^k}\,\alpha^k\,|k\rangle \;\otimes\; \sum_l \frac{(-1)^l\sqrt{\binom{4}{l}}}{(\sin\theta)^l}\,\beta^l\,|l\rangle. \tag{63}$$

Hence,

$$\langle\Psi_{2n}, H_0\Psi_{2n}\rangle \;=\; \frac{1}{64}(\sin\theta)^{8-2n}\sum_{k+l=n}\binom{4}{k}\binom{4}{l}\big(4+(k-l)\cos\theta\big)\,\alpha^{2k}\,\beta^{2l}. \tag{64}$$

5 General SU(N)

We now compute the ground state energy of

$$\widetilde{H}_F \;=\; i\,f_{ABC}\,x_{tC}\,\gamma^t_{\hat\alpha,\hat\beta}\,\Theta_{\hat\alpha,A}\,\Theta_{\hat\beta,B}, \tag{65}$$

for general $N \geq 2$ and in regions of the configuration space where the potential V is zero (see eqs. (2), (3), and recalling that f_{ABC} denote the structure constants of SU(N) in an orthonormal basis). By (10), the vanishing of the potential V means that all X_s are commuting, hence can be written $X_s = UD_sU^\dagger$ where U is unitary and independent of s and the D_s are diagonal. If we look into a particular direction (which corresponds to fixing $\vec{e}$ in the SU(2) case) and choose a basis $\{T_A\}$ accordingly, we may write $X_s = D_s = x_{sA}T_A = x_{s\tilde k}T_{\tilde k}$ and $x_{sa} = 0$, where $\tilde k = 1, \ldots, N-1$ are indices in the Cartan subalgebra and $a, b = N, \ldots, N^2-1$ denote the remaining indices.
Denoting the eigenvalues of X_t by μ^t_k, i.e., $X_t = \mathrm{diag}(\mu^t_1, \ldots, \mu^t_N)$, the eigenvectors $\{e_{kl}\}_{k\neq l}$ of $M^t_{ab} := -if_{abC}x_{tC} = -if_{ab\tilde k}x_{t\tilde k}$ satisfy (cf. e.g. [8])

$$M^t e_{kl} \;=\; (\mu^t_k - \mu^t_l)e_{kl} \;=:\; \mu^t_{kl}e_{kl}, \quad (e^a_{kl})^* \;=\; e^a_{lk}. \tag{66}$$

The crucial observation is that these eigen*vectors* are independent of t. Now,

$$\widetilde{H}_F \;=\; -\gamma^t_{\hat\alpha,\hat\beta}\,M^t_{ab}\,\Theta_{\hat\alpha,A}\,\Theta_{\hat\beta,B}, \;=\; W_{\hat\alpha a,\hat\beta b}\,\Theta_{\hat\alpha,a}\,\Theta_{\hat\beta,b}, \tag{67}$$

where $W := -\sum_t \gamma^t \otimes M^t$. From the above observations we have the ansatz $E_{\mu kl} := v_\mu \otimes e_{kl}$ for the eigenvectors of W, yielding

$$W\,E_{\mu kl} \;=\; -\sum_t \gamma^t\,v \otimes M^t\,e_{kl} \;=\; \gamma(k,l)\,v_\mu \otimes e_{kl}, \tag{68}$$

where $\gamma(k,l) := -\sum_t \mu^t_{kl}\gamma^t$ squares to $\sum_t(\mu^t_{kl})^2$. Denoting by $v_\mu = v_{\pm jkl}$ the corresponding 16 eigenvectors of $\gamma(k,l)$, we find

$$W\, E_{\pm jkl} \;=\; \pm\sqrt{\sum_t (\mu^t_{kl})^2}\; E_{\pm jkl}, \tag{69}$$

and $\widetilde{H}_F$ therefore has

$$E_0 \;:=\; -16\sum_{k<l}\sqrt{\sum_{t=1}^{9}(\mu^t_k - \mu^t_l)^2} \tag{70}$$

as its lowest eigenvalue.
This agrees with the following two previously known cases: [8], where only X_9 is assumed to have large eigenvalues so that $E_0 \to -16\sum_{k<l}|\mu^9_k - \mu^9_l|$; as well as the SU(2)-case studied above and in [7], where (13) with, e.g., $e_A = \delta_{A3}$ gives $E_0 = -16r$. Note also [10], where the eigenvalues of $\widetilde{H}_F$ are stated, with the SU(N) symmetry fixed.

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Equivariant Iwasawa Theory: An Example

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Abstract. The equivariant main conjecture of Iwasawa theory is shown to hold for a Galois extension K/k of totally real number fields with Galois group an l-adic pro-l Lie group of dimension 1 containing an abelian subgroup of index l, provided that Iwasawa's μ-invariant $\mu(K/k)$ vanishes.

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This note justifies a remark made in the introduction of [6] according to which the "main conjecture" of equivariant Iwasawa theory, as formulated in [2, p.564], holds when $G = G(K/k)$ is a pro-l group with an abelian subgroup G' of index l.

We quickly repeat the general set-up and, in doing so, refer the reader to [5,§1] for facts and notation that is taken from our earlier papers on Iwasawa theory. Namely, l is a fixed odd prime number and K/k a Galois extension of totally real number fields, with $k/\mathbb{Q}$ and K/k_∞ finite, where k_∞ is the cyclotomic $\mathbb{Z}_l$-extension of k. Throughout it will be assumed that Iwasawa's μ-invariant $\mu(K/k)$ vanishes. We also fix a finite set S of primes of k containing all primes above ∞ and all those whose ramification index in K/k is divisible by l.

In this situation it is shown in [5] that the "main conjecture" of equivariant Iwasawa theory would follow from two kinds of hypothetical congruences between values of Iwasawa L-functions. One of these kinds, the so-called *torsion congruences* (see [5, Proposition 3.2]), stated as

$$\frac{\mathrm{ver}(\lambda_{K_{\mathrm{ab}}/k})}{\lambda_{K/k'}} \equiv 1 \mod \mathcal{T}'$$

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in the proof of the proposition in §2, has meanwhile been verified in [6].

The purpose of the present paper is to show that the torsion congruences already suffice to obtain the whole conjecture in the special case when G is a pro-l group with an abelian subgroup G' of index l. Before stating the precise theorem we need to recall some notation (compare [5,§1]).

> $\Lambda_\wedge G$ is the l-completion of the localization $\Lambda_\bullet G$ which is obtained from the Iwasawa algebra $\Lambda G = \mathbb{Z}_l[[G]]$ by inverting all central elements which are regular in $\Lambda G/l\Lambda G$; $\mathcal{Q}_\wedge G$ is the total ring of fractions of $\Lambda_\wedge G$;
>
> $T(\mathcal{Q}_\wedge G) = \mathcal{Q}_\wedge G/[\mathcal{Q}_\wedge G, \mathcal{Q}_\wedge G]$ is the quotient of $\mathcal{Q}_\wedge G$ by Lie commutators;
>
> if G is a pro-l group, then (see [5,§2] [2])
>
> $$\text{(LD)}\qquad \begin{array}{ccc} K_1(\Lambda_\wedge G) & \xrightarrow{\mathbb{L}} & T(\mathcal{Q}_\wedge G) \\ \mathrm{Det}\downarrow & & \overset{\mathrm{Tr}}{\simeq}\downarrow \\ \mathrm{HOM}(R_l G, (\Lambda_\wedge^c \Gamma_k)^\times) & \xrightarrow{\mathbf{L}} & \mathrm{Hom}^*(R_l G, \mathcal{Q}_\wedge^c \Gamma_k) \end{array}$$
>
> is the logarithmic diagram defining the *logarithmic pseudomeasure*
>
> $$t_{K/k} \in T(\mathcal{Q}_\wedge G) \quad \text{by} \quad \mathrm{Tr}(t_{K/k}) = \mathbf{L}(L_{K/k})$$
>
> where $L_{K/k} = L_{K/k,S} \in \mathrm{HOM}(R_l G, (\Lambda_\wedge^c \Gamma_k)^\times)$ is the Iwasawa L-function.

Theorem. *With K/k and S as at the beginning and $G = G(K/k)$ a pro-l group, $t_{K/k}$ is integral (i.e., $t_{K/k} \in T(\Lambda_\wedge G)$) whenever G has an abelian subgroup G' of index l.*

As a corollary, by [5, Proposition 3.2] and [6, Theorem], $L_{K/k} \in \mathrm{Det}\, K_1(\Lambda_\wedge G)$, which implies the conjecture (see [3, Theorem A]), up to its uniqueness assertion. However, $SK_1(\mathcal{Q}G) = 1$ because each simple component, after tensoring up with a suitable extension field of its centre, becomes isomorphic to a matrix ring of dimension a divisor of l^2 by the proof of [2, Proposition 6], as the character degrees $\chi(1)$ all divide l. Now apply [7, p.334, Corollary].

The proof of the theorem is carried out in §2; before, in a short §1, we introduce restriction maps

$$\mathrm{Res}_G^{G'} : T(\mathcal{Q}_\wedge G) \to T(\mathcal{Q}_\wedge G')$$

and

$$\mathrm{Res}_G^{G'} : \mathrm{Hom}^*(R_l G, \mathcal{Q}_\wedge^c \Gamma_k) \to \mathrm{Hom}^*(R_l G', \mathcal{Q}_\wedge^c \Gamma_{k'})$$

---

[2] $R_l G$ is the ring of all (virtual) $\mathbb{Q}_l^c$-characters of G with open kernel; $\Gamma_k = G(k_\infty/k)$; $\Lambda_\wedge^c \Gamma_k = \mathbb{Z}_l^c \otimes_{\mathbb{Z}_l} \Lambda_\wedge \Gamma_k$ with $\mathbb{Z}_l^c$ the ring of integers in a fixed algebraic closure $\mathbb{Q}_l^c$ of $\mathbb{Q}_l$

making the diagram

$$\begin{array}{ccccc} K_1(\Lambda_\wedge G) & \xrightarrow{\mathbb{L}} & T(\mathcal{Q}_\wedge G) & \xrightarrow{\mathrm{Tr}} & \mathrm{Hom}^*(R_l G, \mathcal{Q}^{\mathrm{c}}_\wedge \Gamma_k) \\ \mathrm{res}\,_G^{G'} \downarrow & & \mathrm{Res}_G^{G'} \downarrow & & \mathrm{Res}_G^{G'} \downarrow \\ K_1(\Lambda_\wedge G') & \xrightarrow{\mathbb{L}'} & T(\mathcal{Q}_\wedge G') & \xrightarrow{\mathrm{Tr}'} & \mathrm{Hom}^*(R_l G', \mathcal{Q}^{\mathrm{c}}_\wedge \Gamma_{k'}) \end{array}$$

commute [3] for any pair of pro-l groups $G = G(K/k)$ and $G' = G(K/k') \leq G$ such that $[G : G']$ is finite. We remark that replacing $\mathrm{Res}_G^{G'}$ by the "natural" restriction map,

$$(\mathrm{res}\,_G^{G'} f)(\chi') = f(\mathrm{ind}\,_{G'}^{G} \chi')\,,\ f \in \mathrm{Hom}^*(R_l G, \mathcal{Q}^{\mathrm{c}}_\wedge \Gamma_k)\,,\ \chi' \in R_l G'\,,$$

does not work, because induction and Adams operations do not commute.

1. Res

Let $G = G(K/k)$ be a pro-l group and $G' = G(K/k') \leq G$ an open subgroup. Recall that $\Psi : \Lambda^{\mathrm{c}}_\wedge \Gamma_k \to \Lambda^{\mathrm{c}}_\wedge \Gamma_k$ is the map induced by $\Psi(\gamma) = \gamma^l$ for $\gamma \in \Gamma_k$ (compare [5,§1]) and that ψ_l is the l^{th} Adams operation on $R_l(-)$.

Definition. $\mathrm{Res}_G^{G'} : \mathrm{Hom}^*(R_l G, \mathcal{Q}^{\mathrm{c}}_\wedge \Gamma_k) \to \mathrm{Hom}^*(R_l G', \mathcal{Q}^{\mathrm{c}}_\wedge \Gamma_{k'})$ *sends* f *to*

$$\mathrm{Res}_G^{G'} f = \big[\, \chi' \mapsto f(\mathrm{ind}\,_{G'}^{G} \chi') + \sum_{r \geq 1} \frac{\Psi^r}{l^r}\,(f(\psi_l^{r-1}\chi))\,\big]\,,$$

where $\chi = \psi_l(\mathrm{ind}\,_{G'}^{G}\chi') - \mathrm{ind}\,_{G'}^{G}(\psi_l \chi')$.

To justify the definition we must show that the sum $\sum_{r\geq 1}$ is actually a finite sum. For this, let $\{t\}$ be a set of coset representatives of G' in G, so $G = \dot\cup_t tG'$, and define

$$m(g) = \min\{r \geq 0 : g^{l^r} \in G'\} \quad \text{for} \quad g \in G\,.$$

Then

$$\mathrm{ind}\,_{G'}^{G} \chi'(g) = \sum_t \dot\chi'(g^t) = \sum_{\{t : m(g^t) = 0\}} \chi'(g^t)\,,$$

if, as usual, $\dot\chi'$ coincides with χ' on G' and vanishes on $G \setminus G'$. Hence,

$$\begin{aligned} &\chi(g) = (\mathrm{ind}\,_{G'}^{G}\chi')(g^l) - \mathrm{ind}\,_{G'}^{G}(\psi_l\chi')(g) \\ &= \textstyle\sum_{m(g^{lt})=0} \chi'(g^{lt}) - \sum_{m(g^t)=0} \chi'(g^{lt}) = \sum_{m(g^t)=1} \chi'(g^{lt})\,. \end{aligned}$$

If r_0 is such that $G^{l^{r_0}} \subset G'$, then $\psi_l^{r_0-1}\chi = 0$ and $\sum_{r\geq 1} = \sum_{r=1}^{r_0-2}$, because the sum $\sum_{m(g^t)=1}$ is empty when $g \in G^{l^{r_0-1}}$.

[3]For finite G, this $\mathrm{Res}_G^{G'}$, making the left square commute, appears already in [1, p.14]. For us the properties (HD),(TD) of $\mathrm{Res}_G^{G'}$ shown in §1 are equally necessary for the proof of the Theorem above.

It remains to show that $\mathrm{Res}_G^{G'} f \in \mathrm{Hom}^*(R_l G', \mathcal{Q}^c_\wedge \Gamma_{k'})$, i.e., $\mathrm{Res}_G^{G'} f$ is a Galois stable homomorphism, compatible with W-twists (see [5,§1]), and taking values in $\mathcal{Q}^c_\wedge \Gamma_{k'}$. The first property is easily checked and the third follows from the second as in [2, proof of Lemma 9]. We turn to twisting.

Let ρ' be a type-W character of G', so ρ' is inflated from $\Gamma_{k'}$, and write $\rho' = \mathrm{res}_G^{G'} \rho$ with ρ inflated from Γ_k to G. Then

$$f(\mathrm{ind}_{G'}^G(\rho'\chi')) = f(\rho \cdot \mathrm{ind}_{G'}^G \chi') = \rho^\sharp(f(\mathrm{ind}_{G'}^G \chi')) = (\rho')^\sharp(f(\mathrm{ind}_{G'}^G \chi'))$$

as $f(\mathrm{ind}_{G'}^G \chi') \in \mathcal{Q}^c_\wedge \Gamma_{k'}$. Moreover, since ψ_l is multiplicative,

$$\psi_l(\mathrm{ind}_{G'}^G(\rho'\chi')) - \mathrm{ind}_{G'}^G(\psi_l(\rho'\chi')) = \psi_l(\rho \cdot \mathrm{ind}_{G'}^G \chi') - \mathrm{ind}_{G'}^G((\rho')^l \cdot \psi_l \chi') = \rho^l \cdot \chi$$

and thus

$$\begin{aligned} &\tfrac{\Psi^r}{l^r}\,(f(\psi_l^{r-1}(\rho^l \cdot \chi)))) = \tfrac{\Psi^r}{l^r}\, f(\rho^{l^r} \cdot \psi_l^{r-1}\chi) = \\ &\tfrac{\Psi^r}{l^r}\,((\rho^{l^r})^\sharp(f(\psi_l^{r-1}\chi))) = \rho^\sharp(\tfrac{\Psi^r}{l^r}\, f(\psi_l^{r-1}\chi)) = (\rho')^\sharp(\tfrac{\Psi^r}{l^r}\, f(\psi_l^{r-1}\chi)) \,. \end{aligned}$$

Lemma 1. *The diagram below commutes. In it,* $\mathbf{L}$ *and* $\mathbf{L}'$ *are the lower horizontal maps of the logarithmic diagram (LD) for* G *and* G'*, respectively.*

$$\text{(HD)} \qquad \begin{array}{ccc} \mathrm{HOM}(R_l G, (\Lambda^c_\wedge \Gamma_k)^\times) & \xrightarrow{\mathbf{L}} & \mathrm{Hom}^*(R_l G, \mathcal{Q}^c_\wedge \Gamma_k) \\ \mathrm{res}_G^{G'} \downarrow & & \mathrm{Res}_G^{G'} \downarrow \\ \mathrm{HOM}(R_l G', (\Lambda^c_\wedge \Gamma_{k'})^\times) & \xrightarrow{\mathbf{L}'} & \mathrm{Hom}^*(R_l G', \mathcal{Q}^c_\wedge \Gamma_{k'}) \,. \end{array}$$

Indeed, for $f \in \mathrm{HOM}(R_l G, (\Lambda^c_\wedge \Gamma_k)^\times)$ we get

$$\begin{aligned} &(\mathrm{Res}_G^{G'} \mathbf{L} f)(\chi') = (\mathbf{L} f)(\mathrm{ind}_{G'}^G \chi') + \textstyle\sum_{r\ge 1} \frac{\Psi^r}{l^r}\,[(\mathbf{L} f)(\psi_l^{r-1}\chi)] \\ &\doteq (\mathbf{L} f)(\mathrm{ind}_{G'}^G \chi') + \textstyle\sum_{r\ge 1} \frac{\Psi^r}{l^r}\,[\log(f(\psi_l^{r-1}\chi)) - \frac{\Psi}{l}\,\log(f(\psi_l^r \chi))] \\ &= (\mathbf{L} f)(\mathrm{ind}_{G'}^G \chi') + \textstyle\sum_{r\ge 1} \frac{\Psi^r}{l^r}\,\log(f(\psi_l^{r-1}\chi)) - \sum_{r\ge 2} \frac{\Psi^r}{l^r}\,\log(f(\psi_l^{r-1}\chi)) \\ &= (\mathbf{L} f)(\mathrm{ind}_{G'}^G \chi') + \tfrac{\Psi}{l}\,\log(f(\chi)) = \tfrac{1}{l}\,\log \tfrac{f(\mathrm{ind}\,\chi')^l}{\Psi(f(\psi_l \mathrm{ind}\,\chi'))} + \tfrac{\Psi}{l}\,\log \tfrac{f(\psi_l \mathrm{ind}\,\chi')}{f(\mathrm{ind}\,\psi_l \chi')} \\ &= \tfrac{1}{l}\,\log \tfrac{f(\mathrm{ind}\,\chi')^l \cdot \Psi f(\psi_l \mathrm{ind}\,\chi')}{\Psi f(\psi_l \mathrm{ind}\,\chi') \cdot \Psi f(\mathrm{ind}\,\psi_l \chi')} = \tfrac{1}{l}\,\log \tfrac{f(\mathrm{ind}\,\chi')^l}{\Psi f(\mathrm{ind}\,\psi_l \chi')} \\ &= (\mathbf{L}' \mathrm{res}_G^{G'} f)(\chi') \,. \end{aligned}$$

The dotted equality sign, $\doteq$, is due to the congruence $\frac{f(\chi)^l}{\Psi f(\psi_l \chi)} \equiv 1 \mod l\Lambda^c_\wedge \Gamma_k$ (see [5,§1]) and to $\chi(1) = 0$, so $(\psi_l^{r-1}\chi)(1) = 0$ for every r. In fact, with $\tilde{\chi} \overset{\mathrm{def}}{=} \psi_l^{r-1}\chi$, we have

$$\begin{aligned} f(\tilde{\chi})^l \equiv \Psi f(\psi_l \tilde{\chi}) \mod l\Lambda^c_\wedge \Gamma_k \implies \\ f(\tilde{\chi})^{l^s} \equiv \Psi^s f(\psi_l^s \tilde{\chi}) \equiv \Psi^s f(\tilde{\chi}(1)1) = 1 \mod l\Lambda^c_\wedge \Gamma_k \end{aligned}$$

for big enough s. Thus $(\mathbf{L} f)(\tilde{\chi}) = \log(f(\tilde{\chi})) - \frac{\Psi}{l}\,\log(f(\psi_l \tilde{\chi}))$ as 'log' converges on an element a power of which is $\equiv 1 \mod l\Lambda^c_\wedge \Gamma_k$.

The proof of Lemma 1 is complete.

By means of the trace isomorphism $\mathrm{Tr} : T(-) \to \mathrm{Hom}^*(-)$ we next transport $\mathrm{Res}_G^{G'}$ to $\mathrm{Res}_G^{G'} : T(\mathcal{Q}_\wedge G) \to T(\mathcal{Q}_\wedge G')$, i.e., the diagram

$$\text{(TD)} \qquad \begin{array}{ccc} T(\mathcal{Q}_\wedge G) & \xrightarrow{\mathrm{Tr}} & \mathrm{Hom}^*(R_l G, \mathcal{Q}_\wedge^{\mathrm{c}} \Gamma_k) \\ \mathrm{Res}_G^{G'} \downarrow & & \mathrm{Res}_G^{G'} \downarrow \\ T(\mathcal{Q}_\wedge G') & \xrightarrow{\mathrm{Tr}'} & \mathrm{Hom}^*(R_l G', \mathcal{Q}_\wedge^{\mathrm{c}} \Gamma_{k'}) \end{array}$$

commutes.

Lemma 2.
$$\begin{array}{ccc} K_1(\Lambda_\wedge G) & \xrightarrow{\mathbb{L}} & T(\mathcal{Q}_\wedge G) \\ \mathrm{res}_G^{G'} \downarrow & & \mathrm{Res}_G^{G'} \downarrow \\ K_1(\Lambda_\wedge G') & \xrightarrow{\mathbb{L}'} & T(\mathcal{Q}_\wedge G') \end{array} \qquad \textit{commutes and } \mathrm{Res}_G^{G'} t_{K/k} = t_{K/k'} \,.$$

The first claim follows from gluing together the diagrams (LD), (HD), (TD) and applying [2, Lemma 9]; the second claim follows from $\mathrm{res}_G^{G'} L_{K/k} = L_{K/k'}$ [2, Proposition 12].

The next lemma already concentrates on the case when G' is abelian and $[G : G'] = l$. We set $A = G/G' = \langle a \rangle$ and observe that a acts on G' by conjugation.

Lemma 3. *Let $\tau : \Lambda_\wedge G \to T(\Lambda_\wedge G)$ denote the canonical map and $g \in G$. If G' is abelian [4] and of index l in G, then*

$$\mathrm{Res}_G^{G'}(\tau g) = \begin{cases} \sum_{i=0}^{l-1} g^{a^i} & \text{if } g \in G' \\ g^l & \text{if } g \notin G' \,. \end{cases}$$

The lemma is just a special case of

Proposition A. *Let H be an open subgroup of $G = G(K/k)$. For $g \in G$ set $m_G^H(g) = \min\{r \geq 0 : g^{l^r} \in H\}$ [5], and let t run through a set of left representatives of H in G, i.e., $G = \dot{\bigcup} tH$. Then*

1. $\mathrm{Res}_G^H \tau_G(g) = \sum_t \tau_H\big((t^{-1}gt)^{l^{m_G^H(t^{-1}gt)}}\big)/l^{m_G^H(t^{-1}gt)}$,
2. Res *is transitive*,
3. Res_G^H *is integral, i.e.,* $\mathrm{Res}_G^H(T(\Lambda_\wedge G)) \subset T(\Lambda_\wedge H)$ *for $H \leq G$ of finite index.*

Proposition A will be shown in the Appendix.

For the purpose of this paper it is enough to know Lemma 3 which we quickly prove directly on applying Tr' to both sides and employing the formula $\mathrm{Tr}'(\tau' g)(\chi') = \chi'(g)\overline{g}$ with $\overline{g}$ denoting the image of $g \in G'$ in $\Gamma_{k'}$ (see [5,§1]):

[4] whence $\tau' : \Lambda_\wedge G' \to T(\Lambda_\wedge G')$ is the identity map
[5] so $m_G^H(g)$ is the $m(g)$ defined earlier with $H = G'$

1. $(\mathrm{Tr}'\mathrm{Res}_G^{G'}(\tau g))(\chi') = \mathrm{Res}_G^{G'}(\mathrm{Tr}(\tau g))(\chi') = \mathrm{Tr}(\tau g)(\mathrm{ind}_{G'}^{G}\chi') + \frac{\Psi}{l}\mathrm{Tr}(\tau g)(\chi)$ since $G^l \subset G'$. Now, if $g \in G'$, $\mathrm{Tr}(\tau g)(\mathrm{ind}_{G'}^{G}\chi') = \sum_{i=0}^{l-1}\chi'(g^{a^i})\overline{g}$ and $\chi(g) = 0$. On the other hand, if $g \notin G'$, $\mathrm{Tr}(\tau g)(\mathrm{ind}_{G'}^{G}\chi') = 0$ and $\frac{\Psi}{l}\mathrm{Tr}(\tau g)(\chi) = \frac{1}{l}\mathrm{ind}_{G'}^{G}\chi'(g^l)\overline{g}^l = \chi'(g^l)\overline{g}^l$ since we may choose $a = g \mod G'$.
2. $\mathrm{Tr}'(\sum_{i=0}^{l-1} g^{a^i})(\chi') = \sum_{i=0}^{l-1}\chi'(g^{a^i})\overline{g}$, since g^{a^i} and g have the same image in Γ_k and so in $\Gamma_{k'}$. On the other hand, $\mathrm{Tr}'(g^l)(\chi') = \chi'(g^l)\overline{g^l}$.

The lemma is established.

We note that if $\Gamma\,(\simeq \mathbb{Z}_l)$ is a central subgroup of G contained in the abelian subgoup G' of index l, then the elements of $T(\Lambda_\wedge G)$ can uniquely be written as $\sum_g \beta_g\tau(g)$ with $\beta_g \in \Lambda_\wedge\Gamma$ and g running through a set of preimages of conjugacy classes of G/Γ (see [3, Lemma 5]). For each summand we have

$$\mathrm{Res}_G^{G'}(\beta_g\tau(g)) = \begin{cases} \sum_{i=0}^{l-1}\beta_g g^{a^i} & \text{if } g \in G' \\ \Psi(\beta_g)g^l & \text{if } g \notin G' \,. \end{cases}$$

2. Proof of the theorem

In this section $G = G(K/k)$ is a pro-l group and $G' = G(K/k')$ an abelian subgroup of index l (K/k is as in the introduction). As before, $A = G/G' = \langle a\rangle$, and we set $\hat{A} = 1 + a + \cdots + a^{l-1}$.

If G itself is abelian, the theorem holds by [4,§5, Example 1], whence we assume that G is non-abelian.

Lemma 4. *Assume that there exists an element $x \in T(\Lambda_\wedge G)$ such that $\mathrm{defl}_G^{G^{\mathrm{ab}}} x = \mathrm{defl}_G^{G^{\mathrm{ab}}} t_{K/k}$ and $\mathrm{Res}_G^{G'} x = \mathrm{Res}_G^{G'} t_{K/k}$. Then $t_{K/k} \in T(\Lambda_\wedge G)$.*

Denoting by K_{ab} the fixed field of the finite group $[G,G]$, we first observe, because of [5, Lemma 2.1] and Lemma 2, that $\mathrm{defl}_G^{G^{\mathrm{ab}}} t_{K/k} = t_{K_{\mathrm{ab}}/k}$ and $\mathrm{Res}_G^{G'} t_{K/k} = t_{K/k'}$ are integral : indeed, a logarithmic pseudomeasure is integral whenever the group is abelian.

From [4, Proposition 9] we obtain a power l^n of l such that $l^n t_{K/k} \in T(\Lambda_\wedge G)$. Consider the element $\tilde{x} = l^n(x - t_{K/k}) \in T(\Lambda_\wedge G)$. It satisfies $\mathrm{defl}_G^{G^{\mathrm{ab}}}\tilde{x} = 0 = \mathrm{Res}_G^{G'}\tilde{x}$. We are going to prove $\tilde{x} = 0$ which implies $x = t_{K/k}$ because $\mathrm{Hom}^*(R_l G, \mathcal{Q}_\wedge^c\Gamma_k)$, and so $T(\mathcal{Q}_\wedge G)$, is torsionfree; whence the the lemma will be verified.

The proof of $\tilde{x} = 0$ employs the commutative diagram shown in the proof of [5, Proposition 2.2] :

$$\begin{array}{ccccc} 1+\mathfrak{a}_\wedge & \rightarrowtail & (\Lambda_\wedge G)^\times & \overset{\mathrm{defl}_G^{G^{\mathrm{ab}}}}{\twoheadrightarrow} & (\Lambda_\wedge G^{\mathrm{ab}})^\times \\ \downarrow & & \mathbb{L}\downarrow & & \mathbb{L}^{\mathrm{ab}}\downarrow \\ \tau(\mathfrak{a}_\wedge) & \rightarrowtail & T(\Lambda_\wedge G) & \overset{\mathrm{defl}_G^{G^{\mathrm{ab}}}}{\twoheadrightarrow} & \Lambda_\wedge G^{\mathrm{ab}} \end{array} ,$$

in which $\mathbb{L}$ is extended to $(\Lambda_\wedge G)^\times$ by means of the canonical surjection $(\Lambda_\wedge G)^\times \twoheadrightarrow K_1(\Lambda_\wedge G)$ and $\mathfrak{a}_\wedge = \ker(\Lambda_\wedge G \to \Lambda_\wedge G^{\mathrm{ab}})$. The diagram yields a $v \in (\Lambda_\wedge G)^\times$ with $\mathbb{L}(v) = \tilde{x}$, simply because $\mathrm{defl}_G^{G^{\mathrm{ab}}} \tilde{x} = 0$. Combining diagrams (HD), (LD) and (TD), we arrive at

$$\mathbf{L}'(\mathrm{res}_G^{G'}(\mathrm{Det}\, v)) = \mathrm{Res}_G^{G'}(\mathbf{L}(\mathrm{Det}\, v)) = \mathrm{Res}_G^{G'}(\mathrm{Tr}\, \mathbb{L}(v)) = \mathrm{Tr}'(\mathrm{Res}_G^{G'} \tilde{x}) = 0$$

and, with $\mathrm{res}_G^{G'}$ replaced by $\mathrm{defl}_G^{G^{\mathrm{ab}}}$, at

$$\begin{aligned}\mathbf{L}^{\mathrm{ab}}(\mathrm{defl}_G^{G^{\mathrm{ab}}}(\mathrm{Det}\, v)) = \mathrm{defl}_G^{G^{\mathrm{ab}}}(\mathbf{L}(\mathrm{Det}\, v)) = \\ = \mathrm{defl}_G^{G^{\mathrm{ab}}}(\mathrm{Tr}\, \mathbb{L}(v)) = \mathrm{Tr}^{\mathrm{ab}}(\mathrm{defl}_G^{G^{\mathrm{ab}}} \tilde{x}) = 0\,,\end{aligned}$$

since $\mathbf{L}$ and Tr commute with deflation.

The first displayed formula in [3, p.46] now implies that $\mathrm{res}_G^{G'}(\mathrm{Det}\, v)$ and $\mathrm{defl}_G^{G^{\mathrm{ab}}}(\mathrm{Det}\, v)$ are torsion elements in $\mathrm{HOM}(R_l G', (\Lambda_\wedge^{\mathrm{c}} \Gamma_{k'})^\times)$ and $\mathrm{HOM}(R_l(G^{\mathrm{ab}}), (\Lambda_\wedge^{\mathrm{c}} \Gamma_k)^\times)$, respectively. Moreover, the first paragraph of the proof of [5, Proposition 3.2] therefore shows that $\mathrm{Det}\, v$ itself is a torsion element in $\mathrm{HOM}(R_l G, (\Lambda_\wedge^{\mathrm{c}} \Gamma_k)^\times)$. Consequently, for some natural number m, $(\mathrm{Det}\, v)^{l^m} = 1$, so $l^m \mathbf{L}(\mathrm{Det}\, v) = 0 = l^m \mathrm{Tr}(\mathbb{L}v) = \mathrm{Tr}(l^m \tilde{x})$, and $\tilde{x} = 0$ follows, as has been claimed.

We now introduce the commutative diagram

$$\begin{array}{ccccc}
\tau(\mathfrak{a}_\wedge) & \rightarrowtail & T(\Lambda_\wedge G) & \twoheadrightarrow & \Lambda_\wedge G^{\mathrm{ab}} = T(\Lambda_\wedge G^{\mathrm{ab}}) \\
\mathrm{Res} \downarrow & & \mathrm{Res}_G^{G'} \downarrow & & \mathrm{Res} \downarrow \\
\mathfrak{b}'_\wedge = \tau'(\mathfrak{b}'_\wedge) & \rightarrowtail & \Lambda_\wedge G' = T(\Lambda_\wedge G') & \twoheadrightarrow & \Lambda_\wedge(G'/[G,G]) = T(\Lambda_\wedge(G'/[G,G]))
\end{array}$$

with exact rows (of which the upper one has already appeared in the diagram shown in the proof of the preceding lemma). The images of all vertical maps are fixed elementwise by A because of Lemma 3. Thus we can turn the diagram into

$$\text{(D)} \qquad \begin{array}{ccccc}
\tau(\mathfrak{a}_\wedge) & \rightarrowtail & T(\Lambda_\wedge G) & \twoheadrightarrow & \Lambda_\wedge G^{\mathrm{ab}} \\
\mathrm{Res} \downarrow & & \mathrm{Res}_G^{G'} \downarrow & & \mathrm{Res} \downarrow \\
\mathfrak{b}'^{\,A}_\wedge & \rightarrowtail & (\Lambda_\wedge G')^A & \to & (\Lambda_\wedge(G'/[G,G]))^A \\
\downarrow & & \downarrow & & \downarrow \\
\hat{H}^0(A, \mathfrak{b}'_\wedge) & \to & \hat{H}^0(A, \Lambda_\wedge G') & \to & \hat{H}^0(A, \Lambda_\wedge(G'/[G,G]))
\end{array}$$

with exact rows and canonical lower vertical maps.

Lemma 5. *In (D), the left vertical column is exact and the left bottom horizontal map is injective.*

Proof. The ideal $\mathfrak{a}_\wedge$ is (additively) generated by the elements $g(c-1)$ with $g \in G$, $c \in [G,G]$; those with $g \in G'$ generate $\mathfrak{b}'_\wedge$. We compute $\mathrm{Res}_G^{G'} \tau(g(c-1))$, using Lemma 3 :

1. if $g \in G'$, $\mathrm{Res}_G^{G'}\tau(g(c-1)) = \sum_{i=0}^{l-1}((gc)^{a^i} - g^{a^i}) = \sum_{i=0}^{l-1}\Big(g(c-1)\Big)^{a^i} \in \mathrm{tr}_A \mathfrak{b}'_\wedge$,
2. if $g \notin G'$, $\mathrm{Res}_G^{G'}\tau(g(c-1)) = \mathrm{Res}_G^{G'}(\tau(gc)-\tau(g)) = (gc)^l - g^l = g^l c^{\hat{A}} - g^l = 0$, since

   $$(\star) \qquad [G,G]^{\hat{A}} = 1$$

   by $[G,G] \doteq [G,G']$ and $[G,G']^{\hat{A}} = ((G')^{a-1})^{\hat{A}} = 1$ as $(a-1)\hat{A} = 0$. Here, the dotted equality sign, $\doteq$, results from the equation

   $$[bg'_1, b^i g'_2] = (g'_1)^{-1} b^{-1} (g'_2)^{-1} b^{-i} b g'_1 b^i g'_2 = (g'_1)^{-1} ({g'_2}^{-1})^b (g'_1)^{b^i} g'_2 = \Big((g'_1)^{-1}(g'_1)^{b^i}\Big)\Big(({g'_2}^{-1})^b g'_2\Big) \in [G',G]\cdot[G,G'] \le [G,G']$$

   for $g'_1, g'_2 \in G'$ and $b \in G \setminus G'$, because G' is abelian and normal in G.

Thus, $\mathrm{Res}_G^{G'}\tau(\mathfrak{a}_\wedge) = \mathrm{tr}_A \mathfrak{b}'_\wedge$, which proves the first claim of the lemma.
The second claim follows from $\hat{H}^{-1}(A, \Lambda_\wedge(G'/[G,G])) = 0$ and this in turn from the trivial action of A on $G'/[G,G]$ and the torsion freeness of $\Lambda_\wedge(G'/[G,G])$. Lemma 5 is established.

As seen in diagram (D), there is an element $x_1 \in T(\Lambda_\wedge G)$ with $\mathrm{defl}_G^{G^{\mathrm{ab}}} x_1 = t_{K_{\mathrm{ab}}/k}$. We define $x'_1 \in \Lambda_\wedge G'$ by $\mathrm{Res}_G^{G'} x_1 = t_{K/k'} + x'_1$. Because of [5, Lemma 3.1], x'_1 is fixed by A. We want to change x_1 modulo $\tau(\mathfrak{a}_\wedge)$ so that the new x'_1 becomes zero: then we have arrived at an $x \in T(\Lambda_\wedge G)$ as assumed in Lemma 4 and the theorem will have been confirmed.
The above change is possible if, and only if, $x'_1 \in \mathrm{Res}_G^{G'}(\tau(\mathfrak{a}_\wedge))$ and so, because of Lemma 5, if x'_1 is in $\mathcal{T}' \overset{\mathrm{def}}{=} \mathrm{tr}_A(\Lambda_\wedge G')$, the A-trace ideal of the A-action on $\Lambda_\wedge G'$.

Proposition. *There exists an element x_1 of $T(\Lambda_\wedge G)$ with $\mathrm{defl}_G^{G^{\mathrm{ab}}} x_1 = t_{K_{\mathrm{ab}}/k}$ and $x'_1 \in \mathcal{T}'$.*

This is seen as follows. From [5,§1] we recall the existence of *pseudomeasures* $\lambda_{K_{\mathrm{ab}}/k}, \lambda_{K/k'}$ in $K_1(\Lambda_\wedge G^{\mathrm{ab}})$ and $K_1(\Lambda_\wedge G')$, respectively, satisfying $\mathrm{Det}\,\lambda_{K_{\mathrm{ab}}/k} = L_{K_{\mathrm{ab}}/k}$, $\mathrm{Det}\,\lambda_{K/k'} = L_{K/k'}$ (so $\mathbb{L}^{\mathrm{ab}}(\lambda_{K_{\mathrm{ab}}/k}) = t_{K_{\mathrm{ab}}/k}$, , $\mathbb{L}'(\lambda_{K/k'}) = t_{K/k'}$). From [5, 2. of Proposition 3.2] and [6, Theorem] we know that

$$\frac{\mathrm{ver}(\lambda_{K_{\mathrm{ab}}/k})}{\lambda_{K/k'}} \equiv 1 \mod \mathcal{T}'$$

where 'ver' is the map induced from the transfer homomorphism $G^{\mathrm{ab}} \to G'$.
Let $y \in (\Lambda_\wedge G)^\times$ have $\mathrm{defl}_G^{G^{\mathrm{ab}}} y = \lambda_{K_{\mathrm{ab}}/k}$ and set $\mathrm{res}_G^{G'} y = \lambda_{K/k'} \cdot y'$. Then

$$y' = \frac{\mathrm{res}_G^{G'} y}{\lambda_{K/k'}} \equiv \frac{\mathrm{ver}(\lambda_{K_{\mathrm{ab}}/k})}{\lambda_{K/k'}} \equiv 1 \mod \mathcal{T}'$$

(see the proof of [5, Proposition 3.2]). Moreover, $y' \in 1 + \mathfrak{b}'_\wedge$. Now, $x_1 \stackrel{\text{def}}{=} \mathbb{L}(y)$ has $\mathrm{Res}_G^{G'} x_1 = \mathrm{Res}_G^{G'} \mathbb{L}(y) = t_{K/k'} + x'_1$ with $x'_1 \stackrel{\text{def}}{=} \mathbb{L}'(y')$, and $x'_1 \in \mathfrak{b}'_\wedge$ because of the commutativity of

$$\begin{array}{ccccc} 1+\mathfrak{b}'_\wedge & \rightarrowtail & (\Lambda_\wedge G')^\times & \twoheadrightarrow & (\Lambda_\wedge(G'/[G,G]))^\times \\ \mathbb{L}' \downarrow & & \mathbb{L}' \downarrow & & \overline{\mathbb{L}'} \downarrow \\ \mathfrak{b}'_\wedge & \rightarrowtail & \Lambda_\wedge G' & \twoheadrightarrow & \Lambda_\wedge(G'/[G,G]) \end{array} .$$

Hence, the proposition (and therefore the theorem) will be proved, if

$$x'_1 = \mathbb{L}'(y') \in \mathcal{T}' .$$

However, Lemma 5 gives

$$y' \in (1 + \mathfrak{b}'^{\,A}_\wedge) \cap (1 + \mathcal{T}') = 1 + (\mathfrak{b}'^{\,A}_\wedge \cap \mathcal{T}') = 1 + \mathrm{tr}_A \mathfrak{b}'_\wedge$$

and as $\mathbb{L}'(y') = \frac{1}{l} \log \frac{y'^l}{\Psi(y')}$ (compare [3, p.39]), we see that

$$(1) \qquad \mathbb{L}'(y') \in \mathcal{T}' \quad \text{if} \quad \frac{y'^l}{\Psi(y')} \equiv 1 \mod l\mathcal{T}' .$$

So it suffices to show this last congruence.

Write $y' = 1 + \mathrm{tr}_A\beta'$ with $\beta' \in \mathfrak{b}'_\wedge$. Since $(1 + \mathrm{tr}_A\beta')^l \equiv 1 + (\mathrm{tr}_A\beta')^l \mod l\mathcal{T}'$, the congruence in (1) is equivalent to

$$(2) \qquad (\mathrm{tr}_A\beta')^l \equiv \Psi(\mathrm{tr}_A\beta') \mod l\mathcal{T}'.$$

On picking a central subgroup $\Gamma\,(\simeq \mathbb{Z}_l)$ of G and writing $\beta' = \sum_{g',c} \beta_{g',c}\, g'(c-1)$ with elements $\beta_{g',c} \in \Lambda_\wedge\Gamma$, $g' \in G'$, $c \in [G,G]$, we obtain

$$\text{(a)} \qquad \begin{aligned} (\mathrm{tr}_A\beta')^l &= \Big(\textstyle\sum_{g',c} \beta_{g',c}\mathrm{tr}_A(g'(c-1))\Big)^l \\ &\equiv \textstyle\sum_{g',c} (\beta_{g',c})^l \Big(\mathrm{tr}_A(g'(c-1))\Big)^l \\ &\equiv \textstyle\sum_{g',c} \Psi(\beta_{g',c}) \Big((\mathrm{tr}_A(g'c))^l - (\mathrm{tr}_A g')^l\Big) \mod l\mathcal{T}' \end{aligned}$$

and

$$\text{(b)} \qquad \Psi(\mathrm{tr}_A\beta') = \textstyle\sum_{g',c} \Psi(\beta_{g',c}) \Big(\mathrm{tr}_A((g'c)^l) - \mathrm{tr}_A(g'^l)\Big)$$

as Ψ and tr_A commute. Thus congruence (2) will result from Lemma 6 below, since then subtracting (b) from (a) yields the sum

$$\begin{aligned} &\textstyle\sum_{g',c} \Psi(\beta_{g',c}) \Big((\mathrm{tr}_A(g'c))^l - \mathrm{tr}_A((g'c)^l) - (\mathrm{tr}_A g')^l + \mathrm{tr}_A(g'^l)\Big) \equiv \\ &\textstyle\sum_{g',c} \Psi(\beta_{g',c}) \Big(-l(g'c)^{\hat{A}} + l g'^{\hat{A}}\Big) \equiv \\ &\textstyle\sum_{g',c} (-l)\Psi(\beta_{g',c}) g'^{\hat{A}}(c^{\hat{A}} - 1) \equiv 0 \mod l\mathcal{T}' , \end{aligned}$$

by $(\star)$ of the proof of Lemma 5.

Lemma 6. $(\mathrm{tr}_A g')^l - \mathrm{tr}_A(g'^l) \equiv -l g'^{\hat{A}} \mod l\mathcal{T}'$ *for* $g' \in G'$.

Proof. Set $\tilde{A} = \mathbb{Z}/l \times A$ and make $M = \mathrm{Maps}(\mathbb{Z}/l, A)$ into an $\tilde{A}$-set by defining $m^{(z,a^i)}(x) = m(x-z)\cdot a^i$. Then

$$(\mathrm{tr}_A g')^l = (\sum_{i=0}^{l-1} g'^{a^i})^l = \sum_{m\in M}\prod_{z\in\mathbb{Z}/l} g'^{m(z)} = \sum_{m\in M} g'^{\sum_{z\in\mathbb{Z}/l} m(z)}$$

with $\sum_z m(z)$ read in $\mathbb{Z}[A]$.

We compute the subsums of $\sum_m$ in which m is constrained to an $\tilde{A}$-orbit.

If $m \in M$ has stabilizer $\{(0,1)\}$ in $\tilde{A}$, then the $\tilde{A}$-orbit sum is

$$\sum_{(z,a^i)\in\tilde{A}} g'^{\sum_{v\in\mathbb{Z}/l} m^{(z,a^i)}(v)} = \sum_{(z,a^i)\in\tilde{A}} g'^{\sum_{v\in\mathbb{Z}/l} m(v-z)a^i} =$$
$$\sum_{(z,a^i)} g'^{\sum_v m(v)a^i} = l\sum_i (g'^{\sum_v m(v)})^{a^i} = l\cdot \mathrm{tr}_A(g'^{\sum_v m(v)}) \in l\mathcal{T}' \ .$$

Note that no $m \in M$ is stabilized by $(0,a^i)$ with $a^i \neq 1$: for $m(z) = m^{(0,a^i)}(z) = m(z)a^i$ implies $a^i = 1$. It follows that the stabilizers of the elements with stabilizer different from $\{(0,1)\}$ must be cyclic of order l and different from $\{(0,a^i) : 0 \le i \le l-1\}$ and therefore $= \langle(1,a^j)\rangle$ for a unique $j \mod l$.

One now checks that for each j there is exactly one $\tilde{A}$-orbit with stabilizer $\langle(1,a^j)\rangle$ and that it is represented by m_j , $m_j(z) = a^{jz}$. Moreover, $\{(0,a^i) : 0 \le i \le l-1\}$ is a transversal of the stabilizer of m_j in $\tilde{A}$.

For each j, the sum of $g'^{\sum_z m(z)}$ over the $\tilde{A}$-orbit of m_j is $\sum_i g'^{\sum_z m_j^{(0,a^i)}(z)} = \sum_i g'^{\sum_z a^{jz}a^i}$. If $j = 0$, this is $\sum_i g'^{la^i} = \mathrm{tr}_A(g'^l)$, accounting for that term in the claim. If $j \neq 0$, it is $\sum_i g'^{\hat{A}\cdot a^i} = l g'^{\hat{A}}$, and summing over $j \neq 0$ gives $(l-1)l\cdot g'^{\hat{A}} \equiv -l\cdot g'^{\hat{A}} \mod l\mathcal{T}'$ because $l^2\cdot g'^{\hat{A}} = l\cdot \mathrm{tr}_A(g'^{\hat{A}}) \in l\mathcal{T}'$.

This finishes the proof of Lemma 6.

3. Appendix

Proof of Proposition A :

We start from the formula $\chi(g) = \sum_{m_G^H(g^t)=1} \chi'(g^{lt})$ which appeared in the justification of the definition in §1 (now with $H = G'$). It implies $\chi(g^{l^{r-1}}) = \sum_{m_G^H(t^{-1}gt)=r} \chi'((t^{-1}gt)^{l^r})$ for $r \ge 1$. Exploiting [3, Lemma 6] for the equality

$\doteq$ and [3, Proposition 3] for $\ddot{=}$ we obtain

$$\begin{aligned}
&(\mathrm{Res}_G^H \mathrm{Tr}_G \tau_G(g))(\chi') \\
&= (\mathrm{Tr}_G \tau_G(g))(\mathrm{ind}\,_H^G \chi') + \textstyle\sum_{r\geq 1} \frac{\Psi^r}{l^r} [(\mathrm{Tr}_G \tau_G(g))(\psi_l^{r-1}\chi)] \\
&\doteq \mathrm{trace}(g \mid \mathfrak{V}_{\mathrm{ind}\,_H^G \chi'}) + \textstyle\sum_{r\geq 1} \frac{\Psi^r}{l^r} [\mathrm{trace}(g \mid \mathfrak{V}_{\psi_l^{r-1}\chi})] \\
&\ddot{=} (\mathrm{ind}\,_H^G \chi')(g)\overline{g} + \textstyle\sum_{r\geq 1} \frac{\Psi^r}{l^r} [(\psi_l^{r-1}\chi)(g)\overline{g}] \\
&= (\mathrm{ind}\,_H^G \chi')(g)\overline{g} + \textstyle\sum_{r\geq 1} \chi(g^{l^{r-1}}) \frac{\overline{g}^{l^r}}{l^r} \\
&= \textstyle\sum_{m_G^H(t^{-1}gt)=0} \chi'(t^{-1}gt)\overline{g} + \sum_{r\geq 1} \sum_{m_G^H(t^{-1}gt)=r} \chi'((t^{-1}gt)^{l^r}) \frac{\overline{g}^{l^r}}{l^r} \\
&= \textstyle\sum_t \chi'((t^{-1}gt)^{l^{m_G^H(t^{-1}gt)}}) \frac{\overline{g}^{l^{m_G^H(t^{-1}gt)}}}{l^{m_G^H(t^{-1}gt)}} \;.
\end{aligned}$$

On the other hand, if $h \in H$, the same calculation on the H-level gives

$$(\mathrm{Tr}_H \tau_H(h))(\chi') = \mathrm{trace}(h \mid \mathfrak{V}_{\chi'}) = \chi'(h)\overline{h}\,,$$

and therefore, with $\tau_H(\sum_t \frac{(t^{-1}gt)^{l^{m_G^H(t^{-1}gt)}}}{l^{m_G^H(t^{-1}gt)}}) \in T(\mathcal{Q}(H))$, we get

$$\begin{aligned}
&(\mathrm{Tr}_H \tau_H \textstyle\sum_t \frac{(t^{-1}gt)^{l^{m_G^H(t^{-1}gt)}}}{l^{m_G^H(t^{-1}gt)}})(\chi') \\
&= \textstyle\sum_t \frac{1}{l^{m_G^H(t^{-1}gt)}} \chi'((t^{-1}gt)^{l^{m_G^H(t^{-1}gt)}}) \overline{(t^{-1}gt)^{l^{m_G^H(t^{-1}gt)}}} \\
&= \textstyle\sum_t \chi'((t^{-1}gt)^{l^{m_G^H(t^{-1}gt)}}) \frac{\overline{g}^{l^{m_G^H(t^{-1}gt)}}}{l^{m_G^H(t^{-1}gt)}}
\end{aligned}$$

via $\mathcal{Q}^{\mathrm{c}}(\Gamma_{k'}) \hookrightarrow \mathcal{Q}^{\mathrm{c}}(\Gamma_k)$ (where k' is the fixed field of H).

Since these formulae agree for all characters χ' of H (with open kernel) it follows that $\mathrm{Res}_G^H \tau_G(g) = \tau_H(\sum_t \frac{(t^{-1}gt)^{l^{m_G^H(t^{-1}gt)}}}{l^{m_G^H(t^{-1}gt)}})$ by the uniqueness of this element, proving 1. of Proposition A.

For 2. we first consider the situation $H \leq G' \leq G$ with $[G : G'] = l$ and show $\mathrm{Res}_G^H = \mathrm{Res}_{G'}^H \circ \mathrm{Res}_G^{G'}$:

Write $G = \dot{\bigcup}_x xG'$, $G' = \dot{\bigcup}_y yH$, hence $G = \dot{\bigcup}_{x,y} xyH$, and recall [6] from 1. that

$$\mathrm{Res}_G^{G'} g = \begin{cases} \tau_{G'}(\sum_x x^{-1}gx) & ,\ g \in G' \\ \tau_{G'}(g^l) & ,\ g \notin G' \end{cases} \;.$$

[6] if $g \notin G'$, then we may use $\{t\} = \{g^i : 0 \leq i \leq l-1\}$

Then, accordingly,

$$
\begin{aligned}
&\mathrm{Res}_{G'}^{H}\mathrm{Res}_{G}^{G'}\tau_G(g)\\
&= \begin{cases} \sum_x \mathrm{Res}_{G'}^{H}\tau_{G'}(x^{-1}gx)\\ \mathrm{Res}_{G'}^{H}\tau_{G'}(g^l)\end{cases}\\
&= \begin{cases} \sum_x\sum_y \tau_H((y^{-1}x^{-1}gxy)^{l^{m_{G'}^H(y^{-1}x^{-1}gxy)}})/l^{m_{G'}^H(y^{-1}x^{-1}gxy)}\\ \sum_y \tau_H((y^{-1}g^l y)^{l^{m_{G'}^H(y^{-1}g^l y)}})/l^{m_{G'}^H(y^{-1}g^l y)}\end{cases}\\
&= \begin{cases} \sum_t \tau_H((t^{-1}gt)^{l^{m_G^H(t^{-1}gt)}})/l^{m_G^H(t^{-1}gt)}\\ \sum_t \tau_H((t^{-1}g^l t)^{l^{m_{G'}^H(t^{-1}g^l t)}})/l\cdot l^{m_{G'}^H(t^{-1}g^l t)}\end{cases}\\
&= \sum_t \tau_H((t^{-1}gt)^{l^{m_G^H(t^{-1}gt)}}/l^{m_G^H(t^{-1}gt)} = \mathrm{Res}_G^H\tau_G(g)\,,
\end{aligned}
$$

using $G' \lhd G$ and $m_{G'}^H(x^l)+1 = m_G^H(x)$ for $x \notin G'$.

Induction on $[G:H]$ now proves 2., $\mathrm{Res}_G^H = \mathrm{Res}_{G''}^H \circ \mathrm{Res}_G^{G''}$ for $H \le G'' \le G$. Indeed, if $G'' \neq G$, find $G' \geq G''$ with $[G:G'] = l$ and use

$$
\begin{aligned}
\mathrm{Res}_{G''}^H\mathrm{Res}_G^{G''} &= \mathrm{Res}_{G''}^H(\mathrm{Res}_{G'}^{G''}\mathrm{Res}_G^{G'}) =\\
&= (\mathrm{Res}_{G''}^H\mathrm{Res}_{G'}^{G''})\mathrm{Res}_G^{G'} = \mathrm{Res}_{G'}^H\mathrm{Res}_G^{G'} = \mathrm{Res}_G^H\,.
\end{aligned}
$$

Finally, for 3., proceed by induction on $[G:H]$ and use 1. if H has index l in G and 2. when the index is bigger, in which case there is a subgroup G' of G with $H \le G$, $[G:G'] = l$.

The proof of Proposition A is complete.

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Degeneration of Polylogarithms and Special Values of L-Functions for Totally Real Fields

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Abstract. The degeneration of the polylogarithm on the universal abelian scheme over a Hilbert modular variety at the boundary is described in terms of (critical) special values of the L-function of the totally real field defining the variety. This gives a relation between the polylogarithm on abelian schemes and special values of L-functions.

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Introduction

The polylogarithm is a very powerful tool in studying special values of L-functions and subject to many conjectures. Most notably, the Zagier conjecture claims that all values of L-functions of number fields can be described by polylogarithms. The interpretation of the polylogarithm functions in terms of periods of variations of Hodge structures has lead to a motivic theory of the polylog and to generalizations as the elliptic polylog by Beilinson and Levin. Building on this work, Wildeshaus has defined polylogarithms in a more general context and in particular for abelian schemes.
Not very much is known about the extension classes arising from these "abelian polylogarithms". In an earlier paper [K] we were able to show that the abelian

[1]Dedicated to John Coates with admiration and respect.
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polylogarithm, is indeed of motivic origin, i.e., is in the image of the regulator from K-theory.
It was Levin in [L], who started to investigate certain "polylogarithmic currents" on abelian schemes, which are related to the construction by Wildeshaus. In [B1], Blottière could show that these currents actually represent the polylogarithmic extension in the category of Hodge modules.
In this paper we will, following and extending ideas from the case of the elliptic polylog treated in [HK] (which is in turn inspired by [BL]), consider the problem of the degeneration of the abelian polylog on Hilbert modular varieties. The main result will describe this degeneration in terms of (critical) special values of L-functions of the totally real field, which defines the Hilbert modular variety. To describe the theorem more precisely, consider the specialization of the polylog, which gives Eisenstein classes (say in the category of mixed étale sheaves to fix ideas)

$$\mathrm{Eis}^k(\alpha) \in \mathrm{Ext}_S^{2g-1}(\mathbb{Q}_\ell, \mathrm{Sym}^k \mathcal{H}(g)),$$

where S is the Hilbert modular variety of dimension g and $\mathcal{H}$ is the locally constant sheaf of relative Tate-modules of the universal abelian scheme. Let $j : S \to \overline{S}$ be the Baily-Borel compactification of S and $i : \partial S := \overline{S} \setminus S \to \overline{S}$ the inclusion of the cusps. The degeneration or residue map (see 1.5.2 for the precise definition) is then

$$\mathrm{res} : \mathrm{Ext}_S^{2g-1}(\mathbb{Q}_\ell, \mathrm{Sym}^k \mathcal{H}(g)) \to \mathrm{Hom}_{\partial S}(\mathbb{Q}_\ell, \mathbb{Q}_\ell).$$

The target of this map is sitting inside a sum of copies of $\mathbb{Q}_\ell$ and the main result of this paper 1.7.1 describes $\mathrm{res}(\mathrm{Eis}^k(\alpha))$ in terms of special values of (partial) L-functions of the totally real field defining S.
The same result was also obtained by Blottière in [B2] with different methods. His computation uses the explicit description of the polylogarithm in terms of the currents constructed by Levin.
Our method of proof is inspired by [BL] 2.4. and [HK] and follows a different line. Instead of computing directly the degeneration on the base we work with the polylog, which lives on the universal abelian scheme, and use the fact that the universal abelian variety can be written in a neighborhood of the cusps as an extension of real tori. The idea is to view the problem as of topological nature and use the good functorial properties of the topological polylog to compute the degeneration. In fact, we avoid computations by reducing to the situation considered by Nori [N] and Sczech [Sc]. For the convenience of the reader we reproduce then their computations, which lead to the relation with the L-values.
There is a very interesting question raised by the results in this paper. In [HK] we were able to construct extension classes related to non-critical values of Dirichlet-L-functions, if the residue map was zero on the specialization of the polylog. Is there an analogous result here?
The paper is organized as follows: In the first section we review the definition of the Hilbert modular variety, define the residue or degeneration map and

formulate our main theorem. The second section reviews the theory of the polylog and the Eisenstein classes emphasing the topological situation, which is not extensively covered in the literature. In the third section we give the proof of the main theorem.

It is a pleasure to thank David Blottière for a series of interesting and stimulating discussions during his stay in Regensburg. Moreover, I like to thank Sascha Beilinson for making available some time ago his notes about his and A. Levin's interpretation of Nori's work.

1 Polylogarithms and degeneration

We review the definition of a Hilbert modular variety to fix notations and pose the problem of computing the degeneration of the specializations of the polylogarithm at the boundary. The main theorem describes this residue in terms of special values of L-functions.

1.1 Notation

As in [BL] we deal with three different types of sheaves simultaneously. Let X/k be a variety and L a coefficient ring for our sheaf theory, then we consider

i) $k = \mathbb{C}$ the usual topology on $X(\mathbb{C})$ and L any commutative ring

ii) $k = \mathbb{R}$ or $\mathbb{C}$ and $L = \mathbb{Q}$ or $\mathbb{R}$ and we work with the category of mixed Hodge modules

iii) $k = \mathbb{Q}$ and $L = \mathbb{Z}/l^r\mathbb{Z}, \mathbb{Z}_l$ or $\mathbb{Q}_l$ and we work with the category of étale sheaves

1.2 Hilbert modular varieties

We recall the definition of Hilbert modular varieties following Rapoport [R]. To avoid all technicalities, we will only consider the moduli scheme over $\mathbb{Q}$. The theory works over more general base schemes without any modification.
Let F be a totally real field, $g := [F : \mathbb{Q}]$, $\mathcal{O}$ the ring of integers, $\mathfrak{D}^{-1}$ the inverse different and d_F its discriminant. Fix an integer $n \geq 3$. We consider the functor, which associates to a scheme T over $\operatorname{Spec}\mathbb{Q}$ the isomorphism classes of triples (A, α, λ), where A/T is an abelian scheme of dimension g, with real multiplication by $\mathcal{O}$, $\alpha : \underline{\operatorname{Hom}}_{et,\mathcal{O},sym}(\mathcal{A}, \mathcal{A}^*) \to \mathfrak{D}^{-1}$ is a $\mathfrak{D}^{-1}$-polarization in the sense of [R] 1.19, i.e., an $\mathcal{O}$-module isomorphism respecting the positivity of the totally positive elements in $\mathfrak{D}^{-1} \subset F$, and $\lambda : A[n] \cong (\mathcal{O}/n\mathcal{O})^2$ is a level n structure satisfying the compatibility of [R] 1.21. For $n \geq 3$ this functor is represented by a smooth scheme $S := S_n^{\mathfrak{D}^{-1}}$ of finite type over $\operatorname{Spec}\mathbb{Q}$. Let

$$\mathcal{A} \xrightarrow{\pi} S$$

be the universal abelian scheme over S. In any of the three categories of sheaves i)-iii) from 1.1 we let

$$\mathcal{H} := \underline{\mathrm{Hom}}_S(R^1\pi_* L, L)$$

the first homology of $\mathcal{A}/S$. In the étale case and $L = \mathbb{Z}_\ell$, the fiber of $\mathcal{H}$ at a point is the Tate module of the abelian variety over that point.

1.3 Transcendental description

For the later computation we need a description in group theoretical terms of the complex points $S(\mathbb{C})$ and of $\mathcal{H}$.
Define a group scheme $G/\operatorname{Spec}\mathbb{Z}$ by the Cartesian diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d] & \mathrm{Res}_{\mathcal{O}/\mathbb{Z}} \mathrm{Gl}_2 \arrow[d, "\det"] \\ \mathbb{G}_m \arrow[r] & \mathrm{Res}_{\mathcal{O}/\mathbb{Z}} \mathbb{G}_m \end{tikzcd}$$

and let

$$\mathfrak{H}^g_\pm := \{\tau \in F \otimes \mathbb{C} | Im\, \tau \text{ totally positive or totally negative}\}.$$

Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbb{R})$ acts on $\mathfrak{H}^g_\pm$ by the usual formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}$$

and the stabilizer of $1 \otimes i \in \mathfrak{H}^g_\pm$ is

$$K_\infty := (F \otimes \mathbb{C})^* \cap G(\mathbb{R}),$$

so that

$$\mathfrak{H}^g_\pm \cong G(\mathbb{R})/K_\infty.$$

With this notation one has

$$S(\mathbb{C}) = G(\mathbb{Z}) \backslash (\mathfrak{H}^g_\pm \times G(\mathbb{Z}/n\mathbb{Z})).$$

On $S(\mathbb{C})$ acts $G(\mathbb{Z}/n\mathbb{Z})$ by right multiplication. The determinant $\det : G \to \mathbb{G}_m$ induces

$$S(\mathbb{C}) \to \mathbb{G}_m(\mathbb{Z}/n\mathbb{Z})$$

and the fibers are the connected components. Define a subgroup $D \subset G$ isomorphic to $\mathbb{G}_m$ by $D := \{\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \in G : a \in \mathbb{G}_m\big)$. This gives a section of det. Then the action of $D(\mathbb{Z}/n\mathbb{Z})$ by right multiplication is transitive on the set of connected components.

The embedding $G(\mathbb{Z}) \subset \mathrm{Gl}_2(\mathcal{O})$ defines an action of $G(\mathbb{Z})$ on $\mathcal{O}^{\oplus 2}$ and in the topological realization the local system $\mathcal{H}$ is given by the quotient

$$G(\mathbb{Z})\backslash(\mathfrak{H}^g_\pm \times \mathcal{O}^{\oplus 2} \times G(\mathbb{Z}/n\mathbb{Z})).$$

In particular, as a family of real ($2g$-dimensional) tori, the complex points $\mathcal{A}(\mathbb{C})$ of the universal abelian scheme can be written as

$$G(\mathbb{Z})\backslash(\mathfrak{H}^g_\pm \times (F \otimes \mathbb{R}/\mathcal{O})^{\oplus 2} \times G(\mathbb{Z}/n\mathbb{Z}))$$

and the level n structure is given by the subgroup

$$(\frac{1}{n}\mathcal{O}/\mathcal{O})^{\oplus 2} \subset (F \otimes \mathbb{R}/\mathcal{O})^{\oplus 2}.$$

The $\mathcal{O}$-multiplication on $\mathcal{A}(\mathbb{C})$ is in this description given by the natural $\mathcal{O}$-module structure on $F \otimes \mathbb{R}$.

1.4 Transcendental description of the cusps

The following description of the boundary cohomology is inspired by [H]. For further details we refer to [H] 2.1. Let $B \subset G$ the subgroup of upper triangular matrices, $T \subset B$ its maximal torus and $N \subset B$ its unipotent radical. We have an exact sequence

$$1 \to N \to B \xrightarrow{q} T \to 1.$$

We denote by G^1, B^1 and T^1 the subgroups of determinant 1. Note that $G^1 = \mathrm{Res}_{\mathcal{O}/\mathbb{Z}} \mathrm{Sl}_2$. Let $K^B_\infty := B(\mathbb{R}) \cap K_\infty$, then the Cartan decomposition shows that $\mathfrak{H}^g_\pm = B(\mathbb{R})/K^B_\infty$. A pointed neighborhood of the set of all cusps is given by

$$\widetilde{S}_B := B(\mathbb{Z})\backslash\big(B(\mathbb{R})/K^B_\infty \times G(\mathbb{Z}/n\mathbb{Z})\big). \tag{1}$$

In particular, the set of cusps is (cf. also [R] p.305)

$$\partial S(\mathbb{C}) = B^1(\mathbb{Z})\backslash G(\mathbb{Z}/n\mathbb{Z}). \tag{2}$$

The fibres of the map $\partial S(\mathbb{C}) \to \mathbb{G}_m(\mathbb{Z}/n\mathbb{Z})$ induced by the determinant are

$$B^1(\mathbb{Z})\backslash G^1(\mathbb{Z}/n\mathbb{Z}) \cong \Gamma_G\backslash\mathbb{P}^1(\mathcal{O}), \tag{3}$$

where $\Gamma_G := \ker(G^1(\mathbb{Z}) \to G^1(\mathbb{Z}/n\mathbb{Z}))$. In particular, we can think of a cusp represented by $h \in G^1(\mathbb{Z}/n\mathbb{Z})$ as a rank 1 $\mathcal{O}$-module $\mathfrak{b}_h$, which is a quotient

$$\mathcal{O}^2 \xrightarrow{p_h} \mathfrak{b}_h, \tag{4}$$

together with a level structure, i.e., a basis $h \in G^1(\mathbb{Z}/n\mathbb{Z})$. Explicitly, the fractional ideal $\mathfrak{b}_h$ is generated by any representatives $u, v \in \mathcal{O}$ of the second row of h.

On $\widetilde{S}_B$ acts $G(\mathbb{Z}/n\mathbb{Z})$ by multiplication from the right. This action is transitive on the connected components of $\widetilde{S}_B$. Define

$$S_B := B(\mathbb{Z})\backslash\big(B(\mathbb{R})/K_\infty^B \times B(\mathbb{Z}/n\mathbb{Z})\big), \tag{5}$$

then $S_B \subset \widetilde{S}_B$ is a union of connected components of $\widetilde{S}_B$. Let K_∞^T (respectively $T(\mathbb{Z})$) be the image of K_∞^B (respectively $B(\mathbb{Z})$) under $q : B(\mathbb{R}) \to T(\mathbb{R})$. Define

$$S_T := T(\mathbb{Z})\backslash\big(T(\mathbb{R})/K_\infty^T \times T(\mathbb{Z}/n\mathbb{Z})\big), \tag{6}$$

then the map $q : B \to T$ induces a fibration

$$q : S_B \to S_T, \tag{7}$$

whose fibers are $N(\mathbb{Z})\backslash\big(N(\mathbb{R})\times N(\mathbb{Z}/n\mathbb{Z})\big)$ with $N(\mathbb{Z}) := B(\mathbb{Z})\cap N(\mathbb{R})$. Denote by

$$u : S_T \to pt \tag{8}$$

the structure map to a point. For the study of the degeneration, one considers the diagram

$$\begin{tikzcd} S_B \arrow[r, "q"] \arrow[d] & S_T \arrow[r, "u"] & pt \\ S \end{tikzcd} \tag{9}$$

In fact we are interested in the cohomology of certain local systems on these topological spaces. For the computations it is convenient to replace S_B and S_T by homotopy equivalent spaces as follows.

Define $K_\infty^{T^1} := K_\infty^T \cap T^1(\mathbb{R})$ and note that this is the kernel of the determinant $K_\infty^T \to \mathbb{R}^*$. Then the inclusion induces an isomorphism

$$T^1(\mathbb{Z})\backslash\big(T^1(\mathbb{R})/K_\infty^{T^1} \times T(\mathbb{Z}/n\mathbb{Z})\big) \cong S_T.$$

The map $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right)$ defines isomorphisms $(F\otimes\mathbb{R})^* \cong T^1(\mathbb{R})$ and $\mathcal{O}^* \cong T^1(\mathbb{Z})$. Note that $K_\infty^{T^1} \subset (F\otimes\mathbb{R})^*$ is identified with the two torsion subgroup in $(F\otimes\mathbb{R})^*$ and that $K_\infty^{T^1} \cong (\mathbb{Z}/2\mathbb{Z})^g$ permutes the set of connected components of $T^1(\mathbb{R})$.

LEMMA 1.4.1. *Let $(F\otimes\mathbb{R})^1 \subset T^1(\mathbb{R}) = (F\otimes\mathbb{R})^*$ be the subgroup of elements of norm 1 and $\mathcal{O}^{*,1} = \mathcal{O}^* \cap (F\otimes\mathbb{R})^1$. Then*

$$S_T^1 := \mathcal{O}^{*,1}\backslash\big((F\otimes\mathbb{R})^1/K_\infty^{T^1} \cap (F\otimes\mathbb{R})^1 \times T(\mathbb{Z}/n\mathbb{Z})\big)$$

is homotopy equivalent to S_T. Moreover, the inclusion of the totally positive elements $(F\otimes\mathbb{R})^1_+$ into $(F\otimes\mathbb{R})^1$ provides an identification

$$(F\otimes\mathbb{R})^1_+ \cong (F\otimes\mathbb{R})^1/K_\infty^{T^1} \cap (F\otimes\mathbb{R})^1.$$

Proof. The exact sequence

$$0 \to (F \otimes \mathbb{R})^1 \to (F \otimes \mathbb{R})^* \to \mathbb{R}^* \to 0$$

together with the fact that $K_\infty^{T^1}$ is the two torsion in $(F \otimes \mathbb{R})^*$ allows to identify

$$T^1(\mathbb{R})/K_\infty^{T^1} \cong \big((F \otimes \mathbb{R})^1/K_\infty^{T^1} \cap (F \otimes \mathbb{R})^1\big) \times \mathbb{R}_{>0}.$$

The last identity is clear. □

We define S_B^1 to be the inverse image of S_T^1 under q, so that we have a Cartesian diagram

$$\begin{tikzcd} S_B^1 \arrow[r, "q"] \arrow[d] & S_T^1 \arrow[d] \\ S_B \arrow[r, "q"] & S_T. \end{tikzcd} \tag{10}$$

Over S_B the representation $\mathcal{O}^2$ has a filtration

$$0 \to \mathcal{O} \to \mathcal{O}^2 \xrightarrow{p} \mathcal{O} \to 0, \tag{11}$$

where the first map sends $a \in \mathcal{O}$ to the vector $\binom{a}{0}$ and the second map is $\binom{a}{b} \mapsto b$. This induces a filtration on the local system $\mathcal{H}$

$$0 \to \mathcal{N} \to \mathcal{H} \to \mathcal{M} \to 0, \tag{12}$$

where $\mathcal{N}$ and $\mathcal{M}$ are the associated local systems. In particular, over S_B^1 one has a filtration of topological tori

$$0 \to \mathcal{T}_\mathcal{N} \to \mathcal{A}(\mathbb{C}) \xrightarrow{p} \mathcal{T}_\mathcal{M} \to 0, \tag{13}$$

where $\mathcal{T}_\mathcal{N} := \mathcal{N} \otimes \mathbb{R}/\mathbb{Z}$ and $\mathcal{T}_\mathcal{M} := \mathcal{M} \otimes \mathbb{R}/\mathbb{Z}$. By definition of N the fibration in (13) and (10) are compatible, i.e., one has a commutative diagram

$$\begin{tikzcd} \mathcal{A}(\mathbb{C}) \arrow[r, "p"] \arrow[d, "\pi"'] & \mathcal{T}_\mathcal{M} \arrow[d, "\pi_\mathcal{M}"] \\ S_B^1 \arrow[r, "q"] & S_T^1. \end{tikzcd} \tag{14}$$

1.5 The degeneration map

In this section we explain the degeneration problem we want to consider.
The polylogarithm on $\pi : \mathcal{A} \to S$ defines for certain linear combinations α of torsion sections of $\mathcal{A}$ an extension class

$$\mathrm{Eis}^k(\alpha) \in \mathrm{Ext}^{2g-1}_{?,S}(L, \mathrm{Sym}^k \mathcal{H}(g)), \tag{15}$$

where ? can be MHM, et, top. The construction of this class will be given in section 2 definition 2.4.1.
Let $\overline{S}$ be the Baily-Borel compactification of S. Denote by $\partial S := \overline{S} \setminus S$ the set of cusps. We get

$$\partial S \xrightarrow{i} \overline{S} \xleftarrow{j} S.$$

The adjunction map together with the edge morphism in the Leray spectral sequence for Rj_* gives

$$\begin{tikzcd} \operatorname{Ext}^{2g-1}_S(L, \operatorname{Sym}^k \mathcal{H}(g)) \arrow[r] \arrow[dr] & \operatorname{Ext}^{2g-1}_{\partial S}(L, i^* Rj_* \operatorname{Sym}^k \mathcal{H}(g)) \arrow[d] \\ & \operatorname{Hom}_{\partial S}(L, i^* R^{2g-1} j_* \operatorname{Sym}^k \mathcal{H}(g)). \end{tikzcd} \tag{16}$$

There are several possibilities to compute $i^* R^{2g-1} j_* \operatorname{Sym}^k \mathcal{H}(g)$.

Theorem 1.5.1. *Assume that $\mathbb{Q} \subset L$. Then, in any of the categories MHM, et, top, there is a canonical isomorphism*

$$i^* R^{2g-1} j_* \operatorname{Sym}^k \mathcal{H}(g) \cong L,$$

where L has the trivial Hodge structure (resp. the trivial Galois action).

Remark: J. Wildeshaus has pointed out that the determination of the weight on the right hand side is not necessary for our main result, but follows from it. In fact, our main result gives non-zero classes in

$$\operatorname{Hom}_{\partial S}(L, i^* R^{2g-1} j_* \operatorname{Sym}^k \mathcal{H}(g)),$$

so that the rank one sheaf $i^* R^{2g-1} j_* \operatorname{Sym}^k \mathcal{H}(g)$ has to be of weight zero.

Using this identification we define the residue or degeneration map:

Definition 1.5.2. *The map from (16) together with the identification of 1.5.1 define the* residue map

$$\operatorname{res} : \operatorname{Ext}^{2g-1}_S(L, \operatorname{Sym}^k \mathcal{H}(g)) \to \operatorname{Hom}_{\partial S}(L, L).$$

The residue map is equivariant for the $G(\mathbb{Z}/n\mathbb{Z})$ action on both sides.

Proof. (of theorem 1.5.1). In the case of Hodge modules we use theorem 2.9. in Burgos-Wildeshaus [BW] and in the étale case we use theorem 5.3.1 in [P2]. Roughly speaking, both results asserts that the higher direct image can be calculated using group cohomology and the "canonical construction", which associates to a representation of the group defining the Shimura variety a Hodge module resp. an étale sheaf.
More precisely, from a topological point of view, the monodromy at the cusps is exactly the cohomology of $\widetilde{S}_B$. One has

$$H^{2g-1}(\widetilde{S}_B, \operatorname{Sym}^k \mathcal{H}(g)) \cong \operatorname{Ind}^{G(\mathbb{Z}/n\mathbb{Z})}_{B(\mathbb{Z}/n\mathbb{Z})} H^{2g-1}(S_B, \operatorname{Sym}^k \mathcal{H}(g))$$

and

$$H^{2g-1}(S_B, \operatorname{Sym}^k \mathcal{H}(g)) \cong \bigoplus_{r+s=2g-1} H^r(S_T, R^s q_* \operatorname{Sym}^k \mathcal{H}(g))).$$

As the cohomological dimension of Γ_T is $g-1$ and that of Γ_N is g, one has in fact

$$H^{2g-1}(S_B, \operatorname{Sym}^k \mathcal{H}(g)) \cong H^{g-1}(S_T, R^g q_* \operatorname{Sym}^k \mathcal{H}(g))).$$

The exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}^2 \xrightarrow{p} \mathcal{O} \to 0$$

from (11) shows that $R^g q_* \operatorname{Sym}^k \mathcal{H}(g)$ can be identified via p with $\operatorname{Sym}^k \mathcal{O} \otimes L$ with the induced $T(\mathbb{Z})$ action, which maps $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ to d^k. To compute the coinvariants, extend the coefficients to $\mathbb{R}$, so that

$$\mathcal{O} \otimes \mathbb{R} \cong \bigoplus_{\tau : F \to \mathbb{R}} \mathbb{R}$$

and $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \in T(\mathbb{Z})$ acts via $\tau(d)$ on the component indexed by τ. Thus $\operatorname{Sym}^k \mathcal{O} \otimes L$ can only have a trivial quotient, if $k \equiv 0 \bmod g$ and on this one dimensional quotient the action is by the norm map $T(\mathbb{Z}) \to \pm 1$. One gets:

$$H^{g-1}(S_T, \operatorname{Sym}^k \mathcal{O} \otimes L) \cong \begin{cases} L & \text{if } k \equiv 0 \bmod g \\ 0 & \text{else} \end{cases}$$

The above mentioned theorems imply that this topological computation gives also the result in the categories MHM, et, top. The Hodge structure on $H^{g-1}(S_T, \operatorname{Sym}^k \mathcal{O} \otimes L)$ is the trivial one, as one sees from the explicit description of the action of T and the fact that the action of the Deligne torus $\mathbb{S}$, which defines the weight, is induced from the embedding $x \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)$., hence is trivial. The same remark and proposition 5.5.4. in [P2] show that the weight is also zero in the étale case. $\square$

1.6 Partial zeta functions of totally real fields

Let $\mathfrak{b}, \mathfrak{f}$ be relatively prime integral ideals of $\mathcal{O}$, $\epsilon : (\mathbb{R} \otimes F)^* \to \{\pm 1\}$ a sign character. This is a product of characters $\epsilon_\tau : \mathbb{R}^* \to \{\pm 1\}$ for all embeddings $\tau : F \to \mathbb{R}$. Denote by $|\epsilon|$ the number of non-trivial ϵ_τ which occur in this product decomposition of ϵ. Moreover let $x \in \mathcal{O}$ such that $x \not\equiv 0 \bmod \mathfrak{b}^{-1}\mathfrak{f}$ and $\mathcal{O}_\mathfrak{f}^* := \{a \in \mathcal{O} | a \text{ totally positive and } a \equiv 1 \bmod \mathfrak{f}\}$. Define

$$F(\mathfrak{b}, \mathfrak{f}, \epsilon, x, s) := \sum_{\nu \in (x + \mathfrak{f}\mathfrak{b}^{-1})/\mathcal{O}_\mathfrak{f}^*} \frac{\epsilon(\nu)}{|N(\nu)|^s} \tag{17}$$

for $Re\ s > 1$. Here N is the norm. On the other hand let $\operatorname{Tr} : F \to \mathbb{Q}$ be the trace map and define

$$L(\mathfrak{b}, \mathfrak{f}, \epsilon, x, s) := \sum_{\lambda \in \mathfrak{b}(\mathfrak{f}\mathfrak{D})^{-1}/\mathcal{O}_\mathfrak{f}^*} \frac{\epsilon(\lambda) e^{2\pi i \operatorname{Tr}(x\lambda)}}{|N(\nu)|^s} \tag{18}$$

These two L-functions are related by a functional equation. To formulate it we introduce the Γ-factor

$$\Gamma_\epsilon(s) := \pi^{-\frac{1}{2}(sg+|\epsilon|)}\Gamma\left(\frac{s+1}{2}\right)^{|\epsilon|}\Gamma\left(\frac{s}{2}\right)^{g-|\epsilon|}.$$

The functional equation follows directly with Hecke's method for Grössencharacters and was first mentioned for these partial zeta functions by Siegel:

PROPOSITION 1.6.1 (cf.[Si] Formel (10)). *The functional equation reads:*

$$\Gamma_\epsilon(1-s)F(\mathfrak{b},\mathfrak{f},\epsilon,x,1-s) = i^{-|\epsilon|}|d_F|^{-\frac{1}{2}}N(\mathfrak{f}^{-1}\mathfrak{b})\Gamma_\epsilon(s)L(\mathfrak{b},\mathfrak{f},\epsilon,x,s),$$

where d_F is the discriminant of $F/\mathbb{Q}$.

The functional equation shows that $F(\mathfrak{b},\mathfrak{f},\epsilon,x,1-k)$ can be non-zero for $k = 1,2,\ldots$ only if $|\epsilon|$ is either g or 0. Let us introduce

$$\zeta(\mathfrak{b},\mathfrak{f},x,s) := \sum_{\nu\in(x+\mathfrak{f}\mathfrak{b}^{-1})/\mathcal{O}_\mathfrak{f}^*}\frac{1}{N(\nu)^s}.$$

We get:

COROLLARY 1.6.2. *The functional equation shows that $F(\mathfrak{b},\mathfrak{f},\epsilon,x,1-k)$ for $k = 1,2,\ldots$ is non-zero for $|\epsilon| = 0$ and k even or for $|\epsilon| = g$ and k odd. In these cases one has*

$$\zeta(\mathfrak{b},\mathfrak{f},x,1-k) = |d_F|^{-\frac{1}{2}}N(\mathfrak{f}^{-1}\mathfrak{b})\frac{((k-1)!)^g}{(2\pi i)^{kg}}L(\mathfrak{b},\mathfrak{f},\epsilon,x,k).$$

1.7 The main theorem

Here we formulate our main theorem. It computes the residue map from (1.5.2) in terms of the partial L-functions.
The transcendental description of the cusps gives

$$H^0(\partial S(\mathbb{C}),L) = \mathrm{Ind}_{B^1(\mathbb{Z})}^{G(\mathbb{Z}/n\mathbb{Z})}L$$

and $H^0(\partial S, L)$ is the subgroup of elements invariant under $D(\mathbb{Z}/n\mathbb{Z})$. Similarly, the n-torsion sections of $\mathcal{A}[n]$ over $S(\mathbb{C})$ can be identified with functions from $G(\mathbb{Z}/n\mathbb{Z})$ to $(\frac{1}{n}\mathcal{O}/\mathcal{O})^2$, which are equivariant with respect to the canonical $G^1(\mathbb{Z}) := \ker(G(\mathbb{Z})\to\mathbb{Z}^*)$ action. The action of $G(\mathbb{Z}/n\mathbb{Z})$ on S induces via pull-back an action on $\mathcal{A}[n](S(\mathbb{C}))$ and we have:

$$\mathcal{A}[n](S(\mathbb{C})) = \mathrm{Ind}_{G^1(\mathbb{Z})}^{G(\mathbb{Z}/n\mathbb{Z})}(\frac{1}{n}\mathcal{O}/\mathcal{O})^2.$$

The group $\mathcal{A}[n](S)$ consists again of the elements invariant under $D(\mathbb{Z}/nZ)$. Let $D := \mathcal{A}[n](S)$ and consider the formal linear combinations

$$L[D]^0 := \big\{\sum_{\sigma\in D} l_\sigma(\sigma) : l_\sigma \in L \text{ and } \sum_{\sigma\in D} l_\sigma = 0\big\}.$$

The $G(\mathbb{Z}/n\mathbb{Z})$ action on D carries over to an action on $L[D]^0$. For $\alpha \in L[D]^0$ we construct in 2.4.1 a class

$$\mathrm{Eis}^k(\alpha) \in \mathrm{Ext}_S^{2g-1}(L, \mathrm{Sym}^k \mathcal{H}(g)),$$

which depends on α in a functorial way. Thus, the resulting map

$$L[D]^0 \xrightarrow{\mathrm{Eis}^k} \mathrm{Ext}_S^{2g-1}(L, \mathrm{Sym}^k \mathcal{H}(g)) \xrightarrow{\mathrm{res}} \mathrm{Ind}_{B^1(\mathbb{Z})}^{G(\mathbb{Z}/n\mathbb{Z})} L \tag{19}$$

is equivariant for the $G(\mathbb{Z}/n\mathbb{Z})$ action.

Theorem 1.7.1. *Let $L \supset \mathbb{Q}$ and $\alpha = \sum_{\sigma\in D} l_\sigma(\sigma)$. Then* $\mathrm{res}(\mathrm{Eis}^m(\alpha))$ *is non-zero only for $m \equiv 0(g)$ and for every $h \in G(\mathbb{Z}/n\mathbb{Z})$ and $k > 0$*

$$\mathrm{res}(\mathrm{Eis}^{gk}(\alpha))(h) = (-1)^{g-1} \sum_{\sigma\in D} l_\sigma \zeta(\mathcal{O}, \mathcal{O}, p(h\sigma), -k).$$

To use the basis given by the coinvariants in $\mathrm{Sym}^{gk} \mathcal{O} \otimes L$ as we did in the proof of theorem 1.5.1 is not natural. A better description is as follows: For each $h \in G(\mathbb{Z}/n\mathbb{Z})$ choose an element $d_h \in D(\mathbb{Z}/n\mathbb{Z})$ such that $\tilde{h} := hd_h^{-1} \in G^1(\mathbb{Z}/n\mathbb{Z})$. Then, as in (4) we have an ideal $\mathfrak{b}_{\tilde{h}}$ and a projection

$$\mathcal{O}^2 \xrightarrow{p_{\tilde{h}}} \mathfrak{b}_{\tilde{h}}.$$

Now use the identification $H^{g-1}(S_T, \mathrm{Sym}^{gk} \mathfrak{b}_{\tilde{h}} \otimes L) \cong L$ at the cusp h. With this basis the above result reads

Corollary 1.7.2. *In this basis*

$$\mathrm{res}(\mathrm{Eis}^{gk}(\alpha))(h) = (-1)^{g-1} N\mathfrak{b}_{\tilde{h}}^{-k-1} \sum_{\sigma\in D} l_\sigma \zeta(\mathfrak{b}_{\tilde{h}}, \mathcal{O}, p_{\tilde{h}}(\sigma), -k).$$

The theorem and the corollary will be proved in section 3.

2 Polylogarithms

In this section we review the theory of the polylogarithm on abelian schemes. Special emphasis is given the topological case, which will be important in the proof of the main theorem. The elliptic polylogarithm was introduced by Beilinson and Levin [BL] and the generalization to higher dimensional families of

abelian varieties is due to Wildeshaus [W]. The idea to interprete the construction by Nori in terms of the topological polylogarithm is due to Beilinson and Nori (unpublished).
The polylogarithm can be defined in any of the categories MHM, et, top for any abelian scheme $\pi : \mathcal{A} \to S$, with unit section $e : S \to \mathcal{A}$ of constant relative dimension g. If we work in *top*, it even suffices to assume that $\pi : \mathcal{A} \to S$ is a family of topological tori (i.e., fiberwise isomorphic to $(\mathbb{R}/\mathbb{Z})^g$). For more details in the case of abelian schemes, see [W] chapter III part I, or [L]. In the case of elliptic curves one can also consult [BL] or [HK].

2.1 Construction of the polylog

For simplicity we assume $L \supset \mathbb{Q}$ in this section and discuss the necessary modifications for integral coefficients later. Define a lisse sheaf $\mathrm{Log}^{(1)}$ on $\mathcal{A}$, which is an extension

$$0 \to \mathcal{H} \to \mathrm{Log}^{(1)} \to L \to 0$$

together with a splitting $s : e^*L \to e^* \mathrm{Log}^{(1)}$ in any of the three categories MHM, et, top as follows: Consider the exact sequence

$$0 \to \mathrm{Ext}^1_S(L, \mathcal{H}) \xrightarrow{\pi^*} \mathrm{Ext}^1_{\mathcal{A}}(L, \pi^*\mathcal{H}) \to \mathrm{Hom}_S(L, R^1\pi_*\pi^*\mathcal{H}) \to 0,$$

which is split by e^*. Note that by the projection formula $R^1\pi_*\pi^*\mathcal{H} \cong R^1\pi_*L \otimes \mathcal{H}$ so that

$$\mathrm{Hom}_S(L, R^1\pi_*\pi^*\mathcal{H}) \cong \mathrm{Hom}_S(\mathcal{H}, \mathcal{H}).$$

Then $\mathrm{Log}^{(1)}$ is a sheaf representing the unique extension class in $\mathrm{Ext}^1_{\mathcal{A}}(L, \pi^*\mathcal{H})$, which splits when pulled back to S via e^* and which maps to $\mathrm{id} \in \mathrm{Hom}_S(\mathcal{H}, \mathcal{H})$. Define

$$\mathrm{Log}^{(k)} := \mathrm{Sym}^k \mathrm{Log}^{(1)}.$$

Definition 2.1.1. *The* logarithm sheaf *is the pro-sheaf*

$$\mathrm{Log} := \mathrm{Log}_{\mathcal{A}} := \varprojlim \mathrm{Log}^{(k)},$$

where the transition maps are induced by the map $\mathrm{Log}^{(1)} \to L$. *In particular, one has exact sequences*

$$0 \to \mathrm{Sym}^k \mathcal{H} \to \mathrm{Log}^{(k)} \to \mathrm{Log}^{(k-1)} \to 0$$

and a splitting induced by $s : e^*L \to e^* \mathrm{Log}^{(1)}$

$$e^* \mathrm{Log} \cong \prod_{k \geq 0} \mathrm{Sym}^k \mathcal{H}.$$

Any isogeny $\phi : \mathcal{A} \to \mathcal{A}$ of degree invertible in L induces an isomorphism $\mathrm{Log} \cong \phi^* \mathrm{Log}$, which is on the associated graded induced by $\mathrm{Sym}^k \phi : \mathrm{Sym}^k \mathcal{H} \to \mathrm{Sym}^k \mathcal{H}$. For every torsion point $x \in \mathcal{A}(S)_{\mathrm{tors}}$ one gets an isomorphism

$$x^* \mathrm{Log} \cong e^* \mathrm{Log} \cong \prod_{k \geq 0} \mathrm{Sym}^k \mathcal{H}. \tag{20}$$

The most important property of the sheaf Log is the vanishing of its higher direct images except in the highest degree.

Theorem 2.1.2 (Wildeshaus, [W], cor. 4.4., p. 70). *One has*

$$R^i \pi_* \mathrm{Log} = 0 \text{ for } i \neq 2g$$

and the augmentation $\mathrm{Log} \to L$ *induces canonical isomorphisms*

$$R^{2g} \pi_* \mathrm{Log} \cong R^{2g} \pi_* L \cong L(-g).$$

For the construction of the polylogarithm one considers a non-empty disjoint union of torsion sections $i : D \subset \mathcal{A}$, whose orders are invertible in L (more generally, one can also consider D étale over S). Let

$$L[D] := \bigoplus_{\sigma \in D} L$$

and $L[D]^0 \subset L[D]$ the kernel of the augmentation map $L[D] \to L$. Elements $\alpha \in L[D]$ are written as formal linear combinations $\alpha = \sum_{\sigma \in D} l_\sigma(\sigma)$. Similarly, define

$$\mathrm{Log}[D] := \bigoplus_{\sigma \in D} \sigma^* \mathrm{Log}$$

and

$$\mathrm{Log}[D]^0 := \ker\left(\mathrm{Log}[D] \to L\right)$$

to be the kernel of the composition of the sum of the augmentation maps $\mathrm{Log}[D] \to L[D]$ and the augmentation $L[D] \to L$.

Corollary 2.1.3. *The localization sequence for* $U := \mathcal{A} \setminus D$ *induces an isomorphism*

$$\mathrm{Ext}_U^{2g-1}(L[D]^0, \mathrm{Log}(g)) \cong \mathrm{Hom}_S(L[D]^0, \mathrm{Log}[D]^0).$$

Proof. The vanishing result 2.1.2 implies that the localization sequence is of the form

$$0 \to \mathrm{Ext}_U^{2g-1}(L[D]^0, \mathrm{Log}(g)) \to \mathrm{Hom}_S(L[D]^0, i^* \mathrm{Log}) \to \mathrm{Hom}_S(L[D]^0, L) \to 0.$$

Inserting the definition of $\mathrm{Log}[D]^0$ gives the desired result. □

DEFINITION 2.1.4. *The* polylogarithm pol^D *is the extension class*

$$\mathrm{pol}^D \in \mathrm{Ext}_U^{2g-1}(L[D]^0, \mathrm{Log}(g)),$$

which maps to the canonical inclusion $L[D]^0 \to \mathrm{Log}[D]$ *under the isomorphism in 2.1.3. In particular, for every* $\alpha \in L[D]^0$ *we get by pull-back an extension class*

$$\mathrm{pol}_\alpha^D \in \mathrm{Ext}_U^{2g-1}(L, \mathrm{Log}(g)).$$

2.2 Integral version of the polylogarithm, the topological case

In the topological and the étale situation it is possible to define the polylogarithm with integral coefficients. In this section we treat the topological case and the étale case in the next section. The construction presented here is a reinterpretation by Beilinson and Levin (unpublished) of results of Nori and Sczech.

We start by defining the logarithm sheaf for any (commutative) coefficient ring L, in particular for $L = \mathbb{Z}$. In the topological situation, it is even possible to define more generally the polylogarithm for any smooth family of real tori of constant dimension g, which has a unit section.

Let

$$\pi : \mathcal{T} \to S$$

be such a family, $e : S \to \mathcal{T}$ the unit section and let $\mathcal{H}_L := \underline{\mathrm{Hom}}_S(R^1\pi_* L, L)$ be the local system of the homologies of the fibers with coefficients in L. Let $\tilde{\mathcal{H}}_\mathbb{R}$ be the associated vector bundle of $\mathcal{H}_\mathbb{R}$. Then $\mathcal{T} \cong \mathcal{H}_\mathbb{Z} \backslash \tilde{\mathcal{H}}_\mathbb{R}$ and we denote by

$$\tilde{\pi} : \tilde{\mathcal{H}}_\mathbb{R} \to \mathcal{T}$$

the associated map. Let

$$L[\mathcal{H}_\mathbb{Z}] := e^* \tilde{\pi}_! L$$

be the local system of group rings on S, which is stalk-wise the group ring of the stalk of the local system $\mathcal{H}_\mathbb{Z}$ with coefficients in L. The augmentation ideal of $L[\mathcal{H}_\mathbb{Z}] \to L$ is denoted by $\mathcal{I}$ and we define

$$L[[\mathcal{H}_\mathbb{Z}]] := \varprojlim_r L[\mathcal{H}_\mathbb{Z}]/\mathcal{I}^r$$

the completion along the augmentation ideal. Note that $\mathcal{I}^n/\mathcal{I}^{n+1} \cong \mathrm{Sym}^n \mathcal{H}_L$. If $L \supset \mathbb{Q}$, one has even a ring isomorphism

$$L[[\mathcal{H}_\mathbb{Z}]] \xrightarrow{\cong} \prod_{k \geq 0} \mathrm{Sym}^k \mathcal{H}_L, \tag{21}$$

induced by $h \mapsto \sum_{k \geq 0} h^{\otimes k}/k!$ for $h \in \mathcal{H}_\mathbb{Z}$. In the special case $L = \mathbb{Z}, \mathbb{Q}$ the canonical map of group rings $\mathbb{Z}[\mathcal{H}_\mathbb{Z}] \to \mathbb{Q}[\mathcal{H}_\mathbb{Z}]$ induces

$$\mathbb{Z}[[\mathcal{H}_\mathbb{Z}]] \to \mathbb{Q}[[\mathcal{H}_\mathbb{Z}]] \cong \prod_{k \geq 0} \mathrm{Sym}^k \mathcal{H}_\mathbb{Q}. \tag{22}$$

Definition 2.2.1. *The* logarithm sheaf Log *is the local system on $\mathcal{T}$ defined by*

$$\mathrm{Log} := \tilde{\pi}_! L \otimes_{L[\mathcal{H}_{\mathbb{Z}}]} L[[\mathcal{H}_{\mathbb{Z}}]].$$

As a local system of $L[[\mathcal{H}_{\mathbb{Z}}]]$-modules, Log *is of rank* 1.

Any isogeny $\phi : \mathcal{T} \to \mathcal{T}$ of order invertible in L induces an isomorphism $\mathrm{Log} \cong \phi^* \mathrm{Log}$, which is induced by $\phi : \mathcal{H}_{\mathbb{Z}} \to \mathcal{H}_{\mathbb{Z}}$. In particular, if the order of a torsion section $x : S \to \mathcal{T}$ is invertible in L, one has an isomorphism

$$x^* \mathrm{Log} \cong e^* \mathrm{Log} = L[[\mathcal{H}_{\mathbb{Z}}]].$$

To complete the definition of the polylogarithm, one has to compute the cohomology of Log. As $L[[\mathcal{H}_{\mathbb{Z}}]]$ is a flat $L[\mathcal{H}_{\mathbb{Z}}]$-module one gets

$$R^i \pi_* \mathrm{Log} \cong R^i \pi_* \tilde{\pi}_! L \otimes_{L[\mathcal{H}_{\mathbb{Z}}]} L[[\mathcal{H}_{\mathbb{Z}}]]$$

and because $\pi_* = \pi_!$ one has to consider $R^i(\pi \circ \tilde{\pi})_! L$. But the fibers of

$$\pi \circ \tilde{\pi} : \tilde{\mathcal{H}}_{\mathbb{R}} \to S$$

are just g-dimensional vector spaces and the cohomology with compact support lives only in degree g, where it is the dual of $\Lambda^{max} \mathcal{H}_L$. Hence, we have proved:

Lemma 2.2.2. *Denote by $\mu_{\mathcal{T}}^{\vee}$ the L-dual of $\mu_{\mathcal{T}} := \Lambda^{max} \mathcal{H}_L$. Then the higher direct images of* Log *are given by*

$$R^i \pi_* \mathrm{Log} \cong \begin{cases} \mu_{\mathcal{T}}^{\vee} & \text{if } i = g \\ 0 & \text{else.} \end{cases}$$

As in 2.1.3 one obtains

$$\mathrm{Ext}^g_U(L[D]^0, \mathrm{Log} \otimes \mu_{\mathcal{T}}) \cong \mathrm{Hom}_S(L[D]^0, \mathrm{Log}[D]^0)$$

and one defines the polylogarithm

$$\mathrm{pol}^D \in \mathrm{Ext}^{g-1}_U(L[D]^0, \mathrm{Log} \otimes \mu_{\mathcal{T}})$$

in the same way. For $\alpha \in L[D]^0$ one has again

$$\mathrm{pol}^D_\alpha \in \mathrm{Ext}^{g-1}_U(L, \mathrm{Log} \otimes \mu_{\mathcal{T}}) = H^{g-1}(U, \mathrm{Log} \otimes \mu_{\mathcal{T}}).$$

The relation to the polylog defined in 2.1.4 is as follows: If we denote the logarithm sheaf and the polylog from this section by $\mathrm{Log}_{\mathbb{Z}}$ and $\mathrm{pol}^D_{\mathbb{Z}}$ and similarly the ones from 2.1.4 by $\mathrm{Log}_{\mathbb{Q}}$ and $\mathrm{pol}^D_{\mathbb{Q}}$, we get from 22 a canonical map $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Log}_{\mathbb{Z}} \to \mathrm{Log}_{\mathbb{Q}}$ and hence a map

$$\mathrm{Ext}^{g-1}_U(\mathbb{Z}[D]^0, \mathrm{Log}_{\mathbb{Z}} \otimes \mu_{\mathcal{T}}) \to \mathrm{Ext}^{g-1}_U(\mathbb{Q}[D]^0, \mathrm{Log}_{\mathbb{Q}} \otimes \mu_{\mathcal{T}}),$$

which maps $\mathrm{pol}^D_{\mathbb{Z}}$ to
$pol^D_{\mathbb{Q}}$.

2.3 Integral version of the polylogarithm, the étale case

This section will not be used in the rest of the paper and can be omitted by any reader not interested in the integral étale case.
To define an integral étale polylogarithm, one has to modify the definition of the logarithm sheaf as in the topological case. The situation we consider here is again an abelian scheme

$$\pi : \mathcal{A} \to S$$

of constant fiber dimension g and unit section $e : S \to \mathcal{A}$. Let ℓ be a prime number, $L = \mathbb{Z}/\ell^k\mathbb{Z}$ and assume that ℓ is invertible in $\mathcal{O}_S$. Then the ℓ^r-multiplication $[\ell^r] : \mathcal{A} \to \mathcal{A}$ is étale and the sheaves $[\ell^r]_! L$ form a projective system via the trace maps

$$[\ell^r]_! L \to [\ell^{r-1}]_! L.$$

DEFINITION 2.3.1. *The* logarithm sheaf *is the inverse limit*

$$\mathrm{Log}_L := \varprojlim_r [\ell^r]_! L$$

with respect to the above trace maps. The logarithm sheaf with $\mathbb{Z}_\ell$-coefficients is defined by

$$\mathrm{Log}_{\mathbb{Z}_\ell} := \varprojlim_k \mathrm{Log}_{\mathbb{Z}/\ell^k\mathbb{Z}}.$$

Let $\mathcal{H}_\ell := \varprojlim_r \mathcal{A}[\ell^r]$ be the Tate-module of $\mathcal{A}/S$. As ℓ is nilpotent in L, we get that $e^* \mathrm{Log} = L[[\mathcal{H}_\ell]]$ is the Iwasawa algebra of $\mathcal{H}_\ell$ with coefficients in L. Any isogeny $\phi : \mathcal{A} \to \mathcal{A}$ of degree prime to ℓ induces an isomorphism $[\ell^r]_! L \to \phi^*[\ell^r]_! L$, which induces

$$\mathrm{Log} \cong \phi^* \mathrm{Log}.$$

PROPOSITION 2.3.2. *Let $L = \mathbb{Z}/\ell^k\mathbb{Z}$ or $L = \mathbb{Z}_\ell$. The higher direct images of* Log *are given by*

$$R^i\pi_* \mathrm{Log} \cong \begin{cases} L(-g) & \text{if } i = 2g \\ 0 & \text{else.} \end{cases}$$

Proof. It suffices to consider the case $L = \mathbb{Z}/\ell^k\mathbb{Z}$. We will show that the transition maps $R^i\pi_*[\ell^r]_! L \to R^i\pi_*[\ell^s]_! L$ are zero for $i < 2g$ and every s, if r is sufficiently big. By Poincaré duality we may consider the maps

$$R^{2g-i}\pi_![\ell^s]_* L(g) \to R^{2g-i}\pi_![\ell^r]_* L(g). \tag{23}$$

By base change we may assume that S is the spectrum of an algebraically closed field. Denote by $\mathcal{A}_s$ the variety $\mathcal{A}$ considered as covering of $\mathcal{A}$ via $[\ell^s]$. Then

$$R^1\pi_![\ell^s]_* L(g) = H^1(\mathcal{A}, [\ell^s]_* L(g)) = \mathrm{Hom}(\pi_1(\mathcal{A}_s), L(g)).$$

With this description we see that for every $f \in \mathrm{Hom}(\pi_1(\mathcal{A}_s), L(g))$ there is an r, such that the restriction to $\pi_1(\mathcal{A}_r)$ is trivial. This shows that the map in (23) is zero, if r is sufficiently big and $i < 2g$ as the cohomology in degree i is the i-th exterior power of the first cohomology. That (23) is an isomorphism for $i = 2g$ is clear. $\square$

2.4 Eisenstein classes

The Eisenstein classes are specializations of the polylogarithm. The situation is as follows. First let $\alpha \in L[\mathcal{A}[n]]^0$ and assume that $\mathbb{Q} \subset L$. Then one can pull-back the class $\mathrm{pol}_\alpha^{\mathcal{A}[n]} \in \mathrm{Ext}_U^{g-1}(L, \mathrm{Log}(g))$ along e and gets:

$$e^* \mathrm{pol}_\alpha^{\mathcal{A}[n]} \in \mathrm{Ext}_S^{2g-1}(L, e^* \mathrm{Log}(g)) = \prod_{k \geq 0} \mathrm{Ext}_S^{2g-1}(L, \mathrm{Sym}^k \mathcal{H}(g)).$$

Definition 2.4.1. *For any $\alpha \in L[\mathcal{A}[n]]^0$, define the k-th* Eisenstein class *associated to α,*

$$\mathrm{Eis}^k(\alpha) \in \mathrm{Ext}_S^{2g-1}(L, \mathrm{Sym}^k \mathcal{H}(g)),$$

to be the k-th component of $e^* \mathrm{pol}_\alpha^{\mathcal{A}[n]}$.

Note that by the functoriality of the polylogarithm the map

$$L[\mathcal{A}[n]]^0 \xrightarrow{\mathrm{Eis}^k} \mathrm{Ext}_S^{2g-1}(L, \mathrm{Sym}^k \mathcal{H}(g)) \tag{24}$$

is equivariant for the $G(\mathbb{Z}/n\mathbb{Z})$ action on both sides.
These Eisenstein classes should be considered as analogs of Harder's Eisenstein classes (but observe that we have only classes in cohomological degree $2g-1$). The advantage of the above classes is that they are defined by a universal condition, which makes a lot of their properties easy to verify.

3 Proof of the main theorem

In this section we assume that $\mathbb{Q} \subset L$.
The proof of the main theorem will be in several steps. First we reduce to the case of local systems for the usual topology. The second step consists of a trick already used in [HK]: instead of working with the Eisenstein classes directly, we work with the polylogarithm itself. The reason is that the polylog is characterized by a universal property and has a very good functorial behavior. The third step reviews the computations of Nori in [N]. In the fourth step we compute the integral over S_T^1 and the fifth step gives the final result.

3.1 1. Step: Reduction to the classical topology

We distinguish the MHM and the étale case. In the MHM case, the target of the residue map from (1.5.2)

$$\mathrm{res} : \mathrm{Ext}_S^{2g-1}(L, \mathrm{Sym}^k \mathcal{H}(g)) \to \mathrm{Hom}_{\partial S}(L, L). \tag{25}$$

is purely topological and does not depend on the Hodge structure. More precisely, the canonical map "forget the Hodge structure" denoted by rat induces an isomorphism

$$\mathrm{rat} : \mathrm{Hom}_{MHM,\partial S}(L, L) \cong \mathrm{Hom}_{top,\partial S}(L, L).$$

By [Sa] thm. 2.1 we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^{2g-1}_{MHM,S}(L, \mathrm{Sym}^k \mathcal{H}(g)) & \xrightarrow{\mathrm{res}} & \mathrm{Hom}_{MHM,\partial S}(L, L) \\ \downarrow{\scriptstyle \mathrm{rat}} & & \cong\downarrow{\scriptstyle \mathrm{rat}} \\ \mathrm{Ext}^{2g-1}_{top,S}(L, \mathrm{Sym}^k \mathcal{H}(g)) & \xrightarrow{\mathrm{res}} & \mathrm{Hom}_{top,\partial S}(L, L). \end{array} \tag{26}$$

This reduces the computation of the residue map for MHM to the case of local systems in the classical topology.
In the étale case one has an injection

$$\mathrm{Hom}_{et,\partial S}(L, L) \hookrightarrow \mathrm{Hom}_{et,\partial S\times\bar{\mathbb{Q}}}(L, L) \cong \mathrm{Hom}_{top,\partial S(\mathbb{C})}(L, L).$$

and a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^{2g-1}_{et,S}(L, \mathrm{Sym}^k \mathcal{H}(g)) & \xrightarrow{\mathrm{res}} & \mathrm{Hom}_{et,\partial S}(L, L) \\ \downarrow & & \downarrow \\ \mathrm{Ext}^{2g-1}_{top,S(\mathbb{C})}(L, \mathrm{Sym}^k \mathcal{H}(g)) & \xrightarrow{\mathrm{res}} & \mathrm{Hom}_{top,\partial S(\mathbb{C})}(L, L). \end{array} \tag{27}$$

Again, this reduces the residue computation to the classical topology.

3.2 2. Step: Topological degeneration

In this section we reduce the computation of $\mathrm{res}\circ\mathrm{Eis}^k$ to a computation of the polylog on $\mathcal{T}_{\mathcal{M}}$.
We are now in the topological situation and use again the notations ∂S and S instead of $\partial S(\mathbb{C})$ and $S(\mathbb{C})$.
Recall from (19) that $\mathrm{res}\circ\mathrm{Eis}^k$ is $G(\mathbb{Z}/n\mathbb{Z})$ equivariant. In particular,

$$\mathrm{res}(\mathrm{Eis}^k(\alpha))(h) = \mathrm{res}(\mathrm{Eis}^k(h\alpha))(\mathrm{id}),$$

where $h\alpha$ denotes the action of h on α. To compute the residue it suffices to consider the residue at id.
Recall from (14) that we have a commutative diagram of fibrations

$$\begin{array}{ccc} \mathcal{A}(\mathbb{C}) & \xrightarrow{p} & \mathcal{T}_{\mathcal{M}} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{M}}} \\ S^1_B & \xrightarrow{q} & S^1_T. \end{array} \tag{28}$$

The map $p : \mathcal{H} \to \mathcal{M}$ induces $\mathrm{Log}_{\mathcal{A}} \to p^* \mathrm{Log}_{\mathcal{M}}$. Let $D = \mathcal{A}[n]$ and $U := \mathcal{A} \setminus D$ be the complement. Let $p(D) = \mathcal{T}_{\mathcal{M}}[n]$ be the image of D in $\mathcal{T}_{\mathcal{M}}$ and $V := \mathcal{T}_{\mathcal{M}} \setminus p(D)$ be its complement in $\mathcal{T}_{\mathcal{M}}$. Then p induces a map

$$p : U \setminus p^{-1}(p(D)) \to V.$$

We define a trace map

$$p_* : \mathrm{Ext}^{2g-1}_U(L, \mathrm{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}}) \to \mathrm{Ext}^{g-1}_V(L, \mathrm{Log}_{\mathcal{M}} \otimes \mu_{\mathcal{T}_{\mathcal{M}}}) \tag{29}$$

as the composition of the restriction to $U \setminus p^{-1}(p(D))$

$$\mathrm{Ext}^{2g-1}_U(L, \mathrm{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}}) \to \mathrm{Ext}^{2g-1}_{U \setminus p^{-1}(p(D))}(L, \mathrm{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}})$$

with the adjunction map

$$\mathrm{Ext}^{2g-1}_{U \setminus p^{-1}(p(D))}(L, \mathrm{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}}) \to \mathrm{Ext}^{g-1}_V(L, R^g p_* p^* \mathrm{Log}_{\mathcal{M}} \otimes \mu_{\mathcal{A}}).$$

As $\mu_{\mathcal{A}} \cong \mu_{\mathcal{T}_{\mathcal{N}}} \otimes \mu_{\mathcal{T}_{\mathcal{M}}}$, the projection formula gives

$$R^g p_* p^* \mathrm{Log}_{\mathcal{M}} \cong \mathrm{Log}_{\mathcal{M}} \otimes \mu_{\mathcal{T}_{\mathcal{M}}}.$$

The composition of these maps gives the desired p_* in (29). The crucial fact is that the polylogarithm behaves well under this trace map.

Proposition 3.2.1. *With the notations above, let $\alpha \in L[D]^0$ and $\mathrm{pol}^D_{\mathcal{A},\alpha} \in \mathrm{Ext}^{2g-1}_U(L, \mathrm{Log}_{\mathcal{A}})$ be the associated polylogarithm. Denote by $p(\alpha)$ the image of α under the map*

$$p : L[D]^0 \to L[p(D)]^0$$

induced by $p : \mathcal{A}(\mathbb{C}) \to \mathcal{T}_{\mathcal{M}}$. Then

$$p_* \mathrm{pol}^D_{\mathcal{A},\alpha} = \mathrm{pol}^{p(D)}_{\mathcal{T}_{\mathcal{M}}, p(\alpha)}.$$

Proof. This is a quite formal consequence of the definition and the fact that the residue map commutes with the trace map. We use cohomological notation, then one has a commutative diagram

$$\begin{array}{ccc}
H^{2g-1}(U, \mathrm{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}}) & \longrightarrow & H^{2g}_D(\mathcal{A}, \mathrm{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}}) \\
\downarrow & & \downarrow \\
H^{2g-1}(U \setminus p^{-1}(p(D)), \mathrm{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}}) & \longrightarrow & H^{2g}_{p^{-1}(p(D))}(\mathcal{A}, \mathrm{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}}) \\
\downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} \\
H^{g-1}(V, \mathrm{Log}_{\mathcal{M}} \otimes \mu_{\mathcal{T}_{\mathcal{M}}}) & \longrightarrow & H^g_{p(D)}(\mathcal{T}_{\mathcal{M}}, \mathrm{Log}_{\mathcal{M}} \otimes \mu_{\mathcal{T}_{\mathcal{M}}}).
\end{array}$$

We can identify

$$H^{2g}_D(\mathcal{A}, \operatorname{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}}) \cong \bigoplus_{\sigma \in D} \sigma^* \operatorname{Log}_{\mathcal{A}}$$

and

$$H^g_{p(D)}(\mathcal{T}_{\mathcal{M}}, \operatorname{Log}_{\mathcal{M}} \otimes \mu_{\mathcal{T}_{\mathcal{M}}}) \cong \bigoplus_{\sigma \in p(D)} \sigma^* \operatorname{Log}_{\mathcal{T}_{\mathcal{M}}}.$$

With this identification the composition of the vertical arrows on the right is induced by $\operatorname{Log}_{\mathcal{A}} \to p^* \operatorname{Log}_{\mathcal{T}_{\mathcal{M}}}$. The polylog $\operatorname{pol}^D_{\mathcal{A},\alpha}$ belongs to the section $\alpha \in L[D]^0 \subset \bigoplus_{\sigma \in D} \sigma^* \operatorname{Log}_{\mathcal{A}}$. This maps to $p(\alpha) \in L[p(D)]^0 \subset \bigoplus_{\sigma \in p(D)} \sigma^* \operatorname{Log}_{\mathcal{T}_{\mathcal{M}}}$. Thus $\operatorname{pol}^D_{\mathcal{A},\alpha}$ is mapped under p_* to $\operatorname{pol}^{p(D)}_{\mathcal{T}_{\mathcal{M}},p(\alpha)}$. □

We want to prove the same sort of result for the Eisenstein classes themselves. To formulate it properly, we need:

Lemma 3.2.2. *Let $q : S^1_B \to S^1_T$ be the fibration from (28). Then*

$$R^g q_* \operatorname{Sym}^k \mathcal{H} \cong \operatorname{Sym}^k \mathcal{M} \otimes \mu^\vee_{\mathcal{T}_{\mathcal{N}}}.$$

Proof. Recall the exact sequence

$$0 \to \mathcal{N} \to \mathcal{H} \to \mathcal{M} \to 0$$

from (12). By definition of $N(\mathbb{Z})$, the coinvariants of $\operatorname{Sym}^k \mathcal{H}$ for $N(\mathbb{Z})$ are exactly $\operatorname{Sym}^k \mathcal{M}$. The lemma follows, as $R^g q_*$ corresponds by definition of the fibering exactly to the coinvariants under $N(\mathbb{Z})$. □

Define a trace map

$$q_* : \operatorname{Ext}^{2g-1}_{S_B}(L, \operatorname{Sym}^k \mathcal{H} \otimes \mu_{\mathcal{A}}) \to \operatorname{Ext}^{g-1}_{S_T}(L, \operatorname{Sym}^k \mathcal{M} \otimes \mu_{\mathcal{T}_{\mathcal{M}}})$$

by adjunction for q, the isomorphism $R^g q_* \operatorname{Sym}^k \mathcal{H} \cong \operatorname{Sym}^k \mathcal{M} \otimes \mu^\vee_{\mathcal{T}_{\mathcal{N}}}$ from lemma 3.2.2 and the isomorphism $\mu_{\mathcal{A}} \cong \mu_{\mathcal{T}_{\mathcal{N}}} \otimes \mu_{\mathcal{T}_{\mathcal{M}}}$. The behaviour of $\operatorname{Eis}^k(\alpha)$ under q_* is given by:

Theorem 3.2.3. *Let $k > 0$ and $\alpha \in L[D]^0$. Then*

$$q_*(\operatorname{Eis}^k_{\mathcal{A}}(\alpha)) = \operatorname{Eis}^k_{\mathcal{T}_{\mathcal{M}}}(p(\alpha)),$$

where $p : L[D]^0 \to L[p(D)]^0$ is the map from 3.2.1.

Proof. Consider the following diagram in the derived category:

$$\begin{tikzcd}
Rp_* \operatorname{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}} \arrow[r] \arrow[d] & Rp_* e_* e^* \operatorname{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}} \arrow[d, equal] \\
Rp_* p^* \operatorname{Log}_{\mathcal{T}_{\mathcal{M}}} \otimes \mu_{\mathcal{A}} \arrow[d] & e'_* R q_* e^* \operatorname{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}} \arrow[d] \\
\operatorname{Log}_{\mathcal{T}_{\mathcal{M}}} \otimes \mu_{\mathcal{T}_{\mathcal{M}}}[-g] \arrow[r] & e'_* e'^* \operatorname{Log}_{\mathcal{T}_{\mathcal{M}}} \otimes \mu_{\mathcal{T}_{\mathcal{M}}}[-g]
\end{tikzcd} \tag{30}$$

We will show that this diagram is commutative and thereby explain all the maps. First consider the commutative diagram

$$\begin{tikzcd} Rp_* \operatorname{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}} \arrow[r] \arrow[d] & Rp_* e_* e^* \operatorname{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}} \arrow[d] \\ Rp_* p^* \operatorname{Log}_{\mathcal{T_M}} \otimes \mu_{\mathcal{A}} \arrow[r] & Rp_* e_* e^* p^* \operatorname{Log}_{\mathcal{T_M}} \otimes \mu_{\mathcal{A}}, \end{tikzcd}$$

where the horizontal arrows are induced from adjunction $\mathrm{id} \to e_* e^*$ and the vertical arrows from $\operatorname{Log}_{\mathcal{A}} \to p^* \operatorname{Log}_{\mathcal{T_M}}$. One has $p \circ e = e' \circ q$ and hence

$$Rp_* e_* e^* p^* \operatorname{Log}_{\mathcal{T_M}} \otimes \mu_{\mathcal{A}} \cong e'_* Rq_* q^* e'^* \operatorname{Log}_{\mathcal{T_M}} \otimes \mu_{\mathcal{A}}.$$

The projection formula gives

$$e'_* Rq_* q^* e'^* \operatorname{Log}_{\mathcal{T_M}} \otimes \mu_{\mathcal{A}} \cong e'_* e'^* \operatorname{Log}_{\mathcal{T_M}} \otimes \mu_{\mathcal{A}} \otimes Rq_* L.$$

Projection to the highest cohomology gives a commutative diagram

$$\begin{tikzcd} Rp_* p^* \operatorname{Log}_{\mathcal{T_M}} \otimes \mu_{\mathcal{A}} \arrow[r] \arrow[d] & e'_* e'^* \operatorname{Log}_{\mathcal{T_M}} \otimes \mu_{\mathcal{A}} \otimes Rq_* L \arrow[d] \\ \operatorname{Log}_{\mathcal{T_M}} \otimes \mu_{\mathcal{A}} \otimes \mu_{\mathcal{T_N}}^{\vee} \arrow[r] & e'_* e'^* \operatorname{Log}_{\mathcal{T_M}} \otimes \mu_{\mathcal{A}} \otimes \mu_{\mathcal{T_N}}^{\vee}, \end{tikzcd}$$

where the horizontal maps are adjunction maps $\mathrm{id} \to e'_* e'^*$. Finally we use $\mu_{\mathcal{A}} \otimes \mu_{\mathcal{T_N}}^{\vee} \cong \mu_{\mathcal{T_M}}$ to obtain the commutative diagram (30). Applying $\operatorname{Ext}_V^{2g-1}(L, -)$ to this diagram, where $V := \mathcal{T_M} \setminus p(D)$ we get

$$\begin{tikzcd} \operatorname{Ext}_U^{2g-1}(L, \operatorname{Log}_{\mathcal{A}} \otimes \mu_{\mathcal{A}}) \arrow[r] \arrow[d, "p_*"'] & \operatorname{Ext}_S^{2g-1}(L, e^* \operatorname{Log} \otimes \mu_{\mathcal{A}}) \arrow[d, "q_*"] \\ \operatorname{Ext}_V^{g-1}(L, \operatorname{Log}_{\mathcal{T_M}} \otimes \mu_{\mathcal{T_M}}) \arrow[r] & \operatorname{Ext}_S^{g-1}(L, e'^* \operatorname{Log}_{\mathcal{T_M}} \otimes \mu_{\mathcal{T_M}}). \end{tikzcd}$$

Now, as $k > 0$, we may assume that $\alpha \in L[D \backslash e(S)]^0$ and $p(\alpha) \in L[p(D) \backslash e'(S)]^0$. The result follows then from proposition 3.2.1. $\square$

In a similar (but simpler) way one shows:

Theorem 3.2.4. *Let $\phi : \mathcal{T_M} \to \mathcal{T_{M'}}$ be an isogeny of tori, then ϕ induces a morphism $\phi_* : e^* \operatorname{Log}_{\mathcal{M}} \to e^* \operatorname{Log}_{\mathcal{M'}}$ and*

$$\phi_* \operatorname{Eis}^k_{\mathcal{T_M}}(\alpha) = \operatorname{Eis}^k_{\mathcal{T_{M'}}}(\phi(\alpha)).$$

3.3 3. Step: Explicit description of the polylog

In this section we follow Nori [N] to describe the polylog $\mathrm{pol}_\beta^{\mathcal{T}_\mathcal{M}[n]}$ for any $\beta \in L[\mathcal{T}_\mathcal{M}[n]\setminus 0]^0$ explicitly. The presentation is also influenced by unpublished notes of Beilinson and Levin.
In fact it is useful for the connection with L-functions to consider a more general situation and to allow arbitrary fractional ideals $\mathfrak{a}$ instead just $\mathcal{O}$.
We assume $L = \mathbb{C}$. The geometric situation is this: Recall that $T^1(\mathbb{Z}) = \mathcal{O}^*$ and let $\mathfrak{a} \subset F$ be a fractional ideal with the usual $T^1(\mathbb{Z})$-action. We can form as usual the semi direct product

$$\mathfrak{a} \rtimes T^1(\mathbb{Z}),$$

where the multiplication is given by the formula $(v,t)(v',t') = (v + tv', tt')$. Similarly, we can form $\mathfrak{a} \otimes \mathbb{R} \rtimes T^1(\mathbb{R})$ and we define

$$\mathcal{T}_\mathfrak{a} := \mathfrak{a} \rtimes T(\mathbb{Z})\backslash \left(\mathfrak{a} \otimes \mathbb{R} \rtimes T^1(\mathbb{R})\right)/K_\infty^T.$$

We have

$$\pi_\mathfrak{a} : \mathcal{T}_\mathfrak{a} \to S_T^1$$

and we consider the polylog for this real torus bundle of relative dimension g. The case $\mathcal{T}_\mathcal{M}$ is the one where $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \in T^1(\mathbb{Z})$ acts via $d \in \mathcal{O}^*$ on $\mathcal{O}$. Let us describe the logarithm sheaf $\mathrm{Log}_{\mathcal{T}_\mathfrak{a}}$ in this setting. As the coefficients are $L = \mathbb{C}$, we can use the isomorphism from (21)

$$\mathbb{C}[[\mathfrak{a}]] \xrightarrow{\cong} \prod_{k\geq 0} \mathrm{Sym}^k \mathfrak{a}_\mathbb{C} =: \hat{\mathcal{U}}(\mathfrak{a}) \tag{31}$$

$$v \mapsto \exp(v) := \sum_{k=0}^{\infty} \frac{v^{\otimes k}}{k!}$$

The action of $(0,t) \in \mathfrak{a} \rtimes T^1(\mathbb{Z})$ on $\hat{\mathcal{U}}(\mathfrak{a})$ is induced by the action of $T^1(\mathbb{Z})$ on $\mathfrak{a}$. The action of

$$(v, \mathrm{id}) \in \mathfrak{a} \rtimes T^1(\mathbb{Z})$$

on $\hat{\mathcal{U}}(\mathfrak{a})$ is given by multiplication with $\exp(v)$. The logarithm sheaf $\mathrm{Log}_{\mathcal{T}_\mathfrak{a}}$ is just the local system defined by the quotient

$$\mathfrak{a} \rtimes T^1(\mathbb{Z})\backslash \left(\mathfrak{a} \otimes \mathbb{R} \rtimes T^1(\mathbb{R}) \times \hat{\mathcal{U}}(\mathfrak{a})\right)/K_\infty^T.$$

A $\mathcal{C}^\infty$-section f of $\mathrm{Log}_{\mathcal{T}_\mathfrak{a}}$ is a function $f : \mathfrak{a} \otimes \mathbb{R} \rtimes T^1(\mathbb{R}) \to \hat{\mathcal{U}}(\mathfrak{a})$, which has the equivariance property

$$f((v,t)(v',t')) = (v,t)^{-1} f(v',t').$$

In a similar way, we can describe $\mathrm{Log}_{\mathcal{T}_\mathfrak{a}}$-valued currents. The global $\mathcal{C}^\infty$-section

$$\exp(-v) : (v,t) \mapsto \sum_{k=0}^{\infty} \frac{(-v)^{\otimes k}}{k!},$$

with $(v,t) \in \mathfrak{a} \otimes \mathbb{R} \rtimes T^1(\mathbb{R})$ defines a trivialization of $\mathrm{Log}_{\mathcal{T}_\mathfrak{a}}$ as $\mathcal{C}^\infty$-bundle. Every current $\mu(v,t)$ with values in $\mathrm{Log}_{\mathcal{T}_\mathfrak{a}}$ can then be written in the form

$$\mu(v,t) = \nu(v,t)\exp(-v),$$

where $\nu(v,t)$ is now a current with values in the constant bundle $\hat{\mathcal{U}}(\mathfrak{a})$. In particular, $\nu(v,t)$ is invariant under the action of $\mathfrak{a} \subset \mathfrak{a} \rtimes T^1(\mathbb{Z})$.

Lemma 3.3.1. *Let $\mathbf{v} : \mathfrak{a}\otimes\mathbb{R} \to \hat{\mathcal{U}}(\mathfrak{a})$ be the canonical inclusion given by $\mathfrak{a}\otimes\mathbb{R} \subset \mathrm{Sym}^1 \mathfrak{a} \otimes \mathbb{C}$, then the canonical connection ∇ on $\mathrm{Log}_{\mathcal{T}_\mathfrak{a}}$ acts on ν by*

$$\nabla\nu = (d - d\mathbf{v})\nu.$$

Proof. Straightforward computation. □

Following Nori [N] we describe the polylog as a $\mathrm{Log}_{\mathcal{T}_\mathfrak{a}}$-valued current $\mu(v,t)$ on $\mathcal{T}_\mathfrak{a}$, such that

$$\nabla\mu(v,t) = \delta_\beta, \tag{32}$$

where

$$\delta_\beta := \sum_{\sigma\in D} l_\sigma \delta_\sigma$$

and δ_σ are the currents defined by integration over the cycles on $\mathcal{T}_\mathfrak{a}$ given by the section σ. If we write as above

$$\mu(v,t) = \nu(v,t)\exp(-v)$$

we get the equivalent condition

$$(d - d\mathbf{v})\nu(v,t) = \delta_\beta. \tag{33}$$

As $\nu(v,t)$ is invariant under the $\mathfrak{a}$-action, we can develop $\nu(v,t)$ into a Fourier series

$$\nu(v,t) = \sum_{\rho\in\mathfrak{a}^\vee} \nu_\rho(t) e^{2\pi i\rho(v)}. \tag{34}$$

The property (33) reads for the Fourier coefficients $\nu_\rho(t)$:

$$(d + 2\pi i d\rho - d\underline{v})\nu_\rho(t) = (e^{-2\pi i\rho(\beta)})\,\mathrm{vol}, \tag{35}$$

where vol is the unique constant coefficient g-form on $\mathfrak{a}\otimes\mathbb{R}$, such that the integral $\int_{\mathfrak{a}\otimes\mathbb{R}/\mathfrak{a}} \mathrm{vol} = 1$ and

$$e^{-2\pi i\rho(\beta)} := \sum_\sigma l_\sigma e^{-2\pi i\rho(\sigma)}.$$

We do not explain in detail the method of Nori to solve this equation, we just give the result. This suffices, because the cohomology class of the polylogarithm is uniquely determined by the equation (32) and we just need to give a solution for it.
Fix a positive definite quadratic form q on $\mathfrak{a}\otimes\mathbb{R}$, viewed as an isomorphism

$$q : (\mathfrak{a}\otimes\mathbb{R})^\vee \cong \mathfrak{a}\otimes\mathbb{R}.$$

Define a left action of $t\in T^1(\mathbb{R})$ by $q_t(v,w) := q(t^{-1}v, t^{-1}w)$. Consider ρ as element in $(\mathfrak{a}\otimes\mathbb{R})^\vee$. Then $q_t(\rho)$ can be considered as a vector field and we denote by ι_ρ the contraction with this vector field $q_t(\rho)$. We may also consider $q_t(\rho)$ as element in $\hat{\mathcal{U}}(\mathfrak{a})$ and denote this by $\mathbf{q}_t(\rho)$.

THEOREM 3.3.2 (Nori). *With the notations above, one has for $0\neq\rho$*

$$\nu_\rho(t) = \sum_{m=0}^{g-1} \frac{(-1)^m(e^{-2\pi i\rho(\beta)})}{(2\pi i\rho(q_t(\rho)) - \mathbf{q}_t(\rho))^{m+1}}\iota_\rho(d\circ\iota_\rho)^m \operatorname{vol}$$

and

$$\nu_0(t) = 0$$

Proof. Write Φ_ρ for the operator multiplication by $2\pi i d\rho - d\mathbf{v}$ and $\Psi_\rho := d+\Phi_\rho$. One checks that $\Psi_\rho\circ\Psi_\rho = 0 = \iota_\rho\circ\iota_\rho$ and that $\Psi_\rho\circ\iota_\rho + \iota_\rho\circ\Psi_\rho$ is an isomorphism. Indeed $\Phi_\rho\circ\iota_\rho+\iota_\rho\circ\Phi_\rho$ is multiplication by $2\pi i\rho(q_t(\rho)) - \mathbf{q}_t(\rho)$ and $\mathcal{L}_\rho := d\circ\iota_\rho + \iota_\rho\circ$ is the Lie derivative with respect to the vector field $q_t(\rho)$. The formula in the theorem is just

$$\iota_\rho\circ(\Psi_\rho\circ\iota_\rho+\iota_\rho\circ\Psi_\rho)^{-1}(e^{-2\pi i\rho(\beta)})\operatorname{vol}$$

and to check that

$$\Psi_\rho\circ\iota_\rho\circ(\Psi_\rho\circ\iota_\rho+\iota_\rho\circ\Psi_\rho)^{-1} = \mathrm{id}$$

note that $\iota_\rho\circ\Psi_\rho$ commutes with $(\Psi_\rho\circ\iota_\rho+\iota_\rho\circ\Psi_\rho)^{-1}$ and $\iota_\rho\circ\Psi_\rho(e^{-2\pi i\rho(\beta)})\operatorname{vol} = 0$. □

COROLLARY 3.3.3. *The polylogarithm $\mathrm{pol}_\beta^{\mathcal{T}_\mathfrak{a}[n]}$ is given in the topological realization by the current*

$$\mu(v,t) = \nu(v,t)\exp(-v)$$

where $\nu(v,t)$ is the current given by

$$\sum_{m=0}^{g-1}\sum_{k=0}^{\infty}\binom{k+m}{k}\sum_{\rho\in\mathfrak{a}^\vee\setminus 0}\frac{(-1)^m e^{2\pi i\rho(v-\beta)}}{(2\pi i\rho(q_t(\rho)))^{k+m+1}}\mathbf{q}_t(\rho)^{\otimes k}\iota_\rho(d\circ\iota_\rho)^m\operatorname{vol}.$$

Proof. This follows from the formula $\frac{1}{(A-B)^{m+1}} = \sum_{k=0}^{\infty}\frac{B^k}{A^{k+m+1}}\binom{k+m}{k}$. □

The Eisenstein classes are obtained by pull-back of this current along the zero section e. As for $k > 0$ the series over the ρ converges absolutely, this is defined and only terms with $m = g - 1$ survive. We get the following formula for the Eisenstein classes.

Corollary 3.3.4. *Let $\beta \in \mathbb{C}[\mathcal{T}_{\mathfrak{a}}[n]]^0$ and $k > 0$, then the topological Eisenstein class is given by*

$$\mathrm{Eis}^k(\beta) = \frac{(k+g-1)!}{k!} \sum_{\rho \in \mathfrak{a}^\vee \setminus 0} \frac{(-1)^{g-1} e^{-2\pi i \rho(\beta)}}{(2\pi i \rho(q_t(\rho)))^{k+g}} \mathbf{q}_t(\rho)^{\otimes k} q_t(\rho)^* \iota_{\mathcal{E}} \operatorname{vol}.$$

Here, we have written $\mathcal{E}$ for the Euler vector field and $q_t(\rho)$ is considered as a function $q_t(\rho) : S_T \to \mathfrak{a} \otimes \mathbb{R}$, which maps t to the vector $q_t(\rho)$.

Proof. From 3.3.3 we have to compute

$$e^* \iota_\rho (d \circ \iota_\rho)^m \operatorname{vol}.$$

For this remark that the Lie derivative $\mathcal{L}_\rho = d \circ \iota_\rho + \iota_\rho \circ d$ with respect to the vector field $q_t(\rho)$ acts in the same way on vol as $d \circ \iota_\rho$. One sees immediately that $e^* \iota_\rho (d \circ \iota_\rho)^m \operatorname{vol} = 0$, if $m < g - 1$ and a direct computation in coordinates gives that $\iota_\rho (\mathcal{L}_\rho)^{g-1} \operatorname{vol} = (g-1)! q_t(\rho)^* \iota_{\mathcal{E}} \operatorname{vol}$. □

3.4 4. Step: Computation of the integral

To finish the proof of theorem 1.7.1 we have to compute $u_* \mathrm{Eis}^k(\beta)$, where $u : S_T^1 \to pt$ is the structure map. As we need only to compute the corresponding integral for the component of S_T^1 corresponding to id, we let $\Gamma_T \subset T^1(\mathbb{Z})$ be the stabilizer of $\mathrm{id} \in T(\mathbb{Z}/n\mathbb{Z})$ and consider

$$u_{\mathrm{id}} : \Gamma_T \backslash \big(T^1(\mathbb{R}) / K_\infty^{T^1}\big) \to pt.$$

To compute the integral, we introduce coordinates on $T^1(\mathbb{R}) \cong (F \otimes \mathbb{R})^*$ and on the torus $\mathcal{T}_{\mathfrak{a}}$. We identify $F \otimes \mathbb{R} \cong \prod_{\tau : F \to \mathbb{R}} \mathbb{R}$ and denote by $e_1, \ldots, e_g$ the standard basis on the right hand side and by $x_1, \ldots, x_g$ the dual basis. For any element $u = \sum u_i e_i$ or $u = \sum u_i x_i$ we write $Nu := u_1 \cdots u_g$. Let q be the quadratic form given by $\sum x_i^2$. We identify the orbit of q under $T^1(\mathbb{R})$ with $(F \otimes \mathbb{R})_+^*$ by mapping

$$\begin{aligned} (F \otimes \mathbb{R})^* &\to T^1(\mathbb{R}) q \\ t &\mapsto q_t. \end{aligned} \tag{36}$$

This map factors over $(F \otimes \mathbb{R})_+^*$ and the map is compatible with the $T^1(\mathbb{Z})$ action on both sides. We let $t_1, \ldots, t_g$ be coordinates on $(F \otimes \mathbb{R})^1$ so that $t_1^2, \ldots, t_g^2$ are coordinates on $(F \otimes \mathbb{R})_+^1$. If we write $\rho = \sum \rho_i x_i$ and $t_i := x_i(t)$, then

$$\rho(q_t(\rho)) = \sum t_i^2 \rho_i^2$$

and $\mathbf{q}_t(\rho)$ has coordinates $t_i^2\rho_i$. More precisely, if we let $\mathbf{e}_1,\ldots,\mathbf{e}_g$ be the basis $e_1,\ldots,e_g$ considered as elements of $\hat{\mathcal{U}}(\mathfrak{a})$, which identifies $\hat{\mathcal{U}}(\mathfrak{a})$ with the power series ring $\mathbb{C}[[\mathbf{e}_1,\ldots,\mathbf{e}_g]]$, then $\mathbf{q}_t(\rho)=\sum t_i^2\rho_i\mathbf{e}_i$. The volume form is given by

$$\mathrm{vol}=|d_F|^{-1/2}N\mathfrak{a}^{-1}dx_1\wedge\ldots\wedge dx_g$$

and we can write the Euler vector field as $\mathcal{E}=\sum x_i\partial_{x_i}$. One gets (observe that $Nt=1$)

$$q_t(\rho)^*\iota_{\mathcal{E}}\,\mathrm{vol}=|d_F|^{-1/2}2^{g-1}N(\rho)N\mathfrak{a}^{-1}\sum_{k=1}^{g}(-1)^{k-1}t_k dt_1\wedge\ldots\widehat{dt_k}\ldots\wedge dt_g.$$

Explicitly, the Eisenstein class is given as a current on $T^1(\mathbb{R})$ by

$$\mathrm{Eis}^k(\beta)(t)=\frac{(k+g-1)!}{k!}\sum_{\rho\in\mathfrak{a}^\vee\setminus 0}\frac{(-1)^{g-1}e^{-2\pi i\rho(\beta)}(\sum t_i^2\rho_i\mathbf{e}_i)^{\otimes k}}{(2\pi i\sum\rho_i^2t_i^2)^{k+g}}q_t(\rho)^*\iota_{\mathcal{E}}\,\mathrm{vol}\tag{37}$$

Define an isomorphism $(\mathbb{R}\otimes F)^1\times\mathbb{R}^*\cong(\mathbb{R}\otimes F)^*$ by mapping $(t,r)\mapsto y:=rt$. Then we get:

$$\frac{dy_1}{y_1}\wedge\ldots\wedge\frac{dy_g}{y_g}=\frac{dr}{r}\wedge\sum_{k=1}^{g}(-1)^{k-1}t_k dt_1\wedge\ldots\widehat{dt_k}\ldots\wedge dt_g.\tag{38}$$

We use this decomposition to write $\mathrm{Eis}^k(\beta)(t)$ as a Mellin transform:

$$\mathrm{Eis}^k(\beta)(t)=\sum_{\rho\in\mathfrak{a}^\vee\setminus 0}(-1)^{g-1}e^{-2\pi i\rho(\beta)}\int_{\mathbb{R}_{>0}}e^{-u(2\pi i\sum\rho_i^2t_i^2)}\frac{(\sum t_i^2\rho_i\mathbf{e}_i)^{\otimes k}}{k!}u^{k+g}\frac{du}{u}\wedge q_t(\rho)^*\iota_{\mathcal{E}}\,\mathrm{vol}.\tag{39}$$

Substitute $u=r^2=N(y)^{2/g}$ and use (38) to get

$$\mathrm{Eis}^k(\beta)(t)=\sum_{\rho\in\mathfrak{a}^\vee\setminus 0}\frac{(-1)^{g-1}2^ge^{-2\pi i\rho(\beta)}N(\rho)}{|d_F|^{1/2}N\mathfrak{a}}\int_{\mathbb{R}_{>0}}e^{-2\pi i\sum\rho_i^2y_i^2}\frac{(\sum y_i^2\rho_i\mathbf{e}_i)^{\otimes k}}{k!}N(y)dy_1\wedge\ldots\wedge dy_g.\tag{40}$$

The application of $u_{\mathrm{id},*}$ amounts to integration over

$$\Gamma_T\backslash\big(T^1(\mathbb{R})/K_\infty^{T^1}\big)\cong\mathcal{O}_{(n)}^*\backslash(F\otimes\mathbb{R})_+^1,$$

where $\mathcal{O}^*_{(n)}$ are the totally positive units, which are congruent to 1 modulo the ideal generated by (n). This gives with the usual trick

(41) $$u_{\mathrm{id},*}\operatorname{Eis}^k(\beta) = \sum_{\rho\in\mathcal{O}^*_{(n)}\backslash(\mathfrak{a}^\vee\backslash 0)} \frac{(-1)^{g-1}2^g e^{-2\pi i\rho(\beta)}N(\rho)}{|d_F|^{1/2}N\mathfrak{a}} \times \int\limits_{(F\otimes\mathbb{R})^*_+} e^{-2\pi i\sum\rho_i^2y_i^2}\frac{(\sum y_i^2\rho_i\mathbf{e}_i)^{\otimes k}}{k!}N(y)dy_1\wedge\ldots\wedge dy_g.$$

The integral is a product of integrals for $j=1,\ldots,g$:

$$\int\limits_{\mathbb{R}_{>0}} e^{-2\pi i\rho_j^2y_j^2}\rho_j^k\frac{\mathbf{e}_j^{\otimes k}}{k!}y_j^{2k+2}\frac{dy_j}{y_j} = \frac{\mathbf{e}_j^{\otimes k}}{2\rho_j(2\pi i\rho_j)^{k+1}}.$$

We now consider $\operatorname{Eis}^{gk}(\beta)$ instead of $\operatorname{Eis}^k(\beta)$. If we consider $e^*\operatorname{pol}^D_\beta$ as a power series in the $\mathbf{e}_i$ we are interested in the coefficient of $\frac{N\mathbf{e}^{\otimes k}}{k!^g}$. In fact, the integrallity properties of $\operatorname{Eis}^{gk}(\beta)$ are better reflected if we write it in terms of a basis $a_1,\ldots,a_g$ of $\mathfrak{a}$. Then $N\mathbf{e}^{\otimes k} = N\mathfrak{a}^{-k}N\mathbf{a}^{\otimes k}$, where $\mathbf{a}_1,\ldots,\mathbf{a}_g$ denote again the images of $a_1,\ldots,a_g$ in $\hat{\mathcal{U}}(\mathfrak{a})$. We get:

Corollary 3.4.1. *With the above basis $\mathbf{a}_1,\ldots,\mathbf{a}_g$, The integral over the Eisenstein class is given by*

$$u_{\mathrm{id},*}\operatorname{Eis}^{gk}(\beta) = \frac{(-1)^{g-1}(k!)^g}{(2\pi i)^{g(k+1)}|d_F|^{1/2}N\mathfrak{a}^{k+1}}\sum_{\rho\in\mathcal{O}^*_{(n)}\backslash(\mathfrak{a}^\vee\backslash 0)}\frac{e^{-2\pi i\rho(\beta)}}{N(\rho)^{k+1}}\frac{N\mathbf{a}^{\otimes k}}{k!^g}.$$

3.5 5. Step: End of the proof

To finish the proof of the theorem 1.7.1, let $\alpha\in L[\mathcal{A}[n]]^0$ and suppose we want to compute $\operatorname{res}(\operatorname{Eis}^k(\alpha))(h)$. Using the equivariance of $\operatorname{res}\circ\operatorname{Eis}^k$ from (19), this amounts to compute $\operatorname{res}(\operatorname{Eis}^k(h\alpha))(\mathrm{id})$. Theorem 1.5.1 shows that

$$\operatorname{res}(\operatorname{Eis}^k(h\alpha))(\mathrm{id}) = u_{\mathrm{id},*}q_*\operatorname{Eis}^k(h\alpha),$$

where $q: S^1_B\to S^1_T$ and $u_{\mathrm{id}}:\Gamma_T\backslash\big(T^1(\mathbb{R})/K^{T^1}_\infty\big)\to pt$ is the structure map of the component corresponding to $\mathrm{id}\in T^1(\mathbb{Z}/n\mathbb{Z})$. From theorem 3.2.3 we get

$$q_*\operatorname{Eis}^k(h\alpha) = \operatorname{Eis}^k(p(h\alpha)).$$

Using corollary 3.4.1 for $\mathfrak{a}=\mathcal{O}$ and the formula 1.6.2 for $\mathfrak{b}=\mathfrak{f}=\mathcal{O}$ we get

(42)
$$\frac{(-1)^{g-1}(k!)^g}{(2\pi i)^{gk+g}|d_F|^{1/2}}\sum_{\rho\in\mathcal{O}^*_{(n)}\backslash(\mathcal{O}^\vee\backslash 0)}\frac{e^{-2\pi i\rho(p(h\alpha))}}{N(\rho)^{k+1}} = (-1)^{g-1}\sum_{\sigma\in D}l_\sigma\zeta(\mathcal{O},\mathcal{O},p(h\sigma),-k),$$

which is the formula in the main theorem 1.7.1. To prove the corollary, we use that the map of real tori

$$\mathcal{A}(\mathbb{C}) \xrightarrow{p} \mathcal{T}_{\mathcal{M}}$$

factors through $\phi : \mathcal{T}_{\mathfrak{b}_{\widetilde{h}}} \to \mathcal{T}_{\mathcal{M}}$, where ϕ is induced by the inclusion $\mathfrak{b}_{\widetilde{h}} \subset \mathcal{O}$. Using corollary 3.4.1 for $\mathfrak{a} = \mathfrak{b}_{\widetilde{h}}$, we get the desired formula

$$\operatorname{res}(\mathrm{Eis}^{gk}(\alpha))(h) = (-1)^{g-1} N\mathfrak{b}_{\widetilde{h}}^{-k-1} \sum_{\sigma \in D} l_\sigma \zeta(\mathfrak{b}_{\widetilde{h}}, \mathcal{O}, p_{\widetilde{h}}(\sigma), -k),$$

which ends the proof.

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C^*-Algebras Associated to Coverings of k-Graphs

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Abstract. A covering of k-graphs (in the sense of Pask-Quigg-Raeburn) induces an embedding of universal C^*-algebras. We show how to build a $(k+1)$-graph whose universal algebra encodes this embedding. More generally we show how to realise a direct limit of k-graph algebras under embeddings induced from coverings as the universal algebra of a $(k+1)$-graph. Our main focus is on computing the K-theory of the $(k+1)$-graph algebra from that of the component k-graph algebras.

Examples of our construction include a realisation of the Kirchberg algebra $\mathcal{P}_n$ whose K-theory is opposite to that of $\mathcal{O}_n$, and a class of $A\mathbf{T}$-algebras that can naturally be regarded as higher-rank Bunce-Deddens algebras.

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1. Introduction

A directed graph E consists of a countable collection E^0 of vertices, a countable collection E^1 of edges, and maps $r, s : E^1 \to E^0$ which give the edges their direction; the edge e points from $s(e)$ to $r(e)$. Following the convention established in [30], the associated graph algebra $C^*(E)$ is the universal C^*-algebra generated by partial isometries $\{s_e : e \in E^1\}$ together with mutually orthogonal projections $\{p_v : v \in E^0\}$ such that $p_{s(e)} = s_e^* s_e$ for all $e \in E^1$, and $p_v \geq \sum_{e \in F} s_e s_e^*$ for all $v \in E^0$ and finite $F \subset r^{-1}(v)$, with equality when $F = r^{-1}(v)$ is finite and nonempty.

Graph algebras, introduced in [13, 23], have been studied intensively in recent years because much of the structure of $C^*(E)$ can be deduced from elementary

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features of E. In particular, graph C^*-algebras are an excellent class of models for Kirchberg algebras, because it is easy to tell from the graph E whether $C^*(E)$ is simple and purely infinite [22]. Indeed, a Kirchberg algebra can be realised up to Morita equivalence as a graph C^*-algebra if and only if its K_1-group is torsion-free [39]. It is also true that every AF algebra can be realised up to Morita equivalence as a graph algebra; the desired graph is a Bratteli diagram for the AF algebra in question (see [11] or [40]). However, this is the full extent to which graph algebras model simple classifiable C^*-algebras due to the following dichotomy: if E is a directed graph and $C^*(E)$ is simple, then $C^*(E)$ is either AF or purely infinite (see [22, Corollary 3.10], [2, Remark 5.6]).

Higher-rank graphs, or k-graphs, and their C^*-algebras were originally developed by the first two authors [20] to provide a graphical framework for the higher-rank Cuntz-Krieger algebras of Robertson and Steger [35]. A k-graph Λ is a kind of k-dimensional graph, which one can think of as consisting of vertices Λ^0 together with k collections of edges $\Lambda^{e_1}, \ldots, \Lambda^{e_k}$ which we think of as lying in k different dimensions. As an aid to visualisation, we often distinguish the different types of edges using k different colours.

Higher-rank graphs and their C^*-algebras are generalisations of directed graphs and their algebras. Given a directed graph E, its path category E^* is a 1-graph, and the 1-graph C^*-algebra $C^*(E^*)$ as defined in [20] is canonically isomorphic to the graph algebra $C^*(E)$ as defined in [23]. Furthermore, every 1-graph arises this way, so the class of graph algebras and the class of 1-graph algebras are one and the same. For $k \geq 2$, there are many k-graph algebras which do not arise as graph algebras. For example, the original work of Robertson and Steger on higher-rank Cuntz-Krieger algebras describes numerous 2-graphs Λ for which $C^*(\Lambda)$ is a Kirchberg algebra and $K_1(C^*(\Lambda))$ contains torsion.

Recent work of Pask, Raeburn, Rørdam and Sims has shown that one can also realise a substantial class of A$\mathbf{T}$-algebras as 2-graph algebras, and that one can tell from the 2-graph whether or not the resulting C^*-algebra is simple and has real-rank zero [27]. The basic idea of the construction in [27] is as follows. One takes a Bratteli diagram in which the edges are coloured red, and replaces each vertex with a blue simple cycle (there are technical restrictions on the relationship between the lengths of the blue cycles and the distribution of the red edges joining them, but this is the gist of the construction). The resulting 2-graph is called a *rank-2 Bratteli diagram*. The associated C^*-algebra is A$\mathbf{T}$ because the C^*-algebra of a simple cycle of length n is isomorphic to $M_n(C(\mathbf{T}))$ [17]. The results of [27] show how to read off from a rank-2 Bratteli diagram the K-theory, simplicity or otherwise, and real-rank of the resulting A$\mathbf{T}$ algebra.

The construction explored in the current paper is motivated by the following example of a rank-2 Bratteli diagram. For each $n \in \mathbf{N}$, let L_{2^n} be the simple directed loop graph with 2^n vertices labelled $0, \ldots, 2^n - 1$ and 2^n edges $f_0, \ldots, f_{2^n-1}$, where f_i is directed from the vertex labelled $i + 1 \pmod{2^n}$ to the vertex labelled i. We specify a rank-2 Bratteli diagram $\Lambda(2^\infty)$ as follows. The n^{th} level of $\Lambda(2^\infty)$ consists of a single blue copy of $L_{2^{n-1}}$ $(n = 1, 2, \cdots)$. For $0 \leq i \leq 2^n - 1$, there is a single red edge from the vertex labelled i at the

$(n+1)^{\text{st}}$ level to the vertex labelled $i \pmod{2^n}$ at the n^{th} level. The C^*-algebra of the resulting 2-graph is Morita equivalent to the Bunce-Deddens algebra of type 2^∞, and this was one of the first examples of a 2-graph algebra which is simple but neither purely infinite nor AF (see [27, Example 6.7]).

The purpose of this paper is to explore the observation that the growing blue cycles in $\Lambda(2^\infty)$ can be thought of as a tower of *coverings* of 1-graphs (roughly speaking, a covering is a locally bijective surjection — see Definition 2.1), where the red edges connecting levels indicate the covering maps.
In Section 2, we describe how to construct $(k+1)$-graphs from coverings. In its simplest form, our construction takes k-graphs Λ and Γ and a covering map $p : \Gamma \to \Lambda$, and produces a $(k+1)$-graph $\Lambda \overset{p}{\leftarrow} \Gamma$ in which each edge in the $(k+1)^{\text{st}}$ dimension points from a vertex v of Γ to the vertex $p(v)$ of Λ which it covers[†]. Building on this construction, we show how to take an infinite tower of coverings $p_n : \Lambda_{n+1} \to \Lambda_n$, $n = 1, 2, \ldots$ and construct from it an infinite $(k+1)$-graph $\varprojlim(\Lambda_n, p_n)$ with a natural inductive structure (Corollary 2.11). The next step, achieved in Section 3, is to determine how the universal C^*-algebra of $\Lambda \overset{p}{\leftarrow} \Gamma$ relates to those of Λ and Γ. We show that $C^*(\Lambda \overset{p}{\leftarrow} \Gamma)$ is Morita equivalent to $C^*(\Gamma)$ and contains an isomorphic copy of $C^*(\Lambda)$ (Proposition 3.2). We then show that given a system of coverings $p_n : \Lambda_{n+1} \to \Lambda_n$, the C^*-algebra $C^*(\varprojlim(\Lambda_n, p_n))$ is Morita equivalent to a direct limit of the $C^*(\Lambda_n)$ (Theorem 3.8).
In Section 4, we use results of [34] to characterise simplicity of $C^*(\varprojlim(\Lambda_n, p_n))$, and we also give a sufficient condition for this C^*-algebra to be purely infinite.
In Section 5, we show how various existing methods of computing the K-theory of the $C^*(\Lambda_n)$ can be used to compute the K-theory of $C^*(\varprojlim(\Lambda_n, p_n))$. Our results boil down to checking that each of the existing K-theory computations for the $C^*(\Lambda_n)$ is natural in the appropriate sense. Given that K-theory for higher-rank graph C^*-algebras has proven quite difficult to compute in general (see [14]), our K-theory computations are an important outcome of the paper.
We conclude in Section 6 by exploring some detailed examples which illustrate the covering-system construction, and show how to apply our K-theory calculations to the resulting higher-rank graph C^*-algebras. For integers $3 \le n < \infty$, we obtain a 3-graph algebra realisation of Kirchberg algebra $\mathcal{P}_n$ whose K-theory is opposite to that of $\mathcal{O}_n$ (see Section 6.3). We also obtain, using 3-graphs, a class of simple A$\mathbf{T}$-algebras with real-rank zero which cannot be obtained from the rank-2 Bratteli diagram construction of [27] (see Section 6.4), and which we can describe in a natural fashion as higher-rank analogues of the Bunce-Deddens algebras. These are, to our knowledge, the first explicit computations of K-theory for infinite classes of 3-graph algebras.

[†]In its full generality, our construction is more complicated (see Proposition 2.14), enabling us to recover the important example of the irrational rotation algebras discussed in [27]. To keep technical detail in this introduction to a minimum, we discuss only the basic construction here.

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2. Covering systems of k-graphs

For k-graphs we adopt the conventions of [20, 25, 31]; briefly, a *k-graph* is a countable small category Λ equipped with a functor $d : \Lambda \to \mathbf{N}^k$ satisfying the *factorisation property*: for all $\lambda \in \Lambda$ and $m, n \in \mathbf{N}^k$ such that $d(\lambda) = m + n$ there exist unique $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu\nu$. When $d(\lambda) = n$ we say λ has *degree* n. By abuse of notation, we will use d to denote the degree functor in every k-graph in this paper; the domain of d is always clear from context.

The standard generators of $\mathbf{N}^k$ are denoted $e_1, \ldots, e_k$, and for $n \in \mathbf{N}^k$ and $1 \leq i \leq k$ we write n_i for the i^{th} coordinate of n.

If Λ is a k-graph, the *vertices* are the morphisms of degree 0. The factorisation property implies that these are precisely the identity morphisms, and so can be identified with the objects. For $\alpha \in \Lambda$, the *source* $s(\alpha)$ is the domain of α, and the *range* $r(\alpha)$ is the codomain of α (strictly speaking, $s(\alpha)$ and $r(\alpha)$ are the identity morphisms associated to the domain and codomain of α).

For $n \in \mathbf{N}^k$, we write Λ^n for $d^{-1}(n)$. In particular, Λ^0 is the vertex set. For $u, v \in \Lambda^0$ and $E \subset \Lambda$, we write $uE := E \cap r^{-1}(u)$ and $Ev := E \cap s^{-1}(v)$. For $n \in \mathbf{N}^k$, we write

$$\Lambda^{\leq n} := \{\lambda \in \Lambda : d(\lambda) \leq n, s(\lambda)\Lambda^{e_i} = \emptyset \text{ whenever } d(\lambda) + e_i \leq n\}.$$

We say that Λ is *connected* if the equivalence relation on Λ^0 generated by $\{(v, w) \in \Lambda^0 \times \Lambda^0 : v\Lambda w \neq \emptyset\}$ is the whole of $\Lambda^0 \times \Lambda^0$. A *morphism* between k-graphs is a degree-preserving functor.

We say that Λ is *row-finite* if $v\Lambda^n$ is finite for all $v \in \Lambda^0$ and $n \in \mathbf{N}^k$. We say that Λ is *locally convex* if whenever $1 \leq i < j \leq k$, $e \in \Lambda^{e_i}$, $f \in \Lambda^{e_j}$ and $r(e) = r(f)$, we can extend both e and f to paths ee' and ff' in $\Lambda^{e_i+e_j}$.

We next introduce the notion of a covering of one k-graph by another. For a more detailed treatment of coverings of k-graphs, see [25].

Definition 2.1. A *covering* of a k-graph Λ is a surjective k-graph morphism $p : \Gamma \to \Lambda$ such that for all $v \in \Gamma^0$, p maps Γv 1-1 onto $\Lambda p(v)$ and $v\Gamma$ 1-1 onto $p(v)\Lambda$. A covering $p : \Gamma \to \Lambda$ is *connected* if Γ, and hence also Λ, is connected. A covering $p : \Gamma \to \Lambda$ is *finite* if $p^{-1}(v)$ is finite for all $v \in \Lambda^0$.

Remarks 2.2. (1) A covering $p : \Gamma \to \Lambda$ has the unique path lifting property: for every $\lambda \in \Lambda$ and $v \in \Gamma^0$ with $p(v) = s(\lambda)$ there exists a unique γ such that $p(\gamma) = \lambda$ and $s(\gamma) = v$; likewise, if $p(v) = r(\lambda)$ there is a unique ζ such that $p(\zeta) = \lambda$ and $r(\zeta) = v$.

(2) If Λ is connected then surjectivity of p is implied by the unique path-lifting property.

(3) If there is a fixed integer n such that $|p^{-1}(v)| = n$ for all $v \in \Lambda^0$, p is said to be an *n-fold covering*. If Γ is connected, then p is automatically an n-fold covering for some n.

Notation 2.3. For $m \in \mathbf{N} \setminus \{0\}$, we write S_m for the group of permutations of the set $\{1, \dots, m\}$. We denote both composition of permutations in S_m, and the action of a permutation in S_m on an element of $\{1, \dots, m\}$ by juxtaposition; so for $\phi, \psi \in S_m$, $\phi\psi \in S_m$ is the permutation $\phi \circ \psi$, and for $\phi \in S_m$ and $j \in \{1, \dots, m\}$, $\phi j \in \{1, \dots, m\}$ is the image of j under ϕ. When convenient, we regard S_m as (the morphisms of) a category with a single object.

Definition 2.4. Fix $k, m \in \mathbf{N} \setminus \{0\}$, and let Λ be a k-graph. A *cocycle* $\mathfrak{s} : \Lambda \to S_m$ is a functor $\lambda \mapsto \mathfrak{s}(\lambda)$ from the category Λ to the category S_m. That is, whenever $\alpha, \beta \in \Lambda$ satisfy $s(\alpha) = r(\beta)$ we have $\mathfrak{s}(\alpha)\mathfrak{s}(\beta) = \mathfrak{s}(\alpha\beta)$.

We are now ready to describe the data needed for our construction.

Definition 2.5. A *covering system of k-graphs* is a quintuple $(\Lambda, \Gamma, p, m, \mathfrak{s})$ where Λ and Γ are k-graphs, $p : \Lambda \to \Gamma$ is a covering, m is a nonzero positive integer, and $\mathfrak{s} : \Gamma \to S_m$ is a cocycle. We say that the covering system is *row finite* if the covering map p is finite and both Λ and Γ are row finite. When $m = 1$ and $\mathfrak{s}$ is the identity cocycle, we drop references to m and $\mathfrak{s}$ altogether, and say that (Λ, Γ, p) is a covering system of k-graphs.

Given a covering system $(\Lambda, \Gamma, p, m, \mathfrak{s})$ of k-graphs, we will define a $(k+1)$-graph $\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ which encodes the covering map. Before the formal statement of this construction, we give an intuitive description of $\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$. The idea is that $\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ is a $(k+1)$-graph containing disjoint copies $\imath(\Lambda)$ and $\jmath(\Gamma)$ of the k-graphs Λ and Γ in the first k dimensions. The image $\jmath(v)$ of a vertex $v \in \Gamma$ is connected to the image $\imath(p(v))$ of the vertex it covers in Λ by m parallel edges $e(v, 1), \dots, e(v, m)$ of degree e_{k+1}. Factorisations of paths in $\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ involving edges $e(v, l)$ of degree e_{k+1} are determined by the unique path-lifting property and the cocycle $\mathfrak{s}$.
It may be helpful on the first reading to consider the case where $m = 1$ so that $\mathfrak{s}$ is necessarily trivial. To state the result formally, we first establish some notation.

Notation 2.6. Fix $k > 0$. For $n \in \mathbf{N}^k$ we denote by $(n, 0_1)$ the element $\sum_{i=1}^{k} n_i e_i \in \mathbf{N}^{k+1}$ and for $m \in \mathbf{N}$, we denote by $(0_k, m)$ the element $m e_{k+1} \in \mathbf{N}^{k+1}$. We write $(\mathbf{N}^k, 0_1)$ for $\{(n, 0_1) : n \in \mathbf{N}^k\}$ and $(0_k, \mathbf{N})$ for $\{(0_k, m) : m \in \mathbf{N}\}$.
Given a $(k+1)$-graph Ξ, we write $\Xi^{(0_k, \mathbf{N})}$ for $\{\xi \in \Xi : d(\xi) \in (0_k, \mathbf{N})\}$, and we write $\Xi^{(\mathbf{N}^k, 0_1)}$ for $\{\xi \in \Xi : d(\xi) \in (\mathbf{N}^k, 0_1)\}$. When convenient, we regard $\Xi^{(0_k, \mathbf{N})}$ as a 1-graph and $\Xi^{(\mathbf{N}^k, 0_1)}$ as a k-graph, ignoring the distinctions between $\mathbf{N}$ and $(0_k, \mathbf{N})$ and between $\mathbf{N}^k$ and $(\mathbf{N}^k, 0_1)$.

Proposition 2.7. *Let $(\Lambda, \Gamma, p, m, \mathfrak{s})$ be a covering system of k-graphs. There is a unique $(k+1)$-graph $\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ such that:*

(1) *there are injective functors* $\imath : \Lambda \to \Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ *and* $\jmath : \Gamma \to \Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ *such that* $d(\imath(\alpha)) = (d(\alpha), 0_1)$ *and* $d(\jmath(\beta)) = (d(\beta), 0_1)$ *for all* $\alpha \in \Lambda$ *and* $\beta \in \Gamma$*;*
(2) $\imath(\Lambda) \cap \jmath(\Gamma) = \emptyset$ *and* $\imath(\Lambda) \cup \jmath(\Gamma) = \{\tau \in \Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma : d(\tau)_{k+1} = 0\}$*;*
(3) *there is a bijection* $e : \Gamma^0 \times \{1, \dots, m\} \to (\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)^{e_{k+1}}$*;*
(4) $s(e(v,l)) = \jmath(v)$ *and* $r(e(v,l)) = \imath(p(v))$ *for all* $v \in \Gamma^0$ *and* $1 \le l \le m$*; and*
(5) $e(r(\lambda), l)\jmath(\lambda) = \imath(p(\lambda))e(s(\lambda), \mathfrak{s}(\lambda)^{-1}l)$ *for all* $\lambda \in \Gamma$ *and* $1 \le l \le m$*.*

If the covering system $(\Lambda, \Gamma, p, m, \mathfrak{s})$ *is row finite, then* $\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ *is row finite. Moreover,* Λ *is locally convex if and only if* Γ *is locally convex, and in this case* $\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ *is also locally convex.*

Notation 2.8. If $m = 1$ so that $\mathfrak{s}$ is necessarily trivial, we drop all reference to $\mathfrak{s}$. We denote $\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ by $\Lambda \overset{p}{\leftarrow} \Gamma$, and write $(\Lambda \overset{p}{\leftarrow} \Gamma)^{e_{k+1}} = \{e(v) : v \in \Gamma^0\}$. In this case, the factorisation property is determined by the unique path-lifting property alone.

The main ingredient in the proof of Proposition 2.7 is the following fact from [15, Remark 2.3] (see also [31, Section 2]).

Lemma 2.9. *Let* $E_1, \dots, E_k$ *be* 1*-graphs with the same vertex set* E^0*. For distinct* $i, j \in \{1, \dots, k\}$*, let* $E_{i,j} := \{(e, f) \in E_i^1 \times E_j^1 : s(e) = r(f)\}$*, and write* $r\big((e,f)\big) = r(e)$ *and* $s\big((e,f)\big) = s(f)$*. For distinct* $h, i, j \in \{1, \dots, k\}$*, let* $E_{h,i,j} := \{(e, f, g) \in E_h^1 \times E_i^1 \times E_j^1 : (e, f) \in E_{h,i}, (f, g) \in E_{i,j}\}$*.*
Suppose we have bijections $\theta_{i,j} : E_{i,j} \to E_{j,i}$ *such that* $r \circ \theta_{i,j} = r$*,* $s \circ \theta_{i,j} = s$ *and* $\theta_{i,j} \circ \theta_{j,i} = \mathrm{id}$*, and such that*

$$(\theta_{i,j} \times \mathrm{id})(\mathrm{id} \times \theta_{h,j})(\theta_{h,i} \times \mathrm{id}) = (\mathrm{id} \times \theta_{h,i})(\theta_{h,j} \times \mathrm{id})(\mathrm{id} \times \theta_{i,j}) \tag{2.1}$$

as bijections from $E_{h,i,j}$ *to* $E_{j,i,h}$*.*
Then there is a unique k*-graph* Λ *such that* $\Lambda^0 = E^0$*,* $\Lambda^{e_i} = E_i^1$ *for* $1 \le i \le k$*, and for distinct* $i, j \in \{1, \dots, k\}$ *and* $(e, f) \in E_{i,j}$*, the pair* $(f', e') \in E_{j,i}$ *such that* $(f', e') = \theta_{i,j}(e, f)$ *satisfies* $ef = f'e'$ *as morphisms in* Λ*.*

Remark 2.10. Every k-graph arises in this way: Given a k-graph Λ, let $E^0 := \Lambda^0$, and $E_i^1 := \Lambda^{e_i}$ for $1 \le i \le k$, and define $r, s : E_i^1 \to E^0$ by restriction of the range and source maps in Λ. Define bijections $\theta_{i,j} : E_{i,j} \to E_{j,i}$ via the factorisation property: $\theta_{i,j}(e, f)$ is equal to the unique pair $(f', e') \in E_{j,i}$ such that $ef = f'e'$ in Λ. Then condition (2.1) holds by the associativity of the category Λ, and the uniqueness assertion of Lemma 2.9 implies that Λ is isomorphic to the k-graph obtained from the E_i and the $\theta_{i,j}$ using Lemma 2.9.

Lemma 2.9 tells us how to describe a k-graph pictorially. As in [31, 27], the *skeleton* of a k-graph Λ is the directed graph E_Λ with vertices $E_\Lambda^0 = \Lambda^0$, edges $E_\Lambda^1 = \bigcup_{i=1}^k \Lambda^{e_i}$, range and source maps inherited from Λ, and edges of different degrees in Λ distinguished using k different colours in E_Λ: in this paper, we will often refer to edges of degree e_1 as "blue" and edges of degree e_2 as "red." Lemma 2.9 implies that the skeleton E_Λ together with the factorisation rules $fg = g'f'$ where $f, f' \in \Lambda^{e_i}$ and $g, g' \in \Lambda^{e_j}$ completely specify Λ. In practise, we draw E_Λ using solid, dashed and dotted edges to distinguish the different colours, and list the factorisation rules separately.

Proof of Proposition 2.7. The idea is to apply Lemma 2.9 to obtain the $(k+1)$-graph $\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$. We first define sets E^0 and E_i^1 for $1 \le i \le k+1$. As a set, E^0 is a copy of the disjoint union $\Lambda^0 \sqcup \Gamma^0$. We denote the copy of Λ^0 in E^0 by $\{\imath(v) : v \in \Lambda^0\}$ and the copy of Γ^0 in E^0 by $\{\jmath(w) : w \in \Gamma^0\}$ where as yet the $\imath(v)$ and $\jmath(w)$ are purely formal symbols. So

$$E^0 = \{\imath(v) : v \in \Lambda^0\} \sqcup \{\jmath(w) : w \in \Gamma^0\}.$$

For $1 \le i \le k$, we define, in a similar fashion,

$$E_i^1 := \{\imath(f) : f \in \Lambda^{e_i}\} \sqcup \{\jmath(g) : g \in \Gamma^{e_i}\}$$

to be a copy of the disjoint union $\Lambda^{e_i} \sqcup \Gamma^{e_i}$. We define E_{k+1}^1 to be a copy of $\Gamma^0 \times \{1, \dots, m\}$ which is disjoint from E^0 and each of the other E_i^1, and use formal symbols $\{e(v,l) : v \in \Gamma^0, 1 \le l \le m\}$ to denote its elements. For $1 \le i \le k$, define range and source maps $r, s : E_i^1 \to E^0$ by $r(\imath(f)) := \imath(r(f))$, $s(\imath(f)) := \imath(s(f))$, $r(\jmath(g)) := \jmath(r(g))$ and $s(\jmath(g)) := \jmath(s(g))$. Define $r, s : E_{k+1}^1 \to E^0$ as in Proposition 2.7(4).

For distinct $i, j \in \{1, \dots, k+1\}$, define $E_{i,j}$ as in Lemma 2.9. Define bijections $\theta_{i,j} : E_{i,j} \to E_{j,i}$ as follows:

- For $1 \le i, j \le k$ and $(e, f) \in E_{i,j}$, we must have either $e = \imath(a)$ and $f = \imath(b)$ for some composable pair $(a, b) \in \Lambda^{e_i} \times_{\Lambda^0} \Lambda^{e_j}$, or else $e = \jmath(a)$ and $f = \jmath(b)$ for some composable pair $(a, b) \in \Gamma^{e_i} \times_{\Gamma^0} 3\Gamma^{e_j}$. If $e = \imath(a)$ and $f = \imath(b)$, the factorisation property in Λ yields a unique pair $b' \in \Lambda^{e_j}$, $a' \in \Lambda^{e_i}$ such that $ab = b'a'$, and we then define $\theta_{i,j}(e, f) = (\imath(b'), \imath(a'))$. If $e = \jmath(a)$ and $f = \jmath(b)$, we define $\theta_{i,j}(e, f)$ similarly using the factorisation property in Γ.
- For $1 \le i \le k$, and $(e, f) \in E_{k+1,i}$, we have $f = \jmath(b)$ and $e = e(r(b), l)$ for some $b \in \Gamma^{e_i}$ and $1 \le l \le m$. Define $\theta_{k+1,i}(e, f) := (\imath(p(b)), e(s(f), \mathfrak{s}(f)^{-1}l))$.
- For $1 \le i \le k$, to define $\theta_{i,k+1}$, first note that if $(f', e') = \theta_{k+1,i}(e, f)$, then $e' = e(w, l)$ for some $w \in \Gamma^0$ and $l \in \{1, \dots, m\}$ such that $p(w) = s(f')$, f is the unique lift of f' such that $s(f) = \jmath(w)$, and $e = e(r(f), \mathfrak{s}(f)l)$. It follows that $\theta_{k+1,i}$ is a bijection and we may define $\theta_{i,k+1} := \theta_{k+1,i}^{-1}$.

Since Λ and Γ are k-graphs, the maps $\theta_{i,j}$, $1 \le i, j \le k$ are bijections with $\theta_{j,i} = \theta_{i,j}^{-1}$, and we have $\theta_{i,k+1} = \theta_{k+1,i}^{-1}$ by definition, so to invoke Lemma 2.9, we just need to establish equation (2.1).

Equation (2.1) holds when $h, i, j \le k$ because Λ and Γ are both k-graphs. Suppose one of $h, i, j = k+1$. Fix edges $f_h \in E_h^1$, $f_i \in E_i^1$ and $f_j \in E_j^1$. First suppose that $h = k+1$; so $f_h = e(r(f_i), l)$ for some l, and f_i and f_j both belong to $\jmath(\Gamma)$. Apply the factorisation property for Γ to obtain f_j' and f_i' such that $f_i' \in E_i^1$, $f_j' \in E_j^1$ and $f_j' f_i' = f_i f_j$. We then have $\theta_{i,j}(f_i, f_j) = (f_j', f_i')$. If we write $\tilde{p}$ for the map from $\{\jmath(f) : f \in \bigcup_{i=1}^k \Gamma^{e_i}\}$ to $\{\imath(f) : f \in \bigcup_{i=1}^k \Lambda^{e_i}\}$

given by $\tilde{p}(\jmath(\lambda)) := \imath(p(\lambda))$, then the properties of the covering map imply that $\theta_{i,j}(\tilde{p}(f_i), \tilde{p}(f_j)) = (\tilde{p}(f_j'), \tilde{p}(f_i)')$. Now

$$\begin{aligned}
(\theta_{i,j} \times \mathrm{id})&(\mathrm{id} \times \theta_{h,j})(\theta_{h,i} \times \mathrm{id})(f_h, f_i, f_j) \\
&= (\theta_{i,j} \times \mathrm{id})(\mathrm{id} \times \theta_{h,j})(\tilde{p}(f_i), e(s(f_i), \mathfrak{s}(f_i)^{-1} l), f_j) \\
&= (\theta_{i,j} \times \mathrm{id})(\tilde{p}(f_i), \tilde{p}(f_j), e(s(f_j), \mathfrak{s}(f_j)^{-1}(\mathfrak{s}(f_i)^{-1})l)) \\
&= (\tilde{p}(f_j'), \tilde{p}(f_i'), e(s(f_j), \mathfrak{s}(f_i f_j)^{-1} l), \qquad (2.2)
\end{aligned}$$

where, in the last equality, $\mathfrak{s}(f_j)^{-1}\mathfrak{s}(f_i)^{-1} = \mathfrak{s}(f_i f_j)^{-1}$ by the cocycle property. On the other hand,

$$\begin{aligned}
(\mathrm{id} \times \theta_{h,i})&(\theta_{h,j} \times \mathrm{id})(\mathrm{id} \times \theta_{i,j})(f_h, f_i, f_j) \\
&= (\mathrm{id} \times \theta_{h,i})(\theta_{h,j} \times \mathrm{id})(f_h, f_j', f_i') \\
&= (\mathrm{id} \times \theta_{h,i})(\tilde{p}(f_j'), e(s(f_j), \mathfrak{s}(f_j')^{-1} l), f_i') \\
&= \big(\tilde{p}(f_j'), \tilde{p}(f_i'), e(s(f_i), \mathfrak{s}(f_i')^{-1}(\mathfrak{s}(f_j')^{-1} l))\big) \\
&= \big(\tilde{p}(f_j'), \tilde{p}(f_i'), e(s(f_i), \mathfrak{s}(f_j' f_i')^{-1} l)\big).
\end{aligned}$$

Since $f_j' f_i' = f_i f_j$, this establishes (2.1) when $h = k+1$ and $1 \le i, j \le k$. Similar calculations establish (2.1) when $i = k+1$ and when $j = k+1$.

By Lemma 2.9, there is a unique $(k+1)$-graph $\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ with $(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)^0 = E^0$, $(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)^{e_i} = E_i^1$ for all i and with commuting squares determined by the $\theta_{i,j}$. Since the $\theta_{i,j}$, $1 \le i, j \le k$ agree with the factorisation properties in Γ and Λ, the uniqueness assertion of Lemma 2.9 applied to paths consisting of edges in $E_1^1 \cup \dots \cup E_k^1$ shows that $\imath$ and $\jmath$ extend uniquely to injective functors from Λ and Γ to $(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)^{(\mathbf{N}^k, 0_1)}$ which satisfy Proposition 2.7(2). Assertions (3) and (4) of Proposition 2.7 follow from the definition of E_{k+1}^1, and the last assertion (5) is established by factorising λ into edges from the E_i^1, $1 \le i \le k$ and then performing calculations like (2.2).

Now suppose that p is finite. Then Γ is row-finite if and only if Λ is, and in this case, $\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ is also row-finite because p is locally bijective and $m < \infty$. That p is locally bijective shows that Λ is locally convex if and only if Γ is. Suppose that Γ is locally convex. Fix $1 \le i < j \le k+1$, $a \in (\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)^{e_i}$ and $b \in (\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)^{e_j}$ with $r(a) = r(b)$. If $j < k+1$ then a and b can be extended to paths of degree $e_i + e_j$ because Λ and Γ are locally convex. If $j = k+1$, then $b = e(v, l)$ for some $v \in \Gamma^0$ and $1 \le l \le m$. Let a' be the lift of a such that $r(a') = s(v)$, then $ae(s(a'), l)$ and ba' extend a and b to paths of degree $e_i + e_j$. It follows that $\Lambda \overset{p}{\leftarrow} \Gamma$ is locally convex. □

Corollary 2.11. *Fix $N \ge 2$ in $\mathbf{N} \cup \{\infty\}$. Let $(\Lambda_n, \Lambda_{n+1}, p_n, m_n, \mathfrak{s}_n)_{n=1}^{N-1}$ be a sequence of covering systems of k-graphs. Then there is a unique $(k+1)$-graph $\mathbf{\Lambda}$ such that $\mathbf{\Lambda}^{e_i} = \bigsqcup_{n=1}^N \Lambda_n^{e_i}$ for $1 \le i \le k$, $\mathbf{\Lambda}^{e_{k+1}} = \bigsqcup_{n=1}^{N-1} (\Lambda_n \overset{p_n,\mathfrak{s}_n}{\leftarrow} \Lambda_{n+1})^{e_{k+1}}$, and such that range, source and composition are all inherited from the $\Lambda_n \overset{p_n,\mathfrak{s}_n}{\leftarrow} \Lambda_{n+1}$. If each $(\Lambda_n, \Lambda_{n+1}, p_n, m_n, \mathfrak{s}_n)$ is row-finite then $\mathbf{\Lambda}$ is row-finite. If each Λ_n is locally convex, so is $\mathbf{\Lambda}$, and if each Λ_n is connected, so is $\mathbf{\Lambda}$.*

Proof. For the first part we just apply Lemma 2.9; the hypotheses follow automatically from the observation that if h, i, j are distinct elements of $\{1, \ldots, k+1\}$ then each path of degree $e_h + e_i + e_j$ lies in some $\Lambda_n \overset{p_n, \mathfrak{s}_n}{\leftharpoondown} \Lambda_{n+1}$, and these are all $(k+1)$-graphs by Proposition 2.7.
The arguments for row-finiteness, local convexity and connectedness are the same as those in Proposition 2.7. □

Notation 2.12. When N is finite, the $(k+1)$-graph $\mathbf{\Lambda}$ of the previous lemma will henceforth be denoted $\Lambda_1 \overset{p_1, \mathfrak{s}_1}{\leftharpoondown} \cdots \overset{p_{N-1}, \mathfrak{s}_{N-1}}{\leftharpoondown} \Lambda_N$. If $N = \infty$, we instead denote $\mathbf{\Lambda}$ by $\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)$.

2.1. Matrices of covering systems. In this subsection, we generalise our construction to allow for a different covering system $(\Lambda_j, \Gamma_i, p_{i,j}, m_{i,j}, \mathfrak{s}_{i,j})$ for each pair of connected components $\Lambda_j \subset \Lambda$ and $\Gamma_i \subset \Gamma$. The objective is to recover the example of the irrational rotation algebras [27, Example 6.5].

Definition 2.13. Fix nonnegative integers $c_\Lambda, c_\Gamma \in \mathbf{N} \setminus \{0\}$. A *matrix of covering systems* $(\Lambda_j, \Gamma_i, m_{i,j}, p_{i,j}, \mathfrak{s}_{i,j})_{i,j=1}^{c_\Gamma, c_\Lambda}$ consists of:

(1) k-graphs Λ and Γ which decompose into connected components $\Lambda = \bigsqcup_{j=1,\ldots,c_\Lambda} \Lambda_j$ and $\Gamma = \bigsqcup_{i=1,\ldots,c_\Gamma} \Gamma_i$;
(2) a matrix $(m_{i,j})_{i,j=1}^{c_\Gamma, c_\Lambda} \in M_{c_\Gamma, c_\Lambda}(\mathbf{N})$ with no zero rows or columns; and
(3) for each i, j such that $m_{i,j} \neq 0$, a covering system $(\Lambda_i, \Gamma_j, p_{i,j}, m_{i,j}, \mathfrak{s}_{i,j})$ of k-graphs.

Proposition 2.14. *Fix nonnegative integers $c_\Lambda, c_\Gamma \in \mathbf{N} \setminus \{0\}$ and a matrix of covering systems $(\Lambda_j, \Gamma_i, m_{i,j}, p_{i,j}, \mathfrak{s}_{i,j})_{i,j=1}^{c_\Gamma, c_\Lambda}$. Then there is a unique $(k+1)$-graph*

$$\big(\bigsqcup \Lambda_j\big) \overset{p, \mathfrak{s}}{\leftharpoondown} \big(\bigsqcup \Gamma_i\big)$$

such that

$$\Big(\big(\bigsqcup \Lambda_j\big) \overset{p, \mathfrak{s}}{\leftharpoondown} \big(\bigsqcup \Gamma_i\big)\Big)^{e_{k+1}} = \bigsqcup_{i,j} (\Lambda_j \overset{p_{i,j}, \mathfrak{s}_{i,j}}{\leftharpoondown} \Gamma_i)^{e_{k+1}},$$

each $\big(\big(\bigsqcup \Lambda_j\big) \overset{p, \mathfrak{s}}{\leftharpoondown} \big(\bigsqcup \Gamma_i\big)\big)^{e_l}$ for $1 \leq l \leq k$ is equal to $\Lambda^{e_l} \sqcup \Gamma^{e_l}$ and the commuting squares are inherited from the $\Lambda_j \overset{p_{i,j}, \mathfrak{s}_{i,j}}{\leftharpoondown} \Gamma_i$.
If each $(\Lambda_i, \Gamma_j, p_{i,j}, m_{i,j}, \mathfrak{s}_{i,j})$ is row finite then $\big(\bigsqcup \Lambda_j\big) \overset{p, \mathfrak{s}}{\leftharpoondown} \big(\bigsqcup \Gamma_i\big)$ is row finite. If Λ and Γ are locally convex, so is $\big(\bigsqcup \Lambda_j\big) \overset{p, \mathfrak{s}}{\leftharpoondown} \big(\bigsqcup \Gamma_i\big)$.

Proof. We apply Lemma 2.9; since the commuting squares are inherited from the $\Lambda_j \overset{p_{i,j}, \mathfrak{s}_{i,j}}{\leftharpoondown} \Gamma_i$, they satisfy the associativity condition (2.1) because each $\Lambda_j \overset{p_{i,j}, \mathfrak{s}_{i,j}}{\leftharpoondown} \Gamma_i$ is a $(k+1)$-graph. □

Corollary 2.15. *Fix $N \geq 2$ in $\mathbf{N} \cup \{\infty\}$. Let $(c_n)_{n=1}^N \subset \mathbf{N} \setminus \{0\}$ be a sequence of positive integers. For $1 \leq n < N$, let $(\Lambda_{n,j}, \Lambda_{n+1,i}, p_{i,j}^n, m_{i,j}^n, \mathfrak{s}_{i,j}^n)_{i,j=1}^{c_{n+1}, c_n}$ be a matrix of covering systems. Then there exists a unique $(k+1)$-graph $\mathbf{\Lambda}$ such* *that*

$$\mathbf{\Lambda}^{e_i} = \bigcup_{n=1}^{N} \bigcup_{j=1}^{c_n} \Lambda_{n,j}^{e_i} \quad \text{for } 1 \leq i \leq k,$$

$$\mathbf{\Lambda}^{e_{k+1}} = \bigcup_{n=1}^{N-1} \Big(\big(\bigsqcup_{j=1}^{c_n} \Lambda_{n,j} \big) \overset{p^n, \mathfrak{s}^n}{\leftarrow} \big(\bigsqcup_{i=1}^{c_{n+1}} \Lambda_{n+1,i} \big) \Big)^{e_{k+1}},$$

and the range, source and composition functions are all inherited from the $(k+1)$*-graphs* $\big(\bigsqcup_{j=1}^{c_n} \Lambda_{n,j} \big) \overset{p^n, \mathfrak{s}^n}{\leftarrow} \big(\bigsqcup_{i=1}^{c_{n+1}} \Lambda_{n+1,i} \big)$.

If each $(\Lambda_{n,j}, \Lambda_{n+1,i}, p_{i,j}^n, m_{i,j}^n, \mathfrak{s}_{i,j}^n)$ *is row finite, then* $\mathbf{\Lambda}$ *is row finite. If each* Λ_n *is locally convex, so is* $\mathbf{\Lambda}$.

Example 2.16 (The Irrational Rotation algebras). Fix $\theta \in [0,1] \setminus \mathbf{Q}$. Let $[a_1, a_2, \ldots]$ be the simple continued fraction expansion of θ. For each n, let $c_n = 2$, let $\phi_n := \left(\begin{smallmatrix} a_n & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and let $m^n := (m_{i,j}^n)_{i,j=1}^2$ be the matrix product $\phi_{T(n+1)} \cdots \phi_{T(n)+1}$ where $T(n) := n(n+1)/2$ is the n^{th} triangular number. Of all the integers $m_{i,j}^n$ obtained this way, only $m_{1,2}^1$ is equal to zero, so the matrices m^n have no zero rows or columns. Whenever $m_{i,j}^n \neq 0$, let $\mathfrak{s}_{i,j}^n$ be the permutation of the set $\{1, \ldots, m_{i,j}^n\}$ given by $\mathfrak{s}_{i,j}^n l = l+1$ if $1 \leq l < m_{i,j}^n$, and $\mathfrak{s}_{i,j}^n m_{i,j}^n = 1$.

Let $\Lambda_{n,i}$, $n \in \mathbf{N} \setminus \{0\}$, $i = 1, 2$ be mutually disjoint copies of the 1-graph T_1 whose skeleton consists of a single vertex and a single directed edge. For each n, let Λ_n be the 1-graph $\Lambda_{n,1} \sqcup \Lambda_{n,2}$ so that for each n, $(\Lambda_{n,j}, \Lambda_{n+1,i}, p_{i,j}^n, m_{i,j}^n, \mathfrak{s}_{i,j}^n)_{i,j=1}^2$ is a matrix of covering systems.

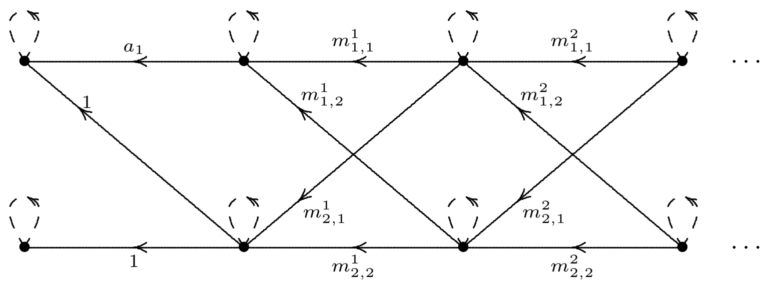

Figure 1. A tower of coverings with multiplicities

Modulo relabelling the generators of $\mathbf{N}^2$, the 2-graph $\varprojlim \Big(\bigsqcup_{j=1}^{c_n} \Lambda_{n,j}; p_{i,j}^n, \mathfrak{s}_{i,j}^n \Big)$ obtained from this data as in Corollary 2.15 is precisely the rank-2 Bratteli diagram of [27, Example 6.5] whose C^*-algebra is Morita equivalent to the irrational rotation algebra A_θ. Figure 1 is an illustration of its skeleton (parallel edges drawn as a single edge with a label indicating the multiplicity). The factorisation rules are all of the form $fg = \sigma(g) f'$ where f and f' are the dashed loops at either end of a solid edge in the diagram, and σ is a transitive permutation of the set of edges with the same range and source as g.

More generally, Section 7 of [27] considers in some detail the structure of the C^*-algebras associated to rank-2 Bratteli diagrams with length-1 cycles. All such rank-2 Bratteli diagrams can be recovered as above from Corollary 2.15.

3. C^*-algebras associated to covering systems of k-graphs

In this section, we describe how a covering system $(\Lambda, \Gamma, p, m, \mathfrak{s})$ induces an inclusion of C^*-algebras $C^*(\Lambda) \hookrightarrow M_m(C^*(\Gamma))$ and hence a homomorphism of K-groups $K_*(C^*(\Lambda)) \to K_*(C^*(\Gamma))$. The main result of the section is Theorem 3.8 which shows how to use these maps to compute the K-theory of $C^*(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n))$ from the data in a sequence $(\Lambda_n, \Lambda_{n+1}, p_n, m_n, \mathfrak{s}_n)_{n=1}^\infty$ of covering systems.

The following definition of the Cuntz-Krieger algebra of a row-finite locally convex k-graph Λ is taken from [31, Definition 3.3].

Given a row-finite, locally convex k-graph (Λ, d), a Cuntz-Krieger Λ-family is a collection $\{t_\lambda : \lambda \in \Lambda\}$ of partial isometries satisfying

(CK1) $\{t_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;
(CK2) $t_\lambda t_\mu = t_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$;
(CK3) $t_\lambda^* t_\lambda = t_{s(\lambda)}$ for all $\lambda \in \Lambda$; and
(CK4) $t_v = \sum_{\lambda \in v\Lambda^{\leq n}} t_\lambda t_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbf{N}^k$.

The *Cuntz-Krieger algebra* $C^*(\Lambda)$ is the C^*-algebra generated by a Cuntz-Krieger Λ-family $\{s_\lambda : \lambda \in \Lambda\}$ which is universal in the sense that for every Cuntz-Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$ there is a unique homomorphism π_t of $C^*(\Lambda)$ satisfying $\pi_t(s_\lambda) = t_\lambda$ for all $\lambda \in \Lambda$.

Remarks 3.1. If Λ has no sources (that is $v\Lambda^n \neq \emptyset$ for all $v \in \Lambda^0$ and $n \in \mathbf{N}^k$), then Λ is automatically locally convex, and the definition of $C^*(\Lambda)$ given above reduces to the original definition [20, Definition 1.5].

By [31, Theorem 3.15] there is a Cuntz-Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$ such that $t_\lambda \neq 0$ for all $\lambda \in \Lambda$. The universal property of $C^*(\Lambda)$ therefore implies that the generating partial isometries $\{s_\lambda : \lambda \in \Lambda\} \subset C^*(\Lambda)$ are all nonzero.

Let Ξ be a k-graph. The universal property of $C^*(\Xi)$ gives rise to an action γ of $\mathbf{T}^k$ on $C^*(\Xi)$, called the *gauge-action* (see, for example [31, §4.1]), such that $\gamma_z(s_\xi) = z^{d(\xi)} s_\xi$ for all $z \in \mathbf{T}^k$ and $\xi \in \Xi$.

Proposition 3.2. *Let $(\Lambda, \Gamma, p, m, \mathfrak{s})$ be a row-finite covering system of locally convex k-graphs. Let γ_Λ and γ_Γ denote the gauge actions of $\mathbf{T}^k$ on $C^*(\Lambda)$ and $C^*(\Gamma)$, and let γ denote the gauge action of $\mathbf{T}^{k+1}$ on $C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)$.*

(1) *The inclusions $\imath : \Lambda \to \Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ and $\jmath : \Gamma \to \Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ induce embeddings of $C^*(\Lambda)$ and $C^*(\Gamma)$ in $C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)$ characterised by*
$$\imath_*(s_\alpha) = s_{\imath(\alpha)} \text{ and } \jmath_*(s_\beta) = s_{\jmath(\beta)} \quad \text{for } \alpha \in \Lambda \text{ and } \beta \in \Gamma.$$
(2) *The sum $\sum_{v \in \jmath(\Gamma^0)} s_v$ converges in the strict topology to a full projection $Q \in \mathcal{M}(C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma))$, and the range of $\jmath_*$ is $QC^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)Q$.*
(3) *For $1 \leq i \leq m$, the sum $\sum_{v \in \Gamma^0} s_{e(v,i)}$ converges strictly to a partial isometry $V_i \in \mathcal{M}(C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma))$. The sum $\sum_{v \in \imath(\Lambda^0)} s_v$, converges strictly to the full projection $P := \sum_{i=1}^m V_i V_i^* \in \mathcal{M}(C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma))$. Moreover, $\imath_*$ is a nondegenerate homomorphism into $PC^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)P$.*

(4) *There is an isomorphism* $\phi : M_m(C^*(\Gamma)) \to PC^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)P$ *such that*

$$\phi\big((a_{i,j})_{i,j=1}^m\big) = \sum_{i,j=1}^m V_i \jmath_*(a_{i,j}) V_j^*.$$

(5) *There is an embedding* $\iota_{p,\mathfrak{s}} : C^*(\Lambda) \to M_m(C^*(\Gamma))$ *such that* $\phi \circ \iota_{p,\mathfrak{s}} = \imath_*$. *The embedding* $\iota_{p,\mathfrak{s}}$ *is equivariant in* γ_Λ *and the action* $\mathrm{id}_m \otimes \gamma_\Gamma$ *of* $\mathbf{T}^k$ *on* $M_m(C^*(\Gamma))$ *by coordinate-wise application of* γ_Γ.

(6) *If we identify* $K_*(C^*(\Gamma))$ *with* $K_*(M_m(C^*(\Gamma)))$, *then the induced homomorphism* $(\iota_{p,\mathfrak{s}})_*$ *may be viewed as a map from* $K_*(C^*(\Lambda)) \to K_*(C^*(\Gamma))$. *When applied to* K_0*-classes of vertex projections, this map satisfies*

$$(\iota_{p,\mathfrak{s}})_*([s_v]) = \sum_{p(u)=v} m \cdot [s_u] \in K_0(C^*(\Gamma)).$$

The proofs of the last three statements require the following general Lemma. This is surely well-known but we include it for completeness.

LEMMA 3.3. *Let* A *be a* C^**-algebra, let* $q \in \mathcal{M}(A)$ *be a projection, and suppose that* $v_1, \dots, v_n \in \mathcal{M}(A)$ *satisfy* $v_i^* v_j = \delta_{i,j} q$ *for* $1 \le i, j \le n$. *Then* $p = \sum_{i=1}^n v_i v_i^*$ *is a projection and* $pAp \cong M_n(qAq)$.

Proof. That $v_i^* v_j = \delta_{i,j} q$ implies that the v_i are partial isometries with mutually orthogonal range projections $v_i v_i^*$. Hence p is a projection in $\mathcal{M}(A)$. Define a map ϕ from pAp to $M_n(qAq)$ as follows: for $a \in pAp$ and $1 \le i, j \le n$, let $a_{i,j} := v_i^* a v_j$, and define $\phi(a)$ to be the matrix $\phi(a) = (a_{i,j})_{i,j=1}^n$.
It is straightforward to check using the properties of the v_i that ϕ is a C^*-homomorphism. It is an isomorphism because the homomorphism $\psi : M_n(qAq) \to pAp$ defined by

$$\psi\big((a_{i,j})_{i,j=1}^n\big) := \sum_{i,j=1}^n v_i a_{ij} v_j^* \in qAq$$

is an inverse for ϕ. □

Proof of Proposition 3.2. (1) The collection $\{s_{\imath(\lambda)} : \lambda \in \Lambda\}$ forms a Cuntz-Krieger Λ-family in $C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)$, and so by the universal property of $C^*(\Lambda)$ induces a homomorphism $\imath_* : C^*(\Lambda) \to C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)$. For $z \in \mathbf{T}^k$, write $(z, 1)$ for the element $(z_1, \dots, z_k, 1) \in \mathbf{T}^{k+1}$. Recall that γ denotes the gauge action of $\mathbf{T}^{k+1}$ on $C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)$. Then the action $z \mapsto \gamma_{(z,1)}$ of $\mathbf{T}^k$ on $C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)$ satisfies

$$\imath_*((\gamma_\Lambda)_z(a)) = \gamma_{(z,1)}(\imath_*(a))$$

for all $a \in C^*(\Lambda)$ and $z \in \mathbf{T}^k$. Since $\imath_*(s_v) = s_{\imath(v)} \neq 0$ for all $v \in \Lambda^0$ it follows from the gauge-invariant uniqueness theorem [20, Theorem 2.1] that $\imath_*$ is injective. A similar argument applies to $\jmath_*$.
(2) As the projections s_v, $v \in \jmath(\Gamma^0)$ are mutually orthogonal, a standard argument shows that the sum $\sum_{v \in \jmath(\Gamma^0)} s_v$ converges to a projection Q in the multiplier algebra (see [30, Lemma 2.1]). The range of $\jmath_*$ is equal to $QC^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)Q$

because $\jmath(\Gamma^0)(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)\jmath(\Gamma^0) = \jmath(\Gamma)$. To see that Q is full, it suffices to show that every generator of $C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)$ belongs to the ideal $I(Q)$ generated by Q. So let $\alpha \in \Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$. Either $s(\alpha) \in \jmath(\Gamma^0)$ or $s(\alpha) \in \imath(\Lambda^0)$. If $s(\alpha) \in \jmath(\Gamma^0)$, then $s_\alpha = s_\alpha Q \in I(Q)$. On the other hand, if $s(\alpha) \in \imath(\Lambda^0)$, the Cuntz-Krieger relation ensures that

$$s_\alpha = \textstyle\sum_{p(w)=s(\alpha)} \sum_{i=1}^m s_\alpha s_{e(w,i)} Q s_{e(w,i)}^*,$$

which also belongs to $I(Q)$.
(3) For fixed i, the partial isometries $s_{e(v,i)}$ have mutually orthogonal range projections and mutually orthogonal source projections. Hence an argument similar that of [30, Lemma 2.1] shows that $\sum_{v\in\Gamma^0} s_{e(v,i)}$ converges strictly to a multiplier $V_i \in \mathcal{M}(C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma))$. A simple calculation shows that $V_i^* V_j = \delta_{i,j} Q$ for all i, j. Hence each V_i is a partial isometry, and P is full because Q is full. The homomorphism $\imath_*$ is nondegenerate because the net

$$\Big(\imath_*\big(\textstyle\sum_{v\in F} s_v\big)\Big)_{F\subset\Lambda^0 \text{ finite}}$$

converges strictly to $P \in \mathcal{M}(C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma))$.
(4) This follows directly from Part (3) and Lemma 3.3.
(5) We define $\iota_{p,\mathfrak{s}} := \phi^{-1} \circ \imath_*$. For the gauge-equivariance, recall that $\imath_*$ (respectively $\jmath_*$) are equivariant in $\gamma|_{(\mathbf{T}^k,1)}$ and γ_Λ (respectively γ_Γ). By definition, ϕ is equivariant in $(\mathrm{id} \otimes \gamma)$ and $\gamma_{(\mathbf{T}^k,1)} \circ \jmath_*$. The equivariance of $\iota_{p,\mathfrak{s}}$ follows.
(6) By (CK4), for $v \in \Lambda^0$ we have $s_{\imath(v)} = \sum_{f\in v(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)^{e_{k+1}}} s_f s_f^*$, so the K_0-class $[s_{\imath(v)}]$ is equal to $\sum_{f\in v(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)^{e_{k+1}}} [s_f s_f^*]$. We can write $v(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)^{e_{k+1}}$ as the disjoint union

$$v(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)^{e_{k+1}} = \bigsqcup_{p(u)=v} \{e(u,i) : 1 \le i \le m\}.$$

In $K_0(C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma))$, we have $[s_{e(u,i)} s_{e(u,i)}^*] = [s_{e(u,i)}^* s_{e(u,i)}] = [s_{\jmath(u)}]$, and the result follows. □

Notation 3.4. As in Notation 2.8, when $m = 1$ so that $\mathfrak{s}$ is trivial, we continue to drop references to $\mathfrak{s}$ at the level of C^*-algebras. So Proposition 3.2(5) gives an inclusion $\iota_p : C^*(\Lambda) \to C^*(\Gamma)$ and the induced homomorphism of K-groups obtained from Proposition 3.2(6) is denoted $(\iota_p)_* : K_*(C^*(\Lambda)) \to K_*(C^*(\Gamma))$. This homomorphism satisfies

$$(\iota_p)_*([s_v]) = \sum_{p(u)=v} [s_u].$$

When no confusion is likely to occur, we will suppress the maps $\imath$, $\jmath$, $\imath_*$ and $\jmath_*$ and regard Λ and Γ as subsets of $\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma$ and $C^*(\Lambda)$ and $C^*(\Gamma)$ as C^*-subalgebras of $C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)$.

Remark 3.5. (1) The isomorphism ϕ of Proposition 3.2(4) extends to an isomorphism $\tilde{\phi} : M_{m+1}(C^*(\Gamma)) \to C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)$ which takes the block diagonal matrix $\left(\begin{smallmatrix} 0_{m\times m} & 0_{m\times 1} \\ 0_{1\times m} & a \end{smallmatrix}\right)$ to $\jmath_*(a)$. To see this, let $V, \ldots, V_m$ be as in Proposition 2.7(3), let $V_{m+1} = Q$, and apply Lemma 3.3.

(2) If $m = 1$ then ϕ is an isomorphism of $C^*(\Gamma)$ onto $PC^*(\Lambda \overset{p}{\leftarrow} \Gamma)P$, and $\iota_p : C^*(\Lambda) \hookrightarrow C^*(\Gamma)$ satisfies
$$\iota_p(s_\lambda) = \textstyle\sum_{p(\tilde\lambda)=\lambda} s_{\tilde\lambda}.$$

Fix $N \geq 2$ in $\mathbf{N}$. Let $(\Lambda_n, \Lambda_{n+1}, p_n, m_n, \mathfrak{s}_n)_{n=1}^{N-1}$ be a sequence of row-finite covering systems of locally convex k-graphs. Recall that in Corollary 2.11 we obtained from such data a $(k+1)$-graph $\Lambda_1 \overset{p_1,\mathfrak{s}_1}{\leftarrow} \cdots \overset{p_{N-1},\mathfrak{s}_{N-1}}{\leftarrow} \Lambda_N$, which for convenience we will denote $\mathbf{\Lambda}_N$ (the subscript is unnecessary here, but will be needed in Proposition 3.7). We now examine the structure of $C^*(\mathbf{\Lambda}_N)$ using Proposition 3.2.

Proposition 3.6. *Continue with the notation established in the previous paragraph. For each $v \in \Lambda_N^0$, list $\mathbf{\Lambda}_N^{Ne_{k+1}} v$ as $\{\alpha(v,i) : 1 \leq i \leq M\}$ where $M = m_1 m_2 \cdots m_{N-1}$.*

(1) *For $1 \leq n \leq N$, the sum $\sum_{v\in\Lambda_n^0} s_v$ converges strictly to a full projection $P_n \in \mathcal{M}(C^*(\mathbf{\Lambda}_N))$.*
(2) *For $1 \leq i \leq M$, the sum $\sum_{v\in\Lambda_N^0} s_{\alpha(v,i)}$ converges strictly to a partial isometry $V_i \in \mathcal{M}(C^*(\mathbf{\Lambda}_N))$ such that $V_i^* V_i = P_N$.*
(3) *We have $\sum_{i=1}^M V_i V_i^* = P_1$, and there is an isomorphism*
$$\phi : M_M(C^*(\Lambda_N)) \to P_1 C^*(\mathbf{\Lambda}_N) P_1$$
such that $\phi((a_{i,j})_{i,j=1}^M) = \sum_{i,j=1}^M V_i a_{i,j} V_j^*$.

Proof. Calculations like those in parts (2) and (3) of Proposition 3.2 show that the sums defining the P_n and the V_i converge in the multiplier algebra of $C^*(\mathbf{\Lambda}_N)$ and that each P_n is full.

Since distinct paths in $\mathbf{\Lambda}_N^{Ne_{k+1}}$ have orthogonal range projections and since paths in $\mathbf{\Lambda}_N^{Ne_{k+1}}$ with distinct sources have orthogonal source projections, each $V_i^* V_i = P_N$, and $\sum_{i=1}^M V_i V_i^* = P_1$.

One checks as in Proposition 3.2(1) that the inclusions $\imath_n : \Lambda_n \hookrightarrow \mathbf{\Lambda}_N$ induce inclusions $(\imath_n)_* : C^*(\Lambda_n) \hookrightarrow P_n C^*(\mathbf{\Lambda}_N) P_n$, and in particular that $(\imath_N)_* : C^*(\Lambda_N) \to P_N C^*(\mathbf{\Lambda}_N) P_N$ is an isomorphism. The final statement follows from Lemma 3.3. $\square$

We now describe the inclusions of the corners determined by P_1 as N increases. To do this, we first need some notation. Given a C^*-algebra A, and positive integers m, n, we denote by $\pi_{m,n} \otimes \mathrm{id}_A : M_m(M_n(A)) \to M_{mn}(A)$ the canonical isomorphism which takes the matrix $a = \big((a_{i,j,j',i'})_{j,j'=1}^n\big)_{i,i'=1}^m$ to the matrix $\pi(a)$ satisfying
$$\pi(a)_{j+n(i-1),j'+n(i'-1)} = a_{i,j,j',i'} \quad \text{for } 1 \leq i, i' \leq m,\ 1 \leq j, j' \leq n.$$

Given C^*-algebras A and B, a positive integer m, and a C^*-homomorphism $\psi : A \to B$, we write $\mathrm{id}_m \otimes \psi : M_m(A) \to M_m(B)$ for the C^*-homomorphism

$$(\mathrm{id}_m \otimes \psi)\big((a_{i,j})_{i,j=1}^m\big) = \big(\psi(a_{i,j})\big)_{i,j=1}^m.$$

Finally, given a matrix algebra $M_m(A)$ over a C^*-algebra A, and given $1 \leq i, i' \leq m$ and $a \in A$, we write $\theta_{i,i'}a$ for the matrix

$$(\theta_{i,i'}a)_{j,j'} = \begin{cases} a & \text{if } j = i \text{ and } j' = i' \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.7. *Fix $N \geq 2$ in $\mathbf{N}$. Let $(\Lambda_n, \Lambda_{n+1}, p_n, m_n, \mathfrak{s}_n)_{n=1}^N$ be a sequence of row-finite covering systems of locally convex k-graphs. We view the $(k+1)$-graph $\mathbf{\Lambda}_N := \Lambda_1 \overset{p_1,\mathfrak{s}_1}{\leftarrow} \cdots \overset{p_{N-1},\mathfrak{s}_{N-1}}{\leftarrow} \Lambda_N$ as a subcategory of $\mathbf{\Lambda}_{N+1} := \Lambda_1 \overset{p_1,\mathfrak{s}_1}{\leftarrow} \cdots \overset{p_N,\mathfrak{s}_N}{\leftarrow} \Lambda_{N+1}$ and likewise regard $C^*(\mathbf{\Lambda}_N)$ as a C^*-subalgebra of $C^*(\mathbf{\Lambda}_{N+1})$. In particular, we view $P_1 = \sum_{v \in \Lambda_1^0} s_v$ as a projection in both $\mathcal{M}(C^*(\mathbf{\Lambda}_N))$ and $\mathcal{M}(C^*(\mathbf{\Lambda}_{N+1}))$.*
Let $M := m_1 m_2 \dots m_{N-1}$, and let $\phi_N : M_M(C^*(\Lambda_N)) \to P_1(C^*(\mathbf{\Lambda}_N)P_1$ and $\phi_{N+1} : M_{Mm_N}(C^*(\Lambda_{N+1})) \to P_1 C^*(\mathbf{\Lambda}_{N+1})P_1$ be the isomorphisms obtained from Proposition 3.6. Then the following diagram commutes.

$$\begin{tikzcd}
P_1 C^*(\mathbf{\Lambda}_N) P_1 \arrow[r, hook, "\subseteq"] & P_1 C^*(\mathbf{\Lambda}_{N+1}) P_1 \\
M_M(C^*(\Lambda_N)) \arrow[u, "\phi_N"'] \arrow[r, "(\pi_{M,m_N} \otimes \mathrm{id}_{C^*(\Lambda_{N+1})}) \circ (\mathrm{id}_M \otimes \iota_{p_N,\mathfrak{s}_N})"] & M_{Mm_N}(C^*(\Lambda_{N+1})) \arrow[u, "\phi_{N+1}"']
\end{tikzcd}$$

Proof. As in Proposition 3.6, write $\mathbf{\Lambda}_N^{Ne_{k+1}} = \{\alpha(v,i) : v \in \Lambda_N^0, i \in \{1, \cdots, M\}\}$. For $i = 1, \dots, M$, let $V_i := \sum_{v \in \Lambda_N^0} s_{\alpha(v,i)}$. For $j = 1, \dots, m_N$, let

$$W_j := \sum_{w \in \Lambda_{N+1}^0} \sum_{i=1}^M s_{\alpha(p_N(w),i)} s_{e(w,j)}.$$

For (i,j) in the cartesian product $\{1, \dots, M\} \times \{1, \dots, m_N\}$, let $U_{j+m_N(i-1)} := \sum_{u \in \Lambda_{N+1}^0} s_{\alpha(p_N(u),i)e(u,j)}$. In what follows, we suppress canonical inclusion maps, and regard $C^*(\Lambda_N)$ as a subalgebra of $C^*(\mathbf{\Lambda}_N)$, and both $C^*(\mathbf{\Lambda}_N)$ and $C^*(\Lambda_{N+1})$ as subalgebras of $C^*(\mathbf{\Lambda}_{N+1})$. The corner $P_1 C^*(\mathbf{\Lambda}_N)P_1$ is equal to the closed span of elements of the form $V_i a V_{i'}^*$ where $a \in C^*(\Lambda_N)$ and $i, i' \in \{1, \dots, M\}$, and $P_1 C^*(\mathbf{\Lambda}_{N+1})P_1$ is equal to the closed span of elements of the form $U_l b U_{l'}^*$ where $b \in C^*(\Lambda_{N+1})$, $l, l' \in \{1, \dots, Mm_N\}$.
We have $\phi_N\big((a_{i,i'})_{i,i'=1}^M\big) = \sum_{i,i'=1}^M V_i a_{i,i'} V_{i'}^*$ by definition. The isomorphism ϕ_{N+1} from $M_{Mm_N}(C^*(\Lambda_{N+1}))$ to $P_1 C^*(\mathbf{\Lambda}_{N+1})P_1$ described in Proposition 3.6 satisfies

$$\phi_{N+1}\big(\textstyle\sum_{l,l'=1}^{Mm_N} U_l b_{l,l'} U_{l'}^*\big) = \big(b_{l,l'}\big)_{l,l'=1}^{Mm_N}.$$

The Cuntz-Krieger relations show that

$$V_iV_{i'}^*W_jW_{j'}^* = U_{j+m_N(i-1)}U_{j'+m_N(i'-1)}^* = W_jW_{j'}^*V_iV_{i'}^*$$

for $1 \le i, i' \le M$, $1 \le j, j' \le m_N$, and this decomposition of the matrix units $U_lU_{l'}^*$ implements π_{M,m_N}. Hence $\phi_{N+1} \circ (\pi_{M,m_N} \otimes \mathrm{id}_{C^*(\Lambda_{N+1})})$ satisfies

$$\begin{aligned}\phi_{N+1} \circ (\pi_{M,m_N} \otimes \mathrm{id}_{C^*(\Lambda_{N+1})})\Big(\big((b_{i,j,j',i'}\big)_{j,j'=1}^{m_N}\big)_{i,i'=1}^{M}\Big) \\ = \textstyle\sum_{i,i'=1}^{M}\sum_{j,j'=1}^{m_N} U_{j+m_N(i-1)}b_{i,j,j',i'}U_{j'+m_N(i'-1)}^*.\end{aligned} \tag{3.1}$$

The Cuntz-Krieger relations also show that $V_i = \sum_{j=1}^{m_N} W_jW_j^*V_i$ for all i, and hence $V_iaV_{i'}^* = \sum_j U_{j+m_N(i-1)}W_j^*aW_jU_{j+m_N(i'-1)}^*$ for all $a \in P_1C^*(\mathbf{\Lambda}_N)P_1$. One now checks that for $\lambda \in \Lambda_N$, we have

$$W_j^*s_\lambda W_j = \textstyle\sum_{p_N(\lambda')=\lambda} s_{e(r(\lambda'),j)}^* s_{e(r(\lambda),\mathfrak{s}_N(\lambda')j)}s_{\lambda'},$$

and hence that $V_is_\lambda V_{i'}^* = \sum_j\sum_{p_N(\lambda')=\lambda} U_{\mathfrak{s}_N(\lambda')j+m_N(i-1)}s_{\lambda'}U_{j+m_N(i'-1)}^*$. Recall that $\theta_{i,i'}s_\lambda \in M_M(C^*(\Lambda_N))$ denotes the matrix

$$\big(\theta_{i,i'}s_\lambda\big)_{j,j'} = \begin{cases} s_\lambda & \text{if } j = i \text{ and } j' = i' \\ 0 & \text{otherwise.}\end{cases}$$

Then $V_is_\lambda V_{i'}^* = \phi_N\big(\theta_{i,i'}s_\lambda\big)$ by definition of ϕ_N, so

$$\phi_N(\theta_{i,i'}s_\lambda) = \textstyle\sum_j\sum_{p_N(\lambda')=\lambda} U_{\mathfrak{s}_N(\lambda')j+m_N(i-1)}s_{\lambda'}U_{j+m_N(i'-1)}^*.$$

Since $(\mathrm{id}_M \otimes \iota_{p_N,\mathfrak{s}_N})(\theta_{i,i'}s_\lambda) = \theta_{i,i'}\sum_{p_N(\lambda')=\lambda}s_{\lambda'}$, we may therefore apply (3.1) to see that

$$\phi_N(\theta_{i,i'}s_\lambda) = \phi_{N+1} \circ (\pi_{M,m_N} \otimes \mathrm{id}_{C^*(\Lambda_{N+1})}) \circ (\mathrm{id}_M \otimes \iota_{p_N,\mathfrak{s}_N})(\theta_{i,i'}s_\lambda).$$

Since elements of the form $\theta_{i,i'}s_\lambda$ generate $M_M(C^*(\Lambda_N))$ this proves the result. $\square$

Theorem 3.8. *Let $(\Lambda_n, \Lambda_{n+1}, p_n, m_n, \mathfrak{s}_n)_{n=1}^\infty$ be a sequence of row-finite coverings of locally convex k-graphs. For each n, let $\mathbf{\Lambda}_n := \Lambda_1 \overset{p_1,\mathfrak{s}_1}{\leftarrow} \cdots \overset{p_{n-1},\mathfrak{s}_{n-1}}{\leftarrow} \Lambda_n$, identify $\mathbf{\Lambda}_n$ with the corresponding subset of $\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)$, and likewise identify $C^*(\mathbf{\Lambda}_n)$ with the corresponding C^*-subalgebra of $C^*(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n))$. Then*

$$C^*(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)) = \overline{\textstyle\bigcup_{n=1}^\infty C^*(\mathbf{\Lambda}_n)}. \tag{3.2}$$

Let $P_1 := \sum_{v\in\Lambda_1^0} s_v$, and for each n, let $M_n := m_1m_2\cdots m_{n-1}$. Then P_1 is a full projection in each $\mathcal{M}(C^*(\mathbf{\Lambda}_n))$, and we have

$$P_1C^*(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n))P_1 \cong \varinjlim\big(M_{M_n}(C^*(\Lambda_n)), \mathrm{id}_{M_n} \otimes \iota_{p_n,\mathfrak{s}_n}\big). \tag{3.3}$$

In particular,

$$\begin{aligned}K_*(C^*(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n))) &= K_*(P_1C^*(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n))P_1) \\ &\cong \varinjlim(K_*(C^*(\Lambda_n)), (\iota_{p_n,\mathfrak{s}_n})_*).\end{aligned}$$

Proof. For the duration of the proof, let $\mathbf{\Lambda} := \varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)$. We have $C^*(\mathbf{\Lambda}) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in \mathbf{\Lambda}\}$, so for the first statement, we need only show that

$$\operatorname{span}\{s_\mu s_\nu^* : \mu, \nu \in \mathbf{\Lambda}\} \subset \textstyle\bigcup_{n=1}^\infty C^*(\mathbf{\Lambda}_n).$$

To see this we simply note that for any finite $F \subset \mathbf{\Lambda}$, the integer $N := \max\{n \in \mathbf{N} : s(F) \cap \Lambda_n^0 \neq \emptyset\}$ satisfies $F \subset \mathbf{\Lambda}_N$.
Since P_1 is full in each $C^*(\mathbf{\Lambda}_n)$ by Proposition 3.2(3), it is full in $C^*(\mathbf{\Lambda})$ by (3.2). Equation 3.3 follows from Proposition 3.7. The final statement then follows from continuity of the K-functor. □

Remark 3.9. Note that if we let γ denote the restriction of the gauge action to $P_1 C^*(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n))P_1$ then $\gamma_{(1,\cdots,1,z)}$ is trivial for all $z \in \mathbf{T}$. Indeed, if $s_\mu s_\nu^*$ is a nonzero element $P_1 C^*(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n))P_1$, then $d(\mu)_{n+1} = d(\nu)_{n+1}$. So γ may be regarded as an action by $\mathbf{T}^k$ rather than $\mathbf{T}^{k+1}$.

We can extend Theorem 3.8 to the situation of matrices of covering systems as discussed in Section 2.1 as follows.

Proposition 3.10. *Resume the notation of Corollary 2.15. Each $C^*(\Lambda_n)$ is canonically isomorphic to $\bigoplus_{j=1}^{c_n} C^*(\Lambda_{n,j})$. There are homomorphisms $(\iota_n)_* : K_*(C^*(\Lambda_n)) \to K_*(C^*(\Lambda_{n+1}))$ such that the partial homomorphism which maps the j^{th} summand in $K_*(C^*(\Lambda_n))$ to the i^{th} summand in $K_*(C^*(\Lambda_{n+1}))$ is equal to 0 if $m_{i,j}^n = 0$, and is equal to $(\iota_{p_{i,j}^n, \mathfrak{s}_{i,j}^n})_*$ otherwise. The sum $\sum_{v \in \Lambda_1^0} s_v$ converges strictly to a full projection $P_1 \in \mathcal{M}(C^*(\mathbf{\Lambda}))$. Furthermore,*

$$K_*(P_1 C^*(\mathbf{\Lambda})P_1) \cong \varinjlim \Big(\bigoplus_{j=1}^{c_n} K_*(C^*(\Lambda_{n,j})), (\iota_n)_* \Big).$$

Proof. For each $\lambda \in \Lambda_n = \bigsqcup_{j=1}^{c_n} \Lambda_{n,j}$, define a partial isometry $t_\lambda \in \bigoplus_{j=1}^{c_n} C^*(\Lambda_{n,j})$ by $t_\lambda := (0, \ldots, 0, s_\lambda, 0, \ldots, 0)$ (the nonzero term is in the j^{th} coordinate when $\lambda \in \Lambda_{n,j}$). These nonzero partial isometries form a Cuntz-Krieger Λ_n-family consisting of nonzero partial isometries. The universal property of $C^*(\Lambda_n)$ gives a homomorphism $\pi_t^n : C^*(\Lambda_n) \to \bigoplus_{j=1}^{c_n} C^*(\Lambda_{n,j})$ which intertwines the direct sum of the gauge actions on the $C^*(\Lambda_{n,j})$ and the gauge action on $C^*(\Lambda_n)$. The gauge-invariant uniqueness theorem [20, Theorem 3.4], and the observation that each generator of each summand in $\bigoplus_{j=1}^{c_n} C^*(\Lambda_{n,j})$ is nonzero and belongs to the image of π_t^n therefore shows that π_t^n is an isomorphism.
The individual covering systems $(\Lambda_{n,j}, \Lambda_{n+1,i}, p^n, m^n, \mathfrak{s}^n)$ induce inclusions $\iota_{p_{i,j}^n, \mathfrak{s}_{i,j}^n} : C^*(\Lambda_{n,j}) \to M_{m_{i,j}^n}(C^*(\Lambda_{n+1,i}))$ as in Proposition 3.2(5). We therefore obtain homomorphisms $(\iota_{p_{i,j}^n, \mathfrak{s}_{i,j}^n})_* : K_*(C^*(\Lambda_{n,j})) \to K_*(C^*(\Lambda_{n+1,i}))$. The statement about the partial homomorphisms of K-groups then follows from the properties of the isomorphism $K_*(\bigoplus_i A_i) \cong \bigoplus_i K_*(A_i)$ for C^*-algebras A_i.
The final statement can then be deduced from arguments similar to those of Theorem 3.8. □

4. Simplicity and pure infiniteness

Theorem 3.1 of [34] gives a necessary and sufficient condition for simplicity of the C^*-algebra of a row-finite k-graph with no sources. Specifically, $C^*(\Lambda)$ is simple if and only if Λ is cofinal and every vertex of Λ receives an aperiodic infinite path (see below for the definitions of cofinality and aperiodicity). In this section we present some means of deciding whether $\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)$ is cofinal (Lemma 4.7), and whether an infinite path in $\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)$ is aperiodic (Lemma 4.3). We also present a condition under which $C^*(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n))$ is purely infinite (Proposition 4.8).
We begin by recalling the notation and definitions required to make sense of the hypotheses of [34, Theorem 3.1]. For more detail, see Section 2 of [31].

Notation 4.1. We write Ω_k for the k-graph such that $\Omega_k^q := \{(m,n) \in \mathbf{N}^k \times \mathbf{N}^k : n - m = q\}$ for each $q \in \mathbf{N}^k$, with $r(m,n) = (m,m)$ and $s(m,n) = (n,n)$. We identify $\Omega_k^0 = \{(m,m) : m \in \mathbf{N}^k\}$ with $\mathbf{N}^k$. An infinite path in a k-graph Ξ is a graph morphism $x : \Omega_k \to \Xi$, and we denote the image $x(0)$ of the vertex $0 \in \Omega_k^0$ by $r(x)$. We write Ξ^∞ for the collection of all infinite paths in Ξ, and for $v \in \Xi^0$ we denote by $v\Xi^\infty$ the collection $\{x \in \Xi^\infty : r(x) = v\}$. For $x \in \Xi^\infty$ and $q \in \mathbf{N}^k$, there is a unique infinite path $\sigma^q(x) \in \Xi^\infty$ such that $\sigma^q(x)(m,n) = x(m+q, n+q)$ for all $m \leq n \in \mathbf{N}^k$.

Definition 4.2. We say that a row-finite k-graph Ξ with no sources is *aperiodic* if for each vertex $v \in \Xi^0$ there is an infinite path $x \in v\Xi^\infty$ such that $\sigma^q(x) \neq \sigma^{q'}(x)$ for all $q \neq q' \in \mathbf{N}^k$. We say that Ξ is *cofinal* if for each $v \in \Xi^0$ and $x \in \Xi^\infty$ there exists $m \in \mathbf{N}^k$ such that $v\Xi x(m) \neq \emptyset$.

We continue to make use in the following of the notation established earlier (see Notation 2.6) for the embeddings of $\mathbf{N}^k$ and of $\mathbf{N}$ in $\mathbf{N}^{k+1}$.
If y is an infinite path in the $(k+1)$-graph Ξ, we write α_y for the infinite path in $\Xi^{(0_k, \mathbf{N})}$ defined by $\alpha_y(p,q) := y((0_k,p),(0_k,q))$ for $p \leq q \in \mathbf{N}$, and we write x_y for the infinite path in $\Xi^{(\mathbf{N}^k, 0_1)}$ defined by $x_y(p,q) := y((p,0_1),(q,0_1))$ where $p \leq q \in \mathbf{N}^k$.

Proposition 4.3. *Let $(\Lambda_n, \Lambda_{n+1}, p_n, m_n, \mathfrak{s}_n)_{n=1}^\infty$ be a sequence of row-finite covering systems of k-graphs with no sources. For $a, b \in \mathbf{N}^{k+1}$, an infinite path $y \in \big(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)\big)^\infty$ satisfies $\sigma^a(y) = \sigma^b(y)$ if and only if $x_{\sigma^a(y)} = x_{\sigma^b(y)}$ and $\alpha_{\sigma^a(y)} = \alpha_{\sigma^b(y)}$.*

Proof. The "only if" implication is trivial. For the "if" implication, note that the factorisation property implies that an infinite path z of $\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)$ is uniquely determined by x_z and the paths $\alpha_{\sigma^{(n,0_1)}(z)}$, $n \in \mathbf{N}^k$. So it suffices to show that each $\alpha_{\sigma^{(n,0_1)}(z)}$ is uniquely determined by $x_z(0_k, n)$ and α_z. Fix $n \in \mathbf{N}^k$ and let $\lambda := x_z(0_k, n) = z(0_{k+1}, (n, 0_1))$. Fix $i \in \mathbf{N}$. We will show that $\alpha_{\sigma^{(n,0_1)}(z)}(0_1, i)$ is uniquely determined by $\alpha_z(0_1, i)$ and λ. Let $v = r(z)$, and let $N \in \mathbf{N}$ be the element such that $v \in \Lambda_N^0$. For $1 \leq j \leq i$, let $w_j = \alpha_z(i) \in \Lambda_{N+j}^0$, and let $1 \leq l_j \leq m_{N+j-1}$ be the integer such that $\alpha_z(j-1, j) = e(w_i, l_j)$. We have $p_N(w_1) = v$, and $p_{N+j-1}(w_j) = w_{j-1}$ for $2 \le j \le i$. For each j, let λ_j be the unique lift of λ such that $r(\lambda_j) = w_j$. By definition of the $(k+1)$-graph $\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)$, the path

$$\lambda e(s(\lambda_1), \mathfrak{s}(\lambda_1)^{-1} l_1) e(s(\lambda_2), \mathfrak{s}(\lambda_2)^{-1} l_2) \dots e(s(\lambda_i), \mathfrak{s}(\lambda_i)^{-1} l_i) = \alpha_z(0_1, i)\lambda_i$$

is the unique minimal common extension of λ and $\alpha_z(0_1, i)$ in $\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)$. Hence

$$\alpha_{\sigma^{(n,0_1)}(z)}(0_1, i) = e(s(\lambda_1), \mathfrak{s}(\lambda_1)^{-1} l_1) e(s(\lambda_2), \mathfrak{s}(\lambda_2)^{-1} l_2) \dots e(s(\lambda_i), \mathfrak{s}(\lambda_i)^{-1} l_i)$$

which is uniquely determined by λ and $\alpha_z(0_1, i)$. □

Corollary 4.4. *Let $(\Lambda_n, \Lambda_{n+1}, p_n, m_n, \mathfrak{s}_n)_{n=1}^{\infty}$ be a sequence of row-finite covering systems of k-graphs with no sources. Suppose that Λ_n is aperiodic for some n. Then so is $\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)$.*

Proof. Since each vertex in Λ_n receives an aperiodic path in Λ_n, Proposition 4.3, guarantees that each vertex in Λ_n receives an aperiodic path in $\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)$. Since the p_n are coverings, it follows that every vertex of $\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)$ receives an infinite path of the form λy or of the form $\sigma^p(y)$ where y is an aperiodic path with range in Λ_n. If y is aperiodic, then λy is aperiodic for any λ and $\sigma^a(y)$ is aperiodic for any a and the result follows. □

Lemma 4.5. *Let $(\Lambda_n, \Lambda_{n+1}, p_n, m_n, \mathfrak{s}_n)_{n=1}^{\infty}$ be a sequence of row-finite covering systems of k-graphs with no sources. Fix $y \in \big(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)\big)^{\infty}$, with $y(0) \in \Lambda_n$ and $a, b \in \mathbf{N}^{k+1}$. Let $\tilde{a}$ and $\tilde{b}$ denote the elements of $\mathbf{N}^k$ determined by the first k coordinates of a and b. For each $m \ge n$, let v_m and i_m be the unique pair such that $\alpha_y(m, m+1) = e(v_m, i_m)$. For each $m \ge n$, let μ_m and ν_m be the unique lifts of $x_y(0, \tilde{a})$ and $x_y(0, \tilde{b})$ such that $r(\mu_m) = r(\nu_m) = v_m$. Then $\alpha_{\sigma^a(y)} = \alpha_{\sigma^b(y)}$ if and only if the following three conditions hold:*

1. *$a_{k+1} = b_{k+1}$;*
2. *$s(\mu_m) = s(\nu_m)$ for all $m \ge n$; and*
3. *$\mathfrak{s}_m(\mu_m) i_m = \mathfrak{s}_m(\nu_m) i_m$ for all $m \ge n$.*

Proof. We have $\alpha_{\sigma^a(y)}(m, m+1) = e(s(\mu_{m+a_{k+1}}), \mathfrak{s}_m(\mu_{m+a_{k+1}}) i_{m+a_{k+1}})$ for all m, and likewise for b and ν. □

Remark 4.6. Lemma 5.4 of [27] implies that an infinite path in a rank-2 Bratteli diagram Λ is aperiodic if and only if the factorisation permutations of its red coordinate-paths are of unbounded order. Lemma 4.5 is the analogue of this result for general systems of coverings. To see the analogy, note that in a rank-2 Bratteli diagram, every x_y is of the form $\lambda\lambda\lambda\dots$ for some blue cycle Λ, so that condition (3) fails for all $a \ne b$ precisely when the orders of the permutations $\mathfrak{s}_m(\mu_m)$ grow arbitrarily large with m.

Lemma 4.7. *Let $(\Lambda_n, \Lambda_{n+1}, p_n, m_n, \mathfrak{s}_n)_{n=1}^{\infty}$ be a sequence of row-finite coverings of k-graphs with no sources. If infinitely many of the Λ_n are cofinal, then $\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n)$ is also cofinal.*

Proof. Fix a vertex v and an infinite path $z \in (\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n))^\infty$. Let $n_1, n_2 \in \mathbf{N}$ be the elements such that $v \in \Lambda^0_{n_1}$ and $r(z) \in \Lambda^0_{n_2}$. Choose $N \geq n_1, n_2$ such that Λ_N is cofinal. Fix $w \in \Lambda^0_N$ such that $p_n \circ p_{n+1} \circ \cdots \circ p_{N-1}(w) = v$; so $v(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n))w \neq \emptyset$. We have $x_{\sigma^{(0_k, N-n_2)}(z)} \in \Lambda^\infty_N$, and since Λ_N is cofinal, it follows that $w\Lambda_N x_{\sigma^{(0_k,N-n_2)}(z)}(q) \neq \emptyset$ for some $q \in \mathbf{N}^k$. Since $x_{\sigma^{(0_k,N-n_2)}(z)}(q) = z(q, N-n_2)$, this completes the proof. □

As in [38], we say that a path λ in a k-graph Λ is a *cycle with an entrance* if $s(\lambda) = r(\lambda)$, and there exists $\mu \in r(\lambda)\Lambda$ with $d(\mu) \leq d(\lambda)$ and $\lambda(0, d(\mu)) \neq \mu$.

Proposition 4.8. *Let $(\Lambda_n, \Lambda_{n+1}, p_n, m_n, \mathfrak{s}_n)_{n=1}^\infty$ be a sequence of row-finite coverings of k-graphs with no sources. There exists n such that Λ_n contains a cycle with an entrance if and only if every Λ_n contains a cycle with an entrance. Moreover, if $C^*(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n))$ is simple and Λ_1 contains a cycle with an entrance, then $C^*(\mathbf{\Lambda})$ is purely infinite.*

Proof. That the presence of a cycle with an entrance in Λ_1 is equivalent to the presence of a cycle with an entrance in every Λ_n follows from the properties of covering maps. Now the result follows from [38, Proposition 8.8] □

5. K-Theory

In this section, we consider the K-theory of $C^*(\Lambda \overset{p,\mathfrak{s}}{\leftarrow} \Gamma)$. Specifically, we show how the homomorphism from $K_*(C^*(\Lambda))$ to $K_*(C^*(\Gamma))$ obtained from Proposition 3.2 behaves with respect to existing calculations of K-theory for various classes of higher-rank graph C^*-algebras. We will use these results later to compute the K-theory of $C^*(\varprojlim(\Lambda_n; p_n, \mathfrak{s}_n))$ for a number of sequences of covering systems.

Throughout this section, given a k-graph Λ, we view the ring $\mathbf{Z}\Lambda^0$ as the collection of finitely supported functions $f : \Lambda^0 \to \mathbf{Z}$. For $v \in \Lambda^0$, we denote the point-mass at v by δ_v. Given a finite covering $p : \Gamma \to \Lambda$ of row-finite k-graphs, we define $p^* : \mathbf{Z}\Lambda^0 \to \mathbf{Z}\Gamma^0$ by $p^*(\delta_v) = \sum_{p(u)=v} \delta_w$; equivalently, $p^*(f)(w) = f(p(w))$.

5.1. Coverings of 1-graphs and the Pimsner-Voiculescu exact sequence.

It is shown in [26, 32] how to compute the K-theory of a graph C^*-algebra using the Pimsner-Voiculescu exact sequence. In this subsection, we show how this calculation interacts with the inclusion of C^*-algebras arising from a covering of 1-graphs.

The K-theory computations for arbitrary graph C^*-algebras [12, 1] are somewhat more complicated than for the C^*-algebras of row-finite graphs with no sources. Moreover, every graph C^*-algebra is Morita equivalent to the C^*-algebra of a row-finite graph with no sources [12]. We therefore restrict out attention here to the simpler setting.

Theorem 5.1. *Let $(E^*, F^*, p, m, \mathfrak{s})$ be a row-finite covering system of 1-graphs with no sources. Let A, B be the vertex connectivity matrices of the underlying* *graphs E and F respectively. Then the diagram*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & K_1(C^*(E^*)) & \longrightarrow & \mathbf{Z}E^0 & \xrightarrow{1-A^t} & \mathbf{Z}E^0 & \longrightarrow & K_0(C^*(E^*)) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle (\iota_{p,\mathfrak{s}})_*} & & \Big\downarrow{\scriptstyle m\cdot p^*} & & \Big\downarrow{\scriptstyle m\cdot p^*} & & \Big\downarrow{\scriptstyle (\iota_{p,\mathfrak{s}})_*} & & \\ 0 & \longrightarrow & K_1(C^*(F^*)) & \longrightarrow & \mathbf{Z}F^0 & \xrightarrow{1-B^t} & \mathbf{Z}F^0 & \longrightarrow & K_0(C^*(F^*)) & \longrightarrow & 0 \end{array} \tag{5.1}$$

commutes and the rows are exact.

The proof of this theorem occupies the remainder of Section 5.1. We fix, for the duration, a finite covering $p : F^* \to E^*$ of row-finite 1-graphs with no sources, a multiplicity m and a cocycle $\mathfrak{s} : F^* \to S_m$.

It is relatively straightforward to prove that the right-hand two squares of (5.1) commute and that the rows are exact.

Lemma 5.2. *Resume the notation of Theorem 5.1. We have $(1 - B^t)p^* = p^*(1 - A^t)$, the right-hand two squares of* (5.1) *commute, and the rows are exact.*

Proof. For the first statement, consider a generator $\delta_v \in \mathbf{Z}E^0$. We have

$$\big(p^* \circ (1 - A^t)\big)(\delta_v) = p^*(\delta_v - \sum_{e\in vE^1} \delta_{s(e)}) = \sum_{p(u)=v} \delta_u - \sum_{e\in vE^1} \sum_{p(f)=e} \delta_{s(f)}.$$

On the other hand,

$$\big((1 - B^t) \circ p^*\big)(\delta_v) = (1 - B^t) \sum_{p(u)=v} \delta_u = \sum_{p(u)=v} \Big(\delta_u - \sum_{f\in uF^1} \delta_{s(f)}\Big).$$

Since p is a covering the double-sums occurring in these two equations each contain exactly one term for each edge $f \in F^1$ such that $p(r(f)) = v$, and it follows that the two are equal.

Multiplying by m throughout the above calculation shows that the middle square of (5.1) commutes.

The identification of $K_0(C^*(E^*))$ with $\operatorname{coker}(1 - A^t)$ takes the class of the projection $s_v \in C^*(E^*)$ to the class of the corresponding generator $\delta_v \in \mathbf{Z}E^0$ (see [30]). That the right-hand square commutes then follows from Proposition 3.2(6).

Exactness of the rows is precisely the computation of K-theory for 1-graph C^*-algebras [8, 26, 32]. □

It remains to prove that the left-hand square of (5.1) commutes. The strategy is to assemble the eight-term commuting diagrams which describe the K-theory of each of $C^*(E^*)$ and $C^*(F^*)$ (see equation (5.3) below) into a sixteen-term diagram, one face of which is the left-hand square of (5.1). We then focus on the cube in the sixteen-term diagram which contains left-hand square of (5.1) as one of its faces, and show that the remaining five faces of this cube commute. A diagram-chase then establishes that the sixth face commutes as well. The majority of the work involved goes into defining the connecting maps needed to write down the sixteen-term diagram in the first place. The proof that the various squares in it commute is then relatively straightforward.

To begin, we recall how one shows that the rows of (5.1) are exact. Let $E^* \times_d \mathbf{Z}$ be the skew-product of E^* by the length functor d (see [20, Section 5]). Let γ be the gauge action of $\mathbf{T}$ on $C^*(E^*)$ satisfying $\gamma_z(s_e) = zs_e$ for $e \in E^1$ and $z \in \mathbf{T}$. Let $(i_{\mathbf{T}}, i_{C^*(E^*)})$ be the universal covariant representation of $(C^*(E^*), \mathbf{T}, \gamma)$ in the crossed product $C^*(E^*) \times_\gamma \mathbf{T}$. By [32, Lemma 3.1], there is an isomorphism

$$\psi_E : C^*(E^* \times_d \mathbf{Z}) \to C^*(E^*) \times_\gamma \mathbf{T} \tag{5.2}$$

satisfying $\psi_E(s_{(\lambda,n)}) = i_{\mathbf{T}}(z)^n i_{C^*(E^*)}(s_\lambda)$.

The C^*-algebra $C^*(E^* \times_d \mathbf{Z})$ is AF with K_0-group $\varinjlim(\mathbf{Z}E^0, A^t)$ (see [26, 32]). Hence one may apply the dual Pimsner-Voiculescu sequence [4, Section 10.6] to the crossed product algebra $C^*(E^*) \times_\gamma \mathbf{T}$ to show that the top row of (5.1) is exact (the bottom row is the same after replacing E with F).

From the point of view of coverings, the skew-product graph $E^* \times_d \mathbf{Z}$ and its C^*-algebra are more natural to work with than the crossed product $C^*(E^*) \times_\gamma \mathbf{T}$. Before proving that the final square of (5.1) commutes, we therefore detail first how coverings $p : F^* \to E^*$ interact with the isomorphisms $\psi_E : C^*(E^* \times_d \mathbf{Z}) \to C^*(E^*) \times_\gamma \mathbf{T}$.

Lemma 5.3. *With the above notation, let $E^* \times_d \mathbf{Z}$ and $F^* \times_d \mathbf{Z}$ be the skew-product graphs by the length functors d, and let ψ_E and ψ_F be the isomorphisms described in* (5.2). *Let γ_E and γ_F denote the gauge actions of $\mathbf{T}$ on $C^*(E^*)$ and $C^*(F^*)$.*

(1) *the formulae $\tilde{p}(\lambda, n) := (p(\lambda), n)$ and $\tilde{\mathfrak{s}}(\lambda, n) := \mathfrak{s}(\lambda)$ determine a covering $\tilde{p} : F^* \times_d \mathbf{Z} \to E^* \times_d \mathbf{Z}$ and a cocycle $\tilde{\mathfrak{s}} : F^* \times_d \mathbf{Z} \to S_m$.*
(2) *the inclusion $\iota_{p,\mathfrak{s}} : C^*(E^*) \to M_m(C^*(F^*))$ is equivariant in the actions γ_E and $\mathrm{id}_m \otimes \gamma_F$, and induces an inclusion $\widetilde{\iota_{p,\mathfrak{s}}} : C^*(E^*) \times_{\gamma_E} \mathbf{T} \to M_m(C^*(F^*)) \times_{\mathrm{id}_m \otimes \gamma_F} \mathbf{T}$.*
(3) *The following diagram commutes.*

$$\begin{tikzcd}
C^*(E^* \times_d \mathbf{Z}) \arrow[r, "\iota_{\tilde{p},\tilde{\mathfrak{s}}}"] \arrow[d, "\psi_E"'] & M_m(C^*(F^* \times_d \mathbf{Z})) \arrow[d, "\mathrm{id}_m \otimes \psi_F"'] \\
C^*(E^*) \times_{\gamma_E} \mathbf{T} \arrow[r, "\widetilde{\iota_{p,\mathfrak{s}}}"] & M_m(C^*(F^*)) \times_{\mathrm{id}_m \otimes \gamma_F} \mathbf{T}
\end{tikzcd}$$

Proof. (1) It is straightforward to check that $\tilde{p}$ is a covering. To see that $\tilde{\mathfrak{s}}$ is a cocycle, note that (μ, m) and (ν, n) are composable in the skew-product precisely when μ and ν are composable, and $n = m - d(\nu)$. So for $i \in \{1, \ldots, m\}$ we may calculate

$$\tilde{\mathfrak{s}}(\mu, m)(\tilde{\mathfrak{s}}(\nu, m - d(\nu))i) = \mathfrak{s}(\mu)(\mathfrak{s}(\nu)i) = \mathfrak{s}(\mu\nu)i = \tilde{\mathfrak{s}}(\mu\nu, m - d(\nu))i.$$

(2) That $\iota_{p,\mathfrak{s}}$ is equivariant in γ_E and $\mathrm{id}_m \otimes \gamma_F$ follows from Proposition 3.2(5). That it induces the desired inclusion $\widetilde{\iota_{p,\mathfrak{s}}}$ of crossed-products follows from the universal properties of the crossed-product algebras.

(3) That the diagram commutes follows from a simple calculation using the definitions of the maps involved. □

Proof of Theorem 5.1. Lemma 5.2 establishes everything except that the left-hand square in the diagram (5.1) commutes. To establish this last claim, recall from [32, Theorem 3.2] (see also [26]) that there is a homomorphism $\phi_E : \mathbf{Z}E^0 \to K_0(C^*(E^*) \times_{\gamma_E} \mathbf{T})$ satisfying $\phi_E(\delta_v) = [i_{\mathbf{T}}(1) i_{C^*(E^*)}(s_v)]$. Moreover, the rows of the following commutative diagram are exact and the left- and right-most vertical maps are isomorphisms (see [30, Lemma 7.15], [26]).

$$(5.3)\qquad \begin{tikzcd} 0 \arrow[r] & \ker(1-A^t) \arrow[r] \arrow[d, "\cong"] & \mathbf{Z}E^0 \arrow[r, "1-A^t"] \arrow[d, "\phi_E"] & \mathbf{Z}E^0 \arrow[r] \arrow[d, "\phi_E"] & \operatorname{coker}(1-A^t) \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & K_1(C^*(E^*)) \arrow[r] & K_0(C^*(E^*) \times_{\gamma_E} \mathbf{T}) \arrow[r, "1-\hat{\gamma}_*^{-1}"] & K_0(C^*(E^*) \times_{\gamma_E} \mathbf{T}) \arrow[r] & K_0(C^*(E^*)) \arrow[r] & 0 \end{tikzcd}$$

A similar commutative diagram holds for F^*, and using the standard isomorphism of $K_*(M_m(C^*(F^*)))$ with $K_*(C^*(F^*))$, we may assemble these two diagrams can into a single three-dimensional diagram by connecting each term in the diagram for E^* to the corresponding term in the diagram for F^* using the appropriate maps induced from $(p, \mathfrak{s})$. The map connecting the K_0-groups of the skew-product graph algebras is induced from the connecting map in the bottom row of the commuting diagram in Lemma 5.3(3) by applying the K-functor and using the canonical isomorphisms

$$K_*(M_m(C^*(F^*) \times_{\gamma_F} \mathbf{T})) \cong K_*(C^*(F^*) \times_{\gamma_F} \mathbf{T}) \quad \text{and}$$
$$M_m(C^*(F^*) \times_{\gamma_F} \mathbf{T}) \cong M_m(C^*(F^*)) \times_{\mathrm{id}_m \otimes \gamma_F} \mathbf{T}.$$

Let η denote the unlabelled inclusion $K_1(C^*(F^*)) \hookrightarrow K_0(C^*(F^*) \times_{\gamma_F} \mathbf{T}))$ in the bottom row of the diagram of the form (5.3) for F^*. Notice that injectivity of the map $m \cdot p^* : \mathbf{Z}E^0 \to \mathbf{Z}F^0$ together with the first statement of Lemma 5.2 ensures that $m \cdot p^*$ restricts to a map from $\ker(1-A^t)$ to $\ker(1-B^t)$; abusing notation, we denote this map $m \cdot p^*$ too. With this notation the diagram (5.4) below is the left-hand cube of the three-dimensional diagram described in the previous paragraph.

$$(5.4)\qquad \begin{tikzcd} & \ker(1-A^t) \arrow[rr, hook] \arrow[dl, "m\cdot p^*"'] \arrow[dd, "\cong"] & & \mathbf{Z}E^0 \arrow[dl, "m\cdot p^*"'] \arrow[dd, "\phi_E"] \\ \ker(1-B^t) \arrow[rr, hook] \arrow[dd, "\cong"] & & \mathbf{Z}F^0 \arrow[dd, "\phi_E"] & \\ & K_1(C^*(E^*)) \arrow[rr] \arrow[dl, "(\iota_{p,\mathfrak{s}})_*"'] & & K_0(C^*(E^*) \times_{\gamma_E} \mathbf{T}) \arrow[dl, "(\widetilde{\iota_{p,\mathfrak{s}}})_*"'] \\ K_1(C^*(F^*)) \arrow[rr, "\eta"] & & K_0(C^*(F^*) \times_{\gamma_F} \mathbf{T}) & \end{tikzcd}$$

We have shown the whole cube because we prove that the left-hand face — which is none other than the left-hand square of (5.1) — commutes by showing that the other five faces commute.

To see why this suffices, suppose that the other five faces do indeed commute. Since η is an injection by the exactness of the rows of (5.3), we just need to show that the two maps from $\ker(1 - A^t)$ into $K_0(C^*(F^*) \times_\gamma \mathbf{T}))$ obtained from the maps in the left-hand face of the cube followed by η agree. A diagram chase shows that this is the case.

It therefore remains only to show that the top, bottom, front, back and right-hand faces of (5.4) commute. The top square commutes by definition. The bottom square commutes by the naturality of the dual Pimsner-Voiculescu exact sequence (see the argument at the beginning of [32, Section 3]). The back and front faces commute because (5.3) commutes.

To see that the right-hand face commutes, recall that $C^*(E^* \times_d \mathbf{Z})$ is AF with K_0-group $\varinjlim(\mathbf{Z}E^0, 1 - A^t)$. Hence there is an inclusion $\varepsilon_E : \mathbf{Z}E^0 \to K_0(C^*(E^* \times_d \mathbf{Z}))$ which takes δ_v to the K_0-class of the vertex projection $s_{(v,0)}$, and likewise for F. Consider the map ψ_E defined in (5.2) and the map ϕ_E appearing in (5.3). It is clear that $\phi_E = (\psi_E)_* \circ \varepsilon_E$ and similarly for F. So it suffices to show that the following diagram commutes.

$$\begin{tikzcd} \mathbf{Z}E^0 \arrow[r, "m\cdot p^*"] \arrow[d, "\varepsilon_E"'] & \mathbf{Z}F^0 \arrow[d, "\varepsilon_F"'] \\ K_0(C^*(E^* \times_d \mathbf{Z})) \arrow[r, "(\iota_{\tilde{p},\tilde{\mathfrak{s}}})_*"] \arrow[d, "(\psi_E)_*"'] & K_0(C^*(F^* \times_d \mathbf{Z})) \arrow[d, "(\psi_F)_*"'] \\ K_0(C^*(E^*) \times_{\gamma_E} \mathbf{T}) \arrow[r, "(\widetilde{\iota_{p,\mathfrak{s}}})_*"] & K_0(C^*(F^*) \times_{\gamma_F} \mathbf{T}) \end{tikzcd} \tag{5.5}$$

If one applies the K-functor to all terms and maps in the diagram of Lemma 5.3(3), and then applies the natural isomorphism

$$K_*(M_m(C^*(E^*) \times_{\gamma_E} \mathbf{T})) \cong K_*(C^*(E^*) \times_{\gamma_E} \mathbf{T})$$

to the terms on the right, one obtains precisely the bottom rectangle of (5.5). The bottom rectangle of (5.5) therefore commutes by naturality of the K-functor together with Lemma 5.3(3).

To see that the top rectangle of (5.5) commutes, recall that ε_E takes the image of the point-mass δ_v in the direct limit $\varinjlim(\mathbf{Z}E^0, A^t)$ to the class of the projection $s_{(v,0)}$. The image of $s_{(v,0)}$ under the homomorphism $\iota_{\tilde{p},\tilde{\mathfrak{s}}}$ is the diagonal matrix in $M_m(C^*(F^* \times_d \mathbf{Z}))$ whose diagonal entries are all equal to $\sum_{p(w)=v} s_{(w,0)}$. Under the standard isomorphism $K_0(M_m(C^*(F^* \times_d \mathbf{Z}))) \cong K_0(C^*(F^* \times_d \mathbf{Z}))$, we therefore obtain the following equality in $K_0(C^*(F^* \times_d \mathbf{Z}))$:

$$[\iota_{\tilde{p},\tilde{\mathfrak{s}}}(s_{(v,0)})] = \sum_{p(w)=v} m \cdot [s_{(w,0)}] = m \cdot \Big(\sum_{p(w)=v} [s_{(w,0)}] \Big).$$

Using once again the characterisation of the maps ε_E and ε_F, we see that this is precisely the statement that the bottom rectangle of (5.5) commutes. $\square$

5.2. Coverings of higher-rank graphs and Kasparov's spectral sequence theorem. We turn to the case where $k > 1$. We invoke the K-theory computations of [14] which are based on Kasparov's spectral sequence theorem for the computation of the K-theory of crossed products by groups for which the Baum-Connes conjecture holds (see [18, Theorem 6.10], [14, Lemma 3.3] and [35]). We are grateful to Gennadi Kasparov for pointing out that the spectral sequence is natural.

The standard notation for spectral sequences is that a spectral sequence (E^r, d^r) has terms $E^r_{p,q}$ and differentials $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ where $r > 0$ and $p, q \in \mathbf{Z}$. This however is problematic in the current situation because p clashes with our notation for a covering map. To avoid this, we replace the indexing variables p, q in the spectral sequence with a, b. That is, our spectral sequences have terms $E^r_{a,b}$ and differentials $d^r : E^r_{a,b} \to E^r_{a-r,b+r-1}$ where $r > 0$ and $a, b \in \mathbf{Z}$.

Since each higher rank graph C^*-algebra $C^*(\Lambda)$ is Morita equivalent to a crossed product by $\mathbf{Z}^k$ [21, Theorem 5.6], Kasparov's result applies to give a spectral sequence which converges to $K_*(C^*(\Lambda))$ with E^2 terms given by the homology of $\mathbf{Z}^k$ with appropriately chosen coefficients. In [14] Evans computes these homology groups using a resolution related to the Koszul complex. It follows that the above spectral sequence may be extended so that the terms of the resolution become the terms $E^1_{a,b}$ for b even.

The main result of this subsection is to show that given a finite covering $p : \Gamma \to \Lambda$ of row-finite k-graphs with no sources, a multiplicity m and a cocycle $\mathfrak{s} : \Gamma \to S_m$, there is a natural morphism of spectral sequences defined on E^1 terms using $m \cdot p^* : \mathbf{Z}\Lambda^0 \to \mathbf{Z}\Gamma^0$ which is compatible (see [41, p. 126]) with $(\iota_{p,\mathfrak{s}})_*$ the induced map on K-theory. This result is specialised to the case $k = 2$ with a view to applications in Section 6.

The following is an immediate Corollary of [18, Theorem 6.10] (see [14, Lemma 3.3] and [35]). For more detail on spectral sequences used in this context, see [35, 14].

Proposition 5.4. *Let $\mathcal{F}$ be a C^*-algebra and let $\alpha : \mathbf{Z}^k \to Aut\mathcal{F}$ be an action of $\mathbf{Z}^k$ on $\mathcal{F}$. Then there is a spectral sequence (E^r, d^r) with differentials $d^r : E^r_{a,b} \to E^r_{a-r,b+r-1}$ which converges to $K_*(\mathcal{F} \times_\alpha \mathbf{Z}^k)$ with $E^2_{a,b} = H_a(\mathbf{Z}^k, K_b(\mathcal{F}))$. Moreover, the spectral sequence is natural with respect to equivariant maps of C^*-algebras.*

Proof. As noted in the proof of [14, Lemma 3.3] this follows immediately from [18, Theorem 6.10] since $\mathbf{Z}^k$ is amenable and the Baum-Connes conjecture is known to hold for amenable groups [16, Theorem 1.1], so the γ part of $K_*(\mathcal{F} \times_\alpha \mathbf{Z}^k)$ exhausts. The naturality of the spectral sequence with respect to equivariant maps follows from the construction in the proof of [18, Theorem 6.10], since every step is functorial. □

Naturality means that given $\mathbf{Z}^k$ actions α_i on $\mathcal{F}_i$, a $\mathbf{Z}^k$-equivariant map $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ induces a morphism of spectral sequences and this morphism is compatible with

$$\widehat{\varphi}_* : K_*(\mathcal{F}_1 \times_{\alpha_1} \mathbf{Z}^k) \to K_*(\mathcal{F}_2 \times_{\alpha_2} \mathbf{Z}^k)$$

where $\widehat{\varphi} : \mathcal{F}_1 \times_{\alpha_1} \mathbf{Z}^k \to \mathcal{F}_2 \times_{\alpha_2} \mathbf{Z}^k$ is the natural map.

Evans applies this when $\mathcal{F} = \mathcal{F}_\Lambda$ is the crossed product $C^*(\Lambda) \times_\gamma \mathbf{T}^k$ of $C^*(\Lambda)$ by the gauge action, and α is the dual action $\hat{\gamma}$ of $\mathbf{Z}^k$. Hence, by Takai duality we have $K_*(C^*(\Lambda)) = K_*(\mathcal{F}_\Lambda \times_\alpha \mathbf{Z}^k)$. In this case we have more specific results (see [14, Lemma 3.3]):

$$E^2_{a,b} = \begin{cases} H_a(\mathbf{Z}^k, K_0(\mathcal{F}_\Lambda)) & \text{if } 0 \le a \le k \text{ and } b \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

In [14, Theorem 3.14]), Evans shows that these homology groups may be computed as the homology of the complex $D^\Lambda_* = \bigwedge^* \mathbf{Z}^k \otimes \mathbf{Z}\Lambda^0$. That is, $D^\Lambda_a = \bigwedge^a \mathbf{Z}^k \otimes \mathbf{Z}\Lambda^0$ for $0 \le a \le k$ and $D^\Lambda_a = 0$ for $a > k$. For $1 \le j \le k$ let M_j denote the vertex connectivity matrix of the coordinate graph $(\Lambda^0, \Lambda^{e_j}, r, s)$. For $1 \le a \le k$ define the differential $\partial_a : D^\Lambda_a \to D^\Lambda_{a-1}$ by

$$\partial_a(\epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_a} \otimes e_v) = \sum_{j=1}^{a} (-1)^{j+1} \epsilon_{i_1} \wedge \cdots \wedge \widehat{\epsilon}_{i_j} \wedge \cdots \wedge \epsilon_{i_a} \otimes (1 - M_j^t) e_v$$

where $\epsilon_1, \ldots, \epsilon_k$ constitute the canonical basis for $\mathbf{Z}^k$, $1 \le i_1 < \cdots < i_a \le k$ and $v \in \Lambda^0$. It is straightforward to verify that D^Λ_* is a complex. The first part of the following theorem is a restatement of [14, Theorem 3.15]).

Theorem 5.5. *Fix $k > 1$. Let Λ be a row-finite k-graph with no sources. With notation as above there is a spectral sequence (E^r, d^r) with differentials $d^r : E^r_{a,b} \to E^r_{a-r,b+r-1}$ which converges to $K_*(C^*(\Lambda)) = K_*(\mathcal{F}_\Lambda \times_\alpha \mathbf{Z}^k)$ with*

$$E^1_{a,b} = D^\Lambda_a := \bigwedge^a \mathbf{Z}^k \otimes \mathbf{Z}\Lambda^0,$$

if $0 \le a \le k$ and b is even, and 0 otherwise. The differential $d^1 : E^1_{a,b} \to E^1_{a-1,b}$ is given by ∂_a if b is even.

Let $(\Lambda, \Gamma, p, m, \mathfrak{s})$ be a row-finite covering system of k-graphs with no sources. There is a morphism f of spectral sequences which is compatible with $(\iota_{p,\mathfrak{s}})_* : K_*(C^*(\Lambda)) \to K_*(C^*(\Gamma))$ such that $f^1 : D^\Lambda_a \to D^\Gamma_a$ is given by $\mathrm{id} \otimes (m \cdot p^*)$.

Proof. Evans computes the homology groups using a Koszul complex (see [41, §4.5]). Set $G = \mathbf{Z}^k = \langle s_1, \ldots s_k \rangle$, $R = \mathbf{Z}G$ and let I be the ideal in R generated by $\{1 - s_a^{-1} : 1 \le a \le k\}$. Let $\epsilon_1, \ldots, \epsilon_k$ constitute the canonical basis for R^k. For each a, define $\partial_a : \bigwedge^a R^k \to \bigwedge^{a-1} R^k$ as follows: for $1 \le i_1 < \cdots < i_a \le k$ so that $\epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_a} \in \bigwedge^a R^k$, define

$$\partial_a(\epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_a}) = \sum_{j=1}^{a} (-1)^{j+1} (1 - {s_j}^{-1}) \epsilon_{i_1} \wedge \cdots \wedge \widehat{\epsilon}_{i_j} \wedge \cdots \wedge \epsilon_{i_a}$$

where the symbol "$\widehat{\cdot}$" denotes deletion of an element (note that $\partial_1(\epsilon_i) = 1 - s_i^{-1}$).

Then $R/I \cong \mathbf{Z}$ and the following sequence of R-modules is exact (see [41, Corollary 4.5.5])

$$0 \to \bigwedge^k R^k \to \cdots \to \bigwedge^1 R^k \to \bigwedge^0 R^k \to \mathbf{Z} \to 0.$$

Note that $\bigwedge^0 R^k = R$ and $\bigwedge^a R^k$ is a free R-module with basis

$$\{\epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_a} : 1 \le i_1 < \cdots < i_a \le k\}.$$

Hence, $\bigwedge^* R^k$ yields a projective resolution of $\mathbf{Z}$. Thus, by [6, §III.1] we have

$$H_*(G, K_0(\mathcal{F}_\Lambda)) \cong H_*(\bigwedge^* R^k \otimes_G K_0(\mathcal{F}_\Lambda)).$$

We follow Evans here but have adopted slightly different notation to make naturality more apparent (see [14, Definition 3.11] and following). Under the isomorphism $\bigwedge^a R^k \otimes_G K_0(\mathcal{F}_\Lambda) \cong \bigwedge^a \mathbf{Z}^k \otimes K_0(\mathcal{F}_\Lambda)$ (as abelian groups), the boundary map $\partial_a : \bigwedge^a \mathbf{Z}^k \otimes K_0(\mathcal{F}_\Lambda) \to \bigwedge^{a-1} \mathbf{Z}^k \otimes K_0(\mathcal{F}_\Lambda)$ is given by

$$\partial_a(\epsilon_{i_1} \wedge \cdots \wedge \epsilon_{i_a} \otimes x) = \sum_{j=1}^{a} (-1)^{a+1} \epsilon_{i_1} \wedge \cdots \wedge \widehat{\epsilon}_{i_j} \wedge \cdots \wedge \epsilon_{i_a} \otimes (1 - s_j)x$$

where $1 \le i_1 < \cdots < i_a \le k$ and $x \in K_0(\mathcal{F}_\Lambda)$.

Let D_a^Λ be given as above. There is a natural map $\varepsilon^\Lambda : C_0(\Lambda^0) \hookrightarrow \mathcal{F}_\Lambda$ which induces a map $\varepsilon_*^\Lambda : \mathbf{Z}\Lambda^0 \to K_0(\mathcal{F}_\Lambda)$. Moreover (see [14, Theorem 3.14]) the natural map

$$\mathrm{id} \otimes \varepsilon_*^\Lambda : \bigwedge^* \mathbf{Z}^k \otimes \mathbf{Z}\Lambda^0 \to \bigwedge^* \mathbf{Z}^k \otimes K_0(\mathcal{F}_\Lambda)$$

is a map of complexes which induces an isomorphism on homology and hence

$$H_*(G, K_0(\mathcal{F}_\Lambda)) \cong H_*(\bigwedge^* \mathbf{Z}^k \otimes \mathbf{Z}\Lambda^0).$$

Therefore, setting

$$E^1_{a,b} = \begin{cases} \bigwedge^a \mathbf{Z}^k \otimes \mathbf{Z}\Lambda^0 & \text{if } 0 \le a \le k \text{ and } b \text{ is even,} \\ 0 & \text{otherwise} \end{cases}$$

and defining $d^1 : E^1_{a,b} \to E^1_{a-1,b}$ to be ∂_a if b is even (and 0 otherwise), yields

$$E^2_{a,b} \cong \begin{cases} H_p(G, K_0(\mathcal{F}_\Lambda)) & \text{if } 0 \le a \le k \text{ and } b \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

It follows by [14, Lemma 3.3] that the spectral sequence converges to $K_*(C^*(\Lambda)) = K_*(\mathcal{F}_\Lambda \times_\alpha \mathbf{Z}^k)$ as required.

For the second part of the theorem, fix $(\Lambda, \Gamma, p, m, \mathfrak{s})$. The embedding $\iota_{p,\mathfrak{s}} : C^*(\Lambda) \to M_m(C^*(\Gamma))$ induces an embedding $\widetilde{\iota_{p,\mathfrak{s}}} : \mathcal{F}_\Lambda \to M_m(\mathcal{F}_\Gamma)$. Functoriality yields a map of complexes

$$\mathrm{id} \otimes (\widetilde{\iota_{p,\mathfrak{s}}})_* : \bigwedge^* \mathbf{Z}^k \otimes K_0(\mathcal{F}_\Lambda) \to \bigwedge^* \mathbf{Z}^k \otimes K_0(\mathcal{F}_\Gamma).$$

Since group homology is a covariant functor of its coefficient module we obtain the functorial maps for each $n = 0, 1, \ldots, k$

$$H_n((\widetilde{\iota_{p,\mathfrak{s}}})_*) : H_n(\mathbf{Z}^k, K_0(\mathcal{F}_\Lambda)) \to H_n(\mathbf{Z}^k, K_0(\mathcal{F}_\Gamma)).$$

Then arguing as in Lemma 5.2 with $p^* : \mathbf{Z}\Lambda^0 \to \mathbf{Z}\Gamma^0$ defined as above we see that

$$(1 - (M_j^\Gamma)^t)(m \cdot p^*) = (m \cdot p^*)(1 - (M_j^\Lambda)^t)$$

for all $j = 1, \ldots, k$. It follows that the natural map

$$\mathrm{id} \otimes (m \cdot p^*) : \bigwedge^* \mathbf{Z}^k \otimes \mathbf{Z}\Lambda^0 \to \bigwedge^* \mathbf{Z}^k \otimes \mathbf{Z}\Gamma^0$$

is a map of complexes.

Arguing as in the proof of Theorem 5.1, we see that $(\widetilde{\iota_{p,\mathfrak{s}}})_* \circ \varepsilon_*^\Lambda = \varepsilon_*^\Gamma \circ (m \cdot p^*)$, so the map on homology induced by $\mathrm{id} \otimes (m \cdot p^*)$ coincides with the functorial map above (under the identifications of the homology groups induced by $\mathrm{id} \otimes \varepsilon_*^\Lambda$ and $\mathrm{id} \otimes \varepsilon_*^\Gamma$). This combined with the naturality of Proposition 5.4 yields a morphism f of spectral sequences compatible with the map

$$\left(\widehat{\widetilde{\iota_{p,\mathfrak{s}}}}\right)_* : K_*(\mathcal{F}_\Lambda \times_\alpha \mathbf{Z}^k) \to K_*(\mathcal{F}_\Gamma \times_\alpha \mathbf{Z}^k)$$

such that $f^1 : D_a^\Lambda \to D_a^\Gamma$ is given by $\mathrm{id} \otimes (m \cdot p^*)$. Under the identifications $K_*(C^*(\Lambda)) = K_*(\mathcal{F}_\Lambda \times_\alpha \mathbf{Z}^k)$ and $K_*(C^*(\Gamma)) = K_*(\mathcal{F}_\Gamma \times_\alpha \mathbf{Z}^k)$, we have $\left(\widehat{\widetilde{\iota_{p,\mathfrak{s}}}}\right)_* = (\iota_{p,\mathfrak{s}})_*$. □

The following corollary is an immediate consequence of the above theorem restricted to the case $k = 2$; for the first assertion see [14, Proposition 3.16] and its proof (see also [35]).

Given a 2-graph Λ, recall that M_1 and M_2 denote the vertex connectivity matrices of the coordinate graphs $(\Lambda^0, \Lambda^{e_1}, r, s)$ and $(\Lambda^0, \Lambda^{e_2}, r, s)$.

Corollary 5.6. *Suppose that $(\Lambda, \Gamma, p, m, \mathfrak{s})$ is a row-finite covering system of 2-graphs with no sources. With the notation of Theorem 5.5, the complex $D_a^\Lambda = \bigwedge^a \mathbf{Z}^2 \otimes \mathbf{Z}\Lambda^0$ may be written as follows:*

$$0 \leftarrow \mathbf{Z}\Lambda^0 \xleftarrow{\partial_1} \mathbf{Z}\Lambda^0 \oplus \mathbf{Z}\Lambda^0 \xleftarrow{\partial_2} \mathbf{Z}\Lambda^0 \leftarrow 0 \tag{5.6}$$

where $\partial_1 = (1 - M_1^t, 1 - M_2^t)$ and $\partial_2 = \begin{pmatrix} M_2^t - 1 \\ 1 - M_1^t \end{pmatrix}$. We have $E_{a,b}^2 = E_{a,b}^\infty$, and

$$\begin{aligned} K_0(C^*(\Lambda)) &\cong \operatorname{coker} \partial_1 \oplus \ker \partial_2 \\ K_1(C^*(\Lambda)) &\cong \ker \partial_1 / \operatorname{Im} \partial_2 \cong H_1(\mathbf{Z}^k, K_0(\mathcal{F}_\Lambda)). \end{aligned} \tag{5.7}$$

Moreover, the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longleftarrow & \mathbf{Z}\Lambda^0 & \xleftarrow{\partial_1^\Lambda} & \mathbf{Z}\Lambda^0 \oplus \mathbf{Z}\Lambda^0 & \xleftarrow{\partial_2^\Lambda} & \mathbf{Z}\Lambda^0 & \longleftarrow & 0 \\ & & \Big\downarrow{\scriptstyle m\cdot p^*} & & \Big\downarrow{\scriptstyle m\cdot p^* \oplus m\cdot p^*} & & \Big\downarrow{\scriptstyle m\cdot p^*} & & \\ 0 & \longleftarrow & \mathbf{Z}\Gamma^0 & \xleftarrow{\partial_1^\Gamma} & \mathbf{Z}\Gamma^0 \oplus \mathbf{Z}\Gamma^0 & \xleftarrow{\partial_2^\Gamma} & \mathbf{Z}\Gamma^0 & \longleftarrow & 0 \end{array} \tag{5.8}$$

and by naturality induces $(\iota_{p,\mathfrak{s}})_* : K_*(C^*(\Lambda)) \to K_*(C^*(\Gamma))$.

The inclusion of $\operatorname{coker} \partial_1$ into $K_0(C^*(\Lambda))$ obtained from (5.7) takes the equivalence class (in the quotient group $\operatorname{coker} \partial_1 = \mathbf{Z}\Lambda^0 / \operatorname{Im}(\partial_1)$) of the generator δ_v of $\mathbf{Z}\Lambda^0$ to the K_0-class of the vertex projection $[s_v]$ in $C^*(\Lambda)$. The proof of this fact can be obtained from the proof of [14, Proposition 4.4]. We thank Gwion Evans for pointing this out to us.

5.3. Product coverings and the Künneth formula. In this section we consider covering systems (Λ_n, p_n) in which each k-graph Λ_n is a cartesian product of two lower-dimensional graphs, and the covering maps p_n respect the product decomposition.

Recall from [20, Proposition 1.8] that given a k-graph (Λ, d) and a k'-graph (Λ', d'), the cartesian-product category $\Lambda \times \Lambda'$ becomes a $(k+k')$-graph when we endow it with the degree functor $d \times d' : (\lambda, \lambda') \mapsto (d(\lambda)_1, \dots, d(\lambda)_k, d'(\lambda')_1, \dots, d'(\lambda')_{k'})$.

Proposition 5.7. *Fix $k, k' \in \mathbf{N} \setminus \{0\}$. Let $(\Lambda, \Gamma, p, m, \mathfrak{s})$ and $(\Lambda', \Gamma', p', m', \mathfrak{s}')$ be row-finite covering systems of k- and k'-graphs with no sources. Then*

$$p \times p' : \Gamma \times \Gamma' \to \Lambda \times \Lambda'$$

is a finite covering of row-finite $(k+k')$-graphs with no sources. Let $f : \{1, \dots, m\} \times \{1 \dots, m'\} \to \{1, \dots, mm'\}$ denote the bijection $f(j, j') := j + (j'-1)m$. There is a cocycle $\mathfrak{s} \times \mathfrak{s}' : \Gamma \times \Gamma' \to S_{mm'}$ determined by $\big((\mathfrak{s} \times \mathfrak{s}')(\alpha, \alpha')\big) f(j, j') := f\big(\mathfrak{s}(\alpha) j, \mathfrak{s}'(\alpha') j'\big)$. Moreover, the following diagram commutes.

$$\begin{array}{ccc}
C^*(\Lambda \times \Lambda') & \xrightarrow{\cong} & C^*(\Lambda) \otimes C^*(\Lambda') \\
\Big\downarrow{\scriptstyle \iota_{p \times p', \mathfrak{s} \times \mathfrak{s}'}} & & \Big\downarrow{\scriptstyle \iota_{p,\mathfrak{s}} \otimes \iota_{p',\mathfrak{s}'}} \\
M_{mm'}(C^*(\Gamma \times \Gamma')) & \xrightarrow{\cong} & M_m(C^*(\Gamma)) \otimes M_{m'}(C^*(\Gamma'))
\end{array}$$

Suppose that at least one of $K_*(C^*(\Lambda))$, $K_*(C^*(\Lambda'))$ and at least one of $K_*(C^*(\Gamma))$, $K_*(C^*(\Gamma'))$ are torsion-free. Then the following diagram commutes and the horizontal connecting maps are zero-graded isomorphisms:

$$\begin{array}{ccc}
K_*(C^*(\Lambda)) \otimes K_*(C^*(\Lambda')) & \xrightarrow{\cong} & K_*(C^*(\Lambda \times \Lambda')) \\
\Big\downarrow{\scriptstyle (\iota_{p,\mathfrak{s}})_* \otimes (\iota_{p',\mathfrak{s}'})_*} & & \Big\downarrow{\scriptstyle (\iota_{p \times p', \mathfrak{s} \times \mathfrak{s}'})_*} \\
K_*(C^*(\Gamma)) \otimes K_*(C^*(\Gamma')) & \xrightarrow{\cong} & K_*(C^*(\Gamma \times \Gamma'))
\end{array}$$

If Γ^0 and $(\Gamma')^0$ (and hence also Λ^0 and $(\Lambda')^0$) are finite then the C^*-algebras are unital, and the horizontal isomorphisms take $[1] \otimes [1]$ to $[1]$.

Proof. It is straightforward to check that $p \times p'$ is a covering using the properties of the covering maps p and p' and the definition of the cartesian-product graph. A simple calculation shows that $\mathfrak{s} \times \mathfrak{s}'$ defines a cocycle.

Theorem 5.5 of [20] shows that $C^*(\Lambda)$, $C^*(\Lambda')$, $C^*(\Gamma)$ and $C^*(\Gamma')$ are nuclear, and so there is just one tensor-product C^*-algebra $C^*(\Lambda) \otimes C^*(\Lambda')$. Corollary 3.5(iv) of [20] shows that the map $s_{(\lambda,\mu)} \mapsto s_\lambda \otimes s_\mu$ is an isomorphism of $C^*(\Lambda \times \Lambda')$ onto $C^*(\Lambda) \otimes C^*(\Lambda')$, and similarly for $C^*(\Gamma)$ and $C^*(\Gamma')$. It is easy to check using the formulae for the maps $\iota_{p,\mathfrak{s}}, \iota_{p',\mathfrak{s}'}$, and $\iota_{p \times p', \mathfrak{s} \times \mathfrak{s}'}$ and using the chain of isomorphisms

$$\begin{aligned} M_{mm'}(C^*(\Gamma \times \Gamma')) &\cong M_{mm'}(\mathbf{C}) \otimes C^*(\Gamma \times \Gamma') \\ &\cong M_m(\mathbf{C}) \otimes C^*(\Gamma) \otimes M_{m'}(\mathbf{C}) \otimes C^*(\Gamma') \\ &\cong M_m(C^*(\Gamma)) \otimes M_{m'}(C^*(\Gamma')) \end{aligned}$$

that the first diagram commutes.

In the presence of the additional hypothesis concerning torsion-free K-groups, the Künneth Theorem of [37] (see also Theorem 23.1.3 of [4]) implies: (1) that

$$K_*(C^*(\Lambda)) \otimes K_*(C^*(\Lambda')) \cong K_*(C^*(\Lambda) \otimes C^*(\Lambda'))$$

and similarly for Γ, Γ'; (2) that these isomorphisms are natural and are zero-graded; and (3) that these isomorphisms take $[1]\otimes[1]$ to $[1]$. The result therefore follows from the naturality of the K-functor. □

Note that in general when no assumption is made about torsion, the Künneth Theorem of [37] gives a short exact sequence which is still natural. The analogue of Proposition 5.7 still holds and gives a (fairly complicated) commuting diagram in which the rows are short exact sequences.

6. Examples

In this section we discuss a number of examples. A recurring theme will be supernatural numbers and the associated dimension groups, so we pause here to establish some notation.

We will think of a supernatural number as an infinite product $\alpha = \prod_{n=1}^\infty \alpha_n$ where each α_n is an integer greater than 1. Any two such expressions in which the same prime factors occur with the same cardinality correspond to the same supernatural number. Given supernatural numbers α, β, we will abuse notation and write $\alpha\beta$ for the supernatural number $\prod_{n=1}^\infty \alpha_n\beta_n$. We write $\alpha[1,n]$ for the product $\prod_{i=1}^n \alpha_i$ of the first n terms in α.

For $z_1, \dots, z_n \in \mathbf{C}$, we write $\mathbf{Z}[z_1, \dots, z_n]$ for the ring obtained by adjoining $z_1, \dots, z_n$ to $\mathbf{Z}$; we regard $\mathbf{Z}[z_1, \dots, z_n]$ as a group under addition. Abusing notation, for a supernatural number α, we write $\mathbf{Z}\big[\frac{1}{\alpha}\big]$ for the dimension group $\varinjlim(\mathbf{Z}, \times\alpha_n)$ which we identify with the group

$$\bigcup_{n=1}^\infty \mathbf{Z}\Big[\frac{1}{\alpha[1,n]}\Big] \subset \mathbf{Q}$$

consisting of all fractions p/q where $p, q \in \mathbf{Z}$, and q is a divisor of some $\alpha[1,n]$.

6.1. Rank-2 Bratteli diagrams. A rank-2 Bratteli diagram is a 2-graph in which the blue edges form a Bratteli diagram and the red edges determine simple cycles so that every vertex lies on precisely one red cycle, and all vertices on a given red cycle are at the same level in the blue Bratteli diagram.

The C^*-algebras of these 2-graphs were studied in [27] and provided the initial motivation for the covering construction. A rank-2 Bratteli diagram Λ can be constructed using Proposition 2.14 and Corollary 2.15 precisely when the length of each red cycle at level n of Λ is divisible by the lengths of all the cycles at level $n-1$ to which it connects. In particular, the 2-graphs whose C^*-algebras are Morita equivalent to the Bunce-Deddens algebras [27, Example 6.7] and the irrational rotation algebras [27, Example 6.5] arise in this fashion.

6.2. Coverings of dihedral graphs $\boldsymbol{D_n}$. For $n \in \mathbf{N} \setminus \{0\}$, let D_n be the directed graph with n vertices $\{v_0, \dots, v_{n-1}\}$ and edges $\{x_i, y_i : 0 \le i \le n-1\}$ where $r(x_i) = v_i = s(y_i)$ and $s(x_i) = v_{i+1} = r(y_i)$ (throughout this section, addition in the subscripts is understood to be evaluated modulo n). More descriptively, D_n is a ring of n vertices, each of which connects to both of its neighbours (see Figure 2). Let D_n^* be the path-category of D_n, regarded as a 1-graph. Note that for $n \in \mathbf{N} \setminus \{0\}$, the graph D_{2n} is the Cayley graph for the

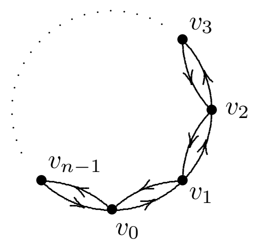

Figure 2. The 1-graph D_n

dihedral group with $2n$ elements.

Example 6.1. For $n, m \ge 1$ there are m-fold covering maps $p_{n,mn} : D_{mn}^* \to D_n^*$ as follows: for $0 \le i \le mn-1$ let $i' = i \mod n$ and define

$$p_{n,mn}(v_i) := v_{i'}, \quad p_{n,mn}(x_i) := x_{i'} \quad \text{and} \quad p_{n,mn}(y_i) := y_{i'}.$$

Hence for each pair of positive integers n, m, we obtain a row-finite covering system $(D_n^*, D_{mn}^*, p_{n,mn})$ of 1-graphs with no sources (see Notation 2.8).
Fix an infinite supernatural number $\alpha = \prod_{i=1}^{\infty} \alpha_i$. Consider the sequence of covering systems $(D^*_{6\alpha[1,n]}, D^*_{6\alpha[1,n+1]}, p_{6\alpha[1,n],6\alpha[1,n+1]})_{n=1}^{\infty}$ as in Notation 2.8. Applying Corollary 2.11, we obtain a 2-graph

$$D := \varprojlim(D^*_{6\alpha[1,n]}, p_{6\alpha[1,n],6\alpha[1,n+1]}).$$

Proposition 6.2. *Consider the situation discussed in Example 6.1. We have* $K_0(C^*(D)) = \mathbf{Z}[\frac{1}{\alpha}] \oplus \mathbf{Z}[\frac{1}{\alpha}]$ *and* $K_1(C^*(D)) = \mathbf{Z} \oplus \mathbf{Z}$. *Let* $P_1 := \sum_{v \in D_6^0} s_v$. *Then* $[P_1]$ *is the 0 element of* $K_0(P_1 C^*(D) P_1)$. *Moreover,* $C^*(D)$ *is simple and purely infinite.*

Before proving the proposition, we describe the K-theory of $C^*(D_n^*)$ in general.

Lemma 6.3. (1) $K_0(C^*(D_n^*))$ *is generated by* $[s_{v_0}]$ *and* $[s_{v_1}]$, *and for each* i, *we have* $[s_{v_i}] = -[s_{v_{i+3}}]$ *in* $K_0(C^*(D_n^*))$.
(2) $K_1(C^*(D_n^*)) \cong \{(a_1, \dots, a_n) \in \mathbf{Z}^n : a_{i+2} = a_{i+1} - a_i$ *for all* $i\}$.

(3) *the following table describes the K-theory of each $C^*(D_n^*)$.*

n mod 6	$K_0(C^*(D_n^*))$	$K_1(C^*(D_n^*))$
0	$\mathbf{Z}^2$	$\mathbf{Z}^2$
1	0	0
2	$\mathbf{Z}/3\mathbf{Z}$	0
3	$\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$	0
4	$\mathbf{Z}/3\mathbf{Z}$	0
5	0	0

Proof. (1) The K_0 group is generated by the classes $[s_{v_0}], \dots, [s_{v_{n-1}}]$ subject to the relations $[s_{v_i}] = [s_{v_{i+1}}] + [s_{v_{i-1}}]$. This relation forces $[s_{v_{i+2}}] = [s_{v_{i+1}}] - [s_{v_i}]$, from which we conclude first that K_0 is generated by $[s_{v_0}]$ and $[s_{v_1}]$ and second that

$$[s_{v_{i+3}}] = [s_{v_{i+2}}] - [s_{v_{i+1}}] = ([s_{v_{i+1}}] - [s_{v_i}]) - [s_{v_{i+1}}] = -[s_{v_i}].$$

(2) Let A_n denote the vertex connectivity matrix of D_n; so $A_n(i,j) = 1$ when $i = j \pm 1 \pmod n$ and zero otherwise. As in Theorem 5.1, we have $K_1(C^*(D_n^*)) \cong \ker(1 - A_n^t)$. For $m \in \mathbf{Z}^n$, $((1 - A_n^t)m)_i = -m_{i-1} + m_i - m_{i+1}$ by definition of A_n, and this establishes (2).
(3) If E is a finite 1-graph with no sinks or sources, then $C^*(E)$ is isomorphic to the Cuntz-Krieger algebra of the adjacency matrix A_E of E [23]. In particular, $K_1(C^*(E))$ is torsion-free and has the same rank as $K_0(C^*(E))$ [9]. Hence it suffices to verify that the first column of the table is correct. To calculate K_0, we use (1) to check by hand that the cases $n = 1, 2, \dots 6$ are as claimed. If $n > 6$, then applying the relations we find that $[s_{v_{i+6}}] = [s_{v_i}]$ for all i which accounts for all remaining cases. $\square$

Proof of Proposition 6.2. Lemma 6.3(1) shows that $K_0(C^*(D^*_{6\alpha[1,n]})$ is generated by $[s_{v_0^n}]$ and $[s_{v_1^n}]$ where the v_i^n are the vertices of $D^*_{6\alpha[1,n]}$. Fix $i \in \{1, 2\}$. We have

$$(\iota_{p_n})_*[s_{v_i^n}] = [s_{v_i^{n+1}}] + [s_{v^{n+1}_{i+6\alpha[1,n]}}] + \cdots + [s_{v^{n+1}_{i+6(\alpha_{n+1}-1)\alpha[1,n]}}]. \tag{6.1}$$

By Lemma 6.3(1), each $[s_{v_i^{n+1}+6k}] = [s_{v_i^{n+1}}]$ in $K_0(C^*(D^*_{6\alpha[1,n+1]}))$, so (6.1) implies $(\iota_{p_n})_*[s_{v_i^n}] = \alpha_n \cdot [s_{v_i^{n+1}}]$. Hence $K_0(\iota_{p_n}) : \mathbf{Z}^2 \to \mathbf{Z}^2$ is multiplication by α_n.
Fix $m \in \mathbf{N} \setminus \{0\}$. By Lemma 6.3(2), $K_1(C^*(D^*_{6m}))$ is identified with the set of sequences $(a_1, \dots, a_{6m})$ which satisfy $a_{i+2} = a_{i+1} - a_i$ for all i. By Lemma 6.3(2), this forces $a_{i+2} = a_{i+1} - a_i$ for all i. Consequently, the map $a = (a_1, \dots, a_{6m}) \mapsto (a_1, a_2)$ yields an isomorphism $\zeta_m : K_1(C^*(D^*_{6m})) \to \mathbf{Z}^2$. As $\zeta_{\alpha[1,n+1]} \circ K_1(\iota_{p_{6\alpha[1,n],6\alpha[1,n+1]}}) = \zeta_{\alpha[1,n]}$, it follows that $K_1(\iota_{p_n}) : \mathbf{Z}^2 \to \mathbf{Z}^2$ is the identity map.
Recall that D denotes $\varprojlim(D^*_{6\alpha[1,n]}, p_{6\alpha[1,n],6\alpha[1,n+1]})$. By Theorem 5.1 the K-groups of $C^*(D)$ are as claimed. To compute the class of the identity, let $P_1 \in C^*(D)$ be the sum of the six vertex projections in the bottom level. The final statement of Lemma 6.3(1) shows that the classes of the vertex projections

in $K_0(C^*(D_6^*))$ cancel, so that the class of the identity in $K_0(C^*(D_6^*))$ is the zero element. It follows that the class of the identity P_1 in $K_0(P_1C^*(D)P_1)$ is also the zero element.

Each D_n^* is aperiodic and cofinal (see Definition 4.2), so we may conclude from Corollary 4.4 and Lemma 4.7 that D is aperiodic and cofinal. Hence Proposition 4.8 of [20] implies that $C^*(D)$ is simple. The path x_1y_1 is a cycle with an entrance (namely y_0) in D_1^*. Proposition 4.8 now shows that $C^*(D)$ is purely infinite. □

6.3. Direct limits of $\mathcal{O}_n \otimes C(\mathbf{T})$.

Example 6.4. Fix $n \geq 3$, and let B_n be the bouquet of n loops. For $m \geq 1$, let L_m denote the loop with m vertices, and let Λ_m be the cartesian-product 2-graph $\Lambda_m = L^*_{(n-1)^m} \times B_n^*$ obtained from the path categories of $L_{(n-1)^m}$ and B_n.

For each m, Let p_m denote the obvious $(n-1)$-fold covering of $L^*_{(n-1)^m}$ by $L^*_{(n-1)^{m+1}}$, and let p' be the identity covering of B_n by B_n.

Proposition 6.5. *Consider the situation of Example 6.4. Let v be a vertex of Λ_1. Then $s_vC^*(\varprojlim(\Lambda_m, p_m \times p'))s_v$ is isomorphic to the Kirchberg algebra $\mathcal{P}_n$ (see* [5]*) whose K-theory is opposite to that of $\mathcal{O}_n$.*

Proof. Since $C^*(B_n)$ is generated by n isometries whose range projections sum to the identity, $C^*(B_n)$ is canonically isomorphic to $\mathcal{O}_n$ [7]. Hence

$$C^*(\Lambda_m) \cong C^*(L^*_{(n-1)^m}) \otimes \mathcal{O}_n$$

by [20, Corollary 3.5(iv)]. As in [17, Lemma 2.4], there is an isomorphism $C^*(L^*_{(n-1)^m}) \cong M_{(n-1)^m}(C(\mathbf{T}))$ for each m, and in particular we have $K_*(C^*(L^*_{(n-1)^m})) \cong (\mathbf{Z}, \mathbf{Z})$. Since $K_*(\mathcal{O}_n) = (\mathbf{Z}/(n-1)\mathbf{Z}, 0)$ [9], the Künneth theorem implies that $K_*(C^*(\Lambda_m)) \cong (\mathbf{Z}/(n-1)\mathbf{Z}, \mathbf{Z}/(n-1)\mathbf{Z})$.

A special case of [27, Equation (4.7)] implies that the covering map p_m induces multiplication by $n-1$ from $K_0(C^*(L^*_{(n-1)^m}))$ to $K_0(C^*(L^*_{(n-1)^{m+1}}))$, and the identity homomorphism from $K_1(C^*(L^*_{(n-1)^m}))$ to $K_1(C^*(L^*_{(n-1)^{m+1}}))$. Clearly p' induces the identity map on $K_*(\mathcal{O}_n)$.

Let $\Lambda = \varprojlim(\Lambda_m, p_m \times p')$. Theorem 3.8 and Proposition 5.7 combine to show that

$$K_*(C^*(\Lambda)) \cong \varinjlim((\mathbf{Z}/(n-1)\mathbf{Z}, \mathbf{Z}/(n-1)\mathbf{Z}), (\times(n-1), \mathrm{id})).$$

Since multiplication by $n-1$ is the 0 homomorphism from $\mathbf{Z}/(n-1)\mathbf{Z}$ to $\mathbf{Z}/(n-1)\mathbf{Z}$, it follows that $K_*(C^*(\Lambda)) \cong (0, \mathbf{Z}/(n-1)\mathbf{Z})$.

Lemma 4.7 proves that Λ is cofinal. For an infinite path $y \in \Lambda^\infty$, Lemma 4.5 combined with the observation that the cycles in the $L^*_{(n-1)^m}$ grow with m shows that if $a, b \in \mathbf{N}^3$ and $\sigma^a(y) = \sigma^b(y)$, then a and b differ only in their first coordinates. It follows from Proposition 4.3 that the aperiodicity of Λ is implied by the well-known aperiodicity of B_n. Hence $C^*(\Lambda)$ is simple by [20, Proposition 4.8]. Moreover, since every vertex of Λ hosts a cycle with an entrance, $C^*(\Lambda)$ is also purely infinite (see [20, Proposition 4.9], [38, Proposition 8.8]). The result therefore follows from the Kirchberg-Phillips classification theorem [28]. □

6.4. Higher-rank Bunce-Deddens algebras. In this subsection we describe a class of simple A$\mathbf{T}$ algebras with real-rank 0 which arise from sequences of covering systems of 2-graphs and which cannot in general be obtained from the construction of [27] (see Example 6.6 and Theorem 6.7). We indicate in Remark 6.12 why we think of these algebras as higher-rank analogues of the Bunce-Deddens algebras.

For $k \geq 1$, let Δ_k be the k-graph with vertices $\mathbf{Z}^k$, morphisms $\{(m,n) \in \mathbf{Z}^k \times \mathbf{Z}^k : m \leq n\}$ where $r(m,n) = m$, $s(m,n) = n$ and $d(m,n) = n-m$. There is a free action of $\mathbf{Z}^k$ on Δ_k given by translation; that is $m \cdot (p,q) = (p+m, q+m)$ for $m \in \mathbf{Z}^k$ and $(p,q) \in \Delta_k$.

Given a finite-index subgroup H of $\mathbf{Z}^k$, we denote by Δ_k/H the quotient of Δ_k by the action of H. That is, for $q \in \mathbf{N}^k$, $(\Delta_k/H)^q = \{[g, g+q] : g \in \mathbf{Z}^k\}$; in particular, $(\Delta_k/H)^0 = \{[g,g] : g \in \mathbf{Z}^k\}$, and we henceforth identify $(\Delta_k/H)^0$ with $\mathbf{Z}^k/H$ via the map $[g,g] \mapsto [g]$ where $[g]$ denotes the class $g+H$ of g in $\mathbf{Z}^k/H$. The range and source maps in Δ_k/H are then given by $r([g, g+q]) = [g]$ and $s([g, g+q]) = [g+q]$. If $H' \subset H$ is a finite-index subgroup of H, then it also has finite index in $\mathbf{Z}^k$, and there is a natural surjection $p : \mathbf{Z}^k/H' \to \mathbf{Z}^k/H$ which induces a finite covering map, also denoted p of Δ_k/H by Δ_k/H'.

Most of the remainder of this section is concerned with the following example of a sequence of covering systems.

Example 6.6. Let $H_1 \supset H_2 \supset H_3 \supset \dots$ be a chain of finite-index subgroups of $\mathbf{Z}^2$. For each n, let $p_n : \Delta_2/H_{n+1} \to \Delta_2/H_n$ be the canonical covering induced by the quotient maps described above, let $m_n = 1$, and let $\mathfrak{s}_n : \Delta_2/H_{n+1} \to S_1$ be the trivial cocycle. This data specifies a sequence $(\Delta_2/H_n, \Delta_2/H_{n+1}, p_n)_{n=1}^\infty$ of row-finite covering systems of 2-graphs with no sources. Applying Corollary 2.11, we obtain a 3-graph $\varprojlim(\Delta_2/H_n; p_n))$. As always, P_1 denotes $\sum_{v \in (\Delta_2/H_1)^0} s_v \in C^*(\Delta_2/H_1) \subset C^*(\varprojlim(\Delta_2/H_n; p_n))$.

Theorem 6.7. *Consider the situation of Example 6.6.*

(1) *We have*
$$K_0(P_1 C^*(\varprojlim(\Delta_2/H_n; p_n))P_1) \cong \varinjlim(\mathbf{Z}, \times[H_n : H_{n+1}]) \oplus \mathbf{Z},$$
and this isomorphism takes $[P_1]$ *to* $(g, 0)$ *where* g *is the image of* $[\mathbf{Z}^2 : H_1]$ *in the direct limit* $\varinjlim(\mathbf{Z}, \times[H_n : H_{n+1}])$.

(2) *For each* n *the homomorphism from* $\mathbf{Z}^2$ *to* $\mathbf{Z}^2$ *determined by coordinate-wise multiplication by the integer* $[H_n : H_{n+1}]$ *restricts to a homomorphism* $m_{H_n, H_{n+1}} : H_n \to H_{n+1}$. *Moreover,*
$$K_1(P_1 C^*(\varprojlim(\Delta_2/H_n; p_n))P_1) \cong \varinjlim(H_n, m_{H_n, H_{n+1}}).$$

(3) $C^*(\varprojlim(\Delta_2/H_n; p_n))$ *is simple if and only if* $\bigcap_{n=1}^\infty H_n = \{0\}$, *and is an A*$\mathbf{T}$ *algebra with real-rank 0 when it is simple.*

The proof of this result will occupy the bulk of this section. Before presenting it, we state a Corollary and use it to formulate some concrete examples.

Corollary 6.8. *Consider the situation of Example 6.6. There are sequences* $(h_1^n)_{n=1}^\infty$ *and* $(h_2^n)_{n=1}^\infty$ *in* $\mathbf{Z}^2$ *such that: (1) for each* n*, the elements* h_1^n *and* h_2^n *generate* H_n*; and (2) the matrix* $M_n = \begin{pmatrix} m_{1,1}^n & m_{1,2}^n \\ m_{2,1}^n & m_{2,2}^n \end{pmatrix}$ *satisfying* $h_1^{n+1} = m_{1,1}^n h_1^n + m_{1,2}^n h_2^n$ *and* $h_2^{n+1} = m_{2,1}^n h_1^n + m_{2,2}^n h_2^n$ *has positive determinant for all* n*. Moreover, if* M_n^{ca} *denotes the classical adjoint* $\begin{pmatrix} m_{2,2}^n & -m_{1,2}^n \\ -m_{2,1}^n & m_{1,1}^n \end{pmatrix}$ *of* M_n *for each* n*, and if we regard these matrices as homomorphisms of* $\mathbf{Z}^2$*, then*

$$K_1(P_1 C^*(\varprojlim(\Delta_2/H_n; p_n))P_1) \cong \varinjlim(\mathbf{Z}^2, M_n^{\mathrm{ca}}). \tag{6.2}$$

Proof. That we can choose the h_i^n so that the matrices M_n all have positive determinant follows from an inductive argument based on the observation that replacing h_i^{n+1} with $-h_i^{n+1}$ reverses the sign of $\det(M_n)$.

For each n, let ψ_n be the isomorphism of $\mathbf{Z}^2$ onto H_n satisfying $\psi_n(e_i) = h_i^n$, and let $m_{H_n,H_{n+1}} : H_n \to H_{n+1}$ be the homomorphism described in Theorem 6.7(2). We claim that $\psi_{n+1} \circ M_n^{\mathrm{ca}} = m_{H_n,H_{n+1}} \circ \psi_n$.

To see this, observe that $m_{H_n,H_{n+1}}$ is multiplication by the determinant of M_n. Hence, as rational transformations, $m_{H_n,H_{n+1}}^{-1} \circ M_n^{\mathrm{ca}} = M_n^{-1}$. Since m_{H_n,H_n+1} commutes with ψ_{n+1}, the desired equality $\psi_{n+1} \circ M_n^{\mathrm{ca}} = m_{H_n,H_{n+1}} \circ \psi_n$ is therefore equivalent to $\psi_{n+1} = \psi_n \circ M_n$, which follows from the definitions of the maps involved. This establishes the claim.

The claim guarantees that $\varinjlim(H_n, m_{H_n,H_{n+1}}) \cong \varinjlim(\mathbf{Z}^2, M_n^{\mathrm{ca}})$, and (6.2) then follows from Theorem 6.7(2). □

Examples 6.9. (1) Let α and β be supernatural numbers. For $n \in \mathbf{N} \setminus \{0\}$, let ϕ_n be the homomorphism of $\mathbf{Z}^2$ determined by the diagonal matrix $M_n := \begin{pmatrix} \alpha_n & 0 \\ 0 & \beta_n \end{pmatrix}$.

For each n, let

$$H_n := \alpha[1,n]\mathbf{Z} \times \beta[1,n]\mathbf{Z} = \phi_n(\mathbf{Z}^2) \subset \mathbf{Z}^2.$$

We deduce from Theorem 6.7 that

$$K_*(P_1 C^*(\varprojlim(\Delta_2/H_n; p_n))P_1) = \Big(\mathbf{Z}\big[\tfrac{1}{\alpha\beta}\big] \oplus \mathbf{Z},\ \mathbf{Z}\big[\tfrac{1}{\alpha}\big] \oplus \mathbf{Z}\big[\tfrac{1}{\beta}\big]\Big),$$

that the position of the unit in K_0 corresponds to the element $(\alpha_1, 0)$, and that $P_1 C^*(\varprojlim(\Delta_2/H_n; p_n))P_1$ is a simple A$\mathbf{T}$ algebra of real-rank 0.

We claim that this is an example of an A$\mathbf{T}$ algebra which cannot be realised using a rank-2 Bratteli diagram as in [27]. To see this, suppose otherwise. Then [27, Theorem 6.1] implies that there exists an injective homomorphism $\phi : \mathbf{Z}\big[\frac{1}{\alpha}\big] \oplus \mathbf{Z}\big[\frac{1}{\beta}\big] \to \mathbf{Z}\big[\frac{1}{\alpha\beta}\big] \oplus \mathbf{Z}$ such that each element of $\operatorname{coker}(\phi)$ has finite order. Hence there exists $(x,y) \in \mathbf{Z}\big[\frac{1}{\alpha}\big] \oplus \mathbf{Z}\big[\frac{1}{\beta}\big]$ such that $\phi(x,y) = (z,m)$ with $m \neq 0$. Since $\mathbf{Z}\big[\frac{1}{\alpha}\big] \oplus \mathbf{Z}\big[\frac{1}{\beta}\big]$ is generated by elements of the form $(x,0)$ and $(0,y)$, we may in fact assume without loss of generality that there is an element $x \in \mathbf{Z}\big[\frac{1}{\alpha}\big]$ such that $\phi(x,0) = (z,m)$. Since α is infinite, there exist $n > m$ and $x' \in \mathbf{Z}\big[\frac{1}{\alpha}\big]$ such that $n \cdot x' = x$, and this forces $n \cdot \phi(x', 0) = (z, m)$ which is impossible by our choice of n.

Since each $\Delta_2/H_n \cong L^*_{\alpha[1,n]} \times L^*_{\beta[1,n]}$, the K-theory calculations for this example can also be verified using the Künneth formula (Theorem 3.8 and Proposition 5.7).

(2) Let ϕ be the homomorphism of $\mathbf{Z}^2$ determined by the integer matrix $M := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Suppose that M is diagonalisable as a real 2×2 matrix, and that its eigenvalues are greater than 1 in modulus. Let $D := ad - bc$ be the determinant of M. For $n \geq 1$, let $H_n := M^n\mathbf{Z}^2$ and $\Lambda_n := \Delta_2/H_n$. Our assumption regarding the eigenvalues of M ensures that $\bigcap_{n=1}^\infty H_n = \{0\}$, so Theorem 6.7 and Corollary 6.8 imply that $C^*(\varprojlim(\Delta_2/H_n; p_n))$ is a simple A$\mathbf{T}$ algebra of real rank zero with

$$K_*(P_1 C^*(\varprojlim(\Delta_2/H_n; p_n))P_1) \cong \Big(\mathbf{Z}\big[\tfrac{1}{D}\big] \oplus \mathbf{Z},\ \varinjlim\big(\mathbf{Z}^2, \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)\big)\Big).$$

In particular, let $M = \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$ with $a^2 + b^2 > 1$. We may identify $\mathbf{Z}^2$ with the group of Gaussian integers $\mathbf{Z}[i]$ by $(m, n) \mapsto m + in$, and then the group homomorphism of $\mathbf{Z}^2$ obtained from multiplication by M coincides with the group homomorphism of $\mathbf{Z}[i]$ obtained from multiplication by $a + ib$. Likewise M^{ca} implements multiplication by the conjugate $a - ib$. With $D := a^2 + b^2$ and $\zeta := \frac{1}{a - ib} = \frac{a + ib}{a^2 + b^2}$, we have

$$K_*(P_1 C^*(\varprojlim(\Delta_2/H_n; p_n))P_1) \cong \Big(\mathbf{Z}\big[\tfrac{1}{D}\big] \oplus \mathbf{Z},\ \mathbf{Z}\big[i, \tfrac{1}{\zeta}\big]\Big).$$

by Theorem 6.7.

(3) More generally, a sequence of Gaussian integers $\zeta_j := a_j + b_j i$ with $|\zeta_j| > 1$ for all j gives rise to a natural notion of a Gaussian supernatural number $\zeta = \prod_{j=1}^\infty \zeta_j$. Generalising the construction of the latter part of example (2) above, let $H_n := (\prod_{j=1}^n \overline{\zeta_j})\mathbf{Z}[i]$ for each n, and identify $\mathbf{Z}[i]$ with $\mathbf{Z}^2$ as a group to obtain a decreasing chain of subgroups of H_n of $\mathbf{Z}^2$ with trivial intersection.

Let α be the supernatural number $\alpha = \prod_{j=1}^\infty |\zeta_j|^2$. Then

$$K_*(P_1 C^*(\varprojlim(\Delta_2/H_n; p_n))P_1) \cong \Big(\mathbf{Z}\big[\tfrac{1}{\alpha}\big] \oplus \mathbf{Z},\ \mathbf{Z}\big[i, \tfrac{1}{\zeta}\big]\Big).$$

by Theorem 6.7 and Corollary 6.8.

We now turn to the proof of Theorem 6.7; in particular, we adopt the notation and conventions of Example 6.6. Our first step is to describe explicitly the K-theory of $C^*(\Delta_2/H_n)$ for a fixed $n \in \mathbf{N} \setminus \{0\}$. We do this using the results of Section 5.2.

For $q \in \mathbf{Z}^k$ we write q_+ and q_- for the positive and negative parts of q. That is to say that q_+ and q_- are the unique elements of $\mathbf{N}^k$ whose coordinate-wise minimum $q_+ \wedge q_-$ is equal to 0, and which satisfy $q = q_+ - q_-$.

For $q \in \mathbf{Z}^k$, a *cycle of degree* q in a k-graph Λ is a pair (μ, ν) where $\mu \in \Lambda^{q_+}$ and $\nu \in \Lambda^{q_-}$ such that $r(\mu) = r(\nu)$ and $s(\mu) = s(\nu)$. When $q \in \mathbf{N}^k$, $q = q_+$

and $q_- = 0$, so ν is a vertex, and μ is a cycle in the usual sense: a path whose range and source coincide.

Let $H \subset \mathbf{Z}^2$ be a finite-index subgroup of $\mathbf{Z}^2$. Let $G = \mathbf{Z}^2/H$. We view the ring $\mathbf{Z}G$ as the collection of functions $f : G \to \mathbf{Z}$. For $X \subseteq G$ we denote the indicator function of X by 1_X. We denote the point-mass at $g \in G$ by δ_g.

Let $\Lambda := \Delta_2/H$. Let E be the skeleton of Λ. That is E is the directed graph with the same vertices as Λ, and edges $\Lambda^{e_1} \cup \Lambda^{e_2}$, with range and source inherited from Λ. The degree map from Λ restricts to a map from E^1 to $\{e_1, e_2\}$. As in [31, 27] we call edges in E *blue* when they are of degree e_1 in Λ, and *red* when they are of degree e_2. We often blur the distinction between concatenation of edges in E and the corresponding factorisation of a path in Λ.

Recall that we are identifying Λ^0 with $G = \mathbf{Z}^2/H$. Hence, given a path $\alpha = a_0 a_1 \cdots a_n$ in E, we define functions f^b_α and f^r_α in $\mathbf{Z}G$ by

$$\begin{aligned} f^b_\alpha(g) &= \#\{0 \le j \le n : r(a_j) = g, d(a_j) = e_1\} \\ f^r_\alpha(h) &= \#\{0 \le k \le n : r(a_k) = h, d(a_k) = e_2\}. \end{aligned}$$

The idea is that $f^b_\alpha(g)$ counts the number of blue edges in α whose range is g, and $f^r_\alpha(g)$ does the same thing for red edges.

We define $f_\alpha \in \mathbf{Z}G \oplus \mathbf{Z}G$ by $f_\alpha = f^b_\alpha \oplus f^r_\alpha$. For a vertex $g \in \Lambda^0 = G$, we define f^b_g and f^r_g to be the zero element of $\mathbf{Z}G$, and $f_g = f^b_g \oplus f^r_g$ is then the zero element of $\mathbf{Z}G \oplus \mathbf{Z}G$.

As $\Lambda = \Delta_2/H$, for each $g \in \Lambda^0 = G$ there is a unique path $[g, g + (1,1)]$ of degree $(1,1)$ with range g. Using the factorisation property, we can express this path as $b_g r_{g+[e_1]} = r_g b_{g+[e_2]}$ where r_g and b_g denote the unique red and blue edges in E with range g (for $n \in \mathbf{Z}^2$, $[n]$ denotes the class of n in the quotient group $G = \mathbf{Z}^2/H$). We write z_g for the function $(\delta_{g+[e_2]} - \delta_g) \oplus (\delta_g - \delta_{g+[e_1]})$ in $\mathbf{Z}G \oplus \mathbf{Z}G$.

Given paths $\alpha = a_0 \cdots a_m$ and $\beta = b_0 \cdots b_n$ in the skeleton E of Λ such that $r(a_0) = r(b_0)$ and $s(a_m) = s(b_n)$, let $f_{\alpha,\beta} := f_\alpha - f_\beta \in \mathbf{Z}G \oplus \mathbf{Z}G$. Fix generators h_1, h_2 for H; so $[h_i] = [0]$ in G. By definition of Λ, there are unique paths $\mu_1^+ \in \Lambda^{(h_1)_+}$ and $\mu_1^- \in \Lambda^{(h_1)_-}$ with $r(\mu_1^\pm) = 0$. Fix factorisations $\alpha_1^\pm$ of $\mu_1^\pm$ into edges from the skeleton E. Since

$$s(\mu_1^+) = [(h_1)_+] = [(h_1)_-] = s(\mu_1^-)$$

in G, the pair (μ_1^+, μ_1^-) is a cycle of degree h_1 in Λ with range $[0]$. The same construction for h_2 gives a cycle (μ_2^+, μ_2^-) of degree h_2 with range $[0]$ and fixed factorisations $\alpha_2^\pm$ of $\mu_2^\pm$ into edges from the skeleton E.

Lemma 6.10. *With the notation established in the preceding paragraphs, the chain complex* (5.6) *can be described as follows:*

(1) *for each* $g \in G$, $\partial_1(\delta_g \oplus 0) = \delta_g - \delta_{g+[e_1]}$, $\partial_1(0 \oplus \delta_g) = \delta_g - \delta_{g+[e_2]}$, *and*

$$\partial_2(\delta_g) = (\delta_{g+[e_2]} - \delta_g) \oplus (\delta_g - \delta_{g+[e_1]}) = z_g.$$

(2) $\operatorname{coker}(\partial_1) \cong \mathbf{Z}$ *is generated by* $\delta_0 + \operatorname{Im}(\partial_1)$*;*
(3) $\ker(\partial_2) \cong \mathbf{Z}$ *is generated by* 1_G*;*

(4) *For each* $h \in G$*, the set* $\{z_g : g \in G \setminus \{h\}\}$ *is a basis for* $\mathrm{Im}(\partial_2) \cong \mathbf{Z}^{|G|-1}$.
(5) *Fix any two factorisations* α *and* β *of a path* μ *in* Λ *into edges from* E*. Then* $f_\alpha - f_\beta \in \mathrm{Im}(\partial_2)$*, and* $\partial_1(f_\alpha) = \partial_1(f_\beta) = \delta_{r(\alpha)} - \delta_{s(\alpha)}$.
(6) $\ker(\partial_1)$ *is the subgroup of* $\mathbf{Z}G \oplus \mathbf{Z}G$ *generated by the elements* $f_{\alpha,\beta}$ *where* α *and* β *are paths in the skeleton* E *with* $r(\alpha) = r(\beta)$ *and* $s(\alpha) = s(\beta)$.
(7) *There is an isomorphism* ψ *of* H *onto* $\ker(\partial_1)/\mathrm{Im}(\partial_2)$ *which takes* $d(\mu) - d(\nu)$ *to* $f_{\alpha,\beta} + \mathrm{Im}(\partial_2)$ *for each cycle* (μ,ν) *in* Λ *and pair of factorisations* α *of* μ *and* β *of* ν*. In particular, for any basis* B *for* $\mathrm{Im}(\partial_2)$*, the set* $B \cup \{f_{\alpha_1^+,\alpha_1^-}, f_{\alpha_2^+,\alpha_2^-}\}$ *is a basis for* $\ker(\partial_1) \cong \mathbf{Z}^{|G|+1}$ *(where* $\alpha_i^\pm$ *are the fixed factorisations of the paths* $\mu_i^\pm$ *of degree* $(h_i)_\pm$ *described above).*

In particular, $K_*(C^*(\Lambda)) \cong (\mathbf{Z}^2, H)$ *where the class of the identity in* K_0 *is identified with the element* $(|G|, 0)$ *of* $\mathbf{Z}^2$.

Proof. (1) The adjacency matrix M_1 associated to $(\Lambda^0, \Lambda^{e_1}, r, s)$ is the permutation matrix determined by translation by $[e_1]$ in G and similarly for M_2. The first statement then follows from the formulae for ∂_1 and ∂_2 in terms of M_1 and M_2.
(2) The formulae for $\partial_1(\delta_g \oplus 0)$ and $\partial_1(0 \oplus \delta_g)$ show that $\delta_g + \mathrm{Im}(\partial_1) = \delta_{g+[e_i]} + \mathrm{Im}(\partial_1)$ in $\mathrm{coker}(\partial_1)$ for $i = 1, 2$ and $g \in G$. Since the action of $\mathbf{Z}^2$ on G by translation is transitive, this establishes (2).
(3) Using the formula for ∂_2 established in (1), one can see that for $f \in \mathbf{Z}G$, $\partial_2(f) = f_1 \oplus f_2$ where

$$f_1(g) = -f(g) + f(g - [e_1]) \qquad \text{and} \qquad f_2(g) = f(g) - f(g - [e_2])).$$

Hence $f \in \ker(\partial_2)$ if and only if $f(g) = f(g - [e_1]) = f(g - [e_2])$ for all $g \in G$, and since the action of $\mathbf{Z}^2$ on G is transitive, this establishes (3).
(4) Part (1) establishes that $\mathrm{Im}(\partial_2)$ is generated by $\{z_g : g \in G\}$. A simple calculation shows that $\sum_{g \in G} z_g = 0$ in $\mathbf{Z}G \oplus \mathbf{Z}G$, and it follows that for any $h \in G$, the set $\{z_g : g \in G \setminus \{h\}\}$ generates $\mathrm{Im}(\partial_2) \cong \mathbf{Z}^{|G|-1}$. Since $\ker(\partial_2)$ has rank 1, the rank of its image is $|G| - 1$, establishing (4).
(5) By part (4), the image of ∂_2 is generated by elements of the form $f_\alpha - f_\beta$ where α and β are the two possible factorisations of a path in $\Lambda^{(1,1)}$. Since $f_{\alpha\beta} = f_\alpha + f_\beta$ when α and β are paths in E which can be concatenated, this establishes the first claim. The second statement follows from a straightforward calculation using that

$$\partial_1(f^b \oplus f^r)(g) = f^b(g) - f^b(g - [e_1]) + f^r(g) - f^r(g - [e_2]). \tag{6.3}$$

(6) If α, β are paths in the skeleton with $r(\alpha) = r(\beta)$ and $s(\alpha) = s(\beta)$ then $f_{\alpha,\beta}$ belongs to $\ker(\partial_1)$ by (5).
We must show that every $f \in \ker(\partial_1)$ can be written as a $\mathbf{Z}$-linear combination of elements of the form $f_{\alpha,\beta}$. First note that it suffices to treat the case where f takes only nonnegative values (this is because $1_G \oplus 1_G$ can be so expressed). So suppose that f takes nonnegative values, and write $f = f^b \oplus f^r$. Let E_f be the directed graph with vertices G and which contains $f^b(g)$ parallel copies of the blue edge in E with range g and $f^r(g)$ copies of the red edge in E with range

g. If E_f contains a *terminal vertex* g which receives at least one edge but emits no edges at all, then $f^b(g) + f^r(g) \neq 0$, but $f^b(g - [e_1]) = f^r(g - [e_2]) = 0$, and (6.3) shows that $\partial_1(f)(g) \neq 0$. Hence E_f contains no such vertex, and therefore must either contain a cycle α or contain no edges at all. In the latter case, the claim is trivial, and in the former case, $f \geq f_\alpha$, and removing the cycle α from E_f produces the graph E_{f-f_α} for the function $f - f_\alpha$. After finitely many such steps, we must obtain a forest with no terminal vertex. The only such forest is the empty graph which corresponds to the function $0 \oplus 0$. That is $f - \sum_{\alpha \in L} f_\alpha = 0 \oplus 0$ for some collection L of cycles, and this proves (6).
(7) Suppose that (μ, ν) is a cycle in Λ. Then

$$s(\mu) - [d(\mu)] = r(\mu) = r(\nu) = s(\nu) - [d(\nu)] = s(\mu) - [d(\nu)]$$

in $G = \Lambda^0 = \mathbf{Z}^2/H$, so $d(\mu) - d(\nu) \in H$. It is clear from the definition of Λ that each element of H arises as $d(\mu) - d(\nu)$ for some cycle (μ, ν) in Λ.
To see that the assignment $d(\mu) - d(\nu) \mapsto f_{\alpha,\beta} + \operatorname{Im}(\partial_2)$ is well defined, we must show two things. First that for distinct factorisations α and α' of μ and distinct factorisations β and β' of ν, the difference $f_{\alpha,\beta} - f_{\alpha',\beta'}$ lies in the image of ∂_2. This follows from (5). Second, we must show that if (μ, ν) and (μ', ν') are cycles in Λ with $d(\mu) - d(\nu) = d(\mu') - d(\nu')$, then there exist factorisations α of μ, β of ν, α' of μ', and β' of ν' such that $f_{\alpha,\beta} - f_{\alpha',\beta'}$ is in $\operatorname{Im}(\partial_2)$. To see this, first note that by factorising $\mu = \mu'\tau$ and $\nu = \nu'\tau$ where $d(\tau) = d(\mu) \wedge d(\nu)$, we can reduce to the case where $d(\mu) \wedge d(\nu) = 0$. Next we claim that it suffices to consider the case where $r(\mu) = r(\nu) = r(\mu') = r(\nu') = [0]$. To see this, fix η in $[0]\Lambda r(\mu)$ and note that the cycle $(\eta\mu, \eta\nu)$ corresponds to the same class as (μ, ν) in $\ker(\partial_1)/\operatorname{Im}(\partial_2)$. Factorise $\eta\mu = \xi\rho$ and $\eta\nu = \omega\sigma$ where $d(\xi) = d(\mu)$, $d(\omega) = d(\nu)$ and $d(\rho) = d(\sigma) = d(\eta)$. Since each $g\Lambda^n$ is a singleton and since $\mathbf{Z}^2$ acts on Λ by translation, (ξ, ω) is a cycle with range $[0]$, and $\rho = \sigma$. Hence the cycle (ξ, ω) corresponds to the same class in $\ker(\partial_1)/\operatorname{Im}(\partial_2)$ as (μ, ν). After shifting (μ', ν') in a similar way we may assume that both cycles have range $[0]$. We now have cycles (μ, ν) and (μ', ν') with range $[0]$ and such that $d(\mu) - d(\nu) = d(\mu') - d(\nu')$ and $d(\mu) \wedge d(\nu) = 0 = d(\mu') \wedge d(\nu')$. Since $[0]\Lambda^n$ is a singleton for any $n \in \mathbf{Z}^2$, this forces $\mu = \mu'$ and $\nu = \nu'$. This completes the proof that $d(\mu) - d(\nu) \mapsto f_{\alpha,\beta} + \operatorname{Im}(\partial_2)$ is well defined.
That $f_{\alpha\beta} = f_\alpha + f_\beta$ ensures that $\psi(g+h) = \psi(g) + \psi(h)$, and that $f_{\beta,\alpha} = -f_{\alpha,\beta}$ shows that $\psi(-g) = -\psi(g)$. Hence ψ is a homomorphism. By part (6), to see that ψ is surjective, we just need to show that each $f_{\alpha,\beta} + \operatorname{Im}(\partial_2)$ is in the range of ψ. This is clear because $f_{\alpha,\beta} + \operatorname{Im}(\partial_2)$ is precisely $\psi(d(\mu) - d(\nu))$ where μ factorises as α and ν factorises as β. Finally, to see that ψ is injective, note that if $f_{\alpha,\beta} \in \operatorname{Im}(\partial_2)$, then $d(\mu) = d(\nu)$ where μ factorises as α and ν factorises as β. This completes the proof that $\psi : H \to \ker(\partial_1)/\operatorname{Im}(\partial_2)$ is an isomorphism. The remaining statement follows from (4) and that (μ_1^+, μ_1^-) and (μ_2^+, μ_2^-) are cycles whose degrees form a basis for H. This proves (7).
The final statement of the Lemma follows from (5.7). □

We now consider two consecutive graphs in the sequence of covering systems described in Example 6.6, and describe the homomorphism of K-invariants obtained from Proposition 3.2(6).

Theorem 6.11. *Consider the situation described in Example 6.6, and fix $n \in \mathbf{N} \setminus \{0\}$. For $i = n, n+1$, let $\Lambda_i := \Delta_2/H_i$, and consider the commuting diagram*

$$\begin{array}{ccccccccc} 0 & \longleftarrow & \mathbf{Z}\Lambda_n^0 & \xleftarrow{\partial_1^{\Lambda_n}} & \mathbf{Z}\Lambda_n^0 \oplus \mathbf{Z}\Lambda_n^0 & \xleftarrow{\partial_2^{\Lambda_n}} & \mathbf{Z}\Lambda_n^0 & \longleftarrow & 0 \\ & & \downarrow{\scriptstyle p_n^*} & & \downarrow{\scriptstyle p_n^* \oplus p_n^*} & & \downarrow{\scriptstyle p_n^*} & & \\ 0 & \longleftarrow & \mathbf{Z}\Lambda_{n+1}^0 & \xleftarrow{\partial_1^{\Lambda_{n+1}}} & \mathbf{Z}\Lambda_{n+1}^0 \oplus \mathbf{Z}\Lambda_{n+1}^0 & \xleftarrow{\partial_2^{\Lambda_{n+1}}} & \mathbf{Z}\Lambda_{n+1}^0 & \longleftarrow & 0 \end{array}$$

(1) *The right-hand vertical map $p_n^* : \mathbf{Z}\Lambda_n^0 \to \mathbf{Z}\Lambda_{n+1}^0$ restricts to a homomorphism $p_n^*|_{\ker(\partial_2^{\Lambda_n})} : \ker(\partial_2^{\Lambda_n}) \to \ker(\partial_2^{\Lambda_{n+1}})$ which is characterised by $p_n^*|_{\ker(\partial_2^{\Lambda_n})}(1_{G_n}) = 1_{G_{n+1}}$.*

(2) *The left-hand vertical map $p_n^* : \mathbf{Z}\Lambda_n^0 \to \mathbf{Z}\Lambda_{n+1}^0$ induces a homomorphism $\widetilde{p_n^*} : \operatorname{coker}(\partial_1^{\Lambda_n}) \to \operatorname{coker}(\partial_1^{\Lambda_{n+1}})$ characterised by*

$$\widetilde{p_n^*}(\delta_0 + \operatorname{Im}(\partial_1^{\Lambda_n})) = [H_n : H_{n+1}] \cdot \delta_0 + \operatorname{Im}(\partial_1^{\Lambda_{n+1}}).$$

(3) *The middle vertical map $p_n^* \oplus p_n^* : \mathbf{Z}\Lambda_n^0 \oplus \mathbf{Z}\Lambda_n^0 \to \mathbf{Z}\Lambda_{n+1}^0 \oplus \mathbf{Z}\Lambda_{n+1}^0$ induces a homomorphism $(p_n^* \oplus p_n^*)^\sim : \ker(\partial_1^{\Lambda_n})/\operatorname{Im}(\partial_2^{\Lambda_n}) \to \ker(\partial_1^{\Lambda_{n+1}})/\operatorname{Im}(\partial_2^{\Lambda_{n+1}})$ such that the following diagram commutes.*

$$\begin{array}{ccc} H_n & \xrightarrow{\psi_n} & \ker(\partial_1^{\Lambda_n})/\operatorname{Im}(\partial_2^{\Lambda_n}) \\ \downarrow{\scriptstyle m_{H_n,H_{n+1}}} & & \downarrow{\scriptstyle (p_n^* \oplus p_n^*)^\sim} \\ H_{n+1} & \xrightarrow{\psi_{n+1}} & \ker(\partial_1^{\Lambda_{n+1}})/\operatorname{Im}(\partial_2^{\Lambda_{n+1}}) \end{array}$$

where ψ_n and ψ_{n+1} are the isomorphisms obtained from Lemma 6.10(7), and $m_{H_n,H_{n+1}}$ is as in Theorem 6.7(2).

Under the isomorphism

$$K_*(C^*(\Lambda_i)) \cong \big(\operatorname{coker}(\partial_1^{\Lambda_i}) \oplus \ker(\partial_2^{\Lambda_i}),\ \ker(\partial_1^{\Lambda_i})/\operatorname{Im}(\partial_2^{\Lambda_i})\big)$$

obtained from Corollary 5.6, the maps described in (1), (2) and (3) determine the map $(\iota_{p_n})_* : K_*(C^*(\Lambda_n)) \to K_*(C^*(\Lambda_{n+1}))$ obtained from Proposition 3.2(6).

Proof. Lemma 6.10(3) ensures that 1_{G_i} generates $\ker(\partial_2^{\Lambda_i})$ for $i = n, n+1$. The formula for p_n^* shows that $p_n^*(1_{G_n}) = 1_{G_{n+1}}$, which gives (1). Statement (2) follows from the formula for p_n^* combined with the observation that for $i = n, n+1$, the δ_g, $g \in G_i$ are all equivalent modulo $\operatorname{Im}(\partial_1^{\Lambda_i})$.

It remains only to prove (3). We first consider the case where $H_n = \mathbf{Z}^2$, so $G_n = \{0\}$ and Λ_n is a copy of the 2-graph $T_2 \cong \mathbf{N}^2$ (as a category) with one vertex and one morphism λ_m of each degree $m \in \mathbf{N}^2$. In this case, ψ_n is just the identity map from $\mathbf{Z}^2$ to $\mathbf{Z} \oplus \mathbf{Z}$. Let h_1, h_2 be a pair of generators for H_{n+1}.

Since H_{n+1} has finite index in $\mathbf{Z}^2$, the assignments $(1,0) \mapsto h_1$ and $(0,1) \mapsto h_2$ determine an endomorphism of H_n which is a rational isomorphism. Hence it suffices to show that $(p_n^* \oplus p_n^*)^\sim \circ \psi_n(h_i) = \psi_{n+1}([\mathbf{Z}^2 : H_{n+1}] \cdot h_i)$ for $i = 1, 2$. We just argue that this happens for $i = 1$ (the case $i = 2$ follows from a symmetric argument).

Writing $h_1 = (x, y)$ where $x, y \in \mathbf{Z}$, the formula for p_n^* ensures that $(p_n^* \oplus p_n^*)^\sim$ takes $\psi_n(h_1)$ to the class of $x1_{G_{n+1}} \oplus y1_{G_{n+1}}$. To see that this is $\psi_{n+1}([\mathbf{Z}^2 : H_{n+1}] \cdot h_1)$, let $f := f_{\alpha_1^+, \alpha_1^-} = \psi_{n+1}(h_1)$ be the function in $\mathbf{Z}G_{n+1} \oplus \mathbf{Z}G_{n+1}$ obtained from Lemma 6.10(7). By definition of f, we have $f = f_b \oplus f_r$ where the entries of f_b sum to x and the entries of f_r sum to y. For $g \in G_{n+1}$, let $g \cdot f_b$ be the function determined by $g \cdot f_b(h) = f_b(h - g)$, and similarly for f_r. Since G_{n+1} acts freely and transitively on $\Lambda_{n+1}^0 = G_{n+1}$, it follows that

$$\sum_{g \in G_{n+1}} g \cdot f = x1_{G_{n+1}} \oplus y1_{G_{n+1}} = (p_n^* \oplus p_n^*)^\sim \circ \psi_n(h_1). \tag{6.4}$$

The proof of statement (7) in Lemma 6.10 shows that each $g \cdot f := g \cdot f_b \oplus g \cdot f_r$ represents the same class as f in $\ker(\partial_1^{\Lambda_{n+1}}) / \operatorname{Im}(\partial_2^{\Lambda_{n+1}})$. Hence the left-hand side of (6.4) has the same class in $\ker(\partial_1^{\Lambda_{n+1}}) / \operatorname{Im}(\partial_2^{\Lambda_{n+1}})$ as $\psi_{n+1}(|G_{n+1}| \cdot h_1)$ as required.

For the general case, first note that we may assume without loss of generality that $H_1 = \mathbf{Z}^2$ so that $\Lambda_1 = T_2$. Let $p_{[1,n]} := p_1 \circ \cdots \circ p_{n-1}$ and $p_{[1,n+1]} := p_1 \circ \cdots \circ p_n$ be the coverings of $\Lambda_1 = T_2$ by Λ_n and Λ_{n+1} obtained by composing the first n and $n+1$ levels of the covering system; we may apply the argument of the previous paragraph to these coverings. Then $p_{[1,n+1]} = p_{[1,n]} \circ p_n$, so $p_{[1,n+1]}^* \oplus p_{[1,n+1]}^* = (p_{[1,n]}^* \oplus p_{[1,n]}^*) \circ (p_n^* \oplus p_n^*)$, and since these maps induce homomorphisms between $\ker(\partial_1^{T_2}) / \operatorname{Im}(\partial_2^{T_2})$ and $\ker(\partial_1^{\Lambda_{n+1}}) / \operatorname{Im}(\partial_2^{\Lambda_{n+1}})$ which are rational isomorphisms, it follows that $(p_n^* \oplus p_n^*)^\sim$ behaves as claimed.

The final statement follows from Corollary 5.6. $\square$

We are now ready to prove Theorem 6.7.

Proof of Theorem 6.7. Proposition 3.2 shows that P_1 is full so that compression by P_1 induces an isomorphism on K-theory. The formulae for the K-groups in statements (1) and (2) follow from Lemma 6.10 and Theorem 6.11 and the continuity of the K-functor.

Since $v(\Delta_2/H_n)w \neq \emptyset$ for all $n \in \mathbf{N} \setminus \{0\}$, and $v, w \in \Delta_2^0/H_n$, the 3-graph $\varprojlim(\Delta_2/H_n, p_n)$ is cofinal. Moreover a given infinite path x in $\varprojlim(\Delta_2/H_n, p_n)$ is periodic with period $m \in \mathbf{Z}^2$ if and only if every infinite path in $\varprojlim(\Delta_2/H_n, p_n)$ is periodic with period m, which in turn is equivalent to the condition that $m \in \bigcap_{n=1}^\infty H_n$. It follows from Lemma 4.5 that $\varprojlim(\Delta_2/H_n, p_n)$ is simple if and only if $\bigcap H_n = \{0\}$; moreover, in this case, the argument of the second part of [27, Section 5] shows that $C^*(\varprojlim(\Delta_2/H_n, p_n))$ has unique trace.

We next claim that each $C^*(\Delta_2/H_n) \cong M_{[\mathbf{Z}^2 : H_n]}(C(\mathbf{T}^2))$. To verify this, one first checks that $h \mapsto s_{[(0,h_+)]} s_{[(0,h_-)]}^*$ is a group isomorphism $H_n \to \mathcal{U}(s_{[0]} C^*(\Delta_2/H_n) s_{[0]})$ for each n. The standard argument used in [27, Lemma 3.9] shows that each $s_{[(0,h_+)]} s_{[(0,h_-)]}^*$ has full spectrum. One can then deduce that $s_{[0]}C^*(\Delta_2/H_n)s_{[0]} \cong C^*(H_n) \cong C^*(\mathbf{Z}^2) \cong C(\mathbf{T}^2)$. For $m \in \mathbf{Z}^2/H_n$, define $V_m := s^*_{[0,m]} \in C^*(\Delta_2/H_n)$. Applying Lemma 3.3 to these partial isometries with $p = s_{[0]}$ and $q = 1_{C^*(\Delta_2/H_n)}$proves that $C^*(\Delta_2/H_n) \cong M_{[\mathbf{Z}^2:H_n]}(C(\mathbf{T}^2))$.

It now follows from [3, Theorem 1.3] that $C^*(\varprojlim(\Delta_2/H_n; p_n))$ has real-rank 0. The classification of such algebras of Dădărlat-Elliott-Gong (see [36, Theorem 3.3.1]), and the K-theory calculations above complete the proof. □

Remark 6.12. *Higher-rank Bunce-Deddens algebras and generalised odometer actions.* We consider a slightly more general version of the situation described in Example 6.6. Let $H_1 \supset H_2 \supset H_3 \supset \dots$ be a chain of finite-index subgroups of $\mathbf{Z}^k$ such that $\bigcap_n H_n = \{0\}$. For each n, let $p_n : \Delta_k/H_{n+1} \to \Delta_k/H_n$ be the canonical covering induced by the quotient maps described above, let $m_n = 1$, and let $\mathfrak{s}_n : \Delta_k/H_{n+1} \to S_1$ be the trivial cocycle. This data specifies a sequence $(\Delta_k/H_n, \Delta_k/H_{n+1}, p_n)_{n=1}^\infty$ of row-finite covering systems of k-graphs with no sources. Applying Corollary 2.11, we obtain a $(k+1)$-graph $\varprojlim(\Delta_k/H_n; p_n)$.

We claim that the corner $P_1C^*(\varprojlim(\Delta_k/H_n; p_n))P_1$ can be thought of as a higher-rank Bunce-Deddens algebra. We justify this by giving a description of $P_1C^*(\varprojlim(\Delta_k/H_n; p_n))P_1$ as a crossed product by a generalised odometer action. We assume here that $H_1 = \mathbf{Z}^k$ so that Δ_k/H_1 is a copy of the k-graph $T_k \cong \mathbf{N}^k$ (as a category) with one vertex and one morphism λ_m of each degree $m \in \mathbf{N}^k$.

One way to realise the Bunce-Deddens algebras is as crossed products of algebras of continuous functions on Cantor sets by generalised odometer actions. Given a supernatural number $\alpha = \alpha_1\alpha_2\cdots$, let $G_n := \mathbf{Z}/\alpha[1,n]\mathbf{Z}$ for all n. Then for each n, since $\alpha[1,n+1]\mathbf{Z} \supset \alpha[1,n]\mathbf{Z}$, there is a natural surjective group homomorphism from G_{n+1} to G_n. Hence, we may form the projective limit group $\varprojlim(G_n, p_n)$. The automorphism $\tau(g_1, g_2, \dots) = (g_1 + [1], g_2 + [1], \dots)$ for $(g_1, g_2, \dots) \in \varprojlim(G_n, p_n)$ can then naturally be regarded as an odometer action on $\varprojlim(G_n, p_n)$. The Bunce-Deddens algebra of type α is the crossed product $C(\varprojlim(G_n, p_n)) \rtimes_{\tilde{\tau}} \mathbf{Z}$ where $\tilde{\tau}$ is the automorphism of $C(\varprojlim(G_n, p_n))$ induced by τ (see [33, Examples 1(3)]).

There is an analogous realisation of $P_1C^*(\varprojlim(\Delta_k/H_n, p_n))P_1$ as follows. Let $\Lambda := \varprojlim(\Delta_k/H_n, p_n)$. Let F denote the fixed-point algebra of $C^*(\Lambda)$ for the gauge action γ of $\mathbf{T}^{k+1}$. Note that by Remark 3.9, the restriction of the gauge action to $P_1C^*(\Lambda)P_1$ is trivial on the last coordinate of $\mathbf{T}^{k+1}$ and therefore becomes an action by $\mathbf{T}^k$ denoted $\tilde{\gamma}$. Recall that Λ^∞ denotes the collection of infinite paths in Λ (see Notation 4.1). It is not hard to see that P_1FP_1 is canonically isomorphic to $C(v\Lambda^\infty)$ where v is the unique vertex of $\Delta_k/H_1 \cong T_k$. Let $G_n := \mathbf{Z}^k/H_n$ for each n, and let $p_n : G_{n+1} \to G_n$ be the induced map $p_n(m + H_{n+1}) := m + H_n$. Observe that $G = \varprojlim(G_n, p_n)$ is a compact abelian group. By functoriality of the projective limit the quotient maps $\mathbf{Z}^k \to \mathbf{Z}^k/H_n$ induce a homomorphism $j : \mathbf{Z}^k \to G$; injectivity of j follows from the fact that $\bigcap_n H_n = \{0\}$. There is an action τ of $\mathbf{Z}^k$ on G given by $\tau_m(g_1, g_2, \dots) = (g_1 + [m], g_2 + [m], \dots)$, which generalises the odometer action discussed above. Since there is just one infinite path in T_k, the arguments of Section 4 show that $v\Lambda^\infty \cong G$ as a topological space. Note that for every $m \in \mathbf{N}^k$, the generator s_{λ_m} associated to the unique path $\lambda_m \in T_k^m$ is a unitary in $P_1 C^*(\Lambda) P_1$ and that under the identification of $P_1 F P_1$ with $C(v\Lambda^\infty) = C(G)$ conjugation by s_{λ_m} implements the automorphism induced by the homeomorphism τ_m of G. It follows that the reduction of the path groupoid (see [20, Section 2]) of Λ to $v\Lambda^\infty$ is isomorphic to the semidirect product groupoid $G \rtimes_\tau \mathbf{Z}^k$. Therefore, standard arguments show that

$$P_1 C^*(\Lambda) P_1 \cong C(G) \rtimes_{\tilde{\tau}} \mathbf{Z}^k$$

where $\tilde{\tau}$ is the action induced by τ. Note that under this identification the restricted gauge action $\tilde{\gamma}$ coincides with the dual action of $\mathbf{T}^k = \widehat{\mathbf{Z}^k}$.
The action of G on $C(G)$ induced by translation in G yields an action of G on $C(G) \rtimes_{\tilde{\tau}} \mathbf{Z}^k$ which commutes with the dual action of $\mathbf{T}^k = \widehat{\mathbf{Z}^k}$. Thus we obtain an action α by the compact abelian group $G \times \mathbf{T}^k$ with fixed point algebra isomorphic to $\mathbf{C}$. Hence, $C(G) \rtimes_{\tilde{\tau}} \mathbf{Z}^k$ (and thus $P_1 C^*(\Lambda) P_1$) admits an ergodic action of a compact abelian group. Such ergodic actions have been classified in [24, 4.5, 6.1]; the invariant is a symplectic bicharacter χ_α on $\widehat{G} \times \mathbf{Z}^k$, the dual of $G \times \mathbf{T}^k$. This gives rise to an alternative description of the C^*-algebra as a twisted group C^*-algebra with the group $\widehat{G} \times \mathbf{Z}^k$ and a 2-cocycle associated to the bicharacter χ_α (only its cohomology class is determined by the bicharacter). It follows that

$$C(G) \rtimes_{\tilde{\tau}} \mathbf{Z}^k \cong C(\mathbf{T}^k) \rtimes \widehat{G}$$

where the action of $\widehat{G}$ on $C(\mathbf{T}^k)$ arises by translation from the embedding $\widehat{G} \to \mathbf{T}^k$ dual to $j : \mathbf{Z}^k \to G$.

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Faces of Generalized Permutohedra

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Abstract. The aim of the paper is to calculate face numbers of simple generalized permutohedra, and study their f-, h- and γ-vectors. These polytopes include permutohedra, associahedra, graph-associahedra, simple graphic zonotopes, nestohedra, and other interesting polytopes.

We give several explicit formulas for h-vectors and γ-vectors involving descent statistics. This includes a combinatorial interpretation for γ-vectors of a large class of generalized permutohedra which are flag simple polytopes, and confirms for them Gal's conjecture on the nonnegativity of γ-vectors.

We calculate explicit generating functions and formulae for h-polynomials of various families of graph-associahedra, including those corresponding to all Dynkin diagrams of finite and affine types. We also discuss relations with Narayana numbers and with Simon Newcomb's problem.

We give (and conjecture) upper and lower bounds for f-, h-, and γ-vectors within several classes of generalized permutohedra.

An appendix discusses the equivalence of various notions of deformations of simple polytopes.

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Contents

1 Introduction

Generalized permutohedra are a very well-behaved class of convex polytopes studied in [Post'05], as generalizations of the classical permutohedra, associahedra, cyclohedra, etc. That work explored their wonderful properties from the point of view of valuations such as volumes, mixed volumes, and number of lattice points. This paper focuses on their further good behavior with respect to face enumeration in the case when they are simple polytopes.

Simple generalized permutohedra include as an important subclass (the duals of) the *nested set complexes* considered by DeConcini and Procesi in their work on *wonderful compactifications* of hyperplane arrangements; see [DP'95, FS'05]. In particular, when the arrangement comes from a Coxeter system, one obtains interesting flag simple polytopes studied by Davis, Januszkiewicz, and

Scott [DJS'03]. These polytopes can be combinatorially described in terms of the corresponding Coxeter graph. Carr and Devadoss [CD'06] studied these polytopes for arbitrary graphs and called them *graph-associahedra.*
We mention here two other recent papers in which generalized permutohedra have appeared. Morton, Pachter, Shiu, Sturmfels, and Wienand [M–W'06] considered generalized permutohedra from the point of view of *rank tests* on ordinal data in statistics. The normal fans of generalized permutohedra are what they called *submodular rank tests.* Agnarsson and Morris [AM'06] investigated closely the 1-skeleton (vertices and edges) in the special case where generalized permutohedra are Minkowski sums of standard simplices.

Let us formulate several results of the present paper. A few definitions are required. A *connected building set* $\mathcal{B}$ on $[n] := \{1, \ldots, n\}$ is a collection of nonempty subsets in $[n]$ such that

1. if $I, J \in \mathcal{B}$ and $I \cap J \neq \emptyset$, then $I \cup J \in \mathcal{B}$,
2. $\mathcal{B}$ contains all singletons $\{i\}$ and the whole set $[n]$;

see Definition 6.1. An interesting subclass of *graphical building sets* $\mathcal{B}(G)$ comes from connected graphs G on $[n]$. The building set $\mathcal{B}(G)$ contains all nonempty subsets of vertices $I \subseteq [n]$ such that the induced graph $G|_I$ is connected.
The *nestohedron* $P_\mathcal{B}$ is defined (see Definition 6.3) as the Minkowski sum

$$P_\mathcal{B} = \sum_{I \in \mathcal{B}} \Delta_I$$

of the coordinate simplices $\Delta_I := \text{ConvexHull}(e_i \mid i \in I)$, where the e_i are the endpoints of the coordinate vectors in $\mathbb{R}^n$. According to [Post'05, Theorem 7.4] and [FS'05, Theorem 3.14] (see Theorem 6.5 below), the nestohedron $P_\mathcal{B}$ is a simple polytope which is dual to a simplicial nested set complex. For a graphical building set $\mathcal{B}(G)$, the nestohedron $P_{\mathcal{B}(G)}$ is called the *graph-associahedron.* In the case when G is the n-path, $P_{\mathcal{B}(G)}$ is the usual associahedron; and in the case when $G = K_n$ is the complete graph, $P_{\mathcal{B}(G)}$ is the usual permutohedron.
Recall that the f-vector and the h-vector of a simple d-dimensional polytope P are $(f_0, f_1, \ldots, f_d)$ and $(h_0, h_1, \ldots, h_d)$, where f_i is the number of i-dimensional faces of P and $\sum h_i\,(t+1)^i = \sum f_i\,t^i$. It is known that the h-vector of a simple polytope is positive and symmetric. Since the h-vector is symmetric, one can define another vector called the γ-vector $(\gamma_1, \gamma_2, \ldots, \gamma_{\lfloor d/2 \rfloor})$ by the relation

$$\sum_{i=0}^{d} h_i\,t^i = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i\,t^i (1+t)^{d-2i}.$$

A simplicial complex Δ is called a *flag complex* (or a *clique complex*) if its simplices are cliques (i.e., subsets of vertices with complete induced subgraphs) of some graph (1-skeleton of Δ). Say that a simple polytope is *flag* if its dual simplicial complex is flag.

Gal conjectured [Gal'05] that the γ-vector has nonnegative entries for any flag simple polytope.

Let us that say a connected building set $\mathcal{B}$ is *chordal* if, for any of the sets $I = \{i_1 < \cdots < i_r\}$ in $\mathcal{B}$, all subsets $\{i_s, i_{s+1}, \ldots, i_r\}$ also belong to $\mathcal{B}$; see Definition 9.2. By Proposition 9.4, graphical chordal building sets $\mathcal{B}(G)$ are exactly building sets coming from chordal graphs. By Proposition 9.7, all nestohedra $P_{\mathcal{B}}$ for chordal building sets are flag simple polytopes. So Gal's conjecture applies to this class of *chordal nestohedra,* which include graph-associahedra for chordal graphs and, in particular, for trees.

For a building set $\mathcal{B}$ on $[n]$, define (see Definition 8.7) the set $\mathfrak{S}_n(\mathcal{B})$ of $\mathcal{B}$-*permutations* as the set of permutations w of size n such that, for any $i = 1, \ldots, n$, there exists $I \in \mathcal{B}$ such that $I \subseteq \{w(1), \ldots, w(i)\}$, and I contains both $w(i)$ and $\max\{w(1), w(2), \ldots, w(i)\}$. It turns out that $\mathcal{B}$-permutations are in bijection with vertices of the nestohedron $P_{\mathcal{B}}$; see Proposition 8.10.

Let $\operatorname{des}(w) = \#\{i \mid w(i) > w(i+1)\}$ denote the number of descents in a permutation w. Let $\widehat{\mathfrak{S}}_n$ be the subset of permutations w of size n without two consecutive descents and without final descent, i.e., there is no $i \in [n-1]$ such that $w(i) > w(i+1) > w(i+2)$, assuming that $w(n+1) = 0$.

Theorem 1.1. (Corollary 9.6 and Theorem 11.6) *Let $\mathcal{B}$ be a connected chordal building set on $[n]$. Then the h-vector of the nestohedron $P_{\mathcal{B}}$ is given by*

$$\sum_i h_i\, t^i = \sum_{w \in \mathfrak{S}_n(\mathcal{B})} t^{\operatorname{des}(w)},$$

and the γ-vector of the nestohedron $P_{\mathcal{B}}$ is given by

$$\sum_i \gamma_i\, t^i = \sum_{w \in \mathfrak{S}_n(\mathcal{B}) \cap \widehat{\mathfrak{S}}_n} t^{\operatorname{des}(w)}.$$

This result shows that Gal's conjecture is true for chordal nestohedra.

The paper is structured as follows. Sections 2, 3, and 4 give some background on face numbers and general results about generalized permutohedra. More specifically, Section 2 reviews polytopes, cones, fans, and gives basic terminology of face enumeration for polytopes (*f-vectors*), simple polytopes (*h-vectors*), and *flag* simple polytopes (*γ-vectors*).

Section 3 reviews the definition of generalized permutohedra, and recasts this definition equivalently in terms of their normal fans. It then sets up the dictionary between preposets, and cones and fans coming from the braid arrangement. In particular, one finds that each vertex in a generalized associahedron has associated to it a *poset* that describes its normal cone. This is used to characterize when the polytope is simple, namely when the associated posets have Hasse diagrams which are trees. In Section 4 this leads to a combinatorial formula for the h-vector in terms of descent statistics on these tree-posets.

The remainder of the paper deals with subclasses of simple generalized permutohedra. Section 5 dispenses quickly with the very restrictive class of simple zonotopal generalized permutohedra, namely the *simple graphic zonotopes.*

Sections 6 through 11 deal with the very interesting class of *nestohedra*, culminating with a proof of Gal's conjecture for chordal nestohedra. More specifically, Section 6 discusses nestohedra $P_{\mathcal{B}}$ coming from a building set $\mathcal{B}$, where the posets associated to each vertex are rooted trees. These include graph-associahedra. Section 7 characterizes the flag nestohedra.

Section 8 discusses $\mathcal{B}$-trees and $\mathcal{B}$-permutations. These trees and permutations are in bijection with each other and with vertices of the nestohedron $P_{\mathcal{B}}$. The h-polynomial of a nestohedron is the descent-generating function for $\mathcal{B}$-trees. Then Section 9 introduces the class of chordal building sets and shows that h-polynomials of their nestohedra are descent-generating functions for $\mathcal{B}$-permutations.

Section 10 illustrates these formulas for h-polynomials by several examples: the classical permutohedron and associahedron, the cyclohedron, the stellohedron (the graph-associahedron for the star graph), and the Stanley-Pitman polytope.

Section 11 gives a combinatorial formula for the γ-vector of all chordal nestohedra as a descent-generating function (or peak-generating function) for a subset of $\mathcal{B}$-permutations. This result implies Gal's nonnegativity conjecture for this class of polytopes. The warm-up example here is the classical permutohedron, and the section concludes with the examples of the associahedron and cyclohedron.

Sections 12 through 14 give some graph-associahedra calculations as well as conjectures. Specifically, Section 12 calculates the generating functions for f-polynomials of the graph-associahedra for all trees with one branching point and discusses a relation with Simon Newcomb's problem. Section 13 deals with graphs that are formed by a path with two small fixed graphs attached to the ends. It turns out that the h-vectors of graph-associahedra for such path-like graphs can be expressed in terms of h-vectors of classical associahedra. The section includes explicit formulas for graph-associahedra for the Dynkin diagrams of all finite and affine Coxeter groups. Section 14 gives some bounds and monotonicity conjectures for face numbers of generalized permutohedra.

The paper ends with an Appendix which clarifies the equivalence between various kinds of deformations of a simple polytope.

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2 Face numbers

This section recalls some standard definitions from the theory of convex polytopes and formulates Gal's extension of the Charney-Davis conjecture.

2.1 Polytopes, cones, and fans

A *convex polytope* P is the convex hull of a finite collection of points in $\mathbb{R}^n$. The *dimension* of a polytope (or any other subset in $\mathbb{R}^n$) is the dimension of its affine span.

A *polyhedral cone* in $\mathbb{R}^n$ is a subset defined by a conjunction of weak inequalities of the form $\lambda(x) \geq 0$ for linear forms $\lambda \in (\mathbb{R}^n)^*$. A *face* of a polyhedral cone is a subset of the cone given by replacing some of the inequalities $\lambda(x) \geq 0$ by the equalities $\lambda(x) = 0$.

Two polyhedral cones σ_1, σ_2 *intersect properly* if their intersection is a face of each. A *complete fan of cones* $\mathcal{F}$ in $\mathbb{R}^n$ is a collection of distinct nonempty polyhedral cones covering $\mathbb{R}^n$ such that (1) every nonempty face of a cone in $\mathcal{F}$ is also a cone in $\mathcal{F}$, and (2) any two cones in $\mathcal{F}$ intersect properly. Cones in a fan $\mathcal{F}$ are also called *faces* of $\mathcal{F}$.

Note that fans can be alternatively defined only in terms of their top dimensional faces, as collections of distinct pairwise properly intersecting n-dimensional cones covering $\mathbb{R}^n$.

A *face* F of a convex polytope P is the set of points in P where some linear functional $\lambda \in (\mathbb{R}^n)^*$ achieves its maximum on P, i.e.,

$$F = \{x \in P \mid \lambda(x) = \max\{\lambda(y) \mid y \in P\}\}.$$

Faces that consist of a single point are called *vertices* and 1-dimensional faces are called *edges* of P.

Given any convex polytope P in $\mathbb{R}^n$ and a face F of P, the *normal cone to P at F*, denoted $\mathcal{N}_F(P)$, is the subset of linear functionals $\lambda \in (\mathbb{R}^n)^*$ whose maximum on P is achieved on all of the points in the face F, i.e.,

$$\mathcal{N}_F(P) := \{\lambda \in (\mathbb{R}^n)^* \mid \lambda(x) = \max\{\lambda(y) \mid y \in P\} \text{ for all } x \in F\}.$$

Then $\mathcal{N}_F(P)$ is a polyhedral cone in $(\mathbb{R}^n)^*$, and the collection of all such cones $\mathcal{N}_F(P)$ as one ranges through all faces F of P gives a complete fan in $(\mathbb{R}^n)^*$ called the *normal fan* $\mathcal{N}(P)$. A fan of the form $\mathcal{N}(P)$ for some polytope P is called a *polytopal fan.*

The combinatorial structure of faces of P can be encoded by the *lattice of faces* of P ordered via inclusion. This structure is also encoded by the normal fan $\mathcal{N}(P)$. Indeed, the map $F \mapsto \mathcal{N}_F(P)$ is an inclusion-reversing bijection between the faces of P and the faces of $\mathcal{N}(P)$.

A cone is called *pointed* if it contains no lines (1-dimensional linear subspaces), or equivalently, if it can be defined by a conjunction of inequalities $\lambda_i(x) \geq 0$ in which the λ_i span $(\mathbb{R}^n)^*$. A fan is called *pointed* if all its faces are pointed.

If the polytope $P \subset \mathbb{R}^n$ is full-dimensional, that is $\dim P = n$, then the normal fan $\mathcal{N}(P)$ is pointed. For polytopes P of lower dimension d, define the $(n-d)$-dimensional subspace $P^\perp \subset (\mathbb{R}^n)^*$ of linear functionals that are constant on P. Then all cones in the normal fan $\mathcal{N}(P)$ contain the subspace $P^\perp$. Thus $\mathcal{N}(P)$ can be reduced to a pointed fan in the space $(\mathbb{R}^n)^*/P^\perp$.

A polytope P is called *simple* if any vertex of P is incident to exactly $d = \dim P$ edges. A cone is called *simplicial* if it can be given by a conjunction of linear inequalities $\lambda_i(x) \geq 0$ and linear equations $\mu_j(x) = 0$ where the covectors λ_i and μ_j form a basis in $(\mathbb{R}^n)^*$. A fan is called *simplicial* if all its faces are simplicial. Clearly, simplicial cones and fans are pointed. A convex polytope $P \subset \mathbb{R}^n$ is simple if and only if its (reduced) normal fan $\mathcal{N}(P)/P^\perp$ is simplicial. The *dual simplicial complex* Δ_P of a simple polytope P is the simplicial complex obtained by intersecting the (reduced) normal fan $\mathcal{N}(P)/P^\perp$ with the unit sphere. Note that i-simplices of Δ_P correspond to faces of P of codimension $i+1$.

2.2 f-vectors and h-vectors

For a d-dimensional polytope P, the *face number* $f_i(P)$ is the number of i-dimensional faces of P. The vector $(f_0(P), \ldots, f_d(P))$ is called the f*-vector*, and the polynomial $f_P(t) = \sum_{i=0}^d f_i(P)\, t^i$ is called the f*-polynomial* of P.
Similarly, for a d-dimensional fan $\mathcal{F}$, $f_i(\mathcal{F})$ is the number of i-dimensional faces of $\mathcal{F}$, and $f_{\mathcal{F}}(t) = \sum_{i=0}^d f_i(\mathcal{F})\, t^i$. Note that face numbers of a polytope P and its (reduced) normal cone $\mathcal{F} = \mathcal{N}(P)/P^\perp$ are related as $f_i(P) = f_{d-i}(\mathcal{F})$, or equivalently, $f_P(t) = t^d\, f_{\mathcal{F}}(t^{-1})$.
We will most often deal with the case where P is a simple polytope, or equivalently, when $\mathcal{F}$ is a simplicial fan. In these situations, there is a more compact encoding of the face numbers $f_i(P)$ or $f_i(\mathcal{F})$ by smaller nonnegative integers. One defines the h*-vector* $(h_0(P), \ldots, h_d(P))$ and h*-polynomial* $h_P(t) = \sum_{i=0}^d h_i(P)\, t^i$ uniquely by the relation

$$f_P(t) = h_P(t+1), \quad \text{or equivalently,} \quad f_j(P) = \sum_i \binom{i}{j} h_i(P),\ j = 0, \ldots, d. \tag{1}$$

For a simplicial fan $\mathcal{F}$, the h-vector $(h_0(\mathcal{F}), \ldots, h_d(\mathcal{F}))$ and the h-polynomial $h_{\mathcal{F}}(t) = \sum_{i=0}^d h_i(\mathcal{F})\, t^i$ are defined by the relation $t^d\, f_{\mathcal{F}}(t^{-1}) = h_{\mathcal{F}}(t+1)$, or equivalently, $f_j(\mathcal{F}) = \sum_i \binom{i}{d-j} h_i(\mathcal{F})$, for $j = 0, \ldots, d$. Thus the h-vector of a simple polytope coincides with the h-vector of its normal fan.
The nonnegativity of $h_i(P)$ for a simple polytope P comes from its well-known combinatorial interpretation [Zieg'94, §8.2] in terms of the 1-skeleton of the simple polytope P. Let us extend this interpretation to arbitrary complete simplicial fans.
For a simplicial fan $\mathcal{F}$ in $\mathbb{R}^d$, construct the graph $G_{\mathcal{F}}$ with vertices corresponding to d-dimensional cones and edges corresponding to $(d-1)$-dimensional cones of $\mathcal{F}$, where two vertices of $G_{\mathcal{F}}$ are connected by an edge whenever the corresponding cones share a $(d-1)$-dimensional face. Pick a vector $g \in \mathbb{R}^d$ that does not belong to any $(d-1)$-dimensional face of $\mathcal{F}$ and orient edges of $G_{\mathcal{F}}$, as follows. Orient an edge $\{\sigma_1, \sigma_2\}$ corresponding to two cones σ_1 and σ_2 in $\mathcal{F}$ as (σ_1, σ_2) if the vector g points from σ_1 to σ_2 (in a small neighborhood of the common face of these cones).

Proposition 2.1. *For a simplicial fan $\mathcal{F}$, the ith entry $h_i(\mathcal{F})$ of its h-vector equals the number of vertices with outdegree i in the oriented graph $G_\mathcal{F}$. These numbers satisfy the Dehn-Sommerville symmetry: $h_i(\mathcal{F}) = h_{d-i}(\mathcal{F})$.*

Corollary 2.2. (cf. [Zieg'94, §8.2]) *Let $P \in \mathbb{R}^n$ be a simple polytope. Pick a generic linear form $\lambda \in (\mathbb{R}^n)^*$. Let G_P be the 1-skeleton of P with edges directed so that λ increases on each edge. Then $h_i(P)$ is the number of vertices in G_P of outdegree i.*

Proof of Proposition 2.1. The graph $G_\mathcal{F}$ has a unique vertex of outdegree 0. Indeed, this is the vertex corresponding to the cone in $\mathcal{F}$ containing the vector g. For any face F of $\mathcal{F}$ (of an arbitrary dimension), let $G_\mathcal{F}(F)$ be the induced subgraph on the set of d-dimensional cones of $\mathcal{F}$ containing F as a face. Then $G_\mathcal{F}(F) \simeq G_{\mathcal{F}'}$, where $\mathcal{F}'$ is the link of the face F in the fan $\mathcal{F}$, which is also a simplicial fan of smaller dimension. Thus the subgraph $G_\mathcal{F}(F)$ also contains a unique vertex of outdegree 0 (in this subgraph).
There is a surjective map $\phi : F \mapsto \sigma$ from all faces of $\mathcal{F}$ to vertices of $G_\mathcal{F}$ (i.e., d-dimensional faces of $\mathcal{F}$) that sends a face F to the vertex σ of outdegree 0 in the subgraph $G_\mathcal{F}(F)$. Now, for a vertex σ of $G_\mathcal{F}$ of outdegree i, the preimage $\phi^{-1}(\sigma)$ contains exactly $\binom{d-i}{d-j}$ faces of dimension j. Indeed, $\phi^{-1}(\sigma)$ is formed by taking all possible intersections of σ with some subset of its $(d-1)$-dimensional faces $\{F_1, \dots, F_{d-i}\}$ on which the vector g is directed towards the interior of σ; there are exactly $d-i$ such faces because σ has indegree i in $G_\mathcal{F}$. Thus a face of dimension j in $\phi^{-1}(\sigma)$ has the form $F_{i_1} \cap \cdots \cap F_{i_{d-j}}$ for a $(d-j)$-element subset $\{i_1, \dots, i_{d-j}\} \subseteq [d-i]$.
Let $\tilde{h}_i$ be the number of vertices of $G_\mathcal{F}$ of outdegree i. Counting j-dimensional faces in preimages $\phi^{-1}(\sigma)$ one obtains the relation $f_j(\mathcal{F}) = \sum_i \binom{d-i}{d-j} \tilde{h}_i$. Comparing this with the definition of $h_i(\mathcal{F})$, one deduces that $h_i(\mathcal{F}) = \tilde{h}_{d-i}$.
Note that the numbers $h_i(\mathcal{F})$ do not depend upon the choice of the vector g. It follows that the numbers $\tilde{h}_i$ of vertices with given outdegrees also do not depend on g. Replacing the vector g with $-g$ reverses the orientation of all edges in the d-regular graph $G_\mathcal{F}$, implying the the symmetry $\tilde{h}_i = \tilde{h}_{d-i}$. □

The Dehn-Sommerville symmetry means that h-polynomials are palindromic polynomials: $t^d\, h_\mathcal{F}(\frac{1}{t}) = h_\mathcal{F}(t)$. In this sense the h-vector encoding is more compact, since it is determined by roughly half of its entries, namely $h_0, h_1, \dots, h_{\lfloor \frac{d}{2} \rfloor}$.
Whenever possible, we will try to either give further explicit combinatorial interpretations or generating functions for the f- and h-polynomials of simple generalized permutohedra.

2.3 Flag simple polytopes and γ-vectors

A simplicial complex Δ is called a *flag simplicial complex* or *clique complex* if it has the following property: a collection C of vertices of Δ forms a simplex in Δ if and only if there is an edge in the 1-skeleton of Δ between any two vertices in C. Thus flag simplicial complexes can be uniquely recovered from their 1-skeleta.

We call a simple polytope P is a *flag polytope* if its dual simplicial complex Δ_P is a flag simplicial complex.

We next discuss γ-vectors of flag simple polytopes, as introduced by Gal [Gal'05] and independently in a slightly different context by Bränden [Brä'04, Brä'06]; see also the discussion in [Stem'07, §1D]. A conjecture of Charney and Davis [ChD'95] led Gal [Gal'05] to define the following equivalent encoding of the f-vector or h-vector of a simple polytope P, in terms of smaller integers, which are conjecturally nonnegative when P is *flag.* Every palindromic polynomial $h(t)$ of degree d has a unique expansion in terms of centered binomials $t^i(1+t)^{d-2i}$ for $0 \leq i \leq d/2$, and so one can define the entries $\gamma_i = \gamma_i(P)$ of the γ-*vector* $(\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor \frac{d}{2} \rfloor})$ and the γ-polynomial $\gamma_P(t) := \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i t^i$ uniquely by

$$h_P(t) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i \, t^i (1+t)^{d-2i} = (1+t)^d \gamma_P \left(\frac{t}{(1+t)^2} \right).$$

Conjecture 2.3. *[Gal'05] The γ-vector has nonnegative entries for any flag simple polytope. More generally, the nonnegativity of the γ-vector holds for every flag simplicial homology sphere.*

Thus we will try to give explicit combinatorial interpretations, where possible, for the γ-vectors of flag simple generalized permutohedra. As will be seen in Section 7.1, any graph-associahedron is a flag simple polytope.

3 Generalized permutohedra and the cone-preposet dictionary

This section reviews the definition of generalized permutohedra from [Post'05]. It then records some observations about the relation between cones and fans coming from the braid arrangement and preposets. (Normal fans of generalized permutohedra are examples of such fans.) This leads to a characterization for when generalized permutohedra are simple, an interpretation for their h-vector in this situation, and a corollary about when the associated toric variety is smooth.

The material in this section and in the Appendix (Section 15) has substantial overlap with recent work on *rank tests of non-parametric statistics* [M–W'06]. We have tried to indicate in places the corresponding terminology used in that paper.

3.1 Generalized permutohedra

Recall that a *usual permutohedron* in $\mathbb{R}^n$ is the convex hull of $n!$ points obtained by permuting the coordinates of any vector $(a_1, \ldots, a_n)$ with strictly increasing coordinates $a_1 < \cdots < a_n$. So the vertices of a usual permutohedron can be labelled $v_w = (a_{w^{-1}(1)}, \ldots, a_{w^{-1}(n)})$ by the permutations w in the symmetric group $\mathfrak{S}_n$. The edges of this permutohedron are $[v_w, v_{ws_i}]$, where $s_i = (i, i+1)$ is an adjacent transposition. Then, for any $w \in \mathfrak{S}_n$ and any s_i, one has

$$v_w - v_{ws_i} = k_{w,i}(e_{w(i)} - e_{w(i+1)}) \tag{2}$$

where the $k_{w,i}$ are some *strictly positive* real scalars, and $e_1, \dots, e_n$ are the standard basis vectors in $\mathbb{R}^n$.
Note that a usual permutohedron in $\mathbb{R}^n$ has dimension $d = n-1$, because it is contained in an affine hyperplane where the sum of coordinates $x_1 + \cdots + x_n$ is constant.

Definition 3.1. *[Post'05, Definition 6.1] A generalized permutohedron P is the convex hull of $n!$ points v_w in $\mathbb{R}^n$ labelled by the permutations w in the symmetric group $\mathfrak{S}_n$, such that for any $w \in \mathfrak{S}_n$ and any adjacent transposition s_i, one still has equation (2), but with $k_{w,i}$ assumed only to be nonnegative, that is, $k_{w,i}$ can vanish.*

The Appendix shows that all $n!$ points v_w in a generalized permutohedron P are actually vertices of P (possibly with repetitions); see Theorem 15.3. Thus a generalized permutohedron P comes naturally equipped with the surjective map $\Psi_P : \mathfrak{S}_n \to \mathrm{Vertices}(P)$ given by $\Psi_P : w \mapsto v_w$, for $w \in \mathfrak{S}_m$.
Definition 3.1 says that a generalized permutohedron is obtained by moving the vertices of the usual permutohedron in such a way that directions of edges are preserved, but some edges (and higher dimensional faces) may degenerate. In the Appendix such deformations of a simple polytope are shown to be equivalent to various other notions of deformation; see Proposition 3.2 below and the more general Theorem 15.3.

3.2 Braid arrangement

Let $x_1, \dots, x_n$ be the usual coordinates in $\mathbb{R}^n$. Let $\mathbb{R}^n/(1,\dots,1)\mathbb{R} \simeq \mathbb{R}^{n-1}$ denote the quotient space modulo the 1-dimensional subspace generated by the vector $(1,\dots,1)$. The *braid arrangement* is the arrangement of hyperplanes $\{x_i - x_j = 0\}_{1 \le i < j \le n}$ in the space $\mathbb{R}^n/(1,\dots,1)\mathbb{R}$. These hyperplanes subdivide the space into the polyhedral cones

$$C_w := \{x_{w(1)} \le x_{w(2)} \le \cdots \le x_{w(n)}\}$$

labelled by permutations $w \in \mathfrak{S}_n$, called *Weyl chambers* (of type A). The Weyl chambers and their lower dimensional faces form a complete simplicial fan, sometimes called the *braid arrangement fan.*
Note that a usual permutohedron P has dimension $d = n-1$, so one can reduce its normal fan modulo the 1-dimensional subspace $P^\perp = (1,\dots,1)\mathbb{R}$. The braid arrangement fan is exactly the (reduced) normal fan $\mathcal{N}(P)/P^\perp$ for a usual permutohedron $P \subset \mathbb{R}^n$. Indeed, the (reduced) normal cone $\mathcal{N}_{v_w}(P)/P^\perp$ of P at vertex v_w is exactly the Weyl chamber C_w. (Here one identifies $\mathbb{R}^n$ with $(\mathbb{R}^n)^*$ via the standard inner product.)

Recall that the *Minkowski sum* $P + Q$ of two polytopes $P, Q \subset \mathbb{R}^n$ is the polytope $P+Q := \{x+y \mid x \in P, y \in Q\}$. Say that P is a *Minkowski summand* of R, if there is a polytope Q such that $P+Q = R$. Say that a fan $\mathcal{F}$ is *refined* by a fan $\mathcal{F}'$ if any cone in $\mathcal{F}$ is a union of cones in $\mathcal{F}'$. The following proposition is a special case of Theorem 15.3.

Proposition 3.2. *A polytope P in $\mathbb{R}^n$ is a generalized permutohedron if and only if its normal fan (reduced by $(1,\dots,1)\mathbb{R}$) is refined by the braid arrangement fan.*
Also, generalized permutohedra are exactly the polytopes arising as Minkowski summands of usual permutohedra.

This proposition shows that generalized permutohedra lead to the study of cones given by some inequalities of the form $x_i - x_j \geq 0$ and fans formed by such cones. Such cones are naturally related to posets and preposets.

3.3 Preposets, equivalence relations, and posets

Recall that a *binary relation* R on a set X is a subset of $R \subseteq X \times X$. A *preposet* is a reflexive and transitive binary relation R, that is $(x,x) \in R$ for all $x \in X$, and whenever $(x,y),(y,z) \in R$ one has $(x,z) \in R$. In this case we will often use the notation $x \preceq_R y$ instead of $(x,y) \in R$. Let us also write $x \prec_R y$ whenever $x \preceq_R y$ and $x \neq y$.
An *equivalence relation* $\equiv$ is the special case of a preposet whose binary relation is also symmetric. Every preposet Q gives rise to an equivalence relation $\equiv_Q$ defined by $x \equiv_Q y$ if and only if both $x \preceq_Q y$ and $y \preceq_Q x$. A *poset* is the special case of a preposet Q whose associated equivalence relation $\equiv_Q$ is the trivial partition, having only singleton equivalence classes.
Every preposet Q gives rise to the poset $Q/{\equiv_Q}$ on the equivalence classes $X/{\equiv_Q}$. A preposet Q is uniquely determined by $\equiv_Q$ and $Q/{\equiv_Q}$, that is, a preposet is just an equivalence relation together with a poset structure on the equivalence classes.
A *preposet* Q on X is *connected* if the undirected graph having vertices X and edges $\{x,y\}$ for all $x \preceq_Q y$ is connected.
A *covering relation* $x \lessdot_Q y$ in a poset Q is a pair of elements $x \prec_Q y$ such that there is no z such that $x \prec_Q z \prec_Q y$. The *Hasse diagram* of a poset Q on X is the directed graph on X with edges (x,y) for covering relations $x \lessdot_Q y$.
We call a poset Q a *tree-poset* if its Hasse diagram is a spanning tree on X. Thus tree-posets correspond to directed trees on the vertex set X.
A *linear extension* of a poset Q on X is a linear ordering $(y_1,\dots,y_n)$ of all elements in X such that $y_1 \prec_Q y_2 \prec_Q \cdots \prec_Q y_n$. Let $\mathcal{L}(Q)$ denote the set of all linear extensions of Q.
The *union* $R_1 \cup R_2$ of two binary relations R_1, R_2 on X is just their union as two subsets of $X \times X$. Given any reflexive binary relation Q, denote by $\overline{Q}$ the preposet which is its transitive closure. Note that if Q_1 and Q_2 are two preposets on the same set X, then the binary relation $Q_1 \cup Q_2$ is not necessarily

a preposet. However, we can obtain a preposet by taking its transitive closure $\overline{Q_1 \cup Q_2}$.

Let $R \subseteq Q$ denote containment of binary relations on the same set, meaning containment as subsets of $X \times X$. Also let R^{op} be the *opposite* binary relation, that is $(x, y) \in R^{op}$ if and only if $(y, x) \in R$.

For two preposets P and Q on the same set, let us say that Q is a *contraction* of P if there is a binary relation $R \subseteq P$ such that $Q = \overline{P \cup R^{op}}$. In other words, the equivalence classes of $\equiv_Q$ are obtained by merging some equivalence classes of $\equiv_P$ *along relations* in P and the poset structure on equivalence classes of $\equiv_Q$ is induced from the poset structure on classes of $\equiv_P$.

For example, the preposet $1 < \{2, 3\} < 4$ (where $\{2, 3\}$ is an equivalence class) is a contraction of the poset $P = (1 < 3, 2 < 3, 1 < 4, 2 < 4)$. However, the preposet $(\{1, 2\} < 3,\ \{1, 2\} < 4)$ is not a contraction of P because 1 and 2 are incomparable in P.

Definition 3.3. *We say that two preposets Q_1 and Q_2 on the same set intersect properly if the preposet $\overline{Q_1 \cup Q_2}$ is both a contraction of Q_1 and of Q_2. A complete fan of posets*[4] *on X is a collection of distinct posets on X which pairwise intersect properly, and whose linear extensions (disjointly) cover all linear orders on X.*

Compare Definition 3.3 to the definitions of properly intersecting cones and complete fans of cones in Section 2.1. Proposition 3.5 will elucidate this connection.

Example 3.4. *The two posets $P_1 := 1 < 2$ and $P_2 := 2 < 1$ on the set $\{1, 2\}$ intersect properly. Here $\overline{P_1 \cup P_2}$ is equal to $\{1 < 2, 2 < 1\}$. These P_1 and P_2 form a complete fan of posets. However, the two posets $Q_1 := 2 < 3$ and $Q_2 := 1 < 2 < 3$ on the set $\{1, 2, 3\}$ do not intersect properly. In this case $\overline{Q_1 \cup Q_2} = Q_2$, which is not a contraction of Q_1.*

3.4 The dictionary

Let us say that a *braid cone* is a polyhedral cone in the space $\mathbb{R}^n/(1, \ldots, 1)\mathbb{R} \simeq \mathbb{R}^{n-1}$ given by a conjunction of inequalities of the form $x_i - x_j \geq 0$. In other words, braid cones are polyhedral cones formed by unions of Weyl chambers or their lower dimensional faces.

There is an obvious bijection between preposets and braid cones. For a preposet Q on the set $[n]$, let σ_Q be the braid cone in the space $\mathbb{R}^n/(1, \ldots, 1)\mathbb{R}$ defined by the conjunction of the inequalities $x_i \leq x_j$ for all $i \preceq_Q j$. Conversely, one can always reconstruct the preposet Q from the cone σ_Q by saying that $i \preceq_Q j$ whenever $x_i \leq x_j$ for all points in σ_Q.

Proposition 3.5. *Let the cones σ, σ' correspond to the preposets Q, Q' under the above bijection. Then*

[4]In [M–W'06], this is called a *convex rank test.*

(1) The preposet $\overline{Q \cup Q'}$ corresponds to the cone $\sigma \cap \sigma'$.

(2) The preposet Q is a contraction of Q' if and only if the cone σ is a face σ'.

(3) The preposets Q, Q' intersect properly if and only if the cones σ, σ' do.

(4) Q is a poset if and only if σ is a full-dimensional cone, i.e., $\dim \sigma = n-1$.

(5) The equivalence relation $\equiv_Q$ corresponds to the linear span $\mathrm{Span}(\sigma)$ *of σ.*

(6) The poset $Q/{\equiv_Q}$ corresponds to a full-dimensional cone inside $\mathrm{Span}(\sigma_Q)$.

(7) The preposet Q is connected if and only if the cone σ is pointed.

(8) If Q is a poset, then the minimal set of inequalities describing the cone σ is $\{x_i \leq x_j \mid i \lessdot_Q j\}$. (These inequalities associated with covering relations in Q are exactly the facet inequalities for σ.)

(9) Q is a tree-poset if and only if σ is a full-dimensional simplicial cone.

(10) For $w \in \mathfrak{S}_n$, the cone σ contains the Weyl chamber C_w if and only if Q is a poset and w is its linear extension, that is $w(1) \prec_Q w(2) \prec_Q \cdots \prec_Q w(n)$.

Proof. (1) The cone $\sigma \cap \sigma'$ is given by conjunction of all inequalities for σ and σ'. The corresponding preposet is obtained by adding all inequalities that follow from these, i.e., by taking the transitive closure of $Q \cup Q'$.
(2) Faces of σ' are obtained by replacing some inequalities $x_i \leq x_j$ defining σ' with equalities $x_i = x_j$, or equivalently, by adding the opposite inequalities $x_i \geq x_j$.
(3) follows from (1) and (2).
(4) σ is full-dimensional if its defining relations do not include any equalities $x_i = x_j$, that is $\equiv_Q$ has only singleton equivalence classes.
(5) The cone associated with the equivalence relation $\equiv_Q$ is given by the equations $x_i = x_j$ for $i \equiv_Q j$, which is exactly $\mathrm{Span}(\sigma)$.
(6) Follows from (4) and (5).
(7) The maximal subspace contained in the half-space $\{x_i \leq x_j\}$ is given by $x_i = x_j$. Thus the maximal subspace contained in the cone σ is given by the conjunction of equations $x_i = x_j$ for $i \leq_Q j$. If Q is disconnected then this subspace has a positive dimension. If Q is connected then this subspace is given by $x_1 = \cdots = x_n$, which is just the origin in the space $\mathbb{R}^n/(1, \ldots, 1)\mathbb{R}$.
(8) The inequalities for the covering relations $i \lessdot_Q j$ imply all other inequalities for σ and they cannot be reduced to a smaller set of inequalities.
(9) By (4) and (7) full-dimensional pointed cones correspond to connected posets. These cones will be simplicial if they are given by exactly $n-1$ inequalities. By (8) this means that the corresponding poset should have exactly $n-1$ covering relations, i.e., it is a tree-poset.
(10) Follows from (4) and definitions. ∎

According to Proposition 3.5, a full-dimensional braid cone σ associated with a poset Q can be described in three different ways (via all relations in Q, via covering relations in Q, and via linear extensions $\mathcal{L}(Q)$ of Q) as

$$\sigma = \{x_i \le x_j \mid i \preceq_Q j\} = \{x_i \le x_j \mid i \lessdot_Q j\} = \bigcup_{w \in \mathcal{L}(Q)} C_w.$$

Let $\mathcal{F}$ be a family of d-cones in $\mathbb{R}^d$ which intersect properly. Since they have disjoint interiors, they will correspond to a complete fan if and only if their closures cover $\mathbb{R}^d$, or equivalently, their spherical volumes sum to the volume of the full $(d-1)$-sphere.

A braid cone corresponding to a poset Q is the union of the Weyl chambers C_w for all linear extensions $w \in \mathcal{L}(Q)$, and every Weyl chamber has the same spherical volume ($\frac{1}{n!}$ of the sphere) due to the transitive Weyl group action. Therefore, a collection of properly intersecting posets $\{Q_1, \ldots, Q_t\}$ on $[n]$ correspond to a complete fan on braid cones if and only if

$$\bigcup_{i=1}^{t} \mathcal{L}(Q_i) = \mathfrak{S}_n (\text{disjoint union}), \text{ or equivalently, if and only if} \sum_{i=1}^{t} |\mathcal{L}(Q_i)| = n!,$$

cf. Definition 3.3.

Corollary 3.6. *A complete fan of braid cones (resp., pointed braid cones, simplicial braid cones) in $\mathbb{R}^n/(1, \ldots, 1)\mathbb{R}$ corresponds to a complete fan of posets (resp., connected posets, tree-posets) on $[n]$.*

Using Proposition 3.2, we can relate Proposition 3.5 and Corollary 3.6 to generalized permutohedra. Indeed, normal cones of a generalized permutohedron (reduced modulo $(1, \ldots, 1)\mathbb{R}$) are braid cones.

For a generalized permutohedron P, define the *vertex poset* Q_v at a vertex $v \in \text{Vertices}(P)$ as the poset on $[n]$ associated with the normal cone $\mathcal{N}_v(P)/(1, \ldots, 1)\mathbb{R}$ at the vertex v, as above.

Corollary 3.7. *For a generalized permutohedron (resp., simple generalized permutohedron) P, the collection of vertex posets $\{Q_v \mid v \in \text{Vertices}(P)\}$ is a complete fan of posets (resp., tree-posets).*

Thus normal fans of generalized permutohedra correspond to certain complete fans of posets, which we call *polytopal.* In [M–W'06], the authors call such fans *submodular rank tests,* since they are in bijection with the faces of the cone of submodular functions. That cone is precisely the deformation cone we discuss in the Appendix.

Example 3.8. *In [M–W'06], the authors modify an example of Studený [Stud'05] to exhibit a non-polytopal complete fan of posets. They also kindly provided us with the following further nonpolytopal example, having* 16 *posets Q_v, all of them tree-posets:* $(1, 2 < 3 < 4)$ *(which means that* $1 < 3$ *and*

$2 < 3$), $(1, 2 < 4 < 3)$, $(3, 4 < 1 < 2)$, $(3, 4 < 2 < 1)$, $(1 < 4 < 2, 3)$, $(4 < 1 < 2, 3)$, $(2 < 3 < 1, 4)$, $(3 < 2 < 1, 4)$, $(1 < 3 < 2 < 4)$, $(1 < 3 < 4 < 2)$, $(3 < 1 < 2 < 4)$, $(3 < 1 < 4 < 2)$, $(2 < 4 < 1 < 3)$, $(2 < 4 < 3 < 1)$, $(4 < 2 < 1 < 3)$, $(4 < 2 < 3 < 1)$. *This gives a complete fan of simplicial cones, but does not correspond to a (simple) generalized permutohedron.*

Recall that $\Psi_P : \mathfrak{S}_n \to \mathrm{Vertices}(P)$ is the surjective map $\Psi_P : w \mapsto v_w$; see Definition 3.1. The previous discussion immediately implies the following corollary.

Corollary 3.9. *Let P be a generalized permutohedron in $\mathbb{R}^n$, and $v \in \mathrm{Vertices}(P)$ be a vertex. For $w \in \mathfrak{S}_n$, one has $\Psi_P(w) = v$ whenever the normal cone $\mathcal{N}_v(P)$ contains the Weyl chamber C_w. The preimage $\Psi_P^{-1}(v) \subseteq \mathfrak{S}_n$ of a vertex $v \in \mathrm{Vertices}(P)$ is the set $\mathcal{L}(Q_v)$ of all linear extensions of the vertex poset Q_v.*

We remark on the significance of this cone-preposet dictionary for toric varieties associated to generalized permutohedra or their normal fans; see Fulton [Ful'93] for further background.
A complete fan $\mathcal{F}$ of polyhedral cones in $\mathbb{R}^d$ whose cones are rational with respect to $\mathbb{Z}^d$ gives rise to a toric variety $X_{\mathcal{F}}$, which is normal, complete and of complex dimension d.
This toric variety is *projective* if and only if $\mathcal{F}$ is the normal fan $\mathcal{N}(P)$ for some polytope P, in which case one also denotes $X_{\mathcal{F}}$ by X_P.
The toric variety $X_{\mathcal{F}}$ is *quasi-smooth* or *orbifold* if and only if $\mathcal{F}$ is a complete fan of simplicial cones; in the projective case, where $\mathcal{F} = \mathcal{N}(P)$, this corresponds to P being a simple polytope.
In this situation, the h-numbers of $\mathcal{F}$ (or of P) have the auxiliary geometric meaning as the (singular cohomology) Betti numbers $h_i = \dim H^i(X_{\mathcal{F}}, \mathbb{C})$. The symmetry $h_i = h_{d-i}$ reflects Poincaré duality for this quasi-smooth variety.
The toric variety $X_{\mathcal{F}}$ is *smooth* exactly when every top-dimensional cone of $\mathcal{F}$ is not only simplicial but *unimodular*, that is, the primitive vectors on its extreme rays form a $\mathbb{Z}$-basis for $\mathbb{Z}^d$. Equivalently, the facet inequalities $\ell_1, \ldots, \ell_d$ can be chosen to form a $\mathbb{Z}$-basis for $(\mathbb{Z}^d)^* = \mathrm{Hom}(\mathbb{Z}^d, \mathbb{Z})$ inside $(\mathbb{R}^d)^*$. One has $X_{\mathcal{F}}$ both smooth and projective if and only if $\mathcal{F} = \mathcal{N}(P)$ for a *Delzant* polytope P, that is, one which is simple and has every vertex normal cone unimodular.

Corollary 3.10. (cf. [Zel'06, §5]) *A complete fan $\mathcal{F}$ of posets gives rise to a complete toric variety $X_{\mathcal{F}}$, which will be projective if and only if $\mathcal{F}$ is associated with the normal fan $\mathcal{N}(P)$ for a generalized permutohedron.*
A complete fan $\mathcal{F}$ of tree-posets gives rise to a (smooth, not just orbifold) toric variety $X_{\mathcal{F}}$, which will be projective if and only $\mathcal{F}$ is associated with the normal fan $\mathcal{N}(P)$ of a simple generalized permutohedron. In other words, simple generalized permutohedra are always Delzant.

Proof. All the assertions should be clear from the above discussion, except for the last one about simple generalized permutohedra being Delzant. However, a tree-poset Q corresponds to a set of functionals $x_i - x_j$ for the edges $\{i,j\}$ of a tree, which are well-known to give a $\mathbb{Z}$-basis for $(\mathbb{Z}^d)^*$, cf. [Post'05, Proposition 7.10]. ☐

4 Simple generalized permutohedra

The goal of this section is to combinatorially interpret the h-vector of any simple generalized permutohedron.

4.1 Descents of tree-posets and h-vectors

Definition 4.1. *Given a poset Q on $[n]$, define the descent set* $\mathrm{Des}(Q)$ *to be the set of ordered pairs (i,j) for which $i \lessdot_Q j$ is a covering relation in Q with $i >_{\mathbb{Z}} j$, and define the statistic number of descents* $\mathrm{des}(Q) := |\mathrm{Des}(Q)|$.

Theorem 4.2. *Let P be a simple generalized permutohedron, with vertex posets $\{Q_v\}_{v\in\mathrm{Vertices}(P)}$. Then one has the following expression for its h-polynomial:*

$$h_P(t) = \sum_{v\in\mathrm{Vertices}(P)} t^{\mathrm{des}(Q_v)}. \tag{3}$$

More generally, for a complete fan $\mathcal{F} = \{Q_v\}$ of tree-posets (see Definition 3.3), one also has $h_{\mathcal{F}}(t) = \sum_v t^{\mathrm{des}(Q_v)}$.

Proof. (cf. proof of Proposition 7.10 in [Post'05]) Let us prove the more general claim about fans of tree-posets, that is, simplicial fans coarsening the braid arrangement fan.

Pick a generic vector $g = (g_1,\ldots,g_n) \in \mathbb{R}^n$ such that $g_1 < \cdots < g_n$ and construct the directed graph $G_{\mathcal{F}}$, as in Proposition 2.1. Let $\sigma = \{x_i \le x_j \mid i \lessdot_{Q_v} j\}$ be the cone of $\mathcal{F}$ associated with poset Q_v; see Proposition 3.5(8). Let σ' be an adjacent cone separated from σ by the facet $x_i = x_j$, $i \lessdot_{Q_v} j$. The vector g points from σ to σ' if and only if $g_i >_{\mathbb{R}} g_j$, or equivalently, $i >_{\mathbb{Z}} j$. Thus the outdegree of σ in the graph $G_{\mathcal{F}}$ is exactly the descent number $\mathrm{des}(Q)$. The claim now follows from Proposition 2.1. ☐

For a usual permutohedron P in $\mathbb{R}^n$, the vertex posets Q_v are just all linear orders on $[n]$. So $h_P(t)$ is the classical *Eulerian polynomial*[5]

$$A_n(t) := \sum_{w\in\mathfrak{S}_n} t^{\mathrm{des}(w)}, \tag{4}$$

where $\mathrm{des}(w) := \#\{i \mid w(i) > w(i+1)\}$ is the *descent number* of a permutation w.

Any element w in the Weyl group $\mathfrak{S}_n$ sends a complete fan $\mathcal{F} = \{Q_i\}$ of tree-posets to another such complete fan $w\mathcal{F} = \{wQ_i\}$, by relabelling all of the posets. Since $w\mathcal{F}$ is an isomorphic simplicial complex, with the same h-vector, this leads to a curious corollary.

[5] Note that a more standard convention is to call $tA_n(t)$ the Eulerian polynomial.

Definition 4.3. *Given a tree-poset Q on $[n]$, define its generalized Eulerian polynomial*

$$A_Q(t) := \sum_{w \in \mathfrak{S}_n} t^{\mathrm{des}(wQ)}.$$

Note that A_Q depends upon Q only as an unlabelled poset.

When Q is a linear order, $A_Q(t)$ is the usual Eulerian polynomial $A_n(t)$.

Corollary 4.4. *The h-polynomial $h_P(t)$ of a simple generalized permutohedron P is the "average" of the generalized Eulerian polynomials of its vertex tree-posets Q_v:*

$$h_P(t) = \frac{1}{n!} \sum_{v \in \mathrm{Vertices}(P)} A_{Q_v}(t).$$

See Example 5.5 below for an illustration of Theorem 4.2 and Corollary 4.4.

4.2 Bounds on the h-vector and monotonicity

It is natural to ask for upper and lower bounds on the h-vectors of simple generalized permutohedra. Some of these follow immediately from an h-vector monotonicity result of Stanley [Stan'92] that applies to complete simplicial fans.

Definition 4.5. *A simplicial complex Δ' is a geometric subdivision of a simplicial complex Δ if they have geometric realizations that are topological spaces on the same underlying set, and every face of Δ' is contained in a single face of Δ.*

Theorem 4.6. (see [Stan'92, Theorem 4.1]) *If Δ' is a geometric subdivision of a Cohen-Macaulay simplicial complex Δ, then the h-vector of Δ' is componentwise weakly larger than that of Δ. In particular this holds when Δ, Δ' come from two complete simplicial fans and Δ' refines Δ, e.g., the normal fans of two simple polytopes P, P' in which P is a Minkowski summand of P.*

Corollary 4.7. *A simple generalized permutohedron P in $\mathbb{R}^n$ has h-polynomial coefficientwise smaller than that of the permutohedron, namely the Eulerian polynomial $A_n(t)$.*

Proof. Proposition 3.2 tells us that the normal fan of P is refined by that of the permutohedron, so the above theorem applies. $\square$

Question 4.8. *Does the permutohedron also provide an upper bound for the f-vectors, flag f- and flag h-vectors, generalized h-vectors, and cd-indices of generalized permutohedra also in the non-simple case? Is there also a monotonicity result for these other forms of face and flag number data when one has two generalized permutohedra P, P' in which P is a Minkowski summand of P'?*

The answer is "Yes" for f-vectors and flag f-vectors, which clearly increase under subdivision. The answer is also "Yes" for generalized h-vectors, which Stanley also showed [Stan'92, Corollary 7.11] can only increase under geometric subdivisions of rational convex polytopes. But for flag h-vectors and cd-indices, this is not so clear.

Later on (Example 6.11, Section 7.2, and Section 14) we will say more about lower bounds for h-vectors of simple generalized permutohedra within various classes.

5 The case of zonotopes

This section illustrates some of the foregoing results in the case where the simple generalized permutohedron is a zonotope; see also [Post'05, §8.6]. Zonotopal generalized permutohedra are exactly the graphic zonotopes, and those that are simple correspond to a very restrictive class of graphs that are easily dealt with.

A *zonotope* is a convex polytope Z which is the Minkowski sum of one-dimensional polytopes (line segments), or equivalently, a polytope whose normal fan $\mathcal{N}(Z)$ coincides with chambers and cones of a hyperplane arrangement. Under this equivalence, the line segments which are the Minkowski summands of Z lie in the directions of the normal vectors to the hyperplanes in the arrangement. Given a graph $G = (V, E)$ without loops or multiple edges, on node set $V = [n]$ and with edge set E, define the associated *graphic zonotope* Z_G to be the Minkowski sum of line segments in the directions $\{e_i - e_j\}_{ij \in E}$.

Proposition 3.2 then immediately implies the following.

Proposition 5.1. *The zonotopal generalized permutohedra are exactly the graphic zonotopes Z_G.*

Simple zonotopes are very special among all zonotopes, and simple graphic zonotopes have been observed [Kim'06] to correspond to a very restrictive class of graphic zonotopes, namely those whose biconnected components are all complete graphs.

Recall that for a graph $G = (V, E)$, there is an equivalence relation on E defined by saying $e \sim e'$ if there is some circuit (i.e., cycle which is minimal with respect to inclusion of edges) of G containing both e, e'. The $\sim$-equivalence classes are then called *biconnected components* of G.

Proposition 5.2. [Kim'06, Remark 5.2] *The graphic zonotope Z_G corresponding to a graph $G = (V, E)$ is a simple polytope if and only if every biconnected component of G is the set of edges of a complete subgraph some subset of the vertices V.*

In this case, if $V_1, \ldots, V_r \subseteq V$ are the node sets for these complete subgraphs, then Z_G is isomorphic to the Cartesian product of usual permutohedra of dimensions $|V_j| - 1$ for $j = 1, 2, \ldots, r$.

Let us give another description for this class of graphs. For a graph F with n edges $e_1, \ldots, e_n$, the *line graph* $\mathrm{Line}(F)$ of F is the graph on the vertex set $[n]$ where $\{i, j\}$ is an edge in $\mathrm{Line}(F)$ if and only if the edges e_i and e_j of F have a common vertex. The following claim is left as an exercise for the reader.

Remark 5.3. *For a graph G, all biconnected components of G are edge sets of complete graphs if and only if G is isomorphic to the line graph* $\mathrm{Line}(F)$ *of some forest F. Biconnected components of* $\mathrm{Line}(F)$ *correspond to non-leaf vertices of F.*

For the sake of completeness, we include a proof of Proposition 5.2.

Proof of Proposition 5.2. If the biconnected components of G induce subgraphs isomorphic to graphs $G_1, \ldots, G_r$ then one can easily check that Z_G is the Cartesian product of the zonotopes Z_{G_i}. Since a Cartesian product of polytopes is simple if and only if each factor is simple, this reduces to the case where $r = 1$. Also note that when $r = 1$ and G is a complete graph, then Z_G is the permutohedron, which is well-known to be simple.
For the reverse implication, assume G is biconnected but not a complete graph, and it will suffice, by Proposition 3.5(9), to construct a vertex v of Z_G whose poset Q_v is not a tree-poset. One uses the fact [GZ'83] that a vertex v in the graphic zonotope Z_G corresponds to an acyclic orientation of G, and the associated poset Q_v on V is simply the transitive closure of this orientation. Thus it suffices to produce an acyclic orientation of G whose transitive closure has Hasse diagram which is not a tree.
Since G is biconnected but not complete, there must be two vertices $\{x, y\}$ that do not span an edge in E, but which lie in some circuit C. Traverse this circuit C in some cyclic order, starting at the node x, passing through some nonempty set of vertices V_1 before passing through y, and then through a nonempty set of vertices V_2 before returning to x. One can then choose arbitrarily a total order on the node set V so that these sets appear as segments in this order:

$$V_1, \quad x, \quad y, \quad V_2, \quad V - (V_1 \cup V_2 \cup \{x, y\}).$$

It is then easily checked that if one orients the edges of G consistently with this total order, then the associated poset has a non-tree Hasse diagram: for any $v_1 \in V_1$ and $v_2 \in V_2$, one has $v_1 < x, y < v_2$ with x, y incomparable. $\square$

Corollary 5.4. *Let Z_G be a simple graphic zonotope, with notation as in Proposition 5.2. Then Z_G is flag, and its f-polynomial, h-polynomial, and γ-polynomial are all equal to products for $j = 1, 2, \ldots, r$ of the f-, h-, or γ-polynomials of $(|V_j| - 1)$-dimensional permutohedra.*

Proof. Use Proposition 5.2 along with the fact that a Cartesian product of simple polytopes is flag if and only if each factor is flag, and has f-, h- and γ-polynomial equal to the product of the same polynomials for each factor. $\square$

Example 5.5. *Consider the graph $G = (V, E)$ with $V = [4] := \{1, 2, 3, 4\}$ and $E = \{12, 13, 23, 14\}$, whose biconnected components are the triangle 123 and the edge 14, which are both complete subgraphs on node sets $V_1 = \{1, 2, 3\}$ and $V_2 = \{1, 4\}$. Hence the graphic zonotope Z_G is simple and flag, equal to the Cartesian product of a hexagon with a line segment, that is, Z_G is a hexagonal prism.*

Its f-, h- and γ-polynomials are

$$\begin{array}{lll} f_{Z_G}(t) & = (2+t)(6+6t+t^2) & = 12+18t+8t^2+t^3 \\ h_{Z_G}(t) & = A_2(t)A_3(t) = (1+t)(1+4t+t^2) & = 1+5t+5t^2+t^3 \\ \gamma_{Z_G}(t) & = (1)(1+2t) & = 1+2t. \end{array}$$

One can arrive at the same h-polynomial using Theorem 4.2. One lists the tree-posets Q_v for each of the 12 vertices v of the hexagonal prism Z_G, coming in 5 isomorphism types, along with the number of descents for each:

type	*poset Q_v*	des	*type*	*poset Q_v*	des
chain:	$2<3<1<4$	1	*wedge:*	$2<3<1$ *and* $4<1$	2
	$3<2<1<4$	2		$3<2<1$ *and* $4<1$	3
	$4<1<2<3$	1	*wye:*	$2<1<3$ *and* $1<4$	1
	$4<1<3<2$	2		$3<1<2$ *and* $1<4$	1
vee:	$1<2<3$ *and* $1<4$	0	*lambda:*	$3<1<2$ *and* $4<1$	2
	$1<3<2$ *and* $1<4$	1		$2<1<3$ *and* $4<1$	2

and finds that $\sum_v t^{\mathrm{des}(Q_v)} = 1+5t+5t^2+t^3$.
Lastly one can get this h-polynomial from Corollary 4.4, by calculating directly that

$$\begin{aligned} A_{chain}(t) &= 1+11t+11t^2+t^3 = A_4(t) \\ A_{vee}(t) &= 3+10t+8t^2+3t^3 \\ A_{wedge}(t) &= 3+8t+10t^2+3t^3 \\ A_{wye}(t) = A_{lambda}(t) &= 2+10t+10t^2+2t^3 \end{aligned}$$

and then the h-polynomial is

$$\frac{1}{4!}\left[4A_{chain}(t)+2A_{vee}(t)+2A_{wedge}(t)+2A_{wye}(t)+2A_{lambda}(t)\right]$$
$$== 1+5t+5t^2+t^3.$$

6 Building sets and nestohedra

This section reviews some results from [FS'05], [Post'05], and [Zel'06] regarding the important special case of generalized permutohedra that arise from building sets. These generalized permutohedra, which we call *nestohedra*, turn out to always be simple, and they include the graph-associahedra, which in turn generalize the associahedron, cyclohedron, and permutohedron; this will be fleshed out in Section 10. Their dual simplicial complexes, the nested set complexes, are defined, and several tools are given for calculating their f- and h-vectors. The notion of nested sets goes back to work of Fulton and MacPherson [FM'94], and DeConcini and Procesi [DP'95] defined building sets and nested set complexes. However, our exposition mostly follows [Post'05] and [Zel'06].

6.1 Building sets, nestohedra, and nested set complexes

Definition 6.1. [Post'05, Definition 7.1] *A collection $\mathcal{B}$ of nonempty subsets of a finite set S is a building set if it satisfies the conditions:*

(B1) If $I, J \in \mathcal{B}$ and $I \cap J \neq \emptyset$, then $I \cup J \in \mathcal{B}$.

(B2) $\mathcal{B}$ contains all singletons $\{i\}$, for $i \in S$.

For a building set $\mathcal{B}$ on S and a subset $I \subseteq S$, define the *restriction* of $\mathcal{B}$ to I as $\mathcal{B}|_I := \{J \in \mathcal{B} \mid J \subseteq I\}$. Let $\mathcal{B}_{\max} \subset \mathcal{B}$ denote the inclusion-maximal subsets of a building $\mathcal{B}$. Then elements of $\mathcal{B}_{\max}$ are pairwise disjoint subsets that partition the set S. Call the restrictions $\mathcal{B}|_I$, for $I \in \mathcal{B}_{\max}$, the *connected components* of $\mathcal{B}$. Say that a building set is *connected* if $\mathcal{B}_{\max}$ has only one element: $\mathcal{B}_{\max} = \{S\}$.

Example 6.2. *Let G be a graph (with no loops nor multiple edges) on the node set S. The graphical building $\mathcal{B}(G)$ is the set of nonempty subsets $J \subseteq S$ such that the induced graph $G|_J$ on node set J is connected. Then $\mathcal{B}(G)$ is a building set.*
The graphical building set $\mathcal{B}(G)$ is connected if and only if the graph G is connected. The connected components of the graphical $\mathcal{B}(G)$ building set correspond to connected components of the graph G. Also each restriction $\mathcal{B}(G)|_I$ is the graphical building set $\mathcal{B}(G|_I)$ for the induced subgraph $G|_I$.

Definition 6.3. *Let $\mathcal{B}$ be a building set on $[n] := \{1, \ldots, n\}$. Faces of the standard coordinate simplex in $\mathbb{R}^n$ are the simplices $\Delta_I :=$ ConvexHull$(e_i \mid i \in I)$, for $I \subseteq [n]$, where the e_i are the endpoints of the coordinate vectors in $\mathbb{R}^n$. Define the nestohedron[6] $P_{\mathcal{B}}$ as the Minkowski sum of these simplices*

$$P_{\mathcal{B}} := \sum_{I \in \mathcal{B}} y_I \Delta_I, \tag{5}$$

where y_I are strictly positive real parameters; see [Post'05, Section 6].

Note that since each of the normal fans $\mathcal{N}(\Delta_I)$ is refined by the braid arrangement fan, the same holds for their Minkowski sum [Zieg'94, Prop. 7.12], and hence the nestohedra $P_{\mathcal{B}}$ are generalized permutohedra by Proposition 3.2.

[6] Called the *nested set polytope* in [Zel'06].

It turns out that $P_{\mathcal{B}}$ is always a simple polytope, whose poset of faces does not depend upon the choice of the positive parameters y_I. To describe this combinatorial structure it is convenient to describe the dual simplicial complex of $P_{\mathcal{B}}$.

Definition 6.4. [Post'05, Definition 7.3] *For a building set $\mathcal{B}$, let us say that a subset $N \subseteq \mathcal{B} \setminus \mathcal{B}_{\max}$ is a nested set if it satisfies the conditions:*

(N1) For any $I, J \in N$ one has either $I \subseteq J$, $J \subseteq I$, or I and J are disjoint.

(N2) For any collection of $k \geq 2$ disjoint subsets $J_1, \ldots, J_k \in N$, their union $J_1 \cup \cdots \cup J_k$ is not in $\mathcal{B}$.

Define the nested set complex $\Delta_{\mathcal{B}}$ as the collection of all nested sets for $\mathcal{B}$.

It is immediate from the definition that the nested set complex $\Delta_{\mathcal{B}}$ is an abstract simplicial complex on node set $\mathcal{B}$. Note that this slightly modifies the definition of a nested set from [Post'05], following [Zel'06], in that one does not include elements of $\mathcal{B}_{\max}$ in nested sets.

Theorem 6.5. [Post'05, Theorem 7.4], [FS'05, Theorem 3.14] *Let $\mathcal{B}$ be a building set on $[n]$. The nestohedron $P_{\mathcal{B}}$ is a simple polytope of dimension $n - |\mathcal{B}_{\max}|$. Its dual simplicial complex is isomorphic to the nested set complex $\Delta_{\mathcal{B}}$.*

An explicit correspondence between faces of $P_{\mathcal{B}}$ and nested sets in $\Delta_{\mathcal{B}}$ is described in [Post'05, Proposition 7.5]. The dimension of the face of $P_{\mathcal{B}}$ associated with a nested set $N \in \Delta_{\mathcal{B}}$ equals $n - |N| - |\mathcal{B}_{\max}|$. Thus vertices of $P_{\mathcal{B}}$ correspond to inclusion-maximal nested sets in $\Delta_{\mathcal{B}}$, and all maximal nested sets contain exactly $n - |\mathcal{B}_{\max}|$ elements.

Remark 6.6. *For a building set $\mathcal{B}$ on $[n]$, it is known [FY'04, Theorem 4] that one can obtain the nested set complex $\Delta_{\mathcal{B}}$ (resp., the nestohedron $P_{\mathcal{B}}$) via the following stellar subdivision (resp., shaving) construction, a common generalization of*

- *the barycentric subdivision of a simplex as the dual of the permutohedron,*
- *Lee's construction of the associahedron [Lee'89, §3].*

Start with an $(n-1)$-simplex whose vertices (resp., facets) have been labelled by the singletons i for $i \in [n]$, which are all in $\mathcal{B}$. Then proceed through each of the non-singleton sets I in $\mathcal{B}$, in any order that reverses inclusion (i.e., where larger sets come before smaller sets), performing a stellar subdivision on the face with vertices (resp., shave off the face which is the intersection of facets) indexed by the singletons in I.

Remark 6.7. Note that if $\mathcal{B}_1, \ldots, \mathcal{B}_k$ are the connected components of a building set $\mathcal{B}$, then $P_{\mathcal{B}}$ is isomorphic to the direct product of polytopes $P_{\mathcal{B}_1} \times \cdots \times P_{\mathcal{B}_k}$. Thus it is enough to investigate generalized permutohedra $P_{\mathcal{B}}$ and nested set complexes $\Delta_{\mathcal{B}}$ only for connected buildings.

Remark 6.8. *The definition* (5) *of the nestohedron $P_{\mathcal{B}}$ as a Minkowski sum should make it clear that whenever one has two building sets $\mathcal{B} \subseteq \mathcal{B}'$, then $P_{\mathcal{B}}$ is a Minkowski summand of $P_{\mathcal{B}'}$. Hence Theorem 4.6 implies the h-vector of $P_{\mathcal{B}'}$ is componentwise weakly larger than that of $P_{\mathcal{B}}$.*

Remark 6.9. *Nestohedra $P_{\mathcal{B}(G)}$ associated with graphical building sets $\mathcal{B}(G)$ are called graph-associahedra, and have been studied in [CD'06, Post'05, Tol'05, Zel'06]. In [CD'06], the sets in $\mathcal{B}(G)$ are called tubes, and the nested sets are called tubings.*
In particular, the h-vector monotonicity discussed in Remark 6.8 applies to graph-associahedra $P_{\mathcal{B}(G)}, P_{\mathcal{B}(G')}$ associated to graphs G, G' where G is an edge-subgraph of G'.

Example 6.10. *(Upper bound for nestohedra: the permutohedron) see [Post'05, Sect. 8.1] For the complete graph K_n, the building set $\mathcal{B}(K_n) = 2^{[n]} \setminus \{\emptyset\}$ consists of all nonempty subsets in $[n]$. Let us call it the complete building set. The corresponding nestohedron (the graph-associahedron of the complete graph) is the usual $(n-1)$-dimensional permutohedron in $\mathbb{R}^n$. The k-th component h_k of its h-vector is the Eulerian number, that is the number of permutations in $\mathfrak{S}_n$ with k descents; and its h-polynomial is the Eulerian polynomial $A_n(t)$; see* (4).
This h-vector gives the componentwise upper bound on h-vectors for all $(d-1)$-dimensional nestohedra. This also implies that the f-vector of the permutohedron gives the componentwise upper bound on f-vectors of nestohedra.

Example 6.11. *(Lower bound for nestohedra: the simplex) The smallest possible connected building set $\mathcal{B} = \{\{1\}, \{2\}, \ldots, \{n\}, [n]\}$ gives rise to the nestohedron $P_{\mathcal{B}}$ which is the $(n-1)$-simplex in $\mathbb{R}^n$. In this case*

$$f(t) = \sum_{i=1}^{n} \binom{n}{i} t^{i-1} = \frac{(1+t)^n - 1}{t} \quad \textit{and} \quad h(t) = 1 + t + t^2 + \cdots + t^{n-1}$$

give trivial componentwise lower bounds on the f-, h-vectors of nestohedra.

6.2 Two recurrences for f-polynomials of nestohedra

There are two useful recurrences for f-polynomials of nestohedra and nested set complexes.
Let $f_{\mathcal{B}}(t)$ be the f-polynomial of the nestohedron $P_{\mathcal{B}}$:

$$f_{\mathcal{B}}(t) := \sum f_i \, t^i = \sum_{N \in \Delta_{\mathcal{B}}} t^{|S| - |\mathcal{B}_{\max}| - |N|},$$

where $f_i = f_i(P_{\mathcal{B}})$ is the number of i-dimensional faces of $P_{\mathcal{B}}$. As usual, it is related to the h-polynomial as $f_{\mathcal{B}}(t) = h_{\mathcal{B}}(t+1)$.

Theorem 6.12. [Post'05, Theorem 7.11] *The f-polynomial $f_{\mathcal{B}}(t)$ is determined by the following recurrence relations:*

1. *If $\mathcal{B}$ consists of a single singleton, then $f_{\mathcal{B}}(t) = 1$.*

2. *If $\mathcal{B}$ has connected components $\mathcal{B}_1, \ldots, \mathcal{B}_k$, then*

$$f_{\mathcal{B}}(t) = f_{\mathcal{B}_1}(t) \cdots f_{\mathcal{B}_k}(t).$$

3. *If $\mathcal{B}$ is a connected building set on S, then*

$$f_{\mathcal{B}}(t) = \sum_{I \subsetneq S} t^{|S|-|I|-1} f_{\mathcal{B}|_I}(t).$$

Another recurrence relation for f-polynomials was derived in [Zel'06], and will be used in Section 12.4 below. It will be more convenient to work with the f-polynomial of nested set complexes

$$\tilde{f}_{\mathcal{B}}(t) := \sum_{N \in \Delta_{\mathcal{B}}} t^{|N|} = t^{|S|-|\mathcal{B}_{\max}|} f_{\mathcal{B}}(t^{-1}),$$

where $\mathcal{B}$ is a building set on S.

For a building set $\mathcal{B}$ on S and a subset $I \subset S$, recall that the restriction of $\mathcal{B}$ to I is defined as $\mathcal{B}|_I = \{J \in \mathcal{B} \mid J \subseteq I\}$. Also define the *contraction* of I from $\mathcal{B}$ as the building set on $S \setminus I$ given by

$$\mathcal{B}/I := \{J \in S \setminus I \mid J \in \mathcal{B} \text{ or } J \cup I \in \mathcal{B}\},$$

see [Zel'06, Definition 3.1]. A link decomposition of nested set complexes was constructed in [Zel'06]. It implies the following recurrence relation for the f-vector.

Theorem 6.13. [Zel'06, Proposition 4.7] *For a building set $\mathcal{B}$ on a nonempty set S, one has*

$$\frac{d}{dt}\tilde{f}_{\mathcal{B}}(t) = \sum_{I \in \mathcal{B} \setminus \mathcal{B}_{\max}} \tilde{f}_{\mathcal{B}|_I}(t) \cdot \tilde{f}_{\mathcal{B}/I}(t) \quad \text{and} \quad \tilde{f}_{\mathcal{B}}(0) = 1.$$

Let G be a simple graph on S and let $I \in \mathcal{B}(G)$, i.e., I is a connected subset of nodes of G. It has already been mentioned that $\mathcal{B}(G)|_I = \mathcal{B}(G|_I)$; see Example 6.2. Let G/I be the graph on the node set $S \setminus I$ such that two nodes $i, j \in S \setminus I$ are connected by an edge in G/I if and only if

1. i and j are connected by an edge in G, or

2. there are two edges (i, k) and (j, l) in G with $k, l \in I$.

Then the contraction of I from the graphical building set $\mathcal{B}(G)$ is the graphical building set associated with the graph G/I, that is $\mathcal{B}(G)/I = \mathcal{B}(G/I)$.

7 Flag nestohedra

This section characterizes the flag nested set complexes and nestohedra, and then identifies those which are "smallest".

7.1 When is the nested set complex flag?

For a graphical building set $\mathcal{B}(G)$ it has been observed ([Post'05, §8,4], [Zel'06, Corollary 7.4]) that one can replace condition (N2) in Definition 6.4 with a weaker condition:

(N2') For a disjoint pair of subsets $I, J \in N$, one has $I \cup J \notin \mathcal{B}$.

This implies that nested set complexes associated to graphical buildings are flag complexes. More generally, one has the following characterization of the nested set complexes which are flag.

Proposition 7.1. *For a building set $\mathcal{B}$, the following are equivalent.*

(i) The nested set complex $\Delta_{\mathcal{B}}$ (or equivalently, the nestohedron $P_{\mathcal{B}}$) is flag.

(ii) The nested sets for $\mathcal{B}$ are the subsets $N \subseteq \mathcal{B} \backslash \mathcal{B}_{\max}$ which satisfy conditions (N1) *and* (N2').

(iii) If $J_1, \ldots, J_\ell \in \mathcal{B}$ with $\ell \geq 2$ are pairwise disjoint and their union $J_1 \cup \cdots \cup J_\ell$ is in $\mathcal{B}$, then one can reindex so that for some k with $1 \leq k \leq \ell - 1$ one has both $J_1 \cup \cdots \cup J_k$ and $J_{k+1} \cup \cdots \cup J_\ell$ in $\mathcal{B}$.

Proof. The equivalence of (i) and (ii) essentially follows from the definitions. We will show here the equivalence of (i) and (iii).
Assume that (iii) fails, and let $J_1, \ldots, J_\ell$ provide such a failure with ℓ minimal. Note that this means $\ell \geq 3$, and minimality of ℓ forces $J_r \cup J_s \notin \mathcal{B}$ for each $r \neq s$; otherwise one could replace the two sets J_r, J_s on the list with the set $J_r \cup J_s$ to obtain a counterexample of size $\ell - 1$. Therefore all of the pairs $\{J_r, J_s\}$ index edges of $\Delta_{\mathcal{B}}$, although $\{J_1, \ldots, J_\ell\}$ does not. Hence $\Delta_{\mathcal{B}}$ is not flag, i.e., (i) fails.
Now assume (i) fails, i.e., $\Delta_{\mathcal{B}}$ is not flag. Let $J_1, \ldots, J_\ell$ be subsets in $\mathcal{B}$, for which each pair $\{J_r, J_s\}$ with $r \neq s$ is a nested set, but the whole collection $M := \{J_1, \ldots, J_\ell\}$ is not, and assume that this violation has ℓ minimal. Because $\{J_r, J_s\}$ are nested for $r \neq s$, it must be that M *does* satisfy condition (N1), and so M must fail condition (N2). By minimality of ℓ, it must be that the $J_1, \ldots, J_\ell$ are pairwise disjoint and their union $J_1 \cup \cdots \cup J_\ell$ is in $\mathcal{B}$. Bearing in mind that $J_r \cup J_s \notin \mathcal{B}$ for $r \neq s$, it must be that $\ell \geq 3$. But then M must give a violation of property (iii), else one could use property (iii) to produce a violation of (i) either of size k or of size $\ell - k$, which are both smaller than ℓ. ∎

Corollary 7.2. *For graphical buildings $\mathcal{B}(G)$, the graph-associahedron $P_{\mathcal{B}(G)}$ and nested set complex $\Delta_{\mathcal{B}(G)}$ are flag.*

7.2 Stanley-Pitman polytopes and their relatives

One can now use Proposition 7.1 to characterize the inclusion-minimal connected building sets $\mathcal{B}$ for which $\Delta_\mathcal{B}$ and $P_\mathcal{B}$ are flag.

For any building set $\mathcal{B}$ on $[n]$ with $\Delta_\mathcal{B}$ flag, one can apply Proposition 7.1(iii) with $\{J_1, \ldots, J_\ell\}$ equal to the collection of singletons $\{\{1\}, \ldots, \{n\}\}$, since they are disjoint and their union $[n]$ is also in $\mathcal{B}$. Thus after reindexing, some initial segment $[k]$ and some final segment $[n] \setminus [k]$ must also be in $\mathcal{B}$. Iterating this, one can assume after reindexing that there is a *plane binary tree* τ with these properties

- the singletons $\{\{1\}, \ldots, \{n\}\}$ label the leaves of τ,
- each internal node of τ is labelled by the set I which is the union of the singletons labelling the leaves of the subtree below it (so $[n]$ labels the root node), and
- the building set $\mathcal{B}$ contains of all of the sets labelling nodes in this tree.

It is not hard to see that these sets labelling the nodes of τ already comprise a building set $\mathcal{B}_\tau$ which satisfies Proposition 7.1(iii), and therefore give rise to a nested set complex $\Delta_{\mathcal{B}_\tau}$ and nestohedron $P_{\mathcal{B}_\tau}$ which are flag. See Figure 1.

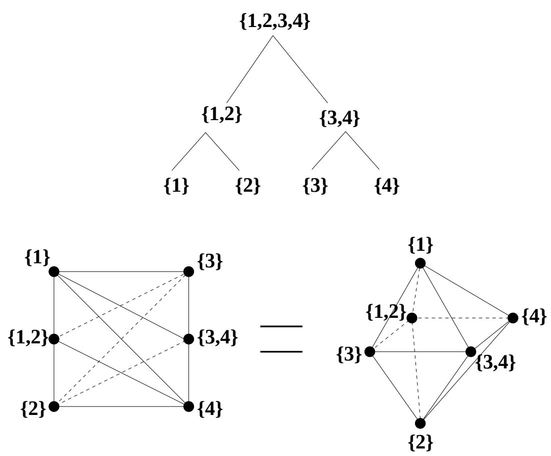

Figure 1: A binary tree τ and building set $\mathcal{B}_\tau$, along with its complex of nested sets $\Delta_{\mathcal{B}_\tau}$, drawn first as in the construction of Remark 6.6, and then redrawn as the boundary of an octahedron.

The previous discussion shows the following.

Proposition 7.3. *The building sets $\mathcal{B}_\tau$ parametrized by plane binary trees τ are exactly the inclusion-minimal building sets among those which are connected and have the nested set complex and nestohedron flag.*

As a special case, when τ is the plane binary tree having leaves labelled by the singletons and internal nodes labelled by all initial segments $[k]$, one obtains the building set $\mathcal{B}_\tau$ whose nestohedron $P_{\mathcal{B}_\tau}$ is the *Stanley-Pitman polytope* from [StPi'02]; see [Post'05, §8.5]. The Stanley-Pitman polytope is shown there to be combinatorially (but not affinely) isomorphic to an $(n-1)$-cube; the argument given there generalizes to prove the following.

Proposition 7.4. *For any plane binary tree τ with n leaves, the nested set complex $\Delta_{\mathcal{B}_\tau}$ is isomorphic to the boundary of a $(n-1)$-dimensional cross-polytope (hyperoctahedron), and $P_{\mathcal{B}_\tau}$ is combinatorially isomorphic to an $(n-1)$-cube.*

Proof. Note that the sets labelling the non-root nodes of τ can be grouped into $n-1$ pairs $\{I_1, J_1\}, \ldots, \{I_{n-1}, J_{n-1}\}$ of siblings, meaning that I_k, J_k are nodes with a common parent in τ. One then checks that the nested sets for $\mathcal{B}_\tau$ are exactly the collections N containing *at most one set from each pair* $\{I_k, J_k\}$. As a simplicial complex, this is the boundary complex of an $(n-1)$-dimensional cross-polytope in which each pair $\{I_k, J_k\}$ indexes an antipodal pair of vertices. ☐

Note that in this case,

$$f_{\mathcal{B}_\tau}(t) = (2+t)^{n-1}, \quad h_{\mathcal{B}_\tau}(t) = (1+t)^{n-1}, \quad \gamma_{\mathcal{B}_\tau}(t) = 1 = 1 + 0 \cdot t + 0 \cdot t^2 + \cdots .$$

which gives a lower bound for the f- and h-vectors of flag nestohedra by Remark 6.8. If one assumes Conjecture 2.3, then it would also give a lower bound for γ-vectors of flag nestohedra (and for flag simplicial polytopes in general). Note that the permutohedron is a graph-associahedron (and hence a flag nestohedra). Therefore, Corollary 4.7 implies that the permutohedron provides the upper bound on the f- and h-vectors among the flag nestohedra.

8 $\mathcal{B}$-trees and $\mathcal{B}$-permutations

This section discusses $\mathcal{B}$-trees and $\mathcal{B}$-permutations, which are two types of combinatorial objects associated with vertices of the nestohedron $P_\mathcal{B}$. The h-polynomial of $P_\mathcal{B}$ equals the descent-generating function for $\mathcal{B}$-trees.

8.1 $\mathcal{B}$-trees and h-polynomials

This section gives a combinatorial interpretation of the h-polynomials of nestohedra. Since nestohedra $P_\mathcal{B}$ are always simple, one should expect some description of their vertex tree-posets Q_v (see Corollaries 3.7 and 3.9) in terms of the building set $\mathcal{B}$.

Recall that a *rooted tree* is a tree with a distinguished node, called its *root*. One can view a rooted tree T as a partial order on its nodes in which $i <_T j$ if j lies on the unique path from i to the root. One can also view it as a directed graph in which all edges are directed towards the root; we will use both viewpoints here.

For a node i in a rooted tree T, let $T_{\leq i}$ denote the set of all descendants of i, that is $j \in T_{\leq i}$ if there is a directed path from the node j to the node i. Note that $i \in T_{\leq i}$. Nodes i and j in a rooted tree are called *incomparable* if neither i is a descendant of j, nor j is a descendant of i.

Definition 8.1. [Post'05, Definition 7.7], cf. [FS'05] *For a connected building set $\mathcal{B}$ on $[n]$, let us define a $\mathcal{B}$-tree as a rooted tree T on the node set $[n]$ such that*

(T1) For any $i \in [n]$, one has $T_{\leq i} \in \mathcal{B}$.

(T2) For $k \geq 2$ incomparable nodes $i_1, \ldots, i_k \in [n]$, one has $\bigcup_{j=1}^{k} T_{\leq i_j} \notin \mathcal{B}$.

Note that, when the nested set complex $\Delta_{\mathcal{B}}$ is flag, that is when $\mathcal{B}$ satisfies any of the conditions of Proposition 7.1, one can define a $\mathcal{B}$-tree by requiring condition (T2) only for $k = 2$.

Proposition 8.2. [Post'05, Proposition 7.8], [FS'05, Proposition 3.17] *For a connected building set $\mathcal{B}$, the map sending a rooted tree T to the collection of sets $\{T_{\leq i} \mid i \text{ is a nonroot vertex}\} \subset \mathcal{B}$ gives a bijection between $\mathcal{B}$-trees and maximal nested sets. (Recall that maximal nested sets correspond to the facets of the nested set complex $\Delta_{\mathcal{B}}$ and to the vertices of the nestohedron $P_{\mathcal{B}}$.) Furthermore, if the $\mathcal{B}$-tree T corresponds to the vertex v of $P_{\mathcal{B}}$ then $T = Q_v$, that is, T is the vertex tree-poset for v in the notation of Corollary 3.7.*

Question 8.3. *Does a simple (indecomposable) generalized permutohedron P come from a (connected) building set if and only if every poset Q_v is a rooted tree, i.e. has a unique maximal element?*

Proposition 8.2 and Theorem 4.2 yield the following corollary.

Corollary 8.4. *For a connected building set $\mathcal{B}$ on $[n]$, the h-polynomial of the generalized permutohedron $P_{\mathcal{B}}$ is given by*

$$h_{\mathcal{B}}(t) = \sum_{T} t^{\mathrm{des}(T)},$$

where the sum is over $\mathcal{B}$-trees T.

The following description of $\mathcal{B}$-trees is straightforward from the definition.

Proposition 8.5. [Post'05, Section 7] *Let $\mathcal{B}$ be a connected building set on S and let $i \in S$. Let $\mathcal{B}_1, \ldots, \mathcal{B}_r$ be the connected components of the restriction $\mathcal{B}|_{S\setminus\{i\}}$. Then all $\mathcal{B}$-trees with root at i are obtained by picking a $\mathcal{B}_j$-tree T_j, for each component $\mathcal{B}_j$, $j = 1, \ldots, r$, and connecting the roots of $T_1, \ldots, T_r$ with the node i by edges.*

In other words, each $\mathcal{B}$-tree is obtained by picking a root $i \in S$, splitting the restriction $\mathcal{B}|_{S\setminus\{i\}}$ into connected components, then picking nodes in all connected components, splitting corresponding restrictions into components, etc.

Recall Definition 3.1 of the surjection $\Psi_{\mathcal{B}} := \Psi_{P_{\mathcal{B}}}$

$$\Psi_{\mathcal{B}} : \mathfrak{S}_n \longrightarrow \mathrm{Vertices}(P_{\mathcal{B}}) = \{\mathcal{B}\text{-trees}\},$$

Here and below one identifies vertices of the nestohedron $P_{\mathcal{B}}$ with $\mathcal{B}$-trees via Proposition 8.2. By Corollary 3.9, for a $\mathcal{B}$-tree T, one has $\Psi_{\mathcal{B}}(w) = T$ if and only if w is a linear extension of T.

Proposition 8.5 leads to an explicit recursive description of the surjection $\Psi_{\mathcal{B}}$.

Proposition 8.6. *Let $\mathcal{B}$ be a connected building set on $[n]$. Given a permutation $w = (w(1), \ldots, w(n)) \in \mathfrak{S}_n$, one recursively constructs a $\mathcal{B}$-tree $T = T(w)$, as follows.*

The root of T is the node $w(n)$. Let $\mathcal{B}_1, \ldots, \mathcal{B}_r$ be the connected components of the restriction $\mathcal{B}|_{\{w(1),\ldots,w(n-1)\}}$. Restricting w to each of the sets $\mathcal{B}_i$ gives a subword of w, to which one can recursively apply the construction and obtain a $\mathcal{B}_i$-tree T_i. Then attach these $T_1, \ldots, T_r$ as subtrees of the root node $w(n)$ in T. This association $w \mapsto T(w)$ is the map $\Psi_{\mathcal{B}}$.

8.2 $\mathcal{B}$-permutations

It is natural to ask for a nice section of the surjection $\Psi_{\mathcal{B}}$; these are the $\mathcal{B}$-permutations defined next.

Definition 8.7. *Let $\mathcal{B}$ be a building set on $[n]$. Define the set $\mathfrak{S}_n(\mathcal{B}) \subset \mathfrak{S}_n$ of $\mathcal{B}$-permutations as the set of permutations $w \in \mathfrak{S}_n$ such that for any $i \in [n]$, the elements $w(i)$ and $\max\{w(1), w(2), \ldots, w(i)\}$ lie in the same connected component of the restricted building set $\mathcal{B}|_{\{w(1),\ldots,w(i)\}}$.*

The following construction of $\mathcal{B}$-permutations is immediate from the definition.

Lemma 8.8. *A permutation $w \in \mathfrak{S}_n$ is a $\mathcal{B}$-permutation if and only if it can be constructed via the following procedure.*

Pick $w(n)$ from the connected component of $\mathcal{B}$ that contains n; then pick $w(n-1)$ from the connected component of $\mathcal{B}|_{[n]\setminus\{w(n)\}}$ that contains the maximal element of $[n]\setminus\{w(n)\}$; then pick $w(n-2)$ from the connected component of $\mathcal{B}|_{[n]\setminus\{w(n),w(n-1)\}}$ that contains the maximal element of $[n]\setminus\{w(n), w(n-1)\}$, etc. Continue in this manner until $w(1)$ has been chosen.

Let T be a rooted tree on $[n]$ viewed as a tree-poset where the root is the unique maximal element. The *lexicographically minimal linear extension* of T is the permutation $w \in \mathfrak{S}_n$ such that $w(1)$ is the minimal leaf of T (in the usual order on $\mathbb{Z}$), $w(2)$ is the minimal leaf of $T - \{w(1)\}$ (the tree T with the vertex $w(1)$ removed), $w(3)$ is the minimal leaf of $T - \{w(1), w(2)\}$, etc. There is the following alternative "backward" construction for the lexicographically minimal linear extension of T.

Lemma 8.9. *Let w be the lexicographically minimal linear extension of a rooted tree T on $[n]$. Then the permutation w can be constructed from T, as follows: $w(n)$ is the root of T; $w(n-1)$ is the root of the connected component of $T - \{w(n)\}$ that contains the maximal vertex of this forest (in the usual order on $\mathbb{Z}$); $w(n-2)$ is the root of the connected component of $T - \{w(n), w(n-1)\}$ that contains the maximal vertex of this forest, etc.*
In general, $w(i)$ is the root of the connected component of the forest

$$T - \{w(n), \ldots, w(i+1)\}$$

that contains the vertex $\max(w(1), \ldots, w(i))$.

Proof. The proof is by induction on the number of vertices in T. Let T' be the rooted tree obtained from T by removing the minimal leaf l. Then the lexicographically minimal linear extension w of T is $w = (l, w')$, where w' is the lexicographically minimal linear extension of T', and both w and w' are written in list notation. By induction, w' can be constructed from T' backwards. When one performs the backward construction for T, the vertex l can never be the root of the connected component of $T - \{w(n), \ldots, w(i+1)\}$ containing the maximal vertex, for $i > 1$. So the backward procedure for T produces the same permutation $w = (l, w')$. $\square$

The next claim gives a correspondence between $\mathcal{B}$-trees and $\mathcal{B}$-permutations.

Proposition 8.10. *Let $\mathcal{B}$ be a connected building set on $[n]$. The set $\mathfrak{S}_n(\mathcal{B})$ of $\mathcal{B}$-permutations is exactly the set of lexicographically minimal linear extensions of the $\mathcal{B}$-trees. (Equivalently, $\mathfrak{S}_n(\mathcal{B})$ is the set of lexicographically minimal representatives of fibers of the map $\Psi_{\mathcal{B}}$.)*
In particular, the map $\Psi_{\mathcal{B}}$ induces a bijection between $\mathcal{B}$-permutations and $\mathcal{B}$-trees, and $\mathfrak{S}_n(\mathcal{B})$ is a section of the map $\Psi_{\mathcal{B}}$.

Proof. Let $w \in \mathfrak{S}_n$ be a permutation and let $T = T(w)$ be the corresponding $\mathcal{B}$-tree constructed as in Proposition 8.6. Note that, for $i = n-1, n-2, \ldots, 1$, the connected components of the forest $T|_{\{w(1),\ldots,w(i)\}} = T - \{w(n), \ldots, w(i+1)\}$ correspond to the connected components of the building set $\mathcal{B}|_{\{w(1),\ldots,w(i)\}}$, and corresponding components have the same vertex sets. According to Lemma 8.9, the permutation w is the lexicographically minimal linear extension of T if and only if w is a $\mathcal{B}$-permutation as described in Lemma 8.8. $\square$

9 Chordal building sets and their nestohedra

This section describes an important class of building sets $\mathcal{B}$, for which the descent numbers of $\mathcal{B}$-trees are equal to the descent numbers of $\mathcal{B}$-permutations. In this case, the h-polynomial of the nestohedron $P_{\mathcal{B}}$ equals the descent-generating function of the corresponding $\mathcal{B}$-permutations.

9.1 Descents in posets vs. descents in permutations

A *descent* of a permutation $w \in \mathfrak{S}_n$ is a pair[7] $(w(i), w(i+1))$ such that $w(i) > w(i+1)$. Let $\mathrm{Des}(w)$ be the set of all descents in w. Also recall that the descent set $\mathrm{Des}(Q)$ of a poset Q is the set of pairs (a, b) such that $a \lessdot_Q b$ and $a >_{\mathbb{Z}} b$; see Definition 4.1.

Lemma 9.1. *Let Q be any poset on $[n]$, and let $w = w(Q)$ be the lexicographically minimal linear extension of Q. Then one has $\mathrm{Des}(w) \subseteq \mathrm{Des}(Q)$.*

Proof. One must show that any descent (a, b) (with $a >_{\mathbb{Z}} b$) in w must come from a covering relation $a \lessdot_Q b$ in the poset Q. Indeed, if a and b are incomparable in Q, then the permutation obtained from w by transposing a and b would be a linear extension of P which is lexicographically smaller than w. On the other hand, if a and b are comparable but not adjacent elements in Q, then they can never be adjacent elements in a linear extension of Q. $\square$

In particular, this lemma implies that, for a $\mathcal{B}$-tree T and the corresponding $\mathcal{B}$-permutation w (i.e., w is the lexicographically minimal linear extension of T), one has $\mathrm{Des}(w) \subseteq \mathrm{Des}(T)$. The rest of this section discusses a special class of building sets for which one always has $\mathrm{Des}(w) = \mathrm{Des}(T)$.

9.2 Chordal building sets

Definition 9.2. *A building set $\mathcal{B}$ on $[n]$ is chordal if it satisfies the following condition: for any $I = \{i_1 < \cdots < i_r\} \in \mathcal{B}$ and $s = 1, \ldots, r$, the subset $\{i_s, i_{s+1}, \ldots, i_r\}$ also belongs to $\mathcal{B}$.*

Recall that a graph is called *chordal* if it has no induced k-cycles for $k \geq 4$. It is well known [FG'65] that chordal graphs are exactly the graphs that admit a *perfect elimination ordering,* which is an ordering of vertices such that, for each vertex v, the neighbors of v that occur later than v in the order form a clique. Equivalently, a graph G is chordal if its vertices can be labelled by numbers in $[n]$ so that G has no induced subgraph $G|_{\{i<j<k\}}$ with the edges (i, j), (i, k) but without the edge (j, k). Let us call such graphs on $[n]$ *perfectly labelled chordal graphs.*[8]

Example 9.3. *A tree on $[n]$ is called decreasing if the labels decrease in the shortest path from the vertex n (the root) to another vertex. It is easy to see that decreasing trees are exactly the trees which are perfectly labelled chordal graphs. Clearly, any unlabelled tree has such a decreasing labelling of vertices.*

The following claim justifies the name "chordal building set."

[7] A more standard convention is say that a descent is an *index* i such that $w(i) > w(i+1)$.

[8] We can also call them 312-*avoiding graphs* because they are exactly the graphs that have no induced 3-path a—b—c with the relative order of the vertices a, b, c as in the permutation 312. Note that, unlike the pattern avoidance in permutations, a 312-avoiding graph is the same thing as a 213-avoiding graph.

Proposition 9.4. *A graphical building set $\mathcal{B}(G)$ is chordal if and only if G is a perfectly labelled chordal graph.*

Proof. Suppose that G contains an induced subgraph $G|_{\{i<j<k\}}$ with exactly two edges (i,j), (i,k). Then $\{i,j,k\} \in \mathcal{B}(G)$ but $\{j,k\} \notin \mathcal{B}(G)$. Thus $\mathcal{B}(G)$ is not a chordal building set.
On the other hand, suppose that $\mathcal{B}(G)$ is not chordal. Then one can find an s and a connected subset $I = \{i_1 < \cdots < i_r\}$ of vertices in G such that $\{i_s, i_{s+1}, \ldots, i_r\} \notin \mathcal{B}(G)$. In other words, the induced graph $G' = G|_{\{i_s,\ldots,i_k\}}$ is disconnected. Let us pick a shortest path P in $G|_{\{i_1,\ldots,i_r\}}$ that connects two different components of G'. Let i be the minimal vertex in P and let j and k be the two vertices adjacent of i in the path P. Clearly, $j > i$ and $k > i$. It is also clear that (i,j) is not an edge of G. Otherwise there is a shorter path obtained from P by replacing the edges (i,j) and (i,k) with the edge (j,k). So one has found a forbidden induced subgraph $G|_{\{i,j,k\}}$. Thus G is not a perfectly labelled chordal graph. ☐

Proposition 9.5. *Let $\mathcal{B}$ be a connected chordal building set. Then, for any $\mathcal{B}$-tree T and the corresponding $\mathcal{B}$-permutation w, one has* $\mathrm{Des}(w) = \mathrm{Des}(T)$.

Proof. Let T be a $\mathcal{B}$-tree and let w be the corresponding $\mathcal{B}$-permutation, which can be constructed backward from T as described in Lemma 8.9. Let us fix $i \in \{n-1, n-2, \ldots, 1\}$. Let $T_1, \ldots, T_r, T'_1, \ldots, T'_s$ be the connected components of the forest $T - \{w(n), w(n-1), \ldots, w(i+1)\}$, where $T_1, \ldots, T_r$ are the subtrees whose roots are the children of the vertex $w(i+1)$, and $T'_1, \ldots, T'_s$ are the remaining subtrees. Let $I = T_{\leq w(i+1)} \subset [n]$ be the set of all descendants of $w(i+1)$ in T. By Definition 8.1(T1), one has $I \in \mathcal{B}$.
Suppose that the vertex $m = \max(w(1), \ldots, w(i))$ appears in one of the subtrees $T_1, \ldots, T_r$, say, in the tree T_1. Then, by Lemma 8.9, $w(i)$ should be the root of T_1. We claim that all vertices in $T_2, \ldots, T_r$ are less than $w(i+1)$. Indeed, this is clear if $w(i+1)$ is the maximal element in I. Otherwise, the set $I' = I \cap \{w(i+1)+1, \ldots, n-1, n\}$ is nonempty, $I' \in \mathcal{B}$ because $\mathcal{B}$ is chordal, and I' contains the maximal vertex m. Since the vertex set J of T_1 should be an element of $\mathcal{B}$, it follows that $I' \subseteq J$. So all vertices of $T_2, \ldots, T_r$ are less than $w(i+1)$.
Thus none of the edges of T joining the vertex $w(i+1)$ with the roots of $T_2, T_3, \ldots, T_r$ can be a descent edge. The only potential descent edge is the edge $(w(i), w(i+1))$ that attaches the subtree T_1 to $w(i+1)$. This edge will be a descent edge in T if and only if $w(i) > w(i+1)$, i.e., exactly when $(w(i), w(i+1))$ is a descent in the permutation w.
Now suppose that the maximal vertex $m = \max(w(1), \ldots, w(i))$ appears in one of the remaining subtrees $T'_1, \ldots, T'_s$, which are not attached to the vertex $w(i+1)$, say, in T'_1. In this case $w(i+1)$ should be greater than all $w(1), \ldots, w(i)$. (Otherwise, if $w(i+1) < m$, then at the previous step of the backward construction for w, T'_1 is the connected component of $T - \{w(n), \ldots, w(i+1)\}$ that contains the vertex $\max(w(1), \ldots, w(i+1)) = m$. So $w(i+1)$ should have been

the root of T_1'.) In this case, none of the edges joining the vertex $w(i+1)$ with the components $T_1, \ldots, T_r$ can be a descent edge and $(w(i), w(i+1))$ cannot be a descent in w.

This proves that descent edges of T are in bijection with descents in w. $\square$

Corollary 8.4 and Proposition 9.5 imply the following formula.

Corollary 9.6. *For a connected chordal building set $\mathcal{B}$, the h-polynomial of the nestohedron $P_{\mathcal{B}}$ equals*

$$h_{\mathcal{B}}(t) = \sum_{w \in \mathfrak{S}_n(\mathcal{B})} t^{\mathrm{des}(w)},$$

where $\mathrm{des}(w)$ *is the usual descent number of a permutation* $w \in \mathfrak{S}_n(\mathcal{B})$.

Let us give an additional nice property of nestohedra for chordal building sets.

Proposition 9.7. *For a chordal building set $\mathcal{B}$, the nestohedron $P_{\mathcal{B}}$ is a flag simple polytope.*

Proof. Let us check that a chordal building set $\mathcal{B}$ satisfies the condition in Proposition 7.1(iii). Using the notation of that proposition, let $J_1 \cup \cdots \cup J_\ell = \{i_1 < \cdots < i_r\}$. Let U_s be the union of those subsets $J_1, ..., J_\ell$ that have a nonempty intersection with $\{i_s, i_{s+1}, \ldots, i_r\}$. Since $\{i_s, i_{s+1}, \ldots, i_n\}$ is in $\mathcal{B}$ (because $\mathcal{B}$ is chordal), the subset U_s should also be in $\mathcal{B}$ (by Definition 6.1(B1)). Clearly, U_1 is the union of all J_i's and U_r consists of a single J_i. It is also clear that U_{j+1} either equals U_j or is obtained from U_j by removing a single subset J_i. It follows that there exists an index s such that $U_s = (J_1 \cup \cdots \cup J_\ell) \setminus J_i$. This gives an index i such that $(J_1 \cup \cdots \cup J_\ell) \setminus J_i$ and J_i are both in $\mathcal{B}$, as needed. $\square$

10 Examples of nestohedra

Let us give several examples that illustrate Corollary 8.4 and Corollary 9.6. The f- and h-numbers for the permutohedron and associahedron are well-known.

10.1 The permutohedron

For the complete building set $\mathcal{B} = \mathcal{B}(K_n)$ the nestohedron $P_{\mathcal{B}}$ is the usual permutohedron; see Example 6.10 and [Post'05, Sect. 8.1]. In this case $\mathcal{B}$-trees are linear orders on $[n]$ and $\mathcal{B}$-permutations are all permutations $\mathfrak{S}_n(\mathcal{B}) = \mathfrak{S}_n$. Thus, as noted before in Example 6.10, the h-polynomial is the usual Eulerian polynomial $A_n(t)$, and the h-numbers are the Eulerian numbers $h_k(P_{\mathcal{B}}) = A(n,k) := \#\{w \in \mathfrak{S}_n \mid \mathrm{des}(w) = k\}$.

10.2 The associahedron

Let $G = \mathrm{Path}_n$ denote the graph which is a path having n nodes labelled consecutively $1, \ldots, n$. The graphical building set $\mathcal{B} = \mathcal{B}(\mathrm{Path}_n)$ consists of all intervals $[i, j]$, for $1 \le i \le j \le n$. The corresponding nestohedron $P_{\mathcal{B}(\mathrm{Path}_n)}$ is the usual Stasheff associahedron; see [CD'06, Post'05].

In this case, the $\mathcal{B}$-trees correspond to unlabelled plane binary trees on n nodes, as follows; see [Post'05, Sect. 8.2] for more details. A *plane binary tree* is a rooted tree with two types of edges (left and right) such that every node has at most one left and at most one right edge descending from it. From Proposition 8.5, one can see that a $\mathcal{B}$-tree is a binary tree with n nodes labelled $1, 2, \ldots, n$ so that, for any node, all nodes in its left (resp., right) branch have smaller (resp., bigger) labels. Conversely, given an unlabelled plane binary tree, there is a unique way to label its nodes $1, 2, \ldots, n$ to create a $\mathcal{B}$-tree, namely in the order of traversal of a *depth-first search*. Furthermore, note that descent edges correspond to right edges.

It is well-known that the number of unlabelled binary trees on n nodes is equal to the *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$, and the number of binary trees on n nodes with $k-1$ right edges is the *Narayana number* $N(n, k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$; see [Stan'99, Exer. 6.19c and Exer. 6.36]. Therefore, the h-numbers of the associahedron $P_{\mathcal{B}(\mathrm{Path}_n)}$ are the Narayana numbers: $h_k(P_{\mathcal{B}(\mathrm{Path}_n)}) = N(n, k+1)$, for $k = 0, \ldots, n-1$.

It is also well-known that the f-numbers of the associahedron are $f_k(P_{\mathcal{B}(\mathrm{Path}_n)}) = \frac{1}{n+1}\binom{n-1}{k}\binom{2n-k}{n}$. This follows from a classical Kirkman-Cayley formula [Cay'1890] for the number of ways to draw k noncrossing diagonals in an n-gon.

In this case, the $\mathcal{B}$-permutations are exactly 312-avoiding permutations $w \in \mathfrak{S}_n$. Recall that a permutation w is 312-*avoiding* if there is no triple of indices $i < j < k$ such that $w(j) < w(k) < w(i)$. Thus Corollary 9.6 says that the h-polynomial of the associahedron $P_{\mathcal{B}(\mathrm{Path}_n)}$ is $\sum_w t^{\mathrm{des}(w)}$ where the sum runs over all 312-avoiding permutations in $\mathfrak{S}_n$. This is consistent with the known fact that the Narayana numbers count 312-avoiding permutations according to their number of descents; see Simion [Sim'94, Theorem 5.4] for a stronger statement.

10.3 The cyclohedron

If $G = \mathrm{Cycle}_n$ is the n-cycle, then the nestohedron $P_{\mathcal{B}(\mathrm{Cycle}_n)}$ is the *cyclohedron* or *Bott-Taubes polytope*; see [CD'06, Post'05]. The h-polynomial of the cyclohedron was computed by Simion [Sim'03, Corollary 1]:

$$h_{\mathcal{B}(\mathrm{Cycle}_n)}(t) = \sum_{k=0}^{n} \binom{n}{k}^2 t^k. \tag{6}$$

Note that the n-cycle (for $n > 3$) is not a chordal graph, so Corollary 9.6 does not apply to this case.

10.4 The stellohedron

Let $m = n-1$. Let $G = K_{1,m}$ be the m-star graph with the central node $m+1$ connected to the nodes $1, \ldots, m$. Let us call the associated polytope $P_{\mathcal{B}(K_{1,m})}$ the *stellohedron.*

From Proposition 8.5 one sees that $\mathcal{B}(K_{1,m})$-trees are in bijection with *partial permutations* of $[m]$, which are ordered sequences $u = (u_1, \ldots, u_r)$ of distinct numbers in $[m]$, where $r = 0, \ldots, m$. The tree T associated to a partial permutation $u = (u_1, \ldots, u_r)$ has the edges

$$(u_r, u_{r-1}), \ldots, (u_2, u_1), (u_1, m+1), (m+1, i_1), \ldots, (m+1, i_{m-r})$$

where $i_1, \ldots, i_{m-r}$ are the elements of $[m] \setminus \{u_1, \ldots, u_r\}$. The root of T is u_r if $r \geq 1$, or $m+1$ if $r = 0$. For $r \geq 1$, one has $\mathrm{des}(T) = \mathrm{des}(u) + 1$, where the descent number of a partial permutation is

$$\mathrm{des}(u) := \#\{i = 1, \ldots, r-1 \mid u_i > u_{i+1}\}.$$

Also for the tree T associated with the empty partial permutation (for $r = 0$) one has $\mathrm{des}(T) = 0$. Corollary 8.4 then says that

$$h_{\mathcal{B}(K_{1,m})}(t) = 1 + \sum_u t^{\mathrm{des}(u)+1} = 1 + \sum_{r=1}^{m} \binom{m}{r} \sum_{k=1}^{r} A(r,k)\, t^k, \tag{7}$$

where the first sum is over nonempty partial permutations w of $[m]$. In particular, the total number of vertices of the stellohedron $P_{\mathcal{B}(K_{1,m})}$ equals

$$f_0(P_{\mathcal{B}(K_{1,m})}) = \sum_{r=0}^{m} \binom{m}{r} \cdot r! = \sum_{r=0}^{m} \frac{m!}{r!}.$$

This sequence appears in [Sloa] as A000522.

In this case, $\mathcal{B}(K_{1,m})$-permutations are permutations $w \in \mathfrak{S}_{m+1}$ such that $m+1$ appears before the first descent. Such permutations w are in bijection with partial permutations u of $[m]$. Indeed, u is the part of w after the entry $m+1$. Since our labelling of $K_{1,m}$ (with the central node labelled $m+1$) is decreasing (see Example 9.3), Corollary 9.6 implies that the h-polynomial of the stellohedron $P_{B(K_{1,m})}$ is $h(t) = \sum_w t^{\mathrm{des}(w)}$, where the sum runs over all such permutations $w \in \mathfrak{S}_{m+1}$. This agrees with the above expression in terms of partial permutations.

10.5 The Stanley-Pitman polytope

Let $\mathcal{B}_{\mathrm{PS}} = \{[i,n], \{i\} \mid i = 1, \ldots, n\}$, the collection of all intervals $[i,n]$ and singletons $\{i\}$. This (non-graphical) building set is chordal. According to [Post'05, §8.5], the corresponding nestohedron $P_{\mathcal{B}_{\mathrm{PS}}}$ is the Stanley-Pitman polytope from [StPi'02].

By Proposition 8.5, $\mathcal{B}_{\mathrm{PS}}$-trees have the following form $T(I)$. For an increasing sequence I of positive integers $i_1 < i_2 < \cdots < i_k = n$, construct the tree $T(I)$ on $[n]$ with the root at i_1 and the chain of edges $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$; also, for each $j \in [n] \setminus I$, one has the edge (i_l, j) where i_l is the minimal element of I such that $i_l > j$.

In this case, $\mathcal{B}_{\mathrm{PS}}$-permutations are permutations $w \in \mathfrak{S}_n$ such that $w(1) < w(2) < \cdots < w(k) > w(k+1) > \cdots > w(n)$, for some $k = 1, \ldots, n$.

Using $\mathcal{B}_{\mathrm{PS}}$-trees or $\mathcal{B}_{\mathrm{PS}}$-permutations one can easily deduce that the h-polynomial of the Stanley-Pitman polytope is $h_{\mathcal{B}_{\mathrm{PS}}}(t) = (1+t)^{n-1}$. This is not surprising since $P_{\mathcal{B}_{\mathrm{PS}}}$ is combinatorially isomorphic to the $(n-1)$-dimensional cube.

11 γ-vectors of nestohedra

Recall that the γ-vector $(\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor d/2 \rfloor})$ of a d-dimensional simple polytope is defined via its h-polynomial as $h(t) = \sum \gamma_i \, t^i (1+t)^{d-2i}$; and the γ-polynomial is $\gamma(t) = \sum \gamma_i \, t^i$; see Section 2.3. The main result of this section is a formula for the γ-polynomial of a chordal nestohedron as a descent-generating function (or peak-generating function) for some set of permutations. This implies that Gal's conjecture (Conjecture 2.3) holds for this class of flag simple polytopes. To prove this, we will employ a certain combinatorial approach that goes back to work of Shapiro, Woan, and Getu [SWG'83], also used by Foata and Strehl, and more recently by Bränden; see [Brä'06] for a thorough discussion.

Suppose P is a simple polytope and one has a combinatorial formula for the h-polynomial $h_P(t) = \sum_{a \in A} t^{f(a)}$, where $f(a)$ is some statistic on the set A. Suppose further that one has a partition of A into *f-symmetric Boolean classes*, i.e. such that the f-generating function for each class is $t^r(1+t)^{2n-r}$ for some r. Let $\widehat{A} \subset A$ be the set of representatives of the classes where $f(a)$ takes its minimal value. Then the γ-polynomial equals $\gamma_P(t) = \sum_{a \in \widehat{A}} t^{f(a)}$. Call $f(a)$ a "generalized descent-statistic." Additionally, define

$$\mathrm{peak}(a) = \min\{f(b) \mid a \text{ and } b \text{ in the same class}\} + 1$$

and call it a "generalized peak statistic."

11.1 A warm up: γ-vector for the permutohedron

We review here the beautiful construction of Shapiro, Woan, and Getu [SWG'83] that leads to a nonnegative formula for the γ-vector of the usual permutohedron. This subsection also serves as a warm-up for a more general construction in the following subsection.

Some notation is necessary. Recall that a *descent* in a permutation $w \in \mathfrak{S}_n$ is a pair $(w(i), w(i+1))$ such that $w(i) > w(i+1)$, where $i \in [n-1]$. A *final descent* is when $w(n-1) > w(n)$, and a *double descent* is a pair of consecutive descents, i.e. a triple $w(i) > w(i+1) > w(i+2)$.

Additionally, define a *peak* of w to be an entry $w(i)$ for $1 \le i \le n$ such that $w(i-1) < w(i) > w(i+1)$. Here (and below) set $w(0) = w(n+1) = 0$ and so a peak can occur in positions 1 or n. On the other hand, a *valley* of w is an entry $w(i)$ for $1 < i < n$ such that $w(i-1) > w(i) < w(i+1)$. The *peak-valley sequence* of w is the subsequence in w formed by all peaks and valleys.

Let $\widehat{\mathfrak{S}}_n$ denote the set of permutations in $\mathfrak{S}_n$ which do not contain any final descents or double descents. Let $\mathrm{peak}(w)$ denote the number of peaks in a permutation w. It is clear that $\mathrm{peak}(w)-1 = \mathrm{des}(w)$, for permutations $w \in \widehat{\mathfrak{S}}_n$ (and only for these permutations).

THEOREM 11.1. (cf. [SWG'83, Proposition 4]) The γ-polynomial of the usual permutohedron $P_{\mathcal{B}(K_n)}$ is

$$\sum_{w \in \widehat{\mathfrak{S}}_n} t^{\mathrm{peak}(w)-1} = \sum_{w \in \widehat{\mathfrak{S}}_n} t^{\mathrm{des}(w)}.$$

EXAMPLE 11.2. *Let us calculate the γ-polynomial of the two dimensional permutohedron $P_{\mathcal{B}(K_3)}$. One has $\widehat{\mathfrak{S}}_3 = \{(1,2,3),(2,1,3),(3,1,2)\}$. Of these, $(1,2,3)$ has one peak (and no descents), and $(2,1,3)$ and $(3,1,2)$ have two peaks (and one descent). Therefore, the γ-polynomial is $1+2t$.*

Say that an entry $w(i)$ of w is an *intermediary* entry if $w(i)$ is not a peak or a valley. Say that $w(i)$ is an *ascent-intermediary entry* if $w(i-1) < w(i) < w(i+1)$ and that it is a *descent-intermediary entry* if $w(i-1) > w(i) > w(i+1)$. (Here again one should assume that $w(0) = w(n+1) = 0$.) Note that the set $\widehat{\mathfrak{S}}_n$ is exactly the set of permutations in $\mathfrak{S}_n$ without descent-intermediary entries.

It is convenient to graphically represent a permutation $w \in \mathfrak{S}_n$ by a piecewise linear "mountain range" M_w obtained by connecting the points $(x_0, 0)$, $(x_1, w(1))$, $(x_2, w(2))$, $\ldots$, $(x_n, w(n))$, $(x_{n+1}, 0)$ on $\mathbb{R}^2$ by straight line intervals, for some $x_0 < x_1 < \cdots < x_{n+1}$; see Figure 2. Then peaks in w correspond to local maxima of M_w, valleys correspond to local minima of M_w, ascent-intermediary entries correspond to nodes on ascending slopes of M_w, and descent-intermediary entries correspond to nodes on descending slopes of M_w. For example, the permutation $w = (6,5,4,10,8,2,1,7,9,3)$ shown in Figure 2 has three peaks $6,10,9$, two valleys $4,1$, one ascent-intermediary entry 7, and four descent-intermediary entries $5,8,2,3$. Its peak-valley sequence is $(6,4,10,1,9)$.

As noted in Section 4.1, the h-polynomial of the permutohedron is the descent-generating function for permutations in $\mathfrak{S}_n$ (the Eulerian polynomial). In order to prove Theorem 11.1, one constructs an appropriate partitioning of $\mathfrak{S}_n$ into equivalence classes, where each class has exactly one element from $\widehat{\mathfrak{S}}_n$. To describe the equivalence classes of permutations, one must introduce some operations on permutations.

DEFINITION 11.3. *Let us define the leap operations L_a and L_a^{-1} that act on permutations. Informally, the permutation $L_a(w)$ is obtained from w by moving*

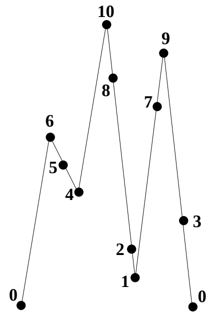

Figure 2: Mountain range M_w for $w = (6, 5, 4, 10, 8, 2, 1, 7, 9, 3)$

an intermediary node a on the mountain range M_w directly to the right until it hits the next slope of M_w. The permutation $L_a^{-1}(w)$ is obtained from w by moving a directly to the left until it hits the next slope of M_w.
More formally, for an intermediary entry $a = w(i)$ in w, the permutation $L_a(w)$ is obtained from w by removing a from the i-th position and inserting a in the position between $w(j)$ and $w(j+1)$, where j is the minimal index such that $j > i$ and a is between $w(j)$ and $w(j+1)$, i.e., $w(j) < a < w(j+1)$ or $w(j) > a > w(j+1)$. The leap operation L_a is not defined if all entries following a in w are less than a.
Similarly, the inverse operation $L_a^{-1}(w)$ is given by removing a from the i-th position in w and inserting a between $w(k)$ and $w(k+1)$, where k is the maximum index such that $k < i$ and a is between $w(k)$ and $w(k+1)$. The operation L_a^{-1} is is not defined if all entries preceding a in w are less than a.

For example, for the permutation w shown on Figure 2, one has $L_2(w) = (6, 5, 4, 10, 8, 1, 2, 7, 9, 3)$ and $L_2^{-1}(w) = (2, 6, 5, 4, 10, 8, 1, 7, 9, 3)$.
Clearly, if a is an ascent-intermediary entry in w then a is a descent-intermediary entry in $L_a^{\pm 1}(w)$, and vice versa. Note that if a is an ascent-intermediary entry in w, then $L_a(w)$ is always defined, and if a is a decent-intermediary entry, then $L_a^{-1}(w)$ is always defined.

Definition 11.4. *Let us also define the hop operations H_a on permutations. For an ascent-intermediary entry a in w, define $H_a(w) = L_a(w)$; and, for a descent-intermediary entry a in w, define $H_a(w) = L_a^{-1}(w)$.*

For example, for the permutation w shown on Figure 2, the permutation $H_2(w) = (2, 6, 5, 4, 10, 8, 1, 7, 9, 3)$ is obtained by moving the descent-intermediary entry 2 to the left to the first ascending slope, and $H_7(w) = (6, 5, 4, 10, 8, 2, 1, 9, 7, 3)$ is obtained by moving the ascent-intermediary entry 7 to the right to the last descending slope.
Note that leaps and hops never change the shape of the mountain range M_w, that is, they never change the peak-valley sequence of w. They just move intermediary nodes from one slope of M_w to another. It is quite clear from the

definition that all leap and hop operations pairwise commute with each other. It is also clear that two hops H_a get us back to the original permutation.

Lemma 11.5. *For intermediary entries a and b in w, one has $(H_a)^2(w) = w$ and $H_a(H_b(w)) = H_b(H_a(w))$.*

Thus the hop operations H_a generate the action of the group $(\mathbb{Z}/2\mathbb{Z})^m$ on the set of permutations with a given peak-valley sequence, where m is the number of intermediary entries in such permutations.

We say that two permutations are *hop-equivalent* if they can be obtained from each other by the hop operations H_a for various a's. The partitioning of $\mathfrak{S}_n$ into hop-equivalence classes allows us to prove Theorem 11.1.

Proof of Theorem 11.1. The number $\mathrm{des}(w)$ of descents in w equals the number of peaks in w plus the number of descent-intermediary entries in w minus 1 (because the last entry is either a peak or a descent-intermediary entry, but it does not contribute a descent). Notice that if a is an ascent-intermediary (resp., descent-intermediary) entry in w then the number of descent-intermediary entries in $H_a(w)$ increases (resp., decreases) by 1 and the number of peaks does not change.

If $w \in \mathfrak{S}_n$ has $p = \mathrm{peak}(w)$ peaks then it has $p-1$ valleys and $n-2p+1$ intermediary entries. Lemma 11.5 implies that the hop-equivalence class C of w involves 2^{n-2p+1} permutations. Moreover, the descent-generating function for these permutations is $\sum_{u\in C} t^{\mathrm{des}(u)} = t^p(t+1)^{n-2p+1}$. Each hop-equivalence class has exactly one representative u without descent-intermediary entries, that is $u \in \widehat{\mathfrak{S}}_n$. Thus, summing the contributions of hop-equivalence classes, one can write the h-polynomial of the permutohedron as

$$h(t) = \sum_{w\in\mathfrak{S}_n} t^{\mathrm{des}(w)} = \sum_{w\in\widehat{\mathfrak{S}}_n} t^{\mathrm{peak}(w)-1}(t+1)^{n+1-2\,\mathrm{peak}(w)}.$$

Comparing this to the definition of the γ-polynomial, one derives the theorem. $\square$

11.2 γ-vectors of chordal nestohedra

According to Proposition 9.7, nestohedra for chordal building sets are flag simple polytopes. Thus Gal's conjecture (Conjecture 2.3) applies. This section proves this conjecture and presents a nonnegative combinatorial formula for γ-polynomials of such nestohedra as peak-generating functions for some subsets of permutations.

Let $\mathcal{B}$ be a connected chordal building set on $[n]$. Recall that $\mathfrak{S}_n(\mathcal{B})$ is the set of $\mathcal{B}$-permutations; see Definition 8.7. Let $\widehat{\mathfrak{S}}_n(\mathcal{B}) := \mathfrak{S}_n(\mathcal{B}) \cap \widehat{\mathfrak{S}}_n$ be the subset of $\mathcal{B}$-permutations which have no final descent or double descent.

The following theorem is the main result of this section.

Theorem 11.6. *For a connected chordal building $\mathcal{B}$ on $[n]$, the γ-polynomial of the nestohedron $P_{\mathcal{B}}$ is the peak-generating function for the permutations in $\widehat{\mathfrak{S}}_n(\mathcal{B})$:*

$$\gamma_{\mathcal{B}}(t) = \sum_{w\in\widehat{\mathfrak{S}}_n(\mathcal{B})} t^{\mathrm{peak}(w)-1} = \sum_{w\in\widehat{\mathfrak{S}}_n(\mathcal{B})} t^{\mathrm{des}(w)}.$$

As noted earlier, $\mathrm{peak}(w) - 1 = \mathrm{des}(w)$ for $w \in \widehat{\mathfrak{S}}_n$.

The proof of Theorem 11.6 will be an extension of the proof given for the γ-vector of the permutohedron in Section 11.1. Recall that Corollary 9.6 interprets the h-polynomial of $P_{\mathcal{B}}$ as the descent-generating function for $\mathcal{B}$-permutations $w \in \mathfrak{S}_n(\mathcal{B})$. Theorem 11.6 will be proven by constructing an appropriate partitioning of the set $\mathfrak{S}_n(\mathcal{B})$ into equivalence classes, where each equivalence class has exactly one representative from $\widehat{\mathfrak{S}}_n(\mathcal{B})$. As before, one uses (suitably generalized) hop operations to describe equivalence classes of elements of $\mathfrak{S}_n(\mathcal{B})$.

One needs powers of the leap operations $L_a^r := (L_a)^r$, for $r \geq 0$, and $L_a^r := (L_a^{-1})^{-r}$, for $r \leq 0$; see Definition 11.3. In other words, for $r > 0$, $L_a^r(w)$ is obtained from w by moving the intermediary entry a to the right until it hits the r-th slope from its original location; and, for $r < 0$, by moving a to the left until it hits $(-r)$-th slope from its original location. Clearly, $L_a^r(w)$ is defined whenever r is in a certain integer interval $r \in [r_{\min}, r_{\max}]$. It is also clear that, if a is an ascent-intermediary entry in w, then a is ascent-intermediary in $L_a^r(w)$ for even r and a is descent-intermediary in $L_a^r(w)$ for odd r, and vice versa if a is descent-intermediary in w.

Note that for a $\mathcal{B}$-permutation $w \in \mathfrak{S}_n(\mathcal{B})$, the permutations $L_a^r(w)$ may no longer be $\mathcal{B}$-permutations. The next lemma ensures that at least some of them will be $\mathcal{B}$-permutations.

Lemma 11.7. *Let $\mathcal{B}$ be a chordal building on $[n]$. Suppose that $w \in \mathfrak{S}_n(\mathcal{B})$ is a $\mathcal{B}$-permutation.*
(1) *If a is an ascent-intermediary letter in w, then there exists an odd positive integer $r > 0$ such that $L_a^r(w) \in \mathfrak{S}_n(\mathcal{B})$ and $L_a^s(w) \notin \mathfrak{S}_n(\mathcal{B})$, for all $0 < s < r$.*
(2) *If a is a descent-intermediary letter in w, then there exists an odd negative integer $r < 0$ such that $L_a^r(w) \in \mathfrak{S}_n(\mathcal{B})$ and $L_a^s(w) \notin \mathfrak{S}_n(\mathcal{B})$, for all $0 > s > r$.*

The proof of Lemma 11.7 will require some preparatory notation and observations.

For a permutation $w \in \mathfrak{S}_n$ and $a \in [n]$ such that $w(i) = a$, let

$$\{w\nwarrow a\} := \{w(j) \mid j \leq i,\, w(j) \geq a\}$$

be the set of all entries in w which are located to the left of a and are greater than or equal to a (including the entry a itself). The arrow in this notation refers to our graphical representation of a permutation as a mountain range M_w: the set $\{w\nwarrow a\}$ is the set of entries in w located to the North-West of the entry a.

According to Definition 8.7, the set $\mathfrak{S}_n(\mathcal{B})$ is the set of permutations w such that, for $i = 1, \ldots, n$, there exists $I \in \mathcal{B}$ such that both $w(i)$ and $\max(w(1), \ldots, w(i))$ are in I and $I \subset \{w(1), \ldots, w(i)\}$. If $\mathcal{B}$ is chordal, then $I' := I \cap [w(i), \infty]$ also belongs to $\mathcal{B}$ (see Definition 9.2) and satisfies the same properties. Clearly $\max(w(1), \ldots, w(i)) = \max\{w \nwarrow w(i)\}$. Thus, for a chordal building set, one can reformulate Definition 8.7 of $\mathcal{B}$-permutations as follows.

Lemma 11.8. *Let $\mathcal{B}$ be a chordal building set. Then $\mathfrak{S}_n(\mathcal{B})$ is the set of permutations $w \in \mathfrak{S}_n$ such that for any $a \in [n]$, the elements a and $\max\{w \nwarrow a\}$ are in the same connected component of $\mathcal{B}|_{\{w \nwarrow a\}}$. Equivalently, there exists $I \in \mathcal{B}$ such that $a \in I$, $\max\{w \nwarrow a\} \in I$, and $I \subset \{w \nwarrow a\}$.*

Let us now return to the setup of Lemma 11.7. There are 2 possible reasons why the permutation $u = L_a^r(w)$ may no longer be a $\mathcal{B}$-permutation, that is, fail to satisfy the conditions in Lemma 11.8:

(A) It is possible that the entry a and the entry $\max\{u \nwarrow a\}$ are in different connected components of $\mathcal{B}|_{\{u \nwarrow a\}}$.

(B) It is also possible that another entry $b \neq a$ in u and $\max\{u \nwarrow b\}$ are in different connected components of $\mathcal{B}|_{\{u \nwarrow b\}}$.

Let us call these two types of failure *A-failure* and *B-failure*. The following auxiliary result is needed.

Lemma 11.9. *Let us use the notation of Lemma 11.7.*
(1) *For left leaps $u = L_a^r(w)$, $r < 0$, one can never have a B-failure.*
(2) *For the maximal left leap $u = L_a^{r_{\min}}(w)$, where the entry a goes all the way to the left, one cannot have an A-failure.*
(3) *For the maximal right leap $u = L_a^{r_{\max}}(w)$, where the entry a goes all the way to the right, one cannot have an A-failure.*
(4) *Let $u = L_a^r(w)$ and $u' = L_a^{r+1}(w)$, for $r \in \mathbb{Z}$, be two adjacent leaps such that a is descent-intermediary in u (and, thus, a is ascent-intermediary in u'). Then there is an A-failure in u if and and only if there is an A-failure in u'.*

Proof. (1) Since $w \in \mathfrak{S}_n(\mathcal{B})$, there is a subset $I \in \mathcal{B}$ that contains both b and $\max\{w \nwarrow b\}$ and such that $I \subset \{w \nwarrow b\}$. The same subset I works for u because $\{u \nwarrow b\} = \{w \nwarrow b\}$ or $\{u \nwarrow b\} = \{w \nwarrow b\} \cup \{a\}$.
(2) In this case, a is greater than all preceding entries in u, so $a = \max\{u \nwarrow a\}$.
(3) In this case, a is greater than all following entries in u. The interval $I = [a, n]$ contains both a and $\max\{u \nwarrow a\}$, $I \subset \{u \nwarrow a\}$, and $I \in \mathcal{B}$ because $\mathcal{B}$ is chordal.
(4) In this case, all entries between the position of a in u and the position of a in u' are less than a. Thus $\{u \nwarrow a\} = \{u' \nwarrow a\}$. So u has an A-failure if and only if u' has an A-failure. $\square$

Proof of Lemma 11.7. It is easier to prove the second part of the lemma.

(2) By parts (1) and (2) of Lemma 11.9, there exists a negative r such that $L_a^r(w) \in \mathfrak{S}_n(\mathcal{B})$. Let us pick such an r with minimal possible absolute value. Then r should be odd, by part (4) of Lemma 11.9, which proves (2).
(1) Suppose that there is an entry $b \neq a$ in the permutation w such that b and $m = \max\{w \nwarrow\!\searrow b\}$ are in different connected components of $\mathcal{B}|_{\{w \nwarrow\searrow b\}\setminus\{a\}}$. In this case, $a \in \{w \nwarrow\!\searrow b\}$, that is $b < a$ and b is located to the right of a in w. (Otherwise, b and m are in different connected components of $\mathcal{B}|_{\{w \nwarrow\searrow b\}}$, which is impossible because w is a $\mathcal{B}$-permutation.) Let us pick the leftmost entry b in w that satisfies this condition. Then the permutation $u = L_a^r(w)$ has a B-failure if the letter a moves to the right of this entry b; and u has no B-failure if a stays to the left of b. By our assumptions, a stays to the left of b in $L_a^1(w)$, so such a u exists.
Let $u = L_a^r(w)$ be the maximal right leap (i.e., with maximal $r > 0$) such that the entry a stays to the left of b. Then all entries in u between the positions of a and b should be less than a. Thus $m = \max\{u \nwarrow\!\searrow a\} = \max\{w \nwarrow\!\searrow b\}$. Since $w \in \mathfrak{S}_n(\mathcal{B})$, there is an $I \in \mathcal{B}$ such that $b, m \in I$ and $I \subset \{w \nwarrow\!\searrow b\}$. This subset I should also contain the entry a. (Otherwise, b and m would be in the same connected component $\mathcal{B}|_{\{w \nwarrow\searrow b\}\setminus\{a\}}$, contrary to our choice of b.) Thus $I' := I \cap [a, +\infty] \in \mathcal{B}$ contains both a and m and $I' \subset \{u \nwarrow\!\searrow a\}$. This means that there is no A-failure in u. Thus $u \in \mathfrak{S}_n(\mathcal{B})$.
If there is no entry b in w as above, then none of the permutations $L_a^r(w)$ has a B-failure. In this case $L_a^{r_{\max}}(w) \in \mathfrak{S}_n(\mathcal{B})$ by part (3) of Lemma 11.9.
In all cases, there exists a positive r such that $L_a^r(w) \in \mathfrak{S}_n(\mathcal{B})$ and only A-failures are possible in $L_a^s(w)$, for $0 < s < r$. Let us pick the minimal such r. Then r should be odd by part (4) of Lemma 11.9, as needed. ☐

Definition 11.10. *Let us define the $\mathcal{B}$-hop operations $\mathcal{B}H_a$. For a $\mathcal{B}$-permutation w with an ascent-intermediary (resp., descent-intermediary) entry a, the permutation $\mathcal{B}H_a(w)$ is the right leap $u = L_a^r(w)$, $r > 0$ (resp., the left leap $u = L_a^r(w)$, $r < 0$) with minimal possible $|r|$ such that u is a $\mathcal{B}$-permutation. Informally, $\mathcal{B}H_a(w)$ is obtained from w by moving the node a on its mountain range M_w directly to the right if a is ascent-intermediary in w, or directly left if a is descent-intermediary in w (possibly passing through several slopes) until one hits a slope and obtain a $\mathcal{B}$-permutation.*

Lemma 11.7 says that the $\mathcal{B}$-hop $\mathcal{B}H_a(w)$ is well-defined for any intermediary entry a in w. It also says that if a is ascent-intermediary in w then a is descent-intermediary in $\mathcal{B}H_a(w)$, and vice versa. Moreover, according to that lemma, $(\mathcal{B}H_a)^2(w) = w$.

Example 11.11. *Let G be the decreasing tree shown on Figure 3. Then the graphical building $\mathcal{B} = \mathcal{B}(G)$ is chordal; see Example 9.3. Figure 3 shows several $\mathcal{B}$-hops of the $\mathcal{B}$-permutation $w = (1, 10, 8, 3, 6, 9, 7, 4, 12, 11, 5, 2)$:*

$$\begin{aligned}
\mathcal{B}H_1(w) &= L_1(w) = (10, 8, 3, 6, 9, 7, 4, 12, 11, 5, 2, 1),\\
\mathcal{B}H_5(w) &= (L_5)^{-5}(w) = (1, 5, 10, 8, 3, 6, 9, 7, 4, 12, 11, 2),\\
\mathcal{B}H_6(w) &= L_6(w) = (1, 10, 8, 3, 9, 7, 6, 4, 12, 11, 5, 2).
\end{aligned}$$

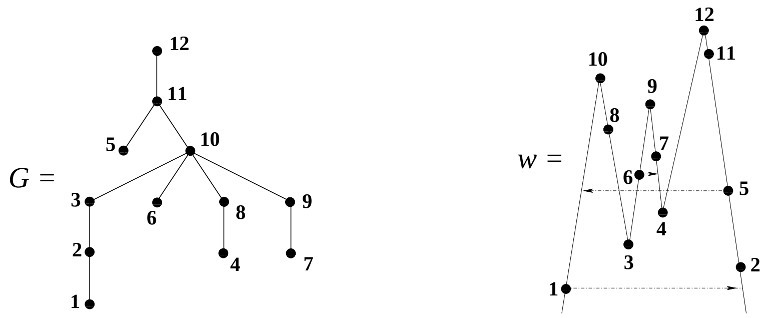

Figure 3: A $\mathcal{B}(G)$-permutation w and some $\mathcal{B}$-hops

Let us now show that the $\mathcal{B}$-hop operations pairwise commute with each other.

Lemma 11.12. *Let a and b be two intermediary entries in a $\mathcal{B}$-permutation w. Then $\mathcal{B}H_a(\mathcal{B}H_b(w)) = \mathcal{B}H_b(\mathcal{B}H_a(w))$.*

Proof. Let us first assume that both a and b are descent-intermediary entries in w. Without loss of generality assume that $a > b$. In this case $\mathcal{B}H_a(w) = L_a^r(w)$ and $\mathcal{B}H_b(w) = L_b^s(w)$ for some *negative* odd r and s, that is the entries a and b of w are moved to the left. According to Lemma 11.9(1), in this case one does not need to worry about B-failures. In other words, $\mathcal{B}H_a(w)$ is the first left leap $L_a^r(w)$ (i.e., with minimal $-r > 0$) that has no A-failure. Similarly, $\mathcal{B}H_b(w)$ is the first left leap $L_b^s(w)$ without A-failures (where A-failures concern the entry b).

Since A-failures for permutations $u = L_a^t(w)$, $t < 0$, are described in terms of the set $\{u\nwarrow a\} \subset [a, \infty]$, moving the entry $b < a$ in w will have no effect on these A-failures. Thus, for the permutation $w' = \mathcal{B}H_b(w)$, one has $\mathcal{B}H_a(w') = L_a^r(w')$ with exactly the same r as in $\mathcal{B}H_a(w) = L_a^r(w)$.

However, for permutations $u = L_b^t(w)$, $t < 0$, the sets $\{u\nwarrow b\}$ might change if one first performs the operation $\mathcal{B}H_a$ to w. Namely, let $\tilde{w} = \mathcal{B}H_a(w)$ and $\tilde{u} = L_b^t(\tilde{w}) = L_b^t(L_a^r(w))$. Then $\{\tilde{u}\nwarrow b\} = \{u\nwarrow b\} \cup \{a\}$ if a is located to the left of b in $\tilde{u}$ and a is located to the right of b in u (and $\{\tilde{u}\nwarrow b\} = \{u\nwarrow b\}$ otherwise). Notice that one always has $m = \max\{u\nwarrow b\} = \max\{\tilde{u}\nwarrow b\}$, since this maximum is the maximal peak preceding b in u (or in $\tilde{u}$), and leaps and hops have no affect on the peaks.

If b and m are in the same connected component of $\mathcal{B}|_{\{u\nwarrow b\}}$ then they are also in the same connected component of $\mathcal{B}|_{\{\tilde{u}\nwarrow b\}}$, that is if there is no A-failure for u then there is no A-failure for $\tilde{u}$.

Suppose that there is no A-failure for $\tilde{u}$ but there is an A-failure for u. Then the sets $\{u\nwarrow b\}$ and $\{\tilde{u}\nwarrow b\}$ have to be different. That means that a is located to the left of b in $\tilde{u}$ and a is located to the right of b in u. Let I be the element

$I \in \mathcal{B}$ such that $b, m \in I$ and $I \subset \{\tilde{u} \nwarrow\!\!\searrow b\}$. Then I should contain the entry a. (Otherwise, $I \subset \{u \nwarrow\!\!\searrow b\}$ and there would be no A-failure for u.)

Let $\hat{w} = L_a^{\hat{t}}(w)$ be the left leap with maximal possible $-\hat{t} \geq 0$ such that the position of a in $\hat{w}$ is located to the right of the position of b in $\tilde{u}$. Since $\tilde{u} = L_b^t(L_a^r(w))$, it follows that $|\hat{t}| < |r|$. In other words, if one starts moving to the right from the node b along the mountain range $M_{\tilde{u}}$, the (ascending) slope that first crosses the level a is the place where the entry a is located in $\hat{w}$. Note that $\hat{t}$ is odd because a should be an ascent-intermediary entry in $\hat{w}$; in particular $\hat{t} < 0$.

Since all entries in $\hat{w}$ located between the position of b in $\tilde{u}$ and the position of a in $\hat{w}$ are less than a, one deduces that $\{\tilde{u} \nwarrow\!\!\searrow b\} \cap [a, \infty] = \{\hat{w} \nwarrow\!\!\searrow a\}$. Thus the subset $\hat{I} = I \cap [a, \infty]$ has three important properties: it lies in $\mathcal{B}$ (because $\mathcal{B}$ is chordal); it contains both a and $m = \max\{\hat{w} \nwarrow\!\!\searrow a\}$; and it is a subset of $\{\hat{w} \nwarrow\!\!\searrow a\}$. It follows that there is no A-failure in $\hat{w}$. This contradicts the fact that $L_a^r(w) \neq L_a^{\hat{t}}(w)$ is the first left leap that has no A-failure.

Thus u has an A-failure if and only if $\tilde{u}$ has an A-failure. It follows that $\mathcal{B}H_b(\tilde{w}) = L_b^s(\tilde{w})$ with exactly the same s as in $\mathcal{B}H_b(w) = L_b^s(w)$.

This proves that $\mathcal{B}H_a(\mathcal{B}H_b(w)) = L_a^r(L_b^s(w)) = L_b^s(L_a^r(w)) = \mathcal{B}H_b(\mathcal{B}H_a(w))$, in the case when both a and b are descent-intermediary in w.

Let us now show that the general case easily follows. Suppose that, say, a is ascent-intermediary and b is descent-intermediary in w. Then, for $w'' = \mathcal{B}H_a(w)$ both a and b are descent-intermediary. One has $\mathcal{B}H_a(\mathcal{B}H_b(w'')) = \mathcal{B}H_b(\mathcal{B}H_a(w''))$. Thus $\mathcal{B}H_a(\mathcal{B}H_b(\mathcal{B}H_a(w))) = \mathcal{B}H_b(\mathcal{B}H_a(\mathcal{B}H_a(w))) = \mathcal{B}H_b(w)$. Applying $\mathcal{B}H_a$ to both sides, one deduces $\mathcal{B}H_b(\mathcal{B}H_a(w)) = \mathcal{B}H_a(\mathcal{B}H_b(w))$. The other cases are similar. □

Thus the $\mathcal{B}$-hop operations $\mathcal{B}H_a$ generate the action of the group $(\mathbb{Z}/2\mathbb{Z})^m$ on the set of $\mathcal{B}$-permutations with a given peak-valley sequence, where m is the number of intermediary entries in such permutations.

We say that two $\mathcal{B}$-permutation are *$\mathcal{B}$-hop-equivalent* if they can be obtained from each other by the $\mathcal{B}$-hop operations $\mathcal{B}H_a$ for various a's. This gives the partitioning of the set of $\mathcal{B}$-permutations into $\mathcal{B}$-hop-equivalence classes.

One can now prove Theorem 11.6 by literally repeating the argument in the proof of Theorem 11.1.

Proof of Theorem 11.6. For a $\mathcal{B}$-permutation $w \in \mathfrak{S}_n(\mathcal{B})$ with $p = \mathrm{peak}(w)$, the descent-generating function of the $\mathcal{B}$-hop-equivalence class C of w is $\sum_{u \in C} t^{\mathrm{des}(u)} = t^p(t+1)^{n-2p+1}$. Each $\mathcal{B}$-hop-equivalence class has exactly one representative without descent-intermediary entries, that is, in the set $\widehat{\mathfrak{S}}_n(\mathcal{B})$. Thus the h-polynomial of the nestohedron $P_{\mathcal{B}}$ (see Corollary 9.6) is

$$h_{P_{\mathcal{B}}}(t) = \sum_{w \in \mathfrak{S}_n(\mathcal{B})} t^{\mathrm{des}(w)} = \sum_{w \in \widehat{\mathfrak{S}}_n(\mathcal{B})} t^{\mathrm{peak}(w)-1}(t+1)^{n+1-2\,\mathrm{peak}(w)}.$$

Comparing this to the definition of the γ-polynomial, one derives the theorem. □

Corollary 11.13. *Gal's conjecture holds for all graph-associahedra corresponding to chordal graphs.*

11.3 γ-vectors for the associahedron and cyclohedron

In Propositions 11.14 and 11.15 we give two explicit formulas which can be derived from the expressions for the corresponding h-polynomials (see Sections 10.2 and 10.3) using standard quadratic transformations of hypergeometric series; e.g., see [RSW'03, Lemma 4.1].

Proposition 11.14. *The γ-polynomial of the associahedron $P_{\mathcal{B}(\mathrm{Path}_n)}$ is*

$$\gamma(t) = \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} C_r \binom{n-1}{2r} t^r,$$

where $C_r = \frac{1}{r+1}\binom{2r}{r}$ is the r-th Catalan number.

Proposition 11.15. *The γ-polynomial of the cyclohedron $P_{\mathcal{B}(\mathrm{Cycle}_n)}$ is*

$$\gamma(t) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{r, r, n-2r} t^r,$$

We now give three combinatorial proofs of Proposition 11.14 as an alternative to using hypergeometric series.

First proof of Proposition 11.14. It is known that the Narayana polynomial which is the h-polynomial of $P_{\mathcal{B}(\mathrm{Path}_n)}$ is also the rank generating function for the well-studied *lattice of noncrossing partitions* $NC(n)$. An explicit *symmetric chain decomposition* for $NC(n)$ was given by Simion and Ullman [SU'91], who actually produced a much stronger decomposition of $NC(n)$ into disjoint Boolean intervals placed symmetrically about the middle rank(s) of $NC(n)$. Their decomposition contains exactly $C_r\binom{n-1}{2r}$ such Boolean intervals of rank $n-(2r+1)$ for each $r = 0, 1, \ldots, \frac{n-1}{2}$, which immediately implies the formula for the γ-polynomial; see [SU'91, Corollary 3.2]. □

Second proof of Proposition 11.14. By Section 10.2, the h-polynomial of $P_{\mathcal{B}(\mathrm{Path}_n)}$ counts plane binary trees on n nodes according to their number of right edges. There is a natural map from binary trees to *full* binary trees, i.e., those in which each node has zero or two children: if a node has a unique child, contract this edge from the node to its child. If the original binary tree T has n nodes, then the resulting full binary tree T' will have $2r+1$ nodes, $2r$ edges and r right edges for some $r = 0, 1, \ldots, \lfloor (n-1)/2 \rfloor$. There are C_r such full binary trees for each r. Given such a full binary tree T', one can produce all of the binary trees in its preimage by inserting $n-(2r+1)$ more nodes and deciding if they create left or right edges. One chooses the locations of these nodes from $2r+1$ choices, either an edge of the full binary tree they

will subdivide or located above the root, giving $\binom{n-(2r+1)+(2r+1)-1}{n-(2r+1)} = \binom{n-1}{2r}$ possible locations. Thus the generating function with respect to the number of right edges for the preimage of T' is $\binom{n-1}{2r} t^r (t+1)^{n-(2r+1)}$, where the term $t^r(t+1)^{n-(2r+1)}$ comes from choosing whether each of the new nodes creates a left or a right edge. It follows that the generating function for all binary trees on n nodes is $h_{\mathrm{Path}_n}(t) = \sum_r C_r \binom{n-1}{2r} t^r (t+1)^{n-(2r+1)}$, where C_r counts full binary trees. This implies the needed expression for the γ-vector of the associahedron $P_{\mathcal{B}(\mathrm{Path}_n)}$.
Equivalently, one can describe the subdivision of all binary trees into classes where two binary trees are in the same class if they can be obtained from each other by switches of left and right edges coming from single child nodes. Then one gets exactly $C_r\binom{n-1}{2r}$ classes having $t^r(t+1)^{n-(2r+1)}$ as its generating function counting number of right edges, for each $r = 0, 1, \ldots, \lfloor (n-1)/2 \rfloor$. □

Third proof of Proposition 11.14. This proof is based on our general approach to γ-vectors of chordal nestohedra. According to Section 10.2, $\mathcal{B}$-permutations for the associahedron are 312-avoiding permutations and h-polynomial is equal to the sum $h_{P_{\mathcal{B}(\mathrm{Path}_n)}}(t) = \sum_w q^{\mathrm{peak}(w)-1}$ over all 312-avoiding permutations $w \in \mathfrak{S}_n$. By Theorem 11.6, $\gamma_r(P_{\mathcal{B}(\mathrm{Path}_n)})$ equals the number of 312-avoiding permutations with no descent-intermediary elements and $r+1$ peaks. The (flattenings of) peak-valley sequences of such permutations are exactly 312-avoiding *alternating permutations* in $\mathfrak{S}_{2r+1}$, that is 312-avoiding permutations w' such that $w'_1 > w'_2 < w'_3 > \cdots < w'_{2r+1}$. It is known that the number of such permutations equals the Catalan number C_r; see [Man'02, Theorem 2.2]. Then there are $\binom{n-1}{2r}$ ways to insert the remaining $n-(2r+1)$ descent-intermediary elements. □

12 Graph-associahedra for single branched trees

Our goal in this section is to compute a generating function that computes the h-polynomials of all graph-associahedra in which the graph is a tree having at most one *branched* vertex (i.e., a vertex of valence 3 or more).

12.1 Associahedra and Narayana polynomials

First recall (see Section 10.2) that the h-numbers of the associahedron $P_{\mathcal{B}(\mathrm{Path}_n)}$ are the Narayana numbers $h_k(P_{\mathcal{B}(\mathrm{Path}_n)}) = N(n,k) := \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$, and the h-polynomial of the associahedron is the *Narayana polynomial*:

$$h_{\mathcal{B}(\mathrm{Path}_n)}(t) = C_n(t) := \sum_{k=1}^{n} N(n,k)\, t^{k-1}. \tag{8}$$

Recall the well-known recurrence and generating function for the Narayana polynomials $C_n(t)$. The recurrence in Theorem 6.12 for the f-polynomials $f_{\mathcal{B}(\mathrm{Path}_n)}(t) = h_{\mathcal{B}(\mathrm{Path}_n)}(t+1) = C_n(t+1)$ can be written as follows. When one removes k vertices from the n-path, it splits into $k+1$ (possibly empty) paths. So one obtains

$$C_n(t) = \sum_{k\geq 1} (t-1)^{k-1} \sum_{m_1+\cdots+m_{k+1}=n-k} C_{m_1}(t)\cdots C_{m_{k+1}}(t), \quad \text{for } n \geq 1, \tag{9}$$

where the sum is over $m_1, \ldots, m_{k+1} \geq 0$ such that $\sum m_i = n-k$. Here one assumes that $C_0(t) = 1$.
Let $C(t,x)$ be the generating function for the Narayana polynomials:

$$\begin{aligned} C(t,x) := \sum_{n\geq 1} C_n(t)\, x^n &= x + (1+t)x^2 + (1+3t+t^2)x^3 + \cdots \\ &= \frac{1-x-tx-\sqrt{(1-x-tx)^2-4tx^2}}{2tx}. \end{aligned} \tag{10}$$

The recurrence relation (9) is equivalent to the following well-known functional equation:

$$C = tx\, C^2 + (1+t)\, x\, C + x, \tag{11}$$

see [Stan'99, Exer. 6.36b].

12.2 Generating function for single branched trees

Trees with at most one branched vertex have the following form. For $a_1, \ldots, a_k \geq 0$, let $T_{a_1,\ldots,a_k}$ be the graph obtained by attaching k chains of lengths $a_1, \ldots, a_k$ to one central node. For example, $T_{0,\ldots,0}$ is the graph with a single node and $T_{1,\ldots,1}$ is the k-star graph $K_{1,k}$.

THEOREM 12.1. *One has the following generating function for the h-polynomials of graph-associahedra $P_{\mathcal{B}(T_{a_1,\ldots,a_k})}$ for the graphs $T_{a_1,\ldots,a_k}$:*

$$\begin{aligned} T(t, x_1, \ldots, x_k) &:= \sum_{a_1,\ldots,a_k\geq 0} h_{T_{a_1,\ldots,a_k}}(t)\, x_1^{a_1+1}\cdots x_k^{a_k+1} \\ &= \frac{(t-1)\,\phi_1\cdots\phi_k}{t-\prod_{i=1}^k (1+(t-1)\,\phi_i)} \end{aligned}$$

where $\phi_i = x_i(1 + t\,C(t,x_i))$, and $C(t,x)$ is the generating function for the Narayana polynomials from (10)*.*

This theorem immediately implies the following formula from [Post'05].

COROLLARY 12.2. [Post'05, Proposition 8.7] *The generating function for the number of vertices in the graph-associahedron $P_{\mathcal{B}(T_{a_1,\ldots,a_k})}$ is*

$$\sum_{a_1,\ldots,a_k} f_0(P_{\mathcal{B}(T_{a_1,\ldots,a_k})})\, x_1^{a_1}\cdots x_k^{a_k} = \frac{\bar{C}(x_1)\cdots\bar{C}(x_k)}{1-x_1\,\bar{C}(x_1)-\cdots-x_k\,\bar{C}(x_k)},$$

where $\bar{C}(x) = \sum_{n\geq 0} C_n\, x^n = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function for the Catalan numbers.

Proof. The claim is obtained from Theorem 12.1 in the limit $t \to 1$. Note however that one needs to use l'Hôpital's rule before plugging in $t = 1$. ☐

The first proof of Theorem 12.1 is fairly direct, using Corollary 8.4 and the solution to Simon Newcomb's problem. The second uses Theorem 6.13 to set up and then solve a system of PDE's; it has the advantage of producing a generating function for the h-polynomials of one further family of graph-associahedra.

12.3 Theorem 12.1 via Simon Newcomb's problem

Let us first review Simon Newcomb's problem and its solution.
Let $w = (w(1), \ldots, w_m)$ be a permutation of the multiset $\{1^{c_1}, \ldots, k^{c_k}\}$, that is, each i appears in w exactly c_i times, for $i = 1, \ldots, k$. A *descent* in w is an index i such that $w(i) > w(i+1)$. Let $\mathrm{des}(w)$ denote the number of descents in w. *Simon Newcomb's Problem* is the problem of counting permutations of a multiset with a given number of descents, see [Mac'17, Sec. IV, Ch. IV] and [GJ'83, Sec. 4.2.13]. Let us define the *multiset Eulerian polynomial* as

$$A_{c_1,\ldots,c_k}(t) := \sum_w t^{\mathrm{des}(w)},$$

where the sum is over all permutations w of the multiset $\{1^{c_1}, \ldots, k^{c_k}\}$. By convention, set $A_{0,\ldots,0}(t) = 1$.
In particular, the polynomial $A_{1,\ldots,1}(t)$ is the usual Eulerian polynomial. It is clear that $A_{c_1,\ldots,c_k}(1) = \binom{m}{c_1,\ldots,c_k}$, the total number of multiset permutations. A solution to Simon Newcomb's problem can be expressed by the following generating function for the $A_{c_1,\ldots,c_k}(t)$.

Proposition 12.3. [GJ'83, Sec. 4.2.13] *One has*

$$\sum_{c_1,\ldots,c_k \geq 0} A_{c_1,\ldots,c_k}(t)\, y_1^{c_1} \cdots y_k^{c_k} = \frac{t-1}{t - \prod_{i=1}^k (1 + (t-1)\, y_i)}.$$

Theorem 12.1 then immediately follows from Proposition 12.3 and the following proposition.

Proposition 12.4. *The generating function for the h-polynomials of the polytopes $P_{\mathcal{B}(T_{a_1,\ldots,a_k})}$ equals*

$$T(t, x_1, \ldots, x_k) = \sum_{c_1,\ldots,c_k \geq 0} A_{c_1,\ldots,c_k}(t)\, \phi_1^{c_1+1} \cdots \phi_k^{c_k+1}.$$

Proof. Let us label nodes of the graph $T_{a_1,\ldots,a_k}$ by integers in $[n]$, where $n = a_1 + \cdots + a_k + 1$, so that the first chain is labelled by $1, \ldots, a_1$, the second chain is labelled by $a_1 + 1, \ldots, a_1 + a_2$, etc., with all labels increasing towards the central node, and finally the central node has the maximal label n.

Let T be a $T_{a_1,\dots,a_k}$-tree. Suppose that the root r of T belongs to the $w(1)$-st chain of the graph $T_{a_1,\dots,a_k}$. If one removes the node r from the graph $T_{a_1,\dots,a_k}$, then the graph decomposes into 2 connected components, one of which is a chain Path_{b_1} and the other is $T_{a_1,\dots,a'_{w(1)},\dots,a_k}$, where $a'_{w(1)} = a_{w(1)} - b_1 - 1$ and all other indices are the same as before. (The first component is empty if $b_1 = 0$.) According to Proposition 8.5, the tree T is obtained by attaching a Path_{b_1}-tree T_1 and a $T_{a_1,\dots,a'_{w(1)},\dots,a_k}$-tree T' to the root r. (Here one assumes that there is one empty Path_0-tree T_1, for $b_1 = 0$.) Let us repeat the same procedure with the tree T'. Assume that its root belongs to the $w(2)$-nd chain and split it into a Path_{b_2}-tree T_2 and a tree T''. Then repeat this procedure with T'', etc. Keep on doing this until one gets a tree $T^{'\cdots'}$ with the root at the central node n. Finally, if one removes the central node n from $T^{'\cdots'}$, then it splits into k trees $\tilde{T}_1,\dots,\tilde{T}_k$ such that $\tilde{T}_j$ is a Path_{d_j}-tree, for $j = 1,\dots,k$.
So each $T_{a_1,\dots,a_k}$-tree T gives us the following data:

1. a sequence $(w(1),\dots,w_m) \in [k]^m$;
2. a Path_{b_i}-tree T_i, for $i = 1,\dots,m$;
3. a Path_{d_j}-tree $\tilde{T}_j$, for $j = 1,\dots,k$.

This data satisfies the following conditions:

1. $m, b_1,\dots,b_m, d_1,\dots,d_k \geq 0$, and
2. $(b_1+1)e_{w(1)} + \cdots + (b_m+1)e_{w_m} + (d_1,\dots,d_k) = (a_1,\dots,a_k)$,

where $e_1,\dots,e_k$ are the standard basis vectors in $\mathbb{R}^k$. Conversely, data of this form gives us a unique $T_{a_1,\dots,a_k}$-tree T. The number of descents in the tree T is

$$\mathrm{des}(T) = \sum_{i=1}^{m} \mathrm{des}(T_i) + \sum_{j=1}^{k} \mathrm{des}(\tilde{T}_j) + l + \mathrm{des}(w),$$

where l is the number of nonempty trees among $T_1, \dots, T_m, \tilde{T}_1, \dots, \tilde{T}_k$. Indeed, all descents in trees T_i and $\tilde{T}_j$ correspond to descents in T, each nonempty tree T_i or $\tilde{T}_j$ gives an additional descent for the edge that attaches this tree, and descents in w correspond to descent edges that attach trees $T', T'', \dots$.
Let us fix a sequence $w = w(1),\dots,w(m)$. For $i \in [k]$, let c_i be the number of times the integer i appears in w. In other words, w is a permutation of the multiset $\{1^{c_1},\dots,k^{c_k}\}$. Then the total contribution to the generating function $T(t,x_1,\dots,x_k)$ of trees T whose data involve w is equal to $t^{\mathrm{des}(w)}\,\phi_1^{c_1+1}\cdots\phi_k^{c_k+1}$. Indeed, the term 1 in $\phi_i = x_i(1 + t\cdot C(t,x_i))$ corresponds to an empty tree, and the term $t\cdot C(t,x_i)$ corresponds to nonempty trees, which contribute one additional descent. The term $\phi_i^{c_i}$ comes from the c_i trees $T_{j_1},\dots,T_{j_{c_i}}$, where $w_{j_1},\dots,w_{j_{c_i}}$ are all occurrences of i in w. Finally, additional 1's in the exponents of ϕ_i's come from the trees $\tilde{T}_1,\dots,\tilde{T}_k$. Summing this expression over all permutations w of the multiset $\{1^{c_1},\dots,k^{c_k}\}$ and then

over all $c_1, \ldots, c_k \geq 0$, one obtains the needed expression for the generating function $T(t, x_1, \ldots, x_k)$. $\square$

Remark 12.5. *One can dualize all definitions, statements, and arguments in this section, as follows. An equivalent dual formulation to Theorem 12.1 says*

$$T(t, x_1, \ldots, x_k) = \frac{(1-t)\,\psi_1 \cdots \psi_k}{1 - t\prod_{i=1}^k (1 + (1-t)\,\psi_i)}$$

where $\psi_i = x_i(1 + C(t, x_i))$. The equivalence to Theorem 12.1 follows from the relation $\phi_i \cdot \psi_i = (t-1)(\phi_i - \psi_i)$, which is a reformulation of the functional equation (11).

The dual multiset Eulerian polynomial is $\bar{A}_{c_1,\ldots,c_k}(t) := \sum_w t^{\mathrm{wdes}(w)+1}$, where the sum is over permutations w of the multiset $M = \{1^{c_1}, \ldots, k^{c_k}\}$, $m = c_1 + \cdots + c_k$, and $\mathrm{wdes}(w)$ *is the number of weak descents in the multiset permutation w, that is, the number of indices i for which $w(i) \geq w(i+1)$. The bijection which reverses the word w shows that $\bar{A}_{c_1,\ldots,c_k}(t) = t^m\, A_{c_1,\ldots,c_k}(t^{-1})$ and consequently one has an equivalent formulation of the solution to Simon Newcomb's problem:*

$$\sum_{c_1,\ldots,c_k \geq 0} \bar{A}_{c_1,\ldots,c_k}(t)\, y_1^{c_1} \cdots y_k^{c_k} = \frac{1-t}{1 - t\prod_{i=1}^k (1 + (1-t)\,y_i)}.$$

Then one can modify the proof of Proposition 12.4, by switching the labels i with $n+1-i$ in the graph $T_{a_1,\ldots,a_k}$, and applying a similar argument to show

$$T(t, x_1, \ldots, x_k) = \sum_{c_1,\ldots,c_k \geq 0} \bar{A}_{c_1,\ldots,c_k}(t)\, \psi_1^{c_1+1} \cdots \psi_k^{c_k+1}.$$

12.4 Proof of Theorem 12.1 via PDE

This section rederives Theorem 12.1 using Theorem 6.13. It also calculates the generating function for f-polynomials of graph-associahedra corresponding to another class of graphs, the *hedgehog graphs* defined below.

Recall that Path_n is the path with n nodes, and $T_{a_1,\ldots,a_k}$ is the graph obtained by attaching the paths $\mathrm{Path}_{a_1}, \ldots, \mathrm{Path}_{a_k}$ to a central node. Let us also define the *hedgehog graph* $H_{a_1,\ldots,a_k}$ as the graph obtained from the disjoint union of the chains $\mathrm{Path}_{a_1}, \ldots, \mathrm{Path}_{a_k}$ by adding edges of the complete graph between the first vertices of all chains. For example, $H_{0,\ldots,0}$ is the empty graph, $H_{1,\ldots,1} = K_k$, and $H_{2,\ldots,2}$ is a graph with $2k$ vertices obtained from the complete graph K_k by adding a "leaf" edge hanging from each of the k original nodes. By convention, for the empty graph, one has $\tilde{f}_{H_{0,\ldots,0}}(t) = 0$.

Theorem 6.13 gives the following recurrence relation for f-polynomials of path graphs:

$$\frac{d}{dt}\tilde{f}_{\mathrm{Path}_n}(t) = \sum_{r=1}^{n-1} (n-r+1) \cdot \tilde{f}_{\mathrm{Path}_r}(t) \cdot \tilde{f}_{\mathrm{Path}_{n-r}}(t).$$

Indeed, there are $n-r+1$ connected r-element subsets I of nodes of Path_n, the deletion $\mathrm{Path}_n|_I$ is isomorphic to Path_r, and the contraction Path_n/I is isomorphic to Path_{n-r}.

For graphs $T_{a_1,\dots,a_k}$, Theorem 6.13 gives the following recurrence relation:

$$\frac{d}{dt}\tilde f_{T_{a_1,\dots,a_k}}(t) = \sum_{i=1}^{k}\sum_{r=1}^{a_i} \tilde f_{\mathrm{Path}_r}(t)\cdot \tilde f_{T_{a_1,\dots,a_i-r,\dots,a_k}}(t)\cdot (a_i-r+1) + \sum \tilde f_{T_{b_1,\dots,b_k}}(t)\cdot \tilde f_{H_{a_1-b_1,\dots,a_k-b_k}}(t),$$

where the second sum is over $b_1,\dots,b_k$ such that $0\le b_i\le a_i$, for $i=1,\dots,k$. Indeed, a connected subset I of vertices of $G=T_{a_1,\dots,a_k}$ either belongs to one of the chains Path_{a_i}, or contains the central node. In the first case, the restriction is $G|_I=\mathrm{Path}_r$ and the contraction is $G/I=T_{a_1,\dots,a_i-r,\dots,a_k}$, where $r=|I|$. In the second case, the restriction $G|_I$ has the form $T_{b_1,\dots,b_k}$ and the contraction is $G/I=H_{a_1-b_1,\dots,a_k-b_k}$. Similarly, for hedgehog graphs $H_{a_1,\dots,a_k}$, one obtains the recurrence relation

$$\frac{d}{dt}\tilde f_{H_{a_1,\dots,a_k}}(t) = \sum_{i=1}^{k}\sum_{r=1}^{a_i} \tilde f_{\mathrm{Path}_r}(t)\cdot \tilde f_{H_{a_1,\dots,a_i-r,\dots,a_k}}(t)\cdot (a_i-r) + \sum \tilde f_{H_{b_1,\dots,b_k}}(t)\cdot \tilde f_{H_{a_1-b_1,\dots,a_k-b_k}}(t),$$

where the second sum is over $b_1,\dots,b_k$ such that $0\le b_i\le a_i$, for $i=1,\dots,k$. In all cases one has the initial conditions $\tilde f_{\mathrm{Path}_n}(0)=\tilde f_{T_{a_1,\dots,a_k}}(0)=\tilde f_{H_{a_1,\dots,a_k}}(0)=1$, except $\tilde f_{\mathrm{Path}_0}(t)=\tilde f_{H_{0,\dots,0}}(t)=0$.

The above recurrence relations can be written in a more compact form using these generating functions:

$$F_A(t,x) := \sum_{n\ge 1}\tilde f_{\mathrm{Path}_n}(t)\,x^{n+1} = x^2+(1+2t)\,x^3+(1+5t+5t^2)\,x^4+\cdots,$$

$$F_T(t,x_1,\dots,x_k) := \sum_{a_1,\dots,a_k\ge 0}\tilde f_{T_{a_1,\dots,a_k}}(t)\,x_1^{a_1+1}\cdots x_k^{a_k+1},$$

$$F_H(t,x_1,\dots,x_k) := \sum_{a_1,\dots,a_k\ge 0}\tilde f_{H_{a_1,\dots,a_k}}(t)\,x_1^{a_1}\cdots x_k^{a_k}.$$

Note that F_A and F_T are related to generating functions from Section 12: $F_A(t,x)=t^{-1}x\,C(t^{-1}+1,tx)$, and $F_T(t,x_1,\dots,x_k)=t^{-k}\,T(t^{-1}+1,tx_1,\dots,tx_k)$.

The above recurrence relations can be expressed as the following partial differential equations with initial conditions at $t = 0$:

$$\frac{\partial F_A}{\partial t} = F_A \cdot \frac{\partial F_A}{\partial x}, \quad F_A|_{t=0} = \frac{x^2}{1-x}, \tag{12}$$

$$\frac{\partial F_T}{\partial t} = \sum_{i=1}^{k} F_A(t, x_i) \frac{\partial F_T}{\partial x_i} + F_T \cdot F_H, \quad F_T|_{t=0} = \frac{x_1 \cdots x_k}{\prod_{i=1}^{k}(1-x_i)}, \tag{13}$$

$$\frac{\partial F_H}{\partial t} = \sum_{i=1}^{k} F_A(t, x_i) \frac{\partial F_H}{\partial x_i} + (F_H)^2, \quad F_H|_{t=0} = \frac{1 - \prod_{i=1}^{k}(1-x_i)}{\prod_{i=1}^{k}(1-x_i)}. \tag{14}$$

One can actually solve these partial differential equations for arbitrary initial conditions, as follows.

Proposition 12.6. *The solutions $F(t,x)$, $G(t, x_1, \ldots, x_k)$, $H(t, x_1, \ldots, x_k)$, and $R(t, x_1, \ldots, x_k)$ to the following system of partial differential equations with initial conditions*

$$\frac{\partial F}{\partial t} = F \cdot \frac{\partial F}{\partial x}, \quad F|_{t=0} = f_0(x), \tag{15}$$

$$\frac{\partial G}{\partial t} = \sum_{i=1}^{k} F(t, x_i) \frac{\partial G}{\partial x_i}, \quad G|_{t=0} = g_0(x_1, \ldots, x_k), \tag{16}$$

$$\frac{\partial H}{\partial t} = \sum_{i=1}^{k} F(t, x_i) \frac{\partial H}{\partial x_i} + H^2, \quad H|_{t=0} = h_0(x_1, \ldots, x_k), \tag{17}$$

$$\frac{\partial R}{\partial t} = \sum_{i=1}^{k} F(t, x_i) \frac{\partial R}{\partial x_i} + R \cdot H, \quad R|_{t=0} = r_0(x_1, \ldots, x_k) \tag{18}$$

are given by

$$\begin{aligned}
&f_0(x + t \cdot F) = F \ \textit{(implicit form)} \\
&G = g_0(\xi_1, \ldots, \xi_k) \\
&H = -(t + (h_0(\xi_1, \ldots, \xi_k))^{-1})^{-1} \\
&R = -r_0(\xi_1, \ldots, \xi_k) \cdot (1 + t \cdot h_0(\xi_1, \cdots, \xi_k))^{-1}
\end{aligned}$$

where $\xi_i = x_i + t \cdot F(t, x_i)$, for $i = 1, \ldots, k$.

Proof. Let us first solve (15). For a constant C, consider the function $x(t)$ given implicitly as $F(t,x) = C$, i.e., the graph of $x(t)$ is a level curve for $F(t,x)$. The tangent vector to the graph of $x(t)$ at some point (t_0, x_0) such that $F(t_0, x_0) = C$ is $(1, \frac{dx(t_0)}{dt})$. The derivative of the function $F(t,x)$ at the point (t_0, x_0) in the direction of this vector should be 0, i.e., $1 \cdot \frac{\partial F(t_0,x_0)}{\partial t} + \frac{dx(t_0)}{dt} \cdot \frac{\partial F(t_0,x_0)}{\partial x} = 0$. This equation, together with the differential equation (15) for F, implies that $\frac{d}{dt}\, x(t) = -C$. Solving this trivial differential equation for $x(t)$ one deduces that $x(t) = -C \cdot t + B(C)$, where B is a function that depends only on the constant C. Since C can be an arbitrary constant, one deduces that

$$x = -F(t,x) \cdot t + B(F(t,x)), \text{ or, equivalently, } B^{\langle -1\rangle}(x + t \cdot F(t,x)) = F(t,x).$$

Plugging the initial condition $F|_{t=0} = f_0(x)$ in the last expression, one gets

$$B^{\langle -1\rangle}(x) = f_0(x).$$

Thus the solution $F(t,x)$ is given by $f_0(x + t \cdot F) = F$, as needed.

Direct verification shows that the function $G = R(F(t,x_1),\ldots,F(t,x_k))$ satisfies the differential equation (16), for an arbitrary $R(y_1,\ldots,y_k)$. The initial condition for $t = 0$ gives $R(f_0(x_1),\ldots,f_0(x_k)) = g_0(x_1,\ldots,x_k)$. Thus $R(y_1,\ldots,y_k) = g_0(B(y_1),\ldots,B(y_k))$, where $B = f_0^{\langle -1\rangle}$, as above. Since $B(F(t,x)) = x + t \cdot F(t,x)$, one deduces that $G = g_0(B(F(t,x_1)),\ldots,B(F(t,x_k))) = g_0(\xi_1,\ldots,\xi_k)$, as needed.

Making the substitution $H = -(t + G(t,x_1,\ldots,x_k))^{-1}$ in differential equation (17) for H, one obtains equation (17) for G with $g_0 = -(h_0)^{-1}$. By the previous calculation, one has $G = -(h_0(\xi_1,\ldots,\xi_k))^{-1}$. Thus the solution for (17) is $H = -(t + (h_0(\xi_1,\ldots,\xi_k))^{-1})^{-1}$.

Making the substitution $R = H \cdot G$ in equation (17) for R, where H is the solution to (17), one obtains equation (16) for G with $g_0 = r_0/h_0$. By the above calculation, one has $G = r_0(\xi_1,\ldots,x_k)/h_0(\xi_1,\ldots,\xi_k)$. Thus,

$$R = -\frac{1}{t + (h_0(\xi_1,\ldots,\xi_k))^{-1}} \cdot \frac{r_0(\xi_1,\ldots,x_k)}{h_0(\xi_1,\ldots,\xi_k)} = -\frac{r_0(\xi_1,\ldots,x_k)}{1 + t \cdot h_0(\xi_1,\cdots,\xi_k)},$$

as needed. □

Applying Proposition 12.6 to differential equation (12) for $F_A(t,x)$, one obtains the implicit solution:

$$\frac{(x + t \cdot F_A)^2}{1 - x - t \cdot F_A} = F_A.$$

This is equivalent to the quadratic equation (11) for $C(t,x)$. Explicitly, one gets

$$F_A(t,x) = \frac{(1 - x - 2tx) - \sqrt{(1 - x - 2tx)^2 - 4t(t+1)x^2}}{2t(t+1)}. \tag{19}$$

Applying Proposition 12.6 to differential equations (13) and (14) for the generating functions F_T and F_H, one obtains the following result.

Theorem 12.7. *The generating functions $F_T(t,x_1,\ldots,x_k)$ and $F_H(t,x_1,\ldots,x_k)$ are given by the following expressions*

$$F_T(t,x_1,\ldots,x_k) = \frac{\xi_1 \cdots \xi_n}{(t+1)(1-\xi_1)\cdots(1-\xi_n) - t},$$

$$F_H(t,x_1,\ldots,x_k) = \frac{1 - (1-\xi_1)\cdots(1-\xi_k)}{(t+1)(1-\xi_1)\cdots(1-\xi_n) - t},$$

where $\xi_i = x_i + t \cdot F_A(t,x_i)$ and F_A is given by (19).

Note that the above expression for F_T is equivalent to Theorem 12.1, using (1).

13 Graph-associahedra for path-like graphs

The goal of this section is to use Theorem 6.12 to compute the h-polynomials of the graph-associahedra of a fairly general infinite family of graphs, including all Dynkin diagrams of finite and affine Coxeter groups.

Let A and B be two connected graphs with a marked vertex in each, and let n_0 be the total number of unmarked vertices in A and B. For $n > n_0$, let $G_n = G_n(A, B)$ be the graph obtained by connecting the marked vertices in A and B by the path Path_{n-n_0} so that the total number of vertices in G_n is n. Call graphs of the form G_n *path-like graphs* because, for large n, they look like paths with some "small" graphs attached to their ends.

The following claim shows that the h-polynomials of the graph-associahedra $P_{\mathcal{B}(G_n)}$ can be expressed as linear combinations (with polynomial coefficients) of the h-polynomials $C_n(t)$ of usual associahedra; see (8).

Theorem 13.1. *There exist unique polynomials $g_0(t), g_1(t), \ldots, g_{n_0}(t) \in \mathbb{Z}[t]$ of degrees $\deg g_i(t) \le i$ such that, for any $n > n_0$ one has*

$$h_{G_n}(t) = g_0(t)\, C_n(t) + g_1(t)\, C_{n-1}(t) + \cdots + g_{n_0}(t)\, C_{n-n_0}(t).$$

The polynomial $g_i(t)$ is a palindromic polynomial, that is $g_i(t) = t^i\, g_i(t^{-1})$, for $i = 0, \ldots, n_0$.

Similarly, one can express the f-polynomials of $P_{\mathcal{B}(G_n)}$ as a linear combination of the f-polynomials of usual associahedra, because $f_G(t) = h_G(t+1)$.

One can rewrite this theorem in terms the generating function $C(t, x)$ for the Narayana numbers; see (10).

Corollary 13.2. *There exists a unique polynomial $g(t, x) \in \mathbb{Z}[t, x]$ such that for any $n > n_0$, $h_{G_n}(t)$ is the coefficient of x^n in $g(t, x)\, C(t, x)$. The polynomial $g(t, x)$ has degree at most n_0 with respect to x, and satisfies $g(t, x) = g(t^{-1}, tx)$.*

Proof. This follows from Theorem 13.1, by setting $g(t, x) = g_0(t) + g_1(t)\, x + \cdots + g_{n_0}(t)\, x^{n_0}$. □

Proof of Theorem 13.1. Let us first prove the existence of the linear expansion. The recurrence from Theorem 6.12 will be used to prove this claim by induction on the total number of vertices in A and B. Suppose that A or B is disconnected, say, A is a disjoint union of graphs A_1 and A_2 where A_1 contains the marked vertex. Let $\tilde{G}_n := G_n(A_1, B)$ and let r be the number of vertices in A_2. Then $h_{G_n}(t) = h_{\tilde{G}_{n-r}}(t)\, h_{A_2}(t)$, where $\deg h_{A_2}(t) \le r - 1$. By induction, $h_{\tilde{G}_{n-r}}(t)$ can be expressed as a linear combination of $C_{n-r}(t), C_{n-r-1}(t), \ldots, C_{n-n_0}(t)$, which produces the needed expression for $h_{G_n}(t)$.

Now assume that both A and B are connected graphs. Theorem 6.12(3) gives the expression for the h-polynomial as the sum $h_{G_n}(t) = \sum_L (t-1)^{|L|-1} h_{G_n \setminus L}(t)$ over nonempty subsets L of vertices of G_n, where $G_n \setminus L$ denotes the graph G_n with removed vertices in L. (Here one has shifted t by -1 to transform f-polynomials into h-polynomials.) Let us write L as a disjoint union $L = I \cup J \cup K$, where I is a subset of unmarked vertices of A, J is a subset of unmarked vertices of B, and K is a subset of vertices in the path connecting the marked vertices. The contribution of the terms with $K = \emptyset$ to the above sum is $\sum_{I,J} (t-1)^{|I|+|J|-1} h_{G_n \setminus (I \cup J)}(t)$. Note that $G_n \setminus (I \cup J) = G_{n-r}(A \setminus I, B \setminus J)$, where $r = |I| + |J|$. By induction, one can express each term $h_{G_n \setminus (I \cup J)}(t)$ as a combination of $C_{n-r}(t), \ldots, C_{n-n_0}(t)$.

The remaining terms involve a nonempty subset K of vertices in the path Path_{n-n_0}. When one removes these $k = |K|$ vertices from the path, it splits into $k+1$ smaller paths $\mathrm{Path}_{m_1}, \ldots, \mathrm{Path}_{m_{k+1}}$ with $m_i \geq 0$; cf. paragraph before (9). Thus the remaining contribution to $h_{G_n}(t)$ can be written as

$$\sum_{I,J} \sum_{m_1,\ldots,m_{k+1} \geq 0} (t-1)^{|I|+|J|+k-1}\, h_{G_p(A \setminus I, \circ)}(t)\, C_{m_2}(t) \cdots C_{m_k}(t)\, h_{G_q(\circ, B \setminus J)}(t),$$

where $\circ$ is the graph with a single vertex,

$$\begin{aligned} p &= m_1 + |A \setminus I| - 1, \\ q &= m_{k+1} + |B \setminus J| - 1, \text{ and} \\ k + \sum m_i &= n - n_0. \end{aligned}$$

By induction, one can express $h_{G_p(A \setminus I, \circ)}(t)$ and $h_{G_q(\circ, B \setminus J)}(t)$ as linear combinations of the $C_{m'}(t)$. So the remaining contribution to $h_{G_n}(t)$ can be written as a sum of several expressions of the form

$$s(t) \sum_{k \geq 1} \sum_{m_1', m_2, \ldots, m_k, m_{k+1}'} (t-1)^{k-1} C_{m_1'}(t)\, C_{m_2}(t) \cdots C_{m_k}(t)\, C_{m_{k+1}'}(t),$$

where the sum is over $m_1', m_2, \ldots, m_k, m_{k+1}'$ such that $m_1' \geq a$, $m_2, \ldots, m_k \geq 0$, $m_{k+1}' \geq b$, $m_1' + m_2 + \cdots + m_k + m_{k+1}' + k = n - c$. This expression depends on nonnegative integers a, b, c such that $a + b + c = n_0$ and a polynomial $s(t)$ of degree $\deg s(t) \leq c$. If one extends the summation to all $m_1', m_{k+1}' \geq 0$, one obtains the expansion (9) for $C_{n-c}(t)$ times $s(t)$. Applying the inclusion-exclusion principle and equation (9), one deduces that the previous sum equals

$$s(t) \left(C_{n-c}(t) - \sum_{m_1'=0}^{a-1} t\, C_{m_1'}(t)\, C_{n-c-m_1'-1}(t) - \cdots \right),$$

which is a combination of $C_n(t), \ldots, C_{n-n_0}(t)$ as needed.

It remains to show the uniqueness of the linear expansion and show that the $g_i(t)$ are palindromic polynomials. (Here one assumes that the graphs A and

B are connected.) According to Corollary 13.2, the polynomial $H(t,x) := \sum_{n>n_0} h_{G_n}(t)\, x^n$ can be written as $H(t,x) = g(t,x)\, C(t,x) + r(t,x)$, where $g(t,x), r(t,x) \in \mathbb{Z}[t,x]$. If $H(t,x) = \tilde{g}(t,x)\, C(t,x) + \tilde{r}(t,x)$ with $\tilde{g}(t,x) \neq g(t,x)$, then this would imply that $C(t,x)$ is a rational function, which contradicts the formula (10) involving a square root. This proves the uniqueness claim.

One has $H(t,x) = H(t^{-1}, tx)/t$ and $C(t,x) = C(t^{-1}, tx)/t$ because h-polynomials are palindromic. Thus

$$H(t,x) = g(t,x)\, C(t,x) + r(t,x) = g(t^{-1}, tx)\, C(t,x) + \frac{1}{t} r(t^{-1}, tx).$$

This implies that $g(t,x) = g(t^{-1}, tx)$. Otherwise, $C(t,x)$ would be a rational function. The equation $g(t,x) = g(t^{-1}, tx)$ says that the coefficients $g_i(t)$ of $g(t,x) = \sum_i g_i(t)\, x^i$ are palindromic. □

Let us illustrate Theorem 13.1 by several examples. For a series of path-like graphs G_n, let $g\{G_n\}$ denote the polynomial $g(t,x)$ that appears in the generating functions $\sum_{n\geq n_0} h_{G_n}(t)\, x^n = g(t,x)\, C(t,x) + r(t,x)$. For instance, the expression $g\{D_n\} = 2 - (t+1)x - tx^2$ (see the example below) is equivalent to the expression $h_{D_n}(t) = 2C_n(t) - (t+1)\, C_{n-1}(t) - t\, C_{n-2}(t)$, for $n > 2$.

Examples 13.3. *Define daisy graphs as* $\mathrm{Daisy}_{n,k} := T_{n-k-1,1^k}$*; see Section 12. (Here* 1^k *means a sequence of* k *ones.) They include type* D *Dynkin diagrams* $D_n := \mathrm{Daisy}_{n,2}$*. For fixed* k*, the* $\mathrm{Daisy}_{n,k}$ *form the series of path-like graphs for* $A = K_{1,k}$ *(the* k*-star with marked central vertex) and* $B = \circ$ *(the graph with a single vertex). Also define kite graphs as* $\mathrm{Kite}_{n,k} := H_{n-k+1,1^{k-1}}$*; see Section 12.4. They are path-like graphs for* $A = K_k$ *and* $B = \circ$*. The affine Dynkin diagram of type* $\widetilde{D}_{n-1}$ *is the* n*th path-like graph in the case when both* A *and* B *are* 3*-paths with marked central vertices.*

Here are the polynomials $g\{G_n\}$ *for several series of such graphs:*

$$\begin{aligned}
g\{D_n\} &= 2 - (t+1)\, x - t\, x^2,\\
g\{\widetilde{D}_{n-1}\} &= 4 - 4(t+1)\, x + (t-1)^2\, x^2 + 2\, t(t+1)\, x^3 + t^2\, x^4,\\
g\{\mathrm{Kite}_{n,3}\} &= 2 - (t+1)\, x,\\
g\{\mathrm{Daisy}_{n,3}\} &= 6 - 6(t+1)\, x + (1 - 5\, t + t^2)\, x^2 - t(t+1)\, x^3,\\
g\{\mathrm{Daisy}_{n,4}\} &= 24 - 36(t+1)\, x + (14 - 16\, t + 14\, t^2)\, x^2 +\\
&\qquad + (-1 + 3\, t + 3\, t^2 - t^3)\, x^3 - (t + t^2 + t^3)\, x^4.
\end{aligned}$$

The formulas for D_n*,* $\mathrm{Kite}_{n,k}$*, and* $\mathrm{Daisy}_{n,k}$ *were derived from Theorems 12.1 and 12.7. The formula for the affine Dynkin diagram* $\widetilde{D}_{n-1}$ *was obtained using the inductive procedure given in the proof of Theorem 13.1.*

Remark 13.4. *The induction from the proof of Theorem 13.1 is quite involved. It is very difficult to calculate by hand other examples for bigger graphs* A *and* B*. It would be interesting to find a simpler procedure for finding the polynomials*

$g\{G_n\}$. Also it would be interesting to find explicit formulas for the polynomials $g\{G_n\}$ for all daisy graphs, kite graphs, and other "natural" series of path-like graphs.

14 Bounds and monotonicity for face numbers of graph-associahedra

Section 7.2 showed that the f- and h-vectors of flag nestohedra coming from connected building sets are bounded below by those of hypercubes and bounded above by those of permutohedra. It is natural to ask for the bounds within the subclass of graph-associahedra corresponding to connected graphs.

Permutohedra are graph-associahedra corresponding to complete graphs, and so they still provide the upper bound for the f- and h-vectors. For lower bounds on f- and h-vectors, the monotonicity discussed in Remark 6.9 implies that the f- and h-vector of any graph-associahedron $P_{\mathcal{B}(G)}$ for a connected graph G is bounded below by the graph-associahedron for any spanning tree inside G. Thus it is of interest to look at bounds for f-, h- and γ-vectors of graph-associahedra for trees.

A glance at Figure 4 suggests that, roughly speaking, trees which are more branched and forked (that is, farther from being a path) tend to have higher entries componentwise in their γ-vectors, and hence also in their f- and h-vectors. In fact, in that figure, which shows all trees on 7 vertices grouped by their degree sequences, one sees several (perhaps misleading) features:

(i) The degree sequences are ordered linearly under the *majorization* (or *dominance*) partial ordering on partitions of $2(n-1)$ $(= 2 \cdot (7-1) = 12$ here).

(ii) The γ-vectors of these trees are linearly ordered under the componentwise partial order.

(iii) Trees whose degree sequence are lower in the majorization order have componentwise smaller γ-vectors.

(iv) The trees are distinguished up to isomorphism by their γ-vectors.

Additionally, it seems that the *Wiener index* [Wie'47] for graphs has some correlation with the γ-vector. The *Wiener index* $W(G)$ of a graph G is defined as the sum of distances $d(i,j)$ over unordered pairs i,j of vertices in G, where $d(i,j)$ is the number of edges in the shortest path from i to j. The Wiener index $W(T)$ of a tree is equal to the number of forbidden 312 patterns (see the remarks following Definition 9.2) provided by the tree T (plus the constant $\binom{n}{2}$). Thus, for two trees on n vertices, if one has $W(T) < W(T')$, then roughly speaking one might expect that the generalized permutohedron $P_{\mathcal{B}(T)}$ has a larger gamma vector than $P_{\mathcal{B}(T')}$.

This is exactly the case for trees on 7 vertices, as shown in Figure 4. It shows that as the γ-vectors decrease, the Wiener indices (weakly) increase. Note that

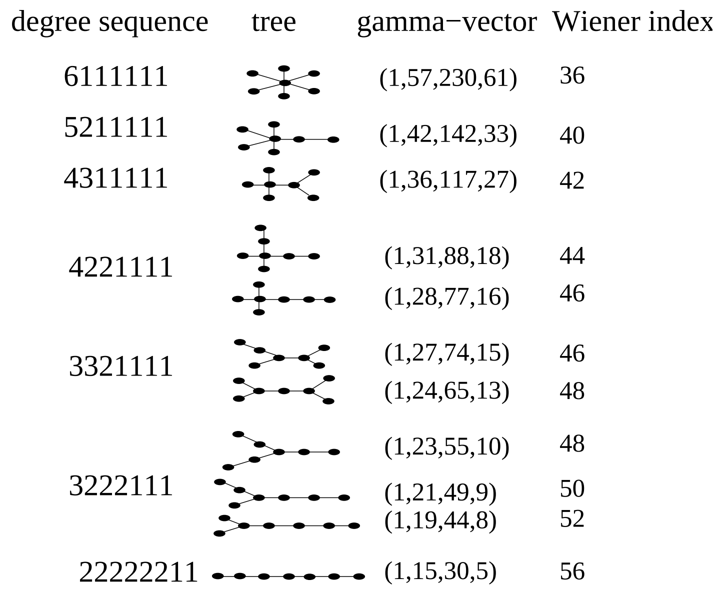

Figure 4: The γ-vectors $(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$ for graph-associahedra of all trees on 7 vertices, grouped by degree sequence.

in this case, the Wiener index together with the degree sequence completely distinguish all equivalence classes of trees.

For trees on n vertices, the maximum and minimum values of the Wiener index are, respectively, $\sum_{i=1}^{n-1} i(n-i) = n(n^2-1)/6$ for Path_n, and $(n-1)^2$ for $K_{1,n-1}$.

None of the properties (i)-(iv) persist for all trees. For example, when looking at trees on $n = 8$ nodes, one finds that

(i) the degree sequences are only partially ordered by the majorization order on partitions of $2(n-1) = 14$:

$$\begin{aligned} 22222211 &< 32222111 \\ &< 33221111 < 33311111, 42221111 \\ &< 43211111 < 44111111, 52211111 \\ &< 53111111 < 62111111 < 71111111 \end{aligned}$$

(ii) there are trees, such as the two shown in Figure 5(a) and (b), whose γ-vectors are incomparable componentwise,

(iii) there are trees, such as the two shown in Figure 5(d) and (c), where the degree sequence of one strictly majorizes that of the other, but its γ-vector is strictly smaller, and

(iv) there are nonisomorphic trees, such as the two shown in Figure 5(d) and (e), having the same γ-vector.

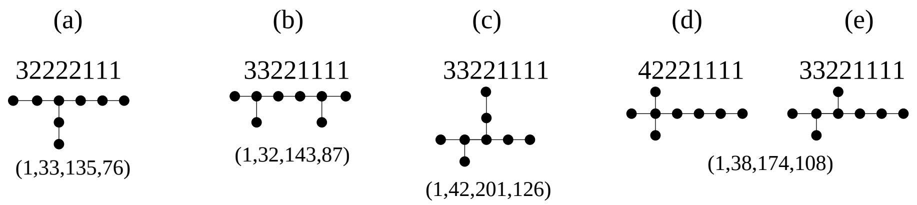

Figure 5: The γ-vectors of the graph-associahedra of some trees on 8 nodes.

Nevertheless, we do make some monotonicity conjectures about the face numbers for graph-associahedra.

CONJECTURE 14.1. *There exists a partial order $\prec$ on the set of (unlabelled, isomorphism classes of) trees with n nodes, having these properties:*

- Path_n *is the unique $\prec$-minimum element,*
- $K_{1,n-1}$ *is the unique $\prec$-maximum element, and*
- $T \prec T'$ *implies* $\gamma_{P_{\mathcal{B}(T)}} \leq \gamma_{P_{\mathcal{B}(T')}}$ *componentwise.*

We suspect that such a partial order $\prec$ can be defined so that T, T' will, in particular, be comparable whenever T, T' are related by one of the *flossing moves* considered in [BR'04, §4.2].
Assuming Conjecture 14.1, the γ-vectors (and hence also the f-, h-vectors) of graph-associahedra for trees on n nodes would have the associahedron $P_{\mathcal{B}(\mathrm{Path}_n)}$ and the stellohedron $P_{\mathcal{B}(K_{1,n-1})}$ giving their lower and upper bounds. This would also imply that the f-, h-vectors of graph-associahedra for connected graphs on n nodes would have associahedra and permutohedra giving their lower and upper bounds. To make a similar assertion for γ-vectors it would be nice to have the following analogue of Stanley's monotonicity result (Theorem 4.6).

CONJECTURE 14.2. *When Δ, Δ' are two flag simplicial complexes and Δ' is a geometric subdivision of Δ, the γ-vector of Δ' is componentwise weakly larger than that of Δ. In particular, when $\mathcal{B}, \mathcal{B}'$ are building sets giving rise to flag nestohedra and $\mathcal{B} \subset \mathcal{B}'$, (such as graphical buildings $\mathcal{B}(G) \subset \mathcal{B}(G')$ for an edge-subgraph $G \subset G'$) then the γ-vector of $P_{\mathcal{B}'}$ is componentwise weakly larger than that of $P_{\mathcal{B}}$.*

We close with a question suggested by the sets of permutations $\mathfrak{S}_n(\mathcal{B})$ and $\widehat{\mathfrak{S}}_n(\mathcal{B})$ for a chordal building set $\mathcal{B}$ which appeared in Corollary 9.6 and Theorem 11.6.

QUESTION 14.3. *Given a (non-chordal) building set $\mathcal{B}$, is there a way to define two sets of permutations $\mathfrak{S}'_n(\mathcal{B})$ and $\widehat{\mathfrak{S}}'_n(\mathcal{B})$ such that:*

- *the h-polynomial for the nestohedron $P_{\mathcal{B}}$ is given by the descent generating function for $\mathfrak{S}'_n(\mathcal{B})$, and*

- *the γ-polynomial is given by the peak generating function for $\widehat{\mathfrak{S}}'_n(\mathcal{B})$?*

15 Appendix: Deformations of a simple polytope

The goal of this section is to clarify the equivalence between various definitions of the deformations of a simple polytope, either by

- deforming vertex positions, keeping edges in the same parallelism class, or
- deforming edge lengths, keeping them nonnegative, or
- altering level sets of facet inequalities, but not allowing facets to move past any vertices.

There will be defined below three cones of such deformations which are all linearly isomorphic. This discussion is essentially implicit in [Post'05, Definition 6.1 and §19], but we hope the explication here clarifies this relationship.
Let P be a simple d-dimensional polytope in $\mathbb{R}^d$. Let V be its set of vertices. Let $E \subseteq \binom{V}{2}$ be its set of edge pairs. Let F be an indexing set for its facets, so that P is defined by facet inequalities $h_f(x) \leq \alpha_f$ for $f \in F$, in which each h_f is a linear functional in $(\mathbb{R}^d)^*$, and $(\alpha_f)_{f \in F} \in \mathbb{R}^F$.

Definition 15.1. *(1) The vertex deformation cone D_P^V of P is the set of points $(x_v)_{v \in V} \in (\mathbb{R}^d)^V$ such that*

$$x_u - x_v \in \mathbb{R}_{\geq 0}(u - v), \quad \textit{for every edge } uv \in E. \tag{20}$$

(2) The edge length deformation cone D_P^E of P is the set of points $(y_e)_{e \in E} \in \mathbb{R}^E$ such that all $y_e \geq 0$, and, for any 2-dimensional face of P with edges $e_1 = v_1v_2$, $e_2 = v_2v_3$, $\ldots$, $e_{k-1} = v_{k-1}v_k$, $e_k = v_kv_1$, one has

$$y_{e_1}(v_1 - v_2) + y_{e_2}(v_2 - v_3) + \cdots + y_{e_k}(v_k - v_1) = 0.$$

(3) For $\beta = (\beta_f)_{f \in F} \in \mathbb{R}^F$, let $P_\beta := \{x \in \mathbb{R}^d \mid h_f(x) \leq \beta_f, \text{ for } f \in F\}$ be the polytope obtained from P by parallel translations of the facets. In particular, $P_\alpha = P$. The open facet deformation cone[9] $D_P^{F,open}$ for P is the set of $\beta \in \mathbb{R}^F$ for which the polytopes P_β and P have the same normal fan $\mathcal{N}(P_\beta) = \mathcal{N}(P)$. (Equivalently, P_β and P have the same combinatorial structure.) The (closed) facet deformation cone is the closure D_P^F of $D_P^{F,open}$ inside $\mathbb{R}^F$.

It is clear that the definitions of D_P^V and D_P^E translate into linear equations and weak linear inequalities. Thus D_P^V and D_P^E are (closed) polyhedral cones in the spaces $(\mathbb{R}^d)^V$ and $\mathbb{R}^E$, correspondingly. The following lemma shows that D_P^F is also a polyhedral cone.

[9]This is linearly isomorphic to the *type-cone* of P described by McMullen [McM'73, §2, p. 88].

Lemma 15.2. *For a simple polytope P, the facet deformation cone $D_P^{F,open}$ is a full $|F|$-dimensional open polyhedral cone inside $\mathbb{R}^F$, that is a nonempty subset in $\mathbb{R}^F$ given by strict linear inequalities. Thus D_P^F is the closed polyhedral cone in $\mathbb{R}^F$ given by replacing the strict inequalities with the corresponding weak inequalities.*

Proof. Since every polytope P_β has facet normals in directions which are a subset of those for P, the rays (=1-dimensional normal cones) in $\mathcal{N}(P_\beta)$ are a subset of those in $\mathcal{N}(P)$. Therefore, one will have $\mathcal{N}(P_\beta) = \mathcal{N}(P)$ if and only if P_β, P have the same face lattices, or equivalently, the same collection of vertex-facet incidences (v, f). This means that one can define the set $D_P^{F,open}$ inside $\mathbb{R}^F$ by a collection of strict linear inequalities on the coordinates $\beta = (\beta_f)_{f \in F}$. It is next explained how to produce one such inequality for each pair (v_0, f_0) of a vertex v_0 and facet f_0 of P such that $v_0 \notin f_0$.
If v_0 lies on the d facets $f_1, \ldots, f_d$ in P, then v_0 is the unique solution to the linear system $h_{f_j}(x) = \alpha_{f_j}$ for $j = 1, ..., d$. Its corresponding vertex x_0 in P_β is then the unique solution to $h_{f_j}(x) = \beta_{f_j}$ for $j = 1, ..., d$. Note that this implies x_0 has coordinates given by linear expressions in the coefficients β. Then the inequality corresponding to the vertex-facet pair (v_0, f_0) asserts that $h_{f_0}(x_0) < \beta_{f_0}$.
Lastly, note that this system of strict linear inequalities has at least one solution, namely the α for which $P_\alpha = P$. Hence this open polyhedral cone is nonempty. ☐

Theorem 15.3 gives several ways to describe deformations of a simple polytope.

Theorem 15.3. *Let P be a d-dimensional simple polytope in $\mathbb{R}^d$, with notation as above. Then the following are equivalent for a polytope P' in $\mathbb{R}^d$:*

(i) The normal fan $\mathcal{N}(P)$ refines the normal fan $\mathcal{N}(P')$.

(ii) The vertices of P' can be (possibly redundantly) labelled x_v, $v \in V$, so that $(x_v)_{v \in V}$ is a point in the vertex deformation cone D_P^V, i.e., the x_v satisfy conditions (20).

(iii) The polytope P' is the convex hull of points x_v, $v \in V$, such that $(x_v)_{v \in V}$ is in the vertex deformation cone D_P^V.

(iv) $P' = P_\beta$ for some β in the closed facet deformation cone D_P^F.

(v) P' is a Minkowski summand of a dilated polytope rP, that is there exist a polytope $Q \subset \mathbb{R}^d$ and a real number $r > 0$ such that $P' + Q = rP$.

Proof. One proceeds by proving the following implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii), (iv) $\Rightarrow$ (v) $\Rightarrow$ (i).
(i) implies (ii). The refinement of normal fans gives rise to the redundant labelling of vertices $(x_v)_{v \in P}$ as follows: given a vertex x of P', label it by x_v for every vertex v in P having its normal cone $\mathcal{N}_v(P)$ contained in the

normal cone $\mathcal{N}_x(P')$. There are then two possibilities for an edge $uv \in E$ of P: either $x_u = x_v$, in which case (20) is trivially satisfied, or else $x_u \neq x_v$ so that $\mathcal{N}_u(P), \mathcal{N}_v(P)$ lie in different normal cones $\mathcal{N}_{x_u}(P') \neq \mathcal{N}_{x_v}(P')$. But then since $\mathcal{N}(P)$ refines $\mathcal{N}(P')$, these latter two cones must share a codimension one subcone lying in the same hyperplane that separates $\mathcal{N}_u(P)$ and $\mathcal{N}_v(P)$. As this hyperplane has normal vector $u - v$, this forces $x_u - x_v$ to be a positive multiple of this vector, as desired.

(ii) implies (iii). Trivial.

(iii) implies (i). Let P' be the convex hull of the points x_v. Fix a vertex $u \in V$. Let $\lambda \in (\mathbb{R}^d)^*$ be a generic linear functional that belongs to the normal cone $\mathcal{N}_u(P)$ of P at the vertex u. Then the maximum of λ on P is achieved at the point u and nowhere else. Let us orient the 1-skeleton of P so that λ increases on each directed edge. This connected graph has a unique vertex of outdegree 0, namely the vertex u. Thus, for any other vertex $v \in V$, there is a directed path $(v_1, \ldots, v_l)$ from $v_1 = v$ to $v_l = u$ in this directed graph. According to the conditions of the lemma, one has have $\lambda(x_{v_1}) \leq \lambda(x_{v_2}) \leq \cdots \leq \lambda(x_{v_l})$, so $\lambda(x_v) \leq \lambda(x_u)$. Thus the maximum of λ on the polytope P' is achieved at the point x_u. This implies that x_u is a vertex of P' and the normal cone $\mathcal{N}_{x_u}(P')$ of P' at this point contains the normal cone $\mathcal{N}_u(P)$ of P at u. Since the same statement is true for any vertex of P, one deduces that $\mathcal{N}(P)$ refines $\mathcal{N}(P')$.

(i) implies (iv). First, note that if $\mathcal{N}(P') = \mathcal{N}(P)$ then $P' = P_\beta$ for some β in the open facet deformation cone $D_P^{F,open}$. Indeed, the facets of P' are orthogonal to the 1-dimensional cones in $\mathcal{N}(P')$, thus they should be parallel to the corresponding facets of P.

Now assume that $\mathcal{N}(P)$ refines $\mathcal{N}(P')$. Recall the standard fact [Zieg'94, Prop. 7.12] that the normal fan $\mathcal{N}(Q_1 + Q_2)$ of a Minkowski sum $Q_1 + Q_2$ is the common refinement of the normal fans $\mathcal{N}(Q_1)$ and $\mathcal{N}(Q_2)$. Thus, for any $\epsilon > 0$, the normal fan of the Minkowski sum $P' + \epsilon P$ coincides with $\mathcal{N}(P)$. By the previous observation, one should have $P' + \epsilon P = P_{\beta(\epsilon)}$ for some $\beta(\epsilon) \in D_P^{F,open}$. Since all coordinates of $\beta(\epsilon)$ linearly depend on ϵ, one obtain $P' = P_\beta$ for $\beta = \lim_{\epsilon \to 0} \beta(\epsilon) \in D_P^F$.

(iv) implies (iii). Given $\beta \in D_P^F$, it is the limit point for some family $\{\beta(\epsilon)\} \subset D_P^{F,open}$. One may assume that $\beta(\epsilon)$ linearly depends on $\epsilon > 0$ and $\lim_{\epsilon \to 0} \beta(\epsilon) = \beta$. Hence $P' = P_\beta$ is the limit of the polytopes $P_{\beta(\epsilon)}$, which each have $\mathcal{N}(P_{\beta(\epsilon)}) = \mathcal{N}(P)$. In particular, the vertices of $P_{\beta(\epsilon)}$ can be labelled by $x_v(\epsilon)$, $v \in V$. These vertices linearly depend on ϵ and satisfy $x_u(\epsilon) - x_v(\epsilon) = \mathbb{R}_{\geq 0}(u - v)$ for any edge $uv \in E$. Taking the limit $\epsilon \to 0$, one obtains that P' is the convex hull of points $x_v = \lim_{\epsilon \to 0} x_v(\epsilon)$ satisfying (20).

(iv) implies (v). Note that $P_\gamma + P_\delta = P_{\gamma+\delta}$, for $\gamma, \delta \in D_P^F$. Let $P' = P_\beta$ for $\beta \in D_P^F$. The point α (such that $P = P_\alpha$) belongs to the *open* cone $D_P^{F,open}$. Thus, for sufficiently large r, the point $\gamma = r\alpha - \beta$ also belongs to the open cone $D_P^{F,open}$. Let $Q = P_\gamma$. Then one has $P' + Q = P_\beta + P_{r\alpha-\beta} = P_{r\alpha} = rP$, as needed.

(v) implies (i). This follows from the standard fact [Zieg'94, Prop. 7.12] on normal fans of Minkowski sums mentioned above. ☐

Remark 15.4. *We are being somewhat careful here, since Theorem 15.3 can fail when one allows a broader interpretation for a simple polytope P to deform into a polytope P' by parallel translations of its facets, e.g. if one allows facets to translate past vertices. For example, letting P' be a regular tetrahedron in $\mathbb{R}^3$, and P the result of "shaving off an edge" from P' with a generically tilted plane in $\mathbb{R}^3$, one finds that $\mathcal{N}(P)$ does not refine $\mathcal{N}(P')$.*

Let us now describe the relationship between the three deformation cones D_P^V, D_P^E, and D_P^F. Let H be the linear subspace in $(\mathbb{R}^d)^V$ given by

$$H := \{(x_v)_{v\in V} \in (\mathbb{R}^d)^V \mid x_u - x_v \in \mathbb{R}(u-v) \text{ for any edge } uv \in E\}.$$

Clearly, the cone D_P^V belongs to the subspace H. Let us define two linear maps

$$\phi : H \to \mathbb{R}^E \quad \text{and} \quad \psi : \mathbb{R}^F \to H.$$

The map ϕ sends $(x_v)_{v\in V} \in H$ to $(y_e)_{e\in E} \in \mathbb{R}^E$, where $x_u - x_v = y_e(u-v)$, for any edge $e = uv \in E$. The map ψ sends $\beta = (\beta_f)_{f\in F}$ to $(x_v)_{v\in V}$, where, for each vertex v of P given as the intersection of the facets of P indexed $f_1, \ldots, f_d$, the point $x_v \in \mathbb{R}^d$ is the unique solution of the linear system $\{h_{f_j}(x) = \beta_{f_j} \mid j = 1, \ldots, d\}$. For $\beta \in D_P^{F,open}$, $\psi(\beta) = (x_v)_{v\in V}$, where the x_v are the vertices of the polytope P_β. One can easily check that $\psi(\beta) \in H$. Indeed, this is clear for $\beta \in D_P^{F,open}$ and thus this extends to all $\beta \in \mathbb{R}^F$ by linearity.
Note that the kernel of the map ϕ is the subspace $\Delta(\mathbb{R}^d) \simeq \mathbb{R}^d$ embedded diagonally into $(\mathbb{R}^d)^V$. This comes from the fact that the 1-skeleton of P is connected. The vertex deformation cone D_P^V can be reduced modulo the subspace $\Delta(\mathbb{R}^d)$ of parallel translations of polytopes. Similarly, the facet deformation cone can be reduced modulo the subspace $\Delta'(\mathbb{R}^d) = \psi^{-1}(\Delta(\mathbb{R}^d)) \simeq \mathbb{R}^d$, where $\Delta'(x) := (h_f(x))_{f\in F}$ for $x \in \mathbb{R}^d$.

Theorem 15.5. *The map ψ gives a linear isomorphism between the cones D_P^F and D_P^V. The map ϕ induces a linear isomorphism between the cones $D_P^V/\Delta(\mathbb{R}^d)$ and D_P^E. Thus one has*

$$D_P^E \simeq D_P^V/\Delta(\mathbb{R}^d) \simeq D_P^F/\Delta'(\mathbb{R}^d).$$

In particular, $\dim D_P^E = \dim D_P^V - d = \dim D_P^F - d = |F| - d$.

Proof. The claim about the map ψ follows immediately from Theorem 15.3. Let us prove the claim about ϕ. Note that, for $(x_v)_{v\in V} \in D_P^V$, the point $(y_e)_{e\in E} = \phi((x_v)_{v\in V})$ satisfies the condition of Definition 15.1(2) because

$$\begin{aligned}
&y_{e_1}(v_1 - v_2) + y_{e_2}(v_2 - v_3) + \cdots + y_{e_k}(v_k - v_1) \\
&= (x_{v_1} - x_{v_2}) + (x_{v_2} - x_{v_3}) + \cdots + (x_{v_k} - x_{v_1}) \\
&= 0.
\end{aligned}$$

It remains to show that for any $(y_e)_{e\in E} \in D_P^E$ there exists a unique (modulo diagonal translations) element $(x_v)_{v\in V} \in H$ such that $x_u - x_v = y_e(u-v)$ for any edge $e = uv \in E$. Let us construct the points $x_v \in \mathbb{R}^d$, as follows. Pick a vertex $v_0 \in V$ and pick any point $x_{v_0} \in \mathbb{R}^d$. For any other $v \in V$, find a path $(v_0, v_1, \ldots, v_l)$ from v_0 to $v_l = v$ in the 1-skeleton of P and define $x_v = x_{v_0} - y_{v_0v_1}(v_0 - v_1) - y_{v_1v_2}(v_1 - v_2) - \cdots - y_{v_{l-1}v_l}(v_{l-1} - v_l)$. This point x_v does not depend on a choice of path from v_0 to v, because any other path in the 1-skeleton can be obtained by switches along 2-dimensional faces of P. These $(x_v)_{v\in V}$ satisfy the needed conditions.
Finally, note that $\dim D_P^F = |F|$ because D_P^F is a full-dimensional cone. $\square$

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Secondary Invariants for Frechet Algebras and Quasihomomorphisms

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Abstract. A Fréchet algebra endowed with a multiplicatively convex topology has two types of invariants: homotopy invariants (topological K-theory and periodic cyclic homology) and secondary invariants (multiplicative K-theory and the non-periodic versions of cyclic homology). The aim of this paper is to establish a Riemann-Roch-Grothendieck theorem relating direct images for homotopy and secondary invariants of Fréchet m-algebras under finitely summable quasihomomorphisms.

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1 Introduction

For a noncommutative space described by an associative Fréchet algebra $\mathscr{A}$ over $\mathbb{C}$, we distinguish two types of invariants. The first type are (smooth) homotopy invariants, for example topological K-theory [27] and periodic cyclic homology [5]. The other type are secondary invariants; they are no longer stable under homotopy and carry a finer information about the "geometry" of the space $\mathscr{A}$. Typical examples of secondary invariants are algebraic K-theory [29] (which will not be used here), multiplicative K-theory [17] and the unstable versions of cyclic homology [18]. The aim of this paper is to define push-forward maps for homotopy and secondary invariants between two Fréchet algebras $\mathscr{A}$ and $\mathscr{B}$, induced by a smooth finitely summable quasihomomorphism [8]. The compatibility between the different types of invariants is expressed through a noncommutative Riemann-Roch-Grothendieck theorem (Theorem 6.3). The present p aper is the first part of a series on secondary characteristic classes. In the second part we will show how to obtain *local formulas* for push-forward maps, following a general principle inspired by renormalization which establishes the link with chiral anomalies in quantum field theory [25]; in order to keep a reasonable size to the present paper, these methods will be published in a separate survey with further examples [26].

We deal with Fréchet algebras endowed with a multiplicatively convex topology, or Fréchet m-algebras for short. These algebras can be presented as inverse limits of sequences of Banach algebras, and as a consequence many constructions valid for Banach algebras carry over Fréchet m-algebras. In particular Phillips [27] defines topological K-theory groups $K_n^{\rm top}(\mathscr{A})$ for any such algebra $\mathscr{A}$ and $n \in \mathbb{Z}$. The fundamental properties of interest for us are (smooth) homotopy invariance and Bott periodicity, i.e. $K_{n+2}^{\rm top}(\mathscr{A}) \cong K_n^{\rm top}(\mathscr{A})$. Hence there are essentially two topological K-theory groups for any Fréchet m-algebra, $K_0^{\rm top}(\mathscr{A})$ whose elements are roughly represented by idempotents in the stabilization of $\mathscr{A}$ by the algebra $\mathscr{K}$ of "smooth compact operators", and $K_1^{\rm top}(\mathscr{A})$ whose elements are represented by invertibles. Fréchet m-algebras naturally arise in many situations related to differential geometry, commutative or not, and the formulation of index problems. In the latter situation one usually encounters an algebra $\mathscr{I}$ of "finitely summable operators", for us a Fréchet m-algebra provided with a continuous trace on its p-th power for some $p \geq 1$. A typical example is the Schatten class $\mathscr{I} = \mathscr{L}^p(H)$ of p-summable operators on an infinite-dimensional separable Hilbert space H. $\mathscr{A}$ can be stabilized by the completed projective tensor product $\mathscr{I}\hat{\otimes}\mathscr{A}$ and its topological K-theory $K_n^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{A})$ is the natural receptacle for indices. Other important topological invariants of $\mathscr{A}$ (as a locally convex algebra) are provided by the periodic cyclic homology groups $HP_n(\mathscr{A})$, which is the correct version sharing the properties of smooth homotopy invariance and periodicity mod 2 with topological K-theory [5]. For any finitely summable algebra $\mathscr{I}$ the Chern character $K_n^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to HP_n(\mathscr{A})$ allows to obtain cohomological formulations of index theorems.

If one wants to go beyond differential topology and detect *secondary* invariants as well, which are no longer stable under homotopy, one has to deal with algebraic K-theory [29] and the unstable versions of cyclic homology [18]. In principle the algebraic K-theory groups $K_n^{\rm alg}(\mathscr{A})$ defined for any $n \in \mathbb{Z}$ provide interesting secondary invariants for any ring $\mathscr{A}$, but are very hard to calculate. It is also unclear if algebraic K-theory can be linked to index theory in a way consistent with topological K-theory, and in particular if it is possible to construct direct images of algebraic K-theory classes in a reasonable context. Instead, we will generalize slightly an idea of Karoubi [16, 17] and define for any Fréchet m-algebra $\mathscr{A}$ the multiplicative K-theory groups $MK_n^{\mathscr{I}}(\mathscr{A})$, $n \in \mathbb{Z}$, indexed by a given finitely summable Fréchet m-algebra $\mathscr{I}$. Depending on the parity of the degree n, multiplicative K-theory classe s are represented by idempotents or invertibles in certain extensions of $\mathscr{I}\hat{\otimes}\mathscr{A}$, together with a transgression of their Chern character in certain quotient complexes. Multiplicative K-theory is by definition a mixture of the topological K-theory $K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A})$ and the non-periodic cyclic homology $HC_n(\mathscr{A})$. It provides a "good" approximation of algebraic K-theory but is much more tractable. In addition, the Jones-Goodwillie Chern character in *negative cyclic homology* $K_n^{\mathrm{alg}}(\mathscr{A}) \to HN_n(\mathscr{A})$ factors through multiplicative K-theory. The precise relations between topological, multiplicative K-theory and the various versions of cyclic homology are encoded in a commutative diagram whose rows are long exact sequences of abelian groups

$$\begin{tikzcd}
K_{n+1}^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \arrow[r] \arrow[d] & HC_{n-1}(\mathscr{A}) \arrow[r, "\delta"] \arrow[d, equal] & MK_n^{\mathscr{I}}(\mathscr{A}) \arrow[r] \arrow[d] & K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \arrow[d] \\
HP_{n+1}(\mathscr{A}) \arrow[r, "S"] & HC_{n-1}(\mathscr{A}) \arrow[r, "\widetilde{B}"] & HN_n(\mathscr{A}) \arrow[r, "I"] & HP_n(\mathscr{A})
\end{tikzcd} \tag{1}$$

The particular case $\mathscr{I} = \mathbb{C}$ was already considered by Karoubi [16, 17] after the construction by Connes and Karoubi of regulator maps on algebraic K-theory [6]. The incorporation of a finitely summable algebra $\mathscr{I}$ is rather straightforward. This diagram describes the primary and secondary invariants associated to the noncommutative "manifold" $\mathscr{A}$. We mention that the restriction to Fréchet m-algebras is mainly for convenience. In principle these constructions could be extended to all locally convex algebras over $\mathbb{C}$, however the subsequent results, in particular the proof of the Riemann-Roch-Grothendieck theorem would become much more involved.

If now $\mathscr{A}$ and $\mathscr{B}$ are two Fréchet m-algebras, it is natural to consider the adequate "morphisms" mapping the primary and secondary invariants from $\mathscr{A}$ to $\mathscr{B}$. Let $\mathscr{I}$ be a p-summable Fréchet m-algebra. By analogy with Cuntz' description of bivariant K-theory for C^*-algebras [8], if $\mathscr{E} \triangleright \mathscr{I}\hat{\otimes}\mathscr{B}$ denotes a Fréchet m-algebra containing $\mathscr{I}\hat{\otimes}\mathscr{B}$ as a (not necessarily closed) two-sided ideal, we define a *p-summable quasihomomorphism* from $\mathscr{A}$ to $\mathscr{B}$ as a continuous homomorphism

$$\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B} \ ,$$

where $\mathscr{E}^s$ and $\mathscr{I}^s$ are certain $\mathbb{Z}_2$-graded algebras obtained from $\mathscr{E}$ and $\mathscr{I}$ by a standard procedure. Quasihomomorphisms come equipped with a parity (even or odd) depending on the construction of $\mathscr{E}^s$ and $\mathscr{I}^s$. In general, we may suppose that the parity is p mod 2. We say that $\mathscr{I}$ is multiplicative if it is provided with a homomorphism $\boxtimes : \mathscr{I}\hat{\otimes}\mathscr{I} \to \mathscr{I}$, possibly defined up to adjoint action of multipliers on $\mathscr{I}$, and compatible with the trace. A basic example of multiplicative p-summable algebra is, once again, the Schatten class $\mathscr{L}^p(H)$. Then it is easy to show that such a quasihomomorphism induces a pushforward map in topological K-theory $\rho_! : K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to K_{n-p}^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{B})$, whose degree coincides with the parity of the quasihomomorphism. This is what one expects from bivariant K-theory and is not really new. Our goal is to extend this map to the entire diagram (1). Direct images for the unstable versions of cyclic homology are necessarily induced by a bivariant *non-periodic* cyclic cohomology class $\mathrm{ch}^p(\rho) \in HC^p(\mathscr{A},\mathscr{B})$. This bivariant Chern character exists only under certain admissibility properties about the algebra $\mathscr{E}$ (note that it is sufficient for $\mathscr{I}$ to be $(p+1)$-summable instead of p-summable). In particular, the bivariant Chern caracter constructed by Cuntz for any quasihomomorphism in [9, 10] cannot be used here because it provides a bivariant periodic cyclic cohomology class, which does not detect the secondary invariants of $\mathscr{A}$ and $\mathscr{B}$. We give the precise definition of an admissible quasihomomorphism and construct the bivariant Chern character $\mathrm{ch}^p(\rho)$ in section 3, on the basis of previous works [23]. An analogous construction was obtained by Nistor [20, 21] or by Cuntz and Quillen [12]. However the bivariant Chern character of [23] is related to other constructions involv ing the heat operator and can be used concretely for establishing local index theorems, see for example [24]. The pushforward map in topological K-theory combined with the bivariant Chern character leads to a pushforward map in multiplicative K-theory $\rho_! : MK_n^{\mathscr{I}}(\mathscr{A}) \to MK_{n-p}^{\mathscr{I}}(\mathscr{B})$. Our first main result is the following non-commutative version of the Riemann-Roch-Grothendieck theorem (see Theorem 6.3 for a precise statement):

Theorem 1.1 *Let $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s \hat{\otimes} \mathscr{B}$ be an admissible quasihomomorphism of parity p mod 2. Suppose that $\mathscr{I}$ is $(p+1)$-summable in the even case and p-summable in the odd case. Then one has a graded-commutative diagram*

$$\begin{tikzcd}
K^{\mathrm{top}}_{n+1}(\mathscr{I}\hat{\otimes}\mathscr{A}) \arrow[r] \arrow[d, "\rho_!"] & HC_{n-1}(\mathscr{A}) \arrow[r] \arrow[d, "\mathrm{ch}^p(\rho)"] & MK^{\mathscr{I}}_n(\mathscr{A}) \arrow[r] \arrow[d, "\rho_!"] & K^{\mathrm{top}}_n(\mathscr{I}\hat{\otimes}\mathscr{A}) \arrow[d, "\rho_!"] \\
K^{\mathrm{top}}_{n+1-p}(\mathscr{I}\hat{\otimes}\mathscr{B}) \arrow[r] & HC_{n-1-p}(\mathscr{B}) \arrow[r] & MK^{\mathscr{I}}_{n-p}(\mathscr{B}) \arrow[r] & K^{\mathrm{top}}_{n-p}(\mathscr{I}\hat{\otimes}\mathscr{B})
\end{tikzcd}$$

compatible with the cyclic homology SBI exact sequences after taking the Chern characters $K^{\mathrm{top}}_*(\mathscr{I}\hat{\otimes}\cdot) \to HP_*$ and $MK^{\mathscr{I}}_* \to HN_*$.

At this point it is interesting to note that the pushforward maps $\rho_!$ and the bivariant Chern character $\mathrm{ch}^p(\rho)$ enjoy some invariance properties with respect to equivalence relations among quasihomomorphisms. Two types of equivalence relations are defined: smooth homotopy and conjugation by invertibles. The second relation corresponds to "compact perturbation" in Kasparov KK-theory for C^*-algebras [2]. In the latter situation, the M_2-stable version of conjugation essentially coincide with homotopy, at least for separable $\mathscr{A}$ and σ-unital $\mathscr{B}$. For Fréchet algebras however, M_2-stable conjugation is *strictly stronger* than homotopy as an equivalence relation. This is indeed in the context of Fréchet algebras that secondary invariants appear. The pushforward maps in topological K-theory and periodic cyclic homology are invariant under homotopy of quasihomomorphisms. The maps in multiplicative K-theory and the non-perio dic versions of cyclic homology

HC_* and HN_* are only invariant under conjugation and not homotopy. Also note that in contrast with the C^*-algebra situation, the Kasparov product of two quasihomomorphisms $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s \hat{\otimes} \mathscr{B}$ and $\rho' : \mathscr{B} \to \mathscr{F}^s \triangleright \mathscr{I}^s \hat{\otimes} \mathscr{C}$ is not defined as a quasihomomorphism from $\mathscr{A}$ to $\mathscr{C}$. The various bivariant K-theories constructed for m-algebras [9, 10] or even for general bornological algebras [11] cannot be used here, again because they are homotopy invariant by construction. We leave the construction of a bivariant K-theory compatible with secondary invariants as an open problem.

In the last part of the paper we illustrate the Riemann-Roch-Grothendieck theorem by constructing assembly maps for certain crossed product algebras. If Γ is a discrete group acting on a Fréchet m-algebra $\mathscr{A}$, under certain conditions the crossed product $\mathscr{A} \rtimes \Gamma$ is again a Fréchet m-algebra and one would like to obtain multiplicative K-theory classes out of a geometric model inspired by the Baum-Connes construction [1]. Thus let $P \xrightarrow{\Gamma} M$ be a principal Γ-bundle over a compact manifold M, and denote by $\mathscr{A}_P$ the algebra of smooth sections of the associated $\mathscr{A}$-bundle. If D is a K-cycle for M represented by a pseudodifferential operator, we obtain a quasihomomorphism from $\mathscr{A}_P$ to $\mathscr{A} \rtimes \Gamma$ and hence a map

$$MK_n^{\mathscr{I}}(\mathscr{A}_P) \to MK_{n-p}^{\mathscr{I}}(\mathscr{A} \rtimes \Gamma)$$

for suitable p and Schatten ideal $\mathscr{I}$. In general this map cannot exhaust the entire multiplicative K-theory of the crossed product but nevertheless interesting secondary invariants arise in this way. In the case where $\mathscr{A}$ is the algebra of smooth functions on a compact manifold, $\mathscr{A}_P$ is commutative and its secondary invaiants are closely related to (smooth) Deligne cohomology. From this point of view the pushforward map in multiplicative K-theory should be considered as a non-commutative version of "integrating Deligne classes along the fibers" of a submersion. We perform the computations for the simple example provided by the noncommutative torus.

The paper is organized as follows. In section 2 we review the Cuntz-Quillen formulation of (bivariant) cyclic cohomology [12] in terms of quasi-free extensions for m-algebras. Nothing is new but we take the opportunity to fix the notations and recall a proof of generalized Goodwillie theorem. In section 3 we define quasihomomorphisms and construct the bivariant Chern character. The formulas are identical to those found in [23] but in addition we carefully establish their adic properties and conjugation invariance. In section 4 we recall Phillips' topological K-theory for Fréchet m-algebras, and introduce the periodic Chern character $K_n^{\mathrm{top}}(\mathscr{I} \hat{\otimes} \mathscr{A}) \to HP_n(\mathscr{A})$ when $\mathscr{I}$ is a finitely summable algebra. The essential point here is to give explicit and simple formulas for subsequent use. Section 5 is devoted to the definition of the multiplicative K-theory groups $MK_n^{\mathscr{I}}(\mathscr{A})$ and the proof of the long exact seque nce relating them with topological K-theory and cyclic homology. We also construct the negative Chern character $MK_n^{\mathscr{I}}(\mathscr{A}) \to HN_n(\mathscr{A})$ and show the compatibility with the SBI exact sequence. Direct images of topological and multiplicative K-theory under quasihomomorphisms are constructed in section 6 and the Riemann-Roch-Grothendieck theorem is proved. The example of assembly maps and crossed products is treated in section 7.

2 Cyclic homology

Cyclic homology can be defined for various classes of associative algebras over $\mathbb{C}$, in particular complete locally convex algebras. For us, a locally convex algebra $\mathscr{A}$ has a topology induced by a family of continuous seminorms $p : \mathscr{A} \to \mathbb{R}_+$, for which the multiplication $\mathscr{A} \times \mathscr{A} \to \mathscr{A}$ is jointly continuous. Hence for any seminorm p there exists a seminorm q such that $p(a_1a_2) \leq q(a_1)q(a_2)$, $\forall a_i \in \mathscr{A}$. For technical reasons however we shall restrict ourselves to *multiplicatively convex* algebras [5], whose topology is generated by a family of submultiplicative seminorms

$$p(a_1a_2) \leq p(a_1)p(a_2) \quad \forall a_i \in \mathscr{A} \ .$$

A complete multiplicatively convex algebra is called m-algebra, and may equivalently be described as a projective limit of Banach algebras. The unitalization $\mathscr{A}^+ = \mathbb{C} \oplus \mathscr{A}$ of an m-algebra $\mathscr{A}$ is again an m-algebra, for the seminorms $\tilde{p}(\lambda 1 + a) = |\lambda| + p(a)$, $\forall \lambda \in \mathbb{C}, a \in \mathscr{A}$. In the same way, if $\mathscr{B}$ is another m-algebra, the direct sum $\mathscr{A} \oplus \mathscr{B}$ is an m-algebra for the topology generated by the seminorms $(p \oplus q)(a, b) = p(a) + q(b)$, where p is a seminorm on $\mathscr{A}$ and q a seminorm on $\mathscr{B}$. Also, the algebraic tensor product $\mathscr{A} \otimes \mathscr{B}$ may be endowed with the projective topology induced by the seminorms

$$(p \otimes q)(c) = \inf \Big\{ \sum_{i=1}^{n} p(a_i)q(b_i) \text{ such that } c = \sum_{i=1}^{n} a_i \otimes b_i \ \in \mathscr{A} \otimes \mathscr{B} \Big\} \ . \qquad (2)$$

The *completion* $\mathscr{A} \hat{\otimes} \mathscr{B} = \mathscr{A} \hat{\otimes}_\pi \mathscr{B}$ of the algebraic tensor product under this family of seminorms is the projective tensor product of Grothendieck [14], and is again an m-algebra.

Cyclic homology, cohomology and bivariant cyclic cohomology for m-algebras can be defined either within the cyclic bicomplex formalism of Connes [5], or the X-complex of Cuntz and Quillen [12]. We will make an extensive use of both formalisms throughout this paper. In general, we suppose that all linear maps or homomorphims between m-algebras are continuous, tensor products are completed projective tensor products, and extensions of m-algebras $0 \to \mathscr{I} \to \mathscr{R} \to \mathscr{A} \to 0$ always admit a continuous linear splitting $\sigma : \mathscr{A} \to \mathscr{R}$.

2.1 Cyclic bicomplex

Non-commutative differential forms. Let $\mathscr{A}$ be an m-algebra. The space of non-commutative differential forms over $\mathscr{A}$ is the algebraic direct sum

$\Omega\mathscr{A} = \bigoplus_{n\geq 0}\Omega^n\mathscr{A}$ of the n-forms subspaces $\Omega^n\mathscr{A} = \mathscr{A}^+\hat{\otimes}\mathscr{A}^{\hat{\otimes}n}$ for $n \geq 1$ and $\Omega^0\mathscr{A} = \mathscr{A}$, where $\mathscr{A}^+$ is the unitalization of $\mathscr{A}$. Each of the subspaces $\Omega^n\mathscr{A}$ is complete but we do not complete the direct sum. It is customary to use the differential notation $a_0da_1\dots da_n$ (resp. $da_1\dots da_n$) for the string $a_0\otimes a_1\dots\otimes a_n$ (resp. $1\otimes a_1\dots\otimes a_n$). A continuous differential $d:\Omega^n\mathscr{A}\to\Omega^{n+1}\mathscr{A}$ is uniquely specified by $d(a_0da_1\dots da_n) = da_0da_1\dots da_n$ and $d^2=0$. A continuous and associative product $\Omega^n\mathscr{A}\times\Omega^m\mathscr{A}\to\Omega^{n+m}\mathscr{A}$ is defined as usual and fulfills the Leibniz rule $d(\omega_1\omega_2) = d\omega_1\omega_2 + (-)^{|\omega_1|}\omega_1 d\omega_2$, where $|\omega_1|$ is the degree of ω_1. This tu rns $\Omega\mathscr{A}$ into a differential graded (DG) algebra. On $\Omega\mathscr{A}$ are defined various operators. First of all, the Hochschild boundary map $b:\Omega^{n+1}\mathscr{A}\to\Omega^n\mathscr{A}$ reads $b(\omega da) = (-)^n[\omega,a]$ for $\omega\in\Omega^n\mathscr{A}$, and $b=0$ on $\Omega^0\mathscr{A}=\mathscr{A}$. One easily shows that b is continuous and $b^2=0$, hence $\Omega\mathscr{A}$ is a complex graded over $\mathbb{N}$. The Hochschild homology of $\mathscr{A}$ (with coefficients in the bimodule $\mathscr{A}$) is the homology of this complex:

$$HH_n(\mathscr{A}) = H_n(\Omega\mathscr{A}, b)\ , \qquad \forall n\in\mathbb{N}\ . \tag{3}$$

Then the Karoubi operator $\kappa:\Omega^n\mathscr{A}\to\Omega^n\mathscr{A}$ is defined by $1-\kappa = db+bd$. Therefore κ is continuous and commutes with b and d. One has $\kappa(\omega\, da) = (-)^n da\,\omega$ for any $\omega\in\Omega^n\mathscr{A}$ and $a\in\mathscr{A}$. The last operator is Connes' $B:\Omega^n\mathscr{A}\to\Omega^{n+1}\mathscr{A}$, equal to $(1+\kappa+\ldots+\kappa^n)d$ on $\Omega^n\mathscr{A}$. It is also continuous and verifies $B^2=0=Bb+bB$ and $B\kappa=\kappa B=B$. Thus $\Omega\mathscr{A}$ endowed with the two anticommuting differentials (b,B) becomes a bicomplex. It splits as a direct sum $\Omega\mathscr{A} = \Omega\mathscr{A}^+\oplus\Omega\mathscr{A}^-$ of even and odd degree differential forms, hence is a $\mathbb{Z}_2$-graded complex for the total boundary map $b+B$. However its homology is trivial [18]. The various versions of cyclic homology are defined using the natural filtrations on $\Omega\mathscr{A}$. Following Cuntz and Quillen [12], we define the *Hodge filtration* on $\Omega\mathscr{A}$ as the decreasing family of $\mathbb{Z}_2$-graded subcomplexes for the total boundary $b+B$

$$F^n\Omega\mathscr{A} = b\Omega^{n+1}\mathscr{A}\oplus\bigoplus_{k>n}\Omega^k\mathscr{A}\ , \qquad \forall n\in\mathbb{Z}\ ,$$

with the convention that $F^n\Omega\mathscr{A}=\Omega\mathscr{A}$ for $n<0$. The completion of $\Omega\mathscr{A}$ is defined as the projective limit of $\mathbb{Z}_2$-graded complexes

$$\widehat{\Omega}\mathscr{A} = \varprojlim_n \Omega\mathscr{A}/F^n\Omega\mathscr{A} = \prod_{n\geq 0}\Omega^n\mathscr{A}\ . \tag{4}$$

Hence $\widehat{\Omega}\mathscr{A} = \widehat{\Omega}^+\mathscr{A}\oplus\widehat{\Omega}^-\mathscr{A}$ is a $\mathbb{Z}_2$-graded complex endowed with the total boundary map $b+B$. It is itself filtered by the decreasing family of $\mathbb{Z}_2$-graded subcomplexes $F^n\widehat{\Omega}\mathscr{A} = \mathrm{Ker}(\widehat{\Omega}\mathscr{A}\to\Omega\mathscr{A}/F^n\Omega\mathscr{A})$, which may be written

$$F^n\widehat{\Omega}\mathscr{A} = b\Omega^{n+1}\mathscr{A}\oplus\prod_{k>n}\Omega^k\mathscr{A}\ , \qquad \forall n\in\mathbb{Z}\ . \tag{5}$$

In particular the quotient $\widehat{\Omega}\mathscr{A}/F^n\widehat{\Omega}\mathscr{A}$ is a $\mathbb{Z}_2$-graded complex isomorphic to $\Omega\mathscr{A}/F^n\Omega\mathscr{A}$, explicitly

$$\widehat{\Omega}\mathscr{A}/F^n\widehat{\Omega}\mathscr{A} = \bigoplus_{k=0}^{n-1}\Omega^k\mathscr{A} \oplus \Omega^n\mathscr{A}/b(\Omega^{n+1}\mathscr{A}) \ , \tag{6}$$

and it vanishes for $n<0$. As a topological vector space, $\widehat{\Omega}\mathscr{A}/F^n\widehat{\Omega}\mathscr{A}$ may fail to be separated because the image $b(\Omega^{n+1}\mathscr{A})$ is not closed in general.

Definition 2.1 *In any degree $n \in \mathbb{Z}$, the* periodic, non-periodic *and* negative *cyclic homologies are respectively the $(b+B)$-homologies*

$$\begin{aligned} HP_n(\mathscr{A}) &= H_{n+2\mathbb{Z}}(\widehat{\Omega}\mathscr{A}) \ , \\ HC_n(\mathscr{A}) &= H_{n+2\mathbb{Z}}(\widehat{\Omega}\mathscr{A}/F^n\widehat{\Omega}\mathscr{A}) \ , \\ HN_n(\mathscr{A}) &= H_{n+2\mathbb{Z}}(F^{n-1}\widehat{\Omega}\mathscr{A}) \ . \end{aligned} \tag{7}$$

Hence $HP_n(\mathscr{A}) \cong HP_{n+2}(\mathscr{A})$ is 2-periodic, $HC_n(\mathscr{A}) = 0$ for $n<0$ and $HN_n(\mathscr{A}) \cong HP_n(\mathscr{A})$ for $n \le 0$. By construction these cyclic homology groups fit into a long exact sequence

$$\ldots \longrightarrow HP_{n+1}(\mathscr{A}) \xrightarrow{S} HC_{n-1}(\mathscr{A}) \xrightarrow{B} HN_n(\mathscr{A}) \xrightarrow{I} HP_n(\mathscr{A}) \longrightarrow \ldots \tag{8}$$

where S is induced by projection, I by inclusion, and the connecting map corresponds to the operator B. The link between cyclic and Hochshild homology may be obtained through *non-commutative de Rham homology* [16], defined as

$$HD_n(\mathscr{A}) := H_{n+2\mathbb{Z}}(\widehat{\Omega}\mathscr{A}/F^{n+1}\widehat{\Omega}\mathscr{A}) \ , \quad \forall n \in \mathbb{Z} \ . \tag{9}$$

This yields for any $n \in \mathbb{Z}$ a short exact sequence of $\mathbb{Z}_2$-graded complexes

$$0 \longrightarrow G_n(\mathscr{A}) \longrightarrow \widehat{\Omega}\mathscr{A}/F^n\widehat{\Omega}\mathscr{A} \longrightarrow \widehat{\Omega}\mathscr{A}/F^{n-1}\widehat{\Omega}\mathscr{A} \longrightarrow 0 \ ,$$

where G_n is $\Omega^n\mathscr{A}/b\Omega^{n+1}\mathscr{A}$ in degree n mod 2, and $b\Omega^n\mathscr{A}$ in degree $n-1$ mod 2. One has $H_{n+2\mathbb{Z}}(G_n) = HH_n(\mathscr{A})$ and $H_{n-1+2\mathbb{Z}}(G_n) = 0$, so that the associated six-term cyclic exact sequence in homology reduces to

$$0 \to HD_{n-1}(\mathscr{A}) \to HC_{n-1}(\mathscr{A}) \to HH_n(\mathscr{A}) \to HC_n(\mathscr{A}) \to HD_{n-2}(\mathscr{A}) \to 0 \ ,$$

and Connes's SBI exact sequence [4] for cyclic homology is actually obtained by splicing together the above sequences for all $n \in \mathbb{Z}$:

$$\ldots \longrightarrow HC_{n+1}(\mathscr{A}) \xrightarrow{S} HC_{n-1}(\mathscr{A}) \xrightarrow{B} HH_n(\mathscr{A}) \xrightarrow{I} HC_n(\mathscr{A}) \longrightarrow \ldots \tag{10}$$

Hence the non-commutative de Rham homology group $HD_n(\mathscr{A})$ may be identified with the image of the periodicity shift $S : HC_{n+2}(\mathscr{A}) \to HC_n(\mathscr{A})$. Clearly the exact sequence (8) can be transformed to (10) by taking the natural maps $HP_n(\mathscr{A}) \to HC_n(\mathscr{A})$ and $HN_n(\mathscr{A}) \to HH_n(\mathscr{A})$.

Passing to the dual theory, let $\mathrm{Hom}(\widehat{\Omega}\mathscr{A}, \mathbb{C})$ be the $\mathbb{Z}_2$-graded complex of linear maps $\widehat{\Omega}\mathscr{A} \to \mathbb{C}$ which are continuous for the adic topology on $\widehat{\Omega}\mathscr{A}$ induced by the Hodge filtration. It is concretely described as the direct sum

$$\mathrm{Hom}(\widehat{\Omega}\mathscr{A}, \mathbb{C}) = \bigoplus_{n\geq 0} \mathrm{Hom}(\Omega^n\mathscr{A}, \mathbb{C}) \ ,$$

where $\mathrm{Hom}(\Omega^n\mathscr{A}, \mathbb{C})$ is the space of continuous linear maps $\Omega^n\mathscr{A} \to \mathbb{C}$. The space $\mathrm{Hom}(\widehat{\Omega}\mathscr{A}, \mathbb{C})$ is endowed with the transposed of the boundary operator $b+B$ on $\widehat{\Omega}\mathscr{A}$. Then the periodic cyclic cohomology of $\mathscr{A}$ is the cohomology of this complex:

$$HP^n(\mathscr{A}) = H^{n+2\mathbb{Z}}(\mathrm{Hom}(\widehat{\Omega}\mathscr{A}, \mathbb{C})) \ . \tag{11}$$

One defines analogously the non-periodic and negative cyclic cohomologies which fit into an IBS long exact sequence.

2.2 X-complex and quasi-free algebras

We now turn to the description of the X-complex. It first appeared in the coalgebra context in Quillen's work [28], and subsequently was used by Cuntz and Quillen in their formulation of cyclic homology [12]. Here we recall the X-complex construction for m-algebras.

Let $\mathscr{R}$ be an m-algebra. The space of non-commutative one-forms $\Omega^1\mathscr{R}$ is a $\mathscr{R}$-bimodule, hence we can take its quotient $\Omega^1\mathscr{R}_\natural$ by the subspace of commutators $[\mathscr{R}, \Omega^1\mathscr{R}] = b\Omega^2\mathscr{R}$. $\Omega^1\mathscr{R}_\natural$ may fail to be separated in general. However, it is automatically separated when $\mathscr{R}$ is *quasi-free*, see below. In order to avoid confusions in the subsequent notations, we always write a one-form $x_0\mathbf{d}x_1 \in \Omega^1\mathscr{R}$ with a bold $\mathbf{d}$ when dealing with the X-complex of $\mathscr{R}$. The latter is the $\mathbb{Z}_2$-graded complex [12]

$$X(\mathscr{R}): \quad \mathscr{R} \underset{\overline{b}}{\overset{\natural\mathbf{d}}{\rightleftarrows}} \Omega^1\mathscr{R}_\natural \ , \tag{12}$$

where $\mathscr{R} = X_+(\mathscr{R})$ is located in even degree and $\Omega^1\mathscr{R}_\natural = X_-(\mathscr{R})$ in odd degree. The class of the generic element $(x_0\mathbf{d}x_1 \mod [,]) \in \Omega^1\mathscr{R}_\natural$ is usually denoted by $\natural x_0\mathbf{d}x_1$. The map $\natural\mathbf{d} : \mathscr{R} \to \Omega^1\mathscr{R}_\natural$ thus sends $x \in \mathscr{R}$ to $\natural\mathbf{d}x$. Also, the Hochschild boundary $b : \Omega^1\mathscr{R} \to \mathscr{R}$ vanishes on the commutator subspace $[\mathscr{R}, \Omega^1\mathscr{R}]$, hence passes to a well-defined map $\overline{b} : \Omega^1\mathscr{R}_\natural \to \mathscr{R}$. Explicitly the image of $\natural x_0\mathbf{d}x_1$ by $\overline{b}$ is the commutator $[x_0, x_1]$. These maps are continuous and satisfy $\natural\mathbf{d}\circ\overline{b} = 0$ and $\overline{b}\circ\natural\mathbf{d} = 0$, so that $(X(\mathscr{R}), \natural\mathbf{d}\oplus\overline{b})$ indeed defines a $\mathbb{Z}_2$-graded complex. We mention that everything can be formulated when $\mathscr{R}$ itself is a $\mathbb{Z}_2$-graded algebra: we just have to replace everywhere the ordinary commutators by graded commutators, and the differentials anticommute with elements of odd degree. In particular one gets $\overline{b}\natural x\mathbf{d}y = (-)^{|x|}[x, y]$, where $|x|$ is the

degree of x and $[x,y]$ is the graded commutator. The X-complex is obviously a functor from m-algebras to $\mathbb{Z}_2$-graded complexes: if $\rho : \mathscr{R} \to \mathscr{S}$ is a continuous homomorphism, it induces a chain map of even degree $X(\rho) : X(\mathscr{R}) \to X(\mathscr{S})$, by setting $X(\rho)(x) = \rho(x)$ and $X(\rho)(\natural x_0 \mathbf{d}x_1) = \natural\, \rho(x_0)\mathbf{d}\rho(x_1)$.
In fact the X-complex may be identified with the quotient of the $(b+B)$-complex $\widehat{\Omega}\mathscr{R}$ by the subcomplex $F^1\widehat{\Omega}\mathscr{R} = b\Omega^2\mathscr{R} \oplus \prod_{k\geq 2}\Omega^k\mathscr{R}$ of the Hodge filtration, i.e. there is an exact sequence

$$0 \to F^1\widehat{\Omega}\mathscr{R} \to \widehat{\Omega}\mathscr{R} \to X(\mathscr{R}) \to 0 \ .$$

It turns out that the X-complex is especially designed to compute the cyclic homology of algebras for which the subcomplex $F^1\widehat{\Omega}\mathscr{R}$ is contractible. This led Cuntz and Quillen to the following definition:

Definition 2.2 ([12]) *An m-algebra $\mathscr{R}$ is called* quasi-free *if there exists a continuous linear map $\phi : \mathscr{R} \to \Omega^2\mathscr{R}$ with property*

$$\phi(xy) = \phi(x)y + x\phi(y) + \mathbf{d}x\mathbf{d}y \ , \quad \forall x,y \in \mathscr{R} \ . \tag{13}$$

We refer to [12, 19] for many other equivalent definitions of quasi-free algebras. Let us just observe that a quasi-free algebra has dimension ≤ 1 with respect to Hochschild cohomology. Indeed, the map ϕ allows to contract the Hochschild complex of $\mathscr{R}$ in dimensions > 1, and this contraction carries over to the cyclic bicomplex. First, the linear map

$$\sigma : \Omega^1\mathscr{R}_\natural \to \Omega^1\mathscr{R} \ , \qquad \natural x\mathbf{d}y \mapsto x\mathbf{d}y + b(x\phi(y))$$

is well-defined because it vanishes on the commutator subspace $[\mathscr{R},\Omega^1\mathscr{R}] = b\Omega^2\mathscr{R}$ by the algebraic property of ϕ. Hence σ is a continuous linear splitting of the exact sequence $0 \to b\Omega^2\mathscr{R} \to \Omega^1\mathscr{R} \to \Omega^1\mathscr{R}_\natural \to 0$. By the way, this implies that for a quasi-free algebra $\mathscr{R}$, the topological vector space $\Omega^1\mathscr{R}$ splits into the direct sum of two closed subspaces $b\Omega^2\mathscr{R}$ and $\Omega^1\mathscr{R}_\natural$. Then, we extend ϕ to a continuous linear map $\phi : \Omega^n\mathscr{R} \to \Omega^{n+2}\mathscr{R}$ in all degrees $n \geq 1$ by the formula

$$\phi(x_0\mathbf{d}x_1\dots\mathbf{d}x_n) = \sum_{i=0}^{n}(-)^{ni}\phi(x_i)\mathbf{d}x_{i+1}\dots\mathbf{d}x_n\mathbf{d}x_0\dots\mathbf{d}x_{i-1} \ .$$

The following proposition gives a chain map $\gamma : X(\mathscr{R}) \to \widehat{\Omega}\mathscr{R}$ which is inverse to the natural projection $\pi : \widehat{\Omega}\mathscr{R} \to X(\mathscr{R})$ up to homotopy. Remark that the infinite sum $(1-\phi)^{-1} := \sum_{n=0}^{\infty}\phi^n$ makes sense as a linear map $\mathscr{R} \to \widehat{\Omega}^+\mathscr{R}$ or $\Omega^1\mathscr{R} \to \widehat{\Omega}^-\mathscr{R}$.

Proposition 2.3 *Let $\mathscr{R}$ be a quasi-free m-algebra. Then*
i) The map $\gamma : X(\mathscr{R}) \to \widehat{\Omega}\mathscr{R}$ defined for $x,y \in \mathscr{R}$ by

$$\begin{aligned} \gamma(x) &= (1-\phi)^{-1}(x) \\ \gamma(\natural x\mathbf{d}y) &= (1-\phi)^{-1}(x\mathbf{d}y + b(x\phi(y))) \end{aligned} \tag{14}$$

is a chain map of even degree from the X-complex to the $(b+B)$-complex.
ii) Let $\pi : \widehat{\Omega}\mathscr{R} \to X(\mathscr{R})$ be the natural projection. There is a contracting homotopy of odd degree $h : \widehat{\Omega}\mathscr{R} \to \widehat{\Omega}\mathscr{R}$ such that

$$\begin{aligned} \pi \circ \gamma &= \mathrm{Id} \quad on \quad X(\mathscr{R}) \ , \\ \gamma \circ \pi &= \mathrm{Id} + [b+B, h] \quad on \quad \widehat{\Omega}\mathscr{R} \ . \end{aligned}$$

Hence $X(\mathscr{R})$ and $\widehat{\Omega}\mathscr{R}$ are homotopy equivalent.

Proof: See the proof of [22], Proposition 4.2. There the result was stated in the particular case of a tensor algebra $\mathscr{R} = T\mathscr{A}$, but the general case of a quasi-free algebra is strictly identical (with the tensor algebra the image of γ actually lands in the subcomplex $\Omega T\mathscr{A} \subset \widehat{\Omega}T\mathscr{A}$ for a judicious choice of ϕ, but for generic quasi-free algebras it is necessary to take the completion $\widehat{\Omega}\mathscr{R}$ of $\Omega\mathscr{R}$). ∎

Extensions. Let $\mathscr{A}$ be an m-algebra. By an extension of $\mathscr{A}$ we mean an exact sequence of m-algebras $0 \to \mathscr{I} \to \mathscr{R} \to \mathscr{A} \to 0$ provided with a continuous linear splitting $\mathscr{A} \to \mathscr{R}$, and the topology of the ideal $\mathscr{I}$ is induced by its inclusion in $\mathscr{R}$. Hence as a topological vector space $\mathscr{R}$ is the direct sum of the closed subspaces $\mathscr{I}$ and $\mathscr{A}$. By convention, the powers $\mathscr{I}^n$ of the ideal $\mathscr{I}$ will always denote the image in $\mathscr{R}$ of the n-th tensor power $\mathscr{I}\hat{\otimes}\ldots\hat{\otimes}\mathscr{I}$ by the multiplication map. For $n \leq 0$, we define $\mathscr{I}^0$ as the algebra $\mathscr{R}$. Now let us suppose that all the powers $\mathscr{I}^n$ are closed and direct summands in $\mathscr{R}$ (this is automatically satisfied if $\mathscr{R}$ is quasi-free). Then the quotients $\mathscr{R}/\mathscr{I}^n$ are m-algebras and give rise to an inverse system with surjective homomorphisms

$$0 \leftarrow \mathscr{A} = \mathscr{R}/\mathscr{I} \leftarrow \mathscr{R}/\mathscr{I}^2 \leftarrow \ldots \leftarrow \mathscr{R}/\mathscr{I}^n \leftarrow \ldots$$

We denote by $\widehat{\mathscr{R}} = \varprojlim_n \mathscr{R}/\mathscr{I}^n$ the projective limit and view it as a *pro-algebra* indexed by the directed set $\mathbb{Z}$ (see [19]). Since the bicomplex of non-commutative differential forms $\widehat{\Omega}\mathscr{R}$ and the X-complex $X(\mathscr{R})$ are functorial in $\mathscr{R}$, we can define $\widehat{\Omega}\widehat{\mathscr{R}}$ and $X(\widehat{\mathscr{R}})$ as the $\mathbb{Z}_2$-graded pro-complexes

$$\begin{aligned} \widehat{\Omega}\widehat{\mathscr{R}} &= \varprojlim_n \widehat{\Omega}(\mathscr{R}/\mathscr{I}^n) = \varprojlim_{m,n} \Omega(\mathscr{R}/\mathscr{I}^n)/F^m\Omega(\mathscr{R}/\mathscr{I}^n) \ , \\ X(\widehat{\mathscr{R}}) &= \varprojlim_n X(\mathscr{R}/\mathscr{I}^n) \ , \end{aligned}$$

endowed respectively with the total boundary maps $b+B$ and $\partial = \natural\mathbf{d} \oplus \overline{b}$. When $\mathscr{R}$ is quasi-free, a refinement of Proposition 2.3 yields a chain map $\gamma : X(\widehat{\mathscr{R}}) \to \widehat{\Omega}\widehat{\mathscr{R}}$ inverse to the projection $\pi : \widehat{\Omega}\widehat{\mathscr{R}} \to X(\widehat{\mathscr{R}})$ up to homotopy, which we call a *generalized Goodwillie equivalence*:

Proposition 2.4 *Let $0 \to \mathscr{I} \to \mathscr{R} \to \mathscr{A} \to 0$ be an extension of m-algebras, with $\mathscr{R}$ quasi-free. Then the chain map $\gamma : X(\mathscr{R}) \to \widehat{\Omega}\mathscr{R}$ extends to a homotopy equivalence of pro-complexes $X(\widehat{\mathscr{R}}) \to \widehat{\Omega}\widehat{\mathscr{R}}$.*

Proof: We recall the proof because it will be useful for establishing Proposition 3.10. Let us introduce the following decreasing filtration of the space $\Omega^m\mathscr{R}$ by the subspaces $H^k\Omega^m\mathscr{R}$, $k\in\mathbb{Z}$:

$$H^k\Omega^m\mathscr{R}=\sum_{k_0+\ldots+k_m\geq k}\mathscr{I}^{k_0}\mathbf{d}\mathscr{I}^{k_1}\ldots\mathbf{d}\mathscr{I}^{k_m}\ .$$

Clearly $H^{k+1}\Omega^m\mathscr{R}\subset H^k\Omega^m\mathscr{R}$, and for $k\leq 0$ $H^k\Omega^m\mathscr{R}=\Omega^m\mathscr{R}$. Morally, $H^k\Omega^m\mathscr{R}$ contains at least k powers of the ideal $\mathscr{I}$. The direct sum $\bigoplus_m H^k\Omega^m\mathscr{R}$ is stable by the operators d, b, κ, B for any k. We have to establish how k changes when the linear map $\phi:\Omega^m\mathscr{R}\to\Omega^{m+2}\mathscr{R}$ is applied. First consider $\phi:\mathscr{R}\to\Omega^2\mathscr{R}$. If $x_1,\ldots,x_k$ denote k elements in $\mathscr{R}$, one has by the algebraic property of ϕ (see [12])

$$\begin{aligned}\phi(x_1\ldots x_k)&=\sum_{i=1}^k x_1\ldots x_{i-1}\phi(x_i)x_{i+1}\ldots x_k\\&\quad+\sum_{1\leq i<j\leq k}x_1\ldots x_{i-1}\mathbf{d}x_ix_{i+1}\ldots x_{j-1}\mathbf{d}x_jx_{j+1}\ldots x_k\ .\end{aligned}$$

Taking the elements x_i in the ideal $\mathscr{I}$ and using that $\phi(\mathscr{I})\subset\Omega^2\mathscr{R}$ yields

$$\phi(\mathscr{I}^k)\subset\sum_{i=1}^k\mathscr{I}^{i-1}\mathbf{d}\mathscr{R}\mathbf{d}\mathscr{R}\mathscr{I}^{k-i}+\sum_{1\leq i<j\leq k}\mathscr{I}^{i-1}\mathbf{d}\mathscr{I}\mathscr{I}^{j-i-1}\mathbf{d}\mathscr{I}\mathscr{I}^{k-j}\ .$$

Therefore $\phi(\mathscr{I}^k)\subset H^{k-1}\Omega^2\mathscr{R}$ for any k. Now from the definition of ϕ on $\Omega^m\mathscr{R}$, one has

$$\phi(\mathscr{I}^{k_0}\mathbf{d}\mathscr{I}^{k_1}\ldots\mathbf{d}\mathscr{I}^{k_m})\subset\sum_{l=0}^m\phi(\mathscr{I}^{k_l})\mathbf{d}\mathscr{I}^{k_{l+1}}\ldots\mathbf{d}\mathscr{I}^{k_{l-1}}\subset H^{k-1}\Omega^{m+2}\mathscr{R}$$

whenever $k=k_0+\ldots+k_m$, hence $\phi(H^k\Omega^m\mathscr{R})\subset H^{k-1}\Omega^{m+2}\mathscr{R}$. Now let us evaluate the even part of the chain map $\gamma:\mathscr{R}\to\widehat{\Omega}^+\mathscr{R}$. The part of γ landing in $\Omega^{2m}\mathscr{R}$ is the m-th power ϕ^m. One gets $\phi^m(\mathscr{I}^k)\subset H^{k-m}\Omega^{2m}\mathscr{R}$, hence ϕ^m sends the quotient algebra $\mathscr{R}/\mathscr{I}^k$ to $\Omega^{2m}(\mathscr{R}/\mathscr{I}^n)$ provided $(1+2m)n\leq k-m$ (indeed $1+2m$ is the maximal number of factors $\mathscr{R}$ in the tensor product $\Omega^{2m}\mathscr{R}$). Passing to the projective limits, ϕ^m induces a well-defined map $\widehat{\mathscr{R}}\to\Omega^{2m}\widehat{\mathscr{R}}$, and summing over all degrees $2m$ yields $\gamma:\widehat{\mathscr{R}}\to\widehat{\Omega}^+\widehat{\mathscr{R}}$.
Let us turn to the odd part of the chain map $\gamma:\Omega^1\mathscr{R}_\natural\to\widehat{\Omega}^-\mathscr{R}$. By construction, it is the composition of the linear map $\sigma:\Omega^1\mathscr{R}_\natural\to\Omega^1\mathscr{R}$ with all the powers $\phi^m:\Omega^1\mathscr{R}\to\Omega^{2m+1}\mathscr{R}$. One has $\natural(\mathscr{I}^k\mathbf{d}\mathscr{R}+\mathscr{R}\mathbf{d}(\mathscr{I}^k))\subset\natural(\mathscr{I}^k\mathbf{d}\mathscr{R}+\mathscr{I}^{k-1}\mathbf{d}\mathscr{I})$, and by the definition of σ,

$$\begin{aligned}\sigma\natural(\mathscr{I}^k\mathbf{d}\mathscr{R}+\mathscr{I}^{k-1}\mathbf{d}\mathscr{I})&\subset\mathscr{I}^k\mathbf{d}\mathscr{R}+\mathscr{I}^{k-1}\mathbf{d}\mathscr{I}+b(\mathscr{I}^k\phi(\mathscr{R})+\mathscr{I}^{k-1}\phi(\mathscr{I}))\\&\subset H^k\Omega^1\mathscr{R}+bH^{k-1}\Omega^2\mathscr{R}\ \subset\ H^{k-1}\Omega^1\mathscr{R}\ .\end{aligned}$$

Therefore $(\phi^m \circ \sigma)\natural(\mathscr{I}^k\mathbf{d}\mathscr{R} + \mathscr{I}^{k-1}\mathbf{d}\mathscr{I}) \subset H^{k-m-1}\Omega^{2m+1}\mathscr{R}$. Since $\mathscr{R}$ is the direct sum of $\mathscr{R}/\mathscr{I}^k$ and $\mathscr{I}^k$ as a topological vector space, the quotient $\Omega^1(\mathscr{R}/\mathscr{I}^k)_\natural$ coincides with $\Omega^1\mathscr{R}/(\mathscr{I}^k\mathbf{d}\mathscr{R} + \mathscr{R}\mathbf{d}(\mathscr{I}^k) + [\mathscr{R}, \Omega^1\mathscr{R}])$, and the map $\phi^m \circ \sigma : \Omega^1(\mathscr{R}/\mathscr{I}^k)_\natural \to \Omega^{2m+1}(\mathscr{R}/\mathscr{I}^n)$ is well-defined provided $(2m+2)n \leq k - m - 1$. Thus passing to the projective limits induces a map $\Omega^1\widehat{\mathscr{R}}_\natural \to \Omega^{2m+1}\widehat{\mathscr{R}}$, and summing over m yields $\gamma : \Omega^1\widehat{\mathscr{R}}_\natural \to \widehat{\Omega}^-\widehat{\mathscr{R}}$.
Finally, the contracting homotopy h of Proposition 2.3 is also constructed from ϕ (see [22] Proposition 4.2), hence extends to a contracting homotopy $h : \widehat{\Omega}\widehat{\mathscr{R}} \to \widehat{\Omega}\widehat{\mathscr{R}}$. The relations $\pi \circ \gamma = \mathrm{Id}$ on $X(\widehat{\mathscr{R}})$ and $\gamma \circ \pi = \mathrm{Id} + [b + B, h]$ on $\widehat{\Omega}\widehat{\mathscr{R}}$ follow immediately. ∎

Adic filtration. Suppose that $\mathscr{I}$ is a (not necessarily closed) two-sided ideal in $\mathscr{R}$, provided with its own topology of m-algebra for which the inclusion $\mathscr{I} \to \mathscr{R}$ is continuous and the multiplication map $\mathscr{R}^+ \times \mathscr{I} \times \mathscr{R}^+ \to \mathscr{I}$ is jointly continuous. As usual we define the powers $\mathscr{I}^n$ as the two-sided ideals corresponding to the image in $\mathscr{R}$ of the n-fold tensor products $\mathscr{I}\hat{\otimes}\ldots\hat{\otimes}\mathscr{I}$ under multiplication. Following [12], we introduce the adic filtration of $X(\mathscr{R})$ by the subcomplexes

$$\begin{aligned} F^{2n}_{\mathscr{I}}X(\mathscr{R}) \quad &: \quad \mathscr{I}^{n+1} + [\mathscr{I}^n, \mathscr{R}] \rightleftarrows \natural\mathscr{I}^n\mathbf{d}\mathscr{R} \\ F^{2n+1}_{\mathscr{I}}X(\mathscr{R}) \quad &: \quad \mathscr{I}^{n+1} \rightleftarrows \natural(\mathscr{I}^{n+1}\mathbf{d}\mathscr{R} + \mathscr{I}^n\mathbf{d}\mathscr{I})\ , \end{aligned} \tag{15}$$

where the commutator $[\mathscr{I}^n, \mathscr{R}]$ is by definition the image of $\mathscr{I}^n\mathbf{d}\mathscr{R}$ under the Hochschild operator b, and $\mathscr{I}^n$ is defined as the unitalized algebra $\mathscr{R}^+$ for $n \leq 0$. This is a decreasing filtration because $F^{n+1}_{\mathscr{I}}X(\mathscr{R}) \subset F^n_{\mathscr{I}}X(\mathscr{R})$, and for $n < 0$ one has $F^n_{\mathscr{I}}X(\mathscr{R}) = X(\mathscr{R})$. Denote by $X_n(\mathscr{R}, \mathscr{I}) = X(\mathscr{R})/F^n_{\mathscr{I}}X(\mathscr{R})$ the quotient complex. It is generally not separated. One gets in this way an inverse system of $\mathbb{Z}_2$-graded complexes $\{X_n(\mathscr{R}, \mathscr{I})\}_{n\in\mathbb{Z}}$ with projective limit $\widehat{X}(\mathscr{R}, \mathscr{I})$.

Now suppose that we start from an extension of m-algebras $0 \to \mathscr{I} \to \mathscr{R} \to \mathscr{A} \to 0$ with continuous linear splitting, and assume that any power $\mathscr{I}^n$ is a direct summand in $\mathscr{R}$. Then, the sequence $X_n(\mathscr{R}, \mathscr{I})$ is related to the X-complexes of the quotient m-algebras $\mathscr{R}/\mathscr{I}^n$:

$$\begin{aligned} &0 \leftarrow X_0(\mathscr{R}, \mathscr{I}) = \mathscr{A}/[\mathscr{A}, \mathscr{A}] \leftarrow X_1(\mathscr{R}, \mathscr{I}) = X(\mathscr{A}) \leftarrow \ldots \\ &\ldots \leftarrow X(\mathscr{R}/\mathscr{I}^{n-1}) \leftarrow X_{2n-1}(\mathscr{R}, \mathscr{I}) \leftarrow X_{2n}(\mathscr{R}, \mathscr{I}) \leftarrow X(\mathscr{R}/\mathscr{I}^n) \leftarrow \ldots \end{aligned}$$

Hence the projective limit of the system $\{X_n(\mathscr{R}, \mathscr{I})\}_{n\in\mathbb{Z}}$ is isomorphic to the X-complex of the pro-algebra $\widehat{\mathscr{R}}$:

$$\widehat{X}(\mathscr{R}, \mathscr{I}) = \varprojlim_n X_n(\mathscr{R}, \mathscr{I}) = \varprojlim_n X(\mathscr{R}/\mathscr{I}^n) = X(\widehat{\mathscr{R}})\ . \tag{16}$$

The pro-complex $\widehat{X}(\mathscr{R}, \mathscr{I})$ is naturally filtered by the family of subcomplexes $F^n\widehat{X}(\mathscr{R}, \mathscr{I}) = \mathrm{Ker}(\widehat{X} \to X_n)$. If $0 \to \mathscr{J} \to \mathscr{S} \to \mathscr{B} \to 0$ is another extension of m-algebras with continuous linear splitting, then the space of linear maps between the two pro-complexes $\widehat{X}(\mathscr{R},\mathscr{I})$ and $\widehat{X}(\mathscr{S},\mathscr{J})$, or between $\widehat{X}$ and $\widehat{X}'$ for short, is given by

$$\mathrm{Hom}(\widehat{X},\widehat{X}') = \varprojlim_m \Big(\varinjlim_n \mathrm{Hom}(X_n, X'_m) \Big) \ , \tag{17}$$

where $\mathrm{Hom}(X_n, X'_m)$ is the space of continuous linear maps between the $\mathbb{Z}_2$-graded complexes $X_n(\mathscr{R},\mathscr{I})$ and $X_m(\mathscr{S},\mathscr{J})$. Thus $\mathrm{Hom}(\widehat{X},\widehat{X}')$ is a $\mathbb{Z}_2$-graded complex. It corresponds to the space of linear maps $\{f : \widehat{X} \to \widehat{X}' \mid \forall k, \exists n : \ f(F^n\widehat{X}) \subset F^k\widehat{X}'\}$; the boundary of an element f of parity $|f|$ is given by the graded commutator $\partial \circ f - (-)^{|f|} f \circ \partial$ with the bounary maps $\partial = \natural\mathbf{d}\oplus\overline{b}$ on $\widehat{X}$ and $\widehat{X}'$. $\mathrm{Hom}(\widehat{X},\widehat{X}')$ itself is filtered by the subcomplexes of linear maps of order $\leq n$ for any $n \in \mathbb{N}$:

$$\mathrm{Hom}^n(\widehat{X},\widehat{X}') = \{f : \widehat{X} \to \widehat{X}' \mid \forall k, \ f(F^{k+n}\widehat{X}) \subset F^k\widehat{X}'\} \ . \tag{18}$$

These Hom-complexes will be used in the various definitions of bivariant cyclic cohomology, once the relation between the adic filtration over the X-complex of a quasi-free algebra $\mathscr{R}$ and the Hodge filtration of the cyclic bicomplex over the quotient algebra $\mathscr{A} = \mathscr{R}/\mathscr{I}$ is established.

2.3 The tensor algebra

Taking $\mathscr{R}$ as the tensor algebra of an m-algebra $\mathscr{A}$ provides the link with cyclic homology [9, 12]. The (non-unital) tensor algebra $T\mathscr{A}$ is the completion of the algebraic direct sum $\bigoplus_{n\geq 1} \mathscr{A}^{\hat{\otimes} n}$ with respect to the family of seminorms

$$\widehat{p} = \bigoplus_{n\geq 1} p^{\otimes n} = p \oplus (p\otimes p) \oplus (p\otimes p\otimes p) \oplus \dots \ ,$$

where p runs through all the submultiplicative seminorms on $\mathscr{A}$. Of course $p^{\otimes n}$ is the projective seminorm on $\mathscr{A}^{\otimes n}$ defined by a generalization of (2). These seminorms are submultiplicative with respect to the tensor product $\mathscr{A}^{\hat{\otimes} n} \times \mathscr{A}^{\hat{\otimes} m} \to \mathscr{A}^{\hat{\otimes} n+m}$ and therefore the completion $T\mathscr{A}$ is an m-algebra. It is free, hence quasi-free: a linear map $\phi : T\mathscr{A} \to \Omega^2 T\mathscr{A}$ with the property $\phi(xy) = \phi(x)y + x\phi(y) + \mathbf{d}x\mathbf{d}y$ may be canonically constructed by setting $\phi(a) = 0$ on the generators $a \in \mathscr{A} \subset T\mathscr{A}$, and then recursively $\phi(a_1 \otimes a_2) = \mathbf{d}a_1\mathbf{d}a_2$, $\phi(a_1\otimes a_2\otimes a_3) = (\mathbf{d}a_1\mathbf{d}a_2)a_3 + \mathbf{d}(a_1\otimes a_2)\mathbf{d}a_3$, and so on...
The multiplication map $T\mathscr{A} \to \mathscr{A}$, sending $a_1\otimes\ldots\otimes a_n$ to the product $a_1 \ldots a_n$, is continuous and we denote by $J\mathscr{A}$ its kernel. Since the inclusion $\sigma_{\mathscr{A}} : \mathscr{A} \to T\mathscr{A}$ is a continuous linear splitting of the multiplication map, the two-sided ideal $J\mathscr{A}$ is a direct summand in $T\mathscr{A}$. This implies a linearly split quasi-free extension $0 \to J\mathscr{A} \to T\mathscr{A} \to \mathscr{A} \to 0$. It is the universal free extension of $\mathscr{A}$ in the following sense: let $0 \to \mathscr{I} \to \mathscr{R} \to \mathscr{A} \to 0$ be any other extension

($\mathscr{R}$ is not necessarily quasi-free), provided with a continuous linear splitting $\sigma : \mathscr{A} \to \mathscr{R}$. Then one gets a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & J\mathscr{A} \arrow[r] \arrow[d, "\rho_*"'] & T\mathscr{A} \arrow[r] \arrow[d, "\rho_*"'] & \mathscr{A} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathscr{I} \arrow[r] & \mathscr{R} \arrow[r] & \mathscr{A} \arrow[r] \arrow[l, bend right, dotted, "\sigma"'] & 0 \end{tikzcd}$$

where $\rho_* : T\mathscr{A} \to \mathscr{R}$ is the continuous algebra homomorphism obtained by setting $\rho_*(a) = \sigma(a)$ on the generators $a \in \mathscr{A} \subset T\mathscr{A}$. Moreover, the homomorphism ρ_* is independent of the linear splitting σ up to homotopy (two splittings can always be connected by a linear homotopy).
As remarked by Cuntz and Quillen [12], the tensor algebra is closely related to a deformation of the algebra of even-degree noncommutative differential forms $\Omega^+\mathscr{A}$. Endow the space $\Omega^+\mathscr{A}$ with the *Fedosov product*

$$\omega_1 \odot \omega_2 := \omega_1\omega_2 - d\omega_1 d\omega_2 \ , \quad \omega_i \in \Omega^+\mathscr{A} \ . \tag{19}$$

Then $(\Omega^+\mathscr{A}, \odot)$ is a dense subalgebra of $T\mathscr{A}$, with the explicit correspondence

$$\Omega^+\mathscr{A} \ni a_0 da_1 \dots da_{2n} \longleftrightarrow a_0 \otimes \omega(a_1, a_2) \otimes \dots \otimes \omega(a_{2n-1}, a_{2n}) \in T\mathscr{A} \ .$$

It turns out that the Fedosov product $\odot$ extends to the projective limit $\widehat{\Omega}^+\mathscr{A}$ and the latter is isomorphic to the pro-algebra

$$\widehat{T}\mathscr{A} = \varprojlim_n T\mathscr{A}/(J\mathscr{A})^n \ . \tag{20}$$

Moreover, $\widehat{\Omega}\mathscr{A}$ and $X(\widehat{T}\mathscr{A}) = \widehat{X}(T\mathscr{A}, J\mathscr{A})$ are isomorphic as $\mathbb{Z}_2$-graded provector spaces [12], and this isomorphism identifies the Hodge filtration $F^n\widehat{\Omega}\mathscr{A}$ with the adic filtration $F^n\widehat{X}(T\mathscr{A}, J\mathscr{A})$. By a fundamental result of Cuntz and Quillen, all these identifications are *homotopy equivalences* of pro-complexes, i.e. the boundary $b + B$ on $\widehat{\Omega}\mathscr{A}$ corresponds to the boundary $\natural\mathbf{d} \oplus \overline{b}$ on $\widehat{X}(T\mathscr{A}, J\mathscr{A})$ up to homotopy and rescaling (see [12]). Hence the periodic and negative cyclic homologies of $\mathscr{A}$ may be computed respectively by $\widehat{X}(T\mathscr{A}, J\mathscr{A})$ and $F^n\widehat{X}(T\mathscr{A}, J\mathscr{A})$. Also, the non-periodic cyclic homology of $\mathscr{A}$ may be computed by the quotient complex $X_n(T\mathscr{A}, J\mathscr{A})$ which is homotopy equivalent to the complex $\widehat{\Omega}\mathscr{A}/F^n\widehat{\Omega}\mathscr{A}$. More generally the same result holds if tensor algebra $T\mathscr{A}$ is replaced by any quasi-free extension of $\mathscr{A}$. Indeed if $0 \to \mathscr{I} \to \mathscr{R} \to \mathscr{A} \to 0$ is a quasi-free extension with c ontinuous linear splitting, the classifying homomorphism $\rho_* : T\mathscr{A} \to \mathscr{R}$ induces a map of pro-complexes $X(\rho_*) : \widehat{X}(T\mathscr{A}, J\mathscr{A}) \to \widehat{X}(\mathscr{R}, \mathscr{I})$ compatible with the adic filtrations induced by the ideals $J\mathscr{A}$ and $\mathscr{I}$. It turns out to be a homotopy equivalence, irrespective to the choice of $\mathscr{R}$:

Theorem 2.5 (Cuntz-Quillen [12]) *For any linearly split extension of m-algebras $0 \to \mathscr{I} \to \mathscr{R} \to \mathscr{A} \to 0$ with $\mathscr{R}$ quasi-free, one has isomorphisms*

$$\begin{aligned} HP_n(\mathscr{A}) &= H_{n+2\mathbb{Z}}(\widehat{X}(\mathscr{R},\mathscr{I}))\ , \\ HC_n(\mathscr{A}) &= H_{n+2\mathbb{Z}}(X_n(\mathscr{R},\mathscr{I}))\ , \\ HD_n(\mathscr{A}) &= H_{n+2\mathbb{Z}}(X_{n+1}(\mathscr{R},\mathscr{I}))\ , \\ HN_n(\mathscr{A}) &= H_{n+2\mathbb{Z}}(F^{n-1}\widehat{X}(\mathscr{R},\mathscr{I}))\ . \end{aligned} \qquad (21)$$

These filtrations also allow to define various versions of bivariant cyclic cohomology, which may be formulated either within the X-complex framework or by means of the $(b+B)$-complex of differential forms.

Definition 2.6 ([12]) *Let $\mathscr{A}$ and $\mathscr{B}$ be m-algebras, and choose arbitrary (linearly split) quasi-free extensions $0 \to \mathscr{I} \to \mathscr{R} \to \mathscr{A} \to 0$ and $0 \to \mathscr{J} \to \mathscr{S} \to \mathscr{B} \to 0$. The bivariant periodic cyclic cohomology of $\mathscr{A}$ and $\mathscr{B}$ is the homology of the $\mathbb{Z}_2$-graded complex (17) of linear maps between the pro-complexes $\widehat{X}(\mathscr{R},\mathscr{I})$ and $\widehat{X}(\mathscr{S},\mathscr{J})$:*

$$HP^n(\mathscr{A},\mathscr{B}) = H_{n+2\mathbb{Z}}\big(\mathrm{Hom}(\widehat{X}(\mathscr{R},\mathscr{I}),\widehat{X}(\mathscr{S},\mathscr{J}))\big)\ , \quad \forall n\in\mathbb{Z}\ . \qquad (22)$$

For any $n\in\mathbb{Z}$, the non-periodic cyclic cohomology group $HC^n(\mathscr{A},\mathscr{B})$ of degree n is the homology, in degree $n \bmod 2$, of the $\mathbb{Z}_2$-graded subcomplex (18) of linear maps of order $\leq n$:

$$HC^n(\mathscr{A},\mathscr{B}) = H_{n+2\mathbb{Z}}\big(\mathrm{Hom}^n(\widehat{X}(\mathscr{R},\mathscr{I}),\widehat{X}(\mathscr{S},\mathscr{J}))\big)\ . \qquad (23)$$

The embedding $\mathrm{Hom}^n \hookrightarrow \mathrm{Hom}^{n+2}$ induces, for any n, the S-operation in bivariant cyclic cohomology $S: HC^n(\mathscr{A},\mathscr{B}) \to HC^{n+2}(\mathscr{A},\mathscr{B})$, and $\mathrm{Hom}^n \hookrightarrow \mathrm{Hom}$ yields a natural map $HC^n(\mathscr{A},\mathscr{B}) \to HP^n(\mathscr{A},\mathscr{B})$.

Of course the bivariant periodic theory has period two: $HP^{n+2} = HP^n$. Let us look at particular cases. The algebra $\mathbb{C}$ is quasi-free hence $\widehat{X}(T\mathbb{C},J\mathbb{C})$ is homotopically equivalent to $X(\mathbb{C}): \mathbb{C} \rightleftarrows 0$, and the periodic cyclic homology of $\mathbb{C}$ is simply $HP_0(\mathbb{C}) = \mathbb{C}$ and $HP_1(\mathbb{C}) = 0$. This implies that for any m-algebra $\mathscr{A}$, we get the usual isomorphisms $HP^n(\mathbb{C},\mathscr{A}) \cong HP_{-n}(\mathscr{A})$ and $HP^n(\mathscr{A},\mathbb{C}) \cong HP^n(\mathscr{A})$ in any degree n. For the non-periodic theory, one has the isomorphism $HC^n(\mathbb{C},\mathscr{A}) \cong HN_{-n}(\mathscr{A})$ with negative cyclic homology, and $HC^n(\mathscr{A},\mathbb{C}) \cong HC^n(\mathscr{A})$ is the non-periodic cyclic cohomology of Connes [4]. Finally, since any class $\varphi \in HC^p(\mathscr{A},\mathscr{B})$ is represented by a chain map sending the subcomplex $F^n\widehat{X}(T\mathscr{A},J\mathscr{A})$ to $F^{n-p}\widehat{X}(T\mathscr{B},J\mathscr{B})$ for any $n\in\mathbb{Z}$, it is not difficult to check that φ induces a transformation of degree $-p$ between the SBI exact sequences for $\mathscr{A}$ and $\mathscr{B}$, i.e. a graded-commutative diagram

$$\begin{tikzcd} HP_{n+1}(\mathscr{A}) \arrow[r, "S"] \arrow[d, "\varphi"] & HC_{n-1}(\mathscr{A}) \arrow[r, "B"] \arrow[d, "\varphi"] & HN_n(\mathscr{A}) \arrow[r, "I"] \arrow[d, "\varphi"] & HP_n(\mathscr{A}) \arrow[d, "\varphi"] \\ HP_{n-p+1}(\mathscr{B}) \arrow[r, "S"] & HC_{n-p-1}(\mathscr{B}) \arrow[r, "B"] & HN_{n-p}(\mathscr{B}) \arrow[r, "I"] & HP_{n-p}(\mathscr{B}) \end{tikzcd}$$

The graded-commutativity comes from the fact that the middle square is actually *anti*commutative when φ is of odd degree, for in this case the connecting morphism B anticommutes with the chain map representing φ.

3 Quasihomomorphisms and Chern character

In this section we define quasihomomorphisms for *metrizable* (or Fréchet) m-algebras and construct a bivariant Chern character. The topology of a Fréchet m-algebra is defined by a countable family of submultiplicative seminorms, and can alternatively be considered as the projective limit of a sequence of Banach algebras [27]. In particular, the projective tensor product of two Fréchet m-algebras is again a Fréchet m-algebra.

We say that a Fréchet m-algebra $\mathscr{I}$ is p-summable (with $p \geq 1$ an integer), if there is a continuous trace $\mathrm{Tr} : \mathscr{I}^p \to \mathbb{C}$ on the pth power of $\mathscr{I}$. Recall that by definition, $\mathscr{I}^p$ is the image in $\mathscr{I}$ of the p-th completed tensor product $\mathscr{I}\hat{\otimes}\ldots\hat{\otimes}\mathscr{I}$ under the multiplication map. Hence the trace is understood as a continuous linear map $\mathscr{I}\hat{\otimes}\ldots\hat{\otimes}\mathscr{I} \to \mathbb{C}$, and the tracial property means that it vanishes on the image of $1-\lambda$, where the operator λ is the backward cyclic permutation $\lambda(i_1 \otimes \ldots \otimes i_p) = i_p \otimes i_1 \ldots \otimes i_{p-1}$. In the low degree $p = 1$ we interpret the trace just as a linear map $\mathscr{I} \to \mathbb{C}$ vanishing on the subspace of commutators $[\mathscr{I}, \mathscr{I}] := b\Omega^1\mathscr{I}$.

Now consider any Fréchet m-algebra $\mathscr{B}$ and form the completed tensor product $\mathscr{I}\hat{\otimes}\mathscr{B}$. Suppose that $\mathscr{E}$ is a Fréchet m-algebra containing $\mathscr{I}\hat{\otimes}\mathscr{B}$ as a (not necessarily closed) two-sided ideal, in the sense that the inclusion $\mathscr{I}\hat{\otimes}\mathscr{B} \to \mathscr{E}$ is continuous. The left and right multiplication maps $\mathscr{E}\times\mathscr{I}\hat{\otimes}\mathscr{B}\times\mathscr{E} \to \mathscr{I}\hat{\otimes}\mathscr{B}$ are then automatically jointly continuous (see [7]). As in [13], we define the semi-direct sum $\mathscr{E} \ltimes \mathscr{I}\hat{\otimes}\mathscr{B}$ as the algebra modeled on the vector space $\mathscr{E} \oplus \mathscr{I}\hat{\otimes}\mathscr{B}$, where the product is such that as many elements as possible are put in the summand $\mathscr{I}\hat{\otimes}\mathscr{B}$. The semi-direct sum is a Fréchet algebra but it may fail to be multiplicatively convex in general. *The situation when $\mathscr{E} \ltimes \mathscr{I}\hat{\otimes}\mathscr{B}$ is a Fréchet m-algebra* will be depicted as

$$\mathscr{E} \triangleright \mathscr{I}\hat{\otimes}\mathscr{B} \tag{24}$$

to stress the analogy with [8]. The definition of quasihomomorphisms involves a $\mathbb{Z}_2$-graded version of $\mathscr{E} \triangleright \mathscr{I}\hat{\otimes}\mathscr{B}$, depending only on a choice of parity (even or odd). It is constructed as follows:

1) Even case: Define $\mathscr{E}^s_+$ as the Fréchet m-algebra $\mathscr{E} \ltimes \mathscr{I}\hat{\otimes}\mathscr{B}$. It is endowed with a linear action of the group $\mathbb{Z}_2$ by automorphisms: the image of an element $(a, b) \in \mathscr{E} \oplus \mathscr{I}\hat{\otimes}\mathscr{B}$ under the generator F of the group is $(a+b, -b)$. We define the $\mathbb{Z}_2$-graded algebra $\mathscr{E}^s$ as the crossed product $\mathscr{E}^s_+ \rtimes \mathbb{Z}_2$. Hence $\mathscr{E}^s$ splits as the direct sum $\mathscr{E}^s_+ \oplus \mathscr{E}^s_-$ where $\mathscr{E}^s_+$ is the subalgebra of even degree elements and $\mathscr{E}^s_- = F\mathscr{E}^s_+$ is the odd subspace.

This definition is rather abstract but there is a concrete description of $\mathscr{E}^s$

in terms of 2×2 matrices. Consider $M_2(\mathscr{E}) = M_2(\mathbb{C})\hat{\otimes}\mathscr{E}$ as a $\mathbb{Z}_2$-graded algebra with grading operator $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Thus diagonal elements are of even degree and off-diagonal elements are odd. $\mathscr{E}^s$ can be identified with a (non-closed) subalgebra of $M_2(\mathbb{C})\hat{\otimes}\mathscr{E}$ in the following way. Any element $x+Fy \in \mathscr{E}^s$ may be decomposed to its even and odd parts $x, y \in \mathscr{E} \oplus \mathscr{I}\hat{\otimes}\mathscr{B}$, with $x = (a, b)$ and $y = (c, d)$. Then $x + Fy$ is represented by the matrix

$$x + Fy = \begin{pmatrix} a+b & c \\ c+d & a \end{pmatrix} \quad \text{with} \quad a, c \in \mathscr{E} \ , \ b, d \in \mathscr{I}\hat{\otimes}\mathscr{B} \ .$$

The action of $\mathbb{Z}_2$ on $\mathscr{E}^s_+$ is implemented by the adjoint action of the following odd-degree multiplier of $M_2(\mathscr{E})$:

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{C}) \ , \quad F^2 = 1 \ . \tag{25}$$

Denote by $\mathscr{I}^s = \mathscr{I}^s_+ \oplus \mathscr{I}^s_-$ the $\mathbb{Z}_2$-graded algebra $M_2(\mathbb{C})\hat{\otimes}\mathscr{I}$, with $\mathscr{I}^s_+$ the subalgebra of diagonal elements and $\mathscr{I}^s_-$ the off-diagonal subspace. We thus have an inclusion of $\mathscr{I}^s\hat{\otimes}\mathscr{B}$ as a (non-closed) two-sided ideal in $\mathscr{E}^s$, with $\mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B}$. The commutator $[F, \mathscr{E}^s_+]$ is contained in $\mathscr{I}^s_-\hat{\otimes}\mathscr{B}$. Finally, we denote by tr_s the supertrace of even degree on $M_2(\mathbb{C})$:

$$\mathrm{tr}_s : M_2(\mathbb{C}) \to \mathbb{C} \ , \qquad \mathrm{tr}_s \begin{pmatrix} a' & c \\ c' & a \end{pmatrix} = a' - a \ .$$

2) ODD CASE: Now regard $M_2(\mathscr{E})$ as a trivially graded algebra and define $\mathscr{E}^s_+$ as the (non-closed) subalgebra

$$\mathscr{E}^s_+ = \begin{pmatrix} \mathscr{E} & \mathscr{I}\hat{\otimes}\mathscr{B} \\ \mathscr{I}\hat{\otimes}\mathscr{B} & \mathscr{E} \end{pmatrix} \tag{26}$$

provided with its own topology of complete m-algebra. Let $C_1 = \mathbb{C} \oplus \varepsilon\mathbb{C}$ be the complex Clifford algebra of the one-dimensional euclidian space. C_1 is the $\mathbb{Z}_2$-graded algebra generated by the unit $1 \in \mathbb{C}$ in degree zero and ε in degree one with $\varepsilon^2 = 1$. We define the $\mathbb{Z}_2$-graded algebra $\mathscr{E}^s$ as the tensor product $C_1\hat{\otimes}\mathscr{E}^s_+$. Hence $\mathscr{E}^s = \mathscr{E}^s_+ \oplus \mathscr{E}^s_-$ where $\mathscr{E}^s_+$ is the subalgebra of even degree and $\mathscr{E}^s_- = \varepsilon\mathscr{E}^s_+$ is the odd subspace. Similarly, define $\mathscr{I}^s = M_2(C_1)\hat{\otimes}\mathscr{I} = \mathscr{I}^s_+ \oplus \mathscr{I}^s_-$. Then $\mathscr{I}^s\hat{\otimes}\mathscr{B}$ is a (non-closed) two-sided ideal of $\mathscr{E}^s$ and one has $\mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B}$. The matrix

$$F = \varepsilon \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in M_2(C_1) \ , \quad F^2 = 1 \tag{27}$$

is an odd multiplier of $\mathscr{E}^s$ and the commutator $[F, \mathscr{E}^s_+]$ is contained in $\mathscr{I}^s_-\hat{\otimes}\mathscr{B}$. Finally, we define a supertrace tr_s of odd degree on C_1 by sending the generators 1 to 0 and ε to $\pm\sqrt{2i}$. The normalization $\pm\sqrt{2i}$ is chosen for compatibility with Bott periodicity, see [22]. We will choose conventionally the "sign" as $-\sqrt{2i}$ in order to simplify the subsequent formulas. One thus has

$$\mathrm{tr}_s : M_2(C_1) \to \mathbb{C} \ , \quad \mathrm{tr}_s \begin{pmatrix} a+\varepsilon a' & b+\varepsilon b' \\ c+\varepsilon c' & d+\varepsilon d' \end{pmatrix} = -\sqrt{2i}\,(a'+d') \ .$$

The objects F, $\mathscr{E}^s$ and $\mathscr{I}^s$ are defined in such a way that we can handle the even and odd case simultaneously. This allows to give the following synthetic definition of quasihomomorphisms.

Definition 3.1 *Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{I}$, $\mathscr{E}$ be Fréchet m-algebras. Assume that $\mathscr{I}$ is p-summable and $\mathscr{E} \triangleright \mathscr{I}\hat{\otimes}\mathscr{B}$. A* quasihomomorphism *from $\mathscr{A}$ to $\mathscr{B}$ is a continuous homomorphism*

$$\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B} \tag{28}$$

sending $\mathscr{A}$ to the even degree subalgebra $\mathscr{E}^s_+$. The quasihomomorphism comes equipped with a degree (even or odd) depending on the degree chosen for the above construction of $\mathscr{E}^s$. In particular, the linear map $a \in \mathscr{A} \mapsto [F, \rho(a)] \in \mathscr{I}^s_-\hat{\otimes}\mathscr{B}$ is continuous.

In other words, a quasihomomorphism of even degree $\rho = \left(\begin{smallmatrix} \rho_+ & 0 \\ 0 & \rho_- \end{smallmatrix}\right)$ is a pair of homomorphisms $(\rho_+, \rho_-) : \mathscr{A} \rightrightarrows \mathscr{E}$ such that the difference $\rho_+(a) - \rho_-(a)$ lies in the ideal $\mathscr{I}\hat{\otimes}\mathscr{B}$ for any $a \in \mathscr{A}$. A quasihomomorphism of odd degree is a homomorphism $\rho : \mathscr{A} \to M_2(\mathscr{E})$ such that the off-diagonal elements land in $\mathscr{I}\hat{\otimes}\mathscr{B}$.

The Cuntz-Quillen formalism for bivariant cyclic cohomology $HC^n(\mathscr{A}, \mathscr{B})$ requires to work with quasi-free extensions of $\mathscr{A}$ and $\mathscr{B}$. Hence let us suppose that we choose such extensions of m-algebras

$$0 \to \mathscr{G} \to \mathscr{F} \to \mathscr{A} \to 0 \;, \qquad 0 \to \mathscr{J} \to \mathscr{R} \to \mathscr{B} \to 0 \;,$$

with $\mathscr{F}$ and $\mathscr{R}$ quasi-free. We always take $\mathscr{F} = T\mathscr{A}$ as the universal free extension of $\mathscr{A}$, but we leave the possibility to take any quasi-free extension $\mathscr{R}$ for $\mathscr{B}$ since the tensor algebra $T\mathscr{B}$ will not be an optimal choice in general. The first step toward the bivariant Chern character is to lift a given quasihomomorphism $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B}$ to a quasihomomorphism from $\mathscr{F}$ to $\mathscr{R}$, compatible with the filtrations by the ideals $\mathscr{G} \subset \mathscr{F}$, $\mathscr{J} \subset \mathscr{R}$. This requires to fix some admissibility conditions on the intermediate algebra $\mathscr{E}$:

Definition 3.2 *Let $0 \to \mathscr{J} \to \mathscr{R} \to \mathscr{B} \to 0$ be a quasi-free extension of $\mathscr{B}$, and let $\mathscr{I}$ be p-summable with trace* $\mathrm{Tr} : \mathscr{I}^p \to \mathbb{C}$. *We say that $\mathscr{E} \triangleright \mathscr{I}\hat{\otimes}\mathscr{B}$ is provided with an* $\mathscr{R}$-admissible extension *if there are algebras $\mathscr{M} \triangleright \mathscr{I}\hat{\otimes}\mathscr{R}$ and $\mathscr{N} \triangleright \mathscr{I}\hat{\otimes}\mathscr{J}$ and a commutative diagram of extensions*

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{N} \arrow[r] & \mathscr{M} \arrow[r] & \mathscr{E} \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{I}\hat{\otimes}\mathscr{J} \arrow[r] \arrow[u] & \mathscr{I}\hat{\otimes}\mathscr{R} \arrow[r] \arrow[u] & \mathscr{I}\hat{\otimes}\mathscr{B} \arrow[r] \arrow[u] & 0
\end{tikzcd} \tag{29}$$

with the following properties:

i) Any power $\mathscr{N}^n$ is a direct summand in $\mathscr{M}$ (as a topological vector space);

ii) For any degree $n \geq \max(1, p-1)$, the linear map $(\mathscr{I}\hat{\otimes}\mathscr{R})^n \mathbf{d}(\mathscr{I}\hat{\otimes}\mathscr{R}) \to \Omega^1\mathscr{R}_\natural$ induced by the trace $\mathscr{I}^{n+1} \to \mathbb{C}$ factors through the quotient

$$\natural(\mathscr{I}\hat{\otimes}\mathscr{R})^n \mathbf{d}(\mathscr{I}\hat{\otimes}\mathscr{R}) = (\mathscr{I}\hat{\otimes}\mathscr{R})^n \mathbf{d}(\mathscr{I}\hat{\otimes}\mathscr{R}) \bmod [\mathscr{M}, \Omega^1\mathscr{M}] \ ,$$

and the chain map $\mathrm{Tr} : F^{2n+1}_{\mathscr{I}\hat{\otimes}\mathscr{R}} X(\mathscr{M}) \to X(\mathscr{R})$ *thus obtained is of order zero with respect to the adic filtration induced by the ideals $\mathscr{N} \subset \mathscr{M}$ and $\mathscr{J} \subset \mathscr{R}$.*

In the following we will say that $\mathscr{E}$ is $\mathscr{R}$-admissible, keeping in mind that the extension $\mathscr{M}$ is given. Condition *i)* is automatically satisfied for example if $\mathscr{M}$ is a quasi-free algebra (this will not always be the case). The chain map Tr of condition *ii)* is constructed as follows. For $n \geq 1$ one has the inclusion $\natural(\mathscr{I}\hat{\otimes}\mathscr{R})^{n+1}\mathbf{d}\mathscr{M} \subset \natural(\mathscr{I}\hat{\otimes}\mathscr{R})^n \mathbf{d}(\mathscr{I}\hat{\otimes}\mathscr{R})$, so that the subcomplex of the $\mathscr{I}\hat{\otimes}\mathscr{R}$-adic filtration reads

$$F^{2n+1}_{\mathscr{I}\hat{\otimes}\mathscr{R}} X(\mathscr{M}) \ : \ (\mathscr{I}\hat{\otimes}\mathscr{R})^{n+1} \rightleftarrows \natural(\mathscr{I}\hat{\otimes}\mathscr{R})^n \mathbf{d}(\mathscr{I}\hat{\otimes}\mathscr{R}) \ .$$

Then, the trace $\mathscr{I}^{n+1} \to \mathbb{C}$ induces a partial trace $(\mathscr{I}\hat{\otimes}\mathscr{R})^{n+1} \to \mathscr{R}$. The latter combined with $\natural(\mathscr{I}\hat{\otimes}\mathscr{R})^n \mathbf{d}(\mathscr{I}\hat{\otimes}\mathscr{R}) \to \Omega^1\mathscr{R}_\natural$ yields a linear map in any degree $n \geq \max(1, p-1)$

$$\mathrm{Tr} : F^{2n+1}_{\mathscr{I}\hat{\otimes}\mathscr{R}} X(\mathscr{M}) \to X(\mathscr{R}) \ , \tag{30}$$

compatible with the inclusions $F^{2n+3}_{\mathscr{I}\hat{\otimes}\mathscr{R}} X(\mathscr{M}) \subset F^{2n+1}_{\mathscr{I}\hat{\otimes}\mathscr{R}} X(\mathscr{M})$. The trace over $\mathscr{I}^{n+1}$ ensures that (30) is a chain map. It is not obvious, however, that it is automatically of degree zero with respect to the $\mathscr{N}$-adic and $\mathscr{J}$-adic filtrations, i.e. that the intersection $F^{2n+1}_{\mathscr{I}\hat{\otimes}\mathscr{R}} X(\mathscr{M}) \cap F^k_{\mathscr{N}} X(\mathscr{M})$ is mapped to $F^k_{\mathscr{J}} X(\mathscr{R})$ for any $k \in \mathbb{Z}$. This should be imposed as a condition.
Remark that the case $p = 1$, $n = 0$ is pathological, since there is no canonical way to map the space $\natural((\mathscr{I}\hat{\otimes}\mathscr{R})\mathbf{d}\mathscr{M} + \mathscr{M}^+\mathbf{d}(\mathscr{I}\hat{\otimes}\mathscr{R}))$ to $\Omega^1\mathscr{R}_\natural$ using only the trace over $\mathscr{I}$. In this situation, it seems preferable to impose directly the existence of a chain map $\mathrm{Tr} : F^1_{\mathscr{I}\hat{\otimes}\mathscr{R}} X(\mathscr{M}) \to X(\mathscr{R})$ in the definition of admissibility.

Example 3.3 When $\mathscr{A}$ is arbitrary and $\mathscr{B} = \mathbb{C}$, a p-summable quasihomomorphism represents a K-homology class of $\mathscr{A}$ in the sense of [4, 5]. Here we take $\mathscr{I} = \mathscr{L}^p(H)$ as the Schatten ideal of p-summable operators on a separable infinite-dimensional Hilbert space H. Recall that $\mathscr{I}$ is a two-sided ideal in the algebra of all bounded operators $\mathscr{L} = \mathscr{L}(H)$. $\mathscr{I}$ is a Banach algebra for the norm $\|x\|_p = (\mathrm{Tr}(|x|^p))^{1/p}$, $\mathscr{L}$ is provided with the operator norm, and the products $\mathscr{I} \times \mathscr{L} \times \mathscr{I} \to \mathscr{I}$ are jointly continuous. Since $\mathscr{L}$ and $\mathscr{I}$ are Banach algebras, the semi-direct sum $\mathscr{L} \ltimes \mathscr{I}$ is automatically a Banach algebra and we can write $\mathscr{L} \triangleright \mathscr{I}$. A p-summable K-homology class of even degree is represented by a pair of continuous homomorphisms $(\rho_+, \rho_-) : \mathscr{A} \rightrightarrows \mathscr{L}$ such that the difference $\rho_+ - \rho_-$ lands to $\mathscr{I}$. We get in this way an even degree quasihomomorph ism $\rho : \mathscr{A} \to \mathscr{L}^s \triangleright \mathscr{I}^s$. Here it is important to note that by a slight modification of the intermediate algebra $\mathscr{L}$, it is always possible to consider $\mathscr{I}$ as a closed ideal [13]. Indeed if we define $\mathscr{E}$ as the Banach algebra

$$\mathscr{E} = \mathscr{L} \ltimes \mathscr{I}$$

then one clearly has $\mathscr{E} \triangleright \mathscr{I}$ and $\mathscr{I}$ is closed in $\mathscr{E}$ by construction. The pair of homomorphisms $(\rho_+, \rho_-) : \mathscr{A} \rightrightarrows \mathscr{L}$ may be replaced with a new pair $(\rho'_+, \rho'_-) : \mathscr{A} \rightrightarrows \mathscr{E}$ by setting $\rho'_+(a) = (\rho_-(a), \rho_+(a) - \rho_-(a))$ and $\rho'_-(a) = (\rho_-(a), 0)$ in $\mathscr{L} \oplus \mathscr{I}$. The above K-homology class is then represented by the new quasihomomorphism $\rho' : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s$.
In the odd case, a p-summable K-homology class is represented by a continuous homomorphism $\rho : \mathscr{A} \to \left(\begin{smallmatrix} \mathscr{L} & \mathscr{I} \\ \mathscr{I} & \mathscr{L} \end{smallmatrix}\right)$, which can be equivalently described as a homomorphism $\rho' : \mathscr{A} \to M_2(\mathscr{E})$ with off-diagonal elements in $\mathscr{I}$.
Concerning cyclic homology, the algebra $\mathbb{C}$ is quasi-free, hence the quasi-free extension $\mathscr{R} = \mathbb{C}$ and $\mathscr{J} = 0$ computes the cyclic homology of $\mathbb{C}$. Therefore by choosing $\mathscr{M} = \mathscr{E}$ and $\mathscr{N} = 0$, the algebra $\mathscr{E} \triangleright \mathscr{I}$ is $\mathbb{C}$-admissible (condition *ii)* is trivial since $\Omega^1\mathbb{C}_\natural = 0$).

Example 3.4 More generally, if $\mathscr{I}$ is a p-summable Fréchet m-algebra contained as a (not necessarily closed) two-sided ideal in a unital Fréchet m-algebra $\mathscr{L}$, with $\mathscr{L} \triangleright \mathscr{I}$, a p-summable quasihomomorphism from $\mathscr{A}$ to $\mathscr{B}$ could be constructed from the generic intermediate algebra $\mathscr{E} = \mathscr{L} \hat{\otimes} \mathscr{B}$, provided that the map $\mathscr{I} \hat{\otimes} \mathscr{B} \to \mathscr{E}$ is injective. If $0 \to \mathscr{J} \to \mathscr{R} \to \mathscr{B} \to 0$ is a quasi-free extension of $\mathscr{B}$, the choice $\mathscr{M} = \mathscr{L} \hat{\otimes} \mathscr{R}$ and $\mathscr{N} = \mathscr{L} \hat{\otimes} \mathscr{J}$ shows that $\mathscr{E}$ is $\mathscr{R}$-admissible provided that the maps $\mathscr{I} \hat{\otimes} \mathscr{R} \to \mathscr{M}$ and $\mathscr{I} \hat{\otimes} \mathscr{J} \to \mathscr{N}$ are injective. In fact it is easy to get rid of these injectivity conditions by redefining the algebra

$$\mathscr{E} = (\mathscr{L} \ltimes \mathscr{I}) \hat{\otimes} \mathscr{B} \ ,$$

which contains $\mathscr{I} \hat{\otimes} \mathscr{B}$ as a closed ideal. Then $\mathscr{E}$ becomes automatically $\mathscr{R}$-admissible by taking $\mathscr{M} = (\mathscr{L} \ltimes \mathscr{I}) \hat{\otimes} \mathscr{R}$ and $\mathscr{N} = (\mathscr{L} \ltimes \mathscr{I}) \hat{\otimes} \mathscr{J}$ (remark that $(\mathscr{L} \ltimes \mathscr{I})^n = \mathscr{L} \ltimes \mathscr{I}$ for any n because $\mathscr{L}$ is unital, hence $\mathscr{N}^n$ is a direct summand in $\mathscr{M}$). The chain map $\mathrm{Tr} : F^{2n+1}_{\mathscr{I} \hat{\otimes} \mathscr{R}} X(\mathscr{M}) \to X(\mathscr{R})$ is obtained by multiplying all the factors in $\mathscr{L}$ and $\mathscr{I}$, and taking the trace on $\mathscr{I}^{n+1}$, while the compatibility between the $\mathscr{N}$-adic and $\mathscr{J}$-adic filtrations is obvious. Although interesting examples arise under this form (see section 7), the algebra $\mathscr{E}$ cannot be decomposed into a tensor product with $\mathscr{B}$ in all situations.

Example 3.5 An important example where $\mathscr{E}$ cannot be taken in the previous form is provided by the Bott element of the real line. Here $\mathscr{A} = \mathbb{C}$ and $\mathscr{B} = C^\infty(0,1)$ is the algebra of smooth functions $f : [0,1] \to \mathbb{C}$ such that f and all its derivatives vanish at the endpoints 0 and 1. Take $\mathscr{I} = \mathbb{C}$ as a 1-summable algebra, and $\mathscr{E} = C^\infty[0,1]$ is the algebra of smooth functions $f : [0,1] \to \mathbb{C}$ with the derivatives vanishing at the endpoints, while f itself takes arbitrary values at 0 and 1. $\mathscr{B}$ and $\mathscr{E}$ provided with their usual Fréchet topology are m-algebras, and one has $\mathscr{E} \triangleright \mathscr{B}$. The Bott element is represented by the quasihomomorphism of *odd degree*

$$\rho : \mathbb{C} \to \mathscr{E}^s \triangleright \mathscr{I}^s \hat{\otimes} \mathscr{B} ,$$

where $\mathscr{I}^s = M_2(C_1)$ and $\mathscr{E}^s \subset M_2(C_1)\hat{\otimes}\mathscr{E}$ by construction. The homomorphism $\rho : \mathbb{C} \to \mathscr{E}^s_+$ is built from an arbitrary real-valued function $\xi \in \mathscr{E}$ such that $\xi(0) = 0$, $\xi(1) = \pi/2$, and sends the unit $e \in \mathbb{C}$ to the matrix

$$\rho(e) = R^{-1} \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} R , \qquad R = \begin{pmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{pmatrix} .$$

The algebra $\mathscr{B}$ is quasi-free hence we can choose $\mathscr{R} = \mathscr{B}$, $\mathscr{J} = 0$ as quasi-free extension. The cyclic homology of $\mathscr{B}$ is therefore computed by $X(\mathscr{B})$. Moreover, setting $\mathscr{M} = \mathscr{E}$ and $\mathscr{N} = 0$ shows that $\mathscr{E}$ is $\mathscr{B}$-admissible. Indeed, $\Omega^1\mathscr{B}_\natural$ is contained in the space $\Omega^1(0,1)$ of ordinary (commutative) complex-valued smooth one-forms over $[0,1]$ vanishing at the endpoints with all their derivatives. The chain map $\mathrm{Tr} : F^{2n+1}_{\mathscr{B}} X(\mathscr{E}) \to X(\mathscr{B})$ is thus well-defined for any $n \geq 1$, and just amounts to project noncommutative forms over $\mathscr{E}$ to ordinary (commutative) differential forms over $[0,1]$.

We shall now construct the bivariant Chern character of a given p-summable quasihomomorphism $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s \hat{\otimes} \mathscr{B}$. We take the universal free extension $0 \to J\mathscr{A} \to T\mathscr{A} \to \mathscr{A} \to 0$ for $\mathscr{A}$, and choose some quasi-free extension $0 \to \mathscr{J} \to \mathscr{R} \to \mathscr{B} \to 0$ for $\mathscr{B}$ with the property that the algebra $\mathscr{E} \triangleright \mathscr{I} \hat{\otimes} \mathscr{B}$ is $\mathscr{R}$-admissible. The bivariant Chern character should be represented by a chain map between the complexes $X(T\mathscr{A})$ and $X(\mathscr{R})$, compatible with the adic filtrations induced by the ideals $J\mathscr{A}$ and $\mathscr{J}$ (section 2). Our task is thus to lift the quasihomomorphism to the quasi-free algebras $T\mathscr{A}$ and $\mathscr{R}$. First, the admissibility condition provides a diagram of extensions (29). From $\mathscr{M} \triangleright \mathscr{I} \hat{\otimes} \mathscr{R}$ define the $\mathbb{Z}_2$-graded algebra $\mathscr{M}^s$ in complete analogy with $\mathscr{E}^s$: depending on the degree of the quasihomomorphism, $\mathscr{M}^s$ is a subalgebra of $M_2(\mathbb{C})\hat{\otimes}\mathscr{M}$ (even case) or $M_2(C_1)\hat{\otimes}\mathscr{M}$ (odd case), with commutator property $[F, \mathscr{M}^s_+] \subset \mathscr{I}^s_- \hat{\otimes} \mathscr{R}$. Also, from $\mathscr{N} \triangleright \mathscr{I} \hat{\otimes} \mathscr{J}$ define $\mathscr{N}^s$ as the $\mathbb{Z}_2$-graded subalgebra of $M_2(\mathbb{C})\hat{\otimes}\mathscr{N}$ or $M_2(C_1)\hat{\otimes}\mathscr{N}$ with commutator $[F, \mathscr{N}^s_+] \subset \mathscr{I}^s_- \hat{\otimes} \mathscr{J}$. The algebras $\mathscr{E}^s$, $\mathscr{M}^s$ and $\mathscr{N}^s$ are gifted with a differential of odd degree induced by the graded commutator $[F,\ ]$ (its square vanishes because $F^2 = 1$). Then we get an extension of $\mathbb{Z}_2$-graded differential algebras

$$0 \to \mathscr{N}^s \to \mathscr{M}^s \to \mathscr{E}^s \to 0 .$$

The restriction to the even-degree subalgebras yields an extension of trivially graded algebras $0 \to \mathscr{N}^s_+ \to \mathscr{M}^s_+ \to \mathscr{E}^s_+ \to 0$, split by a continuous linear map $\sigma : \mathscr{E}^s_+ \to \mathscr{M}^s_+$ (recall the splitting is our basic hypothesis about extensions of m-algebras). The universal properties of the tensor algebra $T\mathscr{A}$ then allows to extend the homomorphism $\rho : \mathscr{A} \to \mathscr{E}^s_+$ to a continuous homomorphism $\rho_* : T\mathscr{A} \to \mathscr{M}^s_+$ by setting $\rho_*(a_1 \otimes \ldots \otimes a_n) = \sigma\rho(a_1) \otimes \ldots \otimes \sigma\rho(a_n)$:

$$
\begin{tikzcd}
0 \arrow[r] & J\mathscr{A} \arrow[r] \arrow[d, "\rho_*"'] & T\mathscr{A} \arrow[r] \arrow[d, "\rho_*"'] & \mathscr{A} \arrow[r] \arrow[d, "\rho"] & 0 \\
0 \arrow[r] & \mathscr{N}^s_+ \arrow[r] & \mathscr{M}^s_+ \arrow[r] & \mathscr{E}^s_+ \arrow[r] \arrow[l, dotted, bend right, "\sigma"'] & 0
\end{tikzcd}
\tag{31}
$$

A priori ρ_* depends on the choice of linear splitting σ, but in a way which will not affect the cohomology class of the bivariant Chern character. This construction may be depicted in terms of a p-summable quasihomomorphism $\rho_* : T\mathscr{A} \to \mathscr{M}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{R}$, compatible with the adic filtration by the ideals in the sense that $J\mathscr{A}$ is mapped to $\mathscr{N}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{J}$. Hence, ρ_* extends to a quasihomomorphism of pro-algebras

$$\rho_* : \widehat{T}\mathscr{A} \to \widehat{\mathscr{M}^s} \triangleright \mathscr{I}^s\hat{\otimes}\widehat{\mathscr{R}} \ , \tag{32}$$

where $\widehat{T}\mathscr{A}$, $\widehat{\mathscr{M}^s}$ and $\widehat{\mathscr{R}}$ are the adic completions of $T\mathscr{A}$, $\mathscr{M}^s$ and $\mathscr{R}$ with respect to the ideals $J\mathscr{A}$, $\mathscr{N}^s$ and $\mathscr{J}$. Next, depending on the degree of the quasihomomorphism, the even supertrace $\mathrm{tr}_s : M_2(\mathbb{C}) \to \mathbb{C}$ or the odd supertrace $\mathrm{tr}_s : M_2(C_1) \to \mathbb{C}$ yields a chain map $X(\mathscr{M}^s) \to X(\mathscr{M})$ by setting $\alpha x \mapsto \mathrm{tr}_s(\alpha)x$ and $\natural\, \alpha x\mathbf{d}(\beta y) \mapsto \pm\mathrm{tr}_s(\alpha\beta)\natural x\mathbf{d}y$ for any $x, y \in \mathscr{M}$ and $\alpha, \beta \in M_2(\mathbb{C})$ or $M_2(C_1)$. The sign $\pm$ is the parity of the matrix β, which has to move across the differential $\mathbf{d}$. Hence composing with the chain map Tr : $F^{2n+1}_{\mathscr{I}\hat{\otimes}\mathscr{R}}X(\mathscr{M}) \to X(\mathscr{R})$ guaranteed by the admissibility condition, we obtain for any integer $n \geq \max(1, p-1)$ a *supertrace* chain map

$$\tau : F^{2n+1}_{\mathscr{I}^s\hat{\otimes}\mathscr{R}}X(\mathscr{M}^s) \xrightarrow{\mathrm{tr}_s} F^{2n+1}_{\mathscr{I}\hat{\otimes}\mathscr{R}}X(\mathscr{M}) \xrightarrow{\mathrm{Tr}} X(\mathscr{R}) \ , \tag{33}$$

of order zero with respect to the $\mathscr{N}^s$-adic filtration on $X(\mathscr{M}^s)$ and the $\mathscr{J}$-adic filtration on $X(\mathscr{R})$. The parity of τ corresponds to the parity of the quasihomomorphism. This allows to construct a chain map $\widehat{\chi}^n : \widehat{\Omega}\mathscr{M}^s_+ \to X(\mathscr{R})$ from the $(b+B)$-complex of the algebra $\mathscr{M}^s_+$, in any degree $n \geq p$ having the same parity as the supertrace τ. Observe that the linear map $x \in \mathscr{M}^s_+ \mapsto [F, x] \in \mathscr{I}^s_-\hat{\otimes}\mathscr{R}$ is continuous by construction.

Proposition 3.6 *Let $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B}$ be a p-summable quasihomomorphism of parity p* mod 2*, with $\mathscr{R}$-admissible algebra $\mathscr{E}$. Given any integer $n \geq p$ of the same parity, consider two linear maps $\widehat{\chi}^n_0 : \Omega^n\mathscr{M}^s_+ \to \mathscr{R}$ and $\widehat{\chi}^n_1 : \Omega^{n+1}\mathscr{M}^s_+ \to \Omega^1\mathscr{R}_\natural$ defined by*

$$\widehat{\chi}^n_0(x_0\mathbf{d}x_1\ldots\mathbf{d}x_n) = (-)^n\frac{\Gamma(1+\frac{n}{2})}{(n+1)!}\sum_{\lambda\in S_{n+1}}\varepsilon(\lambda)\,\tau(x_{\lambda(0)}[F, x_{\lambda(1)}]\ldots[F, x_{\lambda(n)}]) \tag{34}$$

$$\widehat{\chi}_1^n(x_0\mathbf{d}x_1\ldots\mathbf{d}x_{n+1}) = (-)^n\frac{\Gamma(1+\frac{n}{2})}{(n+1)!}\sum_{i=1}^{n+1}\tau\natural(x_0[F,x_1]\ldots\mathbf{d}x_i\ldots[F,x_{n+1}])$$

where S_{n+1} is the cyclic permutation group of $n+1$ elements and ε is the signature. Then $\widehat{\chi}_0^n$ and $\widehat{\chi}_1^n$ define together a chain map $\widehat{\chi}^n : \widehat{\Omega}\mathscr{M}_+^s \to X(\mathscr{R})$ of parity $n \bmod 2$, i.e. fulfill the relations

$$\widehat{\chi}_0^n B = 0\ , \quad \natural\mathbf{d}\widehat{\chi}_0^n - (-)^n\widehat{\chi}_1^n B = 0\ , \quad \overline{b}\widehat{\chi}_1^n - (-)^n\widehat{\chi}_0^n b = 0\ , \quad \widehat{\chi}_1^n b = 0\ . \quad (35)$$

Moreover $\widehat{\chi}^n$ is invariant under the Karoubi operator κ acting on $\Omega^n\mathscr{M}_+^s$ and $\Omega^{n+1}\mathscr{M}_+^s$.

Proof: This follows from purely algebraic manipulations, using the following general properties:
- The graded commutator $[F,\ ]$ is a differential and $\tau([F,\ ]) = 0$;
- $\mathbf{d}F = 0$ so that $[F,\ ]$ and $\mathbf{d}$ are anticommuting differentials;
- $\tau\natural$ is a supertrace.
The computation is lengthy but straightforward. ∎

Thus we have attached to a p-summable quasihomomorphism $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B}$ of parity $p \bmod 2$ a sequence of cocycles $\widehat{\chi}^n$ ($n \geq p$) of the same parity in the $\mathbb{Z}_2$-graded complex $\mathrm{Hom}(\widehat{\Omega}\mathscr{M}_+^s, X(\mathscr{R}))$. They are in fact all cohomologous, and the proposition below gives an explicit transgression formula in terms of the *eta-cochain*:

PROPOSITION 3.7 *Let $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B}$ be a p-summable quasihomomorphism of parity $p \bmod 2$, with $\mathscr{R}$-admissible algebra $\mathscr{E}$. Given any integer $n \geq p+1$ of parity* opposite *to p, consider two linar maps $\widehat{\eta}_0^n : \Omega^n\mathscr{M}_+^s \to \mathscr{R}$ and $\widehat{\eta}_1^n : \Omega^{n+1}\mathscr{M}_+^s \to \Omega^1\mathscr{R}_\natural$ defined by*

$$\begin{aligned}\widehat{\eta}_0^n(x_0\mathbf{d}x_1\ldots\mathbf{d}x_n) &= \frac{\Gamma(\frac{n+1}{2})}{(n+1)!}\frac{1}{2}\tau\Big(Fx_0[F,x_1]\ldots[F,x_n] + \\ &\quad \sum_{i=1}^{n}(-)^{ni}[F,x_i]\ldots[F,x_n]Fx_0[F,x_1]\ldots[F,x_{i-1}]\Big)\end{aligned}$$

$$\begin{aligned}&\widehat{\eta}_1^n(x_0\mathbf{d}x_1\ldots\mathbf{d}x_{n+1}) = \qquad (36)\\ &\quad \frac{\Gamma(\frac{n+1}{2})}{(n+2)!}\sum_{i=1}^{n+1}\frac{1}{2}\tau\natural(ix_0F + (n+2-i)Fx_0)[F,x_1]\ldots\mathbf{d}x_i\ldots[F,x_{n+1}]\ .\end{aligned}$$

Then $\widehat{\eta}_0^n$ and $\widehat{\eta}_1^n$ define together a cochain $\widehat{\eta}^n \in \mathrm{Hom}(\widehat{\Omega}\mathscr{M}_+^s, X(\mathscr{R}))$ of parity $n \bmod 2$, whose coboundary equals the difference of cocycles

$$\widehat{\chi}^{n-1} - \widehat{\chi}^{n+1} = (\natural\mathbf{d}\oplus\overline{b})\widehat{\eta}^n - (-)^n\widehat{\eta}^n(b+B)\ .$$

Expressed in terms of components this amounts to the identities

$$\begin{array}{rclr} \widehat{\chi}_0^{n-1} & = & -(-)^n\widehat{\eta}_0^n B\ , & \overline{b}\widehat{\eta}_1^n-(-)^n(\widehat{\eta}_0^n b+\widehat{\eta}_0^{n+2}B)=0\ , \\ \widehat{\chi}_1^{n-1} & = & \natural\mathbf{d}\widehat{\eta}_0^n-(-)^n\widehat{\eta}_1^n B\ , & \natural\mathbf{d}\widehat{\eta}_0^{n+2}-(-)^n(\widehat{\eta}_1^n b+\widehat{\eta}_1^{n+2}B)=0\ . \end{array} \tag{37}$$

Proof: Direct computation. ∎

Remark 3.8 Using a trick of Connes [4], we may replace the chain map τ by $\tau'=\frac{1}{2}\tau(F[F,\ ])$. This allows to improve the summability condition by requiring the quasihomomorphism to be only $(p+1)$-summable instead of p-summable, while the condition on the degree remains $n\geq p$ for $\widehat{\chi}^n$ and $n\geq p+1$ for $\widehat{\eta}^n$. It is traightforward to write down the new formulas for $\widehat{\chi}^n$ and observe that it involves exactly $n+1$ commutators $[F,x]$. These formulas were actually obtained in [23] in a more general setting where we allow $\mathbf{d}F\neq 0$.

Definition 3.9 *The bivariant Chern character of the quasihomomorphism $\rho:\mathscr{A}\to\mathscr{E}^s\triangleright\mathscr{I}^s\hat{\otimes}\mathscr{B}$ is represented in any degree $n\geq p$ by the composition of chain maps*

$$\mathrm{ch}^n(\rho):X(T\mathscr{A})\xrightarrow{\gamma}\widehat{\Omega}T\mathscr{A}\xrightarrow{\rho_*}\widehat{\Omega}\mathscr{M}_+^s\xrightarrow{\widehat{\chi}^n}X(\mathscr{R})\ , \tag{38}$$

where $\gamma:X(T\mathscr{A})\to\widehat{\Omega}T\mathscr{A}$ is the Goodwillie equivalence constructed in section 2 for any quasi-free algebra and $\rho_*:T\mathscr{A}\to\mathscr{M}_+^s$ is the classifying homomorphism. In the same way, define a transgressed cochain in $\mathrm{Hom}(X(T\mathscr{A}),X(\mathscr{R}))$ *by means of the eta-cochain in any degree:*

$$\not{c}\mathrm{h}^n(\rho):X(T\mathscr{A})\xrightarrow{\gamma}\widehat{\Omega}T\mathscr{A}\xrightarrow{\rho_*}\widehat{\Omega}\mathscr{M}_+^s\xrightarrow{\widehat{\eta}^n}X(\mathscr{R})\ . \tag{39}$$

It fulfills the transgression property $\mathrm{ch}^n(\rho)-\mathrm{ch}^{n+2}(\rho)=[\partial,\not{c}\mathrm{h}^{n+1}(\rho)]$ *where ∂ is the X-complex boundary map.*

Recall that $\gamma(x)=(1-\phi)^{-1}(x)$ and $\gamma(\natural x\mathbf{d}y)=(1-\phi)^{-1}(x\mathbf{d}y+b(x\phi(y)))$ for any $x,y\in T\mathscr{A}$, where the map $\phi:\Omega^nT\mathscr{A}\to\Omega^{n+2}T\mathscr{A}$ is uniquely defined from its restriction to zero-forms. Its existence is guaranteed by the fact that $T\mathscr{A}$ is a free algebra. Several choices are possible, but conventionally we always take $\phi:T\mathscr{A}\to\Omega^2T\mathscr{A}$ as the canonical map obtained by setting $\phi(a)=0$ on the generators $a\in\mathscr{A}\subset T\mathscr{A}$, and then extended to all $T\mathscr{A}$ by the algebraic property $\phi(xy)=\phi(x)y+x\phi(y)+\mathbf{d}x\mathbf{d}y$.
Of course $\mathrm{ch}^n(\rho)$ and $\not{c}\mathrm{h}^n(\rho)$ are not very interesting a priori, because the X-complex of the non-completed tensor algebra $T\mathscr{A}$ is contractible. However, taking into account the adic filtrations induced by the ideals $J\mathscr{A}\subset T\mathscr{A}$ and $\mathscr{J}\subset\mathscr{R}$ yields non-trivial bivariant objects. By virtue of Remark 3.8 we suppose from now on that $\mathscr{I}$ is $(p+1)$-summable.

Proposition 3.10 *Let $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s \hat{\otimes} \mathscr{B}$ be a $(p+1)$-summable quasihomomorphism of parity $p \bmod 2$ with $\mathscr{R}$-admissible extension $\mathscr{E}$, and let $n \geq p$ be an integer of the same parity. The composites $\widehat{\chi}^n \rho_* \gamma$ and $\widehat{\eta}^{n+1} \rho_* \gamma$ are linear maps $X(T\mathscr{A}) \to X(\mathscr{R})$ verifying the adic properties*

$$\begin{aligned} \widehat{\chi}^n \rho_* \gamma \quad &: \quad F^k_{J\mathscr{A}} X(T\mathscr{A}) \to F^{k-n}_{\mathscr{J}} X(\mathscr{R}) \ , \\ \widehat{\eta}^{n+1} \rho_* \gamma \quad &: \quad F^k_{J\mathscr{A}} X(T\mathscr{A}) \to F^{k-n-2}_{\mathscr{J}} X(\mathscr{R}) \ , \end{aligned}$$

for any $k \in \mathbb{Z}$. Consequently the composite $\mathrm{ch}^n(\rho) = \widehat{\chi}^n \rho_* \gamma$ defines a cocycle of parity $n \bmod 2$ in the complex $\mathrm{Hom}^n(\widehat{X}(T\mathscr{A}, J\mathscr{A}), \widehat{X}(\mathscr{R}, \mathscr{J}))$ and the Chern character is a bivariant cyclic cohomology class of degree n:

$$\mathrm{ch}^n(\rho) \in HC^n(\mathscr{A}, \mathscr{B}) \ , \quad \forall n \geq p \ . \tag{40}$$

Moreover, the transgression relation $\mathrm{ch}^n(\rho) - \mathrm{ch}^{n+2}(\rho) = [\partial, \not{c}\mathrm{h}^{n+1}(\rho)]$ holds in the complex $\mathrm{Hom}^{n+2}(\widehat{X}(T\mathscr{A}, J\mathscr{A}), \widehat{X}(\mathscr{R}, \mathscr{J}))$, which implies

$$\mathrm{ch}^{n+2}(\rho) \equiv S\mathrm{ch}^n(\rho) \ \text{in} \ HC^{n+2}(\mathscr{A}, \mathscr{B}) \ . \tag{41}$$

In particular the cocycles $\mathrm{ch}^n(\rho)$ for different n define the same periodic cyclic cohomology class $\mathrm{ch}(\rho) \in HP^n(\mathscr{A}, \mathscr{B})$.

Proof: Let us denote by $0 \to \mathscr{G} \to \mathscr{F} \to \mathscr{A} \to 0$ the universal extension $0 \to J\mathscr{A} \to T\mathscr{A} \to \mathscr{A} \to 0$. Recall that $\mathscr{M}^s$ and its ideal $\mathscr{N}^s$ are $\mathbb{Z}_2$-graded differential algebras on which the graded commutator $[F, \ ]$ acts as a differential of odd degree. Moreover the commutation relations $[F, \mathscr{M}^s_+] \subset \mathscr{I}^s_- \hat{\otimes} \mathscr{R}$ and $[F, \mathscr{N}^s_+] \subset \mathscr{I}^s_- \hat{\otimes} \mathscr{J}$ hold. Now, we have to investigate the adic behaviour of the Goodwillie equivalence $\gamma : X(\mathscr{F}) \to \widehat{\Omega}\mathscr{F}$ with respect to the filtration $F^k_{\mathscr{G}} X(\mathscr{F})$. The first step in that direction was actually done in the proof of Proposition 2.4, where the following filtration of the subspaces $\Omega^n \mathscr{F}$ was introduced:

$$H^k \Omega^n \mathscr{F} = \sum_{k_0 + \ldots + k_n \geq k} \mathscr{G}^{k_0} \mathbf{d} \mathscr{G}^{k_1} \ldots \mathbf{d} \mathscr{G}^{k_n} \ \subset \Omega^n \mathscr{F} \ .$$

Let us look at the image of the latter filtration under the maps $\widehat{\chi}^n \rho_*$ and $\widehat{\eta}^n \rho_* : \widehat{\Omega}\mathscr{F} \to X(\mathscr{R})$ given by Eqs. (34, 36). We know that the homomorphism $\rho_* : \mathscr{F} \to \mathscr{M}^s_+$ respects the ideals $\mathscr{G}$ and $\mathscr{N}^s_+$. Hence if $x_0, \ldots, x_n$ denote $n+1$ elements in $\mathscr{G}^{k_0}, \ldots, \mathscr{G}^{k_n}$ respectively, with $k_0 + \ldots + k_n \geq k$, then $x_{\lambda(0)}[F, x_{\lambda(1)}] \ldots [F, x_{\lambda(n)}] \in (\mathscr{N}^s)^k$ for any permutation $\lambda \in S_{n+1}$. Hence applying the supertrace τ, which is a chain map of order zero with respect to the $\mathscr{N}^s$-adic and $\mathscr{J}$-adic filtrations on $X(\mathscr{M}^s)$ and $X(\mathscr{R})$, yields (from now on we omit to write the homomorphism ρ_*)

$$\widehat{\chi}^n_0(H^k \Omega^n \mathscr{F}) \subset \mathscr{J}^k \ . \tag{42}$$

In the same way, for $n+1$ elements $x_0, \ldots, x_{n+1}$ in $\mathscr{G}^{k_0}, \ldots, \mathscr{G}^{k_{n+1}}$, the one-form $\natural x_0[F, x_1] \ldots \mathbf{d}x_i \ldots [F, x_{n+1}]$ involves $k_0 + \ldots + k_{n+1} \geq k$ powers of the ideal $\mathscr{N}^s$, hence lies in the subspace $\natural((\mathscr{N}^s)^k\mathbf{d}\mathscr{M}^s + (\mathscr{N}^s)^{k-1}\mathbf{d}\mathscr{N}^s)$. Thus applying the supertrace τ one gets

$$\widehat{\chi}_1^n(H^k\Omega^{n+1}\mathscr{F}) \subset \natural(\mathscr{J}^k\mathbf{d}\mathscr{R} + \mathscr{J}^{k-1}\mathbf{d}\mathscr{J}) \ . \tag{43}$$

Proceeding in exactly the same fashion with the maps $\widehat{\eta}_0^n : \Omega^n\mathscr{F} \to \mathscr{R}$ and $\widehat{\eta}_1^n : \Omega^{n+1}\mathscr{F} \to \Omega^1\mathscr{R}_\natural$, it is clear that

$$\begin{aligned} \widehat{\eta}_0^n(H^k\Omega^n\mathscr{F}) &\subset \mathscr{J}^k \ . & (44)\\ \widehat{\eta}_1^n(H^k\Omega^{n+1}\mathscr{F}) &\subset \natural(\mathscr{J}^k\mathbf{d}\mathscr{R} + \mathscr{J}^{k-1}\mathbf{d}\mathscr{J}) \ . & (45)\end{aligned}$$

However these estimates are not optimal concerning the component $\widehat{\chi}_1^n$. We need a refinement of the H-filtration. For any $k \in \mathbb{Z}$, $n \geq 0$, let us define the subspaces

$$G^k\Omega^n\mathscr{F} = \sum_{k_0+\ldots+k_n\geq k} \mathscr{G}^{k_0}(\mathbf{d}\mathscr{F})\mathscr{G}^{k_1}(\mathbf{d}\mathscr{F})\ldots\mathscr{G}^{k_{n-1}}(\mathbf{d}\mathscr{F})\mathscr{G}^{k_n} + H^{k+1}\Omega^n\mathscr{F} \ .$$

Then for fixed n, $G^*\Omega^n\mathscr{F}$ is a decreasing filtration of $\Omega^n\mathscr{F}$, and by convention $G^k\Omega^n\mathscr{F} = \Omega^n\mathscr{F}$ for $k \leq 0$. One has $G^k\Omega^n\mathscr{F} \subset H^k\Omega^n\mathscr{F}$. Now observe the following. Since $[F,\ ]$ and $\mathbf{d}$ are derivations, the map $x_0\mathbf{d}x_1\ldots\mathbf{d}x_i\ldots\mathbf{d}x_{n+1} \mapsto \natural x_0[F,x_1]\ldots\mathbf{d}x_i\ldots[F,x_{n+1}]$ has the property that

$$\begin{aligned}&\mathscr{G}^{k_0}(\mathbf{d}\mathscr{F})\mathscr{G}^{k_1}(\mathbf{d}\mathscr{F})\ldots(\mathbf{d}\mathscr{F})\mathscr{G}^{k_i}\ldots(\mathbf{d}\mathscr{F})\mathscr{G}^{k_{n+1}} \to\\ &\quad \natural(\mathscr{N}^s)^{k_0}[F,\mathscr{M}^s](\mathscr{N}^s)^{k_1}[F,\mathscr{M}^s]\ldots(\mathbf{d}\mathscr{M}^s)(\mathscr{N}^s)^{k_i}\ldots[F,\mathscr{M}^s](\mathscr{N}^s)^{k_{n+1}}\\ &\quad \subset \natural(\mathscr{N}^s)^k\mathbf{d}\mathscr{M}^s \ ,\end{aligned}$$

and because $\widehat{\chi}_1^n(H^{k+1}\Omega^{n+1}\mathscr{F}) \subset \natural(\mathscr{J}^{k+1}\mathbf{d}\mathscr{R} + \mathscr{J}^k\mathbf{d}\mathscr{J}) \subset \natural\mathscr{J}^k\mathbf{d}\mathscr{R}$, one gets the crucial estimate

$$\widehat{\chi}_1^n(G^k\Omega^{n+1}\mathscr{F}) \subset \natural\mathscr{J}^k\mathbf{d}\mathscr{R} \ . \tag{46}$$

Now we have to understand the way γ sends the X-complex filtration

$$\begin{aligned} F_{\mathscr{G}}^{2k}X(\mathscr{F}) &: \ \mathscr{G}^{k+1} + [\mathscr{G}^k,\mathscr{F}] \ \rightleftarrows \ \natural\mathscr{G}^k\mathbf{d}\mathscr{F}\\ F_{\mathscr{G}}^{2k+1}X(\mathscr{F}) &: \ \mathscr{G}^{k+1} \ \rightleftarrows \ \natural(\mathscr{G}^{k+1}\mathbf{d}\mathscr{F} + \mathscr{G}^k\mathbf{d}\mathscr{G}) \ ,\end{aligned}$$

to the filtration $G^*\Omega^n\mathscr{F}$, in all degrees n. Recall that $\gamma(x)|_{\Omega^{2n}\mathscr{F}} = \phi^n(x)$ and $\gamma(\natural x\mathbf{d}y)|_{\Omega^{2n+1}\mathscr{F}} = \phi^n(x\mathbf{d}y + b(x\phi(y)))$ for any $x,y \in \mathscr{F}$, where the map $\phi : \Omega^n\mathscr{F} \to \Omega^{n+2}\mathscr{F}$ is obtained from its restriction to zero-forms as

$$\phi(x_0\mathbf{d}x_1\ldots\mathbf{d}x_n) = \sum_{i=0}^n (-)^{ni}\phi(x_i)\mathbf{d}x_{i+1}\ldots\mathbf{d}x_n\mathbf{d}x_0\ldots\mathbf{d}x_{i-1} \ .$$

Note the following important properties of ϕ. Firstly, it is invariant under the Karoubi operator $\kappa : \Omega^n\mathscr{F} \to \Omega^n\mathscr{F}$ in the sense that $\phi\circ\kappa = \phi$, and vanishes on the image of the boudaries $d, B : \Omega^n\mathscr{F} \to \Omega^{n+1}\mathscr{F}$. Secondly, the relation $\phi b - b\phi = B$ holds on $\Omega^n\mathscr{F}$ whenever $n \geq 1$ (see [22] §4). Since we have to apply successive powers of ϕ on the filtration $G^k\Omega^n\mathscr{F}$, the computation will be greatly simplified by exploiting κ-invariance. Define the linear map $\widetilde{\phi}:\Omega^n\mathscr{F}\to\Omega^{n+2}\mathscr{F}$ by

$$\widetilde{\phi}(x_0\mathbf{d}x_1\ldots\mathbf{d}x_n)=\sum_{i=0}^{n}(-)^i\mathbf{d}x_0\ldots\mathbf{d}x_{i-1}\phi(x_i)\mathbf{d}x_{i+1}\ldots\mathbf{d}x_n\ .$$

Then $\widetilde{\phi}$ coincides with ϕ modulo the image of $1-\kappa$. In particular the relation $\phi^n=\phi\circ\widetilde{\phi}^{n-1}$ holds. The advantage of the map $\widetilde{\phi}$ stems from the fact that it does not involve cyclic permutations of the elements x_i, and verifies the following optimal compatibility with the G-filtration

$$\widetilde{\phi}(G^k\Omega^n\mathscr{F})\subset G^{k-1}\Omega^{n+2}\mathscr{F}\qquad\forall k,n\geq 0\ ,$$

whereas the map ϕ is only compatible with the (coarser) H-filtration:

$$\phi(H^k\Omega^n\mathscr{F})\subset H^{k-1}\Omega^{n+2}\mathscr{F}\qquad\forall k,n\geq 0\ .$$

We shall now evaluate the image of the filtration $F^k_{\mathscr{G}}X(\mathscr{F})$ under the map $\gamma:X(\mathscr{F})\to\widehat{\Omega}\mathscr{F}$. Firstly, one has $\gamma(\mathscr{G}^{k+1})\cap\Omega^{2n}\mathscr{F}=\phi^n(\mathscr{G}^{k+1})$. But $\mathscr{G}^{k+1}\subset G^{k+1}\Omega^0\mathscr{F}$ and $\phi^n=\phi\circ\widetilde{\phi}^{n-1}$, hence

$$\gamma(\mathscr{G}^{k+1})\cap\Omega^{2n}\mathscr{F}\subset\phi(G^{k-n+2}\Omega^{2n-2}\mathscr{F})\ . \tag{47}$$

Secondly, the image of $\natural\mathscr{G}^k\mathbf{d}\mathscr{F}$ in $\Omega^{2n+1}\mathscr{F}$ is given by $\phi^n(\mathscr{G}^k\mathbf{d}\mathscr{F}+b(\mathscr{G}^k\phi(\mathscr{F})))$. One has $b(\mathscr{G}^k\phi(\mathscr{F}))\subset[\mathscr{G}^k\mathbf{d}\mathscr{F},\mathscr{F}]\subset\mathscr{G}^k\mathbf{d}\mathscr{F}$, hence we only need to compute $\phi^n(\mathscr{G}^k\mathbf{d}\mathscr{F})\subset\phi^n(G^k\Omega^1\mathscr{F})$, and

$$\gamma(\natural\mathscr{G}^k\mathbf{d}\mathscr{F})\cap\Omega^{2n+1}\mathscr{F}\subset\phi(G^{k-n+1}\Omega^{2n-1}\mathscr{F})\ . \tag{48}$$

Thirdly, $[\mathscr{G}^k,\mathscr{F}]=\overline{b}\natural\mathscr{G}^k\mathbf{d}\mathscr{F}$ so that $\gamma([\mathscr{G}^k,\mathscr{F}])=(b+B)\gamma(\natural\mathscr{G}^k\mathbf{d}\mathscr{F})$ because γ is a chain map. Therefore, the image of $[\mathscr{G}^k,\mathscr{F}]$ restricted to $\Omega^{2n}\mathscr{F}$ is contained in $B\phi^{n-1}(\mathscr{G}^k\mathbf{d}\mathscr{F})+b\phi^n(\mathscr{G}^k\mathbf{d}\mathscr{F})$. We may estimate coarsly the first term as $B\phi^{n-1}(\mathscr{G}^k\mathbf{d}\mathscr{F})\subset B\phi^{n-1}(H^k\Omega^1\mathscr{F})\subset H^{k-n+1}\Omega^{2n}\mathscr{F}$, and the second term as $b\phi\widetilde{\phi}^{n-1}(\mathscr{G}^k\mathbf{d}\mathscr{F})\subset b\phi(G^{k-n+1}\Omega^{2n-1}\mathscr{F})$. Hence

$$\gamma([\mathscr{G}^k,\mathscr{F}])\cap\Omega^{2n}\mathscr{F}\subset H^{k-n+1}\Omega^{2n}\mathscr{F}+b\phi(G^{k-n+1}\Omega^{2n-1}\mathscr{F})\ . \tag{49}$$

Fourthly, the image of $\natural\mathscr{G}^k\mathbf{d}\mathscr{G}$ in $\Omega^{2n+1}\mathscr{F}$ is given by $\phi^n(\mathscr{G}^k\mathbf{d}\mathscr{G}+b(\mathscr{G}^k\phi(\mathscr{G})))$. We estimate coarsly $\phi^n(\mathscr{G}^k\mathbf{d}\mathscr{G})\subset\phi^n(H^{k+1}\Omega^1\mathscr{F})\subset H^{k-n+1}\Omega^{2n+1}\mathscr{F}$. Then, one has $\mathscr{G}^k\phi(\mathscr{G})\subset\mathscr{G}^k\mathbf{d}\mathscr{F}\mathbf{d}\mathscr{F}\subset G^k\Omega^2\mathscr{F}$, and using repeatedly the relations $\phi b-b\phi=B$, $\phi B=0$ gives $\phi^n b(\mathscr{G}^k\phi(\mathscr{G}))\subset b\phi^n(G^k\Omega^2\mathscr{F})+B\phi^{n-1}(G^k\Omega^2\mathscr{F})\subset b\phi(G^{k-n+1}\Omega^{2n}\mathscr{F})+BH^{k-n+1}\Omega^{2n}\mathscr{F}$. Thus

$$\gamma(\natural\mathscr{G}^k\mathbf{d}\mathscr{G})\cap\Omega^{2n+1}\mathscr{F}\subset H^{k-n+1}\Omega^{2n+1}\mathscr{F}+b\phi(G^{k-n+1}\Omega^{2n}\mathscr{F})\ . \tag{50}$$

Now everything is set to evaluate the adic behaviour of the composites $\widehat{\chi}^n\gamma$ and $\widehat{\eta}^n\gamma$. We shall deal only with even degrees, the odd case is similar.

Hence let us start with the map $\widehat{\chi}_0^{2n}\gamma : \mathscr{F} \to \mathscr{R}$. For any $k \in \mathbb{Z}$, Eq. (47) gives $\widehat{\chi}_0^{2n}\gamma(\mathscr{G}^{k+1}) \subset \widehat{\chi}_0^{2n} \circ \phi(G^{k-n+2}\Omega^{2n-2}\mathscr{F})$. But $\widehat{\chi}_0^{2n}$ is κ-invariant, hence $\widehat{\chi}_0^{2n} \circ \phi = \widehat{\chi}_0^{2n} \circ \widetilde{\phi}$. Therefore, $\widehat{\chi}_0^{2n}\gamma(\mathscr{G}^{k+1}) \subset \widehat{\chi}_0^{2n}(G^{k-n+1}\Omega^{2n}\mathscr{F}) \subset \mathscr{J}^{k-n+1}$ using $G^{k-n+1}\Omega \subset H^{k-n+1}\Omega$ and (42). Now we look at its companion $\widehat{\chi}_1^{2n}\gamma : \Omega^1\mathscr{F}_\natural \to \Omega^1\mathscr{R}_\natural$. From (48) one gets $\widehat{\chi}_1^{2n}(\natural\mathscr{G}^k\mathbf{d}\mathscr{F}) \subset \widehat{\chi}_1^{2n} \circ \phi(G^{k-n+1}\Omega^{2n-1}\mathscr{F})$. But $\widehat{\chi}_1^{2n}$ is also κ-invariant and $\widehat{\chi}_1^{2n} \circ \phi = \widehat{\chi}_1^{2n} \circ \widetilde{\phi}$, thus $\widehat{\chi}_1^{2n}\gamma(\natural\mathscr{G}^k\mathbf{d}\mathscr{F}) \subset \widehat{\chi}_1^{2n}(G^{k-n}\Omega^{2n+1}\mathscr{F}) \subset \natural\mathscr{J}^{k-n}\mathbf{d}\mathscr{R}$ by (46). This allows to estimate the image of $[\mathscr{G}^k,\mathscr{F}] = \overline{b}\natural\mathscr{G}^k\mathbf{d}\mathscr{F}$ under the chain map $\widehat{\chi}^{2n}\gamma$. Indeed $\widehat{\chi}_0^{2n}\gamma(\overline{b}\natural\mathscr{G}^k\mathbf{d}\mathscr{F}) = \overline{b}\widehat{\chi}_1^{2n}\gamma(\natural\mathscr{G}^k\mathbf{d}\mathscr{F}) \subset \overline{b}\natural\mathscr{J}^{k-n}\mathbf{d}\mathscr{R}$, so that $\widehat{\chi}_0^{2n}\gamma([\mathscr{G}^k,\mathscr{F}]) \subset [\mathscr{J}^{k-n},\mathscr{R}]$. Collecting these results shows the effect of the map $\widehat{\chi}^{2n}\gamma$ on the adic filtration in even degree:

$$\left\{\begin{array}{llll} \widehat{\chi}_0^{2n}\gamma : & \mathscr{G}^{k+1} + [\mathscr{G}^k,\mathscr{F}] & \longrightarrow & \mathscr{J}^{k-n+1} + [\mathscr{J}^{k-n},\mathscr{R}] \\ \widehat{\chi}_1^{2n}\gamma : & \natural\mathscr{G}^k\mathbf{d}\mathscr{F} & \longrightarrow & \natural\mathscr{J}^{k-n}\mathbf{d}\mathscr{R} \end{array}\right.$$

hence $\widehat{\chi}^{2n}\gamma : F_{\mathscr{G}}^{2k}X(\mathscr{F}) \to F_{\mathscr{J}}^{2k-2n}X(\mathscr{R})$. To understand the effect on the filtration in odd degree, one has to evaluate $\widehat{\chi}_1^{2n}\gamma$ on $\natural\mathscr{G}^k\mathbf{d}\mathscr{G}$. From (50), one gets $\widehat{\chi}_1^{2n}\gamma(\natural\mathscr{G}^k\mathbf{d}\mathscr{G}) \subset \widehat{\chi}_1^{2n}(H^{k-n+1}\Omega^{2n+1}\mathscr{F} + b\phi(G^{k-n+1}\Omega^{2n}\mathscr{F}))$. But (35) shows $\widehat{\chi}_1^{2n} \circ b = 0$, and (43) implies $\widehat{\chi}_1^{2n}\gamma(\natural\mathscr{G}^k\mathbf{d}\mathscr{G}) \subset \natural(\mathscr{J}^{k-n+1}\mathbf{d}\mathscr{R} + \mathscr{J}^{k-n}\mathbf{d}\mathscr{J})$. One thus gets the adic behaviour of the chain map $\widehat{\chi}^{2n}\gamma$ on the filtration of odd degree:

$$\left\{\begin{array}{llll} \widehat{\chi}_0^{2n}\gamma : & \mathscr{G}^{k+1} & \longrightarrow & \mathscr{J}^{k-n+1} \\ \widehat{\chi}_1^{2n}\gamma : & \natural(\mathscr{G}^{k+1}\mathbf{d}\mathscr{F} + \mathscr{G}^k\mathbf{d}\mathscr{G}) & \longrightarrow & \natural(\mathscr{J}^{k-n+1}\mathbf{d}\mathscr{R} + \mathscr{J}^{k-n}\mathbf{d}\mathscr{J}) \end{array}\right.$$

hence $\widehat{\chi}^{2n}\gamma : F_{\mathscr{G}}^{2k+1}X(\mathscr{F}) \to F_{\mathscr{J}}^{2k-2n+1}X(\mathscr{R})$ and $\widehat{\chi}^{2n}\gamma$ is a map of order $2n$. Using similar methods, one shows that $\widehat{\chi}^{2n+1}\gamma$ is of order $2n+1$.
Now we investigate the eta-cochain. Consider $\widehat{\eta}_0^{2n}\gamma : \mathscr{F} \to \mathscr{R}$. (47) gives $\widehat{\eta}_0^{2n}\gamma(\mathscr{G}^{k+1}) \subset \widehat{\eta}_0^{2n}\phi(G^{k-n+2}\Omega^{2n-2}\mathscr{F})$. However $\widehat{\eta}^{2n}$ is not κ-invariant, so that we cannot replace ϕ by $\widetilde{\phi}$. We are forced to consider $\phi(G^{k-n+2}\Omega^{2n-2}\mathscr{F}) \subset H^{k-n+1}\Omega^{2n}\mathscr{F}$ and consequently $\widehat{\eta}_0^{2n}\gamma(\mathscr{G}^{k+1}) \subset \mathscr{J}^{k-n+1}$ by (44). Similarly, (49) implies $\widehat{\eta}_0^{2n}\gamma([\mathscr{G}^k,\mathscr{F}]) \subset \widehat{\eta}_0^{2n}(H^{k-n+1}\Omega^{2n}\mathscr{F}) + \widehat{\eta}_0^{2n}b\phi(G^{k-n+1}\Omega^{2n-1}\mathscr{F}) \subset \widehat{\eta}_0^{2n}(H^{k-n}\Omega^{2n}\mathscr{F})$ hence $\widehat{\eta}_0^{2n}\gamma([\mathscr{G}^k,\mathscr{F}]) \subset \mathscr{J}^{k-n}$. Its companion $\widehat{\eta}_1^{2n}\gamma : \Omega^1\mathscr{F}_\natural \to \Omega^1\mathscr{R}_\natural$ evaluated on $\natural\mathscr{G}^k\mathbf{d}\mathscr{F}$ uses equation (48) again with $\phi(G^{k-n+1}\Omega^{2n-1}\mathscr{F}) \subset H^{k-n}\Omega^{2n+1}\mathscr{F}$, so that $\widehat{\eta}_1^{2n}\gamma(\natural\mathscr{G}^k\mathbf{d}\mathscr{F}) \subset \widehat{\eta}_1^{2n}(H^{k-n}\Omega^{2n+1}\mathscr{F}) \subset \natural(\mathscr{J}^{k-n}\mathbf{d}\mathscr{R} + \mathscr{J}^{k-n-1}\mathbf{d}\mathscr{J})$ by (45). This shows the effect of $\widehat{\eta}^{2n}\gamma$ on the filtration of even degree

$$\left\{\begin{array}{llll} \widehat{\eta}_0^{2n}\gamma : & \mathscr{G}^{k+1} + [\mathscr{G}^k,\mathscr{F}] & \longrightarrow & \mathscr{J}^{k-n} \\ \widehat{\eta}_1^{2n}\gamma : & \natural\mathscr{G}^k\mathbf{d}\mathscr{F} & \longrightarrow & \natural(\mathscr{J}^{k-n}\mathbf{d}\mathscr{R} + \mathscr{J}^{k-n-1}\mathbf{d}\mathscr{J}) \end{array}\right.$$

hence $\widehat{\eta}^{2n}\gamma : F_{\mathscr{G}}^{2k}X(\mathscr{F}) \to F_{\mathscr{J}}^{2k-2n-1}X(\mathscr{R})$. For the odd filtration, let us compute from (50) $\widehat{\eta}_1^{2n}\gamma(\natural\mathscr{G}^k\mathbf{d}\mathscr{G}) \subset \widehat{\eta}_1^{2n}(H^{k-n+1}\Omega^{2n+1}\mathscr{F}) + \widehat{\eta}_1^{2n}b\phi(G^{k-n+1}\Omega^{2n}\mathscr{F})$. But the identities (37) show that $\widehat{\eta}_1^{2n}b = \widehat{\chi}_1^{2n+1}$, hence using κ-invariance one gets $\widehat{\eta}_1^{2n}b\phi(G^{k-n+1}\Omega^{2n}\mathscr{F}) \subset \widehat{\chi}_1^{2n+1}\widetilde{\phi}(G^{k-n+1}\Omega^{2n}\mathscr{F}) \subset \widehat{\chi}_1^{2n+1}(G^{k-n}\Omega^{2n+2}\mathscr{F})$.

Therefore, (45) and (46) imply $\widehat{\eta}_1^{2n}\gamma(\natural\mathscr{G}^k\mathbf{d}\mathscr{G}) \subset \natural\mathscr{J}^{k-n}\mathbf{d}\mathscr{R}$. These results give the adic behaviour of $\widehat{\eta}^{2n}\gamma$ with respect to the odd filtration

$$\left\{ \begin{array}{lll} \widehat{\eta}_0^{2n}\gamma: & \mathscr{G}^{k+1} & \longrightarrow \quad \mathscr{J}^{k-n+1} \\ \widehat{\eta}_1^{2n}\gamma: & \natural(\mathscr{G}^{k+1}\mathbf{d}\mathscr{F} + \mathscr{G}^k\mathbf{d}\mathscr{G}) & \longrightarrow \quad \natural\mathscr{J}^{k-n}\mathbf{d}\mathscr{R} \end{array} \right.$$

hence $\widehat{\eta}^{2n}\gamma : F_{\mathscr{G}}^{2k+1}X(\mathscr{F}) \to F_{\mathscr{J}}^{2k-2n}X(\mathscr{R})$ and $\widehat{\eta}^{2n}\gamma$ is a map of order $2n+1$. Similarly, one shows that $\widehat{\eta}^{2n+1}\gamma$ is of order $2n+2$. ∎

Note that the chain maps γ and $\widehat{\chi}^n$ extend to the adic completions of all the algebras involved, so that from now on we will consider the bivariant Chern character $\mathrm{ch}^n(\rho) \in \mathrm{Hom}^n(\widehat{X}(T\mathscr{A}, J\mathscr{A}), \widehat{X}(\mathscr{R}, \mathscr{J}))$ as a chain map of pro-complexes

$$\mathrm{ch}^n(\rho) : X(\widehat{T}\mathscr{A}) \xrightarrow{\gamma} \widehat{\Omega}\widehat{T}\mathscr{A} \xrightarrow{\rho_*} \widehat{\Omega}\widehat{\mathscr{M}}_+^s \xrightarrow{\widehat{\chi}^n} X(\widehat{\mathscr{R}}) \ . \tag{51}$$

We would like to introduce some equivalence relations among quasihomomorphisms, and discuss the corresponding invariance properties of the Chern character. The first equivalence relation is (smooth) homotopy. It involves the algebra $C^\infty[0,1]$ of smooth functions $f : [0,1] \to \mathbb{C}$, such that all the derivatives of order ≥ 1 vanish at the enpoints 0 and 1, while the values of f itself remain arbitrary. We have already seen that $C^\infty[0,1]$ endowed with its usual Fréchet topology is an m-algebra. It is moreover nuclear [14], so that its projective tensor product $\mathscr{A}\hat{\otimes}C^\infty[0,1]$ with any m-algebra $\mathscr{A}$ is isomorphic to the algebra of smooth $\mathscr{A}$-valued functions over $[0,1]$, with all derivatives of order ≥ 1 vanishing at the endpoints. We will usually denote by $\mathscr{A}[0,1]$ this m-algebra. The second equivalence relation of interest among quasihomomorphisms is conjugation by an invertible element of the unitalized algebra $(\mathscr{E}_+^s)^+$.

Definition 3.11 *Let $\rho_0 : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B}$ and $\rho_1 : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B}$ be two quasihomomorphisms with same parity. They are called*

i) homotopic *if there exists a quasihomomorphism $\rho : \mathscr{A} \to \mathscr{E}[0,1]^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B}[0,1]$ such that evaluation at the endpoints gives ρ_0 and ρ_1;*

ii) conjugate *if there exists an invertible element in the unitalized algebra $U \in (\mathscr{E}_+^s)^+$ with $U - 1 \in \mathscr{E}_+^s$, such that $\rho_1 = U^{-1}\rho_0 U$ as a homomorphism $\mathscr{A} \to \mathscr{E}_+^s$.*

Remark that the commutators $[F,U]$ and $[F,U^{-1}]$ always lie in the ideal $\mathscr{I}^s\hat{\otimes}\mathscr{B} \subset \mathscr{E}^s$. When the algebra $\mathscr{I}$ is M_2-stable (i.e. $M_2(\mathscr{I}) \cong \mathscr{I}$), two conjugate quasihomomorphisms are also homotopic, but the converse is not true. Hence conjugation is strictly stronger than homotopy as an equivalence relation. The former is an analogue of "compact perturbation" of quasihomomorphisms in Kasparov's bivariant K-theory for C^*-algebras, see [2].
The proposition below describes the compatibility between these equivalence relations and the bivariant Chern character.

Proposition 3.12 *Let $\rho_0 : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s \hat{\otimes} \mathscr{B}$ and $\rho_1 : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s \hat{\otimes} \mathscr{B}$ be two $(p+1)$-summable quasihomomorphisms of parity $p \bmod 2$, with $\mathscr{E}$ admissible with respect to a quasi-free extension $\mathscr{R}$ of $\mathscr{B}$. Let $n \geq p$ be any integer of the same parity.*

i) If ρ_0 and ρ_1 are homotopic, then $S\mathrm{ch}^n(\rho_0) \equiv S\mathrm{ch}^n(\rho_1)$ in $HC^{n+2}(\mathscr{A}, \mathscr{B})$. In particular $\mathrm{ch}^n(\rho_0) \equiv \mathrm{ch}^n(\rho_1)$ whenever $n \geq p+2$.

ii) If ρ_0 and ρ_1 are conjugate, then $\mathrm{ch}^n(\rho_0) \equiv \mathrm{ch}^n(\rho_1)$ in $HC^n(\mathscr{A}, \mathscr{B})$ for all $n \geq p$.

Proof: First observe that if $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s \hat{\otimes} \mathscr{B}$ is a quasihomomorphism with $\mathscr{R}$-admissible algebra $\mathscr{E}$, the lifting homomorphism $\rho_* : T\mathscr{A} \to \mathscr{M}^s_+$ factors through the tensor algebra $T\mathscr{E}^s_+$ by virtue of the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & J\mathscr{A} \arrow[r] \arrow[d, "\varphi"'] & T\mathscr{A} \arrow[r] \arrow[d, "\varphi"'] & \mathscr{A} \arrow[r] \arrow[d, "\rho"'] & 0 \\
0 \arrow[r] & J\mathscr{E}^s_+ \arrow[r] \arrow[d] & T\mathscr{E}^s_+ \arrow[r] \arrow[d] & \mathscr{E}^s_+ \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{N}^s_+ \arrow[r] & \mathscr{M}^s_+ \arrow[r] & \mathscr{E}^s_+ \arrow[r] \arrow[l, dotted, bend right, "\sigma"'] & 0
\end{tikzcd}$$

where the homomorphism $\varphi : T\mathscr{A} \to T\mathscr{E}^s_+$ is $\varphi(a_1 \otimes \ldots \otimes a_n) = \rho(a_1) \otimes \ldots \otimes \rho(a_n)$, and the arrow $T\mathscr{E}^s_+ \to \mathscr{M}^s_+$ maps a tensor product $e_1 \otimes \ldots \otimes e_n \in T\mathscr{E}^s_+$ to the product $\sigma(e_1) \ldots \sigma(e_n)$. By the naturality of the Goodwillie equivalences $\gamma_{\mathscr{A}} : X(T\mathscr{A}) \to \widehat{\Omega} T\mathscr{A}$ and $\gamma_{\mathscr{E}^s_+} : X(T\mathscr{E}^s_+) \to \widehat{\Omega} T\mathscr{E}^s_+$, one immediately sees that the bivariant Chern character coincides with the composition of chain maps

$$\mathrm{ch}^n(\rho) : X(T\mathscr{A}) \xrightarrow{X(\varphi)} X(T\mathscr{E}^s_+) \xrightarrow{\gamma_{\mathscr{E}^s_+}} \widehat{\Omega} T\mathscr{E}^s_+ \xrightarrow{\widehat{\chi}^n} X(\mathscr{R}) \ .$$

Hence, all the information about the homomorphism $\rho : \mathscr{A} \to \mathscr{E}^s_+$ is concentrated in the chain map $X(\varphi) : X(T\mathscr{A}) \to X(T\mathscr{E}^s_+)$. This will simplify the comparison of Chern characters associated to homotopic or conjugate quasihomomorphisms.

i) Homotopy: the cocycles $\mathrm{ch}^n(\rho_0)$ and $\mathrm{ch}^n(\rho_1)$ differ only by the chain maps $X(\varphi_i) : X(T\mathscr{A}) \to X(T\mathscr{E}^s_+)$, $i = 0, 1$. We view $\rho : \mathscr{A} \to \mathscr{E}^s_+[0,1]$ as a smooth family of homomorphisms $\rho_t : \mathscr{A} \to \mathscr{E}^s_+$ parametrized by $t \in [0,1]$, giving a homotopy between the two endpoints ρ_0 and ρ_1. Cuntz and Quillen prove in [12] a Cartan homotopy formula which provides a transgression between the chain maps $X(\varphi_i)$. At any point $t \in [0,1]$, denote by $\dot{\varphi} = \frac{d}{dt}\varphi : T\mathscr{A} \to T\mathscr{E}^s_+$ the derivative of the homomorphism φ_t with respect to t, and define a linear map $\iota : \Omega^m T\mathscr{A} \to \Omega^{m-1} T\mathscr{E}^s_+$ by

$$\iota(x_0 \mathbf{d}x_1 \ldots \mathbf{d}x_m) = (\varphi x_0)(\dot{\varphi} x_1)\mathbf{d}(\varphi x_2) \ldots \mathbf{d}(\varphi x_m) \ .$$

The tensor algebra $T\mathscr{A}$ is quasi-free, hence consider any $\phi : T\mathscr{A} \to \Omega^2 T\mathscr{A}$ verifying $\phi(xy) = \phi(x)y + x\phi(y) + \mathbf{d}x\mathbf{d}y$, and let $h : X(T\mathscr{A}) \to X(T\mathscr{E}^s_+)$ be the linear map of odd degree

$$h(x) = \natural\iota\phi(x) \ , \qquad h(\natural x\mathbf{d}y) = \iota(x\mathbf{d}y + b(x\phi(y)))$$

(the latter is well-defined on $\natural x\mathbf{d}y$). Then Cuntz and Quillen show the following adic properties of h for any $k \in \mathbb{Z}$,

$$\begin{aligned} h(F^k_{J\mathscr{A}}X(T\mathscr{A})) &\subset F^{k-1}_{J\mathscr{E}^s_+}X(T\mathscr{E}^s_+) \quad \text{if} \quad \dot\varphi(J\mathscr{A}) \subset J\mathscr{E}^s_+ \ , \\ h(F^k_{J\mathscr{A}}X(T\mathscr{A})) &\subset F^{k}_{J\mathscr{E}^s_+}X(T\mathscr{E}^s_+) \quad \text{if} \quad \dot\varphi(T\mathscr{A}) \subset J\mathscr{E}^s_+ \ , \end{aligned}$$

and moreover the transgression formula $\frac{d}{dt}X(\varphi) = [\partial, h]$ holds. Hence if we define by integration over $[0,1]$ the odd chain $H = \int_0^1 dt\, h$, one has

$$X(\varphi_1) - X(\varphi_0) = [\partial, H]$$

in the complex $\mathrm{Hom}^1(\widehat{X}(T\mathscr{A}, J\mathscr{A}), \widehat{X}(T\mathscr{E}^s_+, J\mathscr{E}^s_+))$ in case $\dot\varphi(J\mathscr{A}) \subset J\mathscr{E}^s_+$, or in the complex $\mathrm{Hom}^0(\widehat{X}(T\mathscr{A}, J\mathscr{A}), \widehat{X}(T\mathscr{E}^s_+, J\mathscr{E}^s_+))$ in case $\dot\varphi(T\mathscr{A}) \subset J\mathscr{E}^s_+$. For a general homotopy we are in the first case $\dot\varphi(J\mathscr{A}) \subset J\mathscr{E}^s_+$. After composition by the chain map $\widehat{\chi}^n\gamma_{\mathscr{E}^s_+} \in \mathrm{Hom}^n(\widehat{X}(T\mathscr{E}^s_+, J\mathscr{E}^s_+), \widehat{X}(\mathscr{R}, \mathscr{J}))$, this shows the transgression relation

$$\mathrm{ch}^n(\rho_1) - \mathrm{ch}^n(\rho_0) = (-)^n[\partial, \widehat{\chi}^n\gamma_{\mathscr{E}^s_+}H] \in \mathrm{Hom}^{n+1}(\widehat{X}(T\mathscr{A}, J\mathscr{A}), \widehat{X}(\mathscr{R}, \mathscr{J})) \ ,$$

whence $S\mathrm{ch}^n(\rho_1) \equiv S\mathrm{ch}^n(\rho_0)$ in $HC^{n+2}(\mathscr{A}, \mathscr{B})$. The sign $(-)^n$ comes from the parity of the chain map $\widehat{\chi}^n\gamma_{\mathscr{E}^s_+}$.

ii) Conjugation: now $\varphi_0, \varphi_1 : T\mathscr{A} \to T\mathscr{E}^s_+$ are the homomorphism lifts of ρ_0 and $\rho_1 = U^{-1}\rho_0 U$. Introduce the pro-algebra $\widehat{T}\mathscr{E}^s_+ = \varprojlim_k T\mathscr{E}^s_+/(J\mathscr{E}^s_+)^k \cong \prod_{k\geq 0}\Omega^{2k}\mathscr{E}^s_+$, and consider the invertible $U \in (\mathscr{E}^s_+)^+$ as an element $\widehat{U}$ of the unitalization $(\widehat{T}\mathscr{E}^s_+)^+$, via the linear inclusion of zero-forms $\mathscr{E}^s_+ \hookrightarrow \widehat{T}\mathscr{E}^s_+$. By proceeding as in [12], it turns out that $\widehat{U}$ is invertible, with inverse given by the series

$$\widehat{U}^{-1} = \sum_{k\geq 0} U^{-1}(dU\, dU^{-1})^k \ \in (\widehat{T}\mathscr{E}^s_+)^+ \ .$$

Of course the image of $\widehat{U}^{-1}$ under the multiplication map $(\widehat{T}\mathscr{E}^s_+)^+ \to (\mathscr{E}^s_+)^+$ is U^{-1}. We will show that φ_1, viewed as a homomorphism $T\mathscr{A} \to \widehat{T}\mathscr{E}^s_+$, is homotopic to the homomorphism $\widehat{U}^{-1}\varphi_0\widehat{U}$. For any $t \in [0,1]$ define a linear map $\sigma_t : \mathscr{A} \to \widehat{T}\mathscr{E}^s_+$ by

$$\sigma_t(a) = (1-t)\rho_1(a) + t\,\widehat{U}^{-1}\rho_0(a)\widehat{U} \ , \quad \forall a \in \mathscr{A} \ ,$$

where $\rho_0(a)$ and $\rho_1(a)$ are considered as elements of the subspace of zero-forms $\mathscr{E}^s_+ \hookrightarrow \widehat{T}\mathscr{E}^s_+$. Thus σ_t is a linear lifting of the constant homomorphism

$\rho_1 : \mathscr{A} \to \mathscr{E}^s_+$. Then use the universal property of the tensor algebra $T\mathscr{A}$ to build a smooth family of homomorphisms $\varphi(t) : T\mathscr{A} \to \widehat{T}\mathscr{E}^s_+$ by means of the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & J\mathscr{A} \arrow[r] \arrow[d, "\varphi(t)"'] & T\mathscr{A} \arrow[r] \arrow[d, "\varphi(t)"'] & \mathscr{A} \arrow[r] \arrow[d, "\rho_1"] \arrow[dl, dotted, "\sigma_t"'] & 0 \\ 0 \arrow[r] & \widehat{J}\mathscr{E}^s_+ \arrow[r] & \widehat{T}\mathscr{E}^s_+ \arrow[r] & \mathscr{E}^s_+ \arrow[r] & 0 \end{tikzcd}$$

By construction one has $\varphi(0) = \varphi_1$, $\varphi(1) = \widehat{U}^{-1}\varphi_0\widehat{U}$ and the derivative $\dot{\varphi}$ sends $T\mathscr{A}$ to the ideal $\widehat{J}\mathscr{E}^s_+$. Hence from the Cartan homotopy formula of part i) we deduce that the chain maps $X(\varphi_1)$ and $X(\widehat{U}^{-1}\varphi_0\widehat{U})$ are cohomologous in the complex $\mathrm{Hom}^0(\widehat{X}(T\mathscr{A}, J\mathscr{A}), \widehat{X}(T\mathscr{E}^s_+, J\mathscr{E}^s_+))$. Then we have to show that $X(\widehat{U}^{-1}\varphi_0\widehat{U})$ and $X(\varphi_0)$ are cohomologous. Consider the following linear map of odd degree $h : X(T\mathscr{A}) \to X(\widehat{T}\mathscr{E}^s_+) \cong \widehat{X}(T\mathscr{E}^s_+, J\mathscr{E}^s_+)$ defined by

$$h(x) = \natural(\widehat{U}^{-1}\varphi_0(x)\mathbf{d}\widehat{U}) \ , \qquad h(\natural x\mathbf{d}y) = 0 \ .$$

It is easy to see that h defines a cochain of order zero, i.e. lies in the complex $\mathrm{Hom}^0(\widehat{X}(T\mathscr{A}, J\mathscr{A}), \widehat{X}(T\mathscr{E}^s_+, J\mathscr{E}^s_+))$. Moreover, one has the transgression relation $X(\widehat{U}^{-1}\varphi_0\widehat{U}) - X(\varphi_0) = [\partial, h]$. Indeed (we replace $\varphi_0(x)$ by x for notational simplicity)

$$\begin{aligned} X(\widehat{U}^{-1}\varphi_0\widehat{U})(x) - X(\varphi_0)(x) &= \widehat{U}^{-1}x\widehat{U} - x \ = \ [\widehat{U}^{-1}x, \widehat{U}] \\ &= \overline{b}\natural(\widehat{U}^{-1}x\mathbf{d}\widehat{U}) \ = \ \overline{b}h(x) \ , \end{aligned}$$

$$\begin{aligned} X(\widehat{U}^{-1}\varphi_0\widehat{U})(\natural x\mathbf{d}y) - X(\varphi_0)(\natural x\mathbf{d}y) &= \natural\widehat{U}^{-1}x\widehat{U}\mathbf{d}(\widehat{U}^{-1}y\widehat{U}) - \natural x\mathbf{d}y \\ &= \natural(\widehat{U}^{-1}x\widehat{U}\mathbf{d}\widehat{U}^{-1}y\widehat{U} + \widehat{U}^{-1}x\mathbf{d}y\widehat{U} + \widehat{U}^{-1}xy\mathbf{d}\widehat{U} - x\mathbf{d}y) \\ &= \natural(-yx\mathbf{d}\widehat{U}\widehat{U}^{-1} + xy\mathbf{d}\widehat{U}\widehat{U}^{-1}) \ = \natural\widehat{U}^{-1}[x,y]\mathbf{d}\widehat{U} \\ &= h(\overline{b}\natural x\mathbf{d}y) \ , \end{aligned}$$

where in the second computation we use the identity $\mathbf{d}\widehat{U}^{-1} = -\widehat{U}^{-1}\mathbf{d}\widehat{U}\widehat{U}^{-1}$ deduced from $\mathbf{d}1 = 0$. This shows the equality of bivariant cyclic cohomology classes

$$X(\varphi_1) \equiv X(\varphi_0) \ \in HC^0(\mathscr{A}, \mathscr{E}^s_+) \ ,$$

so that after composition with $\widehat{\chi}^n\gamma_{\mathscr{E}^s_+} \in HC^n(\mathscr{E}^s_+, \mathscr{B})$, the equality $\mathrm{ch}^n(\rho_1) \equiv \mathrm{ch}^n(\rho_0)$ holds in $HC^n(\mathscr{A}, \mathscr{B})$. ∎

Part ii) of the above proof also shows the independence of the cohomology class $\mathrm{ch}^n(\rho) \in HC^n(\mathscr{A}, \mathscr{B})$ with respect to the choice of linear splitting $\sigma : \mathscr{E}^s_+ \to \mathscr{M}^s_+$ used to lift the homomorphism ρ, two such splittings being always homotopic. Then, from section 2 we know that any class in $HC^n(\mathscr{A}, \mathscr{B})$

induces linear maps of degree $-n$ between the cyclic homologies of $\mathscr{A}$ and $\mathscr{B}$, compatible with the SBI exact sequence. Hence, if the quasihomomorphism is $(p+1)$-summable and with parity p mod 2, the lowest degree representative of the Chern character $\mathrm{ch}^p(\rho) \in HC^p(\mathscr{A},\mathscr{B})$ carries the maximal information. We collect these results in a theorem:

Theorem 3.13 *Let $\rho : \mathscr{A} \to \mathscr{E}^s_+ \triangleright \mathscr{I}^s_+ \hat{\otimes} \mathscr{B}$ be a $(p+1)$-summable quasihomomorphism of parity p mod 2, with $\mathscr{E}$ admissible with respect to a quasi-free extension of $\mathscr{B}$. The bivariant Chern character $\mathrm{ch}^p(\rho) \in HC^p(\mathscr{A},\mathscr{B})$ induces a graded-commutative diagram*

$$\begin{tikzcd} HP_{n+1}(\mathscr{A}) \arrow[r, "S"] \arrow[d, "\mathrm{ch}^p(\rho)"] & HC_{n-1}(\mathscr{A}) \arrow[r, "B"] \arrow[d, "\mathrm{ch}^p(\rho)"] & HN_n(\mathscr{A}) \arrow[r, "I"] \arrow[d, "\mathrm{ch}^p(\rho)"] & HP_n(\mathscr{A}) \arrow[d, "\mathrm{ch}^p(\rho)"] \\ HP_{n-p+1}(\mathscr{B}) \arrow[r, "S"] & HC_{n-p-1}(\mathscr{B}) \arrow[r, "B"] & HN_{n-p}(\mathscr{B}) \arrow[r, "I"] & HP_{n-p}(\mathscr{B}) \end{tikzcd}$$

invariant under conjugation of quasihomomorphisms. Moreover the arrow in periodic cyclic homology $HP_n(\mathscr{A}) \to HP_{n-p}(\mathscr{B})$ is invariant under homotopy of quasihomomorphisms.

Proof: The fact that $S\mathrm{ch}^p(\rho) \in HC^{p+2}(\mathscr{A},\mathscr{B})$ is homotopy invariant shows its image in the periodic theory $HP^p(\mathscr{A},\mathscr{B})$ is homotopy invariant. ∎

Example 3.14 When $\mathscr{A}$ is arbitrary and $\mathscr{B} = \mathbb{C}$, we saw in Example 3.3 that a $(p+1)$-summable quasihomomorphism $\rho : \mathscr{A} \to \mathscr{L}^s \triangleright \mathscr{I}^s$, represents a K-homology class of $\mathscr{A}$. By hypothesis, the degree of the quasihomomorphism is p mod 2. The Chern character $\mathrm{ch}^p(\rho) \in HC^p(\mathscr{A},\mathbb{C}) \cong HC^p(\mathscr{A})$ is a cyclic cohomology class of degree p over $\mathscr{A}$, represented by a chain map $\widehat{X}(T\mathscr{A},J\mathscr{A}) \to \mathbb{C}$ vanishing on the subcomplex $F^p\widehat{X}(T\mathscr{A},J\mathscr{A})$. Using the pro-vector space isomorphism $\widehat{X}(T\mathscr{A},J\mathscr{A}) \cong \widehat{\Omega}\mathscr{A}$, one finds that $\mathrm{ch}^p(\rho)$ is non-zero only on the subspace of p-forms $\Omega^p\mathscr{A}$, explicitly

$$\mathrm{ch}^p(\rho)(a_0da_1\dots da_p) = \frac{c_p}{2}\,\mathrm{Tr}_s(F[F,a_0]\dots[F,a_p])\ ,$$

where $\mathrm{Tr}_s : (\mathscr{I}^s)^{p+1} \to \mathbb{C}$ is the supertrace of the $(p+1)$-summable algebra $\mathscr{I}$ and c_p is a constant depending on the degree. One has $c_p = (-)^n(n!)^2/p!$ when $p = 2n$ is even, and $c_p = \sqrt{2\pi i}\,(-)^n/2^p$ when $p = 2n+1$ is odd. This coincides with the Chern-Connes character [4, 5], up to a scaling factor accounting for the homotopy equivalence between the X-complex $\widehat{X}(T\mathscr{A},J\mathscr{A})$ and the $(b+B)$-complex $\widehat{\Omega}\mathscr{A}$.

Example 3.15 When $\mathscr{A} = \mathbb{C}$ and $\mathscr{B} = C^\infty(0,1)$, the Bott element (Example 3.5) represented by the odd 1-summable quasihomomorphism $\rho : \mathbb{C} \to \mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B}$ with $\mathscr{B}$-admissible extension $\mathscr{E} = C^\infty[0,1]$, has a Chern character in

$HC^1(\mathbb{C},\mathscr{B}) \cong HP_1(\mathscr{B})$. The periodic cyclic homology of $\mathscr{B}$ is isomorphic to the de Rham cohomology of the open interval $(0,1)$, hence $HP_0(\mathscr{B}) = 0$ and $HP_1(\mathscr{B}) = \mathbb{C}$. Consequently, the Chern character $\mathrm{ch}^1(\rho)$ may be represented by a smooth one-form over $[0,1]$ vanishing at the endpoints. It involves a real-valued function $\xi \in \mathscr{E}$, with $\xi(0) = 0$ and $\xi(1) = \pi/2$, used in the construction of the homomorphism $\rho : \mathbb{C} \to \mathscr{E}^s_+$. One explicitly finds

$$\mathrm{ch}(\rho) = \sqrt{2\pi i}\,\mathbf{d}(\sin^2 \xi) \ ,$$

so that its integral over the interval $[0,1]$ is normalized to $\sqrt{2\pi i}$, and of course does no depend on the chosen function ξ. This is due to the fact that quasihomomorphisms associated to different choices of ξ are homotopic.

4 Topological K-theory

We review here the topological K-theory of Fréchet m-algebras following Phillips [27], and construct various Chern character maps with value in cyclic homology. Topological K-theory for Fréchet m-algebras is defined in analogy with Banach algebras and fulfills the same properties of homotopy invariance, Bott periodicity and excision [27]. For our purposes, only homotopy invariance and Bott periodicity are needed. A basic example of Fréchet m-algebra is provided by the algebra $\mathscr{K}$ of "smooth compact operators". $\mathscr{K}$ is the space of infinite matrices $(A_{ij})_{i,j\in\mathbb{N}}$ with entries in $\mathbb{C}$ and rapid decay, endowed with the family of submultiplicative norms

$$\|A\|_n = \sup_{(i,j)\in\mathbb{N}^2} (1+i+j)^n A_{ij} < \infty \qquad \forall n \in \mathbb{N} \ .$$

The multiplication of matrices makes $\mathscr{K}$ a Fréchet m-algebra. Moreover $\mathscr{K}$ is nuclear as a locally convex vector space [14]. If $\mathscr{A}$ is any Fréchet m-algebra, the completed tensor product $\mathscr{K}\hat{\otimes}\mathscr{A}$ is the *smooth stabilization* of $\mathscr{A}$. Other important examples are the algebras $C^\infty[0,1]$, resp. $C^\infty(0,1)$, of smooth $\mathbb{C}$-valued functions over the interval, with all derivatives of order ≥ 1, resp. ≥ 0, vanishing at the endpoints. As already mentioned in section 3, these are again nuclear Fréchet m-algebras and the completed tensor products $\mathscr{A}[0,1] = \mathscr{A}\hat{\otimes}C^\infty[0,1]$ and $\mathscr{A}(0,1) = \mathscr{A}\hat{\otimes}C^\infty(0,1)$ are isomorphic to the algebras of smooth $\mathscr{A}$-valued functions over the interval, with the appropriate vanishing boundary conditions. In particular $S\mathscr{A} := \mathscr{A}(0,1)$ is the *smooth suspension* of $\mathscr{A}$. We say that two idempotents e_0, e_1 of an algebra $\mathscr{A}$ are smoothly homotopic if there exists an id empotent $e \in \mathscr{A}[0,1]$ whose evaluation at the endpoints gives e_0 and e_1. Similarly for invertible elements.

The definition of topological K-theory involves idempotents and invertibles of the unitalized algebra $(\mathscr{K}\hat{\otimes}\mathscr{A})^+$. Choosing an isomorphism $M_2(\mathscr{K}) \cong \mathscr{K}$ makes $(\mathscr{K}\hat{\otimes}\mathscr{A})^+$ a semigroup for the direct sum $a \oplus b = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$. We denote by p_0 the idempotent $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ of the matrix algebra $M_2(\mathscr{K}\hat{\otimes}\mathscr{A})^+$.

DEFINITION 4.1 (PHILLIPS [27]) *Let $\mathscr{A}$ be a Fréchet m-algebra. The topological K-theory of $\mathscr{A}$ in degree zero and one is defined by*

$$\begin{aligned} K_0^{\mathrm{top}}(\mathscr{A}) &= \{\textit{set of smooth homotopy classes of idempotents } e \in M_2(\mathscr{K}\hat{\otimes}\mathscr{A})^+ \\ &\qquad \textit{such that } e - p_0 \in M_2(\mathscr{K}\hat{\otimes}\mathscr{A})\ \} \\ K_1^{\mathrm{top}}(\mathscr{A}) &= \{\textit{set of smooth homotopy classes of invertibles } g \in (\mathscr{K}\hat{\otimes}\mathscr{A})^+ \\ &\qquad \textit{such that } g - 1 \in \mathscr{K}\hat{\otimes}\mathscr{A}\ \} \end{aligned}$$

$K_0^{\mathrm{top}}(\mathscr{A})$ and $K_1^{\mathrm{top}}(\mathscr{A})$ are semigroups for the direct sum of idempotents and invertibles; in the case of idempotents, the direct sum $e \oplus e' \in M_4(\mathscr{K}\hat{\otimes}\mathscr{A})^+$ has to be conjugated by the invertible matrix

$$c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in M_4(\mathscr{K}\hat{\otimes}\mathscr{A})^+ \ , \quad c^{-1} = c \ , \tag{52}$$

in order to preserve the condition $c(e \oplus e')c - \tilde{p}_0 \in M_4(\mathscr{K}\hat{\otimes}\mathscr{A})$, with $\tilde{p}_0$ the diagonal matrix $\mathrm{diag}(1,1,0,0)$. The proof that $K_0^{\mathrm{top}}(\mathscr{A})$ and $K_1^{\mathrm{top}}(\mathscr{A})$ are actually abelian groups will be recalled in Lemma 5.2. The unit of $K_0^{\mathrm{top}}(\mathscr{A})$ is the class of the idempotent $p_0 \in M_2(\mathscr{K}\hat{\otimes}\mathscr{A})^+$, whereas the unit of $K_1^{\mathrm{top}}(\mathscr{A})$ is represented by $1 \in (\mathscr{K}\hat{\otimes}\mathscr{A})^+$.
The fundamental property of topological K-theory is Bott periodicity [27]. Let $S\mathscr{A} = \mathscr{A}(0,1)$ be the smooth suspension of $\mathscr{A}$. Define two additive maps

$$\alpha : K_1^{\mathrm{top}}(\mathscr{A}) \to K_0^{\mathrm{top}}(S\mathscr{A}) \ , \qquad \beta : K_0^{\mathrm{top}}(\mathscr{A}) \to K_1^{\mathrm{top}}(S\mathscr{A}) \tag{53}$$

as follows. First choose a real-valued function $\xi \in C^\infty[0,1]$ such that $\xi(0) = 0$ and $\xi(1) = \pi/2$ (we recall that all the derivatives of ξ vanish at the endpoints). Let $g \in (\mathscr{K}\hat{\otimes}\mathscr{A})^+$ represent an element of $K_1^{\mathrm{top}}(\mathscr{A})$. Then the idempotent

$$\alpha(g) = G^{-1}p_0G \ , \qquad \alpha(g) - p_0 \in M_2(\mathscr{K}\hat{\otimes}S\mathscr{A})$$

defines an element of $K_0^{\mathrm{top}}(S\mathscr{A})$, where $G : [0,1] \to M_2(\mathscr{K}\hat{\otimes}\mathscr{A})^+$ is the matrix function over $[0,1]$

$$G = R^{-1}\begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} R \quad \text{with} \quad R = \begin{pmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{pmatrix} .$$

Now $z = \exp(4i\xi)$ is a complex-valued invertible function over $[0,1]$ with winding number 1. The functions $z-1$ and $z^{-1}-1$ lie in $C^\infty(0,1)$. Then for any idempotent $e \in M_2(\mathscr{K}\hat{\otimes}\mathscr{A})^+$ representing a class in $K_0^{\mathrm{top}}(\mathscr{A})$, we define the invertible element

$$\beta(e) = (1 + (z-1)e)(1 + (z-1)p_0)^{-1} \ .$$

One has $(1 + (z-1)p_0)^{-1} = (1 + (z^{-1}-1)p_0)$, and the idempotent relations $e^2 = e$, $p_0^2 = p_0$ imply $\beta(e) = 1 + (z-1)e(e-p_0) + (z^{-1}-1)(p_0-e)p_0$, which shows that $\beta(e) - 1$ is an element of the algebra $M_2(\mathscr{K}\hat{\otimes}S\mathscr{A}) \cong \mathscr{K}\hat{\otimes}S\mathscr{A}$, hence $\beta(e)$ defines a class in $K_1^{\mathrm{top}}(S\mathscr{A})$.

Proposition 4.2 (Bott periodicity [27]) *The two maps defined above $\alpha : K_1^{\rm top}(\mathscr{A}) \to K_0^{\rm top}(S\mathscr{A})$ and $\beta : K_0^{\rm top}(\mathscr{A}) \to K_1^{\rm top}(S\mathscr{A})$ are isomophisms of abelian groups.* ∎

Hence Bott periodicity implies $K_i^{\rm top}(S^2\mathscr{A}) = K_i^{\rm top}(\mathscr{A})$ for $i = 0, 1$, so that we may define topological K-theory groups in any degree $n \in \mathbb{Z}$:

$$K_n^{\rm top}(\mathscr{A}) = \left\{ \begin{array}{ll} K_0^{\rm top}(\mathscr{A}) & n \text{ even} \\ K_1^{\rm top}(\mathscr{A}) & n \text{ odd} \ . \end{array} \right. \tag{54}$$

Following Cuntz and Quillen [12], we construct Chern characters with values in periodic cyclic homology $K_n^{\rm top}(\mathscr{A}) \to HP_n(\mathscr{A})$. Recall (section 2) that periodic cyclic homology is computed from any quasi-free extension $0 \to \mathscr{J} \to \mathscr{R} \to \mathscr{A} \to 0$ by the pro-complex

$$\widehat{X}(\mathscr{R}, \mathscr{J}) = X(\widehat{\mathscr{R}}) \ : \ \widehat{\mathscr{R}} \rightleftarrows \Omega^1\widehat{\mathscr{R}}_\natural \ ,$$

where the pro-algebra $\widehat{\mathscr{R}} = \varprojlim_n \mathscr{R}/\mathscr{J}^n$ is the $\mathscr{J}$-adic completion of the quasi-free algebra $\mathscr{R}$. In particular, the universal free extension $0 \to J\mathscr{A} \to T\mathscr{A} \to \mathscr{A} \to 0$ is quasi-free and the universal property of the tensor algebra leads to a classifying homomorphism $T\mathscr{A} \to \mathscr{R}$ compatible with the ideals $J\mathscr{A}$ and $\mathscr{J}$ by means of the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & J\mathscr{A} \arrow[r] \arrow[d] & T\mathscr{A} \arrow[r] \arrow[d] & \mathscr{A} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathscr{J} \arrow[r] & \mathscr{R} \arrow[r] & \mathscr{A} \arrow[r] \arrow[l, dotted, bend right, "\sigma"'] & 0 \end{tikzcd}$$

for any choice of continuous linear section $\sigma : \mathscr{A} \to \mathscr{R}$. The homomorphism $T\mathscr{A} \to \mathscr{R}$ thus extends to a homomorphism of pro-algebras $\widehat{T}\mathscr{A} \to \widehat{\mathscr{R}}$ and the induced morphism of complexes $X(\widehat{T}\mathscr{A}) \to X(\widehat{\mathscr{R}})$ is a homotopy equivalence. The Chern character on topological K-theory requires to lift idempotents and invertible elements from the algebra $\mathscr{K}\hat{\otimes}\mathscr{A}$ to the pro-algebra

$$\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}} = \varprojlim_n \mathscr{K}\hat{\otimes}(\mathscr{R}/\mathscr{J}^n) \ .$$

If $e \in M_2(\mathscr{K}\hat{\otimes}\mathscr{A})^+$ is an idempotent such that $e - p_0 \in M_2(\mathscr{K}\hat{\otimes}\mathscr{A})$, there always exists an idempotent lift $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}})^+$ with $\hat{e} - p_0 \in M_2(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}})$, and two such liftings are always conjugate [12]. A concrete way to construct an idempotent lift is to work first with the tensor algebra and then push forward by the homomorphism $\mathscr{K}\hat{\otimes}\widehat{T}\mathscr{A} \to \mathscr{K}\hat{\otimes}\widehat{\mathscr{R}}$. Using the isomorphism of pro-vector spaces $\widehat{T}\mathscr{A} \cong \widehat{\Omega}^+\mathscr{A}$, the following differential form of even degree defines an idempotent [12]

$$\hat{e} = e + \sum_{k\geq 1} \frac{(2k)!}{(k!)^2}(e - \frac{1}{2})(dede)^k \ \in M_2(\mathscr{K}\hat{\otimes}\widehat{T}\mathscr{A})^+ \ , \tag{55}$$

where concatenation products over $M_2(\mathscr{K})$ are taken. We will refer to (55) as the *canonical lift* of e, but it should be stressed that other choices are possible. Denoting also by $\hat{e}$ its image in $M_2(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}})^+$, the Chern character of e is represented by the cycle of even degree

$$\mathrm{ch}_0(\hat{e}) = \mathrm{Tr}(\hat{e} - p_0) \ \in \widehat{\mathscr{R}} \ , \tag{56}$$

where the partial trace $\mathrm{Tr} : M_2(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}}) \to \widehat{\mathscr{R}}$ comes from the usual trace of matrices with rapid decay. We will show below that the cyclic homology class of $\mathrm{ch}_0(\hat{e})$ is invariant under smooth homotopies of $\hat{e}$. Moreover, the invariance of the trace under similarity implies that ch_0 is additive. Next, if $g \in (\mathscr{K}\hat{\otimes}\mathscr{A})^+$ is an invertible element such that $g - 1 \in \mathscr{K}\hat{\otimes}\mathscr{A}$, we have again to choose an invertible lift $\hat{g} \in (\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}})^+$ with $\hat{g} - 1 \in \mathscr{K}\hat{\otimes}\widehat{\mathscr{R}}$. It turns out that any lifting of g is invertible, and two such liftings are always homotopic [12]. A concrete way to construct an invertible lift is to use the linear inclusion of zero-forms $\mathscr{K}\hat{\otimes}\mathscr{A} \hookrightarrow \mathscr{K}\hat{\otimes}\widehat{T}\mathscr{A} \cong \mathscr{K}\hat{\otimes}\widehat{\Omega}^+\mathscr{A}$ and consider g as an element $\hat{g} = g \in (\mathscr{K}\hat{\otimes}\widehat{T}\mathscr{A})^+$. A simple computation shows that it is invertible, with inverse

$$\hat{g}^{-1} = \sum_{k\geq 0} g^{-1}(dg\, dg^{-1})^k \ \in (\mathscr{K}\hat{\otimes}\widehat{T}\mathscr{A})^+ \ . \tag{57}$$

Here again we shall refer to the above $\hat{g}$ as the *canonical lift* of g, but other choices are possible. Then denoting also by $\hat{g}$ its image in $(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}})^+$, the Chern character of g is represented by the cycle of odd degree

$$\mathrm{ch}_1(\hat{g}) = \frac{1}{\sqrt{2\pi i}}\, \mathrm{Tr}\natural\hat{g}^{-1}\mathbf{d}\hat{g} \ \in \Omega^1\widehat{\mathscr{R}}_\natural \ , \tag{58}$$

with the trace map $\mathrm{Tr} : \Omega^1(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}})_\natural \to \Omega^1\widehat{\mathscr{R}}_\natural$. In this case also we will show that the cyclic homology class of $\mathrm{ch}_1(\hat{g})$ is invariant under smooth homotopies of $\hat{g}$. Clearly ch_1 is additive. The factor $1/\sqrt{2\pi i}$ is chosen for consistency with the bivariant Chern character.

Note the following important property of idempotents and invertibles: two idempotents $\hat{e}_0, \hat{e}_1 \in M_2(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}})^+$ are homotopic if and only if their projections $e_0, e_1 \in M_2(\mathscr{K}\hat{\otimes}\mathscr{A})^+$ are homotopic, and similarly with invertibles [12]. Since the cyclic homology classes of the Chern characters $\mathrm{ch}_0(\hat{e})$ and $\mathrm{ch}_1(\hat{g})$ are homotopy invariant with respect to $\hat{e}$ and $\hat{g}$, one gets well-defined additive maps $\mathrm{ch}_0 : K_0^{\mathrm{top}}(\mathscr{A}) \to HP_0(\mathscr{A})$ and $\mathrm{ch}_1 : K_1^{\mathrm{top}}(\mathscr{A}) \to HP_1(\mathscr{A})$ on the topological K-theory groups. They do not depend on the quasi-free extension $\mathscr{R}$ since we know that the classifying homomorphism $\widehat{T}\mathscr{A} \to \widehat{\mathscr{R}}$ induces a homotopy equivalence of pro-complexes $X(\widehat{T}\mathscr{A}) \xrightarrow{\sim} X(\widehat{\mathscr{R}})$.

To show the homotopy invariance of the Chern characters, we introduce the Cherns-Simons transgressions. Let $\widehat{\mathscr{R}}[0,1]$ be the tensor product $\widehat{\mathscr{R}}\hat{\otimes}C^\infty[0,1]$, and let $\hat{e}$ be any idempotent of $M_2(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}}[0,1])^+$ with $\hat{e} - p_0 \in M_2(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}}[0,1])$. Denote by $s : C^\infty[0,1] \to \Omega^1[0,1]$ the de Rham coboundary map with values in ordinary (commutative) one-forms over the interval. We then define the

Chern-Simons form associated to $\hat{e}$ as the chain of odd degree

$$\mathrm{cs}_1(\hat{e}) = \int_0^1 \mathrm{Tr}\natural(-2\hat{e}+1)\, s\hat{e}\, \mathbf{d}\hat{e} \ \in \Omega^1\widehat{\mathscr{R}}_\natural \ , \tag{59}$$

with obvious notations. Now let $\hat{g} \in (\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}}[0,1])^+$ be any invertible element such that $\hat{g}-1 \in \mathscr{K}\hat{\otimes}\widehat{\mathscr{R}}[0,1]$. The Chern-Simons form associated to $\hat{g}$ is the chain of even degree

$$\mathrm{cs}_0(\hat{g}) = \frac{1}{\sqrt{2\pi i}} \int_0^1 \mathrm{Tr}(\hat{g}^{-1}s\hat{g}) \ \in \widehat{\mathscr{R}} \ . \tag{60}$$

Lemma 4.3 *Let $\hat{e}$ be an idempotent of the algebra $M_2(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}}[0,1])^+$ with $\hat{e}-p_0 \in M_2(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}}[0,1])$. Denote by $\hat{e}_0$ and $\hat{e}_1$ the idempotents of $M_2(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}})^+$ obtained by evaluation at 0 and 1. Then one has*

$$\overline{b}\mathrm{cs}_1(\hat{e}) = \mathrm{ch}_0(\hat{e}_1) - \mathrm{ch}_0(\hat{e}_0) \ \in \widehat{\mathscr{R}} \ . \tag{61}$$

Let $\hat{g} \in (\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}}[0,1])^+$ be an invertible element such that $\hat{g}-1 \in \mathscr{K}\hat{\otimes}\widehat{\mathscr{R}}[0,1]$. Denote by $\hat{g}_0$ and $\hat{g}_1$ the invertibles of $(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}})^+$ obtained by evaluation at 0 and 1. Then one has

$$\natural\mathbf{d}\mathrm{cs}_0(\hat{g}) = \mathrm{ch}_1(\hat{g}_1) - \mathrm{ch}_1(\hat{g}_0) \ \in \Omega^1\widehat{\mathscr{R}}_\natural \ . \tag{62}$$

Proof: First notice that the current $\int_0^1$ is odd, so that

$$\overline{b}\mathrm{cs}_1(\hat{e}) = -\int_0^1 \overline{b}\mathrm{Tr}\natural(-2\hat{e}+1)\, s\hat{e}\, \mathbf{d}\hat{e} \ ,$$

and taking into account the fact that $s\hat{e}$ is also odd, one has

$$\overline{b}\mathrm{Tr}\natural(-2\hat{e}+1)\, s\hat{e}\, \mathbf{d}\hat{e} = -\mathrm{Tr}[(-2\hat{e}+1)\, s\hat{e}, \hat{e}] = \mathrm{Tr}((2\hat{e}-1)s\hat{e}\,\hat{e} - \hat{e}s\hat{e}) \ ,$$

where we use the idempotent property $\hat{e}^2 = \hat{e}$ for the last equality. Since s is a derivation, one has $s\hat{e} = s(\hat{e}^2) = s\hat{e}\,\hat{e} + \hat{e}s\hat{e}$ and $\hat{e}s\hat{e}\,\hat{e} = 0$, whence

$$\overline{b}\mathrm{cs}_1(e) = \int_0^1 \mathrm{Tr}(s\hat{e}\,\hat{e} + \hat{e}s\hat{e}) = \int_0^1 s\mathrm{Tr}\hat{e} = \mathrm{Tr}(\hat{e}_1 - \hat{e}_0) = \mathrm{ch}_0(\hat{e}_1) - \mathrm{ch}_0(\hat{e}_0) \ ,$$

because $\hat{e}_0$ and $\hat{e}_1$ are the evaluations of $\hat{e}$ respectively at 0 and 1. Let us proceed now with invertibles:

$$\natural\mathbf{d}\mathrm{cs}_0(\hat{g}) = \frac{-1}{\sqrt{2\pi i}} \int_0^1 \mathrm{Tr}\natural\mathbf{d}(\hat{g}^{-1}s\hat{g}) \ ,$$

and because $\mathbf{d}$ is an odd derivation anticommuting with s, one has

$$\mathrm{Tr}\natural\mathbf{d}(\hat{g}^{-1}s\hat{g}) = \mathrm{Tr}\natural(\mathbf{d}\hat{g}^{-1}s\hat{g} - \hat{g}^{-1}s\mathbf{d}\hat{g}) \ .$$

Then, $\mathbf{d}1 = 0 = s1$ implies $\mathbf{d}\hat{g}^{-1} = -\hat{g}^{-1}\mathbf{d}\hat{g}\hat{g}^{-1}$ and $s\hat{g}^{-1} = -\hat{g}^{-1}s\hat{g}\hat{g}^{-1}$. But $\mathrm{Tr}\natural$ is a supertrace, hence

$$\mathrm{Tr}\natural(-\hat{g}^{-1}\mathbf{d}\hat{g}\hat{g}^{-1}s\hat{g} - \hat{g}^{-1}s\mathbf{d}\hat{g}) = \mathrm{Tr}\natural(\hat{g}^{-1}s\hat{g}\hat{g}^{-1}\mathbf{d}\hat{g} - \hat{g}^{-1}s\mathbf{d}\hat{g}) = -s\mathrm{Tr}\natural(\hat{g}^{-1}\mathbf{d}\hat{g}) \ .$$

By integration over the interval $[0,1]$, one gets

$$\natural\mathbf{d}\mathrm{cs}_0(\hat{g}) = \frac{1}{\sqrt{2\pi i}}\mathrm{Tr}\natural(\hat{g}_1^{-1}\mathbf{d}\hat{g}_1 - \hat{g}_0^{-1}\mathbf{d}\hat{g}_0) = \mathrm{ch}_1(\hat{g}_1) - \mathrm{ch}_1(\hat{g}_0)$$

as wanted. ∎

Hence the cyclic homology classes of the Chern characters are homotopy invariant as claimed. There is another consequence of the above lemma. Observe that the suspensions $S\mathscr{A} = \mathscr{A}(0,1)$ and $S\widehat{\mathscr{R}} = \widehat{\mathscr{R}}(0,1)$ are subalgebras of $\mathscr{A}[0,1]$ and $\widehat{\mathscr{R}}[0,1]$. If $e \in M_2(\mathscr{K}\hat{\otimes}S\mathscr{A})^+$ is an idempotent representing a class in $K_0^{\mathrm{top}}(S\mathscr{A})$, and $g \in (\mathscr{K}\hat{\otimes}S\mathscr{A})^+$ an invertible representing a class in $K_1^{\mathrm{top}}(S\mathscr{A})$, we choose some lifts $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}S\widehat{\mathscr{R}})^+$ and $\hat{g} \in (\mathscr{K}\hat{\otimes}S\widehat{\mathscr{R}})^+$. Then $\mathrm{cs}_1(\hat{e})$ and $\mathrm{cs}_0(\hat{g})$ are closed and define homology classes in $HP_1(\mathscr{A})$ and $HP_0(\mathscr{A})$ respectively. The following lemma shows the compatibility with Bott periodicity:

Lemma 4.4 *The Chern-Simons forms define additive maps* $\mathrm{cs}_1 : K_0^{\mathrm{top}}(S\mathscr{A}) \to HP_1(\mathscr{A})$ *and* $\mathrm{cs}_0 : K_1^{\mathrm{top}}(S\mathscr{A}) \to HP_0(\mathscr{A})$. *Moreover they are compatible with the Bott isomorphisms* $\alpha : K_1^{\mathrm{top}}(\mathscr{A}) \to K_0^{\mathrm{top}}(S\mathscr{A})$ *and* $\beta : K_0^{\mathrm{top}}(\mathscr{A}) \to K_1^{\mathrm{top}}(S\mathscr{A})$ *and Chern characters, up to multiplication by a factor* $\sqrt{2\pi i}$*:*

$$\begin{aligned} \mathrm{cs}_1\circ\alpha \equiv \sqrt{2\pi i}\,\mathrm{ch}_1 &\quad:\quad K_1^{\mathrm{top}}(\mathscr{A}) \to HP_1(\mathscr{A}) \ , \\ \mathrm{cs}_0\circ\beta \equiv \sqrt{2\pi i}\,\mathrm{ch}_0 &\quad:\quad K_0^{\mathrm{top}}(\mathscr{A}) \to HP_0(\mathscr{A}) \ . \end{aligned}$$

Proof: Let $\hat{e}$ be any idempotent of the pro-algebra $M_2(\mathscr{K}\hat{\otimes}S\widehat{\mathscr{R}})^+$. We have to prove the homotopy invariance of the cyclic homology class determined by the cycle

$$\mathrm{cs}_1(\hat{e}) = \int_0^1 \mathrm{Tr}\natural(-2\hat{e}+1)\,s\hat{e}\,\mathbf{d}\hat{e} \ .$$

To this end, consider a smooth family of idempotents $\hat{e}_t \in M_2(\mathscr{K}\hat{\otimes}S\widehat{\mathscr{R}})^+$ parametrized by $t \in \mathbb{R}$, such that $\hat{e}_t - p_0 \in M_2(\mathscr{K}\hat{\otimes}S\widehat{\mathscr{R}}), \forall t$. Denote by $\dot{\hat{e}}$ the derivative $\partial\hat{e}/\partial t$. The idempotent property of the family $\hat{e}$ implies the following identity:

$$\frac{\partial}{\partial t}\mathrm{Tr}\natural(-2\hat{e}+1)s\hat{e}\mathbf{d}\hat{e} = -\natural\mathbf{d}\mathrm{Tr}(\hat{e}(\dot{\hat{e}}s\hat{e} - s\hat{e}\dot{\hat{e}})) - s\mathrm{Tr}\natural\hat{e}(\dot{\hat{e}}\mathbf{d}\hat{e} - \mathbf{d}\hat{e}\dot{\hat{e}}) \ .$$

Since for any fixed t, the idempotent $\hat{e}_t$ equals p_0 at the boundaries of the suspended algebra $M_2(\mathscr{K}\hat{\otimes}S\widehat{\mathscr{R}})^+$, one gets $\int_0^1 s\mathrm{Tr}\natural\hat{e}(\dot{\hat{e}}\mathbf{d}\hat{e} - \mathbf{d}\hat{e}\dot{\hat{e}}) = 0$ and

$$\frac{\partial}{\partial t}\int_0^1 \mathrm{Tr}\natural(-2\hat{e}+1)\,s\hat{e}\,\mathbf{d}\hat{e} = \natural\mathbf{d}\int_0^1 \mathrm{Tr}(\hat{e}(\dot{\hat{e}}s\hat{e} - s\hat{e}\dot{\hat{e}})) \ .$$

This implies that the cyclic homology class of $\mathrm{cs}_1(\hat{e})$ is a homotopy invariant of $\hat{e}$. Hence if $\hat{e}$ lifts an idempotent $e \in M_2(\mathscr{K}\hat{\otimes}S\mathscr{A})^+$, the cyclic homology class of $\mathrm{cs}_1(\hat{e})$ only depends on the homotopy class of e, and the map $\mathrm{cs}_1 : K_0^{\mathrm{top}}(S\mathscr{A}) \to HP_1(\mathscr{A})$ is well-defined. We now have to show the compatibility with Bott periodicity. Thus let $g \in (\mathscr{K}\hat{\otimes}\mathscr{A})^+$ be an invertible such that $g-1 \in \mathscr{K}\hat{\otimes}\mathscr{A}$, and let $\alpha(g)$ be the idempotent $G^{-1}p_0G \in M_2(\mathscr{K}\hat{\otimes}S\mathscr{A})^+$ constructed by means of a rotation matrix

$$R = \begin{pmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{pmatrix} , \qquad G = R^{-1}\begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} R ,$$

where $\xi \in C^\infty[0,1]$ is a real function with $\xi(0)=0$ and $\xi(1)=\pi/2$. Then, it is clear that the idempotent $\hat{e} = \widehat{G}^{-1}p_0\widehat{G} \in M_2(\mathscr{K}\hat{\otimes}S\widehat{\mathscr{R}})^+$ is a lifting of $\alpha(g)$, where the matrix $\widehat{G} = R^{-1}\left(\begin{smallmatrix} 1 & 0 \\ 0 & \hat{g} \end{smallmatrix}\right)R$ is built from any lifting $\hat{g} \in (\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}})^+$ of g. Hence, the cyclic homology class of $\mathrm{cs}_1(\widehat{\alpha(g)})$ is represented by $\mathrm{cs}_1(\hat{e})$. A direct computation shows the equality

$$\mathrm{Tr}\natural(-2\hat{e}+1)\, s\hat{e}\,\mathbf{d}\hat{e} = s(\cos\xi)\mathrm{Tr}\natural(-\hat{g}^{-1}\mathbf{d}\hat{g} + \frac{1}{2}\mathbf{d}\hat{g} - \frac{1}{2}\mathbf{d}\hat{g}^{-1}) ,$$

so that after integration over $[0,1]$ one gets, modulo boundaries $\natural\mathbf{d}(\cdot)$

$$\mathrm{cs}_1(\widehat{\alpha(g)}) \equiv \mathrm{Tr}\natural(\hat{g}^{-1}\mathbf{d}\hat{g}) \bmod \natural\mathbf{d} \equiv \sqrt{2\pi i}\,\mathrm{ch}_1(\hat{g}) \bmod \natural\mathbf{d} .$$

Next, we turn to the map cs_0. If $\hat{g} \in (\mathscr{K}\hat{\otimes}S\widehat{\mathscr{R}})^+$ is any invertible, one has

$$\mathrm{cs}_0(\hat{g}) = \frac{1}{\sqrt{2\pi i}}\int_0^1 \mathrm{Tr}(\hat{g}^{-1}s\hat{g}) .$$

We have to show that the cyclic homology class of $\mathrm{cs}_0(\hat{g})$ is a homotopy invariant of $\hat{g}$. To this end, consider a smooth one-parameter family of invertibles $\hat{g}_t \in (\mathscr{K}\hat{\otimes}S\widehat{\mathscr{R}})^+$. One has, with $\dot{\hat{g}} = \partial\hat{g}/\partial t$,

$$\frac{\partial}{\partial t}(\hat{g}^{-1}s\hat{g}) = -\hat{g}^{-1}\dot{\hat{g}}\hat{g}^{-1}s\hat{g} + \hat{g}^{-1}s\dot{\hat{g}} = [\hat{g}^{-1}s\hat{g}, \hat{g}^{-1}\dot{\hat{g}}] + s(\hat{g}^{-1}\dot{\hat{g}}) .$$

Since $\mathrm{Tr}[\hat{g}^{-1}s\hat{g}, \hat{g}^{-1}\dot{\hat{g}}] = -\overline{b}\mathrm{Tr}\natural\hat{g}^{-1}s\hat{g}\mathbf{d}(\hat{g}^{-1}\dot{\hat{g}})$, we get

$$\frac{\partial}{\partial t}\int_0^1 \mathrm{Tr}(\hat{g}^{-1}s\hat{g}) = \overline{b}\int_0^1 \mathrm{Tr}\natural\hat{g}^{-1}s\hat{g}\mathbf{d}(\hat{g}^{-1}\dot{\hat{g}}) .$$

Hence the cyclic homology class of $\mathrm{cs}_0(\hat{g})$ is homotopy invariant. In particular if $\hat{g}$ lifts an invertible $g \in (\mathscr{K}\hat{\otimes}S\mathscr{A})^+$, the cyclic homology class of $\mathrm{cs}_0(\hat{g})$ is a homotopy invariant of g and the map $\mathrm{cs}_0 : K_1^{\mathrm{top}}(S\mathscr{A}) \to HP_0(\mathscr{A})$ is well-defined. Now let $e \in M_2(\mathscr{K}\hat{\otimes}\mathscr{A})^+$ be an idempotent, with $e - p_0 \in M_2(\mathscr{K}\hat{\otimes}\mathscr{A})$. Its image under the Bott map β is the invertible element $\beta(e) \in (\mathscr{K}\hat{\otimes}S\mathscr{A})^+$ given by

$$\beta(e) = (1+(z-1)e)(1+(z-1)p_0)^{-1} ,$$

where $z = \exp(4i\xi)$. If $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\widehat{\mathscr{R}})^+$ is any idempotent lift of e, it is clear that the invertible

$$\hat{g} = (1+(z-1)\hat{e})(1+(z-1)p_0)^{-1} \ \in (\mathscr{K}\hat{\otimes}S\widehat{\mathscr{R}})^+$$

is a lifting of $\beta(e)$. Hence the cyclic homology class of $\mathrm{cs}_0(\widehat{\beta(e)})$ is represented by $\mathrm{cs}_0(\hat{g})$. Let us compute explicitly $\mathrm{Tr}(\hat{g}^{-1}s\hat{g})$:

$$\begin{aligned}\mathrm{Tr}\big((1+(z-1)p_0)(1+(z-1)\hat{e})^{-1}s\big((1+(z-1)\hat{e})(1+(z-1)p_0)^{-1}\big)\big)\\ = \mathrm{Tr}\big((1+(z^{-1}-1)\hat{e})sz\,\hat{e} - (1+(z^{-1}-1)p_0)sz\,p_0\big)\\ = \mathrm{Tr}(\hat{e}-p_0)\,z^{-1}sz\ .\end{aligned}$$

Since the integration of $z^{-1}sz$ over the interval $[0,1]$ yields a factor $2\pi i$, one is left with equivalences modulo boundaries $\overline{b}(\cdot)$

$$\mathrm{cs}_0(\widehat{\beta(e)}) \equiv \sqrt{2\pi i}\,\mathrm{Tr}(\hat{e}-p_0) \bmod \overline{b} \equiv \sqrt{2\pi i}\,\mathrm{ch}_0(\hat{e}) \bmod \overline{b}$$

as wanted. ∎

Since our main motivation is index theory we will have to consider the stabilization of $\mathscr{A}$ by a p-summable Fréchet m-algebra $\mathscr{I}$, that is, $\mathscr{I}$ is provided with a continuous trace $\mathrm{Tr} : \mathscr{I}^p \to \mathbb{C}$ as in section 3. Hence it will be convenient to define a Chern character $K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to HP_n(\mathscr{A})$. The difficulty of course is that the trace is not defined on the algebra $\mathscr{K}\hat{\otimes}\mathscr{I}$ but only on its p-th power. To cope with this problem, we shall construct higher analogues of the Chern characters and Chern-Simons forms associated to idempotents and invertibles. Consider the following p-summable quasihomomorphism of even degree, from the algebra $\mathscr{I}\mathscr{A} := \mathscr{I}\hat{\otimes}\mathscr{A}$ to $\mathscr{A}$:

$$\rho : \mathscr{I}\mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\mathscr{A}\ , \qquad \mathscr{E} = \mathscr{I}^+\hat{\otimes}\mathscr{A}\ , \tag{63}$$

where $\mathscr{I}^+$ is the unitalization of $\mathscr{I}$. Because ρ is of even degree, it is entirely specified by a pair of homomorphisms $(\rho_+,\rho_-) : \mathscr{I}\mathscr{A} \rightrightarrows \mathscr{E}$ which agree modulo the ideal $\mathscr{I}\mathscr{A} \subset \mathscr{E}$. Equivalently if we represent $\mathscr{E}^s$ in the $\mathbb{Z}_2$-graded matrix algebra $M_2(\mathscr{E})$ we can write $\rho = \left(\begin{smallmatrix}\rho_+ & 0\\ 0 & \rho_-\end{smallmatrix}\right)$. By definition we set

$$\rho_+ = \mathrm{Id} : \mathscr{I}\mathscr{A} \to \mathscr{I}\mathscr{A} \subset \mathscr{E}\ , \qquad \rho_- = 0\ .$$

Let $0 \to \mathscr{J} \to \mathscr{R} \to \mathscr{A} \to 0$ be any quasi-free extension of $\mathscr{A}$, with continuous linear splitting $\sigma : \mathscr{A} \to \mathscr{R}$. Then choosing $\mathscr{M} = \mathscr{I}^+\hat{\otimes}\mathscr{R}$ and $\mathscr{N} = \mathscr{I}^+\hat{\otimes}\mathscr{J}$, one gets a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{N} \arrow[r] & \mathscr{M} \arrow[r] & \mathscr{E} \arrow[r] & 0 \\ 0 \arrow[r] & \mathscr{I}\mathscr{J} \arrow[r] \arrow[u] & \mathscr{I}\mathscr{R} \arrow[r] \arrow[u] & \mathscr{I}\mathscr{A} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

Since $(\mathscr{I}^+)^n = \mathscr{I}^+$ for any integer n, one sees that $\mathscr{N}^n = \mathscr{I}^+ \hat{\otimes} \mathscr{J}^n$ is a direct summand in $\mathscr{M}$. Moreover, there is an obvious chain map $\mathrm{Tr} : F^{2n+1}_{\mathscr{I}\mathscr{R}} X(\mathscr{M}) \to X(\mathscr{R})$ for any $n \geq p-1$, obtained by taking the trace $\mathscr{I}^{n+1} \to \mathbb{C}$. It follows that the algebra $\mathscr{E} \triangleright \mathscr{I}\mathscr{A}$ is $\mathscr{R}$-admissible (Definition 3.2), hence in all degrees $2n+1 \geq p$ the bivariant Chern characters $\mathrm{ch}^{2n}(\rho) \in HC^{2n}(\mathscr{I}\mathscr{A}, \mathscr{A})$ are defined and related by the S-operation in bivariant cyclic cohomology $\mathrm{ch}^{2n+2}(\rho) \equiv S\mathrm{ch}^{2n}(\rho)$. We recall briefly the construction of $\mathrm{ch}^{2n}(\rho)$. By the universal properties of the tensor algebra $T(\mathscr{I}\mathscr{A})$, the homomorphism $\rho : \mathscr{I}\mathscr{A} \to \left(\begin{smallmatrix} \mathscr{I}\mathscr{A} & 0 \\ 0 & 0 \end{smallmatrix}\right) \subset \mathscr{E}^s_+$ lifts to a classifying homomorphism ρ_* through the commutative diagram (31)

$$\begin{tikzcd}
0 \arrow[r] & J(\mathscr{I}\mathscr{A}) \arrow[r] \arrow[d, "\rho_*"'] & T(\mathscr{I}\mathscr{A}) \arrow[r] \arrow[d, "\rho_*"'] & \mathscr{I}\mathscr{A} \arrow[r] \arrow[d, "\rho"] & 0 \\
0 \arrow[r] & \mathscr{N}^s_+ \arrow[r] & \mathscr{M}^s_+ \arrow[r] & \mathscr{E}^s_+ \arrow[r] \arrow[l, dotted, bend right, "\mathrm{Id}\otimes\sigma"'] & 0
\end{tikzcd}$$

induced by the linear splitting. It extends to a homomorphism of pro-algebras $\rho_* : \widehat{T}(\mathscr{I}\mathscr{A}) \to \left(\begin{smallmatrix} \mathscr{I}\widehat{\mathscr{R}} & 0 \\ 0 & 0 \end{smallmatrix}\right) \subset \widehat{\mathscr{M}^s_+}$. The bivariant Chern character $\mathrm{ch}^{2n}(\rho)$ is the composite of the Goodwillie equivalence $\gamma : X(\widehat{T}(\mathscr{I}\mathscr{A})) \to \widehat{\Omega}\widehat{T}(\mathscr{I}\mathscr{A})$ with the chain maps $\rho_* : \widehat{\Omega}\widehat{T}(\mathscr{I}\mathscr{A}) \to \widehat{\Omega}\widehat{\mathscr{M}^s_+}$ and $\widehat{\chi}^{2n} : \widehat{\Omega}\widehat{\mathscr{M}^s_+} \to X(\widehat{\mathscr{R}})$. The two non-zero components of $\widehat{\chi}^{2n}$ are given by Eqs. (34) and defined on $2n$ and $(2n+1)$-forms respectively:

$$\widehat{\chi}^{2n}_0 : \Omega^{2n}\widehat{\mathscr{M}^s_+} \to \widehat{\mathscr{R}} \ , \qquad \widehat{\chi}^{2n}_1 : \Omega^{2n+1}\widehat{\mathscr{M}^s_+} \to \Omega^1\widehat{\mathscr{R}}_\natural \ .$$

The bivariant Chern character is designed to improve the summability degree and can be used to define the higher Chern characters of idempotents and invertibles via the composition

$$\mathrm{ch}^{2n}_i : K^{\mathrm{top}}_i(\mathscr{I}\hat{\otimes}\mathscr{A}) \to HP_i(\mathscr{I}\hat{\otimes}\mathscr{A}) \xrightarrow{\mathrm{ch}^{2n}_i(\rho)} HP_i(\mathscr{A}) \ , \qquad i = 0,1 \ . \tag{64}$$

We shall now establish very explicit formulas for these higher characters. Let $\hat{e}$ be an idempotent of the algebra $M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$, such that $\hat{e} - p_0 \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})$. It is well-known (see for example [5]) that the differential forms

$$\begin{aligned}
\mathrm{ch}_{2n}(\hat{e}) &= (-)^n \frac{(2n)!}{n!} \mathrm{Tr}\big((\hat{e} - \tfrac{1}{2})(\mathbf{d}\hat{e}\mathbf{d}\hat{e})^n\big) \ \in \Omega^{2n}(\mathscr{I}\widehat{\mathscr{R}}) \quad \text{for } n \geq 1 \\
\mathrm{ch}_0(\hat{e}) &= \mathrm{Tr}(\hat{e} - p_0) \ \in \Omega^0(\mathscr{I}\widehat{\mathscr{R}})
\end{aligned} \tag{65}$$

are the components of a $(b+B)$-cycle of even degree over $\mathscr{I}\widehat{\mathscr{R}}$, i.e. fulfill the relations $B\mathrm{ch}_{2n}(\hat{e}) + b\mathrm{ch}_{2n+2}(\hat{e}) = 0$ for any n. Here Tr is the trace over $\mathscr{K}$. In the odd case, any invertible element $\hat{g} \in (\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$ such that $\hat{g} - 1 \in \mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}$ gives rise to a $(b+B)$-cycle of odd degree with components

$$\mathrm{ch}_{2n+1}(\hat{g}) = \frac{(-)^n}{\sqrt{2\pi i}} n! \, \mathrm{Tr}\big(\hat{g}^{-1}\mathbf{d}\hat{g}(\mathbf{d}\hat{g}^{-1}\mathbf{d}\hat{g})^n\big) \ \in \Omega^{2n+1}(\mathscr{I}\widehat{\mathscr{R}}) \ . \tag{66}$$

The homology classes of these cycles are of course homotopy invariant. If $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$ is a smooth path of idempotents, we define the components of the associated Chern-Simons form as

$$\mathrm{cs}_{2n+1}(\hat{e}) = (-)^n \frac{(2n)!}{n!} \int_0^1 \mathrm{Tr}\big((-2\hat{e}+1)\sum_{i=0}^{2n}(\mathbf{d}\hat{e})^i s\hat{e}(\mathbf{d}\hat{e})^{2n+1-i}\big) \qquad (67)$$

in $\Omega^{2n+1}(\mathscr{I}\widehat{\mathscr{R}})$. Similarly if $\hat{g} \in (\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$ is a smooth path of invertibles, the components of the Chern-Simons form are for $n \geq 1$

$$\mathrm{cs}_{2n}(\hat{g}) = \frac{(-)^n}{\sqrt{2\pi i}}(n-1)! \int_0^1 \mathrm{Tr}\big(\hat{g}^{-1}\mathbf{d}\hat{g}\sum_{i=0}^{n-1}(\mathbf{d}\hat{g}^{-1}\mathbf{d}\hat{g})^i \mathbf{d}\omega(\mathbf{d}\hat{g}^{-1}\mathbf{d}\hat{g})^{n-1-i}\big) \quad (68)$$

in $\Omega^{2n}(\mathscr{I}\widehat{\mathscr{R}})$, where $\omega = \hat{g}^{-1}s\hat{g}$, and for $n=0$ we set as before $\mathrm{cs}_0(\hat{g}) = \frac{1}{\sqrt{2\pi i}}\int_0^1 \mathrm{Tr}(\omega)$. Simple algebraic manipulations show that the $(b+B)$-boundaries of the Chern-Simons forms yield the difference of evaluations of the Chern characters at the endpoints:

$$\begin{aligned} B\mathrm{cs}_{2n-1}(\hat{e}) + b\mathrm{cs}_{2n+1}(\hat{e}) &= \mathrm{ch}_{2n}(\hat{e}_1) - \mathrm{ch}_{2n}(\hat{e}_0)\ , \\ B\mathrm{cs}_{2n}(\hat{g}) + b\mathrm{cs}_{2n+2}(\hat{g}) &= \mathrm{ch}_{2n+1}(\hat{g}_1) - \mathrm{ch}_{2n+1}(\hat{g}_0)\ . \end{aligned} \qquad (69)$$

The higher Chern characters (64) and their associated Chern-Simons forms are obtained by evaluation of these $(b+B)$-chains on the inclusion homomorphism $\iota_* : \mathscr{I}\widehat{\mathscr{R}} \hookrightarrow \left(\begin{smallmatrix}\mathscr{I}\widehat{\mathscr{R}} & 0\\ 0 & 0\end{smallmatrix}\right) \subset \widehat{\mathscr{M}^s_+}$ followed by the chain map $\widehat{\chi}^{2n}$ whenever $2n+1 \geq p$.

Lemma 4.5 *Let $\mathscr{I}$ be p-summable and $2n+1 \geq p$. For any idempotent $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$ such that $\hat{e} - p_0 \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})$, and any invertible $\hat{g} \in (\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$ such that $\hat{g}-1 \in \mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}$, we define the higher Chern characters by the explicit formulas*

$$\begin{aligned} \mathrm{ch}_0^{2n}(\hat{e}) &= \mathrm{Tr}\,(\hat{e}-p_0)^{2n+1}\ , \\ \mathrm{ch}_1^{2n}(\hat{g}) &= \frac{1}{\sqrt{2\pi i}}\frac{(n!)^2}{(2n)!}\mathrm{Tr}\natural\,\hat{g}^{-1}[(\hat{g}-1)(\hat{g}^{-1}-1)]^n\mathbf{d}\hat{g}\ , \end{aligned} \qquad (70)$$

where we take concatenation products over $\mathscr{I}$ and Tr *is the trace over the p-th power of $\mathscr{K}\hat{\otimes}\mathscr{I}$. Then one has $\mathrm{ch}_0^{2n}(\hat{e}) = \widehat{\chi}_0^{2n}\iota_*\mathrm{ch}_{2n}(\hat{e})$ in $\widehat{\mathscr{R}}$ and $\mathrm{ch}_1^{2n}(\hat{g}) = \widehat{\chi}_1^{2n}\iota_*\mathrm{ch}_{2n+1}(\hat{g})$ in $\Omega^1\widehat{\mathscr{R}}_\natural$.*
Similarly, for any idempotent $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$ and any invertible $\hat{g} \in (\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$, we define the higher Chern-Simons forms by the explicit formulas

$$\mathrm{cs}_1^{2n}(\hat{e}) = \int_0^1 \mathrm{Tr}\natural(-2\hat{e}+1)\sum_{i=0}^{n}(\hat{e}-p_0)^{2i}s\hat{e}(\hat{e}-p_0)^{2(n-i)}\mathbf{d}\hat{e}\ , \qquad (71)$$

$$\mathrm{cs}_0^{2n}(\hat{g}) = \frac{1}{\sqrt{2\pi i}}\frac{(n!)^2}{(2n)!}\int_0^1 \mathrm{Tr}\,\hat{g}^{-1}[(\hat{g}-1)(\hat{g}^{-1}-1)]^n s\hat{g}$$

Then $\mathrm{cs}_1^{2n}(\hat{e}) = \widehat{\chi}_1^{2n}\iota_*\mathrm{cs}_{2n+1}(\hat{e})$ *in* $\Omega^1\widehat{\mathscr{R}}_\natural$ *and* $\mathrm{cs}_0^{2n}(\hat{g}) \equiv \widehat{\chi}_0^{2n}\iota_*\mathrm{cs}_{2n}(\hat{g}) \bmod \overline{b}$ *in* $\widehat{\mathscr{R}}$. *Moreover the transgression relations hold:*

$$\overline{b}\mathrm{cs}_1^{2n}(\hat{e}) = \mathrm{ch}_0^{2n}(\hat{e}_1) - \mathrm{ch}_0^{2n}(\hat{e}_0)\ , \qquad \natural\mathbf{d}\mathrm{cs}_0^{2n}(\hat{g}) = \mathrm{ch}_1^{2n}(\hat{g}_1) - \mathrm{ch}_1^{2n}(\hat{g}_0)\ .$$

Proof: Let us briefly explain the computation of the cycles $\widehat{\chi}_0^{2n}\iota_*\mathrm{ch}_{2n}(\hat{e})$ and $\widehat{\chi}_1^{2n}\iota_*\mathrm{ch}_{2n+1}(\hat{g})$ associated to idempotents $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$ and invertibles $\hat{g} \in (\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$. The upper left corner inclusion $\iota_* : \mathscr{I}\widehat{\mathscr{R}} \hookrightarrow \widehat{\mathscr{M}_+^s}$ canonically extends to a unital homomorphism $(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+ \to (\mathscr{K}\hat{\otimes}\widehat{\mathscr{M}_+^s})^+$, and in matrix form we can write

$$\iota_*\hat{e} = \begin{pmatrix} \hat{e} & 0 \\ 0 & p_0 \end{pmatrix}\ , \qquad \iota_*\hat{g} = \begin{pmatrix} \hat{g} & 0 \\ 0 & 1 \end{pmatrix}\ .$$

Consequently the commutators with the odd multiplier $F = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ read

$$[F, \iota_*\hat{e}] = \begin{pmatrix} 0 & p_0 - \hat{e} \\ \hat{e} - p_0 & 0 \end{pmatrix}\ , \qquad [F, \iota_*\hat{g}] = \begin{pmatrix} 0 & 1 - \hat{g} \\ \hat{g} - 1 & 0 \end{pmatrix}\ .$$

It is therefore straightforward to evaluate the differential forms $\mathrm{ch}_{2n}(\hat{e})$ and $\mathrm{ch}_{2n+1}(\hat{g})$ on the chain map $\widehat{\chi}^{2n}$ given by (34). One finds $\widehat{\chi}_0^{2n}\iota_*\mathrm{ch}_{2n}(\hat{e}) = \mathrm{ch}_0^{2n}(\hat{e})$ and $\widehat{\chi}_1^{2n}\iota_*\mathrm{ch}_{2n+1}(\hat{g}) = \mathrm{ch}_1^{2n}(\hat{g})$. Similarly with the Chern-Simons forms one finds $\widehat{\chi}_1^{2n}\iota_*\mathrm{cs}_{2n+1}(\hat{e}) = \mathrm{cs}_1^{2n}(\hat{e})$, whereas by setting $\omega = \hat{g}^{-1}s\hat{g}$

$$\begin{aligned}\widehat{\chi}_0^{2n}\iota_*\mathrm{cs}_{2n}(\hat{g}) = \frac{1}{\sqrt{2\pi i}}\frac{(n!)^2}{(2n+1)!}\int_0^1 \mathrm{Tr}\big(&\omega[(\hat{g}^{-1}-1)(\hat{g}-1)]^n + \\ &(\hat{g}-1)\omega[(\hat{g}^{-1}-1)(\hat{g}-1)]^{n-1}(\hat{g}^{-1}-1) + \ldots + [(\hat{g}^{-1}-1)(\hat{g}-1)]^n\omega\big)\end{aligned}$$

coincides with $\mathrm{cs}_0^{2n}(\hat{g})$ only modulo commutators.
Finally, the transgression relations are an immediate consequence of Eqs. (69) and the fact that $\widehat{\chi}^{2n}$ is a chain map from the $(b+B)$-complex over $\widehat{\mathscr{M}_+^s}$ to the complex $X(\widehat{\mathscr{R}})$. ∎

In any degree $2n+1 \geq p$ the Chern characters $\mathrm{ch}_0^{2n} : K_0^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to HP_0(\mathscr{A})$ and $\mathrm{ch}_1^{2n} : K_1^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to HP_1(\mathscr{A})$ are thus obtained by first lifting idempotents $e \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})^+$ and invertibles $g \in (\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})^+$ to some $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$ and $\hat{g} \in (\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$, and then taking the cyclic homology classes of $\mathrm{ch}_0^{2n}(\hat{e}) \in \widehat{\mathscr{R}}$ and $\mathrm{ch}_1^{2n}(\hat{g}) \in \Omega^1\widehat{\mathscr{R}}_\natural$. Although $\hat{e}$ and $\hat{g}$ are only defined up to homotopy, the above lemma shows these higher Chern characters are well-defined, and moreover independent of the degree $2n$ because the cocycles $\widehat{\chi}^{2n}$ are all related by the transgression relations

$$\widehat{\chi}^{2n} - \widehat{\chi}^{2n+2} = [\partial, \widehat{\eta}^{2n+1}]\ \in \mathrm{Hom}(\widehat{\Omega}\widehat{\mathscr{M}_+^s}, X(\widehat{\mathscr{R}}))\ .$$

Passing to suspensions, lifting any idempotent $e \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}S\mathscr{A})^+$ or invertible $g \in (\mathscr{K}\hat{\otimes}\mathscr{I}S\mathscr{A})^+$ gives rise to an odd cycle $\mathrm{cs}_1^{2n}(\hat{e}) \in \Omega^1\widehat{\mathscr{R}}_\natural$ or an even cycle $\mathrm{cs}_0^{2n}(\hat{g}) \in \widehat{\mathscr{R}}$. As expected, this is well-defined at the K-theory level and compatible with Bott periodicity:

LEMMA 4.6 *Let $\mathscr{I}$ be p-summable. In any degree $2n+1 \geq p$, the Chern-Simons forms define additive maps $\mathrm{cs}_1^{2n} : K_0^{\mathrm{top}}(\mathscr{I}\hat{\otimes}S\mathscr{A}) \to HP_1(\mathscr{A})$ and cs_0^{2n} : $K_1^{\mathrm{top}}(\mathscr{I}\hat{\otimes}S\mathscr{A}) \to HP_0(\mathscr{A})$, independent of n, and compatible with the Bott isomorphisms:*

$$\begin{array}{rcl}\mathrm{cs}_1^{2n}\circ\alpha \equiv \sqrt{2\pi i}\,\mathrm{ch}_1^{2n} & : & K_1^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to HP_1(\mathscr{A})\ , \\ \mathrm{cs}_0^{2n}\circ\beta \equiv \sqrt{2\pi i}\,\mathrm{ch}_0^{2n} & : & K_0^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to HP_0(\mathscr{A})\ .\end{array}$$

Proof: Consider an idempotent $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}S\widehat{\mathscr{R}})^+$. We have to show the homotopy invariance of the cyclic homology class $\mathrm{cs}_1^{2n}(\hat{e})$ with respect to $\hat{e}$. This can be shown by direct computation from Formulas (71). Define the matrix idempotent $\hat{f} = \left(\begin{smallmatrix}\hat{e} & 0\\ 0 & p_0\end{smallmatrix}\right)$. Then one has $s\hat{f} = \left(\begin{smallmatrix}s\hat{e} & 0\\ 0 & 0\end{smallmatrix}\right)$ and $\mathbf{d}\hat{f} = \left(\begin{smallmatrix}\mathbf{d}\hat{e} & 0\\ 0 & 0\end{smallmatrix}\right)$, and $\mathrm{cs}_1^{2n}(\hat{e})$ can be rewritten by means of the operator $F = \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$ and the supertrace τ:

$$\mathrm{cs}_1^{2n}(\hat{e}) = (-)^n\int_0^1 \tau\natural(-2\hat{f}+1)\sum_{i=0}^n [F,\hat{f}]^{2i}s\hat{f}[F,\hat{f}]^{2(n-i)}\mathbf{d}\hat{f}\ .$$

Now suppose that $\hat{e}$ depends smoothly on an additional parameter t. The above integrand may be expressed in terms of the odd differential $\delta = s+\mathbf{d}+dt\frac{\partial}{\partial t}+[F,\ ]$ as

$$\tau\natural(-2\hat{f}+1)\sum_{i=0}^n [F,\hat{f}]^{2i}s\hat{f}[F,\hat{f}]^{2(n-i)}\mathbf{d}\hat{f} = \frac{-1}{n+1}\,\tau\natural\,\hat{f}(\delta\hat{f})^{2n+1}|_{s,\mathbf{d}}$$

where $|_{s,\mathbf{d}}$ means that we select the terms containing only s, $\mathbf{d}$ and not dt. Because $\tau\natural$ is a supertrace, the cocycle property $(s+\mathbf{d}+dt\frac{\partial}{\partial t})\tau\natural\,\hat{f}(\delta\hat{f})^{2n+1} = \tau\natural\,(\delta\hat{f})^{2n+2} = 0$ holds, and projecting this relation on $s,\mathbf{d},dt$ yields

$$s\,\tau\natural\,\hat{f}(\delta\hat{f})^{2n+1}|_{\mathbf{d},dt} + \tau\natural\,\mathbf{d}(\hat{f}(\delta\hat{f})^{2n+1})|_{s,dt} + dt\frac{\partial}{\partial t}\tau\natural\,\hat{f}(\delta\hat{f})^{2n+1}|_{s,\mathbf{d}} = 0\ .$$

This may be rephrased as

$$\begin{aligned}&\frac{\partial}{\partial t}\big(\mathrm{Tr}\natural(-2\hat{e}+1)\sum_{i=0}^n(\hat{e}-p_0)^{2i}s\hat{e}(\hat{e}-p_0)^{2(n-i)}\mathbf{d}\hat{e}\big)\\ &\quad\equiv\quad s\big(\mathrm{Tr}\natural(-2\hat{e}+1)\sum_{i=0}^n(\hat{e}-p_0)^{2i}\frac{\partial\hat{e}}{\partial t}(\hat{e}-p_0)^{2(n-i)}\mathbf{d}\hat{e}\big)\ \mathrm{mod}\ \natural\mathbf{d}\ ,\end{aligned}$$

and integration over the current $\int_0^1$ shows the homotopy invariance of the class $\mathrm{cs}_1^{2n}(\hat{e})$. Hence the map $\mathrm{cs}_1^{2n} : K_0^{\mathrm{top}}(\mathscr{I}\hat{\otimes}S\mathscr{A}) \to HP_1(\mathscr{A})$ is well-defined. Its compatibility with Bott periodicity can be established, without computation, as follows. Let $g \in (\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})^+$ be an invertible element and

$e = \alpha(g) \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}S\mathscr{A})^+$ its idempotent image under the Bott isomorphism. Choose an invertible lift $\widetilde{g} \in (\mathscr{K}\hat{\otimes}S\widehat{T}(\mathscr{I}\mathscr{A}))^+$ of g and an idempotent lift $\widetilde{e} \in M_2(\mathscr{K}\hat{\otimes}S\widehat{T}(\mathscr{I}\mathscr{A}))^+$ of e. The differential forms $\mathrm{ch}_{2n+1}(\widetilde{g})$ and $\mathrm{cs}_{2n+1}(\widetilde{e})$ in $\Omega^{2n+1}\widehat{T}(\mathscr{I}\mathscr{A})$ defined by (66) and (67) are the components of two $(b+B)$-cycles $\mathrm{ch}_*(\widetilde{g})$ and $\mathrm{cs}_*(\widetilde{e})$, whose projections on the odd part of the complex $X(\widehat{T}(\mathscr{I}\mathscr{A}))$ are

$$\natural\mathrm{ch}_1(\widetilde{g}) = \frac{1}{\sqrt{2\pi i}}\mathrm{Tr}\natural\widetilde{g}^{-1}\mathbf{d}\widetilde{g}\ , \qquad \natural\mathrm{cs}_1(\widetilde{e}) = \int_0^1 \mathrm{Tr}\natural(-2\widetilde{e}+1)s\widetilde{e}\mathbf{d}\widetilde{e})\ .$$

By Lemma 4.4, the cycles $\sqrt{2\pi i}\natural\mathrm{ch}_1(\widetilde{g})$ and $\natural\mathrm{cs}_1(\widetilde{e})$ are homologous. But we know that the projection $\widehat{\Omega}\widehat{T}(\mathscr{I}\mathscr{A}) \to X(\widehat{T}(\mathscr{I}\mathscr{A}))$ is a homotopy equivalence, with inverse the Goodwillie map γ. Hence $\sqrt{2\pi i}\,\mathrm{ch}_*(\widetilde{g})$ and $\mathrm{cs}_*(\widetilde{e})$ are $(b+B)$-homologous in $\widehat{\Omega}\widehat{T}(\mathscr{I}\mathscr{A})$. Finally, it remains to observe that under the homomorphism $\rho_* : \widehat{T}\mathscr{A} \to \mathscr{M}^s_+$, the invertible $\rho_*\widetilde{g} = \left(\begin{smallmatrix} \hat{g} & 0 \\ 0 & 1\end{smallmatrix}\right)$ gives a choice of lifting $\hat{g} \in (\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$ and the idempotent $\rho_*(\widetilde{e}) = \left(\begin{smallmatrix} \hat{e} & 0 \\ 0 & p_0\end{smallmatrix}\right)$ gives a choice of lifting $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$. Because the cycles $\sqrt{2\pi i}\,\widehat{\chi}^{2n}\rho_*\mathrm{ch}_*(\widetilde{g})$ and $\widehat{\chi}^{2n}\rho_*\mathrm{cs}_*(\widetilde{e})$ are homologous in $X(\widehat{\mathscr{R}})$, we have $\sqrt{2\pi i}\,\mathrm{ch}_1^{2n}(\hat{g}) \equiv \mathrm{cs}_1^{2n}(\hat{e})$ in $HP_1(\mathscr{A})$.
We proceed similarly with the map cs_0^{2n}. Let $\hat{g} \in (\mathscr{K}\hat{\otimes}\mathscr{I}S\widehat{\mathscr{R}})^+$ be an invertible. We have to show the homotopy invariance of the cyclic homology class $\mathrm{cs}_0^{2n}(\hat{g})$. Define the invertible matrix $\hat{u} = \left(\begin{smallmatrix} \hat{g} & 0 \\ 0 & 1\end{smallmatrix}\right)$. Then $s\hat{u} = \left(\begin{smallmatrix} s\hat{g} & 0 \\ 0 & 0\end{smallmatrix}\right)$ and one has

$$\mathrm{cs}_0^{2n}(\hat{g}) = \frac{(-)^n}{\sqrt{2\pi i}}\frac{(n!)^2}{(2n)!}\int_0^1 \tau(\hat{u}^{-1}([F,\hat{u}][F,\hat{u}^{-1}])^n s\hat{u})\ .$$

Now suppose that $\hat{g}$ is a smooth family of invertibles depending on an additional parameter t. The above integrand may be expressed in terms of the odd derivation $\delta = s + dt\frac{\partial}{\partial t} + [F,\ ]$ as

$$\tau\,\hat{u}^{-1}([F,\hat{u}][F,\hat{u}^{-1}])^n s\hat{u} \equiv \frac{(-)^n}{2n+1}\,\tau(\hat{u}^{-1}\delta\hat{u})^{2n+1}|_s \bmod \overline{b}\ .$$

One has the relation $(s + dt\frac{\partial}{\partial t})\tau(\hat{u}^{-1}\delta\hat{u})^{2n+1} = -\tau(\hat{u}^{-1}\delta\hat{u})^{2n+2} \equiv 0 \bmod \overline{b}$, hence by projection on s, dt

$$s\,\tau(\hat{u}^{-1}\delta\hat{u})^{2n+1}|_{dt} + dt\frac{\partial}{\partial t}\tau(\hat{u}^{-1}\delta\hat{u})^{2n+1}|_s \equiv 0 \bmod \overline{b}\ .$$

This may be rephrased as

$$\frac{\partial}{\partial t}\big(\mathrm{Tr}\,\hat{g}^{-1}[(\hat{g}-1)(\hat{g}^{-1}-1)]^n s\hat{g}\big) \equiv s\big(\mathrm{Tr}\,\hat{g}^{-1}[(\hat{g}-1)(\hat{g}^{-1}-1)]^n\frac{\partial\hat{g}}{\partial t}\big) \bmod \overline{b}\ ,$$

and integration over the current $\int_0^1$ shows the homotopy invariance of the class $\mathrm{cs}_0^{2n}(\hat{g})$. Hence the map $\mathrm{cs}_0^{2n} : K_1^{\mathrm{top}}(\mathscr{I}\hat{\otimes}S\mathscr{A}) \to HP_0(\mathscr{A})$ is well-defined. Its compatibility with Bott periodicity is be established as before, replacing invertibles by idempotents and conversely. ∎

5 Multiplicative K-theory

Let $\mathscr{A}$ and $\mathscr{I}$ be Fréchet m-algebras, $\mathscr{I}$ being p-summable. We shall define the multiplicative K-theory groups $MK_n^{\mathscr{I}}(\mathscr{A})$ in any degree $n \in \mathbb{Z}$. They are intermediate between the topological K-theory $K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A})$ and the non-periodic cyclic homology $HC_n(\mathscr{A})$. Recall from section 2 that if $0 \to \mathscr{J} \to \mathscr{R} \to \mathscr{A} \to 0$ is any quasi-free extension, $HC_n(\mathscr{A})$ is computed by the quotient complex $X_n(\mathscr{R},\mathscr{J}) = X(\mathscr{R})/F^n_{\mathscr{J}}X(\mathscr{R})$ induced by the $\mathscr{J}$-adic filtration:

$$HC_n(\mathscr{A}) = H_{n+2\mathbb{Z}}(X_n(\mathscr{R},\mathscr{J}))\ , \qquad \forall n \in \mathbb{Z}\ .$$

Of course $HC_n(\mathscr{A})$ vanishes whenever $n<0$. Multiplicative K-theory classes are represented by idempotents or invertibles whose higher Chern characters (Lemma 4.5) can be transgressed up to a certain order. As before we adopt the notation $\mathscr{I}\mathscr{A} = \mathscr{I}\hat{\otimes}\mathscr{A}$ and $\mathscr{I}\widehat{\mathscr{R}} = \mathscr{I}\hat{\otimes}\widehat{\mathscr{R}}$, where $\widehat{\mathscr{R}}$ is the $\mathscr{I}$-adic completion of $\mathscr{R}$.

Definition 5.1 *Let $0 \to \mathscr{J} \to \mathscr{R} \to \mathscr{A} \to 0$ be any quasi-free extension of Fréchet m-algebras, and let $\mathscr{I}$ be a p-summable Fréchet m-algebra. Choose an integer q such that $2q+1 \geq p$. We define the multiplicative K-theory $MK_n^{\mathscr{I}}(\mathscr{A})$, in any even degree $n = 2k \in \mathbb{Z}$, as the set of equivalence classes of pairs $(\hat{e},\theta)$ such that $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$ is an idempotent and $\theta \in X_{n-1}(\mathscr{R},\mathscr{J})$ is a chain of odd degree related by the transgression formula*

$$\mathrm{ch}_0^{2q}(\hat{e}) = \overline{b}\theta \ \in X_{n-1}(\mathscr{R},\mathscr{J})\ .$$

Two pairs $(\hat{e}_0,\theta_0)$ and $(\hat{e}_1,\theta_1)$ are equivalent if and only if there exists an idempotent $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$ whose evaluation yields $\hat{e}_0$ and $\hat{e}_1$ at the endpoints, and a chain $\lambda \in X_{n-1}(\mathscr{R},\mathscr{J})$ of even degree such that

$$\theta_1 - \theta_0 = \mathrm{cs}_1^{2q}(\hat{e}) + \natural\mathbf{d}\lambda \ \in X_{n-1}(\mathscr{R},\mathscr{J})\ .$$

In the same way, we define the multiplicative K-theory $MK_n^{\mathscr{I}}(\mathscr{A})$, in any odd degree $n = 2k+1 \in \mathbb{Z}$, as the set of equivalence classes of pairs $(\hat{g},\theta)$ such that $\hat{g} \in (\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$ is an invertible and $\theta \in X_{n-1}(\mathscr{R},\mathscr{J})$ is a chain of even degree related by the transgression formula

$$\mathrm{ch}_1^{2q}(\hat{g}) = \natural\mathbf{d}\theta \ \in X_{n-1}(\mathscr{R},\mathscr{J})\ .$$

Two pairs $(\hat{g}_0,\theta_0)$ and $(\hat{g}_1,\theta_1)$ are equivalent if and only if there exists an invertible $\hat{g} \in (\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$ whose evaluation yields $\hat{g}_0$ and $\hat{g}_1$ at the endpoints, and a chain $\lambda \in X_{n-1}(\mathscr{R},\mathscr{J})$ of odd degree such that

$$\theta_1 - \theta_0 = \mathrm{cs}_0^{2q}(\hat{g}) + \overline{b}\lambda \ \in X_{n-1}(\mathscr{R},\mathscr{J})\ .$$

We will prove in Proposition 5.4 that for any $n \in \mathbb{Z}$ the set $MK_n^{\mathscr{I}}(\mathscr{A})$ depends neither on the degree $2q+1 \geq p$ chosen to represent the Chern characters nor on the quasi-free extension $\mathscr{R}$.

Recall from section 2 that the homology $H_{n-1+2\mathbb{Z}}(X_n(\mathscr{R},\mathscr{J}))$ is the noncommutative de Rham homology $HD_{n-1}(\mathscr{A})$. Hence the transgression relation $\mathrm{ch}_0^{2q}(\hat{e})=\overline{b}\theta\in X_{n-1}(\mathscr{R},\mathscr{J})$ exactly means that the class of $\mathrm{ch}_0^{2q}(\hat{e})$ vanishes in $HD_{n-2}(\mathscr{A})$. Similarly in the odd case, $\mathrm{ch}_1^{2q}(\hat{g})=\natural\mathbf{d}\theta\in X_{n-1}(\mathscr{R},\mathscr{J})$ is equivalent to $\mathrm{ch}_1^{2q}(\hat{g})\equiv 0$ in $HD_{n-2}(\mathscr{A})$. Since the complexes $X_{n-1}(\mathscr{R},\mathscr{J})$ vanish for $n\leq 0$, we immediately deduce that $MK_n^{\mathscr{I}}(\mathscr{A})$ coincides with the topological K-theory $K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A})$ whenever $n\leq 0$.
As in the case of topological K-theory, define an addition on $MK_n^{\mathscr{I}}(\mathscr{A})$ by direct sum of idempotents and invertibles as follows (c is the permutation matrix (52)):

$$\begin{aligned}\text{even case:}\quad & (\hat{e},\theta)+(\hat{e}',\theta')=(c(\hat{e}\oplus\hat{e}')c,\theta+\theta')\ ,\\ \text{odd case:}\quad & (\hat{g},\theta)+(\hat{g}',\theta')=(\hat{g}\oplus\hat{g}',\theta+\theta')\ .\end{aligned}$$

This turns $MK_n^{\mathscr{I}}(\mathscr{A})$ into a semigroup, the unit being represented by $(p_0,0)$ in the even case and $(1,0)$ in the odd case.

Lemma 5.2 *$MK_n^{\mathscr{I}}(\mathscr{A})$ is an abelian group for any $n\in\mathbb{Z}$.*

Proof: We first need to recall the proof that $K_0^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A})$ is a group. Let $e\in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})^+$ be an idempotent, with $e-p_0\in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})$. The idempotent $1-e$ is orthogonal to e, as $e(1-e)=(1-e)e=0$. If $X\in M_2(\mathbb{C})$ is the permutation matrix $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, we claim that the idempotent

$$X(1-e)X\in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})^+\ ,\quad \text{with}\quad X(1-e)X-p_0\in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})\ ,$$

represents the inverse class of e. Indeed, we shall construct a homotopy between the direct sum $c(e\oplus X(1-e)X)c\in M_4(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})^+$ and the unit

$$\tilde{p}_0=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\in M_4(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})^+\ .$$

Choose a smooth real-valued function $\xi\in C^\infty[0,1]$ ranging from $\xi(0)=0$ to $\xi(1)=\pi/2$, and consider the paths of invertible matrices

$$R_{23}=\begin{pmatrix}1&0&0&0\\0&\cos\xi&\sin\xi&0\\0&\sin\xi&-\cos\xi&0\\0&0&0&1\end{pmatrix}\ ,\quad R_{14}=\begin{pmatrix}\cos\xi&0&0&\sin\xi\\0&1&0&0\\0&0&1&0\\\sin\xi&0&0&-\cos\xi\end{pmatrix}\ .$$

A direct computation shows that the idempotents $R_{23}(t)^{-1}\left(\begin{smallmatrix}e&0\\0&0\end{smallmatrix}\right)R_{23}(t)$ and $R_{14}(t)^{-1}\left(\begin{smallmatrix}1-e&0\\0&0\end{smallmatrix}\right)R_{14}(t)$ are orthogonal for any $t\in[0,1]$. Hence the sum

$$f=R_{23}^{-1}\begin{pmatrix}e&0\\0&0\end{pmatrix}R_{23}+R_{14}^{-1}\begin{pmatrix}1-e&0\\0&0\end{pmatrix}R_{14}\ \in M_4(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A}[0,1])^+$$

is an idempotent path such that $f - \tilde{p}_0 \in M_4(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A}[0,1])$, and interpolates $f_0 = \tilde{p}_0$ and $f_1 = c(e \oplus X(1-e)X)c$. This shows that $K_0^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A})$ is a group. It is abelian because a direct sum $e \oplus e'$ can be connected via a smooth path (by conjugation with respect to rotation matrices) to $e' \oplus e$.
Now fix an integer $2q+1 \geq p$ and let $(\hat{e}, \theta)$ represent an element of $MK_n^{\mathscr{I}}(\mathscr{A})$ of even degree $n = 2k$. Hence $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$ is an idempotent such that $\mathrm{ch}_0^{2q}(\hat{e}) = \overline{b}\theta$ in the quotient complex $X_{n-1}(\mathscr{R}, \mathscr{J})$. Consider the smooth idempotent path $\hat{f} \in M_4(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$ constructed as above replacing $\mathscr{A}$ by its extension $\widehat{\mathscr{R}}$ and e by $\hat{e}$. It provides an interpolation between $\hat{f}_0 = \tilde{p}_0$ and $\hat{f}_1 = c(\hat{e} \oplus X(1-\hat{e})X)c$. We guess that the inverse of $(\hat{e}, \theta)$ is represented by the pair $(X(1-\hat{e})X, \mathrm{cs}_1^{2q}(\hat{f}) - \theta)$. Indeed, one has

$$\overline{b}\mathrm{cs}_1^{2q}(\hat{f}) = \mathrm{ch}_0^{2q}(\hat{f}_1) - \mathrm{ch}_0^{2q}(\hat{f}_0) = \mathrm{ch}_0^{2q}(\hat{e}) + \mathrm{ch}_0^{2q}(X(1-\hat{e})X) \ ,$$

so that $\mathrm{ch}_0^{2q}(X(1-\hat{e})X) = \overline{b}(\mathrm{cs}_1^{2q}(\hat{f}) - \theta)$ in the complex $X_{n-1}(\mathscr{R}, \mathscr{J})$ and the pair $(X(1-\hat{e})X, \mathrm{cs}_1^{2q}(\hat{f}) - \theta)$ represents a class in $MK_n^{\mathscr{I}}(\mathscr{A})$. Moreover, the sum

$$(\hat{e}, \theta) + (X(1-\hat{e})X, \mathrm{cs}_1^{2q}(\hat{f}) - \theta) = (c(\hat{e} \oplus X(1-\hat{e})X)c, \mathrm{cs}_1^{2q}(\hat{f}))$$

is equivalent to the unit $(\tilde{p}_0, 0)$ because $\hat{f}$ provides the interpolating idempotent. Hence $MK_n^{\mathscr{I}}(\mathscr{A})$, $n = 2k$ is a group. Abelianity is shown as for topological K-theory, by means of another interpolation between the idempotents $c(\hat{e}\oplus\hat{e}')c$ and $c(\hat{e}' \oplus \hat{e})c$ with the property that its Chern-Simons form cs_1 vanishes.
One proceeds similarly in the odd case. Let $(\hat{g}, \theta)$ represent an element of $MK_n^{\mathscr{I}}(\mathscr{A})$ of odd degree $n = 2k+1$. Hence $\mathrm{ch}_1^{2q}(\hat{g}) = \natural\mathbf{d}\theta$ in $X_{n-1}(\mathscr{R}, \mathscr{J})$. Define an invertible path $\hat{u} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$ by means of the rotation matrix $R = \left(\begin{smallmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{smallmatrix}\right)$:

$$\hat{u} = \begin{pmatrix} \hat{g} & 0 \\ 0 & 1 \end{pmatrix} R^{-1} \begin{pmatrix} \hat{g}^{-1} & 0 \\ 0 & 1 \end{pmatrix} R \ .$$

Then $\hat{u} - 1 \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1])$, and $\hat{u}$ provides a smooth homotopy between the invertibles $\hat{u}_0 = 1$ and $\hat{u}_1 = \left(\begin{smallmatrix} \hat{g} & 0 \\ 0 & \hat{g}^{-1} \end{smallmatrix}\right)$ (the same argument shows that $K_1^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A})$ is an abelian group). We guess that the inverse class of $(\hat{g}, \theta)$ is represented by the pair $(\hat{g}^{-1}, \mathrm{cs}_0^{2q}(\hat{u}) - \theta)$. Indeed, one has

$$\natural\mathbf{d}\mathrm{cs}_0^{2q}(\hat{u}) = \mathrm{ch}_1^{2q}(\hat{u}_1) - \mathrm{ch}_1^{2q}(\hat{u}_0) = \mathrm{ch}_1^{2q}(\hat{g}) + \mathrm{ch}_1^{2q}(\hat{g}^{-1})$$

so that $\mathrm{ch}_1^{2q}(\hat{g}^{-1}) = \natural\mathbf{d}(\mathrm{cs}_0^{2q}(\hat{u}) - \theta)$ in $X_{n-1}(\mathscr{R}, \mathscr{J})$ and $(\hat{g}^{-1}, \mathrm{cs}_0^{2q}(\hat{u}) - \theta)$ represents a class in $MK_n^{\mathscr{I}}(\mathscr{A})$. Moreover, the sum

$$(\hat{g}, \theta) + (\hat{g}^{-1}, \mathrm{cs}_0^{2q}(\hat{u}) - \theta) = (\hat{g} \oplus \hat{g}^{-1}, \mathrm{cs}_0^{2q}(\hat{u}))$$

is equivalent to the unit $(1, 0)$ through the interpolating invertible $\hat{u}$. Hence $MK_n^{\mathscr{I}}(\mathscr{A})$, $n = 2k+1$ is a group as claimed. Abelianity is shown once again by means of rotation matrices. ∎

Remark 5.3 We know that two different liftings of a given idempotent $e \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})^+$ are always homotopic in $M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$. Hence choosing the universal free extension $\mathscr{R} = T\mathscr{A}$ allows to represent any multiplicative K-theory class of even degree by a pair $(\hat{e},\theta)$ where $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{T}\mathscr{A})^+$ is the *canonical lift* of some idempotent e. Moreover, the transgression formula established in the proof of Lemma 4.6 shows that two such pairs $(\hat{e}_0,\theta_0)$ and $(\hat{e}_1,\theta_1)$ are equivalent if and only if e_0 and e_1 can be joined by an idempotent $e \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A}[0,1])^+$ such that $\theta_1 - \theta_0 \equiv \mathrm{cs}_1^{2q}(\hat{e}) \bmod \natural\mathbf{d}$, where $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{T}\mathscr{A}[0,1])^+$ is the canonical lift of e. The same is true with invertibles: any multiplicative K-theory class of odd degree may be represented by a pair $(\hat{g},\theta)$ where $\hat{g} \in (\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{T}\mathscr{A})^+$ is the *canonical lift* of some invertible $g \in (\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})^+$. Two such pairs $(\hat{g}_0,\theta_0)$ and $(\hat{g}_1,\theta_1)$ are equivalent if and only if g_0 and g_1 can be joined by an invertible $g \in (\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A}[0,1])^+$ such that $\theta_1 - \theta_0 \equiv \mathrm{cs}_0^{2q}(\hat{g}) \bmod \overline{b}$, where $\hat{g} \in (\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{T}\mathscr{A}[0,1])^+$ is the canonical lift of g.

The particular case $\mathscr{I} = \mathbb{C}$ is essentially equivalent Karoubi's definition of multiplicative K-theory [16, 17]. The groups $MK_n^{\mathscr{I}}(\mathscr{A})$ are designed to fit in a long exact sequence

$$K_{n+1}^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to HC_{n-1}(\mathscr{A}) \xrightarrow{\delta} MK_n^{\mathscr{I}}(\mathscr{A}) \to K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to HC_{n-2}(\mathscr{A}) \tag{72}$$

The map $K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to HC_{n-2}(\mathscr{A})$ corresponds to the composition of the Chern character $K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to HP_n(\mathscr{A})$ with the natural map $HP_n(\mathscr{A}) \to HC_{n-2}(\mathscr{A})$ induced by the projection $\widehat{X}(\mathscr{R},\mathscr{J}) \to X_{n-2}(\mathscr{R},\mathscr{J})$. The map $MK_n^{\mathscr{I}}(\mathscr{A}) \to K_n^{\mathrm{top}}(\mathscr{A})$ is the forgetful map, which sends a pair $(\hat{e},\theta)$ or $(\hat{g},\theta)$ respectively on its image e or g under the projection homomorphism $\widehat{\mathscr{R}} \to \mathscr{A}$. The connecting map $\delta : HC_{n-1}(\mathscr{A}) \to MK_n^{\mathscr{I}}(\mathscr{A})$ sends a cycle $\theta \in X_{n-1}(\mathscr{R},\mathscr{J})$ to

$$\delta(\theta) = \begin{cases} (p_0, \sqrt{2\pi i}\,\theta) & n \text{ even}, \\ (1, \sqrt{2\pi i}\,\theta) & n \text{ odd}. \end{cases} \tag{73}$$

There is also an additive Chern character map $\mathrm{ch}_n : MK_n^{\mathscr{I}}(\mathscr{A}) \to HN_n(\mathscr{A})$ defined in all degrees $n \in \mathbb{Z}$, with values in *negative* cyclic homology. Recall that the latter is the homology in degree n mod 2 of the subcomplex $F^{n-1}\widehat{X}(\mathscr{R},\mathscr{J}) = \mathrm{Ker}(\widehat{X}(\mathscr{R},\mathscr{J}) \to X_{n-1}(\mathscr{R},\mathscr{J}))$:

$$HN_n(\mathscr{A}) = H_{n+2\mathbb{Z}}(F^{n-1}\widehat{X}(\mathscr{R},\mathscr{J}))\ .$$

Hence in particular $HN_n(\mathscr{A}) = HP_n(\mathscr{A})$ whenever $n \le 0$, and HP_*, HC_* HN_* are related by the SBI long exact sequence (section 2). To define the Chern character $\mathrm{ch}_n : MK_n^{\mathscr{I}}(\mathscr{A}) \to HN_n(\mathscr{A})$ in even degree $n = 2k$, we first have to choose an integer $2q+1 \ge p$. Then, a multiplicative K-theory class of degree n is represented by a pair $(\hat{e},\theta)$, such that the transgression formula $\mathrm{ch}_0^{2q}(\hat{e}) = \overline{b}\theta$ holds in $X_{n-1}(\mathscr{R},\mathscr{J})$. Choose an arbitrary lifting $\tilde{\theta} \in \widehat{X}(\mathscr{R},\mathscr{J})$

of θ, and define the negative Chern character as

$$\mathrm{ch}_n(\hat{e},\theta) = \mathrm{ch}_0^{2q}(\hat{e}) - \overline{b}\tilde{\theta} \ \in F^{n-1}\widehat{X}(\mathscr{R},\mathscr{J}) \ . \tag{74}$$

It is clearly closed, and its negative cyclic homology class does not depend on the choice of lifting $\tilde{\theta}$, since the difference of two such liftings lies in the subcomplex $F^{n-1}\widehat{X}(\mathscr{R},\mathscr{J})$. We will show in the proposition below that it does not depend on the representative $(\hat{e},\theta)$ of the K-theory class, nor on the integer $2q+1 \geq p$. In odd degree $n = 2k+1$, the Chern character $\mathrm{ch}_n : MK_n^{\mathscr{I}}(\mathscr{A}) \to HN_n(\mathscr{A})$ is defined exactly in the same way: take a representative $(\hat{g},\theta)$ of a multiplicative K-theory class, with $\mathrm{ch}_1^{2q}(\hat{g}) = \natural\mathbf{d}\theta$ in $X_{n-1}(\mathscr{R},\mathscr{J})$. Then if $\tilde{\theta} \in \widehat{X}(\mathscr{R},\mathscr{J})$ denotes an arbitrary lifting of θ, the cycle

$$\mathrm{ch}_n(\hat{g},\theta) = \mathrm{ch}_1^{2q}(\hat{g}) - \natural\mathbf{d}\tilde{\theta} \ \in F^{n-1}\widehat{X}(\mathscr{R},\mathscr{J}) \tag{75}$$

defines a negative cyclic homology class. The following proposition shows the compatibility between the K-theory exact sequence (72) and the SBI exact sequence (8), through the various Chern character maps.

PROPOSITION 5.4 *Let $\mathscr{A}$ and $\mathscr{I}$ be Fréchet m-algebras, such that $\mathscr{I}$ is p-summable. Then one has a commutative diagram with long exact rows*

$$\begin{array}{ccccccc} K^{\mathrm{top}}_{n+1}(\mathscr{I}\hat{\otimes}\mathscr{A}) & \longrightarrow & HC_{n-1}(\mathscr{A}) & \xrightarrow{\delta} & MK_n^{\mathscr{I}}(\mathscr{A}) & \longrightarrow & K^{\mathrm{top}}_n(\mathscr{I}\hat{\otimes}\mathscr{A}) \\ \downarrow & & \| & & \downarrow & & \downarrow \\ HP_{n+1}(\mathscr{A}) & \xrightarrow{S} & HC_{n-1}(\mathscr{A}) & \xrightarrow{\widetilde{B}} & HN_n(\mathscr{A}) & \xrightarrow{I} & HP_n(\mathscr{A}) \end{array} \tag{76}$$

where $\widetilde{B}$ is the connecting map of the SBI sequence rescaled by a factor $-\sqrt{2\pi i}$.

Proof: We show the exactness of the sequence (72) in the case of even degree $n = 2k$ (the odd case is completely similar):

$$K_1^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \xrightarrow{\mathrm{ch}_1} HC_{n-1}(\mathscr{A}) \xrightarrow{\delta} MK_n^{\mathscr{I}}(\mathscr{A}) \xrightarrow{\iota} K_0^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \xrightarrow{\mathrm{ch}_0} HC_{n-2}(\mathscr{A}) \ .$$

Fix once and for all an integer $2q+1 \geq p$ to represent the Chern characters. We first have to check that the maps δ and ι are well-defined. Let θ_0 and $\theta_1 = \theta_0 + \natural\mathbf{d}\lambda$ be two homologous odd cycles in $X_{n-1}(\mathscr{R},\mathscr{J})$ representing the same cyclic homology class $[\theta] \in HC_{n-1}(\mathscr{A})$. Their images by δ are respectively $(p_0, \sqrt{2\pi i}\,\theta_0)$ and $(p_0, \sqrt{2\pi i}\,\theta_1)$, which obviously represent the same class in $MK_n^{\mathscr{I}}(\mathscr{A})$ by virtue of the equivalence relation $\theta_1 - \theta_0 = \natural\mathbf{d}\lambda$ (take the constant idempotent $p_0 \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$ as interpolation, with $\mathrm{cs}_1^{2q}(p_0) = 0$). Hence δ is well-defined.
Now take two equivalent pairs $(\hat{e}_0,\theta_0)$ and $(\hat{e}_1,\theta_1)$ representing the same element in $MK_n^{\mathscr{I}}(\mathscr{A})$. In particular, the idempotents $\hat{e}_0$ and $\hat{e}_1$ are smoothly homotopic and their projections $e_0, e_1 \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})^+$ define the same

class in $K_0^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{A})$. Since $\iota(\hat{e}_0,\theta_0)=e_0$ and $\iota(\hat{e}_1,\theta_1)=e_1$, the map ι is well-defined.

Exactness at $HC_{n-1}(\mathscr{A})$: Let $[g]\in K_1^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{A})$ be a class represented by an invertible element $g\in(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})^+$, and consider its idempotent image $\alpha(g)\in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}S\mathscr{A})^+$ under the Bott isomorphism $\alpha: K_1^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{A})\to K_0^{\rm top}(\mathscr{I}\hat{\otimes}S\mathscr{A})$. Choose any invertible lift $\hat{g}\in(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$ and any idempotent lift $\widehat{\alpha(g)}\in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}S\widehat{\mathscr{R}})^+$. By Lemma 4.6, we have the equality of periodic cyclic homology classes

$$\mathrm{ch}_1^{2q}(\hat{g})\equiv\frac{1}{\sqrt{2\pi i}}\mathrm{cs}_1^{2q}(\widehat{\alpha(g)})\ \in HP_1(\mathscr{A})\ ,$$

hence this equality also holds in $HC_{n-1}(\mathscr{A})$. It follows that $\delta(\mathrm{ch}_1(g))$ is represented by

$$\delta\Big(\frac{1}{\sqrt{2\pi i}}\mathrm{cs}_1^{2q}(\widehat{\alpha(g)})\Big)=\big(p_0,\mathrm{cs}_1^{2q}(\widehat{\alpha(g)})\big)\ .$$

But the idempotent path $\widehat{\alpha(g)}$ evaluated at the endpoints is p_0, so that the pairs $(p_0,\mathrm{cs}_1^{2q}(\widehat{\alpha(g)}))$ and $(p_0,0)$ are equivalent in $MK_n^{\mathscr{I}}(\mathscr{A})$. Hence $\delta\circ\mathrm{ch}_1=0$.
Now let a class $[\theta]\in HC_{n-1}(\mathscr{A})$ be in the kernel of δ. It means that the pair $\delta(\theta)=(p_0,\sqrt{2\pi i}\,\theta)$ is equivalent to $(p_0,0)$. Hence there exists an idempotent $\hat{e}\in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}S\widehat{\mathscr{R}})^+$ and a chain λ such that $\sqrt{2\pi i}\,\theta=\mathrm{cs}_1^{2q}(\hat{e})+\natural\mathbf{d}\lambda$ in $X_{n-1}(\mathscr{R},\mathscr{J})$. By Bott periodicity, there exists an element $[g]\in K_1^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{A})$ such that $\mathrm{cs}_1^{2q}(\hat{e})\equiv\sqrt{2\pi i}\,\mathrm{ch}_1^{2q}(\hat{g})$ in $HC_{n-1}(\mathscr{A})$, where $\hat{g}\in(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$ is any invertible lift of g. Whence the equality of cylic homology classes $[\theta]\equiv\mathrm{ch}_1^{2q}(\hat{g})$. It follows that $\mathrm{Ker}\,\delta=\mathrm{Im}\,\mathrm{ch}_1$.

Exactness at $MK_n^{\mathscr{I}}(\mathscr{A})$: Let $[\theta]\in HC_{n-1}(\mathscr{A})$ be any cyclic homology class. Then $\delta(\theta)=(p_0,\sqrt{2\pi i}\,\theta)$, and $\iota(\delta(\theta))=p_0$ is the zero-class in topological K-theory. Therefore $\iota\circ\delta=0$.
Now let $(\hat{e},\theta)\in MK_n^{\mathscr{I}}(\mathscr{A})$ be in the kernel of ι: it means that $\hat{e}$ is smoothly homotopic to p_0. Hence there exists an idempotent $\hat{f}\in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$, with evaluations $\hat{f}_0=\hat{e}$ and $\hat{f}_1=p_0$, and the pair $(\hat{e},\theta)$ is equivalent to $(p_0,\theta+\mathrm{cs}_1^{2q}(\hat{f}))$. Remark that the odd chain $\theta+\mathrm{cs}_1^{2q}(\hat{f})\in X_{n-1}(\mathscr{R},\mathscr{J})$ is closed (indeed, $\overline{b}\theta=\mathrm{ch}_0^{2q}(\hat{e})$ and $\overline{b}\mathrm{cs}_1^{2q}(\hat{f})=-\mathrm{ch}_0^{2q}(\hat{e})$), and we can write

$$(p_0,\theta+\mathrm{cs}_1^{2q}(\hat{f}))=\delta\Big(\frac{1}{\sqrt{2\pi i}}(\theta+\mathrm{cs}_1^{2q}(\hat{f}))\Big)\ .$$

It follows that $\mathrm{Ker}\,\iota=\mathrm{Im}\,\delta$.

Exactness at $K_0^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{A})$: Let $(\hat{e},\theta)\in MK_n^{\mathscr{I}}(\mathscr{A})$ represent any multiplicative K-theory class. Then $\mathrm{ch}_0^{2q}(\hat{e})\equiv 0$ in non-commutative de Rham homology $HD_{n-2}(\mathscr{A})$, and therefore also in $HC_{n-2}(\mathscr{A})$. Thus, the Chern character of $\iota(\hat{e},\theta)=e$ vanishes and $\mathrm{ch}_0\circ\iota=0$.

Now let $[e] \in K_0^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A})$ be in the kernel of ch_0. We know from section 2 that the natural map $HD_{n-2}(\mathscr{A}) \to HC_{n-2}(\mathscr{A})$ is injective, so that $\mathrm{ch}_0^{2q}(\hat{e}) \equiv 0$ in $HD_{n-2}(\mathscr{A})$ for any idempotent lift $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$. Hence there exists an odd chain $\theta \in X_{n-1}(\mathscr{R},\mathscr{J})$ such that $\mathrm{ch}_0^{2q}(\hat{e}) = \overline{b}\theta$, and $e = \iota(\hat{e},\theta)$. This shows that $\mathrm{Ker}\,\mathrm{ch}_0 = \mathrm{Im}\,\iota$.

Let us now show the independence of multiplicative K-theory upon the choice of degree $2q+1 \geq p$. To this end, write $(\hat{e},\theta)^q \in MK_n^{\mathscr{I}}(\mathscr{A})^q$ for a representative of a class obtained using the higher Chern character $\mathrm{ch}_0^{2q}(\hat{e}) = \overline{b}\theta$ of degree $2q$. We shall construct a map $MK_n^{\mathscr{I}}(\mathscr{A})^q \to MK_n^{\mathscr{I}}(\mathscr{A})^{q+1}$ which turns out to be an isomorphism. Let $\rho : \mathscr{I}\mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\mathscr{A}$ be the canonical p-summable quasihomomorphism of even degree considered in section 4, for the construction of the higher Chern characters. Recall that $\mathscr{E} = \mathscr{I}^+\hat{\otimes}\mathscr{A}$ with extension $\mathscr{M} = \mathscr{I}^+\hat{\otimes}\mathscr{R}$. From Proposition 3.10, we know that the chain maps $\chi^{2q} : \widehat{\Omega}(\widehat{\mathscr{M}_+^s}) \to X(\widehat{\mathscr{R}})$ associated to ρ are related in successive degrees by the transgression formula involving the eta-cochain $\widehat{\chi}^{2q} - \widehat{\chi}^{2q+2} = [\partial, \widehat{\eta}^{2q+1}]$. More precisely:

$$
\begin{aligned}
\widehat{\chi}_0^{2q} - \widehat{\chi}_0^{2q+2} &= \overline{b}\widehat{\eta}_1^{2q+1} + \widehat{\eta}_0^{2q+1}(b+B)\ ,\\
\widehat{\chi}_1^{2q} - \widehat{\chi}_1^{2q+2} &= \natural\mathbf{d}\widehat{\eta}_0^{2q+1} + \widehat{\eta}_1^{2q+1}(b+B)\ .
\end{aligned}
$$

The evaluation of the first equation on the $(b+B)$-cycle $\mathrm{ch}_*(\hat{e}) \in \widehat{\Omega}^+(\mathscr{I}\widehat{\mathscr{R}})$ yields (see section 4; we also omit reference to the inclusion homomorphism $\iota_* : \mathscr{I}\widehat{\mathscr{R}} \hookrightarrow \left(\begin{smallmatrix} \mathscr{I}\widehat{\mathscr{R}} & 0 \\ 0 & 0 \end{smallmatrix}\right) \subset \widehat{\mathscr{M}_+^s}$)

$$
\mathrm{ch}_0^{2q}(\hat{e}) - \mathrm{ch}_0^{2q+2}(\hat{e}) = \overline{b}\big(\widehat{\eta}_1^{2q+1}\mathrm{ch}_{2q+2}(\hat{e})\big)\ ,
$$

with $\mathrm{ch}_0^{2q}(\hat{e}) = \widehat{\chi}_0^{2q}\mathrm{ch}_{2q}(\hat{e})$ by Lemma 4.5. Therefore, we guess that the map $MK_n^{\mathscr{I}}(\mathscr{A})^q \to MK_n^{\mathscr{I}}(\mathscr{A})^{q+1}$ should send a pair $(\hat{e},\theta)^q$ to the pair $(\hat{e},\theta')^{q+1}$ with $\theta' = \theta - \widehat{\eta}_1^{2q+1}\mathrm{ch}_{2q+2}(\hat{e})$. Indeed, one has the correct transgression relation

$$
\mathrm{ch}_0^{2q+2}(\hat{e}) = \mathrm{ch}_0^{2q}(\hat{e}) - \overline{b}(\widehat{\eta}_1^{2q+1}\mathrm{ch}_{2q+2}(\hat{e})) = \overline{b}\theta'
$$

in the complex $X_{n-1}(\mathscr{R},\mathscr{J})$. Moreover, this map is well-defined at the level of equivalence classes: let $(\hat{e}_0,\theta_0)^q$ and $(\hat{e}_1,\theta_1)^q$ be two equivalent pairs. Then there exists an interpolating idempotent $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$ and a chain λ such that $\theta_1 - \theta_0 = \mathrm{cs}_1^{2q}(\hat{e}) + \natural\mathbf{d}\lambda$. Hence the respective images $(\hat{e}_0,\theta_0')^{q+1}$ and $(\hat{e}_1,\theta_1')^{q+1}$ verify

$$
\theta_1' - \theta_0' = \theta_1 - \theta_0 - \widehat{\eta}_1^{2q+1}\left(\mathrm{ch}_{2q+2}(\hat{e}_1) - \mathrm{ch}_{2q+2}(\hat{e}_0)\right)\ .
$$

But we know the transgression relation (69)

$$
\mathrm{ch}_{2q+2}(\hat{e}_1) - \mathrm{ch}_{2q+2}(\hat{e}_0) = B\mathrm{cs}_{2q+1}(\hat{e}) + b\mathrm{cs}_{2q+3}(\hat{e})\ ,
$$

so that, using the identities $\widehat{\chi}_1^{2q}\mathrm{cs}_{2q+1}(\hat{e}) = (\natural\mathbf{d}\widehat{\eta}_0^{2q+1} + \widehat{\eta}_1^{2q+1}B)\mathrm{cs}_{2q+1}(\hat{e})$ and $-\widehat{\chi}_1^{2q+2}\mathrm{cs}_{2q+3}(\hat{e}) = \widehat{\eta}_1^{2q+1}b\mathrm{cs}_{2q+3}(\hat{e})$ one gets (recall $\mathrm{cs}_1^{2q}(\hat{e}) = \widehat{\chi}_1^{2q}\mathrm{cs}_{2q+1}(\hat{e})$)

$$\begin{aligned}\theta_1' - \theta_0' &= \mathrm{cs}_1^{2q}(\hat{e}) + \natural\mathbf{d}\lambda \\ &\quad -\widehat{\chi}_1^{2q}\mathrm{cs}_{2q+1}(\hat{e}) + \widehat{\chi}_1^{2q+2}\mathrm{cs}_{2q+3}(\hat{e}) + \natural\mathbf{d}\big(\widehat{\eta}_0^{2q+1}\mathrm{cs}_{2q+1}(\hat{e})\big) \\ &= \mathrm{cs}_1^{2q+2}(\hat{e}) + \natural\mathbf{d}\big(\lambda + \widehat{\eta}_0^{2q+1}\mathrm{cs}_{2q+1}(\hat{e})\big)\ .\end{aligned}$$

Hence $(\hat{e}_0, \theta_0')^{q+1}$ and $(\hat{e}_1, \theta_1')^{q+1}$ are equivalent in $MK_n^{\mathscr{I}}(\mathscr{A})^{q+1}$. It remains to show that the map $MK_n^{\mathscr{I}}(\mathscr{A})^q \to MK_n^{\mathscr{I}}(\mathscr{A})^{q+1}$ is an isomorphism. Consider the following diagram:

$$\begin{tikzcd} HC_{n-1}(\mathscr{A}) \arrow[r, "\delta"] \arrow[d, equal] & MK_n^{\mathscr{I}}(\mathscr{A})^q \arrow[r] \arrow[d] & K_0^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \arrow[d, equal] \\ HC_{n-1}(\mathscr{A}) \arrow[r, "\delta"] & MK_n^{\mathscr{I}}(\mathscr{A})^{q+1} \arrow[r] & K_0^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \end{tikzcd}$$

For any cyclic homology class $[\theta] \in HC_{n-1}(\mathscr{A})$ represented by a closed chain θ, one has $\delta(\theta) = (p_0, \sqrt{2\pi i}\,\theta)^q$ in $MK_n^{\mathscr{I}}(\mathscr{A})^q$. But observe that $\widehat{\eta}_1^{2q+1}\mathrm{ch}_{2q+2}(p_0) = 0$, so that $(p_0, \sqrt{2\pi i}\,\theta)^q$ is mapped to $(p_0, \sqrt{2\pi i}\,\theta)^{q+1}$ in $MK_n^{\mathscr{I}}(\mathscr{A})^{q+1}$. Hence the left square is commutative. Moreover the right square is obviously commutative. The isomorphism $MK_n^{\mathscr{I}}(\mathscr{A})^q \cong MK_n^{\mathscr{I}}(\mathscr{A})^{q+1}$ then follows from the five-lemma.

The negative Chern character $\mathrm{ch}_n : MK_n^{\mathscr{I}}(\mathscr{A})^q \to HN_n(\mathscr{A})$ is also independent of q, and compatible with the SBI exact sequence. Indeed if $(\hat{e}, \theta)^q$ is a representative of a class in $MK_n^{\mathscr{I}}(\mathscr{A})^q$, one has by definition

$$\mathrm{ch}_n(\hat{e}, \theta)^q = \mathrm{ch}_0^{2q}(\hat{e}) - \overline{b}\tilde{\theta}\ ,$$

where $\tilde{\theta} \in \widehat{X}(\mathscr{R}, \mathscr{J})$ is an arbitrary lifting of θ. First remark that ch_n is well-defined at the level of equivalence classes: if $(\hat{e}_0, \theta_0)^q$ and $(\hat{e}_1, \theta_1)^q$ are equivalent, there exists an interpolation $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$ and a chain λ such that $\theta_1 - \theta_0 = \mathrm{cs}_1^{2q}(\hat{e}) + \natural\mathbf{d}\lambda$ in $X_{n-1}(\mathscr{R}, \mathscr{J})$. Let $\tilde{\theta}_0$, $\tilde{\theta}_1$ and $\tilde{\lambda}$ be arbitrary liftings; then there exists a chain $\mu \in F^{n-1}\widehat{X}(\mathscr{R}, \mathscr{J})$ such that $\tilde{\theta}_1 - \tilde{\theta}_0 = \mathrm{cs}_1^{2q}(\hat{e}) + \natural\mathbf{d}\tilde{\lambda} + \mu$ in $\widehat{X}(\mathscr{R}, \mathscr{J})$. Hence the difference

$$\begin{aligned}\mathrm{ch}_n(\hat{e}_1, \theta_1)^q - \mathrm{ch}_n(\hat{e}_0, \theta_0)^q &= \mathrm{ch}_0^{2q}(\hat{e}_1) - \mathrm{ch}_0^{2q}(\hat{e}_0) - \overline{b}(\tilde{\theta}_1 - \tilde{\theta}_0) \\ &= \overline{b}\mathrm{cs}_1^{2q}(\hat{e}) - \overline{b}(\mathrm{cs}_1^{2q}(\hat{e}) + \natural\mathbf{d}\tilde{\lambda} + \mu) = -\overline{b}\mu\end{aligned}$$

is a coboundary of the subcomplex $F^{n-1}\widehat{X}(\mathscr{R}, \mathscr{J})$, and the Chern character $\mathrm{ch}_n : MK_n^{\mathscr{I}}(\mathscr{A})^q \to HN_n(\mathscr{A})$ is well-defined. Now if $(\hat{e}, \theta)^q \in MK_n^{\mathscr{I}}(\mathscr{A})^q$ is any class, its image in $MK_n^{\mathscr{I}}(\mathscr{A})^{q+1}$ is represented by $(\hat{e}, \theta')^{q+1}$ with $\theta' =$

$\theta - \widehat{\eta}_1^{2q+1}\mathrm{ch}_{2q+2}(\hat{e})$. One has

$$\begin{aligned}\mathrm{ch}_n(\hat{e},\theta')^{q+1} &= \mathrm{ch}_0^{2q+2}(\hat{e}) - \overline{b}(\tilde{\theta} - \widehat{\eta}_1^{2q+1}\mathrm{ch}_{2q+2}(\hat{e}))\\ &= \mathrm{ch}_0^{2q+2}(\hat{e}) - \overline{b}(\tilde{\theta}) + \widehat{\chi}_0^{2q}\mathrm{ch}_{2q}(\hat{e}) - \widehat{\chi}_0^{2q+2}\mathrm{ch}_{2q+2}(\hat{e})\\ &= \mathrm{ch}_0^{2q}(\hat{e}) - \overline{b}(\tilde{\theta}) \;=\; \mathrm{ch}_n(\hat{e},\theta)^q\ ,\end{aligned}$$

and the negative Chern character does not depend on the degree q. Finally, for any cyclic homology class $[\theta] \in HC_{n-1}(\mathscr{A})$, one has

$$\mathrm{ch}_n(\delta(\theta)) = \mathrm{ch}_n(p_0, \sqrt{2\pi i}\,\theta) = -\sqrt{2\pi i}\,\overline{b}(\tilde{\theta})\ ,$$

which shows the commutativity of the middle square (76). The compatibility between the negative Chern character and the periodic Chern character on topological K-theory is obvious, whence the commutativity of the right square (76).

Concerning the independence of $MK_n^{\mathscr{I}}(\mathscr{A})$ with respect to the choice of quasi-free extension $\mathscr{R}$, it suffices to consider the universal extension $0 \to J\mathscr{A} \to T\mathscr{A} \to \mathscr{A} \to 0$ together with the classifying homomorphisms $T\mathscr{A} \to \mathscr{R}$ and $J\mathscr{A} \to \mathscr{J}$. The various Chern characters and Chern-Simons forms constructed in $\widehat{X}(\mathscr{R},\mathscr{J})$ are obtained from the universal ones in $\widehat{X}(T\mathscr{A}, J\mathscr{A})$ by applying the chain map $\widehat{X}(T\mathscr{A}, J\mathscr{A}) \to \widehat{X}(\mathscr{R},\mathscr{J})$, which we know is a homotopy equivalence compatible with the adic filtrations. Once again the conclusion follows from the five-lemma.
The case of odd degree $n = 2k+1$ is established along the same lines, replacing idempotents by invertibles. ∎

Before ending this section we need to establish the invariance of topological and multiplicative K-theory with respect to adjoint actions of multipliers on the p-summable Fréchet m-algebra $\mathscr{I}$. We say that U is a multiplier if it defines continuous linear maps (left and right multiplications) $x \mapsto Ux$ and $x \mapsto xU$ on $\mathscr{I}$, which commute and fulfill

$$\begin{aligned}&i)\quad U(xy) = (Ux)y\ ,\ (xU)y = x(Uy)\ ,\ (xy)U = x(yU) \qquad \forall x,y \in \mathscr{I}\ ,\\ &ii)\quad \mathrm{Tr}([U,\mathscr{I}^p]) = 0\ .\end{aligned}$$

U is invertible if there exists a multiplier U^{-1} such that the compositions UU^{-1} and $U^{-1}U$ induce the identity on $\mathscr{I}$, while left and right multiplications by U and U^{-1} commute. In this case the adjoint action of U defined by $\mathrm{Ad}\,U(x) = U^{-1}xU$ is a continuous automorphism of $\mathscr{I}$ preserving the trace on $\mathscr{I}^p$. If $\mathscr{A}$ is any Fréchet m-algebra, the adjoint action of U extends to the tensor product $\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A}$ by acting trivially on the factors $\mathscr{K}$ and $\mathscr{A}$, thus defines an automorphism of $K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A})$. Similarly if $0 \to \mathscr{J} \to \mathscr{R} \to \mathscr{A} \to 0$ is a quasi-free extension, and $(\hat{e},\theta)$ (resp. $(\hat{g},\theta)$) represents a multiplicative

K-theory class of even (resp. odd) degree, the adjoint action of U extends to an automorphism of the pro-algebra $\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}$ and define maps

$$\mathrm{Ad}\,U: \quad (\hat{e},\theta)\mapsto (U^{-1}\hat{e}U,\theta)\ , \quad (\hat{g},\theta)\mapsto (U^{-1}\hat{g}U,\theta)\ . \tag{77}$$

The images represent multiplicative K-theory classes because the invariance of the trace implies $\mathrm{ch}_0^{2q}(U^{-1}\hat{e}U)=\mathrm{ch}_0^{2q}(\hat{e})=\overline{b}\theta$ and $\mathrm{ch}_0^{2q}(U^{-1}\hat{g}U)=\mathrm{ch}_0^{2q}(\hat{g})=\natural\mathbf{d}\theta$. The adjoint action is actually well-defined at the level of K-theory:

Lemma 5.5 *Let U be an invertible multiplier of $\mathscr{I}$. Then the adjoint action $\mathrm{Ad}\,U$ induces the identity on $K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A})$ and $MK_n^{\mathscr{I}}(\mathscr{A})$.*

Proof: First we show that an idempotent $e\in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})^+$, with $e-p_0\in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})$, is smoothly homotopic to its adjoint $U^{-1}eU$. Introduce the idempotent $f_0=\left(\begin{smallmatrix} e & 0\\ 0 & p_0\end{smallmatrix}\right)\in M_4(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})^+$, and choose a smooth real-valued function $\xi\in C^\infty[0,1]$ such that $\xi(0)=0$ and $\xi(1)=\pi/2$. We define a path W of invertible multipliers of $M_4(\mathscr{K}\hat{\otimes}\mathscr{I}\mathscr{A})$ by means of the formula

$$W=R^{-1}\begin{pmatrix}1 & 0\\ 0 & U\end{pmatrix}R\ ,\qquad R=\begin{pmatrix}\cos\xi & \sin\xi\\ -\sin\xi & \cos\xi\end{pmatrix}\ ,$$

where each entry should be viewed as a 2×2 block matrix. Hence, W commutes with the matrix $\tilde{p}_0=\left(\begin{smallmatrix} p_0 & 0\\ 0 & p_0\end{smallmatrix}\right)$. The smooth path of idempotents $f=W^{-1}f_0W$ thus provides an interpolation between f_0 and $f_1=\left(\begin{smallmatrix} U^{-1}eU & 0\\ 0 & p_0\end{smallmatrix}\right)$. Put in another way, cfc interpolates the K-theoretic sums $e+p_0$ and $U^{-1}eU+p_0$. This shows that e and $U^{-1}eU$ define the same topological K-theory class.
Now suppose that $(\hat{e},\theta)\in MK_n^{\mathscr{I}}(\mathscr{A})$ represents a multiplicative K-theory class, with $\hat{e}\in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$, $\theta\in X_{n-1}(\mathscr{R},\mathscr{J})$ and $\mathrm{ch}_0^{2q}(\hat{e})=\overline{b}\theta$. We define as before $\hat{f}_0=\left(\begin{smallmatrix} \hat{e} & 0\\ 0 & p_0\end{smallmatrix}\right)$, and $\hat{f}=W^{-1}\hat{f}_0W$ provides an interpolation between $\hat{f}_0$ and $\hat{f}_1=\left(\begin{smallmatrix} U^{-1}\hat{e}U & 0\\ 0 & p_0\end{smallmatrix}\right)$. If $s:C^\infty[0,1]\to\Omega^1[0,1]$ denotes the differential over $[0,1]$ and $\mathbf{d}:\widehat{\mathscr{R}}\to\Omega^1\widehat{\mathscr{R}}$ the noncommutative differential, the Chern-Simons form (71) associated to $c\hat{f}c$ reads

$$\mathrm{cs}_1^{2q}(c\hat{f}c)=\int_0^1\mathrm{Tr}\natural(-2\hat{f}+1)\sum_{i=0}^{q}(\hat{f}-\tilde{p}_0)^{2i}s\hat{f}(\hat{f}-\tilde{p}_0)^{2(q-i)}\mathbf{d}\hat{f}\ .$$

One has $\mathbf{d}\hat{f}=W^{-1}\mathbf{d}\hat{f}_0W$ and $s\hat{f}=W^{-1}(-sWW^{-1}\hat{f}_0+\hat{f}_0sWW^{-1})W$, hence

$$\begin{aligned}
&\mathrm{Tr}\natural(-2\hat{f}+1)\sum_{i=0}^{q}(\hat{f}-\tilde{p}_0)^{2i}s\hat{f}(\hat{f}-\tilde{p}_0)^{2(q-i)}\mathbf{d}\hat{f}\\
=\ &-\mathrm{Tr}\natural(-2\hat{f}_0+1)\sum_{i=0}^{q}(\hat{f}_0-\tilde{p}_0)^{2i}sWW^{-1}\hat{f}_0(\hat{f}_0-\tilde{p}_0)^{2(q-i)}\mathbf{d}\hat{f}_0\\
&+\mathrm{Tr}\natural(-2\hat{f}_0+1)\sum_{i=0}^{q}(\hat{f}_0-\tilde{p}_0)^{2i}\hat{f}_0sWW^{-1}(\hat{f}_0-\tilde{p}_0)^{2(q-i)}\mathbf{d}\hat{f}_0
\end{aligned}$$

Observe that $\mathrm{Tr}\natural$ is a trace. In the first term of the r.h.s., we can use the identity $\hat f_0(\hat f_0-\tilde p_0)^{2(q-i)}=(\hat f_0-\tilde p_0)^{2(q-i)}\hat f_0$ which holds for any two idempotents $\hat f_0$ and $\tilde p_0$, and then $\hat f_0\mathbf{d}\hat f_0(-2\hat f_0+1)=\hat f_0\mathbf{d}\hat f_0$. In the second term of the r.h.s., we simply write $(-2\hat f_0+1)(\hat f_0-\tilde p_0)^{2i}\hat f_0=(-2\hat f_0+1)\hat f_0(\hat f_0-\tilde p_0)^{2i}=-\hat f_0(\hat f_0-\tilde p_0)^{2i}$. Hence we arrive at

$$
\begin{aligned}
&\mathrm{Tr}\natural(-2\hat f+1)\sum_{i=0}^{q}(\hat f-\tilde p_0)^{2i}s\hat f(\hat f-\tilde p_0)^{2(q-i)}\mathbf{d}\hat f\\
&= -\sum_{i=0}^{q}\mathrm{Tr}\natural\,(\hat f_0-\tilde p_0)^{2i}sWW^{-1}(\hat f_0-\tilde p_0)^{2(q-i)}\hat f_0\mathbf{d}\hat f_0\\
&\qquad -\sum_{i=0}^{q}\mathrm{Tr}\natural\,(\hat f_0-\tilde p_0)^{2i}sWW^{-1}(\hat f_0-\tilde p_0)^{2(q-i)}\mathbf{d}\hat f_0\,\hat f_0\\
&= -\sum_{i=0}^{q}\mathrm{Tr}\natural\,sWW^{-1}(\hat f_0-\tilde p_0)^{2i}\mathbf{d}\hat f_0(\hat f_0-\tilde p_0)^{2(q-i)}
\end{aligned}
$$

by the idempotent property $\hat f_0\mathbf{d}\hat f_0+\mathbf{d}\hat f_0\,\hat f_0=\mathbf{d}\hat f_0$. It remains to show that the latter sum is a boundary:

$$
-\sum_{i=0}^{q}\mathrm{Tr}\natural\,sWW^{-1}(\hat f_0-\tilde p_0)^{2i}\mathbf{d}\hat f_0(\hat f_0-\tilde p_0)^{2(q-i)}=\natural\mathbf{d}\big(\mathrm{Tr}\,sWW^{-1}(\hat f_0-\tilde p_0)^{2q+1}\big)\ .
$$

Indeed $\mathbf{d}$ anticommutes with sWW^{-1}, and $\mathbf{d}(\hat f_0-\tilde p_0)=\mathbf{d}\hat f_0$. Hence if we can show that the terms $\mathrm{Tr}\natural\,sWW^{-1}(\hat f_0-\tilde p_0)^{2i+1}\mathbf{d}\hat f_0(\hat f_0-\tilde p_0)^{2j+1}$ vanish, the conclusion follows. Since $\hat f_0\mathbf{d}\hat f_0\hat f_0=0$, we can write

$$
\begin{aligned}
&\mathrm{Tr}\natural\,sWW^{-1}(\hat f_0-\tilde p_0)^{2i+1}\mathbf{d}\hat f_0(\hat f_0-\tilde p_0)^{2j+1}\\
&\quad= \mathrm{Tr}\natural\,sWW^{-1}(\hat f_0-\tilde p_0)^{2i}(-\hat f_0\mathbf{d}\hat f_0\tilde p_0-\tilde p_0\mathbf{d}\hat f_0\hat f_0+\tilde p_0\mathbf{d}\hat f_0\tilde p_0)(\hat f_0-\tilde p_0)^{2j}\\
&\quad= \mathrm{Tr}\natural\,sWW^{-1}(\hat f_0-\tilde p_0)^{2i}(-\hat f_0\mathbf{d}\hat f_0\tilde p_0-\mathbf{d}\hat f_0\hat f_0\tilde p_0+\mathbf{d}\hat f_0\tilde p_0)(\hat f_0-\tilde p_0)^{2j}\\
&\quad= 0\ ,
\end{aligned}
$$

where we used the fact that $\tilde p_0$ commutes with sWW^{-1} and the even powers of $\hat f_0-\tilde p_0$. Hence $\mathrm{cs}_1^{2q}(c\hat f c)\equiv 0 \bmod \natural\mathbf{d}$, which shows that the pairs $(\hat e,\theta)$ and $(\widehat{U}^{-1}\hat e U,\theta)$ are equivalent. The adjoint action of U on multiplicative K-theory groups in even degrees is thus the identity.

One proceeds in the same fashion with odd groups. Let $g\in(\mathscr{K}\hat\otimes\mathscr{I}\mathscr{A})^+$ be an invertible such that $g-1\in\mathscr{K}\hat\otimes\mathscr{I}\mathscr{A}$. Introduce $u_0=\left(\begin{smallmatrix} g & 0\\ 0 & 1\end{smallmatrix}\right)$ and the invertible path $u=W^{-1}u_0W$, where $W=R^{-1}\left(\begin{smallmatrix} 1 & 0\\ 0 & U\end{smallmatrix}\right)R$ is now viewed as a path of invertible multipliers of $M_2(\mathscr{K}\hat\otimes\mathscr{I}\mathscr{A})$. Hence u interpolates between u_0 and $u_1=\left(\begin{smallmatrix} U^{-1}gU & 0\\ 0 & 1\end{smallmatrix}\right)$. This shows that g and $U^{-1}gU$ define the same topological K-theory class.

Now suppose that $(\hat g,\theta)\in MK_n^{\mathscr{I}}(\mathscr{A})$ represents a multiplicative K-theory class, with $\hat{g} \in (\mathscr{K}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$, $\theta \in X_{n-1}(\mathscr{R}, \mathscr{J})$ and $\mathrm{ch}_1^{2q}(\hat{g}) = \natural \mathbf{d}\theta$. We define $\hat{u}_0 = \left(\begin{smallmatrix} \hat{g} & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $\hat{u} = W^{-1}\hat{u}_0 W$ provides an interpolation between $\hat{u}_0$ and $\hat{u}_1 = \left(\begin{smallmatrix} U^{-1}\hat{g}U & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The Chern-Simons form (71) associated to $\hat{u}$ reads

$$\mathrm{cs}_0^{2q}(\hat{u}) = \frac{1}{\sqrt{2\pi i}}\frac{(q!)^2}{(2q)!}\int_0^1 \mathrm{Tr}\,\hat{u}^{-1}[(\hat{u}-1)(\hat{u}^{-1}-1)]^q s\hat{u}\ .$$

Using $s\hat{u} = W^{-1}(-sWW^{-1}\hat{u}_0 + \hat{u}_0 sWW^{-1})W$, one gets

$$\begin{aligned} \mathrm{Tr}\,\hat{u}^{-1}[(\hat{u}-1)(\hat{u}^{-1}-1)]^q s\hat{u} \quad &= \quad -\mathrm{Tr}\,\hat{u}_0^{-1}[(\hat{u}_0-1)(\hat{u}_0^{-1}-1)]^q sWW^{-1}\hat{u}_0 \\ &\qquad +\mathrm{Tr}\,\hat{u}_0^{-1}[(\hat{u}_0-1)(\hat{u}_0^{-1}-1)]^q \hat{u}_0 sWW^{-1} \\ &\equiv \quad 0 \bmod \overline{b} \end{aligned}$$

Hence $\mathrm{cs}_0^{2q}(\hat{u}) \equiv 0 \bmod \overline{b}$ and the pair $(\widehat{U}^{-1}\hat{g}U, \theta)$ is equivalent to $(\hat{g}, \theta)$. The adjoint action of U on the odd multiplicative K-theory groups is therefore the identity. ∎

Example 5.6 Take $\mathscr{A} = \mathbb{C}$ and $\mathscr{I} = \mathscr{L}^p(H)$ a Schatten ideal. It is known that $K_0^{\mathrm{top}}(\mathscr{I}) = \mathbb{Z}$ and $K_1^{\mathrm{top}}(\mathscr{I}) = 0$. Furthermore $HC_n(\mathbb{C}) = \mathbb{C}$ for $n = 2k \geq 0$ and vanishes otherwise. Hence the exact sequence yields

$$MK_n^{\mathscr{I}}(\mathbb{C}) = \left\{ \begin{array}{ll} \mathbb{Z} & n \leq 0 \text{ even} \\ \mathbb{C}^\times & n > 0 \text{ odd} \\ 0 & \text{otherwise} \end{array} \right.$$

The multiplicative K-theory of $\mathbb{C}$ is the natural target for index maps in even degree, and for regulator maps in odd degree (see [6] and Example 6.4).

Multiplicative K-theory has close connections with higher algebraic K-theory [16, 29]. In fact there exists a morphism $K_n^{\mathrm{alg}}(\mathscr{A}) \to MK_n^{\mathscr{I}}(\mathscr{A})$ in all degrees, and composition with the negative Chern character coincides with the Jones-Goodwillie map $K_n^{\mathrm{alg}}(\mathscr{A}) \to HN_n(\mathscr{A})$ [15]. See [7] for an exact sequence relating topological and algebraic K-theories of locally convex algebras stabilized by operator ideals.

6 Riemann-Roch-Grothendieck theorem

In this section we construct direct images of topological and multiplicative K-theory under quasihomomorphisms and show their compatibility with the K-theory and cyclic homology exact sequences. This provides a noncommutative version of the Riemann-Roch-Grothendieck theorem. If $\mathscr{I}$ is a p-summable Fréchet m-algebra, with trace $\mathrm{Tr} : \mathscr{I}^p \to \mathbb{C}$, the tensor product $\mathscr{I}\hat{\otimes}\mathscr{I}$ is in a natural way a p-sumable algebra with trace $\mathrm{Tr}\hat{\otimes}\mathrm{Tr}$. We demand that $\mathscr{I}$ is provided with an external product as follows.

Definition 6.1 *A p-summable Fréchet m-algebra $\mathscr{I}$ is* multiplicative *if there exists a continuous algebra homomorphism (external product)*

$$\boxtimes : \mathscr{I}\hat{\otimes}\mathscr{I} \to \mathscr{I}$$

such that the composition $\mathrm{Tr}\circ\boxtimes$ *coincides with the trace* $\mathrm{Tr}\hat{\otimes}\mathrm{Tr}$ *on* $(\mathscr{I}\hat{\otimes}\mathscr{I})^p$. *Two external products* $\boxtimes$ *and* $\boxtimes'$ *are* equivalent *if there exists an invertible multiplier U of $\mathscr{I}$ such that* $\boxtimes' = \mathrm{Ad}\,U\circ\boxtimes$ *on* $\mathscr{I}\hat{\otimes}\mathscr{I}$.

Hence if $\mathscr{I}$ is multiplicative the homomorphism $\boxtimes$ induces additive maps $K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{A}) \to K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A})$ and $MK_n^{\mathscr{I}\hat{\otimes}\mathscr{I}}(\mathscr{A}) \to MK_n^{\mathscr{I}}(\mathscr{A})$, clearly compatible with the commutative diagram of Proposition 5.4. Moreover two equivalent products induce the same maps, the adjoint action $\mathrm{Ad}\,U$ being trivial on K-theory by Lemma 5.5. In practice the algebra $\mathscr{I}$ often arises with external products defined only modulo equivalence:

Example 6.2 Let $\mathscr{I} = \mathscr{L}^p(H)$ be the Schatten ideal of p-summable operators on a separable infinite-dimensional Hilbert space H, provided with the operator trace. The tensor product $\mathscr{L}^p(H)\hat{\otimes}\mathscr{L}^p(H)$ is naturally mapped to the algebra $\mathscr{L}^p(H\otimes H)$, and choosing an isomorphism of Hilbert spaces $H\otimes H\cong H$ allows to identify $\mathscr{L}^p(H\otimes H)$ with $\mathscr{L}^p(H)$ modulo the adjoint action of unitary operators $U\in\mathscr{L}(H)$. The product $\boxtimes : \mathscr{L}^p(H)\hat{\otimes}\mathscr{L}^p(H) \to \mathscr{L}^p(H)$ is thus compatible with the traces, and canonically defined modulo the adjoint action of unitary operators.

Let $\mathscr{A}$ and $\mathscr{B}$ be any Fréchet m-algebras. Let $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B}$ be a quasi-homomorphism of parity p mod 2, and suppose that $\mathscr{I}$ is finitely summable (the exact summability degree is irrelevant for the moment). We want to show that ρ induces an additive map

$$\rho_! : K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to K_{n-p}^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{B}) \qquad \forall n\in\mathbb{Z} \tag{78}$$

provided $\mathscr{I}$ is multiplicative. This is has nothing to do with cyclic homology and we don't need to assume $\mathscr{E}$ admissible. Thanks to Bott periodicity, it is sufficient to define $\rho_!$ on $K_1^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A})$, where it is given by very explicit formulas. Suppose first that p is even. Then ρ is described by a pair of homomorphisms $(\rho_+,\rho_-) : \mathscr{A} \rightrightarrows \mathscr{E}$ which coincide modulo $\mathscr{I}\hat{\otimes}\mathscr{B}$. For any invertible element $g\in(\mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{A})^+$ with $g-1\in\mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{A}$ representing a K-theory class in $K_1^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A})$, one has $\rho_\pm(g)\in(\mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{E})^+$ and $\rho_+(g)-\rho_-(g)\in\mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{B}$, where the homomorphisms ρ_+ and ρ_- are extended to the unitalized algebra $(\mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{A})^+$ by acting trivially on the factor $\mathscr{K}\hat{\otimes}\mathscr{I}$ and preserving the unit. set

$$\rho_!(g) = \rho_+(g)\rho_-(g)^{-1} \ \in (\mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{B})^+ \ . \tag{79}$$

Using the homomorphism $\boxtimes : \mathscr{I}\hat{\otimes}\mathscr{I} \to \mathscr{I}$, we may therefore consider $\rho_!(g)$ as an invertible element of $(\mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{B})^+$ such that $\rho_!(g)-1\in\mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{B}$. It is

clear that the homotopy class of $\rho_!(g)$ only depends on the homotopy class of g, hence the map $\rho_! : K_1^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to K_1^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{B})$ is well-defined.
When p is odd, ρ is a homomorphism $\mathscr{A} \to M_2(\mathscr{E})$ such that the off-diagonal terms lie in $\mathscr{I}\hat{\otimes}\mathscr{B}$. For any invertible element $g \in (\mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{A})^+$ as above, one has $\rho(g) \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{E})^+$ with off-diagonal elements in $\mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{B}$. Set

$$\rho_!(g) = \rho(g)^{-1} p_0 \rho(g) \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{B})^+ \ , \tag{80}$$

where $p_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is the trivial matrix idempotent. Again applying the external product $\boxtimes$ we may consider $\rho_!(g)$ as an idempotent of $M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{B})^+$ such that $\rho_!(g) - p_0 \in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}\hat{\otimes}\mathscr{B})$. The homotopy class only depends on the homotopy class of g and we thus obtain $\rho_! : K_1^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to K_0^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{B})$.
To define $\rho_!$ on $K_0^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{A})$ it suffices to pass to the suspensions $S\mathscr{A} = \mathscr{A}(0,1)$ and $S\mathscr{B} = \mathscr{B}(0,1)$, then apply the pushforward map constructed above $\rho_! : K_1^{\rm top}(\mathscr{I}\hat{\otimes}S\mathscr{A}) \to K_{1-p}^{\rm top}(\mathscr{I}\hat{\otimes}S\mathscr{B})$ with trivial action on the factor $C^\infty(0,1)$. The Bott isomorphisms $K_n^{\rm top}(\mathscr{I}\hat{\otimes}\cdot) \cong K_{n+1}^{\rm top}(\mathscr{I}\hat{\otimes}S\cdot)$ allow to define $\rho_!$ for the original algebras through a *graded*-commutative diagram

$$\begin{tikzcd} K_0^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{A}) \arrow[r, "\sim"] \arrow[d, "\rho_!"'] & K_1^{\rm top}(\mathscr{I}\hat{\otimes}S\mathscr{A}) \arrow[d, "\rho_!"] \\ K_{-p}^{\rm top}(\mathscr{I}\hat{\otimes}\mathscr{B}) \arrow[r, "\sim"] & K_{1-p}^{\rm top}(\mathscr{I}\hat{\otimes}S\mathscr{B}) \end{tikzcd} \tag{81}$$

Note the following subtlety concerning graduations: since $K_n^{\rm top}$ has parity n mod 2 by definition, the Bott isomorphisms are *odd*. As a consequence, when p is also odd, the above diagram must be *anti*-commutative. These conventions are necessary if we want to avoid sign problems with the theorem below.

Now choose a quasi-free extension $0 \to \mathscr{J} \to \mathscr{R} \to \mathscr{B} \to 0$ for $\mathscr{B}$, and suppose that the algebra $\mathscr{E} \triangleright \mathscr{I}\hat{\otimes}\mathscr{B}$ is $\mathscr{R}$-admissible. We impose the following compatibility between the parity of the quasihomomorphism ρ and the summability degree of $\mathscr{I}$: in the even case $\mathscr{I}$ is $(p+1)$-summable with p even, and in the odd case $\mathscr{I}$ is p-summable with p odd (this complicated choice is dictated by the theorem below). In both cases the bivariant Chern character $\mathrm{ch}^p(\rho) \in HC^p(\mathscr{A},\mathscr{B})$ constructed in section 3 induces a map

$$\mathrm{ch}^p(\rho) : HC_n(\mathscr{A}) \to HC_{n-p}(\mathscr{B}) \qquad \forall n \in \mathbb{Z} \ . \tag{82}$$

Combining $\rho_!$ with the bivariant Chern character yields a transformation in multiplicative K-theory, compatible with the diagram (76) of Proposition 5.4. This will be detailed in the proof of the following noncommutative version of the Riemann-Roch-Grothendieck theorem:

Theorem 6.3 *Let $\mathscr{A}$, $\mathscr{B}$ be Fréchet m-algebras, and choose a quasi-free extension $0 \to \mathscr{J} \to \mathscr{R} \to \mathscr{B} \to 0$. Let $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}\mathscr{B}$ be a quasihomomorphism of parity p* mod 2*, where $\mathscr{I}$ is multiplicative and $(p+1)$-summable* *in the even case, p-summable in the odd case. Suppose that $\mathscr{E} \triangleright \mathscr{I}\hat{\otimes}\mathscr{B}$ is $\mathscr{R}$-admissible. Then ρ defines a transformation in multiplicative K-theory $\rho_! : MK_n^{\mathscr{I}}(\mathscr{A}) \to MK_{n-p}^{\mathscr{I}}(\mathscr{B})$ compatible with the K-theory exact sequences for $\mathscr{A}$ and $\mathscr{B}$, whence a graded-commutative diagram*

$$
\begin{tikzcd}
K_{n+1}^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \arrow[r] \arrow[d, "\rho_!"] & HC_{n-1}(\mathscr{A}) \arrow[r] \arrow[d, "\mathrm{ch}^p(\rho)"] & MK_n^{\mathscr{I}}(\mathscr{A}) \arrow[r] \arrow[d, "\rho_!"] & K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \arrow[d, "\rho_!"] \\
K_{n+1-p}^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{B}) \arrow[r] & HC_{n-1-p}(\mathscr{B}) \arrow[r] & MK_{n-p}^{\mathscr{I}}(\mathscr{B}) \arrow[r] & K_{n-p}^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{B})
\end{tikzcd}
\tag{83}
$$

The vertical arrows are invariant under conjugation of quasihomomorphisms; the arrow in topological K-theory $K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to K_{n-p}^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{B})$ is also invariant under homotopy of quasihomomorphisms. Moreover (83) is compatible with the commutative diagram of Theorem 3.13 (with connecting map B rescaled by a factor $-\sqrt{2\pi i}$) after taking the Chern characters $MK_n^{\mathscr{I}} \to HN_n$ and $K_n^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\ .\ ) \to HP_n$.

Proof: As a general rule, the bivariant cyclic cohomomology $HC^p(\mathscr{A},\mathscr{B})$ is described as the cohomology of the complex $\mathrm{Hom}^p(\widehat{X}(T\mathscr{A},J\mathscr{A}),\widehat{X}(\mathscr{R},\mathscr{J}))$ of linear maps of order $\leq p$, where we choose the universal free extension $0 \to J\mathscr{A} \to T\mathscr{A} \to \mathscr{A} \to 0$ for $\mathscr{A}$ and the quasi-free extension $0 \to \mathscr{J} \to \mathscr{R} \to \mathscr{B} \to 0$ for $\mathscr{B}$. By hypothesis, the algebra $\mathscr{E} \triangleright \mathscr{I}\hat{\otimes}\mathscr{B}$ is $\mathscr{R}$-admissible (Definition 3.2), i.e. one has a commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & \mathscr{N} \arrow[r] & \mathscr{M} \arrow[r] & \mathscr{E} \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{I}\hat{\otimes}\mathscr{J} \arrow[r] \arrow[u] & \mathscr{I}\hat{\otimes}\mathscr{R} \arrow[r] \arrow[u] & \mathscr{I}\hat{\otimes}\mathscr{B} \arrow[r] \arrow[u] & 0
\end{tikzcd}
$$

verifying adequate properties with respect to the trace over $\mathscr{I}$. The detailed construction of the pushforward map in multiplicative K-theory $\rho_! : MK_n^{\mathscr{I}}(\mathscr{A}) \to MK_{n-p}^{\mathscr{I}}(\mathscr{B})$ depends on the respective parities of n and p.

I) $n = 2k+1$ IS ODD AND $p = 2q$ IS EVEN. Our first task is to understand the composition of the topological Chern character $\mathrm{ch}_1^p : K_1^{\mathrm{top}}(\mathscr{I}\hat{\otimes}\mathscr{A}) \to HP_1(\mathscr{A})$ with the bivariant Chern character $\mathrm{ch}^p(\rho) \in HC^p(\mathscr{A},\mathscr{B})$. For notational simplicity, we write as usual $\mathscr{I}\mathscr{A}$ for the tensor product $\mathscr{I}\hat{\otimes}\mathscr{A}$. Without loss of generality, we may suppose that an element of $K_1^{\mathrm{top}}(\mathscr{I}\mathscr{A})$ is represented by an invertible $g \in (\mathscr{I}\mathscr{A})^+$ such that $g - 1 \in \mathscr{I}\mathscr{A}$ (indeed the algebra $\mathscr{K}$ of smooth compact operators plays a trivial role in what follows). Since the universal free extension $T\mathscr{A}$ is chosen, we can take the *canonical lift* $\hat{g} \in (\mathscr{I}\widehat{T}\mathscr{A})^+$ which corresponds to the image of g under the canonical linear inclusion $\mathscr{A} \hookrightarrow \widehat{\Omega}^+\mathscr{A} \cong \widehat{T}\mathscr{A}$ as the the subspace of zero-forms. Its inverse is given by the series (57), with $\mathscr{K}$ replaced by $\mathscr{I}$. The p-th higher Chern character of $\hat{g}$ is then represented by the cycle (70)

$$\mathrm{ch}_1^p(\hat{g}) = \frac{1}{\sqrt{2\pi i}} \frac{(q!)^2}{p!} \mathrm{Tr}\natural \hat{g}^{-1}[(\hat{g}-1)(\hat{g}^{-1}-1)]^q \mathbf{d}\hat{g} \ \in \Omega^1 \widehat{T}\mathscr{A}_\natural \ ,$$

Observe that $(p+1)$ powers of $\mathscr{I}$ appear in the products of $(\hat{g}-1)$, $(\hat{g}^{-1}-1)$ and $\mathbf{d}\hat{g}$ hence the trace $\mathrm{Tr} : \mathscr{I}^{p+1} \to \mathbb{C}$ is well-defined. On the other hand, the bivariant Chern character of the quasihomomorphism ρ (section 3) is represented by the composition of chain maps $\mathrm{ch}^p(\rho) = \widehat{\chi}^p \rho_* \gamma : X(\widehat{T}\mathscr{A}) \to \widehat{\Omega}\widehat{T}\mathscr{A} \to \widehat{\Omega}\widehat{\mathscr{M}^s_+} \to X(\widehat{\mathscr{R}})$, hence the composite $\mathrm{ch}^p(\rho) \cdot \mathrm{ch}_1^p(\hat{g})$ requires to compute first the image of $\mathrm{ch}_1^p(\hat{g})$ under the Goodwillie equivalence $\gamma : X(\widehat{T}\mathscr{A}) \to \widehat{\Omega}\widehat{T}\mathscr{A}$. This tricky computation can be simplified as follows. We use the isomorphism $\mathscr{I}\mathscr{A} \cong \mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{A}$ to identify $\hat{g}$ with the invertible element

$$\hat{u} = 1 + e \otimes (\hat{g}-1) \ \in (\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{A})^+ \hookrightarrow (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{T}\mathscr{A})^+ \ ,$$

where e is the unit of $\mathbb{C}$. As usual we regard $\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{A}$ as the subspace of zero-forms of the algebra $\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{T}\mathscr{A} \cong \widehat{\Omega}^+\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\Omega}^+\mathscr{A}$. It is not hard to calculate that the inverse of $\hat{u}$ is given by the series

$$\hat{u}^{-1} = \sum_{i\geq 0} \big((dede)^i \otimes [(\hat{g}^{-1}-1)(\hat{g}-1)]^i + e(dede)^i \otimes [(\hat{g}^{-1}-1)(\hat{g}-1)]^i (\hat{g}^{-1}-1) \big)$$

with the convention $(dede)^0 = 1$. Observe that the power of $\mathscr{I}$ is equal to the power of e in each term of this series. Also, recall that the canonical lift of e is the idempotent

$$\hat{e} = e + \sum_{i\geq 1} \frac{(2i)!}{(i!)^2} (e - \frac{1}{2})(dede)^i \ \in \widehat{T}\mathbb{C} \ .$$

We define the *fundamental class* of degree $p = 2q$ as the trace $[2q] : \widehat{T}\mathbb{C} \to \mathbb{C}$ vanishing on all the differential forms $e(dede)^i$ and $(dede)^i$ except $e(dede)^q$, and normalized so that $[2q]\,\hat{e} = 1$. One thus have

$$[2q]\, e(dede)^q = \frac{(q!)^2}{p!} \ , \qquad [2q]\,(\text{anything else}) = 0 \ .$$

Of course, $[2q]$ is the generator in degree p of the cyclic cohomology of $\mathbb{C}$. The fact that it is a trace over $\widehat{T}\mathbb{C}$ is crucial. Indeed, one finds the identity

$$\mathrm{Tr}\natural[2q]\, \hat{u}^{-1}\mathbf{d}\hat{u} = \frac{(q!)^2}{p!} \mathrm{Tr}\natural \hat{g}^{-1}[(\hat{g}-1)(\hat{g}^{-1}-1)]^q \mathbf{d}\hat{g} \ \in \Omega^1 \widehat{T}\mathscr{A}_\natural \ ,$$

so that the Chern character $\mathrm{ch}_1^p(\hat{g})$ is exactly the cycle $\frac{1}{\sqrt{2\pi i}} \mathrm{Tr}\natural[2q]\, \hat{u}^{-1}\mathbf{d}\hat{u}$. This simplifies drastically the computation of $\gamma \mathrm{ch}_1^p(\hat{g})$. The Goodwillie equivalence γ is explicitly constructed in section 2; it is based on the linear map $\phi : \widehat{T}\mathscr{A} \to$

$\Omega^2\widehat{T}\mathscr{A}$ verifying the properties $\phi(xy) = \phi(x)y + x\phi(y) + \mathbf{d}x\mathbf{d}y$ for all $x, y \in \widehat{T}\mathscr{A}$, and $\phi(a) = 0$ whenever $a \in \mathscr{A}$. We extend ϕ to a linear map

$$\phi : (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{T}\mathscr{A})^+ \to \widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\Omega^2\widehat{T}\mathscr{A}$$

acting by the identity on the factor $\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}$ and setting $\phi(1) = 0$. This implies $\phi(\hat{u}\hat{u}^{-1}) = 0 = \phi(\hat{u})\hat{u}^{-1} + \hat{u}\phi(\hat{u}^{-1}) + \mathbf{d}\hat{u}\mathbf{d}\hat{u}^{-1}$. Moreover $\hat{u}$ lies in $(\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{A})^+$ so that $\phi(\hat{u}) = 0$, and one gets

$$\phi(\hat{u}^{-1}) = -\hat{u}^{-1}\mathbf{d}\hat{u}\mathbf{d}\hat{u}^{-1} \ .$$

Now, extending ϕ in all degrees as in section 2 one gets a linear map ϕ : $\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\Omega^i\widehat{T}\mathscr{A} \to \widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\Omega^{i+2}\widehat{T}\mathscr{A}$ for any $i \in \mathbb{N}$. The following computation is then straightforward (remark that $\mathrm{Tr}[2q]$ is a trace hence cyclic permutations are allowed; moreover the fundamental class $[2q]$ selects $(p+1)$ powers of e, hence of $\mathscr{I}$, so that Tr is well-defined):

$$\gamma(\mathrm{Tr}\natural[2q]\,\hat{u}^{-1}\mathbf{d}\hat{u}) = \sum_{i\geq 0}\mathrm{Tr}[2q]\,\phi^i(\hat{u}^{-1}\mathbf{d}\hat{u}) = \sum_{i\geq 0}(-)^i i!\,\mathrm{Tr}[2q]\,\hat{u}^{-1}\mathbf{d}\hat{u}(\mathbf{d}\hat{u}^{-1}\mathbf{d}\hat{u})^i \ .$$

Hence $\gamma\mathrm{ch}_1^p(\hat{g})$ is equal to this $(b+B)$-cycle over $\widehat{T}\mathscr{A}$, divided by a factor $\sqrt{2\pi i}$. It remains to apply the chain map $\chi^p\rho_* : \widehat{\Omega}\widehat{T}\mathscr{A} \to X(\widehat{\mathscr{R}})$ associated to the quasihomomorphism $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\mathscr{B}$. In 2×2 matrix notation, the image of any $x \in \widehat{T}\mathscr{A}$ under the lifted quasihomomorphism $\rho_* : \widehat{T}\mathscr{A} \to \widehat{\mathscr{M}^s} \triangleright \mathscr{I}^s\widehat{\mathscr{R}}$ and the odd multiplier F read

$$\rho_* x = \begin{pmatrix} x_+ & 0 \\ 0 & x_- \end{pmatrix} \in \widehat{\mathscr{M}^s_+} \ , \qquad F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ ,$$

and the difference $x_+ - x_-$ is therefore an element of the pro-algebra $\mathscr{I}\widehat{\mathscr{R}}$. On the other hand, the odd component of the chain map $\widehat{\chi}^p\rho_*$ evaluated on a $(p+1)$-form $x_0\mathbf{d}x_1\ldots\mathbf{d}x_{p+1}$ is given by Eqs. (34):

$$\frac{q!}{(p+1)!}\sum_{i=1}^{p+1}\tau'\natural(\rho_*x_0[F,\rho_*x_1]\ldots\mathbf{d}(\rho_*x_i)\ldots[F,\rho_*x_{p+1}])$$

where $\tau' = \frac{1}{2}\tau(F[F,\ ])$ is the modified supertrace of even degree. Then we extend canonically ρ_* to a unital homomorphism $(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{T}\mathscr{A})^+ \to (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{M}^s_+})^+$ by taking the identity on the factor $\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}$. One thus has $\rho_*\hat{u} = \left(\begin{smallmatrix} \hat{u}_+ & 0 \\ 0 & \hat{u}_- \end{smallmatrix}\right)$ with $\hat{u}_+ - \hat{u}_- \in \widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{I}\widehat{\mathscr{R}}$. A direct computation gives

$$\begin{aligned} \mathrm{ch}^p(\rho)\,(\mathrm{Tr}\natural[2q]\,\hat{u}^{-1}\mathbf{d}\hat{u}) &= (-)^q q!\,\widehat{\chi}_1^p\rho_*\,\mathrm{Tr}[2q](\hat{u}^{-1}\mathbf{d}\hat{u}(\mathbf{d}\hat{u}^{-1}\mathbf{d}\hat{u})^q) \\ &= \frac{(q!)^2}{p!}\,\mathrm{Tr}\natural[2q]\,\widetilde{u}^{-1}[(\widetilde{u}-1)(\widetilde{u}^{-1}-1)]^q\mathbf{d}\widetilde{u} \ , \end{aligned}$$

where $\widetilde{u} = \hat{u}_+\hat{u}_-^{-1} \in (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{I}\widehat{\mathscr{R}})^+$ may be considered as an invertible element of the pro-algebra $(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$ after applying the homomorphism $\boxtimes : \mathscr{I}\hat{\otimes}\mathscr{I} \to \mathscr{I}$. Dividing by a factor $\sqrt{2\pi i}$, the right-hand-side should be defined as the Chern character $\mathrm{ch}_1^p(\widetilde{u})$, cf. (70). One thus gets the identity

$$\mathrm{ch}^p(\rho)\cdot \mathrm{ch}_1^p(\hat{g}) = \mathrm{ch}_1^p(\widetilde{u})$$

at the level of cycles in $X(\widehat{\mathscr{R}})$. Now, observe that the projection of $\widetilde{u}$ onto the quotient algebra $(\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{B})^+$ is $u_+u_-^{-1} = 1 + e\hat{\otimes}(\rho_+(g)\rho_-(g)^{-1} - 1)$. It corresponds to the direct image $\rho_+(g)\rho_-(g)^{-1} = \rho_!(g)$ by virtue of the isomorphism $(\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{B})^+ \cong (\mathscr{I}\mathscr{B})^+$. Hence we expect that $\mathrm{ch}_1^p(\widetilde{u})$ is homologous to the Chern character of any invertible lift $\widehat{\rho_!(g)} \in (\mathscr{I}\widehat{\mathscr{R}})^+$. To see this, consider an invertible path $\hat{v} \in (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$ connecting homotopically $\hat{v}_0 = \widetilde{u}$ to $\hat{v}_1 = 1+\hat{e}\otimes(\widehat{\rho_!(g)}-1)$, and such that its projection onto $(\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{B}[0,1])^+$ is the constant invertible function $1 + e\otimes(\rho_!(g) - 1)$ over $[0,1]$. Such a path always exists, for example the linear interpolation

$$\hat{v}_t = t\big(1 + \hat{e}\otimes(\widehat{\rho_!(g)} - 1)\big) + (1-t)\widetilde{u}\ , \qquad t\in[0,1] \tag{84}$$

works. Since the evaluation of the fundamental class $[2q]$ on the canonical idempotent lift $\hat{e}$ is the unit, a little computation shows the equality

$$\mathrm{ch}_1^p(\hat{v}_1) = \mathrm{ch}_1^p\big(\widehat{\rho_!(g)}\big) \ \in \Omega^1\widehat{\mathscr{R}}_\natural$$

at the level of cycles. Moreover, the Chern-Simons form associated to $\hat{v}$, defined in analogy with formulas (71)

$$\mathrm{cs}_0^p(\hat{v}) = \frac{1}{\sqrt{2\pi i}}\frac{(q!)^2}{p!}\int_0^1 dt\,\mathrm{Tr}[2q]\,\hat{v}^{-1}[(\hat{v}-1)(\hat{v}^{-1}-1)]^q\frac{\partial\hat{v}}{\partial t}$$

fulfills the transgression relation in the complex $X(\widehat{\mathscr{R}})$

$$\natural\mathbf{d}\mathrm{cs}_0^p(\hat{v}) = \mathrm{ch}_1^p(\hat{v}_1) - \mathrm{ch}_1^p(\hat{v}_0) = \mathrm{ch}_1^p(\widehat{\rho_!(g)}) - \mathrm{ch}_1^p(\widetilde{u})$$

as wanted. We are now in a position to define the map $\rho_!$ on multiplicative K-theory. Let a pair $(\hat{g},\theta)$ represent a class in $MK_n^{\mathscr{I}}(\mathscr{A})$ of odd degree $n = 2k+1$. From Remark 5.3, we know that $\hat{g} \in (\mathscr{I}\widehat{T}\mathscr{A})^+$ can be taken as the *canonical lift* of some invertible element $g \in (\mathscr{I}\mathscr{A})^+$. Then, the transgression θ is a chain of even degree in the quotient complex $X_{n-1}(T\mathscr{A}, J\mathscr{A})$, and the relation $\mathrm{ch}_1^p(\hat{g}) = \natural\mathbf{d}\theta$ holds in $X_{n-1}(T\mathscr{A}, J\mathscr{A})$. We set

$$\boxed{\rho_!(\hat{g},\theta) = \big(\widehat{\rho_!(g)}\ ,\ \mathrm{ch}^p(\rho)\cdot\theta + \mathrm{cs}_0^p(\hat{v})\big) \ \in MK_{n-p}^{\mathscr{I}}(\mathscr{B})} \tag{85}$$

where $\widehat{\rho_!(g)} \in (\mathscr{I}\widehat{\mathscr{R}})^+$ is any invertible lift of $\rho_!(g)$ and $\hat{v}$ is an invertible path constructed as above. Let us explain why this defines a multiplicative K-theory class. First, the bivariant Chern character $\mathrm{ch}^p(\rho) \in HC^p(\mathscr{A},\mathscr{B})$ induces a morphism of quotient complexes $X_{n-1}(T\mathscr{A}, J\mathscr{A}) \to X_{n-p-1}(\mathscr{R}, \mathscr{J})$, hence $\mathrm{ch}^p(\rho)\cdot\theta$ is a well-defined chain of even degree in $X_{n-p-1}(\mathscr{R}, \mathscr{J})$. Regarding also $\mathrm{cs}_0^p(\hat{v})$ as an element of $X_{n-p-1}(\mathscr{R}, \mathscr{J})$, we see that the relation

$$\natural\mathbf{d}(\mathrm{ch}^p(\rho)\cdot\theta + \mathrm{cs}_0^p(\hat{v})) = \mathrm{ch}^p(\rho)\cdot\mathrm{ch}_1^p(\hat{g}) + \mathrm{ch}_1^p\big(\widehat{\rho_!(g)}\big) - \mathrm{ch}_1^p(\widetilde{u}) = \mathrm{ch}_1^p\big(\widehat{\rho_!(g)}\big)$$

holds in this quotient complex, hence $\rho_!(\hat{g},\theta)$ represents a class in $MK_{n-p}^{\mathscr{I}}(\mathscr{B})$. In fact, the latter does not depend on the choice of lifting $\widehat{\rho_!(g)}$, nor on the invertible path $\hat{v}$. This can be proved simultaneously with the fact that the equivalence class of $\rho_!(\hat{g},\theta)$ depends only on the equivalence class of $(\hat{g},\theta)$. To show this, consider two equivalent pairs $(\hat{g}_0,\theta_0)$ and $(\hat{g}_1,\theta_1)$ representing the same element of $MK_n^{\mathscr{I}}(\mathscr{A})$. It means there exists a homotopy $\hat{g}\in(\mathscr{I}\widehat{T}\mathscr{A}[0,1]_x)^+$ between $\hat{g}_0$ and $\hat{g}_1$ (we denote by x the variable of this homotopy, which should not be confused with the variable t used in the definition of the interpolation (84)), and a chain $\lambda\in X_{n-1}(T\mathscr{A}, J\mathscr{A})$ such that $\theta_1-\theta_0=\mathrm{cs}_0^p(\hat{g})+\overline{b}\lambda$. From Remark 5.3, we may suppose that $\hat{g}_0$, $\hat{g}_1$ and $\hat{g}$ are respectively the canonical lifts of invertibles $g_0, g_1\in(\mathscr{I}\mathscr{A})^+$ and $g\in(\mathscr{I}\mathscr{A}[0,1]_x)^+$. By definition one has

$$\rho_!(\hat{g}_i,\theta_i) = \big(\widehat{\rho_!(g_i)}\ ,\ \mathrm{ch}^p(\rho)\cdot\theta_i + \mathrm{cs}_0^p(\hat{v}(g_i))\big)\ ,\quad i=0,1\ ,$$

where $\hat{v}(g_i)\in(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1]_t)^+$ is a choice of invertible path associated to g_i, for example by Eq. (84). Choose an invertible path $\widehat{\rho_!(g)}\in(\mathscr{I}\widehat{\mathscr{R}}[0,1]_x)^+$ interpolating $\widehat{\rho_!(g_0)}$ and $\widehat{\rho_!(g_1)}$: it can be chosen as a lift of the path $\rho_!(g)=\rho_+(g)\rho_-(g)^{-1}$. Our goal is to show that the relation

$$\mathrm{ch}^p(\rho)\cdot(\theta_1-\theta_0)+\mathrm{cs}_0^p(\hat{v}(g_1))-\mathrm{cs}_0^p(\hat{v}(g_0))\equiv\mathrm{cs}_0^p(\widehat{\rho_!(g)}) \bmod \overline{b}$$

holds in $X_{n-p-1}(\mathscr{R}, \mathscr{J})$. As before we identify the canonical lift $\hat{g}$ of g with the invertible element

$$\hat{u}=1+e\otimes(\hat{g}-1)\ \in(\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{A}[0,1]_x)^+\hookrightarrow(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{T}\mathscr{A}[0,1]_x)^+\ ,$$

and we remark that the higher Chern-Simons form $\mathrm{cs}_0^p(\hat{g})$ given by Lemma 4.5, Eqs. (71), can be written as

$$\mathrm{cs}_0^p(\hat{g})=\frac{1}{\sqrt{2\pi i}}\int_0^1\mathrm{Tr}[2q]\,\hat{u}^{-1}s\hat{u}$$

where $s=dx\frac{\partial}{\partial x}$ is the de Rham coboundary acting on the space of differential forms $\Omega[0,1]_x$. It follows that the computation of $\mathrm{ch}^p(\rho)\cdot\mathrm{cs}_0^p(\hat{g})$ requires first to evaluate the Goodwillie equivalence γ on the one-form $\hat{\omega}=\hat{u}^{-1}s\hat{u}$. To this end, we extend ϕ to a linear map

$$\phi:\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\Omega^i\widehat{T}\mathscr{A}\hat{\otimes}\Omega[0,1]_x\to\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\Omega^{i+2}\widehat{T}\mathscr{A}\hat{\otimes}\Omega[0,1]_x$$

acting by the identity on the factors $\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}$ and $\Omega[0,1]_x$. The algebraic property of ϕ implies $\phi(\hat{u}^{-1}s\hat{u}) = \phi(\hat{u}^{-1})s\hat{u} + \hat{u}^{-1}s\phi(\hat{u}) + \mathbf{d}\hat{u}^{-1}\mathbf{d}(s\hat{u})$. From $\phi(\hat{u}) = 0$, $\phi(\hat{u}^{-1}) = -\hat{u}^{-1}\mathbf{d}\hat{u}\mathbf{d}\hat{u}^{-1}$ and $\mathbf{d}\hat{u}^{-1} = -\hat{u}^{-1}\mathbf{d}\hat{u}\hat{u}^{-1}$ we deduce

$$\phi(\hat{\omega}) = -\hat{u}^{-1}\mathbf{d}\hat{u}\mathbf{d}\hat{\omega} \ .$$

Then the image of $[2q]\,\hat{\omega}$ under the Goodwillie equivalence is a straightforward computation, taking into account the tracial property of the fundamental class $\mathrm{Tr}[2q]$ and the fact that $\Omega[0,1]_x$ is a commutative algebra:

$$\gamma(\mathrm{Tr}[2q]\,\hat{\omega}) = \sum_{i\geq 0} \mathrm{Tr}[2q]\phi^i(\hat{\omega}) =$$

$$\mathrm{Tr}[2q]\,\hat{\omega} + \sum_{i\geq 1}(-)^i(i-1)!\sum_{j=0}^{i-1}\mathrm{Tr}[2q]\,\hat{u}^{-1}\mathbf{d}\hat{u}(\mathbf{d}\hat{u}^{-1}\mathbf{d}\hat{u})^j\mathbf{d}\hat{\omega}(\mathbf{d}\hat{u}^{-1}\mathbf{d}\hat{u})^{i-j-1} \ .$$

This is a chain in the bicomplex $\widehat{\Omega}\widehat{T}\mathscr{A}\hat{\otimes}\Omega[0,1]_x$ endowed with the boundary maps $(b+B)$ and s. Its is related to the $(b+B)$-cocycle $\gamma(\mathrm{Tr}\natural[2q]\hat{u}^{-1}\mathbf{d}\hat{u})$ via the descent equation

$$(b+B)\gamma(\mathrm{Tr}[2q]\hat{\omega}) + s\gamma(\mathrm{Tr}\natural[2q]\hat{u}^{-1}\mathbf{d}\hat{u}) = 0 \ ,$$

which can be shown either by direct computation, or simply by observing that $\natural\mathbf{d}(\mathrm{Tr}[2q]\omega) + s(\mathrm{Tr}\natural[2q]\hat{u}^{-1}\mathbf{d}\hat{u}) = 0$ and $\gamma\natural\mathbf{d} = (b+B)\gamma$, $\gamma s = s\gamma$. Next we have to evaluate the image of $\gamma(\mathrm{Tr}[2q]\,\hat{\omega})$ by the chain map $\widehat{\chi}^p\rho_* : \widehat{\Omega}\widehat{T}\mathscr{A} \to X(\widehat{\mathscr{R}})$, whose even component evaluated on a p-form $x_0\mathbf{d}x_1\ldots\mathbf{d}x_p$ over $\widehat{T}\mathscr{A}$ reads

$$\frac{q!}{(p+1)!}\sum_{\lambda\in S_{p+1}}\varepsilon(\lambda)\tau'(\rho_*x_{\lambda(0)}[F,\rho_*x_{\lambda(1)}]\ldots[F,\rho_*x_{\lambda(p)}]) \ .$$

Denote as before $\widetilde{u} = \hat{u}_+\hat{u}_-^{-1} \in (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1]_x)^+$, and $\widetilde{\omega} = \widetilde{u}^{-1}s\widetilde{u}$. One finds:

$$\mathrm{ch}^p(\rho)\,(\mathrm{Tr}[2q]\,\hat{\omega}) = \frac{(q!)^2}{(p+1)!}\,\mathrm{Tr}[2q]\,\Big(\hat{u}_-^{-1}\widetilde{\omega}[(\widetilde{u}^{-1}-1)(\widetilde{u}-1)]^q\hat{u}_- +$$
$$(\widetilde{u}-1)\widetilde{\omega}[(\widetilde{u}^{-1}-1)(\widetilde{u}-1)]^{q-1}(\widetilde{u}^{-1}-1) + \ldots + \hat{u}_-^{-1}[(\widetilde{u}^{-1}-1)(\widetilde{u}-1)]^q\widetilde{\omega}\hat{u}_-\Big)$$

After evaluation on the current $\frac{1}{\sqrt{2\pi i}}\int_{x=0}^1$, the right-hand-side may be identified, modulo commutators, with the Chern-Simons form $\mathrm{cs}_0^p(\widetilde{u})$ defined in analogy with (71). One thus gets

$$\mathrm{ch}^p(\rho)\cdot\mathrm{cs}_0^p(\hat{g}) \equiv \mathrm{cs}_0^p(\widetilde{u}) \bmod \overline{b} \ .$$

Now, introduce a parameter $t\in[0,1]$ and choose an invertible interpolation $\hat{v} \in (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1]_x[0,1]_t)^+$ between $\hat{v}_{t=0} = \widetilde{u}$ and $\hat{v}_{t=1} = 1 + \hat{e}\otimes(\widehat{\rho_!(g)-1})$, with the property that it restricts to $\hat{v}(g_0)$ for $x=0$ and to $\hat{v}(g_1)$ for $x=1$. The projection of $\hat{v}$ on the algebra $(\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{B}[0,1]_x[0,1]_t)^+$ may be chosen constant with respect to t. In the proof of Lemma 4.6 we established at any point $(x,t)\in[0,1]^2$ the identity

$$\frac{\partial}{\partial t}\big(\mathrm{Tr}[2q]\,\hat{v}^{-1}[(\hat{v}-1)(\hat{v}^{-1}-1)]^q s\hat{v}\big)\equiv s\big(\mathrm{Tr}[2q]\,\hat{v}^{-1}[(\hat{v}-1)(\hat{v}^{-1}-1)]^q\frac{\partial\hat{v}}{\partial t}\big)\ \mathrm{mod}\ \overline{b}\ ,$$

and integrating over $[0,1]^2$ this implies

$$\mathrm{cs}_0^p(\hat{v}_{t=1})-\mathrm{cs}_0^p(\hat{v}_{t=0})\equiv\mathrm{cs}_0^p(\hat{v}_{x=1})-\mathrm{cs}_0^p(\hat{v}_{x=0})\ \mathrm{mod}\ \overline{b}\ .$$

Taking into account that $\hat{v}_{x=0}=\hat{v}(g_0)$ and $\hat{v}_{x=1}=\hat{v}(g_1)$, we calculate mod $\overline{b}$ in the complex $X_{n-p-1}(\mathscr{R},\mathscr{J})$

$$\begin{aligned}\mathrm{ch}^p(\rho)\cdot(\theta_1-\theta_0)&+\mathrm{cs}_0^p(\hat{v}(g_1))-\mathrm{cs}_0^p(\hat{v}(g_0))\\&\equiv\mathrm{ch}^p(\rho)\cdot\mathrm{cs}_0^p(\hat{g})+\mathrm{cs}_0^p(\hat{v}_{t=1})-\mathrm{cs}_0^p(\hat{v}_{t=0})\ \mathrm{mod}\ \overline{b}\\&\equiv\mathrm{cs}_0^p(\widetilde{u})+\mathrm{cs}_0^p(1+\hat{e}\otimes(\widehat{\rho_!(g)}-1))-\mathrm{cs}_0^p(\widetilde{u})\ \mathrm{mod}\ \overline{b}\\&\equiv\mathrm{cs}_0^p(\widehat{\rho_!(g)})\ \mathrm{mod}\ \overline{b}\ .\end{aligned}$$

Hence the direct images $\rho_!(\hat{g}_0,\theta_0)$ and $\rho_!(\hat{g}_1,\theta_1)$ are equivalent and the map $\rho_!:MK_n^{\mathscr{I}}(\mathscr{A})\to MK_{n-p}^{\mathscr{I}}(\mathscr{B})$ for $n=2k+1$ and $p=2q$ is well-defined. It is obviously compatible with the push-forward map in topological K-theory $\rho_!:K_1^{\mathrm{top}}(\mathscr{I}\mathscr{A})\to K_1^{\mathrm{top}}(\mathscr{I}\mathscr{B})$. The compatibility with the push-forward map in cyclic homology $\mathrm{ch}^p(\rho):HC_{n-1}(\mathscr{A})\to HC_{n-p-1}(\mathscr{B})$ is clear once we remark that the Chern-Simons form $\mathrm{cs}_0^{2q}(\hat{v})$ vanish whenever $\hat{v}=1$. Hence the diagram (83) is commutative.
We have to check the invariance of $\rho_!$ with respect to conjugation of quasihomomorphisms. Let ρ_0 and ρ_1 be two conjugate quasihomomorphisms $\mathscr{A}\to\mathscr{E}^s\triangleright\mathscr{I}^s\mathscr{B}$. Hence there exists an invertible element $U\in(\mathscr{E}_+^s)^+$ such that $\rho_1=U^{-1}\rho_0U$. We follow the proof of Proposition 3.12 and remark that the lifting homomorphisms $\rho_{0*},\rho_{1*}:\widehat{T}\mathscr{A}\to\mathscr{M}_+^s$ factor through homomorphisms $\varphi_0,\varphi_1:\widehat{T}\mathscr{A}\to\widehat{T}\mathscr{E}_+^s$. The maps $\rho_{0!},\rho_{1!}:MK_n^{\mathscr{I}}(\mathscr{A})\to MK_{n-p}^{\mathscr{I}}(\mathscr{B})$ are obtained by composition of the pushforward maps $\varphi_{0!},\varphi_{1!}:MK_n^{\mathscr{I}}(\mathscr{A})\to MK_n^{\mathscr{I}}(\mathscr{E}_+^s)$ induced by the homomorphisms

$$\varphi_{i!}(\hat{g},\theta)=(\varphi_i(\hat{g}),\varphi_i(\theta))$$

with the map $MK_n^{\mathscr{I}}(\mathscr{E}_+^s)\to MK_{n-p}^{\mathscr{I}}(\mathscr{B})$ associated with the natural $(p+1)$-summable quasihomomorphism of even degree $\mathscr{E}_+^s\to\mathscr{E}^s\triangleright\mathscr{I}^s\mathscr{B}$. Hence it is sufficient to check that the maps $\varphi_{0!}$ and $\varphi_{1!}:MK_n^{\mathscr{I}}(\mathscr{A})\to MK_n^{\mathscr{I}}(\mathscr{E}_+^s)$ coincide. From the proof of 3.12 we know that φ_1 is smoothly homotopic to $\widehat{U}^{-1}\varphi_0\widehat{U}$, where $\widehat{U}\in(\widehat{T}\mathscr{E}_+^s)^+$ is a lifting of U, and the interpolating homomorphism $\varphi:\widehat{T}\mathscr{A}\to\widehat{T}\mathscr{E}_+^s[0,1]$ is constant modulo the ideal $\widehat{J}\mathscr{E}_+^s$. Consequently the morphisms $X(\varphi_1)$ and $X(\widehat{U}^{-1}\varphi_0\widehat{U}):X(\widehat{T}\mathscr{A})\to X(\widehat{T}\mathscr{E}_+^s)$ are homotopic,

$$X(\varphi_1)-X(\widehat{U}^{-1}\varphi_0\widehat{U})=[\partial,H]$$

with $H \in \mathrm{Hom}^0(X(T\mathscr{A}, J\mathscr{A}), X(T\mathscr{E}^s_+, J\mathscr{E}^s_+))$ a cochain *of order zero*. Let us now compare the images of $(\hat{g}, \theta) \in MK^{\mathscr{I}}_n(\mathscr{A})$ under the pushworwards $\varphi_{1!}$ and $(\widehat{U}^{-1}\varphi_0\widehat{U})_!$. The images $\varphi_1(\hat{g})$ and $\widehat{U}^{-1}\varphi_0(\hat{g})\widehat{U}$ are smoothly homotopic, with interpolation $\varphi(\hat{g}) \in (\mathscr{I}\widehat{T}\mathscr{E}^s_+[0,1])^+$. If moreover $\hat{g}$ is the canonical lift of an invertible element $g \in (\mathscr{I}\mathscr{A})^+$, Chern-Simons form associated to $\varphi(\hat{g})$ can be written as

$$\begin{aligned} \mathrm{cs}^p_0(\varphi(\hat{g})) &= \frac{1}{\sqrt{2\pi i}} \frac{(q!)^2}{p!} \int_0^1 \mathrm{Tr}\, \varphi(\hat{g}^{-1})\varphi[(\hat{g}-1)(\hat{g}^{-1}-1)]^q s\varphi(\hat{g}) \\ &= \frac{1}{\sqrt{2\pi i}} \int_0^1 \mathrm{Tr}[2q]\, \varphi(\hat{u}^{-1}) s\varphi(\hat{u}) \ , \end{aligned}$$

where as usual $\hat{u} = 1 + e \otimes (\hat{g} - 1)$ is the invertible associated to $\hat{g}$, with $\varphi(\hat{u}) = 1 + e \otimes (\varphi(\hat{g}) - 1) \in (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{T}\mathscr{E}^s_+[0,1])^+$. By construction (Proposition 3.12 i)), the r.h.s. coincides with the evaluation of H on the Chern character $\mathrm{ch}^p_1(\hat{g}) = \frac{1}{\sqrt{2\pi i}} \mathrm{Tr}\natural[2q]\, \hat{u}^{-1}\mathbf{d}\hat{u}$, and because H is of order zero one has

$$\mathrm{cs}^p_0(\varphi(\hat{g})) = H\mathrm{ch}^p_1(\hat{g}) = H(\natural\mathbf{d}\theta) \equiv \varphi_1(\theta) - (\widehat{U}^{-1}\varphi_0\widehat{U})(\theta) \bmod \overline{b}$$

in the complex $X_{n-1}(T\mathscr{E}^s_+, J\mathscr{E}^s_+)$. This proves that $\varphi_{1!}(\hat{g}, \theta)$ is equivalent to $(\widehat{U}^{-1}\varphi_0\widehat{U})_!(\hat{g}, \theta)$. Now it remains to show that the image $(\widehat{U}^{-1}\varphi_0\widehat{U})_!(\hat{g}, \theta) = (\widehat{U}^{-1}\varphi_0(\hat{g})\widehat{U}, \widehat{U}^{-1}\varphi_0(\theta)\widehat{U})$ is equivalent to $\varphi_{0!}(\hat{g}, \theta) = (\varphi_0(\hat{g}), \varphi_0(\theta))$. Here we mimic the proof of Lemma 5.5 and construct a homotopy between the invertible matrices $\left(\begin{smallmatrix} \varphi_0(\hat{g}) & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \widehat{U}^{-1}\varphi_0(\hat{g})\widehat{U} & 0 \\ 0 & 1 \end{smallmatrix}\right)$, whose associated Chern-Simons form is a $\overline{b}$-boundary. Since $\widehat{U}^{-1}\varphi_0(\theta)\widehat{U} \equiv \varphi_0(\theta) \bmod \overline{b}$, we conclude that $\rho_{0!}$ and $\rho_{1!}$ agree on topological and multiplicative K-theories.
Finally we have to check the compatibility with the negative Chern character $MK^{\mathscr{I}}_* \to HN_*$. For any pair $(\hat{g}, \theta) \in MK^{\mathscr{I}}_n(\mathscr{A})$, one has

$$\mathrm{ch}_n(\hat{g}, \theta) = \mathrm{ch}^p_1(\hat{g}) - \natural\mathbf{d}\tilde{\theta} \ \in F^{n-1}\widehat{X}(T\mathscr{A}, J\mathscr{A}),$$

where $\tilde{\theta}$ is any lift of θ in $\widehat{X}(T\mathscr{A}, J\mathscr{A})$. On the other hand, if $\hat{g}$ is the canonical lift of some $g \in (\mathscr{I}\mathscr{A})^+$, its image $\rho_!(\hat{g}, \theta) \in MK^{\mathscr{I}}_{n-p}(\mathscr{B})$ is represented by the pair $(\widehat{\rho_!(g)}, \mathrm{ch}^p(\rho) \cdot \theta + \mathrm{cs}^p_0(\hat{v}))$ constructed above, so that

$$\mathrm{ch}_{n-p}\big(\rho_!(\hat{g}, \theta)\big) = \mathrm{ch}^p_1\big(\widehat{\rho_!(g)}\big) - \natural\mathbf{d}(\mathrm{ch}^p(\rho) \cdot \tilde{\theta} + \mathrm{cs}^p_0(\hat{v})) \ \in F^{n-p-1}\widehat{X}(\mathscr{R}, \mathscr{J}) \ .$$

But we know that the relation $\mathrm{ch}^p_1\big(\widehat{\rho_!(g)}\big) - \natural\mathbf{d}\mathrm{cs}^p_0(\hat{v}) = \mathrm{ch}^p(\rho) \cdot \mathrm{ch}^p_1(\hat{g})$ actually holds in the complex $\widehat{X}(\mathscr{R}, \mathscr{J}) = X(\widehat{\mathscr{R}})$. Therefore

$$\mathrm{ch}_{n-p}\big(\rho_!(\hat{g}, \theta)\big) = \mathrm{ch}^p(\rho) \cdot (\mathrm{ch}^p_1(\hat{g}) - \natural\mathbf{d}\tilde{\theta}) = \mathrm{ch}^p(\rho) \cdot \mathrm{ch}_n(\hat{g}, \theta) \ ,$$

and (83) is compatible with the diagram of Theorem 3.13.

ii) $n=2k$ is even and $p=2q$ is even. As in the case of topological K-theory we pass to the suspensions of $\mathscr{A}$ and $\mathscr{B}$. We shall only sketch the procedure. The multiplicative K-theory group of even degree $MK_n^{\mathscr{I}}(\mathscr{A})$ has an alternative description in terms of the set $MK_n'^{\mathscr{I}}(\mathscr{A})$ of equivalence classes of pairs $(\hat{g},\theta)$, where $\hat{g}\in(\mathscr{K}\hat{\otimes}\mathscr{I}S\widehat{T}\mathscr{A})^+$ is an invertible and $\theta\in X_{n-1}(T\mathscr{A},J\mathscr{A})$ is a chain of odd degree such that $\mathrm{cs}_0^p(\hat{g})=\overline{b}\theta$. The equivalence relation is based on a higher transgression of the Chern-Simons form: $(\hat{g}_0,\theta_0)$ is equivalent to $(\hat{g}_1,\theta_1)$ iff there exists an invertible interpolation $\hat{g}\in(\mathscr{K}\hat{\otimes}\mathscr{I}S\widehat{T}\mathscr{A}[0,1])^+$ and a chain of even degree λ such that

$$\theta_1-\theta_0=\mathrm{cs}_1'^p(\hat{g})+\natural\mathbf{d}\lambda\ \in X_{n-1}(T\mathscr{A},J\mathscr{A})\ ,$$

where the odd chain $\mathrm{cs}_1'^p(\hat{g})\in X(\widehat{T}\mathscr{A})$ is defined modulo $\natural\mathbf{d}$ by the higher transgression formula (see the proof of Lemma 4.6)

$$\overline{b}\mathrm{cs}_1'^p(\hat{g})=\mathrm{cs}_0^p(\hat{g}_1)-\mathrm{cs}_0^p(\hat{g}_0)\ .$$

Like MK, one can show that $MK_n'^{\mathscr{I}}(\mathscr{A})$ is an abelian group inserted between $HC_*(\mathscr{A})$ and $K_*^{\mathrm{top}}(\mathscr{I}S\mathscr{A})$ in an exact sequence. More precisely there is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
K_1^{\mathrm{top}}(\mathscr{I}\mathscr{A}) & \xrightarrow{\mathrm{ch}_1} & HC_{n-1}(\mathscr{A}) & \xrightarrow{\delta} & MK_n^{\mathscr{I}}(\mathscr{A}) & \longrightarrow & K_0^{\mathrm{top}}(\mathscr{I}\mathscr{A}) & \xrightarrow{\mathrm{ch}_0} & HC_{n-2}(\mathscr{A}) \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\times\sqrt{2\pi i}} & & \downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\times\sqrt{2\pi i}}\downarrow \\
K_0^{\mathrm{top}}(\mathscr{I}S\mathscr{A}) & \xrightarrow{\mathrm{cs}_1} & HC_{n-1}(\mathscr{A}) & \xrightarrow{\delta} & MK_n'^{\mathscr{I}}(\mathscr{A}) & \longrightarrow & K_1^{\mathrm{top}}(\mathscr{I}S\mathscr{A}) & \xrightarrow{\mathrm{cs}_0} & HC_{n-2}(\mathscr{A})
\end{array}$$

Because for even n the group MK_n' is constructed from invertibles, it has *odd* parity by convention. The (odd) map $MK_n^{\mathscr{I}}(\mathscr{A})\to MK_n'^{\mathscr{I}}(\mathscr{A})$ sends a pair $(\hat{e},\theta)$ to $(\beta(\hat{e}),\sqrt{2\pi i}\theta+l_1^p(\hat{e}))$, where $\beta(\hat{e})=(1+(z-1)\hat{e})(1+(z-1)p_0)^{-1}$ is the invertible image of $\hat{e}$ under the Bott map, and $l_1^p(\hat{e})$ is the transgressed cochain defined modulo $\natural\mathbf{d}$ by $\overline{b}(l_1^p(\hat{e}))=\mathrm{cs}_0^p(\beta(\hat{e}))-\sqrt{2\pi i}\,\mathrm{ch}_0^p(\hat{e})$. The map $MK_n'^{\mathscr{I}}(\mathscr{A})\to K_1^{\mathrm{top}}(\mathscr{I}S\mathscr{A})$ is the forgetful map, and $HC_{n-1}(\mathscr{A})\to MK_n'^{\mathscr{I}}(\mathscr{A})$ sends a cycle θ to $(1,\sqrt{2\pi i}\,\theta)$. By the five lemma, $MK_n'^{\mathscr{I}}(\mathscr{A})$ is thus isomorphic to $MK_n^{\mathscr{I}}(\mathscr{A})$. One easily checks that the negative Chern character $\mathrm{ch}_n:MK_n'^{\mathscr{I}}(\mathscr{A})\to HN_n(\mathscr{A})$ given by $\mathrm{ch}_n(\hat{g},\theta)=\mathrm{cs}_0^p(\hat{g})-\overline{b}\tilde{\theta}$ coincides with the negative Chern character on $MK_n^{\mathscr{I}}(\mathscr{A})$ up to a factor $\sqrt{2\pi i}$. Hence it suffices to constru ct the pushforward morphism $\rho_!$ for the groups MK_n', whose elements are represented by *invertibles* of the suspended algebras:

$$\begin{array}{ccccccc}
K_0^{\mathrm{top}}(\mathscr{I}S\mathscr{A}) & \longrightarrow & HC_{n-1}(\mathscr{A}) & \longrightarrow & MK_n'^{\mathscr{I}}(\mathscr{A}) & \longrightarrow & K_1^{\mathrm{top}}(\mathscr{I}S\mathscr{A}) \\
\downarrow{\scriptstyle\rho_!} & & \downarrow{\scriptstyle\mathrm{ch}^p(\rho)} & & \downarrow{\scriptstyle\rho_!} & & \downarrow{\scriptstyle\rho_!} \\
K_{-p}^{\mathrm{top}}(\mathscr{I}S\mathscr{B}) & \longrightarrow & HC_{n-1-p}(\mathscr{B}) & \longrightarrow & MK_{n-p}'^{\mathscr{I}}(\mathscr{B}) & \longrightarrow & K_{1-p}^{\mathrm{top}}(\mathscr{I}S\mathscr{B})
\end{array}$$

This can be done explicitly as in case i), with the only difference that the Chern character ch_1^p and Chern-Simons transgression cs_0^p are now replaced respectively by cs_0^p and the higher transgression $\mathrm{cs}_1'^p$. The needed formulas were already established in i): let $g \in (\mathscr{I}S\mathscr{A})^+$ be any invertible with canonical lift $\hat{g} \in (\mathscr{I}S\widehat{T}\mathscr{A})^+$. One can write

$$\mathrm{ch}^p(\rho) \cdot \mathrm{cs}_0^p(\hat{g}) = \mathrm{cs}_0^p(\widetilde{u}) - \overline{b}k_1^p(\hat{u})$$

with the invertibles $\hat{u} = 1 + e \otimes (\hat{g} - 1) \in (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}S\widehat{T}\mathscr{A})^+$ and $\widetilde{u} = \hat{u}_+\hat{u}_-^{-1} \in (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}S\widehat{\mathscr{R}})^+$, and $k_1^p(\hat{u})$ is a chain defined mod $\natural\mathbf{d}$. Let $\widehat{\rho_!(g)} \in (\mathscr{I}S\widehat{\mathscr{R}})^+$ be any invertible lift of $\rho_!(g) = \rho_+(g)\rho_-(g)^{-1}$, and $\hat{v} \in (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}S\widehat{\mathscr{R}}[0,1])^+$ be an invertible interpolation between $\hat{v}_0 = \widetilde{u}$ and $\hat{v}_1 = 1 + \hat{e} \otimes (\widehat{\rho_!(g)} - 1)$. Then one has

$$\overline{b}\mathrm{cs}_1'^p(\hat{v}) = \mathrm{cs}_0^p(\widehat{\rho_!(g)}) - \mathrm{cs}_0^p(\widetilde{u}) \ .$$

Therefore if $(\hat{g}, \theta)$ represents a class in $MK_n'^{\mathscr{I}}(\mathscr{A})$ we define its pushforward as the multiplicative K-theory class over $\mathscr{B}$

$$\boxed{\rho_!(\hat{g}, \theta) = \big(\widehat{\rho_!(g)} \ , \ \mathrm{ch}^p(\rho) \cdot \theta + k_1^p(\hat{u}) + \mathrm{cs}_1'^p(\hat{v})\big) \ \in MK_{n-p}'^{\mathscr{I}}(\mathscr{B})} \tag{86}$$

with the odd chain $\mathrm{ch}^p(\rho) \cdot \theta + k_1^p(\hat{u}) + \mathrm{cs}_1'^p(\hat{v})$ sitting in $X_{n-p-1}(\mathscr{R}, \mathscr{J})$. One shows the consistency of $\rho_!$ with the various equivalence relations using the properties of higher Chern-Simons transgressions. Details are left to the reader.

iii) $n = 2k + 1$ is odd and $p = 2q + 1$ is odd. We first establish an explicit formula for the composition of the topological Chern character $\mathrm{ch}_1^{2q} : K_1^{\mathrm{top}}(\mathscr{I}\mathscr{A}) \to HP_1(\mathscr{A})$ with the bivariant Chern character $\mathrm{ch}^p(\rho) \in HC^p(\mathscr{A}, \mathscr{B})$. Remark that $\mathscr{I}$ is $(2q+1)$-summable by hypothesis hence ch_1^{2q} is well-defined. As in case i) let $g \in (\mathscr{I}\mathscr{A})^+$ be an invertible, $\hat{g} \in (\mathscr{I}\widehat{T}\mathscr{A})^+$ its canonical lift and $\hat{u} = 1 + e \otimes (\hat{g} - 1) \in (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{T}\mathscr{A})^+$ the associated invertible. Recall that

$$\mathrm{ch}_1^{2q}(\hat{g}) = \frac{1}{\sqrt{2\pi i}} \, \mathrm{Tr}\natural[2q] \, \hat{u}^{-1}\mathbf{d}\hat{u} \ \in \Omega^1\widehat{T}\mathscr{A}_\natural \ ,$$

and the image of $\mathrm{Tr}\natural[2q] \, \hat{u}^{-1}\mathbf{d}\hat{u}$ under the Goodwillie equivalence is the $(b+B)$-cycle over $\widehat{T}\mathscr{A}$

$$\gamma(\mathrm{Tr}\natural[2q] \, \hat{u}^{-1}\mathbf{d}\hat{u}) = \sum_{i \geq 0} (-)^i i! \, \mathrm{Tr}[2q] \, \hat{u}^{-1}\mathbf{d}\hat{u}(\mathbf{d}\hat{u}^{-1}\mathbf{d}\hat{u})^i \ .$$

Now the quasihomomorphism $\rho : \mathscr{A} \to \mathscr{E}^s \triangleright \mathscr{I}^s\mathscr{B}$ is of odd degree. Hence, the image of an element $x \in \widehat{T}\mathscr{A}$ under the lifted quasihomomorphism $\rho_* : \widehat{T}\mathscr{A} \to \widehat{\mathscr{M}^s} \triangleright \mathscr{I}^s\widehat{\mathscr{R}}$ is a 2×2 matrix over $\widehat{\mathscr{M}}$ whose off-diagonal entries lie in $\mathscr{I}\widehat{\mathscr{R}}$. Moreover the multiplier F is given by the matrix

$$F = \varepsilon \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \varepsilon(2p_0 - 1)$$

where ε is the odd generator of the Clifford algebra C_1. Thus the commutator $[p_0, \rho_* x]$ lies in the matrix algebra $M_2(\mathscr{I}\widehat{\mathscr{R}})$ for any $x \in \widehat{T}\mathscr{A}$. On the other hand, the component of the chain map $\widehat{\chi}^p \rho_* : \widehat{\Omega}\widehat{T}\mathscr{A} \to X(\widehat{\mathscr{R}})$ evaluated on a p-form $x_0 \mathbf{d}x_1 \ldots \mathbf{d}x_p$ reads

$$-\frac{\Gamma(q+\frac{3}{2})}{(p+1)!} \sum_{\lambda \in S_{p+1}} \varepsilon(\lambda) \tau(\rho_* x_{\lambda(0)}[F, \rho_* x_{\lambda(1)}] \ldots [F, \rho_* x_{\lambda(p)}]) \ ,$$

where $\tau(\varepsilon \cdot) = -\sqrt{2i}\,\mathrm{Tr}(\cdot)$ is the odd supertrace (see section 3). As in case i), let us extend ρ_* to a unital homomorphism $(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{T}\mathscr{A})^+ \to (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{M}^s_+})^+$. Using $\Gamma(q+\frac{3}{2}) = \sqrt{\pi}\, p!/(2^p q!)$ with $p = 2q+1$, one gets by direct computation

$$\begin{aligned} \mathrm{ch}^p(\rho) \cdot \mathrm{ch}^{2q}_1(\hat{g}) &= \frac{1}{2}\mathrm{Tr}[2q]\,(\widetilde{u}^{-1}[p_0, \widetilde{u}])^p + \frac{1}{2}\mathrm{Tr}[2q]\,([p_0, \widetilde{u}]\widetilde{u}^{-1})^p \\ &= \mathrm{Tr}[2q]\,(\widetilde{u}^{-1}[p_0, \widetilde{u}])^p - \overline{b}\frac{1}{2}\mathrm{Tr}\natural[2q]\,(\widetilde{u}^{-1}[p_0, \widetilde{u}])^p \widetilde{u}^{-1}\mathbf{d}\widetilde{u} \end{aligned}$$

where $\widetilde{u} = \rho_* \hat{u}$ is an invertible element of the algebra $(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{M}^s_+})^+$, and the commutator $[p_0, \widetilde{u}] \in M_2(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{I}\widehat{\mathscr{R}})^+$ may be considered as an element of $M_2(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$ after applying the homomorphism $\boxtimes : \mathscr{I}\hat{\otimes}\mathscr{I} \to \mathscr{I}$. The first term of the r.h.s. is recognized as the higher Chern character $\mathrm{ch}^{2q}_0(\widetilde{f}) = \mathrm{Tr}[2q](\widetilde{f} - p_0)^p$ given by (70) for the idempotent $\widetilde{f} = \widetilde{u}^{-1} p_0 \widetilde{u} \in M_2(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{I}\widehat{\mathscr{R}})^+$ (or $M_2(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}})^+$), whence the equality

$$\mathrm{ch}^p(\rho) \cdot \mathrm{ch}^{2q}_1(\hat{g}) = \mathrm{ch}^{2q}_0(\widetilde{f}) - \overline{b}\,\frac{1}{2}\mathrm{Tr}\natural[2q]\,(\widetilde{f} - p_0)^p \widetilde{u}^{-1}\mathbf{d}\widetilde{u}$$

of cycles in $X(\widehat{\mathscr{R}})$. Then, observe that the projection of $\widetilde{f}$ to the algebra $M_2(\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{B})^+$ is the idempotent $p_0 + e \otimes (\rho(g)^{-1} p_0 \rho(g) - p_0)$. Using the isomorphism $(\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{B})^+ \cong (\mathscr{I}\mathscr{B})^+$, this idempotent may be identified with the direct image $\rho_!(g) = \rho(g)^{-1} p_0 \rho(g)$. Hence, it is possible to relate $\mathrm{ch}^{2q}_0(\widetilde{f})$ with the Chern character of a given idempotent lift $\widehat{\rho_!(g)} \in M_2(\mathscr{I}\widehat{\mathscr{R}})^+$, via a homotopy with parameter $t \in [0,1]$. Let $\hat{f} \in M_2(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1])^+$ be an idempotent path lifting the constant family $p_0 + e \otimes (\rho_!(g) - p_0)$ and connecting the two endpoints

$$\hat{f}_0 = \widetilde{f} \ , \qquad \hat{f}_1 = p_0 + \hat{e} \otimes (\widehat{\rho_!(g)} - p_0) \ .$$

$\hat{e} \in \widehat{T}\mathbb{C}$ is the canonical idempotent lift of the unit $e \in \mathbb{C}$ as in case i). The lifting $\hat{f}$ is thus defined up to homotopy (at least after stabilization by the matrix algebra $\mathscr{K}$). The property $[2q]\,\hat{e} = 1$ implies the equality

$$\mathrm{ch}^{2q}_0(\hat{f}_1) = \mathrm{ch}^{2q}_0(\widehat{\rho_!(g)}) \in \widehat{\mathscr{R}}$$

at the level of cycles. Furthermore, in analogy with Eqs. (71) the Chern-Simons form associated to the idempotent $\hat{f}$ is defined by

$$\mathrm{cs}^{2q}_1(\hat{f}) = \int_0^1 dt\, \mathrm{Tr}\natural[2q]\,(-2\hat{f}+1) \sum_{i=0}^{q} (\hat{f} - p_0)^{2i} \frac{\partial \hat{f}}{\partial t} (\hat{f} - p_0)^{2(q-i)} \mathbf{d}\hat{f} \ ,$$

and fulfills the transgression relation in $X(\widehat{\mathscr{R}})$

$$\overline{b}\mathrm{cs}_1^{2q}(\hat{f}) = \mathrm{ch}_0^{2q}(\hat{f}_1) - \mathrm{ch}_0^{2q}(\hat{f}_0) = \mathrm{ch}_0^{2q}(\widehat{\rho_!(g)}) - \mathrm{ch}_0^{2q}(\widetilde{f}) \ .$$

This leads to the definition of the map $\rho_!$ on multiplicative K-theory. Let $(\hat{g},\theta)$ represent a class in $MK_n^{\mathscr{I}}(\mathscr{A})$ of odd degree $n = 2k+1$. By Remark 5.3 we may suppose that $\hat{g}$ is the canonical lift of some invertible $g \in (\mathscr{I}\mathscr{A})^+$, and $\theta \in X_{n-1}(T\mathscr{A}, J\mathscr{A})$ is a transgression of the Chern character $\mathrm{ch}_1^{2q}(\hat{g}) = \natural\mathbf{d}\theta$. We set

$$\boxed{\rho_!(\hat{g},\theta) = \big(\widehat{\rho_!(g)}\ ,\ -\mathrm{ch}^p(\rho)\cdot\theta + h_1^{2q}(\widetilde{u}) + \mathrm{cs}_1^{2q}(\hat{f})\big)\ \in MK_{n-p}^{\mathscr{I}}(\mathscr{B})} \tag{87}$$

where $\widehat{\rho_!(g)} \in M_2(\mathscr{I}\widehat{\mathscr{R}})^+$ is any idempotent lift of $\rho_!(g)$, $h_1^{2q}(\widetilde{u})$ is the chain $\frac{1}{2}\mathrm{Tr}\natural[2q]\,(\widetilde{f}-p_0)^p\widetilde{u}^{-1}\mathbf{d}\widetilde{u}$, and $\hat{f}$ is an idempotent path constructed as above. The minus sign in front of $\mathrm{ch}^p(\rho)\cdot\theta$ is necessary because the bivariant Chern character $\mathrm{ch}^p(\rho)$ is of odd degree $p = 2q+1$. This ensures the correct transgression relation

$$\begin{aligned}
&\overline{b}(-\mathrm{ch}^p(\rho)\cdot\theta + h_1^{2q}(\widetilde{u}) + \mathrm{cs}_1^{2q}(\hat{f}))\\
&= \mathrm{ch}^p(\rho)\cdot\mathrm{ch}_1^{2q}(\hat{g}) + \overline{b}\,\frac{1}{2}\mathrm{Tr}\natural[2q]\,(\widetilde{f}-p_0)^p\widetilde{u}^{-1}\mathbf{d}\widetilde{u} + \mathrm{ch}_0^{2q}(\widehat{\rho_!(g)}) - \mathrm{ch}_0^{2q}(\widetilde{f})\\
&= \mathrm{ch}_0^{2q}(\widehat{\rho_!(g)})
\end{aligned}$$

in the quotient complex $X_{n-p-1}(\mathscr{R},\mathscr{J})$, which shows that $\rho_!(\hat{g},\theta)$ indeed defines an element of $MK_{n-p}^{\mathscr{I}}(\mathscr{B})$. Its class does not dependent on the chosen idempotent lift $\widehat{\rho_!(g)}$ nor on the path $\hat{f}$, and moreover $\rho_!$ is compatible with the equivalence relation on multiplicative K-theory. We proceed as in case i) and let $(\hat{g}_0,\theta_0)$ and $(\hat{g}_1,\theta_1)$ be two equivalent representatives of a class in $MK_n^{\mathscr{I}}(\mathscr{A})$, provided with an interpolation $\hat{g} \in (\mathscr{I}\widehat{T}\mathscr{A}[0,1]_x)^+$ and a chain $\lambda \in X_{n-1}(T\mathscr{A}, J\mathscr{A})$ such that $\theta_1 - \theta_0 = \mathrm{cs}_0^{2q}(\hat{g}) + \overline{b}\lambda$. From Remark 5.3 the elements $\hat{g}_0$, $\hat{g}_1$ and $\hat{g}$ can be taken as the canonical lifts of $g_0, g_1 \in (\mathscr{I}\mathscr{A})^+$ and $g \in (\mathscr{I}\mathscr{A}[0,1]_x)^+$. Denoting by $\rho_*\hat{u}(g_i) = \widetilde{u}(g_i) \in (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{M}_+^s})^+$ the invertible and $\hat{f}(g_i) \in M_2(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{\mathscr{R}}[0,1]_t)^+$ the idempotent path associated to g_i, we have to establish the relation

$$\begin{aligned}
-\mathrm{ch}^p(\rho)\cdot(\theta_1-\theta_0) + h_1^{2q}(\widetilde{u}(g_1)) - h_1^{2q}(\widetilde{u}(g_0)) + \mathrm{cs}_1^{2q}(\hat{f}(g_1)) - \mathrm{cs}_1^{2q}(\hat{f}(g_0))\\
\equiv \mathrm{cs}_1^{2q}\big(\widehat{\rho_!(g)}\big) \bmod \natural\mathbf{d}
\end{aligned}$$

in the complex $X_{n-p-1}(\mathscr{R},\mathscr{J})$, where $\widehat{\rho_!(g)} \in M_2(\mathscr{I}\widehat{\mathscr{R}}[0,1]_x)^+$ is a choice of idempotent interpolation between the liftings $\widehat{\rho_!(g_i)}$'s. As usual, let $\hat{u} = 1 + e\otimes(\hat{g}-1) \in (\mathbb{C}\hat{\otimes}\mathscr{I}\mathscr{A}[0,1]_x)^+ \hookrightarrow (\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}\widehat{T}\mathscr{A}[0,1]_x)^+$ be the invertible identification with $\hat{g}$. We know the equality

$$\mathrm{cs}_0^{2q}(\hat{g}) = \frac{1}{\sqrt{2\pi i}}\int_0^1 \mathrm{Tr}[2q]\,\hat{u}^{-1}s\hat{u}\ ,$$

where s is the de Rham differential on $\Omega[0,1]_x$. Set $\hat\omega = \hat u^{-1}s\hat u$. The computation of $\mathrm{ch}^p(\rho)\cdot \mathrm{cs}_0^{2q}(\hat g)$ involves the formula

$$\gamma(\mathrm{Tr}[2q]\,\hat\omega) =$$
$$\mathrm{Tr}[2q]\,\hat\omega + \sum_{i\geq 1}(-)^i(i-1)!\sum_{j=0}^{i-1}\mathrm{Tr}[2q]\,\hat u^{-1}\mathbf{d}\hat u(\mathbf{d}\hat u^{-1}\mathbf{d}\hat u)^j\mathbf{d}\hat\omega(\mathbf{d}\hat u^{-1}\mathbf{d}\hat u)^{i-j-1}\ ,$$

as well as the component of the chain map $\widehat\chi^p\rho_*$ evaluated on a $(p+1)$-form $x_0\mathbf{d}x_1\ldots\mathbf{d}x_{p+1}$ over $\widehat T\mathscr{A}$:

$$-\frac{\Gamma(q+\frac32)}{(p+1)!}\sum_{i=1}^{p+1}\tau\natural(\rho_*x_0[F,\rho_*x_1]\ldots\mathbf{d}(\rho_*x_i)\ldots[F,\rho_*x_{p+1}])$$

Denote as before $\widetilde u = \rho_*\hat u$ the invertible image in $(\widehat T\mathbb{C}\hat\otimes\mathscr{I}\widehat{\mathscr{M}^s_+}[0,1]_x)^+$, the associated idempotent $\widetilde f = \widetilde u^{-1}p_0\widetilde u \in M_2(\widehat T\mathbb{C}\hat\otimes\mathscr{I}\widehat{\mathscr{R}}[0,1]_x)^+$, and the Maurer-Cartan form $\widetilde\omega = \widetilde u^{-1}s\widetilde u$. One gets

$$\mathrm{ch}^p(\rho)\,(\mathrm{Tr}[2q]\,\hat\omega) =$$
$$-\frac{\sqrt{2\pi i}}{2}\mathrm{Tr}\natural[2q]\Big(\sum_{i=0}^{2q}(\widetilde f-p_0)^i[p_0,\widetilde\omega](\widetilde f-p_0)^{2q-i}\widetilde u^{-1}\mathbf{d}\widetilde u + (\widetilde f-p_0)^p\mathbf{d}\widetilde\omega\Big)\ .$$

Now observe that $\widetilde u_{x=0} = \widetilde u(g_0)$ and $\widetilde u_{x=1} = \widetilde u(g_1)$, so that after integration over the current $\frac{1}{\sqrt{2\pi i}}\int_{x=0}^1$ we get the identity (recall $\mathrm{ch}^p(\rho)$ is odd)

$$\begin{aligned}
&-\mathrm{ch}^p(\rho)\cdot\mathrm{cs}_0^{2q}(\hat g) + h_1^{2q}(\widetilde u(g_1)) - h_1^{2q}(\widetilde u(g_0))\\
&= \frac{1}{\sqrt{2\pi i}}\int_0^1\mathrm{ch}^p(\rho)\,(\mathrm{Tr}[2q]\,\hat\omega) + \frac12\int_0^1 s\,\mathrm{Tr}\natural[2q]\,(\widetilde f-p_0)^p\widetilde u^{-1}\mathbf{d}\widetilde u\\
&= \int_0^1\mathrm{Tr}\natural[2q]\Big(\sum_{i=1}^{q}(\widetilde f-p_0)^{2i-1}[p_0,\widetilde\omega](\widetilde f-p_0)^{2(q-i)+1}\widetilde u^{-1}\mathbf{d}\widetilde u - (\widetilde f-p_0)^p\mathbf{d}\widetilde\omega\Big)
\end{aligned}$$

On the other hand, let us calculate the Chern-Simons form associated to the idempotent $\widetilde f$,

$$\mathrm{cs}_1^{2q}(\widetilde f) = \int_0^1\mathrm{Tr}\natural[2q]\,(-2\widetilde f+1)\sum_{i=0}^{q}(\widetilde f-p_0)^{2i}s\widetilde f(\widetilde f-p_0)^{2(q-i)}\mathbf{d}\widetilde f\ ,$$

in terms of $\widetilde\omega$. Since by definition $\widetilde f = \widetilde u^{-1}p_0\widetilde u$, the structure equation $s\widetilde f = [\widetilde f,\widetilde\omega]$ follows and one finds

$$\begin{aligned}
\mathrm{cs}_1^{2q}(\widetilde f) &= -\natural\mathbf{d}\int_0^1\mathrm{Tr}[2q]\,(\widetilde f-p_0)^p\widetilde\omega\\
&+ \int_0^1\mathrm{Tr}\natural[2q]\Big(\sum_{i=1}^{q}(\widetilde f-p_0)^{2i-1}[p_0,\widetilde\omega](\widetilde f-p_0)^{2(q-i)+1}\widetilde u^{-1}\mathbf{d}\widetilde u - (\widetilde f-p_0)^p\mathbf{d}\widetilde\omega\Big)
\end{aligned}$$

Thus holds the fundamental relation

$$-\mathrm{ch}^p(\rho)\cdot \mathrm{cs}_0^{2q}(\hat g) + h_1^{2q}(\widetilde u(g_1)) - h_1^{2q}(\widetilde u(g_0)) \equiv \mathrm{cs}_1^{2q}(\widetilde f) \bmod \natural\mathbf{d}\ .$$

Now let $\hat f \in M_2(\widehat T\mathbb{C}\hat\otimes\mathscr{I}\mathscr{R}[0,1]_x[0,1]_t)^+$ be an idempotent interpolation between $\hat f_{t=0} = \widetilde f$ and $\hat f_{t=1} = p_0 + \hat e\otimes(\widehat{\rho_!(g)} - p_0)$, with the property that it restricts to $\hat f(g_0)$ for $x=0$ and to $\hat f(g_1)$ for $x=1$. The projection of $\hat f$ to the algebra $M_2(\mathbb{C}\hat\otimes\mathscr{I}\mathscr{B}[0,1]_x[0,1]_t)^+$ may be chosen constant with respect to t. In the proof of Lemma 4.6 we established the following identity at any point $(x,t)\in[0,1]^2$:

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big(\mathrm{Tr}\natural[2q]\,(-2\hat f+1)\sum_{i=0}^{q}(\hat f-p_0)^{2i}s\hat f(\hat f-p_0)^{2(q-i)}\mathbf{d}\hat f\Big)\\
&\equiv\quad s\Big(\mathrm{Tr}\natural[2q]\,(-2\hat f+1)\sum_{i=0}^{q}(\hat f-p_0)^{2i}\frac{\partial\hat f}{\partial t}(\hat f-p_0)^{2(q-i)}\mathbf{d}\hat f\Big) \bmod \natural\mathbf{d}\ ,
\end{aligned}$$

and integration over the square $[0,1]^2$ yields

$$\mathrm{cs}_1^{2q}(\hat f_{t=1}) - \mathrm{cs}_1^{2q}(\hat f_{t=0}) \equiv \mathrm{cs}_1^{2q}(\hat f_{x=1}) - \mathrm{cs}_1^{2q}(\hat f_{x=0}) \bmod \natural\mathbf{d}\ .$$

Since $\hat f_{x=0} = \hat f(g_0)$ and $\hat f_{x=1} = \hat f(g_1)$ we calculate, modulo $\natural\mathbf{d}$ in the complex $X_{n-p-1}(\mathscr{R},\mathscr{J})$

$$\begin{aligned}
&-\mathrm{ch}^p(\rho)\cdot(\theta_1-\theta_0) + h_1^{2q}(\widetilde u(g_1)) - h_1^{2q}(\widetilde u(g_0)) + \mathrm{cs}_1^{2q}(\hat f(g_1)) - \mathrm{cs}_1^{2q}(\hat f(g_0))\\
&\equiv\quad -\mathrm{ch}^p(\rho)\cdot\mathrm{cs}_0^{2q}(\hat g) + h_1^{2q}(\widetilde u(g_1)) - h_1^{2q}(\widetilde u(g_0)) + \mathrm{cs}_1^{2q}(\hat f_{t=1}) - \mathrm{cs}_1^{2q}(\hat f_{t=0})\\
&\equiv\quad \mathrm{cs}_1^{2q}(\widetilde f) + \mathrm{cs}_1^{2q}(p_0+\hat e\otimes(\widehat{\rho_!(g)}-p_0)) - \mathrm{cs}_1^{2q}(\widetilde f)\\
&\equiv\quad \mathrm{cs}_1^{2q}(\widehat{\rho_!(g)}) \bmod \natural\mathbf{d}
\end{aligned}$$

as wanted. Hence $\rho_!(\hat g_0,\theta_0)$ and $\rho_!(\hat g_1,\theta_1)$ are equivalent and the map $\rho_! : MK_n^{\mathscr{I}}(\mathscr{A}) \to MK_{n-p}^{\mathscr{I}}(\mathscr{B})$ for $n=2k+1$ and $p=2q+1$ is well-defined. Its compatibility with the map $\rho_!$ on topological K-theory is obvious. Concerning its compatibility with the map $\mathrm{ch}^p(\rho) : HC_{n-1}(\mathscr{A}) \to HC_{n-p-1}(\mathscr{B})$, we should take care of a minus sign which shows that the middle square of (83) is actually anticommutative; this has to be so because all the maps involved in this square are of odd degree. Hence the diagram (83) is graded commutative. The invariance of $\rho_!$ with respect to conjugation of quasihomomorphisms is proved exactly as in case i), by decomposing $\rho_!$ as the pushforward map $\varphi_! : MK_n^{\mathscr{I}}(\mathscr{A}) \to MK_n^{\mathscr{I}}(\mathscr{E}_+^s)$ induced by the homomorphism $\varphi : \widehat T\mathscr{A} \to \widehat T\mathscr{E}_+^s$, followed by the map $MK_n^{\mathscr{I}}(\mathscr{E}_+^s) \to MK_{n-p}^{\mathscr{I}}(\mathscr{B})$ associated with the natural p-summable quasihomomorphism of odd degree $\mathscr{E}_+^s \to \mathscr{E}^s \triangleright \mathscr{I}^s\mathscr{B}$. Also the compatibility with the negative Chern character is easily established.

iv) $n=2k$ is even and $p=2q+1$ is odd. As in case ii) we pass to the suspensions of $\mathscr{A}$ and $\mathscr{B}$ and work with the group $MK_n'^{\mathscr{I}}(\mathscr{A})$. Hence a multiplicative K-theory class of degree n over $\mathscr{A}$ is represented by a pair $(\hat{g},\theta)$ of an invertible $\hat{g}\in(\mathscr{K}\hat{\otimes}\mathscr{I}S\widehat{T}\mathscr{A})^+$ and an odd chain $\theta\in X_{n-1}(T\mathscr{A},J\mathscr{A})$ such that $\mathrm{cs}_0^{2q}(\hat{g})=\overline{b}\theta$. We are thus led to build a morphism

$$\rho_!: MK_n'^{\mathscr{I}}(\mathscr{A})\to MK_{n-p}'^{\mathscr{I}}(\mathscr{B})\ ,$$

where the group $MK_{n-p}'^{\mathscr{I}}(\mathscr{B})$, for $n-p$ odd, is represented by pairs $(\hat{e},\theta)$ of idempotent $\hat{e}\in M_2(\mathscr{K}\hat{\otimes}\mathscr{I}S\widehat{\mathscr{R}})^+$ and chain of even degree $\theta\in X_{n-p-1}(\mathscr{R},\mathscr{J})$ such that $\mathrm{cs}_1^{2q}(\hat{e})=\natural\mathbf{d}\theta$. Note that the parity of $MK_{n-p}'^{\mathscr{I}}(\mathscr{B})$ is *even*. We already established the needed formulas in case iii): let $g\in(\mathscr{I}S\mathscr{A})^+$ be any invertible with canonical lift $\hat{g}\in(\mathscr{I}S\widehat{T}\mathscr{A})^+$. One can write

$$-\mathrm{ch}^p(\rho)\cdot\mathrm{cs}_0^{2q}(\hat{g})=\mathrm{cs}_1^{2q}(\widetilde{f})-\natural\mathbf{d}k_0^{2q}(\hat{u})$$

with the invertible $\hat{u}=1+e\otimes(\hat{g}-1)\in(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}S\widehat{T}\mathscr{A})^+$, the idempotent $\widetilde{f}=\widetilde{u}^{-1}p_0\widetilde{u}\in M_2(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}S\widehat{\mathscr{R}})^+$ where $\widetilde{u}=\rho_*\hat{u}$, and the chain $k_0^{2q}(\hat{u})=-\int_0^1\mathrm{Tr}[2q]\,(\widetilde{f}-p_0)^p\widetilde{\omega}$ where $\widetilde{\omega}=\widetilde{u}^{-1}s\widetilde{u}$. Let $\widehat{\rho_!(g)}\in M_2(\mathscr{I}S\widehat{\mathscr{R}})^+$ be any idempotent lift of $\rho_!(g)=\rho(g)^{-1}p_0\rho(g)$, and $\hat{f}\in M_2(\widehat{T}\mathbb{C}\hat{\otimes}\mathscr{I}S\widehat{\mathscr{R}}[0,1])^+$ be an idempotent interpolation between $\hat{f}_0=\widetilde{f}$ and $\hat{f}_1=p_0+\hat{e}\otimes(\widehat{\rho_!(g)}-p_0)$. Then one has by means of the higher transgressions (see the proof of Lemma 4.6)

$$\natural\mathbf{d}\mathrm{cs}_0'^{2q}(\hat{f})=\mathrm{cs}_1^{2q}(\widehat{\rho_!(g)})-\mathrm{cs}_1^{2q}(\widetilde{f})\ ,$$

with $\mathrm{cs}_0'^{2q}(\hat{f})$ defined modulo $\overline{b}$. Therefore if $(\hat{g},\theta)$ represents a class in $MK_n'^{\mathscr{I}}(\mathscr{A})$ we define its pushforward as the multiplicative K-theory class over $\mathscr{B}$

$$\boxed{\rho_!(\hat{g},\theta)=\big(\widehat{\rho_!(g)}\ ,\ \mathrm{ch}^p(\rho)\cdot\theta+k_0^{2q}(\hat{u})+\mathrm{cs}_0'^{2q}(\hat{f})\big)\ \in MK_{n-p}'^{\mathscr{I}}(\mathscr{B})}\qquad(88)$$

with the chain $\mathrm{ch}^p(\rho)\cdot\theta+k_0^{2q}(\hat{u})+\mathrm{cs}_0'^{2q}(\hat{f})$ of even degree sitting in the quotient complex $X_{n-p-1}(\mathscr{R},\mathscr{J})$. ∎

Example 6.4 For $\mathscr{B}=\mathbb{C}$ a quasihomomorphism $\mathscr{A}\to\mathscr{E}^s\triangleright\mathscr{I}^s$ induces a map $MK_n^{\mathscr{I}}(\mathscr{A})\to MK_{n-p}^{\mathscr{I}}(\mathbb{C})$. Thus if $\mathscr{I}$ is a Schatten ideal on a Hilbert space, Example 5.6 yields index maps or regulators, depending on the degrees:

$$\begin{array}{rcll} MK_n^{\mathscr{I}}(\mathscr{A}) & \to & \mathbb{Z} & \text{if } n\le p\ ,\ n\equiv p \bmod 2\ ,\\ MK_n^{\mathscr{I}}(\mathscr{A}) & \to & \mathbb{C}^\times & \text{if } n>p\ ,\ n\equiv p+1 \bmod 2\ .\end{array}$$

7 Assembly maps and crossed products

In this section we illustrate the general theory of secondary characteristic classes with the specific example of crossed product algebras, and build an "assembly map" for multiplicative K-theory modelled on the Baum-Connes construction [1].

Let $\mathscr{A}$ be a unital Fréchet m-algebra and Γ a countable discrete group acting on $\mathscr{A}$ from the right by automorphisms. The action of an element $\gamma \in \Gamma$ on $a \in \mathscr{A}$ reads a^γ. We impose the action to be almost isometric in the following sense: for each submultiplicative seminorm $\|\cdot\|_\alpha$ on $\mathscr{A}$ there exists a constant C_α such that

$$\|a^\gamma\|_\alpha \leq C_\alpha \|a\|_\alpha \qquad \forall\, a \in \mathscr{A}\ ,\ \gamma \in \Gamma\ . \tag{89}$$

The algebraic tensor product $\mathscr{A} \otimes \mathbb{C}\Gamma$ is identified with the space of $\mathscr{A}$-valued functions with finite support over Γ. Thus any element of $\mathscr{A} \otimes \mathbb{C}\Gamma$ is a finite linear combination of symbols $a\gamma^*$ with $a_\gamma \in \mathscr{A}$ and $\gamma \in \Gamma$. The star refers to a contravariant notation. We endow $\mathscr{A} \otimes \mathbb{C}\Gamma$ with the crossed product defined in terms of symbols by

$$(a_1\gamma_1^*)(a_2\gamma_2^*) = a_1(a_2)^{\gamma_1}(\gamma_2\gamma_1)^* \qquad \forall\, a_i \in \mathscr{A}\ ,\ \gamma_i \in \Gamma\ .$$

The crossed product algebra $\mathscr{A} \rtimes \Gamma$ is an adequate completion of the above space consisting of $\mathscr{A}$-valued functions with "rapid decay" over Γ. This requires to fix once and for all a right-invariant distance function $d : \Gamma \times \Gamma \to \mathbb{R}_+$. Endow the space $\mathscr{A} \otimes \mathbb{C}\Gamma$ with the seminorms

$$\|b\|_{\alpha,\beta} = C_\alpha \sum_{\gamma\in\Gamma} \sigma_\beta(\gamma)\|b(\gamma)\|_\alpha \qquad \forall\, b \in \mathscr{A} \otimes \mathbb{C}\Gamma\ ,$$

where the $\mathbb{R}_+$-valued function $\sigma_\beta(\gamma) := (1 + d(\gamma, 1))^\beta$, for $\beta \geq 0$, fulfills the property $\sigma_\beta(\gamma_1\gamma_2) \leq \sigma_\beta(\gamma_1)\sigma_\beta(\gamma_2)$. One easily checks that $\|\cdot\|_{\alpha,\beta}$ is submultiplicative with respect to the crossed product, hence the completion of $\mathscr{A} \otimes \mathbb{C}\Gamma$ for the family of seminorms $(\|\cdot\|_{\alpha,\beta})$ yields a Fréchet m-algebra $\mathscr{A} \rtimes \Gamma$.
Multiplicative K-theory classes of $\mathscr{A} \rtimes \Gamma$ may be obtained by adapting the assembly map construction of [1]. The idea is to replace the noncommutative space $\mathscr{A} \rtimes \Gamma$ by a more classical space, for which the secondary invariants are presumably easier to describe. Consider a compact Riemannian manifold M without boundary, and let $P \xrightarrow{\Gamma} M$ be a Γ-covering. Γ acts on P from the left by deck transformations. Denote by

$$\mathscr{A}_P := C^\infty(P; \mathscr{A})^\Gamma$$

the algebra of Γ-invariant smooth $\mathscr{A}$-valued functions over P: any function $a \in \mathscr{A}_P$ verifies $a(\gamma^{-1} \cdot x) = (a(x))^\gamma$, $\forall x \in P, \gamma \in \Gamma$. Thus $\mathscr{A}_P$ is the algebra of smooth sections of a non-trivial bundle with fibre $\mathscr{A}$ over M. It can be represented as a subalgebra of matrices over $C^\infty(M)\hat{\otimes}(\mathscr{A} \rtimes \Gamma) = C^\infty(M; \mathscr{A} \rtimes \Gamma)$ as follows. Let (U_i), $i = 1, \ldots, m$ be a finite open covering of M trivializing the bundle P, via a set of sections $s_i : U_i \to P$ and locally constant transition functions $\gamma_{ij} : U_i \cap U_j \to \Gamma$:

$$\gamma_{ij}\gamma_{jk} = \gamma_{ik} \text{ over } U_i \cap U_j \cap U_k\ , \quad s_i(x) = \gamma_{ij} \cdot s_j(x) \quad \forall x \in U_i \cap U_j\ .$$

Choose a partition of unity $c_i \in C^\infty(M)$ relative to this covering: $\mathrm{supp}\, c_i \subset U_i$ and $\sum_{i=1}^m c_i(x)^2 = 1$. From these data consider the homomorphism $\rho : \mathscr{A}_P \to M_m(C^\infty(M)\hat{\otimes}(\mathscr{A} \rtimes \Gamma))$ sending an element $a \in \mathscr{A}_P$ to the $m \times m$ matrix $\rho(a)$ whose components, as $(\mathscr{A} \rtimes \Gamma)$-valued functions over M, read

$$\rho(a)_{ij}(x) := c_i(x)c_j(x)a(s_i(x))\gamma_{ji}^* \quad \forall\, i,j = 1,\ldots,m \ , \ \forall x \in M \ .$$

Of course ρ depends on the choice of trivialization (U_i, s_i) and partition of unity (c_i), but different choices are related by conjugation in a suitably large matrix algebra. Indeed, if (U'_α, s'_α), $\alpha = 1,\ldots,\mu$ denotes another trivialization with transition functions $\gamma'_{\alpha\beta}$ and partition of unity (c'_α), we get a corresponding homomorphism $\rho' : \mathscr{A}_P \to M_\mu(C^\infty(M)\hat{\otimes}(\mathscr{A} \rtimes \Gamma))$. Introduce the rectangular matrices u, v over $C^\infty(M)\hat{\otimes}(\mathscr{A} \rtimes \Gamma)$ with components

$$u_{i\alpha}(x) = c_i(x)c'_\alpha(x)\gamma_{\alpha i}^* \ , \qquad v_{\alpha i}(x) = c'_\alpha(x)c_i(x)\gamma_{i\alpha}^* \ ,$$

(recall $\mathscr{A}$ is unital by hypothesis hence $\mathbb{C}\Gamma \subset \mathscr{A} \rtimes \Gamma$), where the mixed transition functions $\gamma_{i\alpha}$, $\gamma_{\alpha i}$ are defined by $s_i(x) = \gamma_{i\alpha} \cdot s'_\alpha(x)$ and $s'_\alpha(x) = \gamma_{\alpha i} \cdot s_i(x)$ for any $x \in U_i \cap U'_\alpha$. Then uv and vu are idempotent square matrices, and for any element $a \in \mathscr{A}_P$ one has $\rho(a) = u\rho'(a)v$ and $\rho'(a) = v\rho(a)u$. Moreover, the invertible square matrix of size $m + \mu$

$$W = \begin{pmatrix} 1-uv & -u \\ v & 1-vu \end{pmatrix} \ , \qquad W^{-1} = \begin{pmatrix} 1-uv & u \\ -v & 1-vu \end{pmatrix}$$

verifies $W^{-1}\left(\begin{smallmatrix} \rho(a) & 0 \\ 0 & 0 \end{smallmatrix}\right)W = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \rho'(a) \end{smallmatrix}\right)$, which shows that the homomorphisms ρ and ρ' are stably conjugate.

In order to get a quasihomomorphism from $\mathscr{A}_P$ to $\mathscr{A} \rtimes \Gamma$, we need a K-cycle for the Fréchet algebra $C^\infty(M)$ (see Example 3.3). By a standard procedure [4, 5], such a K-cycle D may be constructed from an elliptic pseudodifferential operator or a Toeplitz operator over M. We shall suppose that D is of parity p mod 2, and of summability degree $p+1$ (even case) or p (odd case). Hence (see Example 3.3) in the even case one has an infinite-dimensional separable Hilbert space H with two continuous representations $C^\infty(M) \rightrightarrows \mathscr{L} = \mathscr{L}(H)$ which agree modulo the Schatten ideal $\mathscr{I} = \mathscr{L}^{p+1}(H)$, whereas in the odd case the algebra $C^\infty(M)$ is represented in the matrix algebra $\left(\begin{smallmatrix} \mathscr{L} & \mathscr{I} \\ \mathscr{I} & \mathscr{L} \end{smallmatrix}\right)$ with $\mathscr{I} = \mathscr{L}^p(H)$. Therefore, upon choosing an isomorphism $H \cong H\hat{\otimes}\mathbb{C}^m$ the composition of $\rho : \mathscr{A}_P \to M_m(C^\infty(M)\hat{\otimes}(\mathscr{A} \rtimes \Gamma))$ with the Hilbert space representation induced by the K-cycle D leads to a quasihomomorphism of parity p mod 2

$$\rho_D : \mathscr{A}_P \to \mathscr{E}^s \triangleright \mathscr{I}^s\hat{\otimes}(\mathscr{A} \rtimes \Gamma) \ ,$$

with intermediate algebra $\mathscr{E} = \mathscr{L}\hat{\otimes}(\mathscr{A} \rtimes \Gamma)$ (or $(\mathscr{L} \ltimes \mathscr{I})\hat{\otimes}(\mathscr{A} \rtimes \Gamma)$, see Example 3.4). Note that $\mathscr{L}$ and $\mathscr{I}$ may be replaced by other suitable operator algebras, if needed. From the discussion above we see that ρ_D depends only on D up to conjugation by an invertible element $W \in \mathscr{E}^s_+$. Taking into account the Chern characters in negative and periodic cyclic homology, the Riemann-Roch-Grothendieck Theorem 6.3 thus yields cube diagrams of the following kind:

COROLLARY 7.1 *A K-cycle D over $C^\infty(M)$ as above yields for any integer $n \in \mathbb{Z}$ a commutative diagram*

$$\begin{tikzcd}
& K_n^{\mathrm{top}}(\mathscr{I}\mathscr{A}_P) \arrow[rr] \arrow[dd] & & K_{n-p}^{\mathrm{top}}(\mathscr{I}(\mathscr{A} \rtimes \Gamma)) \arrow[dd] \\
MK_n^{\mathscr{I}}(\mathscr{A}_P) \arrow[ur] \arrow[rr] \arrow[dd] & & MK_{n-p}^{\mathscr{I}}(\mathscr{A} \rtimes \Gamma) \arrow[ur] \arrow[dd] & \\
& HP_n(\mathscr{A}_P) \arrow[rr] & & HP_{n-p}(\mathscr{A} \rtimes \Gamma) \\
HN_n(\mathscr{A}_P) \arrow[ur] \arrow[rr] & & HN_{n-p}(\mathscr{A} \rtimes \Gamma) \arrow[ur] &
\end{tikzcd}$$

where the horizontal arrows are induced by the quasihomomorphism $\rho_D : \mathscr{A}_P \to \mathscr{E}^s \triangleright \mathscr{I}^s \hat{\otimes}(\mathscr{A} \rtimes \Gamma)$.

The background square describes the topological side of the Riemann-Roch-Grothendieck theorem, namely the compatibility between the push-forward in topological K-theory and the bivariant Chern character in periodic cyclic homology. One may choose D as a representative of the fundamental class in the K-homology of M. If moreover M is a model for the classifying space $B\Gamma$, one may choose P as the universal bundle $E\Gamma$. For torsion-free groups Γ the morphism $K_n^{\mathrm{top}}(\mathscr{I}\mathscr{A}_P) \to K_{n-p}^{\mathrm{top}}(\mathscr{I}(\mathscr{A} \rtimes \Gamma))$ thus obtained is related to the Baum-Connes assembly map [1] and exhausts many (in some cases, all the) interesting topological K-theory classes of $\mathscr{A} \rtimes \Gamma$.
The foreground square provides a lifting of the topological situation at the level of multiplicative K-theory and negative cyclic homology, i.e. secondary characteristic classes. Hence a part of $MK_*^{\mathscr{I}}(\mathscr{A} \rtimes \Gamma)$ may be obtained by direct images of multiplicative K-theory over $\mathscr{A}_P$. Note that in contrast to the topological situation, the push-forward map in multiplicative K-theory does *not* exhaust all the interesting classes over $\mathscr{A} \rtimes \Gamma$.

Let us now deal with the case $\mathscr{A} = C^\infty(N)$, for a compact smooth Riemannian manifold N, endowed with a left action of Γ by diffeomorphisms. We provide $\mathscr{A}$ with its usual Fréchet topology, and condition (89) forces the Γ-action be "almost isometric" on N. The crossed product $\mathscr{A} \rtimes \Gamma$ is then isomorphic to a certain convolution algebra of functions over the smooth étale groupoid $\Gamma \ltimes N$, describing a highly noncommutative space when the action of Γ is not proper. The commutative algebra $\mathscr{A}_P$ is the subalgebra of smooth Γ-invariant functions $a \in C^\infty(P \times N)$, $a(\gamma \cdot x, \gamma \cdot y) = a(x,y)$ for any $(x,y) \in P \times N$, and is thus isomorphic to the algebra of smooth functions over the (compact) quotient manifold $Q = \Gamma \backslash (P \times N)$.

The problem is therefore reduced to the computation of secondary invariants for the classical space Q. The cyclic homology of $\mathscr{A}_P = C^\infty(Q)$ has been determined by Connes [4] and is computable from the de Rham complex of differential forms over Q. We will see that the multiplicative K-theory $MK_*^{\mathscr{I}}(\mathscr{A}_P)$ is closely related (though not isomorphic) to Deligne cohomology. We first recall some definitions. Let $\Omega^n(Q)$ denote the space of complex, smooth differential n-forms over Q, d the de Rham coboundary, $Z^n_{\mathrm{dR}}(Q) = \mathrm{Ker}(d : \Omega^n \to \Omega^{n+1})$ the space of closed n-forms and $B^n_{\mathrm{dR}}(Q) = \mathrm{Im}(d : \Omega^{n-1} \to \Omega^n)$ the space of exact n-forms. By de Rham's theorem, the de Rham cohomology $H^n_{\mathrm{dR}}(Q) = Z^n_{\mathrm{dR}}(Q)/B^n_{\mathrm{dR}}(Q)$ is isomorphic to the Čech cohomology of Q with complex coefficients $\check{H}^n(Q;\mathbb{C})$. For any half-integer q we define the additive group $\mathbb{Z}(q) := (2\pi i)^q\mathbb{Z} \subset \mathbb{C}$ (the square root of $2\pi i$ must be chosen consistentl y with the Chern character on K_1^{top}). Let $\underline{\Omega}^k$ denote the sheaf of differential k-forms over Q and consider for any $n \in \mathbb{N}$ the complex of sheaves

$$0 \longrightarrow \underline{\mathbb{Z}}(n/2) \longrightarrow \underline{\Omega}^0 \xrightarrow{d} \underline{\Omega}^1 \xrightarrow{d} \ldots \xrightarrow{d} \underline{\Omega}^{n-1} \longrightarrow 0 \tag{90}$$

where the constant sheaf $\underline{\mathbb{Z}}(n/2)$ sits in degree 0 and $\underline{\Omega}^k$ in degree $k+1$. The map $\underline{\mathbb{Z}}(n/2) \to \underline{\Omega}^0$ is induced by the natural inclusion of constant functions into complex-valued functions. By definition the (smooth) Deligne cohomology $H^n_{\mathscr{D}}(Q;\mathbb{Z}(n/2))$ is the hyperhomology of (90) in degree n. The natural projection onto the constant sheaf $\underline{\mathbb{Z}}(n/2)$ yields a well-defined map from $H^n_{\mathscr{D}}(Q;\mathbb{Z}(n/2))$ to the Čech cohomology with integral coefficients $\check{H}^n(Q;\mathbb{Z}(n/2))$. On the other extreme, the de Rham coboundary $d : \underline{\Omega}^{n-1} \to \underline{\Omega}^n$ sends a Deligne n-cocycle to a globally defined closed n-form over Q, called its *curvature*, which only depends on the Deligne cohomology class. It follows from the definitions that the image of the curvature in de Rham cohomology coincides with the complexification of the Čech cohomology class of the Deligne cocycle. One thus gets a commutative diagram in any degree n

$$\begin{tikzcd} H^n_{\mathscr{D}}(Q;\mathbb{Z}(n/2)) \arrow[r] \arrow[d, "d"'] & \check{H}^n(Q;\mathbb{Z}(n/2)) \arrow[d, "\otimes\mathbb{C}"] \\ Z^n_{\mathrm{dR}}(Q) \arrow[r] & H^n_{\mathrm{dR}}(Q) \end{tikzcd} \tag{91}$$

This has to be compared with the commutative square involving the multiplicative and topological K-theories of the algebra $\mathscr{A}_P = C^\infty(Q)$, with their Chern characters:

$$\begin{tikzcd} MK_n^{\mathscr{I}}(\mathscr{A}_P) \arrow[r] \arrow[d] & K_n^{\mathrm{top}}(\mathscr{I}\mathscr{A}_P) \arrow[d] \\ HN_n(\mathscr{A}_P) \arrow[r] & HP_n(\mathscr{A}_P) \end{tikzcd} \tag{92}$$

In fact one can construct, at least in low degrees n, an explicit transformation from Deligne cohomology to multiplicative K-theory, and the curvature morphism captures the lowest degree part of the negative Chern character. Let us explain this with some details. Firstly, it is well-known that the bottom line of (91) is included as a direct summand in the bottom line of (92). Since we deal essentially with the X-complex description of cyclic homology (section 2), we recall how the latter is related to the de Rham cohomology of Q. Choose the universal free extension $0 \to J\mathscr{A}_P \to T\mathscr{A}_P \to \mathscr{A}_P \to 0$. The cyclic homology of $\mathscr{A}_P$ is computed by the X-complex $X(\widehat{T}\mathscr{A}_P)$ of the pro-algebra $\widehat{T}\mathscr{A}_P = \varprojlim_n T\mathscr{A}_P/(J\mathscr{A}_P)^n$, together with its filtration by the subcomplexes $F^n\widehat{X}(T\mathscr{A}_P, J\mathscr{A}_P)$. As a pro-vector space, $X(\widehat{T}\mathscr{A}_P)$ is isomorphic to the completed space of noncommutative differential forms $\widehat{\Omega}\mathscr{A}_P$, and the $J\mathscr{A}_P$-adic filtration coincides with the Hodge filtration $F^n\widehat{\Omega}\mathscr{A}_P$. A canonical chain map $X(\widehat{T}\mathscr{A}_P) \to \Omega^*(Q)$ is given by projecting noncommutative differential forms to commutative ones, up to a rescaling:

$$\Omega^n\mathscr{A}_P \ni a_0da_1\dots da_n \to \lambda_n a_0da_1\dots da_n \in \Omega^n(Q)$$

with $\lambda_n = (-)^k\frac{k!}{(2k)!}$ if $n = 2k$ and $\lambda_n = (-)^k\frac{k!}{(2k+1)!}$ if $n = 2k+1$. These factors are fixed in order to get exactly a chain map. Clearly it sends the Hodge filtration of $\widehat{\Omega}\mathscr{A}_P$ onto the natural filtration by degree on $\Omega^*(Q)$. The following proposition is a reformulation of Connes' version of the Hochschild-Kostant-Rosenberg theorem [4].

Proposition 7.2 *The chain map $X(\widehat{T}\mathscr{A}_P) \to \Omega^*(Q)$ is a homotopy equivalence compatible with the filtrations. Hence follow the isomorphisms*

$$\begin{aligned} HP_n(\mathscr{A}_P) &= \bigoplus_{k\in\mathbb{Z}} H^{n+2k}_{\mathrm{dR}}(Q)\ , \\ HC_n(\mathscr{A}_P) &= \frac{\Omega^n(Q)}{B^n_{\mathrm{dR}}(Q)} \oplus \bigoplus_{k<0} H^{n+2k}_{\mathrm{dR}}(Q)\ , \\ HN_n(\mathscr{A}_P) &= Z^n_{\mathrm{dR}}(Q) \oplus \bigoplus_{k>0} H^{n+2k}_{\mathrm{dR}}(Q)\ . \end{aligned} \tag{93}$$

Of course the injections $Z^n_{\mathrm{dR}}(Q) \to HN_n(\mathscr{A}_P)$ and $H^n_{\mathrm{dR}}(Q) \to HP_n(\mathscr{A}_P)$ are compatible with the forgetful maps $Z^n_{\mathrm{dR}}(Q) \to H^n_{\mathrm{dR}}(Q)$ and $HN_n(\mathscr{A}_P) \to HP_n(\mathscr{A}_P)$. It is useful to calculate the image of the Chern character of idempotents and invertibles under the chain map $X(\widehat{T}\mathscr{A}_P) \to \Omega^*(Q)$. Let $e \in M_2(\mathscr{K}\hat{\otimes}\mathscr{A}_P)^+$ be an idempotent such that $e - p_0 \in M_2(\mathscr{K}\hat{\otimes}\mathscr{A}_P)$, and let $\hat{e} \in M_2(\mathscr{K}\hat{\otimes}\widehat{T}\mathscr{A}_P)^+$ be its canonical lift. The image of the Chern character $\mathrm{ch}_0(\hat{e}) = \mathrm{Tr}(\hat{e} - p_0) \in X(\widehat{T}\mathscr{A}_P)$ is the differential form of even degree

$$\mathrm{ch}_{\mathrm{dR}}(e) = \mathrm{Tr}(e - p_0) + \sum_{k\geq 1} \frac{(-)^k}{k!}\,\mathrm{Tr}((e - \frac{1}{2})(dede)^k)\ \in \Omega^+(Q)\ .$$

Let $g \in (\mathscr{K}\hat{\otimes}\mathscr{A}_P)^+$ be an invertible such that $g - 1 \in \mathscr{K}\hat{\otimes}\mathscr{A}_P$, and let $\hat{g} \in (\mathscr{K}\hat{\otimes}\widehat{T}\mathscr{A}_P)^+$ be its canonical lift. The image of the Chern character $\mathrm{ch}_1(\hat{g}) =$

$\frac{1}{\sqrt{2\pi i}}\natural\hat{g}^{-1}\mathbf{d}\hat{g} \in X(\widehat{T}\mathscr{A}_P)$ is the differential form of odd degree

$$\mathrm{ch}_{\mathrm{dR}}(g) = \frac{1}{\sqrt{2\pi i}} \sum_{k\geq 0} (-)^k \frac{k!}{(2k+1)!} \mathrm{Tr}(g^{-1}dg(dg^{-1}dg)^k) \ \in \Omega^-(Q) \ .$$

We now construct explicit morphisms $H^n_{\mathscr{D}}(Q;\mathbb{Z}(n/2)) \to MK^{\mathscr{I}}_n(\mathscr{A}_P)$ in degrees $n = 0, 1, 2$. In fact the ideal $\mathscr{I}$ is irrelevant and the previous morphisms factor through the multiplicative group $MK_n(\mathscr{A}_P) := MK^{\mathbb{C}}_n(\mathscr{A}_P)$ associated to the 1-summable algebra $\mathbb{C}$. Then choosing any rank one injection $\mathbb{C} \to \mathscr{I}$ induces a unique map $MK_n(\mathscr{A}_P) \to MK^{\mathscr{I}}_n(\mathscr{A}_P)$ by virtue of Lemma 5.5.

$n = 0$: Then $\mathbb{Z}(0) = \mathbb{Z}$ and the complex $0 \to \underline{\mathbb{Z}}(0) \to 0$ calculates the Čech cohomology of Q with coefficients in $\mathbb{Z}$. Hence $H^0_{\mathscr{D}}(Q;\mathbb{Z}(0)) = \check{H}^0(Q;\mathbb{Z})$ is the additive group of $\mathbb{Z}$-valued locally constant functions over Q. The map

$$H^0_{\mathscr{D}}(Q;\mathbb{Z}(0)) \to MK_0(\mathscr{A}_P) \cong K^{\mathrm{top}}_0(\mathscr{A}_P) \tag{94}$$

associates to such a function f the K-theory class of the trivial complex vector bundle of rank f over Q.

$n = 1$: Then $\mathbb{Z}(1/2) = \sqrt{2\pi i}\,\mathbb{Z}$ and $H^1_{\mathscr{D}}(Q;\mathbb{Z}(1/2))$ is the hyperhomology in degree 1 of the complex $0 \to \underline{\mathbb{Z}}(1/2) \to \underline{\Omega}^0 \to 0$. Choose a good covering (U_i) of Q. A Deligne 1-cocycle is given by a collection (f_i, n_{ij}) of smooth functions $f_i : U_i \to \mathbb{C}$ and locally constant functions $n_{ij} : U_i \cap U_j \to \mathbb{Z}$ related by the descent equations

$$f_i - f_j = \sqrt{2\pi i}\, n_{ij} \text{ over } U_i \cap U_j \ , \quad n_{jk} - n_{ik} + n_{ij} = 0 \text{ over } U_i \cap U_j \cap U_k \ .$$

The cocycle is trivial if the f_i's are $\mathbb{Z}(1/2)$-valued. Taking the exponentials $g_i = \exp(\sqrt{2\pi i} f_i)$ one gets invertible smooth functions which agree on the overlaps $U_i \cap U_j$, hence define a global invertible function g over Q. The latter is equal to 1 exactly when the cocycle (f_i, n_{ij}) is trivial. Hence $H^1_{\mathscr{D}}(Q;\mathbb{Z}(1/2))$ is the multiplicative group $C^\infty(Q)^\times$ of complex-valued invertible functions over Q. On the other hand, the elements of $MK_1(\mathscr{A}_P)$ are represented by pairs $(\hat{g}, \theta)$ of an invertible $\hat{g} \in (\mathscr{K}\hat{\otimes}\widehat{T}\mathscr{A}_P)^+$ and a cochain $\theta \in X_0(T\mathscr{A}_P, J\mathscr{A}_P) \cong C^\infty(Q)$. We get a map

$$H^1_{\mathscr{D}}(Q;\mathbb{Z}(1/2)) \cong C^\infty(Q)^\times \to MK_1(\mathscr{A}_P) \tag{95}$$

by sending an invertible $g \in C^\infty(Q)^\times$ to the multiplicative K-theory class of $(\hat{g}, 0)$, with $\hat{g}$ the canonical lift of g (to be precise one should replace g by $1 + (g - 1_{\mathscr{A}_P}) \in (\mathscr{A}_P)^+$). This map identifies the curvature morphism $H^1_{\mathscr{D}}(Q;\mathbb{Z}(1/2)) \to Z^1_{\mathrm{dR}}(Q)$ with the lowest degree part of the negative Chern character $\mathrm{ch}_1 : MK_1(\mathscr{A}_P) \to HN_1(\mathscr{A}_P)$. Indeed the curvature of an element $g \in C^\infty(Q)^\times$ is by definition the closed one-form

$$df_i = \frac{1}{\sqrt{2\pi i}}\, g_i^{-1} dg_i = \frac{1}{\sqrt{2\pi i}}\, g^{-1} dg \qquad \forall i \ ,$$

globally defined over Q. But this coincides with the $Z^1_{\mathrm{dR}}(Q)$-component of the negative Chern character $\mathrm{ch}_1(\hat{g},0)$.

$n=2$: Then $\mathbb{Z}(1)=2\pi i\,\mathbb{Z}$ and $H^2_{\mathscr{D}}(Q;\mathbb{Z}(1))$ is the hyperhomology in degree 2 of the complex $0\to\underline{\mathbb{Z}}(1)\to\underline{\Omega}^0\to\underline{\Omega}^1\to 0$. A Deligne cocycle relative to the finite good covering (U_i), $i=1,\ldots,m$ is a collection (A_i,f_{ij},n_{ijk}) of one-forms A_i over U_i, smooth functions $f_{ij}:U_i\cap U_j\to\mathbb{C}$ and locally constant functions $n_{ijk}:U_i\cap U_j\cap U_k\to\mathbb{Z}$, subject to the descent equations

$$A_i-A_j=df_{ij}\ ,\quad -f_{jk}+f_{ik}-f_{ij}=2\pi i\,n_{ijk}\ ,\quad n_{jkl}-n_{ikl}+n_{ijl}-n_{ijk}=0\ .$$

Equivalently, passing to the exponentials $g_{ij}=\exp f_{ij}$ a cocycle is a collection (A_i,g_{ij}) such that $A_i-A_j=g_{ij}^{-1}dg_{ij}$ and $g_{ij}g_{jk}=g_{ik}$. Two cocycles (A_i,g_{ij}) and (A'_i,g'_{ij}) are cohomologous iff there exists a collection of smooth invertible functions (gauge transformations) $\alpha_i:U_i\to\mathbb{C}^\times$ such that

$$A'_i=A_i+\alpha_i^{-1}d\alpha_i\ ,\qquad g'_{ij}=\alpha_i g_{ij}\alpha_j^{-1}\ .$$

One sees that a Deligne cohomology class is nothing else but a complex line bundle over Q, described by the smooth transition functions $g_{ij}:U_i\cap U_j\to\mathbb{C}^\times$, together with a connection given locally by the one-forms A_i, up to gauge transformation. Hence

$$H^2_{\mathscr{D}}(Q;\mathbb{Z}(1))=\{\text{isomorphism classes of complex line bundles with connection}\}$$

The group law is the tensor product of line bundles with connections. The curvature morphism $H^2_{\mathscr{D}}(Q;\mathbb{Z}(1))\to Z^2_{\mathrm{dR}}(Q)$ maps a cocycle (A_i,g_{ij}) to the globally defined closed two-form $dA_i\ \forall i$, i.e. the curvature of the connection of the corresponding line bundle. The construction of the morphism from Deligne cohomology to multiplicative K-theory requires to fix a partition of unity (c_i) relative to the finite covering: $\mathrm{supp}\,c_i\subset U_i$ and $\sum_i c_i(x)^2=1\ \forall x\in Q$. Given a Deligne cocycle (A_i,g_{ij}), we construct the idempotent $e_+\in M_m(C^\infty(Q))$ of rank 1 whose matrix elements are the functions

$$(e_+)_{ij}=c_ig_{ij}c_j\ \in C^\infty_c(U_i\cap U_j)\ ,$$

and let $e_-=1_{\mathscr{A}_P}$ be the unit of $C^\infty(Q)$ (the constant function 1 over Q). Define $e\in M_2(\mathscr{A}_P)^+$ as the idempotent matrix $\left(\begin{smallmatrix}1-e_-&0\\0&e_+\end{smallmatrix}\right)$. The 0th degree of the Chern character $\mathrm{ch}_{\mathrm{dR}}(e)$ is $\mathrm{Tr}(e-p_0)=\mathrm{Tr}(e_+)-1_{\mathscr{A}_P}=0$, so that the class of e in $K_0^{\mathrm{top}}(\mathscr{A}_P)$ is a virtual bundle of rank 0. To get a multiplicative K-theory class $(\hat{e},\theta)\in MK_2(\mathscr{A}_P)$, we must adjoin to the canonical idempotent lift $\hat{e}\in M_2(\widehat{T}\mathscr{A}_P)^+$ an odd cochain $\theta\in X_1(T\mathscr{A}_P,J\mathscr{A}_P)\cong X(\mathscr{A}_P)$. Since $\mathscr{A}_P$ is the commutative algebra of smooth functions over a compact manifold, its X-complex reduces to the de Rham complex of Q truncated in degrees ≥ 2, $X(\mathscr{A}_P):C^\infty(Q)\xrightarrow{d}\Omega^1(Q)$. The fact that e is of virtual rank zero insures that any choice of one-form $\theta \in \Omega^1(Q)$ satisfies the correct transgression relation $\mathrm{ch}_0(\hat{e}) = \overline{b}\theta = 0$ in the complex $X(\mathscr{A}_P)$. We s et

$$\theta = -\sum_i c_i^2 A_i \ \in \Omega^1(Q) \ .$$

Lemma 7.3 *The assignement $(A_i, g_{ij}) \mapsto (\hat{e}, \theta)$ yields a well-defined morphism*

$$H^2_{\mathscr{D}}(Q; \mathbb{Z}(1)) \to MK_2(\mathscr{A}_P) \ , \tag{96}$$

and the curvature of (A_i, g_{ij}) corresponds to the $Z^2_{\mathrm{dR}}(Q)$ component of the negative Chern character $\mathrm{ch}_2(\hat{e}, \theta) \in HN_2(\mathscr{A}_P)$.

Proof: We have to show that the multiplicative K-theory class of $(\hat{e}, \theta)$ only depends on the Deligne cohomology class of (A_i, g_{ij}). Thus let (A'_i, g'_{ij}) be another representative, with $A'_i = A_i + \alpha_i^{-1} d\alpha_i$ and $g'_{ij} = \alpha_i g_{ij} \alpha_j^{-1}$. This yields a new pair $(\hat{e}', \theta')$ with $(e'_+)_{ij} = c_i g'_{ij} c_j$ and $\theta' = -\sum_i c_i^2 A'_i$. We show that $(\hat{e}, \theta)$ and $(\hat{e}', \theta')$ represent the same class in $MK_2(\mathscr{A}_P)$ by using the following general fact: if e and $e' = W^{-1} e W$ are conjugate by an invertible W, then $(\hat{e}, \theta)$ is equivalent to $(\hat{e}', \theta + \mathrm{cs}_1(f))$, where f is the idempotent interpolation between the matrices $\left(\begin{smallmatrix} e & 0 \\ 0 & p_0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} W^{-1}eW & 0 \\ 0 & p_0 \end{smallmatrix}\right)$ constructed as in Lemma 5.5. One has

$$\mathrm{cs}_1(f) \equiv \mathrm{Tr}(W^{-1}(e - p_0) dW) \bmod d \quad \text{in } \Omega^1(Q) \ ,$$

so that finally $(\hat{e}, \theta)$ is equivalent to $(\hat{e}', \theta + \mathrm{Tr}(W^{-1}(e - p_0) dW))$. In the present situation $g'_{ij} = \alpha_i g_{ij} \alpha_j^{-1}$, hence the idempotent e'_+ is conjugate to e_+ via the diagonal matrix $W_+ = \mathrm{diag}\ (\alpha_1^{-1} \ldots, \alpha_m^{-1})$, and of course $e'_- = e_- = 1_{\mathscr{A}_P}$ so one can choose $W_- = 1$. One calculates

$$\begin{aligned} \mathrm{Tr}(W^{-1}(e - p_0) dW) &= \mathrm{Tr}(W_+^{-1} e_+ dW_+) \ = \ \sum_i \alpha_i c_i^2 d(\alpha_i^{-1}) \\ &= \ -\sum_i c_i^2 (A'_i - A_i) \ = \ \theta' - \theta \ , \end{aligned}$$

hence $(\hat{e}, \theta)$ and $(\hat{e}', \theta')$ represent the same multiplicative K-theory class.
Now we leave the cocycle (A_i, g_{ij}) fixed and change the partition of unity (c_i) to (c'_i), $\sum_i (c'_i)^2 = 1$, whence a new idempotent $(e'_+)_{ij} = c'_i g_{ij} c'_j$ and a new cochain $\theta' = -\sum_i (c'_i)^2 A_i$. Introduce the matrices $u_{ij} = c_i g_{ij} c'_j$ and $v_{ij} = c'_i g_{ij} c_j$. Then one has $e_+ = uv$, $e'_+ = vu$, and the invertible matrix $W_+ = \left(\begin{smallmatrix} 1-uv & -u \\ v & 1-vu \end{smallmatrix}\right)$ stably conjugates e_+ and e'_+ in the sense that $W_+^{-1} \left(\begin{smallmatrix} e_+ & 0 \\ 0 & 0 \end{smallmatrix}\right) W_+ = \left(\begin{smallmatrix} 0 & 0 \\ 0 & e'_+ \end{smallmatrix}\right)$. A direct computation yields

$$\mathrm{Tr}(W_+^{-1} \left(\begin{smallmatrix} e_+ & 0 \\ 0 & 0 \end{smallmatrix}\right) dW_+) = \sum_{i,j} (c'_i)^2 c_j^2 g_{ij} dg_{ji} = \sum_{i,j} (c'_i)^2 c_j^2 (A_j - A_i) = \theta' - \theta \ ,$$

hence the multiplicative K-theory class of $(\hat{e}, \theta)$ does not depend on the choice of partition of unity. A similar argument shows that it does not depend on the

good covering (U_i).
It remains to calulate the lowest component of the negative Chern character. By definition $\mathrm{ch}_2(\hat{e},\theta)$ is the cycle of even degree in $X(\widehat{T}\mathscr{A}_P)$ given by $\mathrm{Tr}(\hat{e}-p_0)-\overline{b}\tilde{\theta}$, where $\tilde{\theta}$ is an arbitrary lifting of $\theta\in X(\mathscr{A}_P)$. Thus, the image of $\mathrm{ch}_2(\hat{e},\theta)$ under the chain map $X(\widehat{T}\mathscr{A}_P)\to\Omega^*(Q)$ has a component of degree two given by

$$-\mathrm{Tr}((e-\frac{1}{2})dede)-d\theta=-\sum_i d(c_i^2)A_i+d\sum_i c_i^2A_i=\sum_i c_i^2dA_i=dA_i\ ,$$

and coincides with the curvature of the line bundle (A_i,g_{ij}). ∎

If one forgets the connection A_i, the morphism $H^2_{\mathscr{D}}(Q;\mathbb{Z}(1))\to MK_2(\mathscr{A}_P)$ just reduces to the elementary map $\check{H}^2(Q;\mathbb{Z}(1))\to K_2^{\mathrm{top}}(\mathscr{A}_P)$ which associates to an isomorphism class of line bundles over Q its topological K-theory class.

Example 7.4 The simplest non-trivial example is provided by the celebrated non-commutative torus [3]. Here $\mathscr{A}$ is the algebra of smooth functions over the circle $N=S^1=\mathbb{Z}\backslash\mathbb{R}$. Conventionnally we parametrize the points of N by the variable y. The group $\Gamma=\mathbb{Z}$ acts on N by rotations of angle $\alpha\in\mathbb{R}$:

$$y\mapsto y+n\alpha\qquad\forall\ y\in N\ ,\ n\in\mathbb{Z}\ .$$

When $\mathbb{Z}$ is provided with its natural distance, the crossed product $\mathscr{A}\rtimes\mathbb{Z}$ is isomorphic to the algebra $\mathscr{A}_\alpha$ of the non-commutative torus, presented for example in [3] by generators and relations: let $V_1\in\mathscr{A}$ be the function $V_1(y)=e^{2\pi iy}$ over N and $V_2=1^*\in\mathbb{C}\mathbb{Z}$ be the element corresponding to the generator $1\in\mathbb{Z}$. Then V_1 and V_2 are invertible elements of $\mathscr{A}_\alpha$ and fulfill the noncommutativity relation

$$V_2V_1=e^{2\pi i\alpha}V_1V_2\ . \tag{97}$$

Moreover any element of $\mathscr{A}_\alpha$ is a power series $\sum_{(n_1,n_2)\in\mathbb{Z}^2}a_{n_1n_2}V_1^{n_1}V_2^{n_2}$ with coefficients $a_{n_1n_2}\in\mathbb{C}$ of rapid decay. For $\alpha\in\mathbb{Q}$ this algebra is Morita equivalent to the smooth functions over an ordinary (commutative) 2-torus, and its multiplicative K-theory in any degree turns out to be completely determined by Deligne cohomology in this case. The situation is more interesting for $\alpha\notin\mathbb{Q}$. Following the discussion above we introduce the universal principal $\mathbb{Z}$-bundle $P=E\mathbb{Z}=\mathbb{R}$ over the classifying space $M=B\mathbb{Z}=\mathbb{Z}\backslash\mathbb{R}$. Conventionnally we parametrize the points of P by the variable x. Thus, $\mathscr{A}_P=C^\infty(P;\mathscr{A})^{\mathbb{Z}}$ is the mapping torus algebra

$$\mathscr{A}_P=\{a\in C^\infty(P\times N)\mid a(x+1,y+\alpha)=a(x,y)\ ,\ \forall\ x\in P\ ,\ y\in N\}\ .$$

Equivalently it is the algebra of smooth functions over the commutative 2-torus $Q=\mathbb{Z}\backslash(P\times N)$, quotient of $\mathbb{R}^2$ by the lattice generated by the vectors $(1,\alpha)$ and $(0,1)$. Now to get a quasihomomorphism from $\mathscr{A}_P$ to $\mathscr{A}_\alpha$ we need a K-cycle

D for the circle manifold M. Let $H = L^2(M)$ be the Hilbert space of square-integrable complex-valued functions. The algebra $C^\infty(M)$ is represented in the algebra of bounded operators $\mathscr{L}(H)$ by pointwise multiplication. D will be represented by the Toeplitz operator in $\mathscr{L}(H)$ which projects H onto the Hardy space $H_+ \subset H$:

$$D(e^{2\pi i\, nx}) = \begin{cases} e^{2\pi i\, nx} & \text{if} \quad n \geq 0 \\ 0 & \text{if} \quad n < 0 \end{cases}$$

One thus obtains a polarization of the Hilbert space $H = H_+ \oplus H_-$, with H_- the kernel of D. In the Fourier basis $e^{2\pi i\, nx}$ of H, the representation $C^\infty(M) \to \mathscr{L}(H)$ is easily seen to factor through the matrix subalgebra

$$\begin{pmatrix} \mathscr{T} & \mathscr{K} \\ \mathscr{K} & \mathscr{T} \end{pmatrix} \subset \begin{pmatrix} \mathscr{L}(H_+) & \mathscr{L}(H_-, H_+) \\ \mathscr{L}(H_+, H_-) & \mathscr{L}(H_-) \end{pmatrix} = \mathscr{L}(H) \ ,$$

where $\mathscr{T}$ is the *smooth Toeplitz algebra* (the elementary non-trivial extension of $C^\infty(S^1)$ by th ealgebra $\mathscr{K}$ of smooth compact operators, see [9]). The induced quasihomomorphism $\rho_D : \mathscr{A}_P \to \mathscr{E}^s \triangleright \mathscr{K}^s \hat{\otimes} \mathscr{A}_\alpha$, with $\mathscr{E} = \mathscr{T} \hat{\otimes} \mathscr{A}_\alpha$, is therefore 1-summable and of odd parity. Theorem 6.3 yields a graded-commutative diagram (remark that $MK_n^{\mathscr{K}} \cong MK_n$)

$$\begin{tikzcd} K^{\rm top}_{n+1}(\mathscr{A}_P) \arrow[r] \arrow[d] & HC_{n-1}(\mathscr{A}_P) \arrow[r] \arrow[d, "\mathrm{ch}^1(\rho_D)"] & MK_n(\mathscr{A}_P) \arrow[r] \arrow[d] & K^{\rm top}_n(\mathscr{A}_P) \arrow[d] \\ K^{\rm top}_n(\mathscr{A}_\alpha) \arrow[r] & HC_{n-2}(\mathscr{A}_\alpha) \arrow[r] & MK_{n-1}(\mathscr{A}_\alpha) \arrow[r] & K^{\rm top}_{n-1}(\mathscr{A}_\alpha) \end{tikzcd} \tag{98}$$

in any degree $n \in \mathbb{Z}$. The group $K_n^{\rm top}(\mathscr{A}_P)$ is isomorphic to the topological K-theory of the 2-torus Q. Hence in even degree, $K_0^{\rm top}(\mathscr{A}_P) = \mathbb{Z} \oplus \mathbb{Z}$ is generated by the trivial line bundle over Q together with the Bott class, whereas in odd degree $K_1^{\rm top}(\mathscr{A}_P) = \mathbb{Z} \oplus \mathbb{Z}$ is generated by the invertible functions $g_1(x,y) = e^{2\pi i\, x}$ and $g_2(x,y) = e^{2\pi i(\alpha x - y)}$. The pushforward map $K_n^{\rm top}(\mathscr{A}_P) \to K_{n-1}^{\rm top}(\mathscr{A}_\alpha)$ is known to be an isomorphism (Baum-Connes). In particular the Bott class and the trivial line bundle over Q are mapped respectively to the classes of the invertible elements V_1 and V_2 in $K_1^{\rm top}(\mathscr{A}_\alpha)$. For multiplicative K-theory the situation is more involved. In degree $n = 1$ the map

$$MK_1(\mathscr{A}_P) \cong H^1_{\mathscr{D}}(Q; \mathbb{Z}(1/2)) \cong C^\infty(Q)^\times \to MK_0(\mathscr{A}_\alpha) \cong K_0^{\rm top}(\mathscr{A}_\alpha)$$

simply factors through the topological K-theory group $K_1^{\rm top}(\mathscr{A}_P)$. In degree $n = 2$ one still has an isomorphism $MK_2(\mathscr{A}_P) \cong H^2_{\mathscr{D}}(Q; \mathbb{Z}(1))$, and (98) amounts to

$$\begin{tikzcd} K^{\rm top}_1(\mathscr{A}_P) \arrow[r, "\mathrm{ch}_1"] \arrow[d] & \frac{\Omega^1(Q)}{d\Omega^0(Q)} \arrow[r] \arrow[d, "\mathrm{ch}^1(\rho_D)"] & H^2_{\mathscr{D}}(Q;\mathbb{Z}(1)) \arrow[r] \arrow[d] & K^{\rm top}_0(\mathscr{A}_P) \arrow[d] & \\ K^{\rm top}_0(\mathscr{A}_\alpha) \arrow[r] & HC_0(\mathscr{A}_\alpha) \arrow[r] & MK_1(\mathscr{A}_\alpha) \arrow[r] & K^{\rm top}_1(\mathscr{A}_\alpha) \arrow[r] & 0 \end{tikzcd} \tag{99}$$

The map $\Omega^1/d\Omega^0 \to H^2_{\mathscr{D}}(Q;\mathbb{Z}(1))$ associates to a one-form $\frac{1}{\sqrt{2\pi i}}A$ over Q the isomorphism class of the trivial line bundle with connection $-A$, while the range of $H^2_{\mathscr{D}}(Q;\mathbb{Z}(1)) \to K_0^{\rm top}(\mathscr{A}_P)$ is generated by the Bott class. For generic values of $\alpha \notin \mathbb{Q}$ the commutator subspace $[\mathscr{A}_\alpha,\mathscr{A}_\alpha]$ may not be closed in $\mathscr{A}_\alpha$, therefore the quotient $HC_0(\mathscr{A}_\alpha) = HH_0(\mathscr{A}_\alpha) = \mathscr{A}_\alpha/[\mathscr{A}_\alpha,\mathscr{A}_\alpha]$ may not be separated. However the quotient $\overline{HC}_0(\mathscr{A}_\alpha)$ by the *closure* of the commutator subspace turns out to be isomorphic to $\mathbb{C}$, via the canonical trace

$$\mathscr{A}_\alpha \to \mathbb{C}\ , \qquad V_1^{n_1}V_2^{n_2} \mapsto \left\{ \begin{array}{ll} 1 & \text{if } n_1 = n_2 = 0\ , \\ 0 & \text{otherwise}\ . \end{array} \right.$$

With these identifications the evaluation of the Chern character $\mathrm{ch}^1(\rho_D)$: $\Omega^1/d\Omega^0 \to \overline{HC}_0(\mathscr{A}_\alpha) \cong \mathbb{C}$ on a one-form $A = A_x dx + A_y dy$ is easily performed and one finds

$$\mathrm{ch}^1(\rho_D)\left(\frac{A}{\sqrt{2\pi i}}\right) = \frac{1}{2\pi i}\int_0^1 dy \int_0^1 dx\, A_x(x,y)\ .$$

In particular $\mathrm{ch}^1(\rho_D)\cdot \mathrm{ch}_1(g_1) = 1$ and $\mathrm{ch}^1(\rho_D)\cdot \mathrm{ch}_1(g_2) = \alpha$, and one recovers the well-known fact ([4]) that the image of $K_0^{\rm top}(\mathscr{A}_\alpha)$ in $\overline{HC}_0(\mathscr{A}_\alpha)$ is the subgroup $\mathbb{Z}+\alpha\mathbb{Z} \subset \mathbb{C}$.
We may analogously define a new multiplicative K-theory group $\overline{MK}_1(\mathscr{A}_\alpha)$ whose elements are represented by pairs $(\hat{g},\theta)$ with $\theta \in \mathbb{C}$ instead of $\theta \in \mathscr{A}_\alpha/[\mathscr{A}_\alpha,\mathscr{A}_\alpha]$. Because $K_1^{\rm top}(\mathscr{A}_\alpha)$ is generated by the invertibles V_1 and V_2 any class in $\overline{MK}_1(\mathscr{A}_\theta)$ is represented by a pair $(V_1^{n_1}V_2^{n_2},\theta)$ for some integers n_1,n_2 and a complex number θ. Using a homotopy one shows that this pair is equivalent to $(e^{-\sqrt{2\pi i}\theta}V_1^{n_1}V_2^{n_2},0)$, and by exactness $\overline{MK}_1(\mathscr{A}_\alpha)$ is the quotient of the multiplicative group $\mathbb{C}^\times\langle V_1\rangle\langle V_2\rangle \subset \mathrm{GL}_1(\mathscr{A}_\alpha)$ by its commutator subgroup $\langle e^{2\pi i\alpha}\rangle \subset \mathbb{C}^\times$, or equivalently the abelianization

$$\overline{MK}_1(\mathscr{A}_\alpha) = (\mathbb{C}^\times\langle V_1\rangle\langle V_2\rangle)_{\rm ab}\ . \tag{100}$$

Since the Bott class of $K_0(\mathscr{A}_P)$ is sent to $[V_1] \in K_1^{\rm top}(\mathscr{A}_\alpha)$, one sees that the range of $H^2_{\mathscr{D}}(Q;\mathbb{Z}(1)) \to \overline{MK}_1(\mathscr{A}_\alpha)$ coincides with the subgroup $\mathbb{C}^\times\langle V_1\rangle$.

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Schur Class Operator Functions and Automorphisms of Hardy Algebras

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Abstract. Let E be a W^*-correspondence over a von Neumann algebra M and let $H^\infty(E)$ be the associated Hardy algebra. If σ is a faithful normal representation of M on a Hilbert space H, then one may form the dual correspondence E^σ and represent elements in $H^\infty(E)$ as $B(H)$-valued functions on the unit ball $\mathbb{D}(E^\sigma)^*$. The functions that one obtains are called Schur class functions and may be characterized in terms of certain Pick-like kernels. We study these functions and relate them to system matrices and transfer functions from systems theory. We use the information gained to describe the automorphism group of $H^\infty(E)$ in terms of special Möbius transformations on $\mathbb{D}(E^\sigma)$. Particular attention is devoted to the H^∞-algebras that are associated to graphs.

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1 Introduction

Let M be a W^*-algebra and let E be a W^*-correspondence over M. In [31] we built an operator algebra from this data that we called the Hardy algebra of E and which we denoted $H^\infty(E)$. If $M = E = \mathbb{C}$ - the complex numbers, then $H^\infty(E)$ is the classical Hardy algebra consisting of all bounded analytic functions on the open unit disc, $\mathbb{D}$ (see Example 2.4 below.) If $M = \mathbb{C}$ again, but $E = \mathbb{C}^n$, then $H^\infty(E)$ is the free semigroup algebra $\mathcal{L}_n$ studied by Davidson and Pitts [17], Popescu [32] and others (see Example 2.5.) One of the principal discoveries made in [31], and the source of inspiration for the present paper, is that attached to each faithful normal representation σ of M there is a *dual* correspondence E^σ, which is a W^*-correspondence over the commutant of $\sigma(M)$, $\sigma(M)'$, and the elements of $H^\infty(E)$ define functions on the open unit ball of E^σ, $\mathbb{D}(E^\sigma)$. Further, the value distribution theory of these functions turns out to be linked through our generalization of the Nevanlinna-Pick interpolation theorem [31, Theorem 5.3] with the positivity properties of certain Pick-like kernels of *mappings between operator spaces.*

In the setting where $M = E = \mathbb{C}$ and σ is the 1-dimensional representation of $\mathbb{C}$ on itself, then E^σ is $\mathbb{C}$ again. The representation of $H^\infty(E)$ in terms of functions on $\mathbb{D}(E^\sigma) = \mathbb{D}$ is just the usual way we think of $H^\infty(E)$. In this setting, our Nevanlinna-Pick theorem is exactly the classical theorem. If, however, σ is a representation of $\mathbb{C}$ on a Hilbert space H, $\dim(H) > 1$, then E^σ may be identified with $B(H)$ and then $\mathbb{D}(E^\sigma)$ becomes the space of strict contractions on H, i.e., all those operators of norm strictly less than 1. In this case, the value of an $f \in H^\infty(E)$ at a $T \in \mathbb{D}(E^\sigma)$ is simply $f(T)$, defined through the usual holomorphic functional calculus. Our Nevanlinna-Pick theorem gives a solution to problems such as this: given k operators $T_1, T_2, \ldots, T_k$ all of norm less than 1 and k operators, $A_1, A_2, \ldots, A_k$, determine the circumstances under which one can find a bounded analytic function f on the open unit disc of sup norm at most 1 such that $f(T_i) = A_i$, $i = 1, 2, \ldots, k$ (See [31, Theorem 6.1].) On the other hand, when $M = \mathbb{C}$, $E = \mathbb{C}^n$, and σ is one dimensional, the space E^σ is $\mathbb{C}^n$ and $\mathbb{D}(E^\sigma)$ is the unit ball $\mathbb{B}^n$. Elements in $H^\infty(E) = \mathcal{L}_n$ are realized as holomorphic functions on $\mathbb{B}^n$ that lie in a multiplier space studied in detail by Arveson [5]. More accurately, the functional representation of $H^\infty(E) = \mathcal{L}_n$ in terms of these functions expresses this space as a *quotient* of $H^\infty(E) = \mathcal{L}_n$. The Nevanlinna-Pick theorem of [31] contains those of Davidson and Pitts [18], Popescu [34], and Arias and Popescu [4], which deal with interpolation problems for these spaces of functions (possibly tensored with the bounded operators on an auxiliary Hilbert space). It also contains some of the results of Constaninescu and Johnson in [16] which treats elements of $\mathcal{L}_n$ as functions on the ball of strict row contractions with values in the operators on a Hilbert space. (See their Theorem 3.4 in particular.) This situation arises when one takes $M = \mathbb{C}$ and $E = \mathbb{C}^n$, but takes σ to be scalar multiplication on an auxiliary Hilbert space.

Our objective in the present note is basically two fold. First, we wish to identify those functions on $\mathbb{D}(E^\sigma)$ that arise from evaluating elements of $H^\infty(E)$. For this purpose, we introduce a family of functions on $\mathbb{D}(E^\sigma)$ that we call Schur class operator functions (see Definition 3.1). Roughly speaking, these functions are defined so that a Pick-like kernel that one may attach to each one is completely positive definite in the sense of Barreto, Bhat, Liebscher and Skeide [14]. In Theorem 3.3 we use their Theorem 3.2.3 to give a Kolmogorov-type representation of the kernel, from which we derive an analogue of a unitary system matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ whose transfer function

$$A + B(I - L_\eta^* D)^{-1} L_\eta^* C$$

turns out to be the given Schur class operator function. We then prove in Theorem 3.6 that each such transfer function arises by evaluating an element in $H^\infty(E)$ at points of $\mathbb{D}(E^\sigma)$ and conversely, each function in $H^\infty(E)$ has a representation in terms of a transfer function. The meaning of the notation will be made precise below, but we use it here to highlight the connection between our analysis and realization theory as it comes from mathematical systems theory. The point to keep in mind is that functions on $\mathbb{D}(E^\sigma)$ that come from elements of $H^\infty(E)$ are not, *a priori*, analytic in any ordinary sense and it is not at all clear what analytic features they have. Our Theorems 3.1 and 3.6 together with [31, Theorem 5.3] show that the Schur class operator functions are precisely the functions one obtains when evaluating functions in $H^\infty(E)$ (of norm at most 1) at points of $\mathbb{D}(E^\sigma)$. The fact that each such function may be realized as a transfer function exhibits a surprising level of analyticity that is not evident in the definition of $H^\infty(E)$.

Our second objective is to connect the usual holomorphic properties of $\mathbb{D}(E^\sigma)$ with the automorphisms of $H^\infty(E)$. As a space, $\mathbb{D}(E^\sigma)$ is the unit ball of a J^*-triple system. Consequently, every holomorphic automorphism of $\mathbb{D}(E^\sigma)$ is the composition of a Möbius transformation and a linear isometry [20]. Each of these implements an automorphism of the algebra of all bounded, *complex-valued analytic* functions on $\mathbb{D}(E^\sigma)$, but in our setting only certain of them implement automorphisms of $H^\infty(E)$ - those for which the Möbius part is determined by a "central" element of E^σ (see Theorem 4.21). Our proof requires the fact that the evaluation of functions in $H^\infty(E)$ (of norm at most 1) at points of $\mathbb{D}(E^\sigma)$ are precisely the Schur class operator functions on $\mathbb{D}(E^\sigma)$. Indeed, the whole analysis is an intricate "point - counterpoint" interplay among elements of $H^\infty(E)$, Schur class functions, transfer functions and "classical" function theory on $\mathbb{D}(E^\sigma)$. In the last section, we apply our general analysis of the automorphisms of $H^\infty(E)$ to the special case of H^∞-algebras coming from directed graphs.

In concluding this introduction, we want to note that a preprint of the present paper was posted on the arXiv on June 27, 2006. Recently, inspired in part by our preprint, Ball, Biswas, Fang and ter Horst [8] were able to realize the Fock space that we describe here in terms of the theory of completely positive definite kernels advanced by Barreto, Bhat, Liebscher and Skeide [14] that we also use (See Section 3 and, in particular, the proof of Theorem 3.3.) The analysis of Ball *et al.* makes additional ties between the theory of abstract Hardy algebras that we develop here and classical function theory on the unit disc.

2 Preliminaries

We start by introducing the basic definitions and constructions. We shall follow Lance [24] for the general theory of Hilbert C^*-modules that we shall use. Let A be a C^*-algebra and E be a right module over A endowed with a bi-additive map $\langle \cdot, \cdot \rangle : E \times E \to A$ (referred to as an A-valued inner product) such that, for $\xi, \eta \in E$ and $a \in A$, $\langle \xi, \eta a \rangle = \langle \xi, \eta \rangle a$, $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$, and $\langle \xi, \xi \rangle \geq 0$, with $\langle \xi, \xi \rangle = 0$ only when $\xi = 0$. Also, E is assumed to be complete in the norm $\|\xi\| := \|\langle \xi, \xi \rangle\|^{1/2}$. We write $\mathcal{L}(E)$ for the space of continuous, adjointable, A-module maps on E. It is known to be a C^*-algebra. If M is a von Neumann algebra and if E is a Hilbert C^*-module over M, then E is said to be *self-dual* in case every continuous M-module map from E to M is given by an inner product with an element of E. Let A and B be C^*-algebras. A *C^*-correspondence from A to B* is a Hilbert C^*-module E over B endowed with a structure of a left module over A via a nondegenerate $*$-homomorphism $\varphi : A \to \mathcal{L}(E)$.

When dealing with a specific C^*-correspondence, E, from a C^*-algebra A to a C^*-algebra B, it will be convenient sometimes to suppress the φ in formulas involving the left action and simply write $a\xi$ or $a \cdot \xi$ for $\varphi(a)\xi$. This should cause no confusion in context.

If E is a C^*-correspondence from A to B and if F is a correspondence from B to C, then the balanced tensor product, $E \otimes_B F$ is an A, C-bimodule that carries the inner product defined by the formula

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle_{E \otimes_B F} := \langle \eta_1, \varphi(\langle \xi_1, \xi_2 \rangle_E) \eta_2 \rangle_F$$

The Hausdorff completion of this bimodule is again denoted by $E \otimes_B F$.

In this paper we deal mostly with correspondences over von Neumann algebras that satisfy some natural additional properties as indicated in the following definition. (For examples and more details see [31]).

Definition 2.1 *Let M and N be von Neumann algebras and let E be a Hilbert C^*-module over N. Then E is called a* Hilbert W^*-module *over N in case E is self-dual. The module E is called a* W^*-correspondence from M to N *in case E is a self-dual C^*-correspondence from M to N such that the $*$-homomorphism $\varphi : M \to \mathcal{L}(E)$, giving the left module structure on E, is normal. If $M = N$ we shall say that E is a W^*-correspondence* over M.

We note that if E is a Hilbert W^*-module over a von Neumann algebra, then $\mathcal{L}(E)$ is not only a C^*-algebra, but is also a W^*-algebra. Thus it makes sense to talk about normal homomorphisms into $\mathcal{L}(E)$.

Definition 2.2 *An* isomorphism *of a W^*-correspondence E_1 over M_1 and a W^*-correspondence E_2 over M_2 is a pair (σ, Ψ) where $\sigma : M_1 \to M_2$ is an isomorphism of von Neumann algebras, $\Psi : E_1 \to E_2$ is a vector space isomorphism preserving the σ-topology and for $e, f \in E_1$ and $a, b \in M_1$, we have $\Psi(aeb) = \sigma(a)\Psi(e)\sigma(b)$ and $\langle \Psi(e), \Psi(f)\rangle = \sigma(\langle e, f\rangle)$.*

When considering the tensor product $E \otimes_M F$ of two W^*-correspondences, one needs to take the closure of the C^*-tensor product in the σ-topology of [6] in order to get a W^*-correspondence. However, we will not distinguish notationally between the C^*-tensor product and the W^*-tensor product. Note also that given a W^*-correspondence E over M and a Hilbert space H equipped with a normal representation σ of M, we can form the Hilbert space $E \otimes_\sigma H$ by defining $\langle \xi_1 \otimes h_1, \xi_2 \otimes h_2\rangle = \langle h_1, \sigma(\langle \xi_1, \xi_2\rangle)h_2\rangle$. Thus, H is viewed as a correspondence from M to $\mathbb{C}$ via σ and $E \otimes_\sigma H$ is just the tensor product of E and H as W^*-correspondences.

Note also that, given an operator $X \in \mathcal{L}(E)$ and an operator $S \in \sigma(M)'$, the map $\xi \otimes h \mapsto X\xi \otimes Sh$ defines a bounded operator on $E \otimes_\sigma H$ denoted by $X \otimes S$. The representation of $\mathcal{L}(E)$ that results when one lets $S = I$, is called the representation of $\mathcal{L}(E)$ *induced by* σ and is often denoted by σ^E. The composition, $\sigma^E \circ \varphi$ is a representation of M which we shall also say is induced by σ, but we shall usually denote it by $\varphi(\cdot) \otimes I$.

Observe that if E is a W^*-correspondence over a von Neumann algebra M, then we may form the tensor powers $E^{\otimes n}$, $n \geq 0$, where $E^{\otimes 0}$ is simply M viewed as the identity correspondence over M, and we may form the W^*-direct sum of the tensor powers, $\mathcal{F}(E) := E^{\otimes 0} \oplus E^{\otimes 1} \oplus E^{\otimes 2} \oplus \cdots$ to obtain a W^*-correspondence over M called the *(full) Fock space* over E. The actions of M on the left and right of $\mathcal{F}(E)$ are the diagonal actions and, when it is convenient to do so, we make explicit the left action by writing φ_∞ for it. That is, for $a \in M$, $\varphi_\infty(a) := diag\{a, \varphi(a), \varphi^{(2)}(a), \varphi^{(3)}(a), \cdots\}$, where for all n, $\varphi^{(n)}(a)(\xi_1 \otimes \xi_2 \otimes \cdots \xi_n) = (\varphi(a)\xi_1) \otimes \xi_2 \otimes \cdots \xi_n$, $\xi_1 \otimes \xi_2 \otimes \cdots \xi_n \in E^{\otimes n}$. The *tensor algebra* over E, denoted $\mathcal{T}_+(E)$, is defined to be the *norm-closed* subalgebra of $\mathcal{L}(\mathcal{F}(E))$ generated by $\varphi_\infty(M)$ and the *creation operators* T_ξ, $\xi \in E$, defined by the formula $T_\xi \eta = \xi \otimes \eta$, $\eta \in \mathcal{F}(E)$. We refer the reader to [28] for the basic facts about $\mathcal{T}_+(E)$.

Definition 2.3 *([31]) Given a W^*-correspondence E over the von Neumann algebra M, the ultraweak closure of the tensor algebra of E, $\mathcal{T}_+(E)$, in $\mathcal{L}(\mathcal{F}(E))$, is called the* Hardy Algebra *of E, and is denoted $H^\infty(E)$.*

Example 2.4 *If $M = E = \mathbb{C}$, then $\mathcal{F}(E)$ can be identified with $\ell^2(\mathbb{Z}_+)$ or, through the Fourier transform, $H^2(\mathbb{T})$. The tensor algebra then is isomorphic to the disc algebra $A(\mathbb{D})$ viewed as multiplication operators on $H^2(\mathbb{T})$ and the Hardy algebra is realized as the classical Hardy algebra $H^\infty(\mathbb{T})$.*

Example 2.5 *If $M = \mathbb{C}$ and $E = \mathbb{C}^n$, then $\mathcal{F}(E)$ can be identified with the space $l_2(\mathbb{F}_n^+)$, where $\mathbb{F}_n^+$ is the free semigroup on n generators. The tensor* *algebra then is what Popescu refers to as the "non commutative disc algebra" $\mathcal{A}_n$ and the Hardy algebra is its w^*-closure. It was studied by Popescu [32] and by Davidson and Pitts who denoted it by $\mathcal{L}_n$ [17].*

We need to review some basic facts about the representation theory of $H^\infty(E)$ and of $\mathcal{T}_+(E)$. See [28, 31] for more details.

Definition 2.6 *Let E be a W^*-correspondence over a von Neumann algebra M. Then:*

1. *A* completely contractive covariant representation *of E on a Hilbert space H is a pair (T, σ), where*
   - *(a) σ is a normal $*$-representation of M in $B(H)$.*
   - *(b) T is a linear, completely contractive map from E to $B(H)$ that is continuous in the σ-topology of [6] on E and the ultraweak topology on $B(H)$.*
   - *(c) T is a bimodule map in the sense that $T(S\xi R) = \sigma(S)T(\xi)\sigma(R)$, $\xi \in E$, and $S, R \in M$.*
2. *A completely contractive covariant representation (T, σ) of E in $B(H)$ is called* isometric *in case*
$$T(\xi)^*T(\eta) = \sigma(\langle \xi, \eta \rangle) \tag{1}$$
   for all $\xi, \eta \in E$.

It should be noted that the operator space structure on E to which Definition 2.6 refers is that which E inherits when viewed as a subspace of its linking algebra.
As we showed in [28, Lemmas 3.4–3.6] and in [31], if a completely contractive covariant representation, (T, σ), of E in $B(H)$ is given, then it determines a contraction $\tilde{T} : E \otimes_\sigma H \to H$ defined by the formula $\tilde{T}(\eta \otimes h) := T(\eta)h$, $\eta \otimes h \in E \otimes_\sigma H$. The operator $\tilde{T}$ intertwines the representation σ on H and the induced representation $\sigma^E \circ \varphi = \varphi(\cdot) \otimes I_H$ on $E \otimes_\sigma H$; i.e.
$$\tilde{T}(\varphi(\cdot) \otimes I) = \sigma(\cdot)\tilde{T}. \tag{2}$$

In fact we have the following lemma from [31, Lemma 2.16].

Lemma 2.7 *The map $(T, \sigma) \to \tilde{T}$ is a bijection between all completely contractive covariant representations (T, σ) of E on the Hilbert space H and contractive operators $\tilde{T} : E \otimes_\sigma H \to H$ that satisfy equation (2). Given such a $\tilde{T}$ satisfying this equation, T, defined by the formula $T(\xi)h := \tilde{T}(\xi \otimes h)$, together with σ is a completely contractive covariant representation of E on H. Further, (T, σ) is isometric if and only if $\tilde{T}$ is an isometry.*

The importance of the completely contractive covariant representations of E (or, equivalently, the intertwining contractions $\tilde{T}$ as above) is that they yield all completely contractive representations of the tensor algebra. More precisely, we have the following.

THEOREM 2.8 *Let E be a W^*-correspondence over a von Neumann algebra M. To every completely contractive covariant representation, (T,σ), of E there is a unique completely contractive representation ρ of the tensor algebra $\mathcal{T}_+(E)$ that satisfies*

$$\rho(T_\xi) = T(\xi) \quad \xi \in E$$

and

$$\rho(\varphi_\infty(a)) = \sigma(a) \quad a \in M.$$

The map $(T,\sigma) \mapsto \rho$ is a bijection between the set of all completely contractive covariant representations of E and all completely contractive (algebra) representations of $\mathcal{T}_+(E)$ whose restrictions to $\varphi_\infty(M)$ are continuous with respect to the ultraweak topology on $\mathcal{L}(\mathcal{F}(E))$.

DEFINITION 2.9 *If (T,σ) is a completely contractive covariant representation of a W^*-correspondence E over a von Neumann algebra M, we call the representation ρ of $\mathcal{T}_+(E)$ described in Theorem 2.8 the* integrated form *of (T,σ) and write $\rho = \sigma \times T$.*

REMARK 2.10 *One of the principal difficulties one faces in dealing with $\mathcal{T}_+(E)$ and $H^\infty(E)$ is to decide when the integrated form, $\sigma \times T$, of a completely contractive covariant representation (T,σ) extends from $\mathcal{T}_+(E)$ to $H^\infty(E)$. This problem arises already in the simplest situation, vis. when $M = \mathbb{C} = E$. In this setting, T is given by a single contraction operator on a Hilbert space, $\mathcal{T}_+(E)$ "is" the disc algebra and $H^\infty(E)$ "is" the space of bounded analytic functions on the disc. The representation $\sigma \times T$ extends from the disc algebra to $H^\infty(E)$ precisely when there is no singular part to the spectral measure of the minimal unitary dilation of T. We are not aware of a comparable result in our general context but we have some sufficient conditions. One of them is given in the following lemma. It is not a necessary condition in general.*

LEMMA 2.11 *[31, Corollary 2.14] If $\|\tilde{T}\| < 1$ then $\sigma \times T$ extends to a ultraweakly continuous representation of $H^\infty(E)$.*

In [31] we introduced and studied the concepts of duality and of point evaluation (for elements of $H^\infty(E)$). These play a central role in our analysis here.

DEFINITION 2.12 *Let E be a W^*-correspondence over a von Neumann algebra M and let $\sigma : M \to B(H)$ be a faithful normal representation of M on a Hilbert space H. Then the σ*-dual *of E, denoted E^σ, is defined to be*

$$\{\eta \in B(H, E \otimes_\sigma H) \mid \eta\sigma(a) = (\varphi(a) \otimes I)\eta,\ a \in M\}.$$

An important feature of the dual E^σ is that it is a W^*-correspondence, *but over the commutant* of $\sigma(M)$, $\sigma(M)'$.

PROPOSITION 2.13 *With respect to the action of $\sigma(M)'$ and the $\sigma(M)'$-valued inner product defined as follows, E^σ becomes a W^*-correspondence over $\sigma(M)'$: For Y and X in $\sigma(M)'$, and $\eta \in E^\sigma$, $X\cdot\eta\cdot Y := (I\otimes X)\eta Y$, and for $\eta_1, \eta_2 \in E^\sigma$, $\langle \eta_1, \eta_2\rangle_{\sigma(M)'} := \eta_1^*\eta_2$.*

In the following remark we explain what we mean by "evaluating an element of $H^\infty(E)$ at a point in the open unit ball of the dual".

REMARK 2.14 *The importance of this dual space, E^σ, is that it is closely related to the representations of E. In fact, the operators in E^σ whose norm does not exceed* 1 *are precisely the adjoints of the operators of the form $\tilde{T}$ for a covariant pair (T, σ). In particular, every η in the* open *unit ball of E^σ (written $\mathbb{D}(E^\sigma)$) gives rise to a covariant pair (T, σ) (with $\eta = \tilde{T}^*$) such that $\sigma \times T$ extends to a representation of $H^\infty(E)$.*
Given $X \in H^\infty(E)$ we can apply the representation associated to η to it. The resulting operator in $B(H)$ will be denoted by $\widehat{X}(\eta^*)$. Thus

$$\widehat{X}(\eta^*) = (\sigma \times \eta^*)(X).$$

In this way, we view every element in the Hardy algebra as a $B(H)$-valued function

$$\widehat{X} : \mathbb{D}(E^\sigma)^* \to B(H)$$

on the open unit ball of $(E^\sigma)^*$. One of our primary objectives is to understand the range of the transform $X \to \widehat{X}$, $X \in H^\infty(E)$.

EXAMPLE 2.15 *Suppose $M = E = \mathbb{C}$ and σ the representation of $\mathbb{C}$ on some Hilbert space H. Then it is easy to check that E^σ is isomorphic to $B(H)$. Fix an $X \in H^\infty(E)$. As we mentioned above, this Hardy algebra is the classical $H^\infty(\mathbb{T})$ and we can identify X with a function $f \in H^\infty(\mathbb{T})$. Given $S \in \mathbb{D}(E^\sigma) = B(H)$, it is not hard to check that $\widehat{X}(S^*)$, as defined above, is the operator $f(S^*)$ defined through the usual holomorphic functional calculus.*

EXAMPLE 2.16 *In [17] Davidson and Pitts associate to every element of the free semigroup algebra $\mathcal{L}_n$ (see Example 2.5) a function on the open unit ball of $\mathbb{C}^n$. This is a special case of our analysis when $M = \mathbb{C}$, $E = \mathbb{C}^n$ and σ is a one dimensional representation of $\mathbb{C}$. In this case $\sigma(M)' = \mathbb{C}$ and $E^\sigma = \mathbb{C}^n$. Note, however, that our definition allows us to take σ to be the representation of $\mathbb{C}$ on an arbitrary Hilbert space H. If we do so, then E^σ is isomorphic to $B(H)^{(n)}$, the nth column space over $B(H)$, and elements of $\mathcal{L}_n$ define functions on the open unit ball of this space viewed as a correspondence over $B(H)$ with values in $B(H)$. This is the perspective adopted by Constantinescu and Johnson in [16]. In the analysis of [17] it is possible that a non zero element of $\mathcal{L}_n$ will give rise to the zero function. We shall show in Lemma 3.8 that, by choosing an appropriate H we can insure that this does not happen.*

Example 2.17 *Part of the recent work of Popescu in [35] may be cast in our framework. We will follow his notation. Fix a Hilbert space K, and let E be the column space $B(K)^n$. Take, also, a Hilbert space H and let $\sigma : B(K) \to B(K \otimes H)$ be the representation which sends $a \in B(K)$ to $a \otimes I_H$. Then, since the commutant of $\sigma(B(K))$ is naturally isomorphic to $B(H)$, it is easy to see that E^σ is the column space over $B(H)$, $B(H)^n$. It follows that $\mathbb{D}(E^\sigma)$ is the open unit ball in $B(H)^n$. A* free formal power series *with coefficients from $B(K)$ is a formal series $F = \sum_{\alpha\in\mathbb{F}_n^+} A_\alpha \otimes Z_\alpha$ where $\mathbb{F}_n^+$ is the free semigroup on n generators, the A_α are elements of $B(K)$ and where Z_α is the monomial in noncommuting indeterminates $Z_1, Z_2, \ldots, Z_n$ determined by α. If F has radius of convergence equal to 1, then one may evaluate F at points of $\mathbb{D}(E^\sigma)^*$ to get a function on $\mathbb{D}(E^\sigma)^*$ with values in $B(K\otimes H)$,* vis., *$F((S_1, S_2, \cdots S_n)) = \sum_{\alpha\in\mathbb{F}_n^+} A_\alpha \otimes S_\alpha$. See [35, Theorem 1.1]. In fact, under additional restrictions on the coefficients A_α, F may be viewed as a function X in $H^\infty(B(K)^n)$ in such a way that $F((S_1, S_2, \cdots S_n)) = \widehat{X}(S_1, S_2, \cdots S_n)$ in the sense defined in [31, p. 384] and discussed above in Remark 2.14. The space that Popescu denotes by $H^\infty(B(\mathcal{X})_1^n)$ arises when $K = \mathbb{C}$, and is naturally isometrically isomorphic to $\mathcal{L}_n$ [35, Theorem 3.1]. We noted in the preceding example that $\mathcal{L}_n$ is $H^\infty(\mathbb{C}^n)$. The point of [35], at least in part, is to study $H^\infty(B(\mathcal{X})_1^n) \simeq \mathcal{L}_n = H^\infty(\mathbb{C}^n)$ through all the representations σ of $\mathbb{C}$ on Hilbert spaces H, that is, through evaluating functions in $H^\infty(B(\mathcal{X})_1^n)$ at points the unit ball of $B(H)^n$ for all possible H's. The space $B(K)^n$ is Morita equivalent to $\mathbb{C}^n$ in the sense of [30], at least when* $\dim(K) < \infty$, *and, in that case the tensor algebras $\mathcal{T}_+(B(K)^n)$ and $\mathcal{T}_+(\mathbb{C}^n)$ are Morita equivalent in the sense described by [15]. The tensor algebra $\mathcal{T}_+(\mathbb{C}^n)$, in turn, is naturally isometrically isomorphic to Popescu's noncommutative disc algebra $\mathcal{A}_n$ [33]. The analysis in [15] suggests a sense in which $\mathbb{C}^n$ and $B(K)^n$ are Morita equivalent even when* $\dim(K) = \infty$, *and that together with [30] suggests that $H^\infty(B(K)^n)$ should be Morita equivalent to $H^\infty(B(\mathcal{X})_1^n) \simeq H^\infty(\mathbb{C}^n)$. This would suggest an even closer connection between Popescu's free power series, and all that goes with them, and the perspective we have taken in this paper, which, as we shall see, involves generalized Schur functions and transfer functions. The connection seems like a promising avenue to explore.*

In [31] we exploited the perspective of viewing elements of the Hardy algebra as $B(H)$-valued functions on the open unit ball of the dual correspondence to prove a Nevanlinna-Pick type interpolation theorem. In order to state it we introduce some notation: For operators B_1 and B_2 in $B(H)$, we write $Ad(B_1, B_2)$ for the map from $B(H)$ to itself that sends S to $B_1 S B_2^*$. Also, given elements η_1, η_2 in $\mathbb{D}(E^\sigma)$, we let θ_{η_1,η_2} denote the map, from $\sigma(M)'$ to itself that sends a to $\langle \eta_1, a\eta_2\rangle$. That is, $\theta_{\eta_1,\eta_2}(a) := \langle \eta_1, a\eta_2\rangle = \eta_1^* a \eta_2$, $a \in \sigma(M)'$.

Theorem 2.18 *([31, Theorem 5.3]) Let E be a W^*-correspondence over a von Neumann algebra M and let $\sigma : M \to B(H)$ be a faithful normal representation of M on a Hilbert space H. Fix k points $\eta_1, \ldots \eta_k$ in the disk $\mathbb{D}(E^\sigma)$ and choose*

$2k$ operators $B_1, \ldots B_k, C_1, \ldots C_k$ in $B(H)$. Then there exists an X in $H^\infty(E)$ such that $\|X\| \leq 1$ and

$$B_i \widehat{X}(\eta_i^*) = C_i$$

for $i = 1, 2, \ldots, k$, if and only if the map from $M_k(\sigma(M)')$ into $M_k(B(H))$ defined by the $k \times k$ matrix

$$\big((Ad(B_i, B_j) - Ad(C_i, C_j)) \circ (id - \theta_{\eta_i, \eta_j})^{-1}\big) \tag{3}$$

is completely positive.

That is, the map T, say, given by the matrix (3) is computed by the formula

$$T((a_{ij})) = (b_{ij}),$$

where

$$b_{ij} = B_i((id - \theta_{\eta_i, \eta_j})^{-1}(a_{ij})B_j^* - C_i((id - \theta_{\eta_i, \eta_j})^{-1}(a_{ij})C_j^*$$

and

$$(id - \theta_{\eta_i, \eta_j})^{-1}(a_{ij}) = a_{ij} + \theta_{\eta_i, \eta_j}(a_{ij}) + \theta_{\eta_i, \eta_j}(\theta_{\eta_i, \eta_j}(a_{ij})) + \cdots$$

We close this section with two technical lemmas that will be needed in our analysis. Let M and N be W^*-algebras and let E be a W^*-correspondence from M to N. Given a σ-closed subcorrespondence E_0 of E we know that the orthogonal projection P of E onto E_0 is a right module map. (See [6, Consequences 1.8 (ii)]). In the following lemma we show that P also preserves the left action.

Lemma 2.19 *Let E be a W^*-correspondence from the von Neumann algebra M to the von Neumann algebra N, and let E_0 be a sub W^*-correspondence E_0 of E that is closed in the σ-topology of [6, Consequences 1.8 (ii)]. If P is the orthogonal projection from E onto E_0, then P is a bimodule map; i.e., $P(a\xi b) = aP(\xi)b$ for all $a \in M$ and $b \in N$.*

Proof. It suffices to check that $P(e\xi) = eP(\xi)$ for all $\xi \in E$ and projections $e \in M$. For $\xi, \eta \in E$ and a projection $e \in M$, we have

$$\|e\xi + f\eta\|^2 = \|\langle e\xi, e\xi\rangle + \langle f\eta, f\eta\rangle\| \leq \|\langle e\xi, e\xi\rangle\| + \|\langle f\eta, f\eta\rangle\| = \|e\xi\|^2 + \|f\eta\|^2,$$

where $f = 1 - e$. So, for every $\lambda \in \mathbb{R}$ we have

$$\begin{aligned}(\lambda+1)^2\|fP(e\xi)\|^2 = \|fP(e\xi + \lambda fP(e\xi))\|^2 &\leq \|e\xi + \lambda fP(e\xi)\|^2 \\ &\leq \|e\xi\|^2 + \lambda^2\|fP(e\xi)\|^2.\end{aligned}$$

Hence, for every $\lambda \in \mathbb{R}$,

$$(2\lambda + 1)\|fP(e\xi)\|^2 \leq \|e\xi\|^2$$

and, thus,

$$(I-e)P(e\xi) = fP(e\xi) = 0.$$

Replacing e by $f = I - e$ we get $eP((I-e)\xi) = 0$ and, therefore,

$$P(e\xi) = eP(e\xi) = eP(\xi).$$

Since M is spanned by its projections, we are done. □

Lemma 2.20 *Let E be a W^*-correspondence over M, let σ be a faithful normal representation of M on the Hilbert space $\mathcal{E}$, and let E^σ be the σ-dual correspondence over $N := \sigma(M)'$. Then*

(i) The left action of N on E^σ is faithful if and only if E is full (i.e. if and only if the ultraweakly closed ideal generated by the inner products $\langle \xi_1, \xi_2 \rangle$, $\xi_1, \xi_2 \in E$, is all of M).

(ii) The left action of M on E is faithful if and only if E^σ is full.

Proof. We shall prove (i). Part (ii) then follows by duality (using [31, Theorem 3.6]). Given $S \in N$, $S\eta = 0$ for every $\eta \in E^\sigma$ if and only if for all $\eta \in E^\sigma$ and $g \in \mathcal{E}$, $(I \otimes S)\eta(g) = 0$. Since the closed subspace spanned by the ranges of all $\eta \in E^\sigma$ is all of $E \otimes_M \mathcal{E}$ ([31]), this is equivalent to the equation $\xi \otimes Sg = 0$ holding for all $g \in \mathcal{E}$ and $\xi \in E$. Since $\langle \xi \otimes Sg, \xi \otimes Sg \rangle = \langle g, S^* \langle \xi, \xi \rangle Sg \rangle$, we find that $SE^\sigma = 0$ if and only if $\sigma(\langle E, E \rangle)S = 0$, where $\langle E, E \rangle$ is the ultraweakly closed ideal generated by all inner products. If this ideal is all of M we find that the equation $SE^\sigma = 0$ implies that $S = 0$. In the other direction, if this is not the case, then this ideal is of the form $(I-q)M$ for some central nonzero projection q and then $S = \sigma(q)$ is different from 0 but vanishes on E^σ. □

3 Schur class operator functions and realization

Throughout this section, E will be a fixed W^*-correspondence over the von Neumann algebra M and σ will be a faithful representation of M on a Hilbert space $\mathcal{E}$. We then form the σ-dual of E, E^σ, which is a correspondence over $N := \sigma(M)'$, and we write $\mathbb{D}(E^\sigma)$ for its open unit ball. Further, we write $\mathbb{D}(E^\sigma)^*$ for $\{\eta^* \mid \eta \in \mathbb{D}(E^\sigma)\}$.
The following definition is clearly motivated by the condition appearing in Theorem 2.18 and Schur's theorem from classical function theory.

Definition 3.1 *Let Ω be a subset of $\mathbb{D}(E^\sigma)$ and let $\Omega^* = \{\omega^* \mid \omega \in \Omega\}$. A function $Z : \Omega^* \to B(\mathcal{E})$ will be called a* Schur class operator function *(with values in $B(\mathcal{E})$) if, for every k and every choice of elements $\eta_1, \eta_2, \ldots, \eta_k$ in Ω, the map from $M_k(N)$ to $M_k(B(\mathcal{E}))$ defined by the $k \times k$ matrix of maps,*

$$((id - Ad(Z(\eta_i^*), Z(\eta_j^*))) \circ (id - \theta_{\eta_i, \eta_j})^{-1}),$$

is completely positive.

Note that, when $M = E = B(\mathcal{E})$ and σ is the identity representation of $B(\mathcal{E})$ on $\mathcal{E}$, $\sigma(M)'$ is $\mathbb{C}I_{\mathcal{E}}$, E^σ is isomorphic to $\mathbb{C}$ and $\mathbb{D}(E^\sigma)^*$ can be identified with the open unit disc $\mathbb{D}$ of $\mathbb{C}$. In this case our definition recovers the classical Schur class functions. More precisely, these functions are usually defined as analytic functions Z from an open subset Ω of $\mathbb{D}$ into the closed unit ball of $B(\mathcal{E})$ but it is known that such functions are precisely those for which the Pick kernel $k_Z(z,w) = (I - Z(z)Z(w)^*)(1 - z\bar{w})^{-1}$ is positive semi-definite on Ω. The argument of [31, Remark 5.4] shows that the positivity of this kernel is equivalent, in our case, to the condition of Definition 3.1. This condition, in turn, is the same as asserting that the kernel

$$k_Z(\zeta^*, \omega^*) := (id - Ad(Z(\zeta^*), Z(\omega^*)) \circ (id - \theta_{\zeta,\omega})^{-1} \tag{4}$$

is a completely positive definite kernel on Ω^* in the sense of Definition 3.2.2 of [14].

For the sake of completeness, we record the fact that every element of $H^\infty(E)$ of norm at most one gives rise to a Schur class operator function.

Theorem 3.2 *Let E be a W^*-correspondence over a von Neumann algebra M and let σ be a faithful normal representation of M in $B(H)$ for some Hilbert space H. If X is an element of $H^\infty(E)$ of norm at most one, then the function $\eta^* \to \widehat{X}(\eta^*)$ defined in Remark 2.14 is a Schur class operator function on $\mathbb{D}((E^\sigma))^*$ with values in $B(H)$.*

Proof. One simply takes $B_i = I$ for all i and $C_i = \widehat{X}(\eta_i^*)$ in Theorem 2.18. $\square$

Theorem 3.3 *Let E be a W^*-correspondence over a von Neumann algebra M. Suppose also that σ a faithful normal representation of M on a Hilbert space $\mathcal{E}$ and that q_1 and q_2 are projections in $\sigma(M)$. Finally, suppose that Ω is a subset of $\mathbb{D}((E^\sigma))$ and that Z is a Schur class operator function on Ω^* with values in $q_2B(\mathcal{E})q_1$. Then there is a Hilbert space H, a normal representation τ of $N := \sigma(M)'$ on H and operators A, B, C and D fulfilling the following conditions:*

(i) The operator A lies in $q_2\sigma(M)q_1$.

(ii) The operators C, B, and D, are in the spaces $B(\mathcal{E}_1, E^\sigma \otimes_\tau H)$, $B(H, \mathcal{E}_2)$, and $B(H, E^\sigma \otimes_\tau H)$, respectively, and each intertwines the representations of $N = \sigma(M)'$ on the relevant spaces (i.e. , for every $S \in N$, $CS = (S \otimes I_H)C$, $B\tau(S) = SB$ and $D\tau(S) = (S \otimes I_H)D$).

(iii) The operator matrix

$$V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{5}$$

viewed as an operator from $\mathcal{E}_1 \oplus H$ to $\mathcal{E}_2 \oplus (E^\sigma \otimes_\tau H)$, is a coisometry, which is unitary if E is full.

(iv) For every η^* in Ω^*,

$$Z(\eta^*) = A + B(I - L_\eta^* D)^{-1} L_\eta^* C \tag{6}$$

where $L_\eta : H \to E^\sigma \otimes H$ is defined by the formula $L_\eta h = \eta \otimes h$ (so $L_\eta^*(\theta \otimes h) = \tau(\langle \eta, \theta \rangle)h$).

Remark 3.4 *Before giving the proof of Theorem 3.3, we want to note that the result bears a strong resemblance to standard results in the literature. We call special attention to [1, 2, 7, 9, 10, 11, 12, 13]. Indeed, we recommend [7], which is a survey that explains the general strategy for proving the theorem. What is novel in our approach is the adaptation of the results in the literature to accommodate completely positive definite kernels.*

Since the matrix in equation (5) and the function in equation (6) are familiar constructs in mathematical systems theory, more particularly from H^∞-control theory (see, e.g., [38]), we adopt the following terminology.

Definition 3.5 *Let E be a W^*-correspondence over a von Neumann algebra M. Suppose that σ is a faithful normal representation of M on a Hilbert space $\mathcal{E}$ and that q_1 and q_2 are projections in $\sigma(M)$. Then an operator matrix $V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where the entries A, B, C, and D, satisfy conditions (i) and (ii) of Theorem 3.3 for some normal representation τ of $\sigma(M)'$ on a Hilbert space H, is called a* system matrix *provided V is a coisometry (that is unitary, if E is full). If V is a system matrix, then the function $A + B(I - L_\eta^* D)^{-1} L_\eta^* C$, $\eta^* \in \mathbb{D}(E^\sigma)^*$ is called the* transfer function determined by *V.*

Proof. As we just remarked, the hypothesis that Z is a Schur class function on Ω^* means that the kernel k_Z in equation (4) is completely positive definite in the sense of [14]. Consequently, we may apply Theorem 3.2.3 of [14], which is a lovely extension of Kolmogorov's representation theorem for positive definite kernels, to find an N-$B(\mathcal{E})$ W^*-correspondence F and a function ι from Ω^* to F such that F is spanned by $N\iota(\Omega^*)B(\mathcal{E})$ and such that for every η_1 and η_2 in Ω^* and every $a \in N$,

$$(id - Ad(Z(\eta_1^*), Z(\eta_2^*))) \circ (id - \theta_{\eta_1,\eta_2})^{-1}(a) = \langle \iota(\eta_1), a\iota(\eta_2) \rangle.$$

It follows that for every $b \in N$ and every η_1, η_2 in Ω^*,

$$\begin{aligned} b - Z(\eta_1^*) b Z(\eta_2^*)^* &= \langle \iota(\eta_1), b\iota(\eta_2) \rangle - \langle \iota(\eta_1), \langle \eta_1, b\eta_2 \rangle \iota(\eta_2) \rangle \\ &= \langle \iota(\eta_1), b\iota(\eta_2) \rangle - \langle \eta_1 \otimes \iota(\eta_1), b\eta_2 \otimes \iota(\eta_2) \rangle. \end{aligned}$$

Thus,

$$b + \langle \eta_1 \otimes \iota(\eta_1), b\eta_2 \otimes \iota(\eta_2) \rangle = \langle \iota(\eta_1), b\iota(\eta_2) \rangle + Z(\eta_1^*) b Z(\eta_2^*)^*. \tag{7}$$

Set

$$G_1 := \overline{span}\{bZ(\eta^*)^* q_2 T \oplus b\iota(\eta) q_2 T \mid b \in N,\ \eta \in \Omega^*,\ T \in B(\mathcal{E})\ \}$$

and

$$G_2 := \overline{span}\{bq_2 T \oplus (b\eta \otimes \iota(\eta) q_2 T) \mid b \in N,\ \eta \in \Omega^*,\ T \in B(\mathcal{E})\ \}.$$

Then G_1 is a sub N-$B(\mathcal{E})$ W^*-correspondence of $B(\mathcal{E}) \oplus F$ (where we use the assumption that $q_2 Z(\eta^*) = q_2 Z(\eta^*) q_1$) and G_2 is a sub N-$B(\mathcal{E})$ W^*-correspondence of $B(\mathcal{E}) \oplus (E^\sigma \otimes_N F)$. (The closure in the definitions of G_1, G_2 is in the σ-topology of [6]. It then follows that G_1 and G_2 are W^*-correspondences [6, Consequences 1.8 (i)]). Define $v : G_1 \to G_2$ by the equation

$$v(bZ(\eta^*)^* q_2 T \oplus b\iota(\eta) q_2 T) = bq_2 T \oplus (b\eta \otimes \iota(\eta) q_2 T).$$

It follows from (7) that v is an isometry. It is also clear that it is a bimodule map. We write P_i for the orthogonal projection onto G_i, $i = 1, 2$ and $\tilde{V}$ for the map

$$\tilde{V} := P_2 v P_1 : q_1 B(\mathcal{E}) \oplus F \to q_2 B(\mathcal{E}) \oplus (E^\sigma \otimes_N F).$$

Then $\tilde{V}$ is a partial isometry and, since P_1, v and P_2 are all bimodule maps (see Lemma 2.19), so is $\tilde{V}$. We write $\tilde{V}$ matricially:

$$\tilde{V} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where $\alpha : q_1 B(\mathcal{E}) \to q_2 B(\mathcal{E})$, $\beta : F \to q_2 B(\mathcal{E})$, $\gamma : q_1 B(\mathcal{E}) \to E^\sigma \otimes F$ and $\delta : F \to E^\sigma \otimes F$ and all these maps are bimodule maps. Let H_0 be the Hilbert space $F \otimes_{B(\mathcal{E})} \mathcal{E}$ and note that $B(\mathcal{E}) \otimes_{B(\mathcal{E})} \mathcal{E}$ is isomorphic to $\mathcal{E}$ (and the isomorphism preserves the left N-action). Tensoring on the right by $\mathcal{E}$ (over $B(\mathcal{E})$) we obtain a partial isometry

$$V_0 := \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} : \begin{pmatrix} \mathcal{E}_1 \\ H_0 \end{pmatrix} \to \begin{pmatrix} \mathcal{E}_2 \\ E^\sigma \otimes H_0 \end{pmatrix}.$$

Here $A_0 = \alpha \otimes I_{\mathcal{E}}$, $B_0 = \beta \otimes I_{\mathcal{E}}$, $C_0 = \gamma \otimes I_{\mathcal{E}}$ and $D_0 = \delta \otimes I_{\mathcal{E}}$. These maps are well defined because the maps α, β, γ and δ are right $B(\mathcal{E})$-module maps. Since these maps are also left N-module maps, so are A_0, B_0, C_0 and D_0.
By the definition of V_0, its initial space is $G_1 \otimes \mathcal{E}$ and its final space is $G_2 \otimes \mathcal{E}$. In fact, V_0 induces an equivalence of the representations of N on $G_1 \otimes \mathcal{E}$ and on $G_2 \otimes \mathcal{E}$.
It will be convenient to use the notation $K_1 \preceq_N K_2$ if the Hilbert spaces K_1 and K_2 are both left N-modules and the representation of N on K_1 is equivalent to a subrepresentation of the representation of N on K_2. This means, of course, that there is an isometry from K_1 into K_2 that intertwines the two representations. If the two representations are equivalent we write $K_1 \simeq_N K_2$.

Using this notation, we can write $G_1 \otimes \mathcal{E} \simeq_N G_2 \otimes \mathcal{E}$. Form $\mathcal{M}_2 := (\mathcal{E}_2 \oplus (E^\sigma \otimes H_0)) \ominus (G_2 \otimes \mathcal{E})$, which is a left N-module, and note that $L := \mathcal{F}(E^\sigma) \otimes \mathcal{M}_2$ also is a left N-module, where the representation of N on L is the induced representation. Since $L = \mathcal{F}(E^\sigma) \otimes \mathcal{M}_2 = \bigoplus_{n=0}^{\infty}((E^\sigma)^{\otimes n} \otimes (\mathcal{M}_2))$, it is evident that $(E^\sigma \otimes L) \oplus \mathcal{M}_2 \simeq_N L$. Indeed, the isomorphisms are just the natural ones that give the associativity of the tensor products involved. Thus, $\mathcal{E}_2 \oplus (E^\sigma \otimes (H_0 \oplus L)) = \mathcal{E}_2 \oplus (E^\sigma \otimes H_0) \oplus (E^\sigma \otimes L) = G_2 \otimes \mathcal{E} \oplus \mathcal{M}_2 \oplus E^\sigma \otimes L \simeq_N G_2 \otimes \mathcal{E} \oplus L \simeq_N G_1 \otimes \mathcal{E} \oplus L \preceq_N \mathcal{E}_1 \oplus (H_0 \oplus L)$. Consequently, we obtain a coisometric operator $V : \mathcal{E}_1 \oplus (H_0 \oplus L) \to \mathcal{E}_2 \oplus E^\sigma \otimes (H_0 \oplus L)$ that intertwines the representations of N and extends V_0. Note that, if V_0 were known to be an isometry (so that $G_2 \otimes \mathcal{E} \simeq_N G_1 \otimes \mathcal{E} = \mathcal{E}_1 \oplus H_0$), then we would have equivalence above and V can be chosen to be unitary.

Assume that E is full. We also write $\mathcal{M}_1$ for $(\mathcal{E}_1 \oplus H_0) \ominus G_1 \otimes \mathcal{E}$. Since E is full, the representation ρ of N on $E^\sigma \otimes \mathcal{E}$ is faithful (Lemma 2.20) and it follows that every representation of N is quasiequivalent to a subrepresentation of ρ. Write $\mathcal{E}_\infty$ for the direct sum of infinitely many copies of $\mathcal{E}$. Then $E^\sigma \otimes \mathcal{E}_\infty$ is the direct sum of infinitely many copies of $E^\sigma \otimes \mathcal{E}$ and, thus, every representation of N is equivalent to a subrepresentation of the representation of N on $E^\sigma \otimes \mathcal{E}_\infty$. In particular, we can write $\mathcal{M}_1 \oplus \mathcal{E}_\infty \preceq_N E^\sigma \otimes \mathcal{E}_\infty$. Thus $\mathcal{E}_1 \oplus (H_0 \oplus \mathcal{E}_\infty) = (G_1 \otimes \mathcal{E}) \oplus \mathcal{M}_1 \oplus \mathcal{E}_\infty \preceq_N \mathcal{E}_2 \oplus (E^\sigma \otimes H_0) \oplus (E^\sigma \otimes \mathcal{E}_\infty) = \mathcal{E}_2 \oplus (E^\sigma \otimes (H_0 \oplus \mathcal{E}_\infty))$. So, replacing H_0 by $H_0 \oplus \mathcal{E}_\infty$, we can replace V_0 by an isometry and, using the argument just presented, we conclude that the resulting V is a unitary operator intertwining the representations of N and extending V_0.

So we let V be the coisometry just constructed (and treat it as unitary when E is full). Writing $H := H_0 \oplus L$, we can express V in the matricial form as in part (iii) of the statement of the theorem. Conditions (i) and (ii) then follow from the fact that V intertwines the indicated representations of N. It is left to prove (iv).

Setting $b = T = I$ in the definition of v above and writing v in a matricial form we see that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} Z(\eta^*)^* q_2 \\ \iota(\eta) q_2 \end{pmatrix} = \begin{pmatrix} q_2 \\ \eta \otimes \iota(\eta) q_2 \end{pmatrix}.$$

Tensoring by $I_\mathcal{E}$ on the right and identifying $B(\mathcal{E}) \otimes_{B(\mathcal{E})} \mathcal{E}$ with $\mathcal{E}$ as above, we find that

$$\begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} \begin{pmatrix} Z(\eta^*)^* g \\ \iota(\eta) \otimes g \end{pmatrix} = \begin{pmatrix} g \\ \eta \otimes (\iota(\eta) \otimes g) \end{pmatrix},$$

for $g \in \mathcal{E}_2$. Since A, B, C and D extend A_0, B_0, C_0 and D_0 respectively, we can drop the subscript 0. We also use the fact that the matrix we obtain is a coisometry, and thus its adjoint equals its inverse on its range. We conclude that

$$\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} g \\ \eta \otimes (\iota(\eta) \otimes g) \end{pmatrix} = \begin{pmatrix} Z(\eta^*)^* g \\ \iota(\eta) \otimes g \end{pmatrix}. \tag{8}$$

Thus $\iota(\eta) \otimes g = B^* g + D^*(\eta \otimes (\iota(\eta) \otimes g)) = B^* g + D^* L_\eta(\iota(\eta) \otimes g)$ and

$$\iota(\eta) \otimes g = (I - D^* L_\eta)^{-1} B^* g.$$

Combining this equality with the other equation that we get from (8), we have

$$Z(\eta^*)^*g = A^*g + C^*L_\eta(I - D^*L_\eta)^{-1}B^*g\ ,\ \ g \in \mathcal{E}.$$

Taking adjoints yields (iv). $\square$

Thus, Theorem 3.3 asserts that every Schur class function determines a system matrix whose transfer function represents the function. The system matrix is not unique in general, but as the proof of Theorem 3.3 shows, it arises through a series of natural choices. Of course, equation (6) suggests that every Schur class function represents an element in $H^\infty(E)$. This is indeed the case, as the following converse shows.

Theorem 3.6 *Let E be a W^*-correspondence over a W^*-algebra M, and let σ be a faithful normal representation of M on a Hilbert space $\mathcal{E}$. If $V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a system matrix determined by a normal representation τ of $N := \sigma(M)'$ on a Hilbert space H, then there is an $X \in H^\infty(E)$, $\|X\| \leq 1$, such that*

$$\widehat{X}(\eta^*) = A + B(I - L_\eta^* D)^{-1} L_\eta^* C,$$

for all $\eta^* \in \mathbb{D}(E^\sigma)^*$ and, conversely, every $X \in H^\infty(E)$, $\|X\| \leq 1$, may be represented in this fashion for a suitable system matrix $V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

Proof. For every $n \geq 0$ we define an operator K_n from $\mathcal{E}$ to $(E^\sigma)^{\otimes n} \otimes \mathcal{E}$ as follows. For $n = 0$, we set $K_0 = A$ - an operator in $B(\mathcal{E})$. For $n = 1$, we define K_1, mapping $\mathcal{E}$ to $E^\sigma \otimes \mathcal{E}$, to be $(I_1 \otimes B)C$, where for all $k \geq 1$, I_k denotes the identity operator on $(E^\sigma)^{\otimes k}$. For $n \geq 2$, we set

$$K_n := (I_n \otimes B)(I_{n-1} \otimes D) \cdots (I_1 \otimes D)C.$$

Note, first, that it follows from the properties of A, B, C and D that, for every $n \geq 0$ and every $a \in N$, $K_n a = (\varphi_n(a) \otimes I_{\mathcal{E}})K_n$ where φ_n defines the left multiplication on $(E^\sigma)^{\otimes n}$. Thus, writing ι for the identity representation of N on $\mathcal{E}$, K_n lies in the ι-dual of $(E^\sigma)^{\otimes n}$ which, by Theorem 3.6 and Lemma 3.7 of [31], is isomorphic to $E^{\otimes n}$. Hence, for every $n \geq 0$, K_n defines a unique element ξ_n in $E^{\otimes n}$.

For every $n \geq 0$ and $\eta \in E^\sigma$ we shall write $L_n(\eta)$ for the operator from $(E^\sigma)^{\otimes n} \otimes \mathcal{E}$ to $(E^\sigma)^{\otimes (n+1)} \otimes \mathcal{E}$ given by tensoring on the left by η. Also note that, for $k \geq 1$ and $n \geq 0$, $I_k \otimes K_n$ is an operator from $(E^\sigma)^{\otimes k} \otimes \mathcal{E}$ to $(E^\sigma)^{\otimes (k+n)} \otimes \mathcal{E}$. With this notation, it is easy to see that, for all $k \geq 1$ and $n \geq 0$,

$$(I_{k+1} \otimes K_n)L_k(\eta) = L_{k+n}(\eta)(I_k \otimes K_n). \tag{9}$$

Note, too, that we can write

$$\mathcal{F}(E^\sigma) \otimes \mathcal{E} = \mathcal{E} \oplus (E^\sigma \otimes \mathcal{E}) \oplus \cdots \oplus ((E^\sigma)^{\otimes m} \otimes \mathcal{E}) \oplus \cdots$$

and every operator on $\mathcal{F}(E^\sigma)\otimes\mathcal{E}$ can be written in a matricial form with respect to this decomposition (with indices starting at 0). For every m, $0 \le m \le \infty$, we let S_m be the operator defined by the matrix whose i,j entry is $I_j \otimes K_{i-j}$, if $0 \le j \le i \le m$, and is 0 otherwise. (For $m = \infty$, it is not clear yet that the matrix so constructed represents a bounded operator, but this will be verified later).

So far we have not used the assumption that V is a coisometry. But if we take this into account, form the product VV^*, and set it equal to $I_{\mathcal{E}\oplus(E^\sigma\otimes H)}$, we find that

$$I_{\mathcal{E}} - AA^* = BB^* \tag{10}$$

$$CC^* = I_{E^\sigma\otimes_\tau H} - DD^* \tag{11}$$

$$AC^* = -BD^* \tag{12}$$

We claim that, for $1 \le j \le i \le m$, the following equations hold,

$$(I - S_mS_m^*)_{i,j} = (I_i \otimes B)(I_{i-1}\otimes D)\cdots DD^*\cdots(I_{j-1}\otimes D^*)(I_j\otimes B^*); \tag{13}$$

that for $0 < i \le m$,

$$(I - S_mS_m^*)_{i,0} = (I_i \otimes B)(I_{i-1}\otimes D)\cdots DB^*, \tag{14}$$

and that for $i = j = 0$,

$$(I - S_mS_m^*)_{0,0} = BB^*. \tag{15}$$

Equation (15) follows immediately from (10) since $(S_m)_{0,0} = A$. For $0 < i \le m$ we compute $(I - S_mS_m^*)_{i,0} = -(S_m)_{i,0}(S_m)_{0,0}^* = -(I_i\otimes B)(I_{i-1}\otimes D)\cdots(I_1\otimes D)CA^* = (I_i\otimes B)(I_{i-1}\otimes D)\cdots(I_1\otimes D)DB^*$ where, in the last equality we used (12). It is left to prove (13). Let us write $R_{i,j}$ for the left hand side of (13). (For $j = 0 < i$ we have $R_{i,0} = (I_i\otimes B)(I_{i-1}\otimes D)\cdots DB^*$ and when both are 0, $R_{0,0} = BB^*$). We have $K_0K_0^* = AA^* = I - BB^* = I - R_{0,0}R_{0,0}^*$. For $0 = j < i \le m$ we have $K_iK_0^* = (I_i\otimes B)(I_{i-1}\otimes D)\cdots(I_1\otimes D)CA^* = -(I_i\otimes B)(I_{i-1}\otimes D)\cdots(I_1\otimes D)DB^* = -R_{i,0}$ and for $0 < j \le i \le m$, $K_iK_j^* = (I_i\otimes B)(I_{i-1}\otimes D)\cdots(I_1\otimes D)CC^*(I_1\otimes D^*)\cdots(I_{j-1}\otimes D^*)(I_j\otimes B^*) = (I_i\otimes B)(I_{i-1}\otimes D)\cdots(I_1\otimes D)(I-DD^*)(I_1\otimes D^*)\cdots(I_{j-1}\otimes D^*)(I_j\otimes B^*) = (I_i\otimes B)(I_{i-1}\otimes D)\cdots(I_1\otimes D)(I_1\otimes D^*)\cdots(I_{j-1}\otimes D^*)(I_j\otimes B^*) - (I_i\otimes B)(I_{i-1}\otimes D)\cdots(I_1\otimes D)DD^*(I_1\otimes D^*)\cdots(I_{j-1}\otimes D^*)(I_j\otimes B^*) = I_1\otimes R_{i-1,j-1} - R_{i,j}$.

We have

$$(S_mS_m^*)_{i,j} = \sum_{k=0}^{j}(S_m)_{i,k}(S_m)_{j,k} = \sum_{k=0}^{j} I_k\otimes K_{i-k}K_{j-k}^* = \sum_{l=0}^{j} I_{j-l}\otimes K_{i-j+l}K_l^*.$$

Using the computation above, we get, for $i = j \le m$,

$$(S_mS_m^*)_{i,i} = I_i\otimes(I - R_{0,0}R_{0,0}^*) + \sum_{l=1}^{i}(I_{i-l+1}\otimes R_{l-1,l-1} - I_{i-l}\otimes R_{l,l}) = I - R_{i,i}$$

and, for $j < i \le m$,

$$(S_mS_m^*)_{i,j} = -I_j \otimes R_{i-j,0} + \sum_{l=1}^{j}(I_{j-l+1}\otimes R_{i-j+l-1,l-1} - I_{j-l}\otimes R_{i-j+l,l}) = -R_{i,j}.$$

This completes the proof of the claim. If we let R be the operator whose matrix is $(R_{i,j})$ (letting $R_{i,j} = 0$ if i or j is larger than m) then we get $R = I - S_mS_m^*$. But it is easy to verify that R is a positive operator and, thus, $\|S_m\| \le 1$. This holds for every m and, therefore, we can find a weak limit point of the sequence $\{S_m\}$. But this limit point it clearly equal to S_∞, showing that S_∞ is indeed a bounded operator, with norm at most 1.

Recall that the induced representation of $H^\infty(E)$ on $\mathcal{F}(E)\otimes_\sigma \mathcal{E}$ is the representation that maps $X \in H^\infty(E)$ to $\sigma^{\mathcal{F}(E)}(X) := X \otimes I_{\mathcal{E}}$. The representation is faithful and is a homeomorphism with respect to the ultraweak topologies. Its image is the ultraweakly closed subalgebra of $B(\mathcal{F}(E)\otimes\mathcal{E})$ generated by the operators $T_\xi \otimes I_{\mathcal{E}}$ and $\varphi_\infty(a)\otimes I_{\mathcal{E}}$ for $\xi \in E$ and $a \in M$. Similarly one defines the induced representation $\iota^{\mathcal{F}(E^\sigma)}$ of $H^\infty(E^\sigma)$ on $\mathcal{F}(E^\sigma)\otimes\mathcal{E}$ and its image is generated by the operators $T_\eta \otimes I$ and $\varphi_\infty(b)\otimes I$ for $\eta \in E^\sigma$ and $b \in N$. Recall also, from [31, Theorem 3.9], that there is a unitary operator $U : \mathcal{F}(E^\sigma)\otimes\mathcal{E} \to \mathcal{F}(E)\otimes\mathcal{E}$ such that

$$(\iota^{\mathcal{F}(E^\sigma)}(H^\infty(E^\sigma)))' = U^*\sigma^{\mathcal{F}(E)}(H^\infty(E))U.$$

That is, U gives an explicit representation of $H^\infty(E^\sigma)$ as the commutant of the induced algebra $\sigma^{\mathcal{F}(E)}(H^\infty(E))$. Thus, to show that $S_\infty = U^*(X\otimes I)U$ for an $X \in H^\infty(E)$, we need only show that S_∞ lies in the commutant of $\iota^{\mathcal{F}(E^\sigma)}(H^\infty(E^\sigma))$. And for this, we only have to show that it commutes with the operators $\varphi_\infty(b)\otimes I$, $b \in N$, and $T_\eta \otimes I$, $\eta \in E^\sigma$. Note that, matricially, $\varphi_\infty(b)\otimes I$ is a diagonal operator whose i,i entry is $\varphi_i(b)$. For S_∞ to commute with it we should have, for all $j \le i$,

$$(I_j \otimes K_{i-j})(\varphi_j(b)\otimes I) = (\varphi_i(b)\otimes I)(I_j\otimes K_{i-j}).$$

This equality is obvious for $j > 0$. For $j = 0$ it amounts to the equality

$$K_i b = (\varphi_i(b)\otimes I_{\mathcal{E}})K_i$$

and, this, as was mentioned above, follows immediately from the properties of A, B, C and D. To show that S_∞ commutes with every $T_\eta \otimes I$, $\eta \in E^\sigma$, note that, matricially, the i,j entry of $T_\eta\otimes I$ vanishes unless $i = j+1$ and, in this case the entry is $L_j(\eta)$. Equation (9) then ensures that S_∞ and $T_\eta \otimes I$ commute.

Thus, by [31, Theorem 3.9], there is an element $X \in H^\infty(E)$ such that $S_\infty = U^*(X\otimes I)U$ $(= U^*\sigma^{\mathcal{F}(E)}(X)U)$. Since S_∞ has norm at most one, so does X. It remains to show that X is given by the transfer function built from V. To this end, fix $\xi \in E$ and recall that ξ defines a map $W(\xi) : \mathcal{E} \to E^\sigma \otimes \mathcal{E}$

via the formula $W(\xi)^*(\eta \otimes h) = L_\xi^* \eta(h)$, $\eta \otimes h \in E^\sigma \otimes \mathcal{E}$ (See [31, Theorem 3.6].), and that W maps E onto the ι-dual of E^σ. The desired properties follow easily from the definition of W. For every $k \geq 0$, $I_k \otimes W(\xi)^*$ is a map from $(E^\sigma)^{\otimes k+1} \otimes \mathcal{E}$ into $(E^\sigma)^{\otimes k} \otimes \mathcal{E}$. An easy computation shows that it is equal to the restriction of $U^*(T_\xi^* \otimes I_\mathcal{E})U$ to $(E^\sigma)^{\otimes k+1} \otimes \mathcal{E}$. (Recall from [31, Lemma 3.8] that the restriction of U to $(E^\sigma)^{\otimes k+1} \otimes \mathcal{E}$ is defined by the equation $U(\eta_1 \otimes \cdots \otimes \eta_{k+1} \otimes h) = (I_k \otimes \eta_1) \cdots (I_1 \otimes \eta_k)\eta_{k+1}(h)$.)
It then follows that the i, j entry of the matrix associated with $U^*(T_\xi \otimes I_\mathcal{E})U$ vanishes unless $i = j + 1$ and

$$(U^*(T_\xi \otimes I_\mathcal{E})U)_{j+1,j} = I_j \otimes W(\xi).$$

Similarly one can show that, for $\xi \in E^{\otimes k}$, the i, j entry of the matrix associated with $U^*(T_\xi \otimes I_\mathcal{E})U$ vanishes unless $i = j + k$ and

$$(U^*(T_\xi \otimes I_\mathcal{E})U)_{j+k,j} = I_j \otimes W(\xi).$$

In the last equation, $W(\xi)$, $\xi \in E^{\otimes k}$, is a map from $\mathcal{E}$ to $(E^\sigma)^{\otimes k} \otimes \mathcal{E}$.
Recall that we defined ξ_n to be the vectors in $E^{\otimes n}$ with $W(\xi_n) = K_n$. Thus we see that the n^{th} lower diagonal in the matricial form of S_∞ is the matricial form of $U^*(T_{\xi_n} \otimes I_\mathcal{E})U$.
Recall from the discussion at the end of Section 2 in [31] that S_∞ is the ultraweak limit of the sequence Σ_k where

$$\Sigma_k = \sum_{j=0}^{k-1} (1 - \frac{j}{k}) U^*(T_{\xi_j} \otimes I)U.$$

Hence X is the ultraweak limit of X_k where

$$X_k = \sum_{j=0}^{k-1} (1 - \frac{j}{k}) T_{\xi_j}$$

and, for $\eta \in E^\sigma$, $\widehat{X}(\eta^*)$ is the ultraweak limit of $\widehat{X}_k(\eta^*) = \sum_{j=0}^{k-1}(1 - \frac{j}{k})\widehat{T_{\xi_j}}(\eta^*)$.
Fix $\eta \in E^\sigma$ and $k \geq 1$. Then it is easy to check that, in the notation of the theorem, $L_\eta^*(I_k \otimes B) = (I_{k-1} \otimes B)L_\eta^*$ and $L_\eta^*(I_k \otimes D) = (I_{k-1} \otimes D)L_\eta^*$, all as operators on $(E^\sigma)^{\otimes k} \otimes H$. It then follows that for $n \geq 1$,

$$(L_\eta^*)^n W(\xi_n) = (L_\eta^*)^n K_n = B(L_\eta^* D)^{n-1} L_\eta^* C$$

and

$$A + B(I - L_\eta^* D)^{-1} L_\eta^* C = A + \sum_{n=1}^\infty B(L_\eta^* D)^{n-1} L_\eta^* C = \sum_{n=0}^\infty (L_\eta^*)^n W(\xi_n).$$

(Note that the last series converges in norm). It follows from [31, Proposition 5.1] that $\widehat{T_{\xi_n}}(\eta^*) = (L_\eta^*)^n W(\xi_n)$ and, thus, we finally conclude that $\widehat{X}(\eta^*) = A + B(I - L_\eta^* D)^{-1} L_\eta^* C$.
The 'converse' portion of the Theorem is immediate from Theorems 3.2 and 3.3. □

Corollary 3.7 *Every Schur class operator function defined on a subset Ω^* of $\mathbb{D}(E^\sigma)^*$ with values in some $B(\mathcal{E})$ can be extended to a Schur class operator function defined on all of $\mathbb{D}(E^\sigma)^*$.*

Proof. Let Z be a Schur class function on Ω^* and apply Theorem 3.3 to represent Z as the restriction to Ω^* of a transfer function. The result then follows from the evident combination of Theorems 3.6 and 3.2. □

Recall that every element X in $H^\infty(E)$ with $\|X\| \leq 1$ defines a Schur class operator function by evaluation at η^* for $\eta \in \mathbb{D}(E^\sigma)$ (where σ is a suitable prescribed faithful normal representation of M) . We usually suppress reference to σ and write $\widehat{X}$ for this Schur class operator function. In general, however, the map $X \to \widehat{X}$ is not one-to-one, and whether it is or not depends on the choice of σ. Indeed, in the particular case when $M = \mathbb{C}$ and $E = \mathbb{C}^n$, so $H^\infty(E)$ is $\mathcal{L}_n$, and when σ is the identity representation of $\mathbb{C}$, Davidson and Pitts showed that the kernel of the map $X \mapsto \widehat{X}$ is precisely the commutator ideal in $\mathcal{L}_n$ [17]. We shall show in the next lemma that given E, if σ is chosen to be faithful and have *infinite uniform multiplicity*, meaning that σ is an infinite multiple of another faithful normal representation of M, then the map $X \mapsto \widehat{X}$ will be one-to-one. It will be convenient to write $K(\sigma)$ for the kernel of the map determined by σ, so that

$$\begin{aligned} K(\sigma) &= \{X \in H^\infty(E) : \widehat{X}(\eta^*) = 0, \;\; \eta \in \mathbb{D}(E^\sigma)\} \\ &= \{X \in H^\infty(E) : \sigma \times \eta^*(X) = 0, \;\; \eta \in \mathbb{D}(E^\sigma)\}. \end{aligned} \tag{16}$$

Lemma 3.8 *If σ is a faithful normal representation of M on a Hilbert space H of infinite multiplicity, then $K(\sigma) = 0$. Moreover, if $\{X_\beta\}$ is a bounded net in $H^\infty(E)$ and if there is an element $X \in H^\infty(E)$ such that for every $\eta \in \mathbb{D}(E^\sigma)$, $\widehat{X}_\beta(\eta^*) \to \widehat{X}(\eta^*)$ in the weak operator topology, then $X_\beta \to X$ ultraweakly.*

Proof. It follows from the structure of isomorphisms of von Neumann algebras that any two infinite multiples of faithful representations of a von Neumann algebra are unitarily equivalent. It follows, therefore, that to prove the lemma, we can pick a special representation with this property that is convenient for our purposes. So let π be the representation of M on $\mathcal{F}(E) \otimes_\sigma H$ defined by $\pi = \varphi_\infty \otimes I_H$. We shall see that $K(\pi) = \{0\}$. For $\xi \in E$ let $V(\xi) = T_\xi \otimes I_H$. Then (V, π) is a representation of E on $\mathcal{F}(E) \otimes_\sigma H$. The integrated form of this representation is the induced representation $\pi^{\mathcal{F}(E)}$ restricted to $H^\infty(E)$. It is a faithful representation of $H^\infty(E)$. For $0 \leq r \leq 1$, (rV, π) is also a representation of E. It follows from [31, Lemma 7.11] that, for every $X \in H^\infty(E)$, the limit in the strong operator topology of $(\pi \times rV)(X)$, as $r \to 1$, is $(\pi \times V)(X)$. Thus, for $X \neq 0$ in $H^\infty(E)$, there is an r, $0 \leq r < 1$, such that $(\pi \times rV)(X) \neq 0$. Since for such r the inequality $\|rV\| < 1$ holds, and we conclude that $K(\pi) = \{0\}$.

For the second assertion of the lemma, suppose a bounded net $\{X_\beta\}$ in $H^\infty(E)$ has the property that for every $\eta \in \mathbb{D}(E^\pi)$, $\widehat{X}_\beta(\eta^*) \to 0$. Since the net is

bounded, it has a ultraweak limit point X_0 in $H^\infty(E)$. Since "evaluation at η^*" is the same as applying a ultraweakly continuous representation, we see that $\widehat{X}_\beta(\eta^*) \to \widehat{X}_0(\eta^*)$ for every $\eta \in \mathbb{D}(E^\pi)$. But then, $\widehat{X}_0(\eta^*) = 0$ for every $\eta \in \mathbb{D}(E^\pi)$ and, consequently, $X_0 = 0$ by the first assertion of the lemma. $\Box$
With this lemma in hand, we summarize the results of this section for future reference in the following corollary.

COROLLARY 3.9 *Let E be a W^*-correspondence over the W^*-algebra M, let σ be a faithful normal representation of M on the Hilbert space $\mathcal{E}$ and assume that σ has infinite multiplicity. Then the map $X \to \widehat{X}$ is a bijection from the closed unit ball of $H^\infty(E)$ onto the space of Schur class $B(\mathcal{E})$-valued functions on $\mathbb{D}(E^\sigma)^*$. Further, for each X in the closed unit ball of $H^\infty(E)$, $\widehat{X}$ is the transfer function associated with a system matrix $V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ defined in terms of a suitable auxiliary normal representation τ of $\sigma(M)'$ on a Hilbert space H, and conversely, each such transfer function on $\mathbb{D}(E^\sigma)^*$,*

$$\eta^* \to A + B(I - L_\eta^* D)^{-1} L_\eta^* C,$$

is of the form $\widehat{X}$ for a uniquely determined $X \in H^\infty(E)$: $\widehat{X}(\eta^*) = A + B(I - L_\eta^* D)^{-1} L_\eta^* C$ for all $\eta \in \mathbb{D}(E^\sigma)$.

PROOF. The proof is just the evident combination of Lemma 3.8 and Theorems 3.2, 3.3, and 3.6. $\Box$

REMARK 3.10 *One may well wonder why not stipulate at the outset that all σ's have uniform infinite multiplicity. It turns out that in many interesting examples, such as those coming from graphs, which we discuss in the last section, the principal σ's one wants to consider fail to have this property.*

4 APPLICATIONS TO AUTOMORPHISMS OF THE HARDY ALGEBRA

In this section we apply the analysis of Schur class functions to study automorphisms of $H^\infty(E)$. Our first goal is to show that under very general assumptions, the automorphisms are obtained by composition with (certain) biholomorphic automorphisms of the open unit ball of the dual correspondence. For the case were $E = \mathbb{C}^n$, so that $H^\infty(E)$ is the algebra $\mathcal{L}_n$ studied by Davidson and Pitts and by Popescu, this was shown for the dual correspondence associated with the one dimensional representation σ of $\mathbb{C}$ by Davidson and Pitts in [17].
Throughout this section we will focus on automorphisms α of $H^\infty(E)$ that are *completely isometric* and *w^*-homeomorphisms*. Also, we shall usually assume that the restriction of α to $\varphi_\infty(M)$ is the identity.
It is known that, in various settings, one can assume much less. In [17], the authors begin by assuming that α is simply an algebraic automorphism but, to get the one-to-one correspondence with automorphisms of the unit ball of

the dual, they need to impose also the assumption that the automorphism is contractive. It then follows from their results that it is, in fact, completely isometric and a w^*-homeomorphism. In [22], Katsoulis and Kribs show that in the setting when E is determined by a directed graph, G say, so $H^\infty(E)$ is the algebra they denote by $\mathcal{L}_G$, an algebraic automorphism is always norm-continuous and w^*-continuous.

As for the assumption that the restriction of α to $\varphi_\infty(M)$ is the identity, we shall see that for many purposes this is no significant restriction. However, in some situations, it can be a significant technical headache to sort out what happens if we don't impose the assumption. We will comment on this further, as we proceed. (See, in particular, Remark 4.10).

So, for the remainder of this section, unless specified otherwise, E will be a fixed W^*-correspondence over a W^*-algebra M and α will be a fixed automorphism of $H^\infty(E)$ that is completely isometric, w^*-homeomorphic and fixes $\varphi_\infty(M)$ element-wise. Also, σ will be a faithful normal $*$-representation of M on a Hilbert space H.

We think about elements of $H^\infty(E)$ as functions on $\mathbb{D}(E^\sigma)^*$ via the functional representation developed in the preceding section and we want to study the transposed action of α on $\mathbb{D}(E^\sigma)^*$. For every $\eta \in \mathbb{D}(E^\sigma)$, let $\tau(\eta) : H \to E\otimes_\sigma H$ be defined by the equation

$$\tau(\eta)^*(\xi \otimes h) = \widehat{\alpha(T_\xi)}(\eta^*)h \ (= (\sigma \times \eta^*)(\alpha(T_\xi))h\ ), \qquad (17)$$

$\xi\otimes h \in E\otimes_\sigma H$. (Observe that if α is the identity automorphism of $H^\infty(E)$, then this equation implies that τ is the identity map, as it should.) The next lemma shows that $\tau(\eta)$ is well defined and is an element in the closed unit ball of E^σ. Thus τ is a map from $\mathbb{D}(E^\sigma)$ into $\overline{\mathbb{D}(E^\sigma)}$. What we would really like to show, however, is that τ carries $\mathbb{D}(E^\sigma)$ into $\mathbb{D}(E^\sigma)$, not the closure. At this stage, we can only arrange for this under special circumstances: Theorem 4.7 below. The restriction on circumstances, however, is not so limiting as to eliminate many interesting examples. We also want to show that τ is holomorphic on $\mathbb{D}(E^\sigma)$ in the usual sense of infinite dimensional holomorphy [21].

Lemma 4.1 *For each $\eta \in \mathbb{D}(E^\sigma)$, $\tau(\eta)$ is well defined and lies in the closed unit ball of E^σ.*

Proof. For $\xi \in E$, let $S(\xi) := (\sigma \times \eta^*)(\alpha(T_\xi))$. For every $a, b \in M$, $S(a\xi b) = (\sigma\times\eta^*)(\alpha(T_{a\xi b})) = (\sigma\times\eta^*)(\alpha(\varphi_\infty(a)T_\xi\varphi_\infty(b))) = (\sigma\circ\alpha)(\varphi_\infty(a))(\sigma\times\eta^*)(\alpha(T_\xi))(\sigma\circ\alpha)(\varphi_\infty(b))$. By our assumption, $\sigma\circ\alpha\circ\varphi_\infty = \sigma\circ\varphi_\infty$ and, thus, (S, σ) is a covariant pair. Also, S is a completely contractive map of E into $B(H)$ as a composition of three completely contractive maps. Thus $\tilde{S}^* = \tau(\eta)$ lies in the closed unit ball of E^σ. $\Box$

To determine circumstances under which τ maps $\mathbb{D}(E^\sigma)$ into $\mathbb{D}(E^\sigma)$, we fix $\eta \in \mathbb{D}(E^\sigma)$ and determine circumstances under which $\tau(z\eta) \in \mathbb{D}(E^\sigma)$, for every $z \in \mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$. This will prove that τ maps $\mathbb{D}(E^\sigma)$ into itself.

So for $z \in \mathbb{D}$, we define

$$F(z) := \tau(\bar{z}\eta)^*. \tag{18}$$

Thus, $F(z)(\xi \otimes h) = (\sigma \times z\eta^*)(\alpha(T_\xi))h$ for $\xi \in E$ and $h \in H$.

Lemma 4.2 *F is an analytic function from $\mathbb{D}$ into $B(E \otimes H, H)$.*

Proof. Fix $\xi \otimes h \in E \otimes H$ with $\|\xi\| \leq 1$ and $k \in H$, and consider the expression

$$\langle F(z)(\xi \otimes h), k\rangle = \langle \widehat{\alpha(T_\xi)}(z\eta^*)h, k\rangle.$$

Since $\alpha(T_\xi) \in H^\infty(E)$ and $\|\alpha(T_\xi)\| \leq 1$, we know from Theorem 3.6 that we can write $\widehat{\alpha(T_\xi)}(z\eta^*) = A + B(I - zL_\eta^* D)^{-1} zL_\eta^* C$ for some system matrix. Thus

$$\widehat{\alpha(T_\xi)}(z\eta^*) = A + zBL_\eta^* C + \sum_{k=2}^{\infty} z^k B(L_\eta^*)^{k-1} L_\eta^* C.$$

Hence, for every $\xi \otimes h \in E \otimes H$ (even when $\|\xi\| > 1$) and $k \in H$, the function $z \mapsto \langle F(z)(\xi \otimes h), k\rangle$ is analytic. Since $\|F(z)\| \leq 1$ by Lemma 4.1, $|\langle F(z)g, k\rangle| \leq \|g\|\|k\|$ for every $g \in E \otimes H$ and $k \in H$ and it follows that, for each such g, k, the function $f_{g,k}(z) := \langle F(z)g, k\rangle$ is analytic in $\mathbb{D}$ and $|f_{g,k}(z)| \leq \|g\|\|k\|$. We can then write $f_{g,k}$ as a convergent power series $f_{g,k}(z) = \sum_{k=0}^{\infty} a_n(g,k)z^n$ and, for every $n \geq 0$, $|a_n(g,k)| \leq \|g\|\|k\|$. But then there are operators $A_n \in B(E \otimes H, H)$ with $\|A_n\| \leq 1$ such that $a_n(g,k) = \langle A_n g, k\rangle$ for $g \in E \otimes H$ and $k \in H$. Hence $F(z) = \sum_{k=0}^{\infty} z^n A_n$ where the sum converges in the weak operator topology. Since $|z| < 1$ and the norms of $\{A_n\}$ are bounded by 1, the series converges to $F(z)$, for $z \in \mathbb{D}$, in the norm topology. We conclude that $F(z)$ is analytic. $\square$

If we were dealing with scalar-valued functions, we would be able to assert that $|F(z)| < 1$ for all $z \in \mathbb{D}$, unless F is constant, by the maximum modulus theorem. Unfortunately, an unalloyed version of the maximum modulus theorem does not hold in our setting. This is what leads to the special hypotheses on τ in Theorem 4.7. The next few results, then, which lead up to Theorem 4.7 come out of our efforts to find a serviceable replacement for the maximum modulus theorem. Our first theorem in this direction, Theorem 4.4, is closely related to [36, Proposition V.2.1]. It does not seem to follow directly from this result, however. Instead, we appeal to the following lemma, which in turn is an immediate application of an operator form of the classical Pick criterion for interpolating operators at pre-assigned points by operator-valued analytic functions. As such, it may be traced back to Sz.-Nagy and Koranyi's influential paper [37]. It also is a consequence of Theorem 6.2 in [31], where it is presented as a corollary of our Nevanlinna-Pick Theorem.

Lemma 4.3 *If K, H are Hilbert spaces and if $F : \mathbb{D} \to B(K, H)$ is an analytic function satisfying $\|F(z)\| \leq 1$ for all $z \in \mathbb{D}$, then, for every $z_1, z_2 \in \mathbb{D}$, the matrix*

$$\begin{pmatrix} \frac{I_H - F(z_1)F(z_1)^*}{1-|z_1|^2} & \frac{I_H - F(z_1)F(z_2)^*}{1-z_1\bar{z}_2} \\ \frac{I_H - F(z_2)F(z_1)^*}{1-z_2\bar{z}_1} & \frac{I_H - F(z_2)F(z_2)^*}{1-|z_2|^2} \end{pmatrix}$$

is positive. In particular (setting $z_1 = z$ and $z_2 = 0$), for every $z \in \mathbb{D}$,

$$\begin{pmatrix} \frac{I_H - F(z)F(z)^*}{1-|z|^2} & I_H - F(z)F(0)^* \\ I_H - F(0)F(z)^* & I_H - F(0)F(0)^* \end{pmatrix} \geq 0. \tag{19}$$

THEOREM 4.4 *Suppose H and K are Hilbert spaces and suppose $F : \mathbb{D} \to B(K, H)$ is an analytic function that satisfies the following conditions:*

(1) $\|F(z)\| \leq 1$ for all $z \in \mathbb{D}$.

(2) There are projections P_1, P_2 in $B(H)$ that sum to I_H and projections Q_1, Q_2 in $B(K)$ that sum to I_K and satisfy:

(i) $P_1F(0)Q_2 = 0$ and $P_2F(0)Q_1 = 0$.

(ii) $P_1F(0)F(0)^*P_1 = P_1$.

(iii) $P_2F(0)F(0)^*P_2 \leq rP_2$ for some $0 < r < 1$.

Then, for every $z \in \mathbb{D}$,

(1) $P_1F(z)Q_2 = 0$.

(2) $P_1F(z)Q_1 = P_1F(0)Q_1$.

(3) There is a function $q_0(z)$ on $\mathbb{D}$, such that $0 < q_0(z) < 1$ for all $z \in \mathbb{D}$, and such that $P_2F(z)F(z)^*P_2 \leq q_0(z)P_2$.

PROOF. It will be convenient to use the projections P_1, P_2 and Q_1, Q_2 to write $F(z)$ matricially as

$$F(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

so that, by assumption,

$$F(0) = \begin{pmatrix} A(0) & 0 \\ 0 & D(0) \end{pmatrix}$$

where $A(0)A(0)^* = P_1$ and $D(0)D(0)^* \leq rP_2$.
Since F satisfies the conditions of Lemma 4.3, Equation 19 holds for all $z \in \mathbb{D}$. Compressing each entry of the matrix in (19) to the range of P_1 and using the fact that $A(0)A(0)^* = P_1$ and that $P_1F(0)Q_2 = 0$, we get

$$\begin{pmatrix} \frac{P_1 - P_1F(z)F(z)^*P_1}{1-|z|^2} & P_1 - P_1F(z)Q_1A(0)^* \\ P_1 - A(0)Q_1F(z)^*P_1 & 0 \end{pmatrix} \geq 0. \tag{20}$$

It follows that $P_1 = P_1F(z)Q_1A(0)^*$. Thus $0 \leq (P_1F(z)Q_1 - A(0))(Q_1F(z)^*P_1 - A(0)^*) = P_1F(z)Q_1F(z)^*P_1 + A(0)A(0)^* -$

$P_1F(z)Q_1A(0)^* - A(0)Q_1F(z)^*P_1 \leq 0$. Consequently, $A(0) = P_1F(z)Q_1$ (for every $z \in \mathbb{D}$).
But then $P_1F(z)Q_1F(z)^*P_1 = P_1$ and, since $P_1F(z)F(z)^*P_1 \leq P_1$, $P_1F(z)Q_2 = 0$. This proves (1) and (2).
Compress each entry of (19) to the range of P_2 to get

$$\begin{pmatrix} \frac{P_2-P_2F(z)F(z)^*P_2}{1-|z|^2} & P_2 - P_2F(z)Q_2D(0)^* \\ P_2 - D(0)Q_2F(z)^*P_2 & P_2 - D(0)D(0)^* \end{pmatrix} \geq 0. \tag{21}$$

Write Δ for the positive square root of $P_2 - D(0)D(0)^*$ and note that Δ is invertible as an operator on the range of P_2. Equation (21) implies that

$$(P_2 - D(0)D(z)^*)\Delta^{-2}(P_2 - D(z)D(0)^*) \leq (\frac{P_2 - P_2F(z)F(z)^*P_2}{1-|z|^2}).$$

Since $D(0)D(z)^*$ lies in $B(P_2(H))$ and has norm strictly less than 1 (as $\|D(0)\| < 1$), $P_2 - D(0)D(z)^*$ is invertible in $B(P_2(H))$ and so, therefore, is $(P_2 - D(0)D(z)^*)\Delta^{-2}(P_2 - D(z)D(0)^*)$. Hence, for each $z \in \mathbb{D}$ there is a $q(z) > 0$, such that $\frac{P_2-P_2F(z)F(z)^*P_2}{1-|z|^2} \geq (P_2 - D(0)D(z)^*)\Delta^{-2}(P_2 - D(z)D(0)^*) \geq q(z)P_2$. Thus,

$$P_2 - P_2F(z)F(z)^*P_2 \geq (1 - |z|^2)q(z)P_2,$$

which yields $P_2F(z)F(z)^*P_2 \leq (1 - q(z)(1 - |z|^2))P_2$. So, if we set $q_0(z) = (1 - q(z)(1 - |z|^2))$, we obtain a function with the desired properties. □
We return to our analysis of the special function $F : \mathbb{D} \to B(E \otimes_\sigma H, H)$ defined in equation (18).

Lemma 4.5 *The function F defined by equation (18) satisfies:*

(1) For every $z \in \mathbb{D}$ and $a \in M$, $F(z)(\varphi_E(a) \otimes I_H) = \sigma(a)F(z)$ and $F(z)F(z)^*$ commutes with $\sigma(M)$.

(2) For every $b \in \sigma(M)'$, $bF(0) = F(0)(I_E \otimes b)$ and $F(0)F(0)^* \in \mathfrak{Z}(\sigma(M))$.

Proof. Since $F(z)^* \in E^\sigma$ by Lemma 4.1, (1) holds. For (2), simply note that $bF(0)(\xi \otimes h) = b\alpha(T_\xi)(0)h = \alpha(T_\xi)(0)bh = F(0)(\xi \otimes bh) = F(0)(I_E \otimes b)(\xi \otimes h)$, where we used the fact that for every $X \in H^\infty(E)$, $X(0) \in \sigma(M)$. □

Definition 4.6 *Let τ be the map defined by equation (17). We say that $\tau(0)$ splits if there are projections P_1, P_2 in $\sigma(M)'$ such that*

(i) $P_1 + P_2 = I$,

(ii) $P_1\tau(0)^*\tau(0)P_1 = P_1$ and

(iii) $P_2\tau(0)^*\tau(0)P_2 \leq rP_2$ for some $r < 1$.

Note that $\tau(0) = F(0)^*$ so that, although F depends on a choice of $\eta \in \mathbb{D}(E^\sigma)$, $F(0)$ does not. It follows from Lemma 4.5, therefore, that $\tau(0)^*\tau(0)$ lies in the center of $\sigma(M)$, $\mathfrak{Z}(\sigma(M)) = \sigma(\mathfrak{Z}(M))$.

Note also that, if the center of M, $\mathfrak{Z}(M)$, is an atomic abelian von Neumann algebra, then $\tau(0)$ always splits. This is the case, in particular, if M is a factor or if $M = \mathbb{C}^n$. It is also the case, therefore, when E is the correspondence associated with a (countable) directed graph.

When $\tau(0)$ splits we have the following.

THEOREM 4.7 *Assume that the left action map of M on E, φ_E, is injective and that $\tau(0)$ splits. Then the map τ defined in equation (17)) maps $\mathbb{D}(E^\sigma)$ into itself and satisfies the following equation*

$$(\widehat{\alpha(X)})(\eta^*) = \widehat{X}(\tau(\eta)^*),$$

for every $X \in H^\infty(E)$ and $\eta \in \mathbb{D}(E^\sigma)$.

PROOF. Fix $\eta \in \mathbb{D}(E^\sigma)$ and let F be the map defined in (18). Since $\tau(0) = F(0)^*$ splits, there are projections P_1, P_2 as in Definition 4.6. Using Lemma 4.5, we see that the conditions of Theorem 4.4 are satisfied with $K = E \otimes H$ and $Q_i = I_E \otimes P_i$, $i = 1, 2$. Thus,

$$P_1 F(z) = P_1 F(z)(I_E \otimes P_1) = P_1 F(0)(I_E \otimes P_1) = P_1 F(0)$$

for all $z \in \mathbb{D}$. Consequently, for all $\xi \in E$, $P_1(\sigma \times z\eta^*)(\alpha(T_\xi)) = P_1\sigma(\alpha(T_\xi)_0)$ where, for $X \in H^\infty(E)$, X_0 is the image of X under the conditional expectation onto $\varphi_\infty(M)$. Since the representation $\sigma \times z\eta^*$ is w^*-continuous and α is surjective, we have for all $X \in H^\infty(E)$,

$$P_1(\sigma \times z\eta^*)(X) = P_1\sigma(X_0).$$

In particular, letting $X = T_\xi$, we see that $P_1(\sigma \times z\eta^*)(T_\xi) = 0$. Since, for $h \in H$, $(\sigma \times z\eta^*)(T_\xi)h = P_1\eta^*(\xi \otimes h) = 0$ we have $\eta P_1 = 0$. (Recall that $P_1 \in \sigma(M)'$ and, thus, ηP_1 is well defined since E^σ is a right module over $\sigma(M)'$).

Since η is arbitrary in $\mathbb{D}(E^\sigma)$, $E^\sigma P_1 = 0$. If $P_1 \neq 0$, it follows that E^σ is not full and, using Lemma 2.20, the map φ_E is not injective, contradicting our assumption. Thus $P_1 = 0$ and it follows from Theorem 4.4 that $\|F(z)\| < 1$ for every z. since this holds for all $\eta \in \mathbb{D}(E^\sigma)$, the conclusion of the theorem follows. $\square$

Next we show that the map τ is holomorphic on $\mathbb{D}(E^\sigma)$. We view it as a map into $B(H, E \otimes H)$. To be holomorphic is the same as being Frechet-differentiable. If we use [21, Theorem 3.17.1] and the fact, proved in Lemma 4.1, that τ is bounded, it suffices to show that τ is (G)-differentiable in the sense of [21, Definition 3.16.2]. But if we apply [21, Theorem 3.16.1], this means that we have to show that for every $\eta_0, \eta \in \mathbb{D}(E^\sigma)$, the function $G(z) := \tau(\eta_0 + z\eta)$,

defined on $D(\eta,\eta_0) := \{z \in \mathbb{C} | |z| < (1-\|\eta_0\|)/\|\eta\|\}$ is holomorphic in the sense of [21, Definition 3.10.1].
Since the set of all functionals on $B(H, E\otimes H)$ that are w^*-continuous is a determining manifold for $B(H, E\otimes H)$ in the sense of [21, Definition 2.8.2], it suffices to show that for every w^*-continuous functional w, the map $z \mapsto w(\tau(\eta_0 + z\eta))$ is holomorphic on $D(\eta,\eta_0)$. It is enough, in fact, to consider all functionals of the form $T \mapsto \langle Th, \xi\otimes k\rangle$ for $h,k \in H$ and ξ in the unit ball of E.
So we fix $\eta_0, \eta \in E^\sigma$, $h,k\in H$ and $\xi \in E$ with $\|\xi\| < 1$ and write $f(z) = \langle \tau(\eta_0 + z\eta)h, \xi\otimes k\rangle$ for $z \in D(\eta,\eta_0)$. We have

$$f(z) = \langle h, \tau(\eta_0 + z\eta)^*(\xi\otimes k)\rangle = \langle h, \widehat{\alpha(T_\xi)}(\eta_0^* + \bar{z}\eta^*)k\rangle.$$

Note that by Theorem 3.6, we can write

$$\widehat{\alpha(T_\xi)}(\eta_0^* + z\eta^*) = A + \sum_{m=1}^{\infty} B((L_{\eta_0}^* + \bar{z}L_\eta^*)D)^{m-1}(L_{\eta_0}^* + \bar{z}L_\eta^*)C$$

where A,B,C,D are from some system matrix and the sum converges in norm. Thus

$$f(z) = \langle A^*h, k\rangle + \sum_{m=1}^{\infty} \langle C^*(L_{\eta_0} + zL_\eta)(D^*(L_{\eta_0} + zL_\eta))^{m-1}B^*h, k\rangle$$

and this function is clearly holomorphic.
We can conclude:

Corollary 4.8 *The function τ is a holomorphic map from $\mathbb{D}(E^\sigma)$ to its closure.*

Theorem 4.9 *Let E be a faithful W^*-correspondence over M, let α be an automorphism of $H^\infty(E)$ that is completely isometric, is a w^*-homeomorphism and leaves $\varphi_\infty(M)$ elementwise fixed, and let σ be a faithful representation of M. Write τ for the transpose of α defined in equation (17) and write θ for the map associated similarly with α^{-1}. If both $\tau(0)$ and $\theta(0)$ split (as in Definition 4.6) then τ is a biholomorphic map of the open unit ball of E^σ, $\tau^{-1} = \theta$, and, for every $X \in H^\infty(E)$,*

$$(\widehat{\alpha(X)})(\eta^*) = \widehat{X}(\tau(\eta)^*)\ ,\ \eta \in \mathbb{D}(E^\sigma). \tag{22}$$

Proof. We already know that, under the conditions of the theorem, both τ and θ are holomorphic maps of the open unit ball. It follows from equation (17) that, for every $\xi \in E$, $h \in H$ and $\eta \in \mathbb{D}(E^\sigma)$, $\widehat{\alpha(T_\xi)}(\eta^*) = \tau(\eta)^*(\xi\otimes h)$. But $\tau(\eta)^*(\xi\otimes h) = \widehat{T_\xi}(\tau(\eta)^*)$, so that equation (22) holds for T_ξ. It also holds for $\varphi_\infty(a)$, $a\in M$, since $\alpha(\varphi_\infty(a)) = \varphi_\infty(a)$. Therefore it holds for every X in a w^*-dense subalgebra of $H^\infty(E)$. By the w^*-continuity of α, equation (22) holds for every $X \in H^\infty(E)$. Since a similar claim holds for α^{-1} and θ, we conclude that for all $X \in H^\infty(E)$, $\widehat{X}(\eta^*) = \alpha^{-1}\widehat{(\alpha(X)})(\eta^*) = \widehat{\alpha(X)}(\theta(\eta)^*) = \widehat{X}(\tau(\theta(\eta))^*)$. Thus $\tau^{-1} = \theta$. $\square$

A biholomorphic map τ is said to *implement* α if equation (22) holds.

Remark 4.10 *If α is implemented by τ in the sense of equation (22), then, writing this equation when $X = \varphi_\infty(a)$, $a \in M$, shows that α leaves $\varphi_\infty(M)$ elementwise fixed. Also, inspecting the proof of Lemma 4.1, one sees that, if α does not have this property, the map τ, defined in equation (17) would map the unit ball of E^σ into the unit ball of E^π where $\pi = \sigma \circ \varphi_\infty^{-1} \circ \alpha \circ \varphi_\infty$. One can study such automorphisms by studying these maps but the situation becomes quite complicated, unless one makes a global assumption to begin with,* vis., *that σ has uniform infinite multiplicity. In that event, by properties of normal representations of von Neumann algebras, σ and π are unitarily equivalent. Say $\pi(\cdot) = u\sigma(\cdot)u^*$ for some Hilbert space isomorphism from the Hilbert space of σ to the Hilbert space of π. Then it is a straightforward calculation to see that $E^\pi = (I \otimes u)E^\sigma u^*$. It is then a straightforward matter to incorporate u into our formulas.*

As we have remarked before, $\mathbb{D}(E^\sigma)$ is the unit ball of a J^*-triple system. It results, therefore, from well-known theory [20] that the biholomorphic maps of $\mathbb{D}(E^\sigma)$ are determined by Möbius transformations (and "isometric multipliers"). As we shall, however, the Möbius transformations of $\mathbb{D}(E^\sigma)$ that implement automorphisms of $H^\infty(E)$ have to have a special form: They must be parametrized by "central" elements of $\mathbb{D}(E^\sigma)$ in the sense of the following definition. (See also Remark 2.1.3 of [14]).

Definition 4.11 *Let E be a W^*-correspondence over a W^*-algebra M. The* center of E, *denoted $\mathfrak{Z}(E)$, is the set of $\xi \in E$ such that $a\xi = \xi a$ for all $a \in M$.*

Lemma 4.12 *(1) The center $\mathfrak{Z}(E)$ of a W^*-correspondence E over M is a W^*-correspondence over the center $\mathfrak{Z}(M)$ of M.*

(2) Let σ be a faithful normal representation of M on the Hilbert space $\mathcal{E}$, and for $\xi \in E$, define $\Phi(\xi) := L_\xi$ where L_ξ maps $\mathcal{E}$ to $E \otimes \mathcal{E}$ via the formula $L_\xi(h) = \xi \otimes h$. Then the pair (σ, Φ) defines an isomorphism of $\mathfrak{Z}(E)$ onto $\mathfrak{Z}(E^\sigma)$ in the sense of Definition 2.2. (Here, $\mathfrak{Z}(E)$ is a correspondence over $\mathfrak{Z}(M)$ and $\mathfrak{Z}(E^\sigma)$ is a correspondence over $\mathfrak{Z}(\sigma(M)') = \mathfrak{Z}(\sigma(M)) = \sigma(\mathfrak{Z}(M))$).

(3) Given a faithful representation σ of M on the Hilbert space $\mathcal{E}$ and $\gamma \in \mathbb{D}(E^\sigma)$, then γ lies in the center of E^σ if and only if the representation $\sigma \times \gamma^*$ maps $H^\infty(E)$ into $\sigma(M)$.

Proof. It is clear that $\mathfrak{Z}(E)$ is a bimodule over $\mathfrak{Z}(M)$ and, to prove (1), we need only show that the inner product of two elements in $\mathfrak{Z}(E)$ lies in $\mathfrak{Z}(M)$.

For $a \in M$, $\xi_1, \xi_2 \in \mathfrak{Z}(E)$ we have

$$a\langle \xi_1, \xi_2 \rangle = \langle \xi_1 a^*, \xi_2 \rangle = \langle a^* \xi_1, \xi_2 \rangle = \langle \xi_1, a\xi_2 \rangle = \langle \xi_1, \xi_2 a \rangle = \langle \xi_1, \xi_2 \rangle a.$$

Hence the inner product lies in the center of M, proving (1). We fix a faithful representation σ of M on $\mathcal{E}$. For $\xi \in \mathfrak{Z}(E)$, $a \in M$ and $h \in \mathcal{E}$ we have $L_\xi \sigma(a) h = \xi \otimes_\sigma \sigma(a) h = \xi a \otimes h = a\xi \otimes h = (a \otimes I) L_\xi h$. Hence, $L_\xi \in E^\sigma$. Given $b \in \sigma(M)'$ and $h \in \mathcal{E}$ we have $L_\xi b h = \xi \otimes bh = (I_E \otimes b) L_\xi h$. Thus L_ξ lies in $\mathfrak{Z}(E^\sigma)$.
For $\xi \in \mathfrak{Z}(E)$, $a, b \in \mathfrak{Z}(M)$, and $h \in \mathcal{E}$, $L_{a\xi b} h = a\xi b \otimes h = \xi ab \otimes h = \xi \otimes \sigma(a)\sigma(b) h = (I \otimes \sigma(a)) L_\xi \sigma(b) h$ hence

$$\Phi(a\xi b) = \sigma(a)\Phi(\xi)\sigma(b).$$

For $\xi_1, \xi_2 \in \mathfrak{Z}(E)$ we have $L_{\xi_1}^* L_{\xi_2} = \sigma(\langle \xi_1, \xi_2 \rangle)$. Therefore the pair (σ, Φ) is an isomorphism of $\mathfrak{Z}(E)$ into $\mathfrak{Z}(E^\sigma)$.
To prove that the map Φ is onto, fix an $\eta \in \mathfrak{Z}(E^\sigma)$. Then, η is a map from $\mathcal{E}$ to $E \otimes_\sigma \mathcal{E}$ satisfying

$$\eta\sigma(a) = (a \otimes I)\eta \tag{23}$$

and

$$\eta b = (I \otimes b)\eta, \tag{24}$$

for $a \in M$ and $b \in \sigma(M)'$. Define the map $\psi : E \to B(\mathcal{E})$ by $\psi(\zeta) = \eta^* L_\zeta$ and note that for $b \in \sigma(M)'$ and $h \in \mathcal{E}$, $\eta^* L_\zeta b h = \eta^*(\zeta \otimes bh) = \eta^*(I \otimes b) L_\zeta h$. Using (24) the latter is equal to $b\eta^* L_\zeta h$. Hence $\psi(\zeta)$ lies in $\sigma(M)$. Also $\psi(\zeta a) = \psi(\zeta)\sigma(a)$ for all $a \in M$ and it then follows from the self duality of E that there is an $\xi \in E$ with $\langle \xi, \zeta \rangle = \sigma^{-1}(\psi(\zeta))$. Thus, for all $\zeta \in E$, $L_\xi^* L_\zeta = \sigma(\langle \xi, \zeta \rangle) = \eta^* L_\zeta$ and we conclude that $\eta = L_\xi$.
It follows from (23) that, for all $a \in M$, $L_{\xi a} = \eta\sigma(a) = (a \otimes I)\eta = L_{a\xi}$, showing that ξ lies in $\mathfrak{Z}(E)$.
Finally, to prove (3), fix an $\eta \in \mathbb{D}(E^\sigma)$ and write (T, σ) for the covariant pair associated with $\sigma \times \eta^*$ (so that, $\tilde{T} = \eta^*$). Then the representation maps $H^\infty(E)$ into $\sigma(M)$ if and only if, for each $\xi \in E$, $T(\xi) \in \sigma(M)$. This holds iff, for all $b \in \sigma(M)'$, $\xi \in E$ and $h \in \mathcal{E}$, $\tilde{T}(I_\mathcal{E} \otimes b)(\xi \otimes h) = T(\xi) bh = bT(\xi)h = b\tilde{T}(\xi \otimes h)$; that is, if and only if $\tilde{T}(I_\mathcal{E} \otimes b) = b\tilde{T}$ for every $b \in \sigma(M)'$. But the last statement says that η lies in the center of E^σ. $\square$
The following example may help to show that the center of a correspondence is much less "inert" than the center of a von Neumann algebra.

Example 4.13 *Let M be a von Neumann algebra and let α be an endomorphism of M. Then we obtain a W^*-correspondence over M, denoted ${}_\alpha M$, by taking M with its usual right action and inner product give by the formula, $\langle \xi, \eta \rangle = \xi^* \eta$ and by letting α implement the left action. Then an element ξ in ${}_\alpha M$ lies in the center of ${}_\alpha M$ if and only if ξ intertwines α and the identity endomorphism; i.e., $\xi \in \mathfrak{Z}({}_\alpha M)$ if and only if $\alpha(a)\xi = \xi a$ for all $a \in M$. $\mathfrak{Z}({}_\alpha M)$ is a much studied object in the literature and the preceding lemma spells out some of its important elementary properties.*

Our goal now is to develop the properties of Möbius transformations of $\mathbb{D}(E^\sigma)$ and to identify those that implement automorphisms of $H^\infty(E)$. To this end, fix a faithful representation σ of M on a Hilbert space $\mathcal{E}$. Set $N = \sigma(M)'$, write $K = \mathcal{E} \oplus (E \otimes_\sigma \mathcal{E})$, and define the (necessarily faithful) representation ρ of N on K by the formula

$$\rho(S) = \begin{pmatrix} S & 0 \\ 0 & I \otimes S \end{pmatrix}, \quad S \in N.$$

For $\gamma \in \mathbb{D}(E^\sigma)$ we set $\Delta_\gamma := (I_{\mathcal{E}} - \gamma^*\gamma)^{1/2}$ - an element in $B(\mathcal{E})$ - and $\Delta_{\gamma^*} := (I_{E\otimes\mathcal{E}} - \gamma\gamma^*)^{1/2}$ - an element in $B(E \otimes \mathcal{E})$. When γ is understood, then we shall simply write Δ for Δ_γ and Δ_* for Δ_{γ^*}. Given $\gamma \in \mathbb{D}(E^\sigma)$ we define the map g_γ on $\mathbb{D}(E^\sigma)^*$ by the formula,

$$g_\gamma(z^*) = \Delta_\gamma(I - z^*\gamma)^{-1}(\gamma^* - z^*)\Delta_{\gamma^*}^{-1}, \tag{25}$$

$z \in \mathbb{D}(E^\sigma)$. Then g_γ is a biholomorphic automorphism of $\mathbb{D}(E^\sigma)^*$ that maps 0 to γ^* and γ^* to 0. Further, $g_\gamma^2 = id$, and every biholomorphic map g of $\mathbb{D}(E^\sigma)^*$ is of the form

$$g = w \circ g_\gamma$$

where w is an isometry on $(E^\sigma)^*$ and $\gamma^* = w^{-1}g(0)$ [20]. When γ lies in the center of E^σ, we see that g_γ maps the center onto itself and it follows that every biholomorphic automorphism of the open unit ball of $(E^\sigma)^*$ that preserves the center is of the form

$$g = w \circ g_\gamma$$

where γ lies in the center and w is an isometry on $(E^\sigma)^*$ that preserves the center.

If $z \in \mathbb{D}(E^\sigma)$, then the series $\sum_{n=0}^\infty (z^*\gamma)^n$ converges in norm to the operator in N, $(I - z^*\gamma)^{-1} = \sum_{n=0}^\infty (z^*\gamma)^n$. One easily calculates, then, that

$$g_\gamma(z^*) = \Delta\gamma^*\Delta_*^{-1} - \Delta(I - z^*\gamma)^{-1}z^*\Delta_*.$$

Recall that the equation $U(z \otimes h) = z(h)$ defines a Hilbert space isomorphism $U : E^\sigma \otimes \mathcal{E} \to E \otimes \mathcal{E}$ [31, p. 369]. Consequently, as maps on $\mathcal{E}$, $UL_z = z$ and $z^* = L_z^* U^*$. Thus we may write

$$g_\gamma(z^*) = \Delta\gamma^*\Delta_*^{-1} - \Delta(I - L_z^*U^*\gamma)^{-1}L_z^*U^*\Delta_*.$$

We write $K_1 = E \otimes_\sigma \mathcal{E}$ for the second summand in $K = \mathcal{E} \oplus (E \otimes_\sigma \mathcal{E})$ and we let q_1 denote the projection from K onto K_1. Likewise, we set $K_2 = \mathcal{E}$ with projection q_2. Corresponding to the direct sum decomposition, we define V by the formula

$$V := \begin{pmatrix} \Delta\gamma^*\Delta_*^{-1} & -\Delta \\ U^*\Delta_* & U^*\gamma \end{pmatrix} : \begin{pmatrix} K_1 \\ \mathcal{E} \end{pmatrix} \to \begin{pmatrix} K_2 \\ E^\sigma \otimes \mathcal{E} \end{pmatrix}. \tag{26}$$

If we calculate VV^*, we find that the off diagonal terms vanish and the terms on the diagonal are $\Delta\gamma^*\Delta_*^{-2}\gamma\Delta+\Delta^2$ and $U^*(\Delta_*^2+\gamma\gamma^*)U$. Since $\Delta_*^2+\gamma\gamma^* = I_{E\otimes\mathcal{E}}$, the latter expression is $U^*U = I_{E^\sigma\otimes\mathcal{E}} = q_2$. For the first expression, we note that $\gamma^*\Delta_*^{-2}\gamma = \gamma^*(I-\gamma\gamma^*)^{-1}\gamma = (I-\gamma^*\gamma)^{-1}-1$ and $\Delta\gamma^*\Delta_*^{-2}\gamma\Delta+\Delta^2 = \Delta((I-\gamma^*\gamma)^{-1}-I)\Delta+\Delta^2 = I_{\mathcal{E}}$. This shows that V is a coisometry. Similar computations show that it is, in fact, a unitary operator. Thus V is a transfer operator.

We want to apply Theorem 3.6 to obtain an element $X \in H^\infty(E)$ with $\widehat{X}(\eta^*) = g_\gamma(\eta^*)$, for $\eta \in \mathbb{D}(E^\sigma)$. To do this, we first let F be the correspondence E^σ and then F^ρ is a correspondence over $\rho(N)'$. In order to apply Theorem 3.6 we let M, in that theorem, be the von Neumann algebra $\rho(N)'$ and let σ there be the identity representation of $\rho(N)'$ on K (so that $\mathcal{E}$ there is K). E in that theorem will be F^ρ and N there (the commutant of $\sigma(M)$) will be $\rho(N)$. The representation τ of N then will be the map ρ^{-1} of $\rho(N)$ on $\mathcal{E}$ (so that $\mathcal{E}$ will play the role of H there). Also, q_1 will be as above. We set $A = \Delta\gamma^*\Delta_*^{-1}$, $B = -\Delta$, $C = U^*\Delta_*$ and $D = U^*\gamma$. These A, B, C and D give rise to the matricial operator V of equation (26). In order to show that the assumptions of Theorem 3.6 are satisfied, we have to show that these operators (A, B, C and D) all have the required intertwining properties. (Note that we have already checked that V is a unitary operator).

The required intertwining properties are:

(a) $A = \Delta\gamma^*\Delta_*^{-1}$ lies in $q_2\rho(N)'q_1$.

(b) $B = -\Delta$ lies in N'.

(c) For every $S \in N$, $U^*\Delta_*(I_E \otimes S) = (S \otimes I_{\mathcal{E}})U^*\Delta_*$ on $E \otimes \mathcal{E}$.

(d) For every $S \in N$, $U^*\gamma S = (I_E \otimes S)U^*\gamma$ on $\mathcal{E}$.

Indeed, recall that γ lies in the center of E^σ and, thus, for $S \in N$, $\gamma S = (I\otimes S)\gamma$. Therefore Δ commutes with N and Δ_* commutes with $I \otimes S$ for $S \in N$. This implies (a) and (b). Recall that, for $h \in \mathcal{E}$, $U^*\gamma h = \gamma \otimes h$ and, thus, $U^*\gamma S h = \gamma \otimes Sh = (I \otimes S)(\gamma \otimes h) = (I \otimes S)U^*\gamma h$ proving (d). For (c), it suffices to note that $U(S \otimes I)U^* = I \otimes S$ and Δ_* commutes with $I \otimes S$ for all $S \in N$.

We can now apply Theorem 3.6. Since F^ρ plays the role of E in that theorem and the identity representation of $\rho(N)'$, id, plays the role of σ, E^σ in that theorem is replaced by $(F^\rho)^{id}$ which, by the duality theorem [31, Theorem 3.6] is isomorphic to $F = E^\sigma$. We therefore conclude:

Lemma 4.14 *For every $\gamma \in \mathbb{D}(\mathfrak{Z}(E^\sigma))$, there is an X in $H^\infty(F^\rho)$ with $\|X\| \le 1$ such that, for all $z \in \mathbb{D}(E^\sigma)$, $\widehat{X}(z^*) = g_\gamma(z^*)$.*

Note that $g_\gamma(z^*)$ is an operator from $E \otimes \mathcal{E}$ into $\mathcal{E}$ and can be viewed as an operator in $B(K)$ which is where the values of X, as an element of $H^\infty(F^\rho)$, lie.

We can now use [31, Theorem 5.3] to prove the following.

Corollary 4.15 *Fix $\gamma \in \mathbb{D}(\mathfrak{Z}(E^\sigma))$ as above. Then, for every $z_1, z_2, \ldots, z_k$ in $\mathbb{D}(E^\sigma)$, the map on $M_k(\sigma(M)')$ defined by the $k \times k$ matrix*

$$((id - \theta_{g_\gamma(z_i^*)^*, g_\gamma(z_j^*)^*}) \circ (id - \theta_{z_i,z_j})^{-1})$$

is completely positive.

Proof. Applying [31, Theorem 5.3] to X of Lemma 4.14, we get the complete positivity of the map defined by the matrix

$$((I - Ad(g_\gamma(z_i^*), g_\gamma(z_j^*))) \circ (id - \theta_{z_i,z_j})^{-1}).$$

But note that, for every $b \in \sigma(M)'$, $Ad(g_\gamma(z_i^*), g_\gamma(z_j^*))(\rho(b)) = g_\gamma(z_i^*)\rho(b)g_\gamma(z_j^*)^* = \langle g_\gamma(z_i^*)^*, bg_\gamma(z_j^*)^*\rangle = \theta_{g_\gamma(z_i^*)^*, g_\gamma(z_j^*)^*}(b)$. □

Corollary 4.16 *Let $Z : \mathbb{D}(E^\sigma)^* \to B(\mathcal{E})$ be a Schur class operator function and let γ be in $\mathbb{D}(\mathfrak{Z}(E^\sigma))$. Then the function $Z_\gamma : \mathbb{D}((E^\sigma)^*) \to B(\mathcal{E})$ defined by*

$$Z_\gamma(\eta^*) = Z(g_\gamma(\eta^*))$$

is also a Schur class operator function.

Proof. For every η_i, η_j in $\mathbb{D}(E^\sigma)$ we have $(id - Ad(Z(g_\gamma(\eta_i^*)), Z(g_\gamma(\eta_j^*)))) \circ (id - \theta_{\eta_i,\eta_j})^{-1} = ((id - Ad(Z(g_\gamma(\eta_i^*)), Z(g_\gamma(\eta_j^*)))) \circ (id - \theta_{g_\gamma(\eta_i^*)^*, g_\gamma(\eta_j^*)^*})^{-1}) \circ (id - \theta_{g_\gamma(\eta_i^*)^*, g_\gamma(\eta_j^*)^*}) \circ (id - \theta_{\eta_i,\eta_j})^{-1})$. Hence the map associated with Z_γ is a composition of two completely positive maps and is, therefore, completely positive. □

For the statement of the next lemma, recall from [31, end of Section 2] that every $X \in H^\infty(E)$ has a "Fourier series" expansion given by a sequence of "Fourier coefficient operators" $\{\mathbb{E}_j\}$. (In [31] we wrote $\{\Phi_j\}$ for this sequence). Each map $\mathbb{E}_j$ is completely contractive, w^*-continuous and $\mathbb{E}_j(T_{\xi_1}T_{\xi_2}\cdots T_{\xi_k}) = T_{\xi_1}T_{\xi_2}\cdots T_{\xi_k}$ if $j = k$ and is zero otherwise. The Cesaro means of the "Fourier series" of X converge to X in the w^*-topology.

Lemma 4.17 *Let σ be a normal, faithful, representation of M on a Hilbert space H and let $K(\sigma)$ denote the kernel of the map $X \to \widehat{X}$ defined in equation (16).*

- *(i) $K(\sigma) \subseteq \{X \in H^\infty(E) \mid \mathbb{E}_0(X) = \mathbb{E}_1(X) = 0\}$.*
- *(ii) If, for every $k \in \mathbb{N}$, $\vee\{(\eta^{\otimes k})(H) \mid \eta \in \mathbb{D}(E^\sigma)\} = E^{\otimes k} \otimes H$, then $K(\sigma) = \{0\}$.*
- *(iii) Every completely isometric automorphism α of $H^\infty(E)$ that is a w^*-homeomorphism and is implemented by a biholomorphic map of $\mathbb{D}(E^\sigma)$ in the sense of (22) leaves $K(\sigma)$ invariant. In particular, $K(\sigma)$ is invariant under the action of the gauge group and, thus, under the maps $\mathbb{E}_k$, $k \geq 0$.*

Proof. Write C_1 for $\{X \in H^\infty(E) \mid \mathbb{E}_0(X) = \mathbb{E}_1(X) = 0\}$. Then for every $X \in H^\infty(E)$, $X = \mathbb{E}_0(X) + \mathbb{E}_1(X) + X_1$ where $X_1 \in C_1$. Note that for every $\eta \in \mathbb{D}(E^\sigma)$, every $0 < t \leq 1$ and every $k \geq 0$, $\mathbb{E}_k(X)((t\eta)^*) = t^k\mathbb{E}(X)(\eta^*)$. Thus, for $X \in K(\sigma)$, $0 = X((t\eta)^*) = \mathbb{E}_0(X)(\eta^*) + t\mathbb{E}_1(X)(\eta^*) + t^2 S$ where S is some bounded operator on H. Since this holds for every $0 < t \leq 1$, we have (by differentiation) $\mathbb{E}_0(X) = 0$ and $\mathbb{E}_1(X)(\eta^*) = 0$ for all $\eta \in \mathbb{D}(E^\sigma)$. Write $\mathbb{E}_1(X) = T_\xi$ (for some $\xi \in E$). Then, for all $h \in H$ and $\eta \in \mathbb{D}(E^\sigma)$, $0 = \mathbb{E}_1(X)(\eta^*)h = \eta^*(\xi \otimes h)$. Since $\vee\{\eta(H)| \ \eta \in \mathbb{D}(E^\sigma)\} = E \otimes H$ ([31, Lemma 3.5]), we find that $\xi \otimes h = 0$ for all $h \in H$. Since E is faithful, this implies that $\xi = 0$, completing the proof of (i).

We can also write $0 = X((t\eta)^*) = \mathbb{E}_0(X)(\eta^*) + t\mathbb{E}_1(X)(\eta^*) + \cdots + t^k\mathbb{E}_k(X)(\eta^*) + t^{k+1}S$ and conclude that $\mathbb{E}_j(X)(\eta^*) = 0$ for all $j \leq k$. We can then continue as above but to be able to conclude that $\mathbb{E}_k(X) = 0$ we need the condition in part (ii) (to replace the use of [31, Lemma 3.5] in the argument above).

To prove (iii), note that the invariance of $K(\sigma)$ under an automorphism α as in (iii) follows from (22). The invariance under the gauge group (and under $\mathbb{E}_k$) is then immediate. $\square$

The following proposition is obvious if $K(\sigma) = \{0\}$. But, in fact, it holds for every faithful, normal representation σ. The argument uses an idea from [17, Proof of Theorem 4.11].

Proposition 4.18 *Let σ be a faithful, normal representation of M and let α, β be two homomorphisms of $H^\infty(E)$ into itself such that β is completely isometric, surjective and a w^*-homeomorphism, while α is completely contractive and w^*-continuous. Suppose they satisfy the equation*

$$\widehat{\alpha(X)}(\eta^*) = \widehat{\beta(X)}(\eta^*)$$

for all $X \in H^\infty(E)$ and $\eta \in \mathbb{D}(E^\sigma)$. Then $\alpha = \beta$.

Proof. It is clearly enough to assume $\beta = id$ and $\widehat{\alpha(X)}(\eta^*) = \widehat{X}(\eta^*)$. Note that α, viewed as a representation of $H^\infty(E)$ on $\mathcal{F}(E) \otimes_\sigma H$ (whose restriction to $\varphi_\infty(M)$ is $\varphi_\infty(\cdot) \otimes I_H$), can be written as $(\varphi_\infty(\cdot) \otimes I_H) \times \zeta^*$ for some ζ in the closed unit ball of the $\varphi_\infty(\cdot) \otimes I_H$-dual of E. Thus, for $k \in \mathcal{F}(E) \otimes_\sigma H$, $\alpha(T_\xi)k = (\zeta^*)(\xi \otimes k)$ and $\|\alpha(T_\xi)k\| \leq \|\xi \otimes k\| = \|T_\xi k\|$.

Fix $h \in H$ viewed as the zeroth summand of $\mathcal{F}(E) \otimes_\sigma H$. Then for every $\xi \in E$,

$$\|\alpha(T_\xi)h\| \leq \|T_\xi h\|.$$

By construction $\alpha(T_\xi) - T_\xi \in K(\sigma)$. But also, by Lemma 4.17(i), for every $X \in K(\sigma)$, Xh is orthogonal to $T_\xi h$. Thus

$$\|\alpha(T_\xi)h\|^2 = \|(\alpha(T_\xi) - T_\xi)h\|^2 + \|T_\xi h\|^2 \geq \|T_\xi h\|^2.$$

We conclude that for every $h \in H$, $(\alpha(T_\xi) - T_\xi)h = 0$. It follows that $\alpha(T_\xi) = T_\xi$ for all $\xi \in E$. Since α is a w^*-continuous homomorphism, $\alpha(X) = X$ for all $X \in H^\infty(E)$. $\square$

The following lemma will prove very useful when we deal with a representation σ for which $K(\sigma) \neq \{0\}$. It relates the σ-dual with the π-dual where π is the representation defined in the proof of Lemma 3.8 (for which $K(\pi) = \{0\}$).

Lemma 4.19 *Let σ be a faithful representation of M on H and π be the representation $\varphi_\infty \otimes I_H$ of M on $K := \mathcal{F}(E) \otimes H$. Let $\psi : \sigma(M)' \to (\varphi_\infty(M) \otimes I_H)'$ be defined by $\psi(b) = I_E \otimes b$ and let $\Psi : E^\sigma \to E^\pi$ be defined by $\Psi(\eta) = I_{\mathcal{F}(E)} \otimes \eta$. Then we have the following.*

1. *The pair (ψ, Ψ) is an isomorphism of E^σ into (not necessarily onto) E^π satisfying*
$$\Psi(\eta)P_H = P_{E\otimes H}\Psi(\eta) = \eta \ , \ \eta \in E^\sigma$$
where P_H is the projection from K to H (viewed as a subspace) and $P_{E\otimes H}$ is the projection of $E \otimes K$ onto $E \otimes H$.

2. *For every $X \in H^\infty(E)$ and $\zeta \in E^\pi$ that satisfies $\zeta P_H = P_{E\otimes H}\zeta$, we have $\zeta|H \in E^\sigma$ and the restriction of $\widehat{X}(\zeta^*)$ to H (viewed as a summand of $\mathcal{F}(E) \otimes H = H \oplus E \otimes H \oplus \cdots$) is $\widehat{X}((\zeta|H)^*)$.*

3. *There is an isomorphism Φ of $\mathfrak{Z}(E^\sigma)$ onto $\mathfrak{Z}(E^\pi)$ satisfying*
$$\Phi(\gamma)P_H = P_{E\otimes H}\Phi(\gamma) = \gamma \ , \ \gamma \in \mathfrak{Z}(E^\sigma).$$

4. *For $\eta \in E^\sigma$ and $\gamma \in \mathfrak{Z}(E^\sigma)$,*
$$g_{\Phi(\gamma)}(\Psi(\eta)^*)P_{E\otimes H} = P_H g_{\Phi(\gamma)}(\Psi(\eta)^*) = g_\gamma(\eta^*).$$

Proof. It is clear that ψ is indeed an isomorphism into $(\varphi_\infty(M) \otimes I_H)'$. Note that it follows from the intertwining property of $\eta \in E^\sigma$ that $\Psi(\eta)$ is a well defined bounded operator. To show that Ψ maps E^σ into E^π, fix $\eta \in E^\sigma$, $\theta \otimes h \in \mathcal{F}(E) \otimes H$ and $a \in M$ and compute $(I_{\mathcal{F}(E)} \otimes \eta)\pi(a)(\theta \otimes h) = (I_{\mathcal{F}(E)} \otimes \eta)(\varphi_\infty(a)\theta \otimes h) = \varphi_\infty(a)\theta \otimes \eta(h)$, where we view $\mathcal{F}(E) \otimes E$ as the subspace of $\mathcal{F}(E)$ consisting of all the positive tensor powers of E. But the last expression is equal to $(\varphi_\infty(a) \otimes I_H)(I_{\mathcal{F}(E)} \otimes \eta)(\theta \otimes h)$, showing that $\Psi(\eta) \in E^\pi$.
To show that the map is a bimodule map, fix $\eta \in E^\sigma$, $b, c \in \sigma(M)'$ and $\theta \otimes h \in \mathcal{F}(E) \otimes H$. Then $\Psi(c\eta b)(\theta \otimes h) = \theta \otimes (c\eta b)h = \theta \otimes (I_E \otimes c)\eta bh = \psi(c)(\theta \otimes \eta bh) = \psi(c)\Psi(\eta)(\theta \otimes bh) = \psi(c)\Psi(\eta)\psi(b)(\theta \otimes h)$, proving that the image of Ψ lies in E^π. Regarding the inner product, we have: $\langle \Psi(\eta_1), \Psi(\eta_2)\rangle = \Psi(\eta_1)^*\Psi(\eta_2) = (I_{\mathcal{F}(E)} \otimes \eta_1)^*(I_{\mathcal{F}(E)} \otimes \eta_2) = (I_{\mathcal{F}(E)} \otimes \eta_1^*\eta_2) = \psi(\langle \eta_1, \eta_2\rangle)$ for all $\eta_1, \eta_2 \in E^\sigma$. Thus (ψ, Ψ) is an isomorphism of E^σ into E^π. The proof of the equation $\Psi(\eta)P_H = P_{E\otimes H}\Psi(\eta) = \eta$ for $\eta \in E^\sigma$ is easy. This proves (1).
To prove (2), let $\zeta \in E^\pi$ satisfy $\zeta P_H = P_{E\otimes H}\zeta$ and fix $a \in M$ and $h \in H$. Then $(\zeta|H)\sigma(a)h = \zeta(\varphi_\infty(a) \otimes I_H)h = (\varphi_E(a) \otimes I_K)P_{E\otimes H}\zeta h = (\varphi_E(a) \otimes I_H)(\zeta|H)h$. Thus, $\zeta|H \in E^\sigma$. To prove that $\widehat{X}((\zeta|H)^*) = \widehat{X}(\zeta^*)|H$, let, first, consider $X = \varphi_\infty(a)$ for $a \in M$. Then $\widehat{X}(\zeta^*) = \varphi_\infty(a) \otimes I_H$ and $\widehat{X}((\eta|H)^*) = \sigma(a)$ and (2)

holds in this case. Take $X = T_\xi$ for some $\xi \in E$. Then, for $h \in H \subseteq \mathcal{F}(E) \otimes H$, $\widehat{X}(\zeta^*)h = \zeta^*(\xi \otimes h) = (\zeta|H)^*(\xi \otimes h) = \widehat{X}((\zeta|H)^*)h$. In particular, we see that H is invariant for all $\widehat{X}(\zeta^*)$ where X runs over a set of generators. Thus, H is invariant under $\widehat{X}(\zeta^*)$ for all $X \in H^\infty(E)$ and (2) holds for all X's in a w^*-dense subalgebra of $H^\infty(E)$. Since the map $X \mapsto \widehat{X}(\zeta^*)$ is w^*-continuous, we are done.

To prove (3), recall from Lemma 4.12 (2) that both $\mathfrak{Z}(E^\sigma)$ and $\mathfrak{Z}(E^\pi)$ are isomorphic to $\mathfrak{Z}(E)$. Combining these two isomorphisms, we get Φ. More precisely, every $\eta \in \mathfrak{Z}(E^\sigma)$ is equal to L_ξ for some $\xi \in \mathfrak{Z}(E)$ (that is, $\eta(h) = \xi \otimes h$, $h \in H$). Then we set $\Phi(\eta)k = \xi \otimes k$ for $k \in K = \mathcal{F}(E) \otimes H$. The equation $\Phi(\gamma)P_H = P_{E\otimes H}\Phi(\gamma) = \gamma$, $\gamma \in \mathfrak{Z}(E^\sigma)$ follows easily.

Part (4) follows from (1) and (3). $\square$

Fix $X \in H^\infty(E)$ with $\|X\| \leq 1$, let $\pi = \varphi_\infty \otimes I_H$, as in Lemma 3.8, and let γ be an element of $\mathbb{D}(\mathfrak{Z}(E^\pi))$. Then if $\widehat{X}$ is the Schur class operator function on $\mathbb{D}((E^\pi)^*)$ determined by X then by Corollary 4.16, $\widehat{X} \circ g_\gamma$ also is a Schur class operator function on $\mathbb{D}((E^\pi)^*)$. By Corollary 3.9 there is an element $\alpha_\gamma(X)$ in $H^\infty(E)$, whose norm does not exceed 1, such that $\widehat{\alpha_\gamma(X)} = \widehat{X} \circ g_\gamma$. Further, by Lemma 3.8, this element is uniquely defined. We can, of course, extend this to a map, α_γ, from $H^\infty(E)$ to itself such that, for $X \in H^\infty(E)$ and $\eta \in \mathbb{D}((E^\pi)^*)$,

$$\widehat{\alpha_\gamma(X)}(\eta^*) = \widehat{X}(g_\gamma(\eta^*)). \tag{27}$$

Lemma 4.20 *Let σ and π be as in Lemma 4.19. Then:*

(i) For every $\gamma \in \mathbb{D}(\mathfrak{Z}(E^\pi))$, α_γ, defined by equation (27) is an automorphism of the algebra $H^\infty(E)$ that is completely isometric and is a homeomorphism with respect to the ultraweak topology.

(ii) For every $\gamma \in \mathbb{D}(\mathfrak{Z}(E^\sigma))$ let α_γ be defined to be $\alpha_{\Phi(\gamma)}$ (with Φ as in Lemma 4.19). Then, for every $X \in H^\infty(E)$ and $\eta \in E^\sigma$,

$$\widehat{\alpha_\gamma(X)}(\eta^*) = \widehat{X}(g_\gamma(\eta^*)). \tag{28}$$

Proof. We first prove (i). Linearity and multiplicativity of α_γ are easy to check. Since $g_\gamma^2 = id$, α_γ is invertible (with $\alpha_\gamma^{-1} = \alpha_\gamma$). So it is an automorphism. Since α_γ maps the closed unit ball of $H^\infty(E)$ into itself (as does the inverse map), α_γ is isometric. It is, in fact, completely isometric. To see this, consider, for $n \in \mathbb{N}$, the algebra $H^\infty(M_n(E))$, associated with the W^*-correspondence $M_n(E)$ over the von Neumann algebra $M_n(M)$. The corresponding Fock space is $M_n(\mathcal{F}(E))$ and the algebra can be identified with $M_n(H^\infty(E))$. The representation σ of M gives rise to a representation σ_n of $M_n(M)$ on $H^{(n)} = \mathbb{C}^n \otimes H$ (with $\sigma_n(M_n(M))' = I_{\mathbb{C}^n} \otimes \sigma(M)' \cong \sigma(M)'$). One can check that $E^\sigma \cong (M_n(E))^{\sigma_n}$. For $\gamma \in \mathfrak{Z}(E^\sigma)$, write γ' for the corresponding element of $\mathfrak{Z}(M_n(E^\sigma))$. Then $\alpha_{\gamma'}$ acts on $M_n(H^\infty(E))$ by applying α_γ to each entry. Since we know that $\alpha_{\gamma'}$ is an isometry, it follows that α_γ is a complete isometry.

It is left to show that α_γ is continuous with respect to the ultraweak topology. For this, let $\{X_\beta\}$ be a net in the closed unit ball of $H^\infty(E)$ that converges ultraweakly to X. Since evaluating at η^* (for η in the open unit ball) amounts to applying a ultraweakly continuous representation , we have, for every such η, $\widehat{X_\beta}(\eta^*) \to \widehat{X}(\eta^*)$ in the weak operator topology. Since this holds for $g_\gamma(\eta^*)$ in place of η, we see that, for every η in the open unit ball of E^σ,

$$\widehat{\alpha_\gamma(X_\beta)}(\eta^*) \to \widehat{\alpha_\gamma(X)}(\eta^*).$$

Using Lemma 3.8, we find that $\alpha_\gamma(X_\beta) \to \alpha_\gamma(X)$ in the ultraweak topology. This proves (i).

Part (ii) of the lemma results from the following computation

$$\widehat{\alpha_\gamma(X)}(\eta^*) = \widehat{\alpha_{\Phi(\gamma)}(X)}(\Psi(\eta)^*)|H = \widehat{X}(g_{\Phi(\gamma)}(\Psi(\eta)^*))|H$$

$$= \widehat{X}(g_{\Phi(\gamma)}(\Psi(\eta)^*)|E \otimes H) = \widehat{X}(g_\gamma(\eta)^*),$$

where we used equation (27) and Lemma 4.19. □

Note that we needed to use the representation π in order to define, for every $X \in H^\infty(E)$, the element $\alpha_\gamma(X)$ in $H^\infty(E)$ satisfying (27). That is, we used the fact that $K(\pi) = 0$. Once we defined it, it may be more convenient to work with the original representation σ (which can be chosen to be an arbitrary faithful representation) and invoke (28). Note that, using Proposition 4.18, we see that there is only one automorphism that satisfies (28).

Theorem 4.21 *Let E be a W^*-correspondence over M and let σ be a faithful normal representation of M on a Hilbert space H. Let α be an isometric automorphism of $H^\infty(E)$ and assume that $g : \mathbb{D}(E^\sigma)^* \to \mathbb{D}(E^\sigma)^*$ is a biholomorphic automorphism of $\mathbb{D}(E^\sigma)^*$ such that*

$$\widehat{\alpha(X)}(\eta^*) = \widehat{X}(g(\eta^*)),$$

for all $X \in H^\infty(E)$ and all $\eta \in E^\sigma$. Then:

(i) $g(\mathbb{D}\mathfrak{Z}((E^\sigma)^*)) \subseteq \mathbb{D}\mathfrak{Z}((E^\sigma)^*)$.

(ii) There is a $\gamma \in \mathbb{D}\mathfrak{Z}((E^\sigma))$ and a unitary operator u in $\mathcal{L}(E)$ such that $u(\mathfrak{Z}(E)) = \mathfrak{Z}(E)$ and such that

$$g(\eta^*) = g_\gamma(\eta^*) \circ (u \otimes I_{\mathcal{E}})$$

(as a map from $E \otimes_\sigma H$ to H).

(iii) With u as in (ii), there is an automorphism α_u of $H^\infty(E)$ such that $\alpha_u(T_\xi) = T_{u\xi}$ for every $\xi \in E$.

(iv) *With u and γ as in (ii),*

$$\alpha = \alpha_\gamma \circ \alpha_u$$

where α_γ is the automorphism defined in equation (27) (and satisfies (28)).

(v) *For every $\eta_1, \eta_2, \ldots, \eta_k$ in the open unit ball of E^σ, the map defined by the $k \times k$ matrix*

$$((id - \theta_{g(\eta_i^*)^*, g(\eta_j^*)^*}) \circ (id - \theta_{\eta_i,\eta_j})^{-1})$$

is completely positive.

PROOF. Note first that, since α is an isometric automorphism, it maps $\varphi_\infty(M)$ onto itself.

Suppose η lies in $\mathbb{D}(\mathfrak{Z}(E^\sigma)^*)$. Then, by part (3) of Lemma 4.12, $\widehat{X}(\eta^*) \in \sigma(M)$ for every $X \in H^\infty(E)$. But then, for every X, $\widehat{X}(g(\eta^*))$ lies in $\sigma(M)$, showing that $g(\eta^*) \in \mathfrak{Z}(E^\sigma)$. This proves (i).

The discussion following Lemma 4.12 shows that we can write $g = w \circ g_\gamma$ for some γ in $\mathbb{D}\mathfrak{Z}((E^\sigma))$ and an isometry w on $(E^\sigma)^*$ that preserves the center. Let α_γ be the automorphism described in Lemma 4.20(ii) and write $\beta = \alpha_\gamma^{-1} \circ \alpha$. Then it follows that

$$\widehat{\beta(X)}(\eta^*) = \widehat{X}(w\eta^*)$$

for $X \in H^\infty(E)$ and $\eta \in \mathbb{D}(E^\sigma)$.

For $\eta = 0$ and $Y \in H^\infty(E)$ we have $\widehat{Y}(0) = \sigma(\mathbb{E}_0(Y))$ where $\mathbb{E}_0$ is the conditional expectation of $H^\infty(E)$ onto M (where M is viewed as the "zeroth term"). Thus, $\sigma(\mathbb{E}_0(\beta(X))) = \widehat{\beta(X)}(0) = \widehat{X}(0) = \sigma(\mathbb{E}_0(X))$ for every $X \in H^\infty(E)$. Since σ is faithful, $\mathbb{E}_0(\beta(X)) = \mathbb{E}_0(X)$. Thus, for every $\xi \in E$, $\mathbb{E}_0(\beta(T_\xi)) = 0$ and we can write

$$\beta(T_\xi) = T_\theta + Y \tag{29}$$

where Y lies in $(T_E)^2 H^\infty(E)$. Write C for $(T_E)^2 H^\infty(E)$. Since (29) holds for all $\xi \in E$, $\beta(C) \subseteq C$. We can apply the same arguments to β^{-1}, in place of β, and find that $\beta^{-1}(C) \subseteq C$. Applying β^{-1} to (29), we find that

$$\beta^{-1}(T_\theta) = T_\xi + Z \tag{30}$$

for some $Z \in C$.

Arguing as in the proof of Proposition 4.18, we find that, for every $h \in H$, $\|\beta(T_\xi)h\| \leq \|T_\xi h\|$ and $\|\beta(T_\xi)\|^2 = \|Yh\|^2 + \|T_\theta h\|^2 \geq \|T_\theta h\|^2$. Thus $\|T_\xi h\| \geq \|T_\theta h\|$. Applying the same arguments to β^{-1} (using (30) in place of (29)) we find that $\|T_\theta h\| \geq \|T_\xi h\|$ and, thus, $\|T_\xi h\| = \|T_\theta h\|$ and, consequently, $Yh = 0$ for all $h \in H$. Thus $Y = 0$ and $\beta(T_\xi) = T_\theta$. Since β is isometric, $\|T_\xi\| = \|T_\theta\|$. It follows that $\|\xi\| = \|\theta\|$. If we write $\theta = u\xi$ (and recall that then $\beta(T_\xi) = T_{u\xi}$) then u is a linear isometry. We also have, for $a \in M$, $T_{u(\xi a)} = \beta(T_{\xi a}) = \beta(T_\xi a) = \beta(T_\xi)a = T_{u(\xi)}a = T_{u(\xi)a}$. Hence u is an isometric (right) module map and, therefore, u lies in $\mathcal{L}(E)$. Since β is an automorphism, u is a unitary operator. We also have $\beta(T_\xi) = T_{u\xi}$, so $\beta = \alpha_u$ (in the notation of (iii)). This proves (iii) and (iv).

Recall that $\widehat{\beta(X)}(\eta^*) = \widehat{X}(w\eta^*)$ and set $X = T_\xi$ to get $\widehat{T_{u\xi}}(\eta^*) = \widehat{\beta(T_\xi)}(\eta^*) = \widehat{T_\xi}(w\eta^*)$. Hence $\eta^* L_{u\xi} = (w\eta^*)L_\xi$. Applying this to $h \in \mathcal{E}$ we get $\eta^*(u\xi \otimes h) = (w\eta^*)(\xi \otimes h)$. Hence $w\eta^* = \eta^* \circ (u \otimes I)$, proving $g(\eta^*) = g_\gamma(\eta^*) \circ (u \otimes I_\mathcal{E})$.

To prove (ii) we need only to show that u preserves the center of E. So fix $\xi \in \mathfrak{Z}(E)$. By Lemma 4.12, L_ξ^* lies in the center of $(E^\sigma)^*$. Thus wL_ξ^* lies in $\mathfrak{Z}((E^\sigma)^*)$. But $wL_\xi^* = L_\xi^* \circ (u \otimes I) = L_{u^*\xi}$. Thus $L_{u^*\xi}$ lies in $\mathfrak{Z}((E^\tau)^*)$. Using Lemma 4.12 again we get $u^*\xi \in \mathfrak{Z}(E)$. This shows that $u^*\mathfrak{Z}(E) \subseteq \mathfrak{Z}(E)$ and, applying the same argument to β^{-1}, we complete the proof of (ii).

To prove (v), fix $b \in \sigma(M)'$ and η_i, η_j in $\mathbb{D}(E^\sigma)$ and compute $\langle g(\eta_i^*), b \cdot g(\eta_j^*)\rangle = g(\eta_i^*)(I_E \otimes b)g(\eta_j^*)^* = g_\gamma(\eta_i^*)(u \otimes I_\mathcal{E})(I_E \otimes b)(u^* \otimes I_\mathcal{E})g_\gamma(\eta_j)^* = g_\gamma(\eta_i^*)(I_E \otimes b)g_\gamma(\eta_j^*)^* = \langle g_\gamma(\eta_i^*), b \cdot g_\gamma(\eta_j^*)\rangle$. Thus (v) follows from Corollary 4.15. $\square$

Combining Theorem 4.21 with Theorem 4.9, we get the following.

THEOREM 4.22 *Let E be a faithful W^*-correspondence over M where $\mathfrak{Z}(M)$ is atomic. Let α be an automorphism of $H^\infty(E)$ that is completely isometric and a w^*-homeomorphism and leaves $\varphi_\infty(M)$ elementwise fixed and let σ be a faithful representation of M.*

Then there is a $\gamma \in \mathbb{D}\mathfrak{Z}((E^\sigma))$ and a unitary operator u in $\mathcal{L}(E)$, satisfying $u(\mathfrak{Z}(E)) = \mathfrak{Z}(E)$, such that

$$\alpha = \alpha_\gamma \circ \alpha_u,$$

where α_γ is the automorphism defined in Lemma 4.20 and $\alpha_u(T_\xi) = T_{u\xi}$ for every $\xi \in E$.

In particular, if $\mathfrak{Z}(E) = \{0\}$, every such automorphism is α_u for some unitary operator $u \in \mathcal{L}(E)$.

Theorem 4.22 provides another perspective on the results from [26, 27]. The analytic crossed products discussed there are of the form $H^\infty(E)$, where E is the correspondence ${}_\alpha M$ associated with a von Neumann algebra M and an automorphism α that is properly outer. This means that $\mathfrak{Z}(E) = \{0\}$. Theorem 4.22 implies that all automorphisms of $H^\infty(E)$ are given by automorphisms of M.

5 EXAMPLES : GRAPH ALGEBRAS

In this section we consider some examples that come from directed graphs. We shall assume for simplicity that our graphs have finitely many vertices and edges. We write $\mathcal{Q}$ both for the graph and for its set of edges. The space of vertices will be denoted V. We shall write s and r for the source and range maps on $\mathcal{Q}$, mapping $\mathcal{Q}$ to V, and we shall think of an edge e in $\mathcal{Q}$ as "pointing" from $s(e)$ to $r(e)$. For simplicity, we shall also assume that r is surjective, i.e., we shall assume that $\mathcal{Q}$ is *without sources*. Write $\mathcal{Q}^*$ for the set of all finite paths

in $\mathcal{Q}$, i.e., the path category generated by $\mathcal{Q}$. An element in $\mathcal{Q}$ will be written $\alpha = e_1 e_2 \cdots e_k$, where $s(e_i) = r(e_{i+1})$. We set $s(\alpha) = s(e_k)$, $r(\alpha) = r(e_1)$, and $|\alpha| = k$, the length of α. We will also view vertex $v \in V$ as a "path of length 0", and we extend r and s to V simply by setting $r(v) = s(v) = v$.
Let M be $C(V)$, the set of complex-valued functions on V. Of course, M is a finite dimensional commutative von Neumann algebra. Likewise, we let E be $C(\mathcal{Q})$, the set of complex-valued functions on $\mathcal{Q}$. Then we define an M-bimodule structure on E as follows: for $f \in E$, $\psi \in M$ and $e \in \mathcal{Q}$,

$$(f\psi)(e) := f(e)\psi(s(e)),$$

and

$$(\psi f)(e) := \psi(r(e))f(e).$$

Note that the "no sources" assumption implies that the left action of M is faithful. An M-valued inner product on E will be given by the formula

$$\langle f, g\rangle(v) = \sum_{s(e)=v} \overline{f(e)}g(e),$$

for $f, g \in E$ and $v \in V$. With these operations, E becomes a W^*-correspondence over M. The algebra $H^\infty(E)$ in this case will be written $H^\infty(\mathcal{Q})$. In the literature, $H^\infty(\mathcal{Q})$ is sometimes denoted $\mathcal{L}_{\mathcal{Q}}$. It is the ultraweak closure of the tensor algebra $\mathcal{T}_+(E(\mathcal{Q}))$ acting on the Fock space of $\mathcal{F}(E(\mathcal{Q}))$. For $e \in \mathcal{Q}$, let δ_e be the δ-function at e, i.e., $\delta_e(e') = 1$ if $e = e'$ and is zero otherwise. Then T_{δ_e} is a partial isometry that we denote by S_e. Also, for $v \in V$, P_v is defined to be $\varphi_\infty(\delta_v)$. Then each P_v is a projection and it is an easy matter to see that the families $\{S_e : e \in \mathcal{Q}\}$ and $\{P_v : v \in V\}$ form a *Cuntz-Toeplitz family* in the sense that the following conditions are satisfied:

(i) $P_v P_u = 0$ if $u \neq v$,

(ii) $S_e^* S_f = 0$ if $e \neq f$

(iii) $S_e^* S_e = P_{s(e)}$ and

(iv) $\sum_{r(e)=v} S_e S_e^* \leq P_v$ for all $v \in V$.

In fact, these particular families yield a faithful representation of the Cuntz-Toeplitz algebra $\mathcal{T}(E(\mathcal{Q}))$ [19]. The algebra $\mathcal{T}_+(E(\mathcal{Q}))$ is the norm-closed (unstarred) algebra that they generate inside $\mathcal{T}(E(\mathcal{Q}))$ and $H^\infty(\mathcal{Q})$ is the ultraweak closure of $\mathcal{T}_+(E(\mathcal{Q}))$. The algebra $\mathcal{T}_+(E(\mathcal{Q}))$ was first defined and studied in [25], providing examples of the theory developed in [28]. It was called a quiver algebra there because in pure algebra, graphs of the form $\mathcal{Q}$ are called quivers. (Hence the notation we use here.) The properties of quiver algebras were further developed in [29]. In [23], the focus was on $H^\infty(\mathcal{Q})$ and the authors called this algebra a free semigroupoid algebras. Both algebras are often represented as algebras of operators on $l_2(\mathcal{Q}^*)$, and it will be helpful to understand how

from the perspective of this note. Let H_0 be a Hilbert space whose dimension equals the number of vertices, let $\{e_v|\ v \in V\}$ be a fixed orthonormal basis for H_0 and let σ_0 be the diagonal representation of $M = C(V)$ on H_0. Then $l_2(\mathcal{Q}^*)$ is isomorphic to $\mathcal{F}(E(\mathcal{Q}))\otimes_{\sigma_0} H_0$ where the isomorphism maps an element ξ_α of the standard orthonormal basis of $l_2(\mathcal{Q}^*)$ to $\delta_\alpha \otimes e_{s(e)}$ (where, for $\alpha = e_1 \cdots e_k$, $\delta_\alpha = \delta_{e_1} \otimes \cdots \otimes \delta_{e_k} \in E^{\otimes k}$). The partial isometries S_e can then be viewed as the shift operators $S_e\xi_\alpha = \xi_{e\alpha}$. Thus, the representations of $\mathcal{T}_+(E(\mathcal{Q}))$ and $H^\infty(\mathcal{Q})$ on $l_2(\mathcal{Q}^*)$ are just the representations induced by σ_0.
Quite generally, a completely contractive covariant representation of $E(\mathcal{Q})$ on a Hilbert space H is given by a representation σ of $M = C(V)$ on H and by a contractive map $\tilde{T} : E \otimes_\sigma H \to H$ satisfying equation (2). The representation σ is given by the projections $Q_v = \sigma(\delta_v)$ whose sum is I. Also, from $\tilde{T}$ we may define maps $T(e) \in B(H)$ by the equation $T(e)h = \tilde{T}(\delta_e \otimes h)$ and it is easy to check that $\tilde{T}\tilde{T}^* = \sum_e T(e)T(e)^*$ and $T(e) = Q_{r(e)}T(e)Q_{s(e)}$. Thus to every completely contractive representation of the quiver algebra $\mathcal{T}_+(E(\mathcal{Q}))$ we associate a family $\{T(e)|e \in \mathcal{Q}\}$ of maps on H that satisfy $\sum_e T(e)T(e)^* \leq I$ and $T(e) = Q_{r(e)}T(e)Q_{s(e)}$. Conversely, every such family defines a representation, written $\sigma \times T$ (or $\sigma \times \tilde{T}$), satisfying $(\sigma \times T)(S_e) = T(e)$ and $(\sigma \times T)(P_v) = Q_v$. We fix σ to be σ_0 and write H in place of H_0. So that, in this case, each projection Q_v is one dimensional (with range equal to $\mathbb{C}e_v$). Then obviously $\sigma(M)' = \sigma(M)$. To describe the σ-dual of E, write $\mathcal{Q}^{-1}$ for the directed graph obtained from $\mathcal{Q}$ by reversing all arrows, so that $s(e^{-1}) = r(e)$ and $r(e^{-1}) = s(e)$. Sometimes $\mathcal{Q}^{-1}$ is denoted $\mathcal{Q}^{op}$ and is called the opposite graph. Note that the Hilbert space $E\otimes_\sigma H_0$ is spanned by the orthonormal basis $\{\delta_e\otimes e_{s(\alpha)}\}$. Fix $\eta \in E^\sigma$ and note that its covariance property implies that, for every $e \in \mathcal{Q}$, $\eta^*(\delta_e \otimes e_{s(e)}) = \eta^*(\delta_{r(e)}\delta_e \otimes e_{s(e)}) = Q_{r(e)}\eta^*(\delta_e \otimes e_{s(e)}) = \overline{\eta(e^{-1})}e_{r(e)}$ for some $\overline{\eta(e^{-1})} \in \mathbb{C}$. The reason for the "strange" way of writing that scalar is that we can view η as an element of $E(\mathcal{Q}^{-1})$ and the correspondence structure on E^σ, as described in Proposition 2.13, fits the correspondence structure of $E(\mathcal{Q}^{-1})$. Consequently, we can identify the two and write

$$E^\sigma = E(\mathcal{Q}^{-1}).$$

(See Example 4.3 in [31] for a description of the structure of the dual correspondence for more general representations σ). It will also be convenient to write η matricially with respect to the orthonormal bases $\{\delta_v \mid v \in V\}$ of H_0 and $\{\delta_e \otimes e_{s(e)}\}_{e\in\mathcal{Q}}$ of $E \otimes H_0$ as

$$(\eta)_{e,r(e)} = \eta(e^{-1}). \tag{31}$$

Suppose $\eta \in \mathbb{D}(E^\sigma)$. For every $X \in H^\infty(\mathcal{Q})$, we have defined $X(\eta^*)$ as an element of $B(H)$ in Remark 2.14. For the generators of $H^\infty(\mathcal{Q})$, the definition yields the equations,

$$\widehat{P_v}(\eta^*) = \theta_{v,v}\ ,\ v \in V \tag{32}$$

and

$$\widehat{S_e}(\eta^*) = \overline{\eta(e^{-1})}\theta_{r(e),s(e)}\ ,\ e \in \mathcal{Q} \tag{33}$$

where $\theta_{v,w}$ is the partial isometry operator on H that maps e_w to e_v and vanishes on $(e_w)^{\perp}$. For a general $X \in H^{\infty}(\mathcal{Q})$, $\widehat{X}(\eta^*)$ is obtained by using the linearity, multiplicativity and w^*-continuity of the map $X \mapsto \widehat{X}(\eta^*)$.
The proof of the next lemma is straightforward and is omitted.

Lemma 5.1 *The centers of the correspondences $E(\mathcal{Q})$ and $E(\mathcal{Q}^{-1})$ are given by the formulae*

$$\mathfrak{Z}(E(\mathcal{Q})) = span\{\delta_e \mid s(e) = r(e)\}$$

and

$$\mathfrak{Z}(E(\mathcal{Q}^{-1})) = span\{\delta_{e^{-1}} \mid s(e) = r(e)\}.$$

The following proposition is immediate from Theorem 4.22.

Proposition 5.2 *If there is no $e \in \mathcal{Q}$ with $s(e) = r(e)$, then every automorphism α of $H^{\infty}(\mathcal{Q})$ that is completely isometric, w^*-homeomorphic and leaves $\varphi_{\infty}(C(V))$ elementwise fixed (that is, does not permute the vertices) is of the form α_u for some unitary $u \in \mathcal{L}(E(\mathcal{Q}))$. That is,*

$$\alpha(S_e) = \sum_{s(f)=s(e)} u_{f,e} S_f$$

where the scalars $u_{f,e}$ are given by $u_{f,e} = (u(\delta_e))(f)$. (Note that this is zero if $s(f) \neq s(e)$, since $u(\delta_e) = u(\delta_e \delta_{s(e)}) = u(\delta_e)\delta_{s(e)}$).

We note, as we did at the beginning of Section 4, that the assumptions made on the automorphism can be weakened using arguments of [22] but we shall not elaborate on this here.

Example 5.3 *Let $\mathcal{Q}$ be an n-cycle (for $n > 1$) ; that is $V = \{v_1, v_2, \ldots, v_n\}$ and $\mathcal{Q} = \{e_1, \ldots, e_n\}$ where e_i is the arrow from v_1 to v_{i+1} (or to v_1 when $i = n$). Then, for every α as in Proposition 5.2, there are $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ with $|\lambda_i| = 1$, such that $\alpha(S_{e_i}) = \lambda_i S_{e_i}$ for all i.*

The rest of this section will be devoted to the study of the following example, which is very simple, yet provides a full array of the structures we have been studying.

Example 5.4 *Let the vertex set of the graph have two elements: $V = \{v, w\}$. Suppose the edge set consists of three elements $\mathcal{Q} = \{e, f, g\}$, where e is the arrow from v to w, so $s(e) = v$, $r(e) = w$; f is an arrow from w to v; and g is a loop based at w, $s(g) = r(g) = w$.*

Then by Lemma 5.1, $\mathfrak{Z}(E(\mathcal{Q})) = \mathbb{C}\delta_g$. We know from Theorem 4.22 that every automorphism α is the composition of an automorphism, written α_u associated with a unitary in $\mathcal{L}(E(\mathcal{Q}))$ that maps δ_g into $\lambda_3\delta_g$ (with $|\lambda_3| = 1$) and an automorphism associated with a “Möbius transformation".

As noted in Proposition 5.2, $(u(\delta_{e'}))(f') = 0$ unless $s(e') = s(f')$, so that $u(\delta_e) \in \mathbb{C}\delta_e$ and $u(\delta_f) \in span\{\delta_f, \delta_g\}$. Since u^* is unitary, we have that $u(\delta_f) = \lambda_f \delta_f$. Thus

$$\alpha_u(S_e) = \lambda_e S_e, \quad \alpha_u(S_f) = \lambda_f S_f \tag{34}$$

and

$$\alpha_u(S_g) = \lambda_g S_g$$

for $\lambda_e, \lambda_f, \lambda_g$ with absolute value 1.

It is left to analyze the Möbius transformations and the corresponding automorphisms. Since the center of E^σ are scalar multiples of $\delta_{g^{-1}}$, the Möbius transformations are associated with scalars $\lambda \in \mathbb{D}$ (in fact, with $\lambda\delta_{g^{-1}}$) and will be denoted τ_λ, $\lambda \in \mathbb{D}$. We have

$$\tau_\lambda(\eta^*) = \Delta_\lambda (I - \eta^*(\lambda\delta_{g^{-1}}))^{-1} (\bar{\lambda}\delta_{g^{-1}} - \eta^*) \Delta_{\lambda *}^{-1} \tag{35}$$

where $\Delta_\lambda = (I_H - (\lambda\delta_{g^{-1}})^*(\lambda\delta_{g^{-1}}))^{1/2}$ and $\Delta_{\lambda *} = (I_{E\otimes H} - (\lambda\delta_{g^{-1}})(\lambda\delta_{g^{-1}})^*)^{1/2}$. It will be convenient to write $\tau_\lambda(\eta^*)$ matricially as a map from $E \otimes H$, with the ordered orthonormal basis $\{\delta_e \otimes \delta_v, \delta_f \otimes \delta_w, \delta_g \otimes \delta_w\}$, to H, with the ordered orthonormal basis $\{\delta_v, \delta_w\}$. Using the formula (31), we see that

$$\eta = \begin{pmatrix} 0 & \eta(e^{-1}) \\ \eta(f^{-1}) & 0 \\ 0 & \eta(g^{-1}) \end{pmatrix}$$

and

$$\lambda\delta_{g^{-1}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \lambda \end{pmatrix}.$$

The computation of the expression in (35) yields

$$\tau_\lambda(\eta^*) = \begin{pmatrix} 0 & -\overline{\eta(f^{-1})} & 0 \\ \frac{-\overline{\eta(e^{-1})}(1-|\lambda|^2)^{1/2}}{1-\lambda\overline{\eta(g^{-1})}} & 0 & \frac{\bar{\lambda}-\overline{\eta(g^{-1})}}{1-\lambda\overline{\eta(g^{-1})}} \end{pmatrix}.$$

Thus

$$\overline{\tau_\lambda(\eta^*)^*(e^{-1})} = \frac{-\overline{\eta(e^{-1})}(1-|\lambda|^2)^{1/2}}{1-\lambda\overline{\eta(g^{-1})}} = -\overline{\eta(e^{-1})}(1-|\lambda|^2)^{1/2} \sum_{k=0}^{\infty} (\lambda\overline{\eta(g^{-1})})^k,$$

$$\overline{\tau_\lambda(\eta^*)^*(f^{-1})} = -\overline{\eta(f^{-1})},$$

and

$$\overline{\tau_\lambda(\eta^*)^*(g^{-1})} = \frac{\bar{\lambda} - \overline{\eta(g^{-1})}}{1-\lambda\overline{\eta(g^{-1})}} = (\bar{\lambda} - \overline{\eta(g^{-1})}) \sum_{k=0}^{\infty} (\lambda\overline{\eta(g^{-1})})^k.$$

This suggests setting

$$T(e) = -(1-|\lambda|^2)^{1/2}\sum_{k=0}^{\infty}(\lambda S_g)^k S_e,$$

$$T(f) = -S_f$$

and

$$T(g) = -(\bar{\lambda}P_w - S_g)\sum_{k=0}^{\infty}(\lambda S_g)^k.$$

Using (32), (33) and the fact that the map $X \mapsto \widehat{X}(\eta^*)$ is a continuous homomorphism, we get

$$\widehat{T(e)}(\eta^*) = \overline{\tau_\lambda(\eta^*)^*(e^{-1})}\theta_{w,v}$$

,

$$\widehat{T(f)}(\eta^*) = \overline{\tau_\lambda(\eta^*)^*(f^{-1})}\theta_{v,w}$$

and

$$\widehat{T(g)}(\eta^*) = \overline{\tau_\lambda(\eta^*)^*(g^{-1})}\theta_{w,w}.$$

Using Theorem 4.9, Theorem 4.22, Equation (34) and Theorem 4.18, we conclude the following.

Theorem 5.5 *(1) For every $\lambda \in \mathbb{D}$, there is a unique automorphism α_λ of $H^\infty(\mathcal{Q})$ such that, for every $e' \in \{e, f, g\}$, $\alpha_\lambda(S_{e'}) - T(e') \in K(\sigma)$.*

(2) Every completely isometric, w^*-homeomorphic automorphism α of $H^\infty(\mathcal{Q})$ can be written

$$\alpha = \alpha_u \circ \alpha_\lambda$$

where $\lambda \in \mathbb{D}$ and $\alpha_u(S_{e'}) = \lambda_{e'}S_{e'}$ for every $e' \in \{e, f, g\}$ (where λ_e, λ_f and λ_g are complex numbers of absolute value 1).

Proof. The only thing that we need to clarify here is that, in part (2), we do not have to require that α fixes P_v and P_w. Indeed, assume that α satisfies $\alpha(P_v) = P_w$ and $\alpha(P_w) = P_v$. Then $\alpha(S_e) = P_v\alpha(S_e)P_w$ and, thus, $\mathbb{E}_0(\alpha(S_e)) = 0$ and $\mathbb{E}_1(\alpha(S_e)) \in \mathbb{C}S_f$. Similarly, we get $\mathbb{E}_0(\alpha(S_f)) = \mathbb{E}_1(\alpha(S_g)) = 0$, $\mathbb{E}_1(\alpha(S_f)) \in \mathbb{C}S_e$ and $\mathbb{E}_0(\alpha(S_g)) \in \mathbb{C}P_v$. Thus, S_g is not in the range of α, contradicting the surjectivity of α. $\square$

Finally, we note the following.

Proposition 5.6 *In this example, $K(\sigma)$ is the ideal generated by the commutator $[S_g, S_eS_f]$.*

Proof. Since we shall not use this result, we only sketch the idea of the proof. It follows from Lemma 4.17 that it suffices to analyze $\mathbb{E}_k(K(\sigma))$ for a given k. Since $K(\sigma)$ is an ideal, it suffices to consider $P_{v'}\mathbb{E}_k(K(\sigma))P_{v''}$ for fixed $v', v'' \in \{v, w\}$. Evaluating an element of $P_{v'}\mathbb{E}_k(K(\sigma))P_{v''}$ in η^* yields a polynomial in three the variables $z_1 = \overline{\eta(e^{-1})}, z_2 = \overline{\eta(f^{-1})}$ and $z_3 = \overline{\eta(f^{-1})}$. This polynomial is defined on a small enough neighborhood of 0 and, from the definition of $K(\sigma)$, it vanishes there. It follows that its coefficients are all 0. This shows that an element in $P_{v'}\mathbb{E}_k(K(\sigma))P_{v''}$ is a linear combination of sums of the form $\sum a_i S_{\alpha_i}$ (for some paths α_i) where $\sum a_i = 0$ and for every i, j, the paths α_i and α_j satisfy $s(\alpha_i) = s(\alpha_j) = v''$, $r(\alpha_i) = r(\alpha_j) = v'$ and both paths contain the same edges (with the same multiplicities) but in a different order. A moment's reflection shows that this can happen only if the two paths are identical except that, at some points, one path travels along g and then along ef while the other path "chooses" to travel first along ef and then along g. This shows that the element in $P_{v'}\mathbb{E}_k(K(\sigma))P_{v''}$ lies in the ideal generated by $[S_g, S_e S_f]$. $\square$

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On the 2-Typical De Rham-Witt Complex

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Abstract. In this paper we introduce the 2-typical de Rham-Witt complex for arbitrary commutative, unital rings and log-rings. We describe this complex for the rings $\mathbb{Z}$ and $\mathbb{Z}_{(2)}$, for the log-ring $(\mathbb{Z}_{(2)}, M)$ with the canonical log-structure, and we describe its behaviour under polynomial extensions. In an appendix we also describe the p-typical de Rham-Witt complex of $(\mathbb{Z}_{(p)}, M)$ for p odd.

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1. Introduction

The p-typical de Rham-Witt complex was introduced by Bloch, Deligne, and Illusie for $\mathbb{F}_p$-algebras (see [1], [6]). The definition was generalized by Hesselholt and Madsen to $\mathbb{Z}_{(p)}$-algebras, for p odd (see [4], [5], [3]). Hesselholt and Madsen's construction was motivated by the effort to understand TR, an object that appears in algebraic topology and is related to topological cyclic homology and to higher algebraic K-theory. More precisely, for a fixed prime p and a $\mathbb{Z}_{(p)}$-algebra A, one defines:

$$\mathrm{TR}_q^n(A;p) = \pi_q(T(A)^{C_{p^{n-1}}}),$$

where $T(A)$ is the topological Hochschild spectrum associated to A, and $C_r \subset S^1$ is the cyclic group of order r. As n and q vary these groups are related by certain operators F, V, R, d, ι which satisfy several relations. One notes that ι is induced by the multiplication with the element $\eta \in \pi_1^s S^0$ from stable homotopy. This element has order 2, so the operator ι is trivial if 2 is invertible. This is the case if A is a $\mathbb{Z}_{(p)}$-algebra with p odd, and this explains why the case $p = 2$ is different from p odd.

A first step in understanding TR is to understand the universal example of an object that has the same algebraic structure as TR. The algebraic structure of

TR is captured by the notion of a Witt complex, that we will give shortly. The fact that TR is a Witt complex was proved by Hesselholt in [3]. Before giving the definition we make precise what we mean by a pro-object and a strict map of pro-objects. We let $\mathbb{Z}$ be the category associated with the poset $(\mathbb{Z}, \geq)$; a pro-object in a category $\mathcal{C}$ is a covariant functor $X : \mathbb{Z} \to \mathcal{C}$, in other words a sequence of objects $\{X_n\}_{n\in\mathbb{Z}}$ and of morphisms $R : X_n \to X_{n-1}$. A strict map of pro-objects is a natural transformation of functors, that is a sequence of maps $f_n : X_n \to Y_n$ that commutes with the maps R.

DEFINITION 1.1. *A 2-typical Witt complex over a commutative ring A consists of:*

(i) *a graded-commutative pro-graded ring $\{E_n^*,\ R : E_n^* \to E_{n-1}^*\}_{n\in\mathbb{Z}}$, such that $E_n^* = 0$ for all $n \leq 0$. The index n is called the level.*

(ii) *a strict map of pro-rings $\lambda : W_\bullet(A) \to E_\bullet^0$ from the pro-ring of 2-typical Witt vectors of A.*

(iii) *a strict map of pro-graded rings*

$$F : E_\bullet^* \to E_{\bullet-1}^*,$$

such that $\lambda F = F\lambda$.

(iv) *a strict map of pro-graded $E_\bullet^*$-modules*

$$V : F_* E_\bullet^* \to E_{\bullet+1}^*$$

such that $\lambda V = V\lambda$ and $FV = 2$. The linearity of V means that $V(x)y = V(xF(y)), \forall x \in E_n^*, y \in E_{n+1}^*$.

(vi) *a strict map of pro-graded abelian groups $d : E_\bullet^* \to E_\bullet^{*+1}$, which is a derivation, in the sense that*

$$d(xy) = d(x)y + (-1)^{\deg(x)} x d(y)$$

The operator $\iota : E_\bullet^* \to E_\bullet^{*+1}$ is by definition multiplication by the element $\frac{d\lambda[-1]_n}{\lambda[-1]_n}$, where $[a]_n = (a, 0, \ldots, 0) \in W_n(A)$ is the multiplicative representative.

The operators F, V, d, and ι are required to satisfy the following relations:

$$FdV = d + \iota,$$

$$dd = d\iota = \iota d,$$

$$Fd\lambda([a]_n) = \lambda([a]_{n-1})d\lambda([a]_{n-1}]),\ \textit{for all } a \in A.$$

$$2\iota = 0$$

Visually, a Witt complex is a two dimensional array:

$$\begin{tikzcd}
 & \vdots \arrow[d, dotted, no head] & \vdots \arrow[d, dotted, no head] & \vdots \arrow[d, dotted, no head] & \vdots \arrow[d, dotted, no head] & \\
 & W_2(A) \arrow[r, "\lambda"] \arrow[d, shift right=2, "F"'] \arrow[d, "R"] & E_2^0 \arrow[r, shift left, "d"] \arrow[r, shift right, "\iota"'] \arrow[d, shift right=2, "F"'] \arrow[d, "R"] & E_2^1 \arrow[r, shift left, "d"] \arrow[r, shift right, "\iota"'] \arrow[d, shift right=2, "F"'] \arrow[d, "R"] & E_2^2 \arrow[r, dotted, no head] \arrow[d, shift right=2, "F"'] \arrow[d, "R"] & {} \\
A \arrow[r, "{[-]_1}"] \arrow[ur] \arrow[uur, "{[-]_n}"] & W_1(A) \arrow[r, "\lambda"] \arrow[u, bend right, "V"'] & E_1^0 \arrow[r, shift left, "d"] \arrow[r, shift right, "\iota"'] \arrow[u, bend right, "V"'] & E_1^1 \arrow[r, shift left, "d"] \arrow[r, shift right, "\iota"'] \arrow[u, bend right, "V"'] & E_1^2 \arrow[r, dotted, no head] \arrow[u, bend right, "V"'] & {}
\end{tikzcd}$$

A map of 2-typical Witt complexes is a map $f : E^*_\bullet \to E'^*_\bullet$ of pro-graded rings such that $\lambda' = f\lambda$, $fd = df$, $F'f = fF$, and $V'f = fV$.

The paper is organized as follows. In Section 2 we discuss Witt vectors, the de Rham complex, and Witt complexes in general. We also derive the identity that expresses the Teichmüller representative of an integer as a combination of a system of generators:

$$[a]_n = a[1]_n + \sum_{i=1}^{n-1} \frac{a^{2^i} - a^{2^{i-1}}}{2^i} V^i[1]_{n-i}.$$

In the third section we prove, using category theory, that the category of 2-typical Witt complexes over a given ring admits an initial object, and that is the de Rham-Witt complex of the ring. A similar result holds for the more general notion of a log-ring.

Section 4 contains several calculations. The first result of this section is the structure theorem of the de Rham-Witt complex of the ring of rational integers $\mathbb{Z}$. It states that in degree zero it is the pro-ring of Witt vectors of the integers, in degree one it is generated by the elements $dV^i(1)$, and in degrees above one it vanishes:

$$W_n\Omega^0_{\mathbb{Z}} \cong \bigoplus_{i=0}^{n-1} \mathbb{Z} \cdot V^i(1), \tag{1}$$

$$W_n\Omega^1_{\mathbb{Z}} \cong \bigoplus_{i=1}^{n-1} \mathbb{Z}/2^i\mathbb{Z} \cdot dV^i(1), \tag{2}$$

$$W_n\Omega^i_{\mathbb{Z}} = 0, \quad \text{for } i \geq 2. \tag{3}$$

The product rule and the action of the operators are given in Theorem 4.1 below. We note that, additively, the formula for the 2-typical de Rham-Witt complex is similar to the one for the p-typical de Rham-Witt complex for p odd. Differences appear in the product rule and the action of the operators d, F, and, of course, ι. In a remark at the end of the section we note that a very similar result holds for the de Rham-Witt complex of the ring $\mathbb{Z}_{(2)}$.

In Section 4 we also describe the behaviour of the de Rham-Witt complex under polynomial extensions. Again the result is similar to the one in the p-typical case, for p odd, which is found in Section 4.2 in [3]. The de Rham-Witt complex of the ring $A[X]$ consists of formal sums of four types of elements:

- Type 1: elements of the form $a[X]^j$, where $a \in W_n\Omega_A^q$,
- Type 2: elements of the form $b[X]^{k-1}d[X]$, where $b \in W_n\Omega_A^{q-1}$,
- Type 3: elements of the form $V^r(c[X]^l)$, where $r > 0$, $c \in W_{n-r}\Omega_A^q$, and l is odd,
- Type 4: elements of the form $dV^s(e[X]^m)$, where $s > 0$, $e \in W_{n-s}\Omega_A^{q-1}$, and m is odd.

The product rule and the action of the operators are given explicitely.
In the last part of the fourth section we define the notion of a Witt complex for log-rings and we compute the 2-typical de Rham-Witt complex of the log-ring $(\mathbb{Z}_{(2)}, M)$, where $M = \mathbb{Q}^* \cap \mathbb{Z}_{(2)} \hookrightarrow \mathbb{Z}_{(2)}$ is the canonical log-structure. The difference from the 2-typical de Rham-Witt complex of $\mathbb{Z}_{(2)}$ and of $(\mathbb{Z}_{(2)}, M)$ is the element $d\log[2]$:

$$W_n\Omega^0_{(\mathbb{Z}_{(2)},M)} \cong W_n\Omega^0_{\mathbb{Z}_{(2)}} \cong W_n(\mathbb{Z}_{(2)}), \tag{4}$$

$$W_n\Omega^1_{(\mathbb{Z}_{(2)},M)} \cong W_n\Omega^1_{\mathbb{Z}_{(2)}} \oplus \mathbb{Z}/2^n\mathbb{Z}d\log[2]_n, \tag{5}$$

$$W_n\Omega^i_{(\mathbb{Z}_{(2)},M)} = 0, \text{for all } i \geq 2. \tag{6}$$

An interesting formula in this context is:

$$V(d\log[2]_n) = 2d\log[2]_{n+1} + dV[1]_n - dV^2[1]_{n-1} + 4dV^3[1]_{n-2}.$$

The paper has two appendices. In the first one we describe the structure of the p-typical de Rham-Witt complex of the log ring $(\mathbb{Z}_{(p)}, M)$, with p odd. This result is very similar to the one for $p = 2$, the difference being in the product formulas and the action of the operator V. We note here that there are two distinct cases for p odd, namely $p = 3$ and $p \geq 5$. For example, the mentioned formula becomes:

$$V(d\log[p]_n) = \begin{cases} 3d\log[3]_{n+1} + dV[1]_n + 3dV[1]_{n-1}, & \text{if } p = 3, \\ p\,d\log[p]_{n+1} + dV[1]_n, & \text{if } p \geq 5. \end{cases}$$

In the second appendix, which is rather technical, we verify the associativity of the multiplication defined in Section 4, subsection 4.2.
In this paper all rings are associative, commutative, and unital. Graded rings are graded commutative, or anti-symmetric, meaning that, for every two elements x, y of degrees $|x|, |y|$, respectively, one has

$$xy = (-1)^{|x||y|}yx.$$

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2. Generalities: Witt vectors, the de Rham complex, and Witt complexes

In this section we recall the Witt vectors and the de Rham complex. The standard references for these are [12], [11], respectively. Then we derive some elementary results for Witt vectors and Witt complexes.

The de Rham complex of a ring A is the exterior algebra on the module of Kähler differentials over A. More precisely, if I is the kernel of the multiplication $A \otimes A \to A$, the module of Kahler differentials is defined to be $\Omega^1_A = I/I^2$; the map $d : A \to \Omega^1_A$ defined by $da = a \otimes 1 - 1 \otimes a + I^2$ is the universal derivation from A to an A-module. The de Rham-complex is the exterior algebra

$$\Omega^*_A = \Lambda^*_A \Omega^1_A$$

with differential

$$d(a_0 da_1 \dots da_n) = da_0 da_1 \dots da_n,$$

where the exterior algebra of an A-module M is

$$\Lambda^*(M) = T_A(M)/\langle m \otimes m \mid m \in M \rangle.$$

In this paper we will need a related construction, that of a universal anti-symmetric differential graded algebra over the ring A. By this we mean a graded algebra over A which is commutative in the graded sense and is endowed with a $\mathbb{Z}$-linear differential of degree 1, which is also a derivation. We will denote this by $\tilde{\Omega}^*_A$. Explicitly,

$$\tilde{\Omega}^*_A = \tilde{\Lambda}^*_A \Omega^1_A,$$

where:

$$\tilde{\Lambda}^*(M) = T(M)/\langle m \otimes n + n \otimes m \mid m, n \in M \rangle$$

is the universal anti-symmetric graded A-algebra generated by the A-module M. When 2 is invertible in A the two constructions give the same result as the ideals $\langle m \otimes m \mid m \in M \rangle$ and $\langle m \otimes n + n \otimes m \mid m, n \in M \rangle$ are the same. In this paper we cannot assume that 2 is invertible and this is why we need the second construction.

The ring $W_n(A)$ of Witt vectors of length n in A is the set of n-tuples in A with the following ring structure. One defines the "ghost" map

$$w : W_n(A) \to A^n$$

with components:

$$w_i(a_0, \dots, a_n) = a_0^{2^i} + 2a_1^{2^{i-1}} + \dots 2^i a_i.$$

To add or multiply two vectors a and b one maps them via w in A^n, adds or multiplies them componentwise, then uses w^{-1} to map them back in $W_n(A)$. Of course one has to check that the sum or product of $w(a)$ and $w(b)$ are in the image of the ghost map and that their preimage is unique. That they are in the image follows from a lemma of Dwork; the uniqueness of the preimage is true only when A has no 2-torsion, which will be the case for the rings

considered in this paper. When A has 2-torsion one has to give a canonical element in the preimage, and this is done requiring that the ghost map be a natural transformation of functors from rings to rings.
The projection onto the first $n-1$ factors is a ring homomorphism

$$R : W_n(A) \to W_{n-1}(A),$$

called *restriction*, and this makes $W_\bullet(A)$ a pro-ring. There is a second ring homomorphism, the *Frobenius*,

$$F : W_n(A) \to W_{n-1}(A),$$

such that F satisfies the following relation with respect to the "ghost" map:

$$w(F(a_0, \dots, a_{n-1})) = (w_1(a), \dots, w_{n-1}(a)),$$

and a $W_n(A)$-linear map, *Verschiebung*,

$$V : F_* W_{n-1}(A) \to W_n(A)$$
$$V(a_0, \dots, a_{n-2}) = (0, a_0, \dots, a_{n-2})$$

The notation $F_*W_{n-1}(A)$ indicates that $W_{n-1}(A)$ is considered a $W_n(A)$-module via the Frobenius map $F : W_n(A) \to W_{n-1}(A)$. The linearity of V means therefore that $xV(y) = V(F(x)y)$, for all $x \in W_n(A)$ and $y \in W_{n-1}(A)$, a formula known as Frobenius reciprocity. Both Frobenius and Verschiebung commute with the restriction maps. The *Teichmüller* map is the multiplicative map

$$[\ ]_n : A \to W_n(A),$$
$$[a]_n = (a, 0, \dots, 0).$$

We list now a few numerical results, some of which are not available in the odd prime case.

PROPOSITION 2.1. *In the ring of 2-typical Witt vectors of length n, $W_n(A)$,*

$$[-1]_n = -[1]_n + V([1]_{n-1}).$$

Proof. It is enough to prove this relation for $A = \mathbb{Z}$. In ghost coordinates,

$$w([-1]_n) = (-1, 1, \dots, 1),$$
$$w([1]_n) = (1, 1, \dots, 1),$$
$$w(V[1]_{n-1}) = (0, 2, \dots, 2).$$

The relation follows from the fact that addition is done component-wise in these coordinates and the ghost map is injective for $A = \mathbb{Z}$. □

PROPOSITION 2.2. *In the ring of 2-typical Witt vectors of length n, $W_n(\mathbb{Z})$, there are 4 square roots of unity, $[1]_n, [-1]_n, -[1]_n, -[-1]_n$.*

Proof. Let $a = (a_0, \dots, a_{n-1}) \in W_n(A)$ be a square root of unity. Let $(w_0, \dots, w_{n-1})$ be its ghost coordinates. Then $(w_0^2, \dots, w_{n-1}^2) = (1, \dots, 1)$. From $w_0^2 = 1$ we get $a_0^2 = 1$, hence $a_0 = \pm 1$. Equating the second ghost coordinate we obtain: $(a_0^2 + 2a_1)^2 = 1 \Rightarrow (1 + 2a_1)^2 = 1 \Rightarrow a_1 = 0$ or $a_1 = -1$. We will prove by induction that for $s \geq 1$, $a_s = a_1$. Assume this is true for $s-1$. We have two cases, $a_1 = 0$ and $a_1 = -1$.

(i) $a_1 = 0$: $w_s^2 = (a_0^{2^s} + \cdots + 2^{s-1}a_{s-1}^2 + 2^s a_s)^2 = 1$, so $(1+2^s a_s)^2 = 1$ and the unique integral solution is $a_s = 0$.
(ii) $a_1 = -1$: $w_s^2 = (a_0^{2^s} + \cdots + 2^{s-1}a_{s-1}^2 + 2^s a_s)^2 = 1 \Rightarrow (1+2+\cdots+2^{s-1} + 2^s a_s)^2 = 1 \Rightarrow (2^s - 1 + 2^s a_s)^2 = 1$ and the unique integral solution is $a_s = -1$.

Therefore the solutions of the equation $a^2 = 1$ are the vectors $(\pm 1, 0, \ldots, 0)$ and $(\pm 1, -1, \cdots, -1)$. An examination of these vectors shows that they are exactly those listed in the statement. □

Proposition 2.3. *In the ring of p-typical Witt vectors of length n, $W_n(\mathbb{Z})$, the vectors $\{[1]_n, V([1]_{n-1}), \ldots, V^{n-1}([1]_1)\}$ form a $\mathbb{Z}$-basis. A vector $a = (a_0, \ldots, a_{n-1}) \in W_n(\mathbb{Z})$ with ghost coordinates $(w_0, \ldots, w_{n-1})$ can be written in this basis as:*

$$a = \sum_{s=0}^{n-1} c_s V^s([1]_{n-s}),$$

where

$$c_s = \begin{cases} w_0 & \text{if } s = 0 \\ p^{-s}(w_s - w_{s-1}) & \text{if } 1 \leq s \leq n-1. \end{cases}$$

The multiplication in this basis is given by the rule:

$$V^i([1]_{n-i})V^j([1]_{n-j}) = p^i V^j([1]_{n-j}), \quad \text{if} \quad i \leq j.$$

Proof. In ghost coordinates, $V^s[1]_{n-s} = (0, \ldots, 0, p^s, \ldots, p^s)$, the first s coordinates being zero. Since the addition is component-wise it follows that these vectors are linearly independent. The multiplication is also component-wise and the product formula follows.

We show that they form a system of generators. For a vector $a = (a_0, \ldots, a_{n-1}) \in W_n(\mathbb{Z})$ with ghost coordinates $(w_0, \ldots, w_{n-1})$ we find the coefficients c_i by induction. Equating the first ghost coordinate we get $c_0 = w_0 = a_0$. Assume we have found $c_0, \ldots, c_{s_1}$. We equate the s-th ghost coordinate

$$w_s = \sum_{i=0}^{s} c_i p^i = \sum_{i=0}^{s-1} c_i p^i + c_s p^s = w_{s-1} + p^s c_s$$

and therefore, $c_s = p^{-s}(w_s - w_{s-1})$. These numbers are a priori rational. To finish the proof we need to show that they are integers.

$$\begin{aligned}
c_s &= p^{-s}(w_s - w_{s-1}) \\
&= p^{-s}(\sum_{i=0}^{s} p^i a_i^{p^{s-i}} - \sum_{i=0}^{s-1} p^i a_i^{p^{s-1-i}}) \\
&= p^{-s}(p^s a_s + \sum_{i=0}^{s-1} p^i (a_i^{p^{s-i}} - a_i^{p^{s-1-i}})) \\
&= a_s + \sum_{i=0}^{s-1} p^{i-s} a_i^{p^{s-1-i}} (a_i^{p^{s-i}-p^{s-1-i}} - 1)
\end{aligned}$$

It remains to show that for every integer a and every non-negative integer n:

$$a^{p^{n-1}}(a^{p^n-p^{n-1}} - 1) \equiv 0 \pmod{p^n}$$

There are two cases. If $v_p(a) \geq 1$, then $v_p(a^{p^{n-1}}) \geq p^{n-1} \geq n$, and if $v_p(a) = 0$, then $a^{p^n-p^{n-1}} - 1 = a^{\phi(p^n)} - 1 \equiv 0 \pmod{p^n}$. □

Corollary 2.4. *In the ring of* 2*-typical Witt vectors of length* n*,* $W_n(\mathbb{Z})$*, for every integer* a*, one has:*

$$[a]_n = c_0[1]_n + c_1 V[1]_{n-1} + \cdots + c_{n-1} V^{n-1}[1]_1,$$

where $c_0 = a$ *and* $c_i = 2^{-i}(a^{2^i} - a^{2^{i-1}})$*.*

Proposition 2.5. *In every 2-typical Witt complex* $E^*_\bullet$ *the following relations hold:*

$$\begin{aligned}
Vd &= 2dV, \\
dF &= 2Fd, \\
V(x)dV(y) &= V(xdy) + \iota V(xy).
\end{aligned}$$

Proof. We will use the relations from the definition of a Witt complex:

$$V(xF(y)) = V(x)y, \quad FdV = d + \iota, \quad FV = 2.$$

We have:

$$\begin{aligned} Vd(x) &= V(d+\iota+\iota)(x) = V(FdV+\iota)(x) \\ &= V(1)dV(x) + V\iota(x) \\ &= d(V(1)V(x)) - dV(1)V(x) + V\iota(x) \\ &= d(V(FV(1)x)) - V(FdV(1)x) + V\iota(x) \\ &= dV(2x) - V((d+\iota)(1)x) + V\iota(x) \\ &= 2dV(x) - V(d(1)x) - V(\iota x) + V(\iota x) \\ &= 2dV. \end{aligned}$$

$$\begin{aligned} dF(x) &= (d+\iota)F(x) - \iota F(x) = FdVF(x) - \iota F(x) \\ &= Fd(V(1)x) - F\iota(x) = F(dV(1)x + V(1)d(x)) - F\iota(x) \\ &= FdV(1)F(x) + FV(1Fdx) - F\iota(x) \\ &= (d+\iota)(1)F(x) + FVFdx - F\iota(x) \\ &= F(2dx) = 2Fdx. \end{aligned}$$

$$\begin{aligned} V(x)dV(y) &= V(xFdV(y)) \\ &= V(x(d+\iota)y) = V(xdy) + \iota V(xy). \end{aligned}$$

□

Proposition 2.6. *In every 2-typical Witt complex $E^*_\bullet$ we have:*

$$\iota([1]_n) = \sum_{s=1}^{n-1} 2^{s-1} dV^s([1]_{n-s}).$$

Proof. Since $2\iota([1]_n) = 0$, we we'll prove that $\iota([1]_n) = -\sum_{s=1}^{n-1} 2^{s-1} dV^s([1]_{n-s})$. The proof is by induction on n, starting with the case $n = 1$ which is trivial. Assume the statement for $n-1$. We will use the relations $d([1]_n) = 0$ and $([-1]_n)^2 = [1]_n$.

$$\begin{aligned} [-1]_n &= -[1]_n + V([1]_{n-1}), \\ d([-1]_n) &= dV([1]_{n-1}), \\ \frac{d([-1]_n)}{[-1]_n} &= \frac{dV([1]_{n-1})}{[-1]_n} = [-1]_n dV([1]_{n-1}), \\ \iota([1]_n) &= [-1]_n dV([1]_{n-1}) \\ &= (-[1]_n + V([1]_{n-1}))dV([1]_{n-1}) \\ &= -dV([1]_{n-1}) + V(FdV([1]_{n-1})) \\ &= -dV([1]_{n-1}) + V((d+\iota)([1]_{n-1})) \\ &= -dV([1]_{n-1}) + V(\iota([1]_{n-1})) \\ &= -dV([1]_{n-1}) - \sum_{s=1}^{n-2} 2^{s-1} VdV^s([1]_{n-1-s}). \end{aligned}$$

The statement now follows from $Vd = 2dV$. $\square$

PROPOSITION 2.7. $\iota^2 = 0$.

Proof. Again it is enough to prove $\iota^2([1]_n) = 0$. We do this by induction. The case $n = 1$ is trivial. Assume the statement for $n-1$.

$$\begin{aligned}
\iota^2([1]_n) &= \frac{d([-1]_n)}{[-1]_n}\frac{d([-1]_n)}{[-1]_n} \\
&= (d([-1]_n))^2 \\
&= (d(-[1]_n + V([1]_{n-1}))^2 \\
&= dV([1]_{n-1})dV([1]_{n-1}) \\
&= d(V([1]_{n-1})dV([1]_{n-1})) - V([1]_{n-1})ddV([1]_{n-1}) \\
&= d(V(FdV([1]_{n-1}))) - V(FdV\iota([1]_{n-1})) \\
&= d(V((d+\iota)([1]_{n-1}))) - V((d+\iota)\iota([1]_{n-1}))) \\
&= dV\iota([1]_{n-1})) - V(\iota([1]_{n-1})\iota([1]_{n-1})).
\end{aligned}$$

The second summand is zero by induction. We show that the first summand is also zero. For this we will use the previous lemma and the relations $Vd = 2dV$, $2\iota = 0$, and $dd = d\iota$:

$$\begin{aligned}
dV\iota([1]_{n-1})) &= dV(\sum_{s=1}^{n-2} 2^{s-1} dV^s([1]_{n-s-1}) \\
&= \sum_{s=1}^{n-2} 2^s ddV^{s+1}([1]_{n-s-1}) \\
&= \sum_{s=1}^{n-2} (2^s\iota)dV^{s+1}([1]_{n-s-1}) = 0.
\end{aligned}$$

This completes the proof. $\square$

3. THE DE RHAM-WITT COMPLEX

3.1. EXISTENCE. The Witt complexes over a ring A form a category $\mathcal{W}_A$. We will prove that this category has an initial object. We call this object the de Rham-Witt complex of A and denote it $W_\bullet\Omega_A^*$. To prove the existence of an initial object we use the Freyd adjoint functor theorem [10, p.116].

THEOREM 3.1. *The category $\mathcal{W}_A$ of Witt complexes over A has an initial object.*

Proof. The category $\mathcal{W}_A$ has all small limits, so we need to prove that the solution set condition is verified. First we note that at each level a Witt complex is also a DG-ring. The differential is defined as follows:

$$D : E_\bullet^n \to E_\bullet^{n+1}, D = \begin{cases} d, & \text{if } n = \text{even}; \\ d+\iota, & \text{if } n = \text{odd}. \end{cases}$$

Proposition 3.2. *The operator D is both a differential and a derivation.*

Proof. We show first that D is a differential, that is $D^2 : E^n_\bullet \to E^{n+2}_\bullet$ is zero. If n is even, $D^2 = (d+\iota)d = dd + d\iota = 2d\iota = 0$, the same if n is odd. Let's see that D is a derivation, that is $D(xy) = D(x)y + (-1)^{\deg(x)} xD(y)$. There are three cases.

(1) Both $\deg(x)$ and $\deg(y)$ even:

$$D(xy) = d(xy) = d(x)y + xd(y) = D(x)y + xD(y)$$

(2) $\deg(x)$ even, $\deg(y)$ odd:

$$\begin{aligned} D(xy) &= (d+\iota)(xy) = d(xy) + \iota xy \\ &= (d(x)y + xd(y)) + \iota xy = d(x)y + (xd(y) + \iota xy) \\ &= D(x)y + xD(y) \end{aligned}$$

(3) Both $\deg(x)$ and $\deg(y)$ odd:

$$\begin{aligned} D(xy) &= d(xy) = d(x)y + xd(y) = d(x)y + xd(y) + 2\iota xy \\ &= (d(x)y + \iota xy) + (xd(y) + \iota xy) \\ &= D(x)y + xD(y). \end{aligned}$$

For the last case we used the relation $2\iota = 0$. $\square$

To prove that the category $\mathcal{W}_A$ has an initial object we have to show that the solution set condition is satisfied. That means we have to find a *set* of objects $\{O_i\}_{i\in I}$, such that for any other object X in the category, there is an index $i \in I$ and a map $\phi : O_i \to X$, not necessarily unique. Since at each level, a Witt complex $E = E^*_\bullet$ is also a differential graded ring, there is a map $\lambda : \tilde{\Omega}^*_{W_\bullet(A)} \to E^*_\bullet$ which in degree zero is the map $\lambda : W_\bullet(A) \to E^0_\bullet$ prescribed in the definition of a Witt complex. We prove that the image of λ is a sub-Witt complex of $E^*_\bullet$. Since the isomorphisms classes of such objects form a set (they are all quotients of $\tilde{\Omega}^*_{W_\bullet(A)}$), the solution set condition is satisfied and the proposition is proved.
First of all we have to see that $\iota([1]_n) \in \mathrm{Im}(\lambda)$. But this is so because

$$\begin{aligned} \iota([1]_n) &= \frac{d(\lambda[-1]_n)}{\lambda[-1]_n} = \frac{D(\lambda[-1]_n)}{\lambda[-1]_n} \\ &= \frac{\lambda d[-1]_n}{\lambda[-1]_n} \in \mathrm{Im}(\lambda). \end{aligned}$$

Since $\iota([1]_n) \in \mathrm{Im}(\lambda)$ we see that $\mathrm{Im}(\lambda)$ is closed under ι. It is also closed under d because it is closed under D and $d = D$ or $d = D - \iota$ depending on the degree. It remains to see that it is closed under F and V.
We start with F. The Frobenius operator is multiplicative and each element in the image of λ is of the form $\lambda(a^0 da^1 \dots da^n) = \lambda(a^0)d(\lambda(a^1)) \dots d(\lambda a^n)$, so it suffices to show that $F(\lambda(a))$ and $F(d(\lambda(a))$ are in the image of the canonical

map. Part of the definition of a Witt complex is that $\lambda F = F\lambda$ for all $a \in A$. So $F(\lambda(a)) \in \mathrm{Im}(\lambda)$. Let us prove that $F(d(\lambda(a))) \in \mathrm{Im}(\lambda)$. We use the formula:

$$a = [a_0]_n + V([a_1]_{n-1}) + \cdots + V^{n-1}([a_{n-1}]_1),$$

which shows that

$$F(d\lambda(a)) = F(d\lambda([a_0]_n)) + F(d\lambda(V([a_1]_{n-1}))) + \cdots + F(d\lambda(V^{n-1}([a_{n-1}]_1))).$$

Recall from the definition of a Witt complex that $Fd\lambda([a]_n) = \lambda([a]_{n-1})d\lambda([a]_{n-1}])$. and that both F and V commute with λ in degree 0.

$$\begin{aligned} F(d\lambda(a)) &= F(d\lambda([a_0]_n)) + F(dV\lambda([a_1]_{n-1})) + \cdots + F(d(V^{n-1}\lambda([a_{n-1}]_1)) \\ &= \lambda([a_0]_{n-1})d\lambda([a_0]_{n-1}]) + (d+\iota)(\lambda([a_1]_{n-1}) + \cdots + \\ &\quad\ (d+\iota)V^{n-2}\lambda([a_{n-1}]_1); \end{aligned}$$

and this sum clearly is in the image of λ.
The fact that $\mathrm{Im}\lambda$ is closed under V follows from Proposition 2.5. $\square$

DEFINITION 3.3. *The initial object in the category $\mathcal{W}_A$ of Witt complexes over A is called the* de Rham Witt complex of A *and is denoted* $W_\bullet\Omega^*_A$.

PROPOSITION 3.4. *For every ring A the following assertions hold:*

(i) *the canonical map $\tilde{\Omega}^*_{W_\bullet(A)} \to W_\bullet\Omega^*_A$ is surjective,*

(ii) *the canonical map $\lambda : W_\bullet(A) \to W_\bullet\Omega^0_A$ is an isomorphism.*

Proof. Denote for the moment by $E^*_\bullet$ the image of the map $\lambda : \tilde{\Omega}^*_{W_\bullet(A)} \to W_\bullet\Omega^*_A$. It is a sub-Witt complex of $W_\bullet\Omega^*_A$, in particular it is an object of the category $\mathcal{W}_A$. Therefore it admits a unique map from the initial object. We consider the composition:

$$W_\bullet\Omega^*_A \to E^*_\bullet \to W_\bullet\Omega^*_A.$$

Being an endomorphism of the initial object, it has to be the identity map. So the second map is surjective, which amounts to the same thing as the map $\tilde{\Omega}^*_{W_\bullet(A)} \to W_\bullet\Omega^*_A$ being surjective.
In degree zero this means that the map $W_\bullet(A) \to W_\bullet\Omega^0_A$ is surjective. To prove that it is also injective, we consider the Witt complex $E^*_\bullet$ defined by $E^0_n = W_n(A)$ and $E^i_n = 0$, for all $i \geq 1$. As $W_\bullet\Omega^*_A$ is initial in the category $\mathcal{W}_A$, there is a morphism $\mu : W_\bullet\Omega^*_A \to E^*_\bullet$. The fact that μ is a morphism means among other conditions that the diagram

$$\begin{tikzcd} & W_n\Omega^0_A \arrow[dd, "\mu^0"] \\ W_n(A) \arrow[ur, "\lambda"] \arrow[dr, "\lambda"'] & \\ & E^0_n \end{tikzcd}$$

commutes. By the definition of $E_\bullet^*$ the corresponding λ is the identity morphism, so $\mu^0 \circ \lambda = 1$ and it follows that λ is injective. □

3.2. The standard filtration. On every Witt complex $E_\bullet^*$ there is a standard filtration by graded abelian groups (see also [6]):

$$\mathrm{Fil}^s E_n^q = V^s E_{n-s}^q + dV^s E_{n-s}^{q-1}.$$

This filtration can be used to set up inductive arguments when computing de Rham-Witt complexes. The important result that allows this is the following.

Lemma 3.5. *The following sequence is exact:*

$$0 \to \mathrm{Fil}^s W_n\Omega_A^q \to W_n\Omega_A^q \xrightarrow{R^{n-s}} W_s\Omega_A^q \to 0$$

Proof. First we show that the composition of the two morphisms is zero. Actually the composition is zero for all Witt complexes. This is so because R commutes with the other operators and any Witt complex is by definition zero in levels zero and below:

$$\begin{aligned} R^{n-s}(\mathrm{Fil}^s E_n^q) &= R^{n-s}(V^s E_{n-s}^q + dV^s E_{n-s}^{q-1}) \\ &= V^s R^{n-s} E_{n-s}^q + dV^s R^{n-s} E_{n-s}^{q-1} \\ &\subset V^s E_0^i + dV^s E_0^{q-i} = 0 \end{aligned}$$

Once we know that this composition is zero it follows that R^{n-s} induces a morphism

$$E_n^q/\mathrm{Fil}^s(E_n^q) \longrightarrow R^{n-s}E_n^q.$$

To end the proof of the lemma we need to show that this morphism is an isomorphism for $E_\bullet^* = W_\bullet\Omega^*$. Fix a value of $n-s$ and define

$$W_s'\Omega_A^i = W_n\Omega_A^q/\mathrm{Fil}^s W_n\Omega_A^q.$$

We prove that this is a Witt complex over A. We only need to check that the operators are well defined, then the relations are automatically satisfied. To show that R and F induce operators $R, F : W_s'\Omega_A^q \to W_{s-1}'\Omega_A^q$ we need to show $R(\mathrm{Fil}^s W_n\Omega_A^q) \subset \mathrm{Fil}^{s-1} W_{n-1}\Omega_A^q$ and $F(\mathrm{Fil}^s W_n\Omega_A^q) \subset \mathrm{Fil}^{s-1} W_{n-1}\Omega_A^q$. The first relation follows from $VR = RV$ and $dR = Rd$ and the second from $FV = 2$ and $FdV = d+\iota$. Similarly V induces an operator on $W_\bullet'\Omega_A^*$ if $V(\mathrm{Fil}^s W_n\Omega_A^q) \subset \mathrm{Fil}^{s+1} W_{n+1}\Omega_A^q$ and this follows from $Vd = 2dV$. ι and d induce operators if $\iota(\mathrm{Fil}^s W_n\Omega_A^q) \subset \mathrm{Fil}^s W_n\Omega_A^{q+1}$ and $d(\mathrm{Fil}^s W_n\Omega_A^q) \subset \mathrm{Fil}^s W_n\Omega_A^{q+1}$. The first follows from $\iota V = V\iota$ and $\iota d = d\iota$ and the second from $dd = d\iota$.

We show now that $W_\bullet'\Omega_R^*$ is an initial object in the category of 2-typical Witt complexes over A and hence the morphism induced by R^{n-s}, $W_\bullet'\Omega_R^* \to W_\bullet\Omega_R^*$ is an isomorphism and hence the sequence is exact.

Consider $E_\bullet^*$ a 2-typical Witt complex over A. We construct a morphism $W_\bullet'\Omega_A^* \to E_\bullet^*$ and show that it is unique. Since the standard filtration is natural we have maps:

$$W_s'\Omega_A^i = W_n\Omega_A^i/\mathrm{Fil}^s W_n\Omega_A^q \to E_n^i/\mathrm{Fil}^s E_n^i \xrightarrow{R^{n-s}} E_s^i.$$

To show that this homomorphism of Witt complexes is unique we first show that the map $\tilde{\Omega}^i_{W_s(A)} \to W'_s\Omega^i_A$ is surjective. This is immediate from the diagram

$$\begin{tikzcd} \tilde{\Omega}^i_{W_n(A)} \arrow[r, two heads] \arrow[d, two heads] & W_n\Omega^i_A \arrow[d, two heads] \\ \tilde{\Omega}^i_{W_s(A)} \arrow[r] & W'_s\Omega^i_A \end{tikzcd}$$

Now in the diagram

$$\begin{tikzcd} \tilde{\Omega}^*_{W_\bullet(A)} \arrow[r] \arrow[d, two heads] & E^*_\bullet \\ W'_\bullet\Omega^*_A \arrow[ur] & \end{tikzcd}$$

considered in the category of pro-differential-graded-rings the top map is unique, therefore the oblique map is unique. $\square$

3.3. An additivity result. As far as we know, the relations in the definition of a 2-typical Witt complex are independent. However, one relation can be partially deduced from the others. We make this precise in the following Lemma.

Lemma 3.6. *Let A be an arbitrary ring, and $E^*_\bullet$ a pro-graded ring. Assume that $E^*_\bullet$ is endowed with all the operators in the definition of a 2-typical Witt complex, and all the relations are sastisfied with the exception of the last relation. Assume that this relation holds for two given elements $f, g \in A$, that is*

$$Fd\lambda([f]_n) = \lambda([f]_{n-1})d\lambda([f]_{n-1}),$$
$$Fd\lambda([g]_n) = \lambda([g]_{n-1})d\lambda([g]_{n-1}).$$

Then it also holds for their sum, $f + g \in A$:

$$Fd\lambda([f+g]_n) = \lambda([f+g]_{n-1})d\lambda([f+g]_{n-1}).$$

Proof. The proof is inspired by the proof of Proposition 1.3 in [9].
Since there is no danger of confusion, we omit λ.
We prove the statement by induction on the level n. If $n = 1$ the relation holds trivially. Assume we have proved the Lemma for $n-1$. We know that

$$Fd[f]_n = [f]_{n-1}d[f]_{n-1},$$
$$Fd[g]_n = [g]_{n-1}d[g]_{n-1},$$

and if we apply R to these relations we obtain

$$Fd[f]_{n-1} = [f]_{n-2}d[f]_{n-2},$$
$$Fd[g]_{n-1} = [g]_{n-2}d[g]_{n-2}.$$

By the induction hypothesis, we have

$$Fd[f+g]_{n-1} = [f+g]_{n-2}d[f+g]_{n-2}.$$

We define $\tau \in W_{n-1}(A)$ by the formula:

$$[f+g]_n = [f]_n + [g]_n + V\tau.$$

We apply R to both sides of this identity:

$$[f+g]_{n-1} = [f]_{n-1} + [g]_{n-1} + VR\tau.$$

We square both sides:

$$[f+g]_{n-1}^2 = [f]_{n-1}^2 + [g]_{n-1}^2 + (VR\tau)^2 + 2([f]_{n-1} + [g]_{n-1})VR\tau + 2[fg]_{n-1},$$

and, since $(VR\tau)^2 = V(R\tau FVR\tau) = 2VR(\tau^2)$, we get:

$$F([f+g]_n - [f]_n - [g]_n) = 2(VR(\tau^2) + [fg]_{n-1} + ([f]_{n-1} + [g]_{n-1})VR\tau).$$

The left hand side of this identity is $F(V\tau) = 2\tau$, thus we obtain:

$$2\tau = 2(VR(\tau^2) + [fg]_{n-1} + ([f]_{n-1} + [g]_{n-1})VR\tau).$$

This identity is true in the ring $W_{n-1}(R)$ for any ring R, in particular for the ring of polynomials in two variables $R = \mathbb{Z}[f,g]$. For this ring the multiplication by 2 in $W_{n-1}(R)$ is injective, therefore the following identity is true for this particular ring:

$$\tau = VR(\tau^2) + [fg]_{n-1} + ([f]_{n-1} + [g]_{n-1})VR\tau.$$

Taking Witt vectors of length $n-1$ is functorial, and it follows that the identity is true for arbitrary rings R and elements f and g.
Once we proved this formula we return to the identity we want to prove:

$$Fd[f+g]_n = [f+g]_{n-1}d[f+g]_{n-1},$$

or equivalently:

$$Fd([f]_n + [g]_n + V\tau) = ([f]_{n-1} + [g]_{n-1} + VR\tau)(d[f]_{n-1} + d[g]_{n-1} + dVR\tau).$$

We expand the right hand side:

$$\begin{aligned} Fd[f]_n + Fd[g]_n + FdV\tau = {} & [f]_{n-1}d[f]_{n-1} + [g]_{n-1}d[g]_{n-1} \\ & + d([fg]_{n-1} + ([f]_{n-1} + [g]_{n-1})VR\tau) + (VR\tau)(dVR\tau). \end{aligned}$$

Using the hypothesis that $Fd[f]_n = [f]_{n-1}d[f]_{n-1}$ and $Fd[g]_n = [g]_{n-1}d[g]_{n-1}$, and the formula for τ, the previous identity becomes equivalent to:

$$FdV\tau = d(\tau - VR(\tau^2)) + (VR\tau)(dVR\tau)$$

or:

$$d\tau + \iota\tau = d\tau - dVR(\tau^2) + (VR\tau)(dVR\tau).$$

We reduced the problem to proving the following formula:

$$\iota\tau = -dVR(\tau^2) + (VR\tau)(dVR\tau).$$

We prove this separately in the next Lemma. □

Lemma 3.7. *For every ring A and every element $\tau \in W_k(A)$, the following identity holds:*

$$dVR(\tau^2) = (VR\tau)(dVR\tau) + \iota\tau$$

Remark: This lemma says that ι measures the failure of d to be a PD-derivation. To explain this we need to recall what a PD-structure on a ring is and what a PD-derivation is.
If A is a commutative and unital ring and I is and ideal, a PD-structure on (A, I) is a family of maps $\gamma_n : I \to A$, $n \in \mathbb{N}$ which morally behave like dividing the n'th power by $n!$. More precisely they are required to satisfy five conditions:

(i) $\gamma_0(x) = 1$, $\gamma_1(x) = x$, $\gamma_n(x) \in I$, $\forall x \in I$,
(ii) $\gamma_n(x + y) = \sum_{i=0}^{n} \gamma_i(x)\gamma_{n-i}(y), \forall x, y \in I$,
(iii) $\gamma_n(ax) = a^n\gamma_n(x), \forall a \in A, x \in I$,
(iv) $\gamma_p(x)\gamma_q(x) = \binom{p+q}{p}\gamma_{p+q}(x), \forall p, q \in \mathbb{N}, x \in I$,
(v) $\gamma_p(\gamma_q(x)) = \frac{(pq)!}{p!(q!)^p}\gamma_{pq}(x), \forall p, q \in \mathbb{N}, x \in I$.

If A is a $\mathbb{Z}_{(p)}$-algebra, there is a canonical PD-structure on $(W_n(A), VW_{n-1}(A))$, namely:

$$\gamma_m : VW_{n-1}(A) \to W_n(A)$$

$$\gamma_m(Vx) = \begin{cases} 1, & \text{if } m = 0, \\ \frac{p^{m-1}}{m!}V(x^m), & \text{if } m \geq 1 \end{cases}$$

If (A, I) is a ring with a PD-structure and $d : A \to M$ is a derivation of A into an A-module M, then d is called a PD-derivation if $d(\gamma_n(x)) = \gamma_{n-1}(x)dx$, for all $x \in I$. If we consider $W(A)$, the inverse limit of the pro-ring $W_\bullet(A)$, elements in it are sequences of elements $x_n \in W_n(A)$, and $R : W(A) \to W(A)$ becomes the identity morphism. So the identity in the lemma reads: if $x \in VW(A) \subseteq W(A)$, $x = V\tau$, then:

$$d(\gamma_2(x)) = \gamma_1(x)dx + \iota\tau.$$

If we did not have $\iota\tau$ in this relation, then Lemma 1.2 of Langer, Zink [9] would show that d is a PD-derivation. We will now prove the lemma.

Proof. The proof is in three steps. First we show that the identity holds for all elements $\tau = [\phi]_k$, for $\phi \in A$. Then we show that once it holds for τ it also holds for $V\tau$. Finally we show that if the identity holds for two elements τ_1, τ_2 it also holds for their sum $\tau_1 + \tau_2$.
We begin with $\tau = [\phi]_k$ for some $\phi \in A$. We manipulate the left hand side of the identity that we want to prove:

$$\begin{aligned} dV[\phi]_{k-1}^2 &= dV(F([\phi]_k)) = d(V([1]_{k-1})[\phi]_k) \\ &= dV([1]_{k-1})[\phi]_k + V([1]_{k-1})d[\phi]_k \\ &= dV([1]_{k-1})[\phi]_k + V(Fd[\phi]_k), \end{aligned}$$

and using the induction hypothesis that $Fd[\phi]_k = [\phi]_{k-1}d[\phi]_{k-1}$ for $k \leq n-1$:

$$dV[\phi]_{k-1}^2 = (dV[1]_{k-1})[\phi]_k + V([\phi]_{k-1}d[\phi]_{k-1}).$$

The right hand side of the identity is

$$\begin{aligned} V[\phi]_{k-1}dV[\phi]_{k-1} + \iota[\phi]_k &= V([\phi]_{k-1}FdV[\phi]_{k-1}) + \iota[\phi]_k \\ &= V([\phi]_{k-1}d[\phi]_{k-1}) + V(\iota[\phi]_{n-1}^2) + \iota[\phi]_k. \end{aligned}$$

Therefore the equality of the two terms is equivalent to the equality

$$dV[1]_{k-1}[\phi]_k = V(\iota[1]_{k-1})[\phi]_n + \iota[\phi]_k,$$

which is certainly true if

$$dV[1]_{k-1} = V(\iota[1]_{k-1}) + \iota[1]_k.$$

This last identity follows from Lemma 2.6 which gives the formula for $\iota[1]_k$.
Now we assume we know the relation for τ and we want to prove it for $V(\tau)$.
We know:

$$dVR(\tau^2) = (VR\tau)(dVR\tau) + \iota\tau,$$

we apply V to this:

$$V(dVR(\tau^2)) = V((VR\tau)(dVR\tau)) + V(\iota\tau);$$

The left hand side of this equation is:

$$\begin{aligned} V(dVR(\tau^2)) &= 2dV^2R(\tau^2) = dV^2(R\tau FVR\tau) \\ &= dV(V(R\tau)V(R\tau)) = dVR(V(\tau)^2) \end{aligned}$$

We want to prove:

$$dVR(V(\tau)^2) = VRV\tau dVRV\tau + \iota V\tau.$$

We notice that the left hand side of this equation is equal to the left hand side of the previous equation, so it is sufficient for us to prove:

$$V((VR\tau)(dVR\tau)) + V(\iota\tau) = VRV\tau dVRV\tau + \iota V\tau,$$

or:

$$\begin{aligned} V((VR\tau)(dVR\tau)) &= VRV\tau dVRV\tau \\ V((VR\tau)(dVR\tau)) &= V((RV\tau)FdVRV\tau) \\ V((VR\tau)(dVR\tau)) &= V((RV\tau)dRV\tau) + V((RV\tau)\iota(RV\tau)) \\ 0 &= \iota(V((VR\tau)^2)) \\ 0 &= \iota(V(R\tau FVR\tau)) \end{aligned}$$

which is true since $FV = 2$ and $2\iota = 0$.

Finally we want to prove that if the relation holds for τ_1 and τ_2, it also holds for $\tau_1 + \tau_2$. We know:

$$RdV\tau_1^2 = RV\tau_1 dRV\tau_1 + \iota\tau_1,$$
$$RdV\tau_2^2 = RV\tau_2 dRV\tau_2 + \iota\tau_2$$

and we want:

$$RdV(\tau_1+\tau_2)^2 = RV(\tau_1+\tau_2)dRV(\tau_1+\tau_2) + \iota(\tau_1+\tau_2),$$

or equivalently:

$$\begin{aligned} RdV\tau_1^2 + RdV\tau_2^2 + 2RdV(\tau_1\tau_2) = {} & RV\tau_1 dRV\tau_1 + RV\tau_2 dRV\tau_2 \\ & + d(RV\tau_1 RV\tau_2) + \iota\tau_1 + \iota\tau_2. \end{aligned}$$

After cancelling six terms the equation reduces to:

$$2RdV\tau_1\tau_2 = RdV\tau_1 V\tau_2,$$
$$2RdV\tau_1\tau_2 = RdV(\tau_1 FV\tau_2),$$

which is true since $FV = 2$. □

4. Computations

4.1. The 2-typical de Rham-Witt vectors of the integers. Before we state the structure theorem for $W_\bullet\Omega^*_{\mathbb{Z}}$ we introduce a bit of notation. We denote by $V^i(1) \in W_n\Omega^0_A$ the element $V^i([1]_{n-i})$.

Theorem 4.1. *The structure of $W_\bullet\Omega^*_{\mathbb{Z}}$ is as follows*

(i) *As abelian groups*

$$W_n\Omega^0_{\mathbb{Z}} = \bigoplus_{i=0}^{n-1} \mathbb{Z}\cdot V^i(1), \tag{7}$$

$$W_n\Omega^1_{\mathbb{Z}} = \bigoplus_{i=1}^{n-1} \mathbb{Z}/2^i\mathbb{Z}\cdot dV^i(1), \tag{8}$$

$$W_n\Omega^i_{\mathbb{Z}} = 0, \quad \text{for } i \geq 2. \tag{9}$$

(ii) *The product is given by*

$$V^i(1)\cdot V^j(1) = 2^i V^j(1), \quad \text{if} \quad i \leq j, \tag{10}$$

$$V^i(1)\cdot dV^j(1) = \begin{cases} 2^i dV^j(1) + \sum_{s=j+1}^{n-1} 2^{s-1} dV^s(1), & \text{if } 1 \leq i < j \\ \sum_{s=i+1}^{n-1} 2^{s-1} dV^s(1), & \text{if } 1 \leq j \leq i. \end{cases} \tag{11}$$

(12)

(iii) *The operator V acts as follows*

$$V(V^i(1)) = V^{i+1}(1), \tag{13}$$

$$V(dV^i(1)) = 2dV^{i+1}(1). \tag{14}$$

(iv) *The operator F acts as follows*

$$F(V^i(1)) = 2V^{i-1}(1), \tag{15}$$

$$F(dV^i(1)) = dV^{i-1}(1) + \sum_{s=i}^{n-2} 2^{s-1} dV^s(1). \tag{16}$$

(v) *The operator d acts by $d(V^i(1)) = dV^i(1)$ when $i \geq 1$ and $d(1) = 0$, and the action of the operator ι is given by*

$$\iota(V^i(1)) = \sum_{s=i+1}^{n-1} 2^{s-1} dV^s(1). \tag{17}$$

(vi) *The operator $R : W_n\Omega^*_{\mathbb{Z}} \to W_{n-1}\Omega^*_{\mathbb{Z}}$ acts as follows*

$$RV^i(1) = \begin{cases} V^i(1) & \text{if } i \leq n-2 \\ 0 & \text{if } i = n-1 \end{cases} \tag{18}$$

$$RdV^i(1) = \begin{cases} dV^i(1) & \text{if } i \leq n-2 \\ 0 & \text{if } i = n-1. \end{cases} \tag{19}$$

Proof. We begin with the fifth assertion and then we prove the others in the stated order.
(v) We already know the formula (see 2.6):

$$\iota[1]_n = \sum_{s=1}^{n-1} 2^{s-1} dV^s(1),$$

which is the particular case for the relation we want to prove when $i = 0$. For other i we have:

$$\begin{aligned} \iota V^i([1]_{n-i}) &= V^i(\iota[1]_{n-i}) \\ &= V^i(\sum_{s=1}^{n-i-1} 2^{s-1} dV^s([1]_{n-i-s-1}) \\ &= \sum_{s=1}^{n-i-1} 2^{s+i-1} dV^{s+i}([1]_{n-i-s-1}) \\ &= \sum_{s=i+1}^{n-2} 2^{s-1} dV^s([1]_{n-s-1}). \end{aligned}$$

(i),(ii)The isomorphism described in the first relation follows from the previous theorem. The second relation follows from the fact that the map $\lambda : \tilde{\Omega}^*_{W_\bullet(A)} \to W_\bullet\Omega^*_A$ is surjective and from the product relations that we now prove. The vanishing of the de Rham-Witt complex in higher degrees will be proven after verifying (iii) and (iv).
The first product relation is the product rule described in 2.3 in the case $p = 2$. The second product relation:

- If $1 \le i \le j$:

$$\begin{aligned} V^i(1)\cdot dV^j(1) &= V^i(F^i dV^j(1)) = V^i((d+\iota)V^{j-i}(1)) \\ &= V^i dV^{j-i}(1) + V^j(\iota(1)) \\ &= 2^i dV^j(1) + V^j(\sum_{s=1}^{n-j-1} 2^{s-1} dV^s(1)) \\ &= 2^i dV^j(1) + \sum_{s=1}^{n-j-1} 2^{s+j-1} dV^{j+s}(1) \\ &= 2^i dV^j(1) + \sum_{s=j+1}^{n-1} 2^{s-1} dV^s(1). \end{aligned}$$

- If $1 \le j \le i$:

$$\begin{aligned} V^i(1)\cdot dV^j(1) &= d(V^i(1)V^j(1)) - dV^i(1)V^j(1) \\ &= d(2^j V^i(1)) - V^j(1)dV^i(1) \\ &= 2^j dV^i(1) - 2^j dV^i(1) - \sum_{s=i+1}^{n-1} 2^{s-1} dV^s(1) \\ &= \sum_{s=i+1}^{n-1} 2^{s-1} dV^s(1). \end{aligned}$$

We note here that one can give a unified product relation for $V^i(1)\cdot dV^j(1)$, namely:

$$V^i(1)\cdot dV^j(1) = 2^i dV^j(1) + \sum_{s=\max(i,j)+1}^{n-1} 2^{s-1} dV^s(1)$$

(iii) The first relation is trivial and the second follows from $Vd = 2dV$.
(iv) The first relation follows from $FV = 2$. The second relation:

$$\begin{aligned} F(dV^i(1)) &= (d+\iota)V^{i-1}(1) \\ &= dV^{i-1}(1) + \iota(V^{i-1}(1) \\ &= dV^{i-1}(1) + \sum_{s=i-1}^{n-2} 2^s dV^{s+1}(1). \end{aligned}$$

Once we have these relations and the fact that $2^i dV^i(1) = V^i d(1) = 0$, it follows that λ factors through a surjective map $\bigoplus_{i=1}^{n-1} \mathbb{Z}/2^i\mathbb{Z}\cdot dV^i(1) \to W_\bullet\Omega^1_{\mathbb{Z}}$. We prove now that $W_\bullet\Omega^i_{\mathbb{Z}} = 0$, for $i \ge 2$. We will prove this by induction on the level using the standard filtration. The first step of the induction, that $W_1\Omega^q_{\mathbb{Z}} = 0$, forall $q \ge 2$ follows from the surjectivity of the map $\lambda : \Omega^q_{W_1(\mathbb{Z})} = \Omega^q_{\mathbb{Z}} \to W_1\Omega^q_{\mathbb{Z}}$, and fact that the domain of the map is zero whenever $q \ge 1$. Assuming that $W_n\Omega^q_{\mathbb{Z}} = 0$ for all $q \ge 2$ we prove that $W_{n+1}\Omega^q_{\mathbb{Z}} = 0$, for all

$q \geq 2$. This is so because in the short exact sequence

$$0 \to \mathrm{Fil}^{n-1} W_n\Omega^q_{\mathbb{Z}} \to W_n\Omega^q_{\mathbb{Z}} \xrightarrow{R} W_{n-1}\Omega^q_{\mathbb{Z}} \to 0$$

the right term is zero by induction and the left term is zero because $\mathrm{Fil}^{n-1} W_n\Omega^q_{\mathbb{Z}} = V^{n-1}W_1\Omega^2_{\mathbb{Z}} + dV^{n-1}W_1\Omega^1_{\mathbb{Z}}$ and both $W_1\Omega^1_{\mathbb{Z}}$ and $W_1\Omega^2_{\mathbb{Z}}$ are zero as seen above.

To finish the description of the groups that form the de Rham-Witt complex $W_\bullet\Omega^*_{\mathbb{Z}}$ we consider the pro-graded ring $G^*_\bullet$ defined by the groups on the right hand side of the relations (1) – (3), that is:

$$\begin{aligned} G^0_n &= \bigoplus_{i=0}^{n-1} \mathbb{Z} \cdot V^i(1), \\ G^1_N &= \bigoplus_{i=1}^{n-1} \mathbb{Z}/2^i\mathbb{Z} \cdot dV^i(1), \\ G^i_n &= 0, \quad \text{for } i \geq 2. \end{aligned}$$

The product is defined by the relations in (ii) , the operators F, V, d, ι, R are given by the relations in (iii)- (vi). We check that with these definitions $G^*_\bullet$ is indeed a Witt complex.

The only non-trivial relation to verify is that $Fd\lambda([a]_n) = \lambda([a]_{n-1})d\lambda([a]_{n-1}])$, for all integers a. Using the additivity result 3.6, we see that we need to check this relation only for the integers $a = 1$ and $a = -1$. It is trivially satisfied in the first case, and easy to see in the second, once we recall from 2.1 that $[-1]_n = -[1]_n + V[1]_{n-1}$:

$$\begin{aligned} Fd[-1]_n &= Fd(-[1]_n + V[1]_{n-1}) = FdV[1]_{n-1} \\ &= (d + \iota)[1]_{n-1} = \iota[1]_{n-1} \\ &= [-1]_{n-1}d[-1]_{n-1}. \end{aligned}$$

To prove now that $W_\bullet\Omega^*_{\mathbb{Z}} \cong G^*_\bullet$ we define a morphism of Witt complexes

$$\begin{aligned} G^*_\bullet &\longrightarrow W_\bullet\Omega^*_{\mathbb{Z}} \\ V^i(1) &\longmapsto V^i(1) \\ dV^i(1) &\longmapsto dV^i(1) \end{aligned}$$

The composition $W_\bullet\Omega^*_{\mathbb{Z}} \to G^*_\bullet \to W_\bullet\Omega^*_{\mathbb{Z}}$ is an endomorphism of the initial object in the category $\mathcal{W}_A$ and as so it is the identity. The composition $G^*_\bullet \to W_\bullet\Omega^*_{\mathbb{Z}} \to G^*_\bullet$ is an endomorphism of $G^*_\bullet$; it is not hard to see that the only endomorphism of $G^*_\bullet$ is the identity: being a morphism of pro-rings it maps $[1]_n$ to itself, and since it commutes with V and d it will also map $V^i(1)$ and $dV^i(1)$ to themselves. □

Remark: The same proof works to give us the structure of $W_\bullet\Omega^*_{\mathbb{Z}_{(2)}}$:

(i) As abelian groups:

$$W_n\Omega^0_{\mathbb{Z}_{(2)}} = \bigoplus_{i=0}^{n-1} \mathbb{Z}_{(2)} \cdot V^i(1), \tag{20}$$

$$W_n\Omega^1_{\mathbb{Z}_{(2)}} = \bigoplus_{i=1}^{n-1} \mathbb{Z}/2^i\mathbb{Z} \cdot dV^i(1), \tag{21}$$

$$W_n\Omega^i_{\mathbb{Z}(2)} = 0, \quad \text{for } i \geq 2. \tag{22}$$

(ii) The product formulas and the actions of the various operators are the same as in Theorem 4.1.

Indeed, the only thing we have to check is that $\Omega^i_{\mathbb{Z}_{(2)}} = 0$ for all $i \geq 2$. To see this we need to prove that $d(\frac{1}{m}) = 0$ for all $m \in \mathbb{Z}$ odd. This follows from $0 = d(1) = d(m\frac{1}{m}) = md(\frac{1}{m})$, since $m \in \mathbb{Z}_{(2)}$ is a unit.

4.2. The 2-typical de Rham-Witt complex for polynomial extensions. In this subsection we describe the relationship between the 2-typical de Rham-Witt complex of the ring of polynomials in one variable over a $\mathbb{Z}_{(2)}$-algebra A and the 2-typical de Rham-Witt complex of the ring A. In order to do that we will identify the left adjoint of the forgetful functor $\mathcal{W}_{A[X]} \to \mathcal{W}_A$. We call this functor $P : \mathcal{W}_A \to \mathcal{W}_{A[X]}$, and since it commutes with colimits, it will carry $W_\bullet\Omega^*_A$ into $W_\bullet\Omega^*_{A[X]}$.
In order to define the functor P we first analyze the Witt complex $W_\bullet\Omega^*_{A[X]}$. Inside it we find the image of the map $W_\bullet\Omega^*_A \to W_\bullet\Omega^*_{A[X]}$ induced by the inclusion $A \to A[X]$. Besides this image we can certainly identify the elements $[X]^i_n$. If we play with the multiplication and with the operators R, F, V, d, and ι we will find new elements, but because of the relations that hold in every Witt complex, we will see that all these elements can be classified in four types. The first obvious type is the elements of the form $a[X]^i_n$, where $a \in \mathrm{Im}(W_\bullet\Omega^*_A)$, $i \in \mathbb{N}$, and $n \geq 1$. When there is no danger of confusion, we omit the subscript n, and also write $a \in W_\bullet\Omega^*_A$. This type is closed under multiplication and also under the action of R, F, and ι. If we apply d and V we will get two new types: elements of the form $b[X]^{k-1}d[X]$ and elements of the form $V^r(c[X]^l)$, where $b, c \in W_\bullet\Omega^*_A$, and $k, r, l > 0$. Special attention has to be paid to the latter type, as some elements of that form were already listed as elements of the first type. An example is $V(c[X]^2) = V(cF([X])) = V(c)[X]$. The restriction that we have to impose is l be odd. Finally, if we apply V and then d we obtain a new type, of elements of the form $dV^s(e[X]^m)$, where $s > 0$, $e \in W_\bullet\Omega^*_A$, and m is odd. If we multiply elements of any of these two types together we will get a sum of elemtents of these types. The key observation is the following:

Lemma 4.2. *In any Witt complex over $A[X]$ the following relation holds:*

$$d[X]d[X] = \iota([1])[X]d[X].$$

Proof.

$$\begin{aligned} d[X]d[X] &= d([X]d[X]) - [X]dd[X] = d(Fd[X]) - [X]d\iota[X] \\ &= 2Fdd[X] + \iota([1])[X]d[X] = 2Fd\iota[X] + \iota([1])[X]d[X] \\ &= \iota([1])[X]d[X]. \end{aligned}$$

□

Using this observation we can see for example that the product of two elements of the second type is again an element of second type:

$$b[X]^{k-1}d[X]b'[X]^{k'-1}d[X] = \iota(bb')[X]^{k+k'-1}d[X].$$

The other products and the action of the different operators on the elements can also be derived. The formulas that we obtain will be exactly the formulas that we plug in the definition of the functor P.
Before we define the functor P we need to recall a result of Hesselholt, Madsen that describes the ring of p-typical Witt vectors over the ring $A[X]$: every element $f \in W_n(A[X])$ can be written uniquely as a sum:

$$f = \sum_{j\in\mathbb{N}} a_{0,j}[X]_n^j + \sum_{s=1}^{n-1} \sum_{(j,p)=1} V^s(a_{s,j}[X]_{n-s}^j),$$

with $a_{s,j} \in W_{n-s}(A)$ and all but finitely many $a_{s,j}$ zero (see Lemma 4.1.1 in [3]). In the case $p = 2$ these results read: every element $f \in W_n(A[X])$ can be written uniquely as a finite sum of elements of two types, that we will call type 1 and type 3, for reasons that will soon become clear:

- Type 1: elements of the form $a[X]^j$, where $a \in W_n(A)$,
- Type 3: elements of the form $V^r(c[X]^l)$, where $r > 0$, $c \in W_{n-r}(A)$, and l is odd.

Now we are ready to define the functor $P : \mathcal{W}_A \to \mathcal{W}_{A[X]}$. On objects it is defined as follows: for a Witt complex $E^*_\bullet \in \mathcal{W}_A$, $P(E)^q_n$ consists of formal sums of four types of elements:

- Type 1: elements of the form $a[X]^j$, where $a \in E^q_n$,
- Type 2: elements of the form $b[X]^{k-1}d[X]$, where $b \in E^{q-1}_n$,
- Type 3: elements of the form $V^r(c[X]^l)$, where $r > 0$, $c \in E^q_{n-r}$, and l is odd,
- Type 4: elements of the form $dV^s(e[X]^m)$, where $s > 0$, $e \in E^{q-1}_{n-s}$, and m is odd.

The product is graded commutative, and is given by the following ten formulas:

P1.1: $a[X]^j a'[X]^{j'} = aa'[X]^{j+j'}$,
P1.2: $a[X]^j b[X]^{k-1}d[X] = ab[X]^{j+k-1}d[X]$,
P1.3: $a[X]^j V^r(c[X]^l) = V^r(F^r(a)c[X]^{2^r j+l})$,

P1.4:

$$\begin{aligned}a[X]^j dV^s(e[X]^m) =&(-1)^{|a|}\frac{m}{2^s j+m} dV^s(F^s(a)e[X]^{2^s j+m})\\ &-(-1)^{|a|}V^s((F^s(da)e - \frac{j}{2^s j+m} d(F^s(a)e))[X]^{2^s j+m}),\end{aligned}$$

P2.2: $b[X]^{k-1}d[X]b'[X]^{k'-1}d[X] = \iota(bb')[X]^{k+k'-1}d[X]$,
P2.3:

$$\begin{aligned}b[X]^{k-1}d[X]V^r(c[X]^l) =& -(-1)^{|b|}\frac{1}{2^r k+l}V^r(d(F^r(b)c)[X]^{2^r k+l})\\ &+(-1)^{|b|}\frac{2^r}{2^r k+l}dV^r(F^r(b)c[X]^{2^r k+l}),\end{aligned}$$

P2.4:

$$\begin{aligned}b[X]^{k-1}d[X]dV^s(e[X]^m) =& -(-1)^{|b|}\frac{1}{2^s k+m}V^s(F^s(db+k\iota(b))de[X]^{2^s k+m})\\ &+(-1)^{|b|}\frac{1}{2^s k+m}dV^s(F^s(b)de[X]^{2^s k+m}),\end{aligned}$$

P3.3:

$$\begin{aligned}&V^r(c[X]^l)V^{r'}(c'[X]^{l'}) =\\ &= \begin{cases} 2^{r'}V^r(cF^{r-r'}(c')[X]^{2^{r-r'}l'+l}), & \text{if } r > r',\\ 2^r V^{r-v}(V^v(cc')[X]^{2^{-v}(l+l')}), & \text{if } r = r' \text{ and } v = v_2(l+l'), v \le r,\\ 2^r V^r(cc')[X]^{2^{-r}(l+l')}, & \text{if } r = r' \text{ and } v > r,\end{cases}\end{aligned}$$

P3.4:

$$\begin{aligned}&V^r(c[X]^l)dV^s(e[X]^m) =\\ &= \begin{cases} (-1)^{|c|}\dfrac{2^r m}{2^{s-r}l+m}dV^s(F^{s-r}(c)e[X]^{2^{s-r}l+m}), & \text{if } r < s,\\ V^{r-v}(V^v(c(d+\iota)(e))[X]^{2^{-v}(l+m)}) & \\ \quad +(-1)^{|c|}\frac{2^r m}{l+m}dV^r(ce[X]^{2^{-v}(l+m)}) & \\ \quad -(-1)^{|c|}\frac{2^v m}{l+m}V^{r-v}(dV^v(ce)[X]^{2^{-v}(l+m)}), & \text{if } r = s,\ v = v_2(l+m) < r,\\ V^r((c(d+\iota)(e))[X]^{2^{-r}(l+m)}) & \\ \quad +(-1)^{|e|}mV^r(ce)[X]^{2^{-r}(l+m)-1}d[X], & \text{if } r = s,\ v = v_2(l+m) \ge r,\\ V^r(cF^{r-s}((d+\iota)(e))[X]^{2^{r-s}m+l}) & \\ \quad +(-1)^{|c|}\dfrac{2^r m}{2^{r-s}m+l}dV^r(cF^{r-s}(e)[X]^{2^{r-s}m+l}) & \\ \quad -(-1)^{|c|}\dfrac{m}{2^{r-s}m+l}V^r(d(cF^{r-s}(e))[X]^{2^{r-s}m+l}), & \text{if } r > s,\end{cases}\end{aligned}$$

P4.4:

$$dV^s(e[X]^m)dV^{s'}(e'[X]^{m'}) =$$

$$= \begin{cases} -(-1)^{|e|}dV^{s'}((F^{s'-s}((d+\iota)(e)e') + \dfrac{m}{2^{s'-s}m+m'}d(F^{s'-s}(e)e'))[X]^{2^{s'-s}m+m'}) \\ \qquad +V^{s'}((F^{s'-s}(de)\iota(e') + F^{s'-s}(e)d\iota(e'))[X]^{2^{s'-s}m+m'}), \quad \text{if } s < s', \\ V^{s-v}((V^v(ed\iota(e')) + dV^v(e\iota(e')))[X]^{2^{-v}(m+m')}) \\ \qquad +dV^{s-v}((V^v(e(d+\iota)(e')) + dV^v(ee'))[X]^{2^{-v}(m+m')}), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } s = s',\ v = v_2(l+m) < s, \\ (V^s(e\iota(e')) + (-1)^{e'}m'dV^s(ee')[X]^{2^{-s}(m+m')-1}d[X]) \\ \qquad +V^s(ed\iota(e')[X]^{2^{-s}(m+m')}) \\ \qquad +dV^s(e(d+\iota)(e')[X]^{2^{-s}(m+m')}), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } s = s',\ v = v_2(l+m) \geq s. \end{cases}$$

The definition of λ, R, and ι are obvious, the action of V is given by the following four formulas:

V1:

$$V(a[X]^j) = \begin{cases} V(a[X]^j), & \text{if } j \text{ odd}, \\ V(a)[X]^{j/2}, & \text{if } j \text{ even}, \end{cases}$$

V2:

$$V(b[X]^{k-1}d[X]) = \begin{cases} (-1)^{|b|}\frac{1}{k}V((db)[X]^k) - (-1)^{|b|}\frac{2}{k}dV(b[X]^k), & \text{if } k \text{ odd}, \\ V(b)[X]^{k/2-1}d[X], & \text{if } k \text{ even}, \end{cases}$$

V3: $V(V^r(c[X]^l)) = V^{r+1}(c[X]^l))$,
V4: $V(dV^s(e[X]^m)) = 2dV^{s+1}(e[X]^m))$.

The action of F is given by:

F1: $F(a[X]^j) = F(a)[X]^{2j}$,
F2: $F(b[X]^{k-1}d[X]) = F(b)[X]^{2k-1}d[X]$,
F3: $F(V^r(c[X]^l)) = 2V^{r-1}(c[X]^l)$,
F4: $F(dV^s(e[X]^m)) = dV^{s-1}(e[X]^m) + V^{s-1}(\iota(e)[X]^m$.

The action of d is given by:

d1: $d(a[X]^j) = d(a)[X]^j + (-1)^{|a|}ja[X]^{j-1}d[X]$,
d2: $d(b[X]^{k-1}d[X]) = d(b)[X]^{k-1}d[X] + k\iota(b)[X]^{k-1}d[X]$,
d3: $d(V^r(c[X]^l) = dV^r(c[X]^l$,
d4: $d(dV^s(e[X]^m)) = dV^s(\iota(e)[X]^m)$.

On morphisms the functor P is defined in the obvious way: if $\theta : E^*_\bullet \to F^*_\bullet$ is a morphisms of Witt complexes over A, then $P(\theta) : P(E^*_\bullet) \to P(F^*_\bullet)$ is defined on elements of type 1 by the formula $P(\theta)(a[X]^i) = \theta(a)[X]^i$ and similarly for elements of the other three types.

Theorem 4.3. *The functor $P : \mathcal{W}_A \to \mathcal{W}_{A[X]}$ is well defined and is a left adjoint of the forgetful functor $\mathcal{W}_{A[X]} \to \mathcal{W}_A$.*

Proof. The fact that the functor P is well defined means that for any Witt complex E over the ring A, the complex $P(E)$ is indeed a Witt complex over $A[X]$. We need to prove that the six conditions in the definition of a Witt complex are satisfied. Only two of these conditions and relations are hard to verify, the associativity and the relation $Fd\lambda([f]_n) = \lambda([f]_{n-1})d\lambda([f]_{n-1}]),$ for all $f \in A[X]$. The associativity requires a straightforward verification, that we will do in an appendix.

We will prove the relation $Fd\lambda([f]_n) = \lambda([f]_{n-1})d\lambda([f]_{n-1}]),$ for all $f \in A[X]$ using induction by the level. For the level $n = 1$ the identity is trivial, as both sides are equal to zero. Assume we know that the identity is true for the level $n-1$. We notice that the relation is easily verified for monomials, that is elements of the form $f = aX^m \in A[X]$.

Lemma 4.4. *The relation*

$$Fd\lambda([aX^m]_n) = \lambda([aX^m]_{n-1})d\lambda([aX^m]_{n-1})$$

holds for all $aX^m \in A[X]$.

Proof. Because there is no danger of confusion we will drop λ and the subscript index indicating the level.

$$\begin{aligned} Fd[aX^m] &= Fd([a][X]^m) = F((d[a])[X]^m + [a]d[X]^m) \\ &= F(d[a])F([X]^m) + F([a])F(d[X]^m) \\ &= ([a]d[a])[X]^{2m} + m[a]^2F([X]^{m-1}d[X]) \end{aligned}$$

and using the formula F2:

$$\begin{aligned} Fd[aX^m] &= [a][X]^{2m}d[a] + m[a]^2[X]^{2m-1}d[X] \\ &= [a][X]^m([X]^m d[a] + [a]d[X]^m) = [aX^m]d[aX^m], \end{aligned}$$

which is what we wanted to prove. □

The relation follows for arbitrary polynomials from the additivity result 3.6. With this we proved that $P : \mathcal{W}_A \to \mathcal{W}_{A[X]}$ is well defined. To prove that it is the left adjoint of the forgetful functor $U : \mathcal{W}_{A[X]} \to \mathcal{W}_A$ we need to show that:

$$\mathrm{Hom}_{\mathcal{W}_{A[X]}}(P(E), E') \cong \mathrm{Hom}_{\mathcal{W}_A}(E, U(E')).$$

The morphism from left to right takes a map $f : P(E) \to E'$ to its restriction to $E \rightarrowtail U(P(E))$. The morphism from right to left takes $g : E \to U(E')$ to its unique extension $\tilde{g} : P(E) \to E'$ defined such that $g([X]_n) = \lambda'([X]_n)$. The two morphisms are inverse to each other. □

4.3. The 2-typical de Rham-Witt complex of the log-ring $\mathbb{Z}_{(2)}$ with the canonical log-structure. In this section we define the notion of a 2-typical de Rham-Witt complex associated to a log-ring and we compute this complex for the ring $\mathbb{Z}_{(2)}$ with the canonical log-structure. We first recall the notions of log-rings and of differentials with log-structures. The standard reference is [8].

Definition 4.5. *A log-ring is a ring R together with a map of monoids*

$$\alpha : M \to R,$$

where R is considered a monoid under the multiplication. We will denote this log-ring by (R, M).

The map α itself is called a "pre-log structure".

Definition 4.6. *A derivation of a log-ring (R, M) into an R-module E is a pair of maps*

$$(D, D\log) : (R, M) \to E,$$

where $D : R \to E$ is a derivation and $D\log : M \to E$ is a map of monoids such that for all $a \in M$,

$$\alpha(a) D\log(a) = D\alpha(a).$$

There is a universal example of a derivation of a log-ring (R, M) given by the R-module

$$\Omega^1_{(R,M)} = (\Omega^1_R \oplus (R \otimes_{\mathbb{Z}} M^{\mathrm{gp}})) / < d\alpha(a) - \alpha(a) \otimes a >,$$

where M^{gp} is the group completion of the monoid M. The structure maps are :

$$d : R \to \Omega^1_{(R,M)}, \quad da = da \oplus 0,$$
$$d\log : M \to \Omega^1_{(R,M)}, \quad d\log\, a = 0 \oplus (1 \otimes a).$$

Definition 4.7. *A log-differential graded ring (E^*, M) consists of a differential graded ring E^*, a pre-log structure $\alpha : M \to E^0$, and a derivation $(D, D\log) : (E^0, M) \to E^1$ such that D is equal to the differential $d : E^0 \to E^1$ and such that $d \circ D\log = 0$.*

The universal example of an anti-symmetric log-differential graded ring is:

$$\tilde{\Omega}^*_{(R,M)} = \tilde{\Lambda}^*_R(\Omega^1_{(R,M)}).$$

Here $\tilde{\Lambda}^*_R(N) = T_R(N)/\langle m \otimes n + n \otimes m \mid m, n \in N\rangle$ is the universal anti-symmetric graded R-algebra generated by the R-module N.
If (R, M) is a log-ring, then for each $n \in \mathbb{N}$ the ring of length-n Witt vectors, $W_n(R)$ over R becomes part of the data that gives a log-ring $(W_n(R), M)$: the map of monoids $M \to W_n(R)$ is just the composition of the map $\alpha : M \to R$ and the Teichmüller map $[-]_n : R \to W_n(R)$.

Definition 4.8. *A Witt complex $(E^*_\bullet, M_E)$ over a log-ring (R, M) is a Witt complex $E^*_\bullet$ over R together with pre-log structures $\alpha_n : M_E \to E^0_n$, an extension of $\lambda : W_\bullet(R) \to E^0_\bullet$ to a strict map of pro-log-rings $\lambda : (W_\bullet(R), M) \to (E^0_\bullet, M_E)$,, and a derivation $(E^0_\bullet, M_E) \to E^1_\bullet$ such that:*

(i) *$d \circ d\log[a]_n = 0$, for all $a \in M$,*
(ii) *$F\, d\log[a]_n = d\log[a]_{n-1}$, for all $a \in M$.*

Proposition 4.9. *The category of (2-typical) Witt complexes over a log-ring* (R, M) *has an initial object* $W_\bullet\Omega^*_{(R,M)}$, *called the (2-typical) de Rham-Witt complex of* (R, M).

Proof. The proof is an application of the Freyd's adjoint functor theorem, entirely similar to the proof of Theorem 3.1 that asserts the existence of an initial object in the category of 2-typical Witt complexes over a ring. $\square$

Remark: We note that if M is the trivial monoid, then the 2-typical de Rham-Witt complex associated to (R, M) is the 2-typical de Rham-Witt complex associated to R, so the notion of a 2-typical de Rham-Witt complex over a log-ring is a generalization of the notion of a 2-typical de Rham-Witt complex.

In this section we will describe the 2-typical de Rham-Witt complex of $(\mathbb{Z}_{(2)}, M)$, where $M = \mathbb{Q}^* \cap \mathbb{Z}_{(2)} \hookrightarrow \mathbb{Z}_{(2)}$ is the canonical log-structure. The strategy is the same as in the previous calculations of de Rham-Witt complexes: we find a candidate $G^*_\bullet$ described explicitely by generators and relations and by formulas for the product and the actions of the various operators, and we prove that this candidate is isomorphic to $W_\bullet\Omega^*_{(R,M)}$.
In degree zero the de Rham-Witt complex is again the Witt vectors of the ring R, this following from a proof similar to the proof of Proposition 3.4. In degree one, the only new generator that we have in the de Rham-Witt complex of $(\mathbb{Z}_{(2)}, M)$, which is not in the de Rham-Witt complex of $\mathbb{Z}_{(2)}$ is $d\log[2]$. The product formulas are the same for the elements that already existed in the de Rham-Witt complex of $\mathbb{Z}_{(2)}$, so the only product formulas that we have to derive are $V^i(1)d\log[2]$.

Proposition 4.10. *i)The element* $d\log[2]_n \in W_n\Omega^1_{(\mathbb{Z}_{(2)},M)}$ *is annihilated by* 2^n.
ii) $V[1]_{n-1}d\log[2]_n \equiv 2d\log[2]_n \pmod{dV(W_{n-1}\Omega^0_{(\mathbb{Z}_{(2)},M)})}$.

Proof. The proof of both assertions is by induction. i)The case $n = 1$: $2d\log[2]_1 = d(2) = 0$. Assuming $2^i d\log[2]_i = 0$ for all $i \leq n$ we will prove that $2^{n+1}d\log[2]_{n+1} = 0$. We use the formula $[2]_{n+1}d\log[2]_{n+1} = d[2]_{n+1}$ and the Corollary 2.4 which says that

$$[2]_{n+1} = \sum_{i=0}^{n} c_i V^i(1),$$

where $c_i = 2^{-i}(2^{2^i} - 2^{2^{i-1}})$.
We have:

$$\sum_{i=0}^{n} c_i V^i(1)d\log[2]_{n+1} = \sum_{i=1}^{n} c_i dV^i(1)$$

We use that $V^i(1)d\log[2]_{n+1} = V^i(F^i(d\log[2]_{n+1})) = V^i(d\log[2]_{n+1-i})$:

$$2d\log[2]_{n+1} = \sum_{i=1}^{n}(-V^i(d\log[2]_{n+1-i}) + c_i dV^i(1)).$$

Now if we multiply this relation by 2^n we obtain:

$$2^{n+1}d\log[2]_{n+1} = \sum_{i=1}^{n}(-V^i(2^{n+1}d\log[2]_{n+1-i}) + 2^{n+1}c_i dV^i(1)),$$

which is zero by the induction hypothesis and the fact that $2^i dV^i(1) = 0$.
ii) For $n = 1$ the congruence is trivial as both members are zero. Assuming that the congruence holds for n, we want to prove that it holds for $n+1$. First, by an easy induction, we see that:

$$V^i[1]_{n-i}d\log[2]_n \equiv 2^i d\log[2]_n \pmod{dV}$$

Then we use the formula $[2]_{n+1}d\log[2]_{n+1} = d[2]_{n+1}$ combined with Corollary 2.4:

$$2d\log[2]_{n+1} + \sum_{i=1}^{n} c_i V^i[1]n+1-id\log[2]_{n+1} = \sum_{i=1}^{n} c_i dV^i[1]_{n+1-i} \equiv 0 (\bmod\ dV).$$

This gives:

$$2d\log[2]_{n+1} + \sum_{i=1}^{n} V(c_i V^{i-1}[1]n-(i-1)d\log[2]_n) \equiv 0 \pmod{dV},$$

or, using the formula described above for $V^{i-1}[1]n-(i-1)d\log[2]_n$:

$$2d\log[2]_{n+1} + V(\sum_{i=1}^{n} c_i 2^{i-1}d\log[2]_n) \equiv 0 \pmod{dV},$$

$$2d\log[2]_{n+1} + V((2^{2^{n-1}} - 1)d\log[2]_n) \equiv 0 \pmod{dV}.$$

Since $2^{2^{n-1}}$ is always divisible by 2^n, which annihilates $d\log[2]_n$, we get the desired result.

$\square$

The second part of this Proposition together with Proposition 2.3, which describes a basis of $W_n(\mathbb{Z})$, tell us that the product formula we are trying to derive is of the form:

$$V[1]_{n-1}d\log[2]_n = 2d\log[2]_n + a_1 dV[1]_{n-1} + a_2 dV^2[1]_{n-2} + \cdots + a_{n-1}dV^{n-1}[1]_1.$$

We note that the coefficients a_i, don't depend on n, as we can apply R to the relation in level $n+1$ to obtain the relation in level n.

Lemma 4.11. *Assuming the previous product formula, the following formulas hold:*

$$V^i(1)d\log_n(2) = 2^i d\log_n(2) + 2^{i-1}(a_1 + \cdots + a_i)dV^i(1) + \cdots$$
$$+ 2^{i-1}(a_{n-i} + \cdots + a_{n-1})dV^{n-1}(1).$$

Proof. The proof is by induction. The case $i = 1$ is obvious. We prove that if the formula is true for i then it must be true for $i + 1$.

$$\begin{aligned}
V^{i+1}(1)d\log_n(2) &= V(V^i(1)d\log_{n-1}(2)) \\
&= V(2^i d\log_{n-1}(2) + 2^{i-1}(a_1 + \cdots + a_i)dV^i(1) + \cdots \\
&\quad \cdots + 2^{i-1}(a_{n-i-1} + \cdots + a_{n-2})dV^{n-2}(1)) \\
&= 2^i V(1)d\log_n(2) + 2^{i-1}(a_1 + \cdots + a_i)VdV^i(1) + \\
&\quad \cdots + 2^{i-1}(a_{n-i-1} + \cdots + a_{n-2})VdV^{n-2}(1) \\
= 2^{i+1}d\log_n(2) &+ 2^i a_1 dV(1) + \cdots + 2^i a_i dV^i(1) + 2^i a_{i+1}dV^{i+1}(1) + \cdots \\
&\cdots + 2^i a_{n-1}dV^{n-1}(1) + 2^i(a_1 + \cdots + a_i)dV^{i+1}(1) + \cdots \\
&\cdots + 2^i(a_{n-i-1} + \cdots + a_{n-2})dV^{n-1}(1) \\
= 2^{i+1}d\log_n(2)&2^i(a_1 + \cdots + a_{i+1})dV^{i+1}(1) + \cdots \\
&\cdots + 2^i(a_{n-i-1} + \cdots + a_{n-1})dV^{n-1}(1).
\end{aligned}$$

We used the fact that $2^i dV(1) = \cdots = 2^i dV^i(1) = 0$, which follows from $2dV = Vd$ and $d(1) = 0$. $\square$

We will now compute the coefficients a_i. We start with the relation

$$[2]_n d\log[2]_n = d[2]_n$$

This gives:

$$\sum_{i=0}^{n-1} c_i(V^i(1)d\log_n(2)) = \sum_{i=1}^{n-1} c_i dV^i(1),$$

and we use the formula that we just derived for $V^i(1)d\log_n(2)$:

$$c_0 d\log_n(2) + \sum_{i=0}^{n-1} c_i(2^i d\log_n(2) + \sum_{j=i}^{n-1}(\sum_{k=j-i+1}^{j} a_k)dV^j(1)) = \sum_{i=1}^{n-1} c_i dV^i(1).$$

We regroup the sums and we obtain:

$$(\sum_{i=0}^{n-1} 2^i c_i)d\log_n(2) + \sum_{j=1}^{n-1}\sum_{i=1}^{j}(2^i c_i \sum_{k=j-i+1}^{j} a_k)dV^j(1) = \sum_{j=1}^{n-1} c_j dV^j(1).$$

The first term in the left hand side member is zero because $\sum_{i=0}^{n-1} 2^i c_i = 2^{2^{n-1}}$ and $d\log_n(2)$ is annihilated by 2^n. We equate the coefficients of $dV^j(1)$ modulo 2^j and obtain:

$$\sum_{i=1}^{j}(2^i c_i \sum_{k=j-i+1}^{j} a_k) \equiv c_j \pmod{2^j},$$

or:

$$\sum_{k=1}^{j}(\sum_{i=j-k+1}^{j} 2^{i-1}c_i)a_k \equiv c_j \pmod{2^j}.$$

Let us call $B_{jk} = \sum_{i=j-k+1}^{j} 2^{i-1}c_i$, if $j \geq k$, and $B_{jk} = 0$, if$j < k$. We obtain therefore a system of equations:

$$\sum_{k=1}^{j} B_{jk}a_k \equiv c_j \pmod{2^j}.$$

We need to make a comment about this system. A priori the unknowns a_k are in different rings, namely $a_k \in \mathbb{Z}/2^k\mathbb{Z}$. So the system as it stands doesn't really make sense. However we can think of $2^{j-k}a_k$ as an element of $\mathbb{Z}/2^j\mathbb{Z}$, and we notice that B_{jk} is divisible by 2^{j-k}, because $B_{jk} = \sum_{i=j-k+1}^{j} 2^{i-1}c_i = 2^{2^j-1} - 2^{2^{j-k}-1}$, if $j \geq k$, and $B_{jk} = 0$, if$j < k$. To solve this system, we lift it to a system over the ring of integers $\mathbb{Z}$, we solve that system, and then take classes of congruence modulo the corresponding power of 2.
We observe that $B_{jk} = 2^{2^j-1} - 2^{2^{j-k}-1} \equiv -2^{2^{j-k}-1} \pmod{2^j}$, and that $c_1 = 1$, $c_2 \equiv -1 \pmod{2^2}$, $c_3 \equiv -2 \pmod{2^3}$, and $c_i \equiv 0 \pmod{2^i}$, for $i > 3$. We find thus convenient to lift the previous system of equation to the following system over $\mathbb{Z}$:

$$\sum_{k=1}^{j} b_{jk}a_k = c'_j,$$

with $b_{jk} = -2^{2^{j-k}-1}$ for $j \geq k$, $b_{jk} = 0$ for $j < k$, $c'_1 = 1$, $c'_2 = -1$, $c'_3 = -2$, and $c_i = 0$, for $i > 3$.
The matrix of the system is lower triangular and it has only -1 on the diagonal. We can invert it using, for example, Gauss-Jordan's method. The inverse matrix F is also lower triangular and it has entries:

$$f_{ij} = \begin{cases} 0, & \text{if } i < j, \\ -1, & \text{if } i = j, \\ \sum_{i=i_0>i_1>\cdots>i_s=j} b_{i_0i_1}b_{i_1i_2}\cdots b_{i_{s-1}i_s}, & \text{if } i > j. \end{cases}$$

A direct computation shows that $b_{21} = b_{32} = b_{43} = -2$, $b_{31} = b_{42} = -2^3$, and using this that $f_{21} = f_{32} = f_{43} = 2$ and $f_{31} = f_{42} = -2^2$. These entries of the matrix F are all we need to compute the first three coeficients in the product formula. We obtain: $a_1 \equiv 1 \pmod 2, a_2 \equiv -1 \pmod 4, a_3 \equiv 4 \pmod 8$. We will prove that all the other coefficients are zero.
For all $i \geq 3$ we have:

$$a_i \equiv f_{i1}c'_1 + f_{i2}c'_2 + f_{i3}c'_3 = f_{i1} - f_{i2} - 2f_{i3} \pmod{2^i}$$

We remark first that $v_2(b_{ij}) \geq i-j$, with equality if and only if $i = j+1$. Using this observation we see that all the terms that add up to give f_{i1} are divisible by 2^{i-1}, and the only one that is not divisible by 2^i is $b_{ii-1}b_{i-1i-2}\cdots b_{21} = (-1)^{i-1}2^{i-1}$. Therefore $f_{i1} \equiv (-1)^{i-1}2^{i-1} \pmod{2^i}$. Similarly, f_{i2} is divisible by 2^{i-2} and the terms in the sum that makes up f_{i2} that are not divisible by 2^i are of two forms:

– one product $b_{ii-1}b_{i-1i-2}\cdots b_{32} = (-1)^{i-2}2^{i-2}$,

– $(i-3)$ products of the form $b_{ii-1}b_{i-1i-2}\cdots b_{k+1k}b_{kk-2}b_{k-2k-3}\cdots b_{32} = (-1)^{i-3}2^{i-1}$.

We obtain $f_{i2} \equiv (-1)^{i-2}2^{i-2} + (i-3)(-1)^{i-3}2^{i-1} \pmod{2^i}$.
We treat f_{i3} in the same way:

$$\begin{aligned}2f_{i3} &\equiv 2(b_{ii-1}b_{i-1i-2}\cdots b_{32} + \sum_k b_{ii-1}b_{i-1i-2}\cdots b_{k+1k}b_{kk-2}b_{k-2k-3}\cdots b_{32}\\ &= 2[(-1)^{i-3}2^{i-3} + (i-4)2^{i-2}] \pmod{2^i}\end{aligned}$$

With these formulas we can compute a_i for $i > 3$:

$$\begin{aligned}a_i &= f_{i1} - f_{i2} - 2f_{i3}\\ &\equiv (-1)^{i-1}2^{i-1} - (-1)^{i-2}2^{i-2}\\ &\qquad - (i-3)(-1)^{i-3}2^{i-1} - 2[(-1)^{i-3}2^{i-3} + (i-4)2^{i-2}]\\ &\equiv 0 \pmod{2^i}\end{aligned}$$

We have therefore proved:

Lemma 4.12. *The following product formula holds for all n:*

$$V[1]_{n-1}d\log[2]_n = 2d\log[2]_n + dV[1]_{n-1} - dV^2[1]_{n-2} + 4dV^3[1]_{n-3}.$$

Now we can state the structure theorem for the 2-typical de Rham-Witt complex of $(\mathbb{Z}_{(2)}, M)$, where $M = \mathbb{Z}^*_{(2)}$.

Theorem 4.13. *The structure of $W_\bullet\Omega^*_{(\mathbb{Z}_{(2)},M)}$ is:*

(i) *As abelian groups*

$$W_n\Omega^0_{(\mathbb{Z}_{(2)},M)} = \bigoplus_{i=0}^{n-1} \mathbb{Z}_{(2)} \cdot V^i(1), \tag{23}$$

$$W_n\Omega^1_{(\mathbb{Z}_{(2)},M)} = \mathbb{Z}/2^n\mathbb{Z}\, d\log[2]_n \oplus \bigoplus_{i=1}^{n-1} \mathbb{Z}/2^i\mathbb{Z} \cdot dV^i(1), \tag{24}$$

$$W_n\Omega^i_{(\mathbb{Z}_{(2)},M)} = 0, \textit{for all } i \geq 2. \tag{25}$$

The product relations and the actions of the various operators are the ones from Theorem 4.1, in addition to which we also have:

(ii) *the product formulas:*

$$V[1]_{n-1}d\log[2]_n = 2d\log[2]_n + dV[1]_{n-1} - dV^2[1]_{n-2} + 4dV^3[1]_{n-3},$$

$$V^i[1]_{n-i}d\log[2]_n = 2^i d\log[2]_n - 2^{i-1}dV^{i+1}[1]_{n-i-1} + 2^{i+1}dV^{i+2}[1]_{n-i-2},$$

(iii) *the action of the operator V:*

$$V(d\log[2]_n) = 2d\log[2]_{n+1} + dV[1]_n - dV^2[1]_{n-1} + 4dV^3[1]_{n-2},$$

(iv) *the action of the operator F:*

$$F(d\log[2]_n) = d\log[2]_{n-1}.$$

Proof. The proof is similar to the proof of Theorem 4.1, which describes the structure of the 2-typical de Rham-Witt complex of the integers. More precisely we define the pro-graded ring $G^*_\bullet$ to be:

$$G^0_n = \bigoplus_{i=0}^{n-1} \mathbb{Z}_{(2)} \cdot V^i(1), \tag{26}$$

$$G^1_n = \mathbb{Z}/2^n\mathbb{Z}\, d\log[2]_n \oplus \bigoplus_{i=1}^{n-1} \mathbb{Z}/2^i\mathbb{Z} \cdot dV^i(1), \tag{27}$$

$$G^i_n = 0, \text{for all } i \geq 2, \tag{28}$$

with the product rule and the action of the operators as in the theorem. We want to prove that $W_\bullet\Omega^*_{(\mathbb{Z}_{(2)},M)} \cong G^*_\bullet$. We will show that we have morphisms

$$\phi : W_\bullet\Omega^*_{(\mathbb{Z}_{(2)},M)} \to G^*_\bullet,$$
$$\psi : G^*_\bullet \to W_\bullet\Omega^*_{(\mathbb{Z}_{(2)},M)},$$

and that they are inverse to each other.
The existence (and uniqueness) of the morphism ϕ follows from the fact that $G^*_\bullet$ is a 2-typical Witt complex. The definition of ψ is forced by the requirements that it is a morphism of 2-typical Witt complexes: $[1]_n \mapsto [1]_n$ (since ψ is a morphism of rings), $V^i[1]_{n-i} \mapsto V^i[1]_{n-i}$, $dV^i[1]_{n-i} \mapsto dV^i[1]_{n-i}$ (ψ commutes with V and d), $[2]_n \mapsto [2]_n$ (because $[2]_n = \sum_{i=0}^{n-1} c_i V^i[1]_{n-i}$ by Prop 2.4 and ψ is additive), $d\log[2]_n \mapsto d\log[2]_n$ (because ψ commutes with $d\log$).
In order to see that ψ is well defined we need to show that $W_n\Omega^i_{(\mathbb{Z}_{(2)},M)} = 0$ for $i \geq 2$. This is proven by induction on n. The first step of the iduction, that $W_1\Omega^i_{(\mathbb{Z}_{(2)},M)} = 0$ follows from the fact that $\Omega^i_{(\mathbb{Z}_{(2)},M)} = 0$ for all $i \geq 2$ (which follows from $d \circ d\log = 0$) and the fact that $\Omega^i_{(\mathbb{Z}_{(2)},M)} \to W_1\Omega^i_{(\mathbb{Z}_{(2)},M)}$ is surjective.
Assuming that $W_{n-1}\Omega^i_{(\mathbb{Z}_{(2)},M)} = 0$ for $i \geq 2$, we want to prove that $W_n\Omega^i_{(\mathbb{Z}_{(2)},M)} = 0$ for $i \geq 2$. We use the standard filtration:

$$\operatorname{Fil}^s W_n\Omega^i_{(\mathbb{Z}_{(2)},M)} = V^s W_{n-i}\Omega^i_{(\mathbb{Z}_{(2)},M)} + dV^s W_n\Omega^{i-1}_{(\mathbb{Z}_{(2)},M)}.$$

The sequence:

$$0 \to \operatorname{Fil}^{n-1} W_n\Omega^i_{(\mathbb{Z}_{(2)},M)} \to W_n\Omega^i_{(\mathbb{Z}_{(2)},M)} \to W_{n-1}\Omega^i_{(\mathbb{Z}_{(2)},M)} \to 0$$

is exact by exactly the same argument used to prove Lemma 3.5. For $i \geq 2$ the last term in this short exact sequence is zero by the induction hypothesis. The first term is:

$$\begin{aligned}\operatorname{Fil}^{n-1} W_n\Omega^i_{(\mathbb{Z}_{(2)},M)} &= V^{n-1}W_{n-i}\Omega^i_{(\mathbb{Z}_{(2)},M)} + dV^{n-1}W_n\Omega^{i-1}_{(\mathbb{Z}_{(2)},M)} \\ &= V^{n-1}(0) + dV^{n-1}(\mathbb{Z}/2\mathbb{Z}\, d\log[2]_1).\end{aligned}$$

This is zero if $dV^{n-1}(d\log[2]_1) = 0$. We have:

$$\begin{aligned} dV^{n-1}(d\log[2]_1) &= d(V^{n-1}([1]_1)d\log[2]_n) \\ &= d(2^{n-1}d\log[2]_n + dV(x)) \\ &= 2^{n-1}d \circ d\log[2]_n + ddV(x) \\ &= ddV(x), \end{aligned}$$

where $x \in W_{n-1}\Omega^{i-2}_{(\mathbb{Z}_{(2)},M)}$. Too see that $ddV(x) = 0$ we use the following trick: $W_\bullet\Omega^*_{(\mathbb{Z}_{(2)},M)}$ is a Witt complex over the log-ring $(\mathbb{Z}_{(2)}, M)$, so in particular it is a Witt complex over the ring $\mathbb{Z}_{(2)}$, and as such it is the target of a unique homomorphism from the de Rham-Witt complex $W_\bullet\Omega^*_{\mathbb{Z}_{(2)}}$. The element $ddV(x)$ is in the image of this homomorphism, but $W_\bullet\Omega^i_{\mathbb{Z}_{(2)}} = 0$ for $i \geq 2$, therefore $ddV(x) = 0$. This finishes the proof that $W_n\Omega^i_{(\mathbb{Z}_{(2)},M)} = 0$ if $i \geq 2$, and thus ψ is well defined.
Too see that ϕ and ψ are inverse to each other we check that $\phi \circ \psi = 1$ and $\psi \circ \phi = 1$. The first follows from the fact that $\phi \circ \psi$ is a morphism of Witt complexes, and therefore $[1]_n \mapsto [1]_n$, $V^i[1]_{n-i} \mapsto V^i[1]_{n-i}$, $dV^i[1]_{n-i} \mapsto dV^i[1]_{n-i}$, $[2]_n \mapsto [2]_n$, and $d\log[2]_n \mapsto d\log[2]_n$, and the second from the fact that $\psi \circ \phi$ is an endomorphism of an initial object in a category. □

5. Appendix A: The p-typical de Rham-Witt complex of $\mathbb{Z}_{(p)}$ with the canonical log-structure, p odd

In this appendix we give the structure of the p-typical de Rham-Witt complex of the log-ring $(\mathbb{Z}_{(p)}, M)$, for p odd. Here $M = \mathbb{Q}^* \cap \mathbb{Z}_{(p)} \hookrightarrow \mathbb{Z}_{(p)}$ is the canonical log-structure on $\mathbb{Z}_{(p)}$. The computations are exactly the same as for the case $p = 2$, but the results are a little different. We first recall the structure of the p-typical de Rham-Witt complex of $\mathbb{Z}_{(p)}$ from Example 1.2.4 of [2].

PROPOSITION 5.1. *The structure of $W_\bullet\Omega^*_{\mathbb{Z}_{(p)}}$ for p odd is:*

(i) *As abelian groups:*

$$W_n\Omega^0_{\mathbb{Z}_{(p)}} = \bigoplus_{i=0}^{n-1} \mathbb{Z}_{(p)} \cdot V^i(1), \tag{29}$$

$$W_n\Omega^1_{\mathbb{Z}_{(p)}} = \bigoplus_{i=1}^{n-1} \mathbb{Z}/p^i\mathbb{Z} \cdot dV^i(1), \tag{30}$$

$$W_n\Omega^i_{\mathbb{Z}(p)} = 0, \quad \textit{for } i \geq 2. \tag{31}$$

(ii) *The product is given by:*

$$V^i(1)V^j(1) = p^iV^j(1), \textit{if } i \leq j, \tag{32}$$

$$V^i(1)dV^j(1) = \begin{cases} p^i dV^j(1), & \textit{if } i < j, \\ 0, & \textit{if } i \geq j. \end{cases} \tag{33}$$

(ii) *The action of the operators F and V is given by:*

$$FV^i(1) = pV^{i-1}(1), \tag{34}$$

$$FdV^i(1) = dV^{i-1}(1), \tag{35}$$

$$V(V^i(1)) = V^{i+1}(1), \tag{36}$$

$$V(dV^i(1)) = pdV^{i+1}(1). \tag{37}$$

The structure of $W_\bullet\Omega^*_{(\mathbb{Z}_{(p)},M)}$ is different for $p = 3$ and $p \geq 5$.

Theorem 5.2. *The structure of $W_\bullet\Omega^*_{(\mathbb{Z}_{(p)},M)}$ with p odd is:*

(i) *As additive groups:*

$$W_n\Omega^0_{(\mathbb{Z}_{(p)},M)} = W_n\Omega^0_{\mathbb{Z}_{(p)}} \cong W_n(\mathbb{Z}_{(p)}), \tag{38}$$

$$W_n\Omega^1_{(\mathbb{Z}_{(p)},M)} = W_n\Omega^1_{\mathbb{Z}_{(p)}} \oplus \mathbb{Z}/p^n\mathbb{Z}d\log[p]_n, \tag{39}$$

$$W_n\Omega^i_{(\mathbb{Z}_{(p)},M)} = 0, \textit{ for all } i \geq 2. \tag{40}$$

(ii) *The product is given by the formulas in the previous theorem and the following formula that involves $d\log[p]$:*

$$V^i[1]_{n-i}d\log[p]_n = \begin{cases} 3^i d\log[3]_n + 3^{i-1}dV^i[1]_{n-i} + 3^i dV^{i+1}[1]_{n-i-1}, & \textit{if } p = 3, \\ p^i \log[p]_n + p^{i-1}dV^i[1]_{n-i}, & \textit{if } p \geq 5. \end{cases}$$

(ii) *The action of F and V on $d\log[p]_n$ is:*

$$F(d\log[p]_n) = d\log[p]_{n-1}, \tag{41}$$

$$V(d\log[p]_n) = \begin{cases} 3d\log[3]_{n+1} + dV[1]_n + 3dV[1]_{n-1}, & \textit{if } p = 3, \\ p\,d\log[p]_{n+1} + dV[1]_n, & \textit{if } p \geq 5. \end{cases} \tag{42}$$

The proof of this theorem is entirely similar to the proof of the structure theorem for $W_\bullet\Omega^*_{(Z_{(2)},M)}$.

6. Appendix B: Associativity

In this appendix we discuss the associativity of the multiplication rule defined in Section 4. We recall that the functor $P : \mathcal{W}_A \to \mathcal{W}_{A[X]}$ is defined on objects as follows: for a a Witt complex $E^*_\bullet \in \mathcal{W}_A$, $P(E)^q_n$ consists of formal sums of four types of elements:

- Type 1: elements of the form $a[X]^j$, where $a \in E^q_n$,
- Type 2: elements of the form $b[X]^{k-1}d[X]$, where $b \in E^{q-1}_n$,
- Type 3: elements of the form $V^r(c[X]^l)$, where $r > 0$, $c \in E^q_{n-r}$, and l is odd,
- Type 4: elements of the form $dV^s(e[X]^m)$, where $s > 0$, $e \in E^{q-1}_{n-s}$, and m is odd.

The product is given by ten formulas, from P1.1 to P1.4.

We make now the convention that, for example, A1.3.4 means the statement that says that $(xy)z = x(yz)$, where x is an element of the first type, y an element of the third type, and z an element of the fourth type. In order to prove the associativity one has to check twenty relations like this, from A1.1.1 to A4.4.4.

Since there are three product formulas given in "cases" format, the associativity relations involving these formulas will be a little more tedious to verify. Ten out of the twenty relations that we want to check contain at least a product given in cases format. Out of the remaining ten, five are more or less trivial, namely A1.1.1, A1.1.2, A1.1.3, A1.2.2, and A2.2.2. The five formulas that don't involve products with the cases format are A1.1.4, A1.2.3, A1.2.4, A2.2.3, and A2.2.4. The hardest seems to be the first, even if it doesn't involve the operator ι. We will show how it is derived, and then we will also show A1.2.4, where ι is involved.

Among the ten cases where at least one product is in the cases format, one is almost trivial, A1.3.3. The other nine require all some extensive computations. The most dificult of them are A3.3.4 and A3.4.4.We will show how A3.3.4 is derived.

We start with the relation A1.1.4. Let $x = aX^j$, $y = a'X^{j'}$, and $z = dV^s(eX^m)$. Then:

$$\begin{aligned}(xy)z &= aa'X^{j+j'}dV^s(eX^m)\\ &= (-1)^{|aa'|}\frac{m}{2^s(j+j')+m}dV^s(F^s(aa')eX^{2^s(j+j')+m})\\ -(-1)^{|aa'|}&V^s((F^s(d(aa'))e - \frac{j+j'}{2^s(j+j')+m}d(F^s(aa')e))X^{2^s(j+j')+m}).\end{aligned}$$

On the other hand:

$$\begin{aligned}x(yz) =& aX^j\{(-1)^{|a'|}\frac{m}{2^sj'+m}dV^s(F^s(a')eX^{2^sj'+m})\\ &- (-1)^{|a'|}V^s((F^s(da')e - \frac{j'}{2^sj'+m}d(F^s(a')e))X^{2^sj'+m})\}.\end{aligned}$$

If we denote $E = F^s(a')e$, $M = 2^sj'+m$, and $P = 2^s(j+j')+m$, we obtain:

$$\begin{aligned}x(yz) =& aX^j\{(-1)^{|a'|}\frac{m}{M}dV^s(EX^M) - (-1)^{|a'|}V^s((E - \frac{j'}{M}d(F^s(a')e))X^M)\}\\ =&(-1)^{|a'|}\frac{m}{M}\{(-1)^{|a|}\frac{M}{P}dV^s(F^s(a)EX^P)\\ &- (-1)^{|a|}V^s((F^s(da)E - \frac{j}{P}d(F^s(a)E))X^P)\}\\ &- (-1)^{|a'|}V^s((F^s(ada')e - \frac{j'}{M}F^s(a)d(F^s(a')e))X^P)\\ =&(-1)^{|aa'|}\frac{m}{P}dV^s(F^s(aa')eX^P) + V^s(UX^P),\end{aligned}$$

where U is the expression:

$$\begin{aligned} U =&(-1)^{|aa'|}\frac{m}{M}F^s(da\,a')e+(-1)^{|aa'|}\frac{m}{M}\frac{j}{P}d(F^s(aa')e)\\ &-(-1)^{|a'|}F^s(ada')e+(-1)^{|a'|}F^s(a)d(F^s(a')e)). \end{aligned}$$

Using the fact that d is a derivation and that $dF^s=2^sF^sd$, we obtain:

$$\begin{aligned} U =&-(-1)^{|aa'|}\frac{m}{P}F^s(da\,a')e-(-1)^{|a'|}\frac{m}{P}F^s(ada')e\\ &+\frac{j+j'}{P}F^s(aa')de, \end{aligned}$$

which agrees with the expression we find inside V^s for the elemtent $(xy)z$. We prove now A1.2.4. Let $x=aX^j$, $y=bX^{k-1}dX$, and $z=dV^s(eX^m)$. We have:

$$\begin{aligned} (xy)z =&(abX^{j+k-1}dX)dV^s(eX^m)\\ =&-(-1)^{|ab|}\frac{1}{2^s(j+k)+m}V^s(F^s(d(ab)+(j+k)\iota(ab)deX^{2^s(j+k)+m}))\\ &+(-1)^{|ab|}\frac{1}{2^s(j+k)+m}dV^s(F^s(ab)deX^{2^s(j+k)+m}). \end{aligned}$$

On the other hand:

$$\begin{aligned} x(yz) =&aX^j\{-(-1)^{|b|}\frac{1}{2^sk+m}V^s(F^s(db+(j+k)\iota(b)deX^{2^sk+m}))\\ &+(-1)^{|b|}\frac{1}{2^sk+m}dV^s(F^s(b)deX^{2^sk+m})\}. \end{aligned}$$

We denote by $M=2^sk+m$, and $P=2^s(j+k)+m$, we obtain:

$$\begin{aligned} x(yz) =&-(-1)^{|b|}\frac{1}{M}V^s(F^s(a\,db+k\iota(ab))deX^P)\\ &+(-1)^{|b|}\frac{1}{M}\{(-1)^{|a|}\frac{M}{P}dV^s(F^s(ab)deX^P)\\ &-(-1)^{|a|}V^s((F^s(da)F^s(b)de-\frac{j}{P}d(F^s(ab)de))X^P)\}\\ =&(-1)^{|ab|}\frac{1}{P}dV^s(F^s(ab)deX^P)+V^s(WX^P). \end{aligned}$$

We compute separately the expression W :

$$\begin{aligned} W =&-(-1)^{|b|}\frac{1}{M}F^s(a\,db)de+kF^s(\iota(ab))de\\ &-(-1)^{|ab|}\frac{1}{M}F^s(da\,b)de+(-1)^{|ab|}\frac{j}{MP}d(F^s(ab)de)\\ =&-(-1)^{|ab|}\frac{1}{M}F^s(d(ab))de+F^s(k\iota(ab))de\\ &+(-1)^{|ab|}\frac{2^sj}{MP}F^s(d(ab))de+\frac{j}{MP}F^s(ab)dde. \end{aligned}$$

The last term in this sum is $\frac{j}{MP}F^s(ab)dde = \frac{j}{MP}F^s(\iota(ab))de$. We make the observation that for any odd number $m \in \mathbb{Z}$ we have $\frac{1}{m}\iota = \iota$, so the last term in the sum becomes simply $jF^s(\iota(ab))de$. Therefore:

$$\begin{aligned} W =& -(-1)^{|ab|}\frac{1}{P}F^s(d(ab))de + F^s((j+k)\iota(ab))de \\ =& -(-1)^{|ab|}\frac{1}{P}F^s(d(ab) + (j+k)\iota(ab))de, \end{aligned}$$

which agrees with the expression we find inside V^s in the product $(xy)z$.
We show now how to derive the associativity relation A3.3.4. Let $x = V^r(cX^l)$, $y = V^{r'}(c'X^{l'})$, and $z = dV^s(eX^m)$. Since the product formulas depend on the ordering of the exponents r, r', and s, it follows that verifying this relation involves checking 13 different cases, from $r < r' < s$ to $s < r' < r$. We will verify the case $s < r' < r$.
We make the notations $C = cF^{r-r'}(c')$, $C' = c'F^{r'-s}((d+\iota)(e))$, $E = c'F^{r'-s}(e)$, $L = 2^{r-r'}l' + l$, $M = 2^{r'-s}m + l'$, and $P = 2^{r-s}m + 2^{r-r'}l' + l$. We have:

$$\begin{aligned} (xy)z =&(V^r(cX^l)V^{r'}(c'X^{l'}))dV^s(eX^m) \\ =&2^{r'}V^r(cF^{r-r'}(c')X^{2^{r-r'}l'+l})dV^s(eX^m) \\ =&2^{r'}V^r(CX^L)dV^s(eX^m) \\ =&2^{r'}\{V^r(CF^{r-s}((d+\iota)(e))X^P) + (-1)^{|cc'|}\frac{2^r m}{P}dV^r(CF^{r-s}(e)X^P) \\ &-(-1)^{|cc'|}\frac{m}{P}V^r(d(CF^{r-s}(e))X^P)\} \\ =&(-1)^{|cc'|}\frac{2^{r+r'}m}{P}dV^r(cF^{r-r'}(c')F^{r-s}(e)X^P) + V(UX^P), \end{aligned}$$

where the expression U is:

$$U = 2^{r'}\{cF^{r-r'}(c')F^{r-s}((d+\iota)(e)) - (-1)^{|cc'|}\frac{m}{P}d(cF^{r-r'}(c')F^{r-s}(e))\}$$

On the other hand:

$$\begin{aligned}
x(yz) =& V^r(cX^l)(V^{r'}(c'X^{l'})dV^s(eX^m))\\
=& V^r(cX^l)\{V^{r'}(c'F^{r'-s}((d+\iota)(e))X^{2^{r'-s}+l'})\\
&+(-1)^{|c'|}\frac{2^{r'}m}{2^{r'-s}m+l'}dV^{r'}(c'F^{r'-s}(e)X^{2^{r'-s}+l'})\\
&-(-1)^{|c'|}\frac{m}{2^{r'-s}m+l'}V^{r'}(d(c'F^{r'-s}(e))X^{2^{r'-s}+l'})\}\\
=& V^r(cX^l)\{V^{r'}(C'X^M)+(-1)^{|c'|}\frac{2^{r'}m}{M}dV^{r'}(EX^M)\\
&-(-1)^{|c'|}\frac{m}{M}V^{r'}(dE\,X^M)\}\\
=& 2^{r'}V^r(cF^{r-r'}(C')X^P)+(-1)^{|c'|}\frac{2^{r'}m}{M}\{V^r(cF^{r-r'}((d+\iota)E)X^P)\\
&+(-1)^{|c|}\frac{2^rM}{P}dV^r(cF^{r-r'}(E)X^P)-(-1)^{|c|}\frac{M}{P}V^r(d(cF^{r-r'}(E))X^P)\}\\
&-(-1)^{|c'|}\frac{m}{M}2^{r'}V^r(cF^{r-r'}(dE)X^P).
\end{aligned}$$

The second and the fifth term in this sum cancel each other, since $2^{r'}\iota = 0$. We have:

$$x(yz) = (-1)^{|cc'|}\frac{2^{r+r'}m}{P}dV^r(cF^{r-r'}(c')F^{r-s}(e)X^P) + V(WX^P),$$

where the expression W is:

$$\begin{aligned}
W =& 2^{r'}\{cF^{r-r'}(c')F^{r-s}(d(e)) + (-1)^{|c'|}\frac{2^{r'}m}{M}cF^{r-r'}(d(c'F^{r'-s}(e)))\\
&-(-1)^{|cc'|}\frac{m}{P}d(cF^{r-r'}(c'F^{r'-s}(e))) - (-1)^{|c'|}\frac{2^{r'}m}{M}cF^{r-r'}(d(c'F^{r'-s}(e))).
\end{aligned}$$

The second and the fourth term cancel, and we see that $W = U$, hence $(xy)z = x(yz)$.

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Andreotti–Mayer Loci and the Schottky Problem

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Abstract. We prove a lower bound for the codimension of the Andreotti-Mayer locus $N_{g,1}$ and show that the lower bound is reached only for the hyperelliptic locus in genus 4 and the Jacobian locus in genus 5. In relation with the intersection of the Andreotti-Mayer loci with the boundary of the moduli space $\mathcal{A}_g$ we study subvarieties of principally polarized abelian varieties (B, Ξ) parametrizing points b such that Ξ and the translate Ξ_b are tangentially degenerate along a variety of a given dimension.

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1. Introduction

The Schottky problem asks for a characterization of Jacobian varieties among all principally polarized abelian varieties. In other words, it asks for a description of the Jacobian locus $\mathcal{J}_g$ in the moduli space $\mathcal{A}_g$ of all principally polarized abelian varieties of given dimension g. In the 1960's Andreotti and Mayer (see [2]) pioneered an approach based on the fact that the Jacobian variety of a non-hyperelliptic (resp. hyperelliptic) curve of genus $g \geq 3$ has a singular locus of dimension $g-4$ (resp. $g-3$). They introduced the loci $N_{g,k}$ of principally polarized abelian varieties (X, Θ_X) of dimension g with a singular locus of Θ_X of dimension $\geq k$ and showed that $\mathcal{J}_g$ (resp. the hyperelliptic locus $\mathcal{H}_g$) is an irreducible component of $N_{g,g-4}$ (resp. $N_{g,g-3}$). However, in general there are more irreducible components of $N_{g,g-4}$ so that the dimension of the singular locus of Θ_X does not suffice to characterize Jacobians or hyperelliptic Jacobians. The locus $N_{g,0}$ of abelian varieties with a singular theta divisor has codimension 1 in $\mathcal{A}_g$ and in a beautiful paper (see [27]) Mumford calculated its class. But in general not much is known about these Andreotti-Mayer loci $N_{g,k}$. In particular, we do not even know their codimension. In this paper we give estimates for the codimension of these loci. These estimates are in general not sharp, but we think that the following conjecture gives the sharp bound.

Conjecture 1.1. *If $1 \leq k \leq g-3$ and if N is an irreducible component of $N_{g,k}$ whose general point corresponds to an abelian variety with endomorphism ring $\mathbb{Z}$ then* $\mathrm{codim}_{\mathcal{A}_g}(N) \geq \binom{k+2}{2}$. *Moreover, equality holds if and only if one of the following happens:*

(i) *$g = k+3$ and $N = \mathcal{H}_g$;*
(ii) *$g = k+4$ and $N = \mathcal{J}_g$.*

We give some evidence for this conjecture by proving the case $k = 1$. In our approach we need to study the behaviour of the Andreotti-Mayer loci at the boundary of the compactified moduli space. A principally polarized $(g-1)$-dimensional abelian variety (B, Ξ) parametrizes semi-abelian varieties that are extensions of B by the multiplicative group $\mathbb{G}_m$. This means that B maps to a part of the boundary of the compactified moduli space $\tilde{\mathcal{A}}_g$ and we can intersect B with the Andreotti-Mayer loci. This motivates the definition of loci $N_k(B, \Xi) \subset B$ for a principally polarized $(g-1)$-dimensional abelian variety (B, Ξ). They are formed by the points b in B such that Ξ and its translate Ξ_b are 'tangentially degenerate' (see Section 11 below) along a subvariety of dimension k. These intrinsically defined subvarieties of an abelian variety are interesting in their own right and deserve further study. The conjecture above then leads to a boundary version that gives a new conjectural answer to the Schottky problem for simple abelian varieties.

Conjecture 1.2. *Let $k \in \mathbb{Z}_{\geq 1}$. Suppose that (B, Ξ) is a simple principally polarized abelian variety of dimension g not contained in $N_{g,i}$ for all $i \geq k$. Then there is an irreducible component Z of $N_k(B, \Xi)$ with* $\mathrm{codim}_B(Z) = k+1$ *if and only if one of the following happens:*

(i) *either $g \geq 2$, $k = g-2$ and B is a hyperelliptic Jacobian,*
(ii) *or $g \geq 3$, $k = g-3$ and B is a Jacobian.*

In our approach we will use a special compactification $\tilde{\mathcal{A}}_g$ of $\mathcal{A}_g$ (see [29, 28, 5]). The points of the boundary $\partial\tilde{\mathcal{A}}_g = \tilde{\mathcal{A}}_g - \mathcal{A}_g$ correspond to suitable compactifications of g-dimensional semi-abelian varieties. We prove Conjecture 1.1 for $k = 1$ by intersecting with the boundary. For higher values of k the intersection with the boundary looks very complicated.

2. The universal theta divisor

Let $\pi : \mathcal{X}_g \to \mathcal{A}_g$ be the universal principally polarized abelian variety of relative dimension g over the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g over $\mathbb{C}$. In this paper we will work with orbifolds and we shall identify $\mathcal{X}_g$ (resp. $\mathcal{A}_g$) with the orbifold $\mathrm{Sp}(2g, \mathbb{Z}) \ltimes \mathbb{Z}^{2g} \backslash \mathbb{H}_g \times \mathbb{C}^g$ (resp. with $\mathrm{Sp}(2g, \mathbb{Z}) \backslash \mathbb{H}_g$), where

$$\mathbb{H}_g = \{(\tau_{ij}) \in \mathrm{Mat}(g \times g, \mathbb{C}) : \tau = \tau^t, \mathrm{Im}(\tau) > 0\}$$

is the usual *Siegel upper-half space* of degree g. The τ_{ij} with $1 \le i \le j \le g$ are coordinates on $\mathbb{H}_g$ and we let $z_1, ..., z_g$ be coordinates on $\mathbb{C}^g$.
The *Riemann theta function* $\vartheta(\tau, z)$, given on $\mathbb{H}_g \times \mathbb{C}^g$ by

$$\vartheta(\tau, z) = \sum_{m \in \mathbb{Z}^g} e^{\pi i[m^t \tau m + 2m^t z]},$$

is a holomorphic function and its zero locus is an effective divisor $\tilde{\Theta}$ on $\mathbb{H}_g \times \mathbb{C}^g$ which descends to a divisor Θ on $\mathcal{X}_g$. If the abelian variety X is a fibre of π, then we let Θ_X be the restriction of Θ to X. Note that since $\theta(\tau, z)$ satisfies $\theta(\tau, -z) = \theta(\tau, z)$, the divisor Θ_X is symmetric, i.e., $\iota^*(\Theta_X) = \Theta_X$, where $\iota = -1_X : X \to X$ is multiplication by -1 on X. The divisor Θ_X defines the line bundle $\mathcal{O}_X(\Theta_X)$, which yields the principal polarization on X. The isomorphism class of the pair (X, Θ_X) represents a point ζ of $\mathcal{A}_g$ and we will write $\zeta = (X, \Theta_X)$. Similarly, it will be convenient to identify a point ξ of Θ with the isomorphism class of a representative triple (X, Θ_X, x), where $\zeta = (X, \Theta_X)$ represents $\pi(\xi) \in \mathcal{A}_g$ and $x \in \Theta_X$.
The tangent space to $\mathcal{X}_g$ at a point ξ, with $\pi(\xi) = \zeta$, will be identified with the tangent space $T_{X,x} \oplus T_{\mathcal{A}_g,\zeta} \cong T_{X,0} \oplus \mathrm{Sym}^2(T_{X,0})$. If $\xi = (X, \Theta_X, x)$ corresponds to the $\mathrm{Sp}(2g, \mathbb{Z}) \ltimes \mathbb{Z}^{2g}$-orbit of a point $(\tau_0, z_0) \in \mathbb{H}_g \times \mathbb{C}^g$, then the tangent space $T_{\mathcal{X}_g,\xi}$ to $\mathcal{X}_g$ at ξ can be identified with the tangent space to $\mathbb{H}_g \times \mathbb{C}^g$ at (τ_0, z_0), which in turn is naturally isomorphic to $\mathbb{C}^{g(g+1)/2+g}$, with coordinates (a_{ij}, b_ℓ) for $1 \le i, j \le g$ and $1 \le \ell \le g$ that satisfy $a_{ij} = a_{ji}$. We thus view the a_{ij}'s as coordinates on the tangent space to $\mathbb{H}_g$ at τ_0 and the b_ℓ's as coordinates on the tangent space to X or its universal cover.
An important remark is that by identifying the tangent space to $\mathcal{A}_g$ at $\zeta = (X, \Theta_X)$ with $\mathrm{Sym}^2(T_{X,0})$, we can view the projectivized tangent space $\mathbb{P}(T_{\mathcal{A}_g,\zeta}) \cong \mathbb{P}(\mathrm{Sym}^2(T_{X,0}))$ as the linear system of all quadrics in the dual of $\mathbb{P}^{g-1} = \mathbb{P}(T_{X,0})$. In particular, the matrix (a_{ij}) can be interpreted as the matrix defining a dual quadric in the space $\mathbb{P}^{g-1}$ with homogeneous coordinates $(b_1 : \ldots : b_g)$. Quite naturally, we will often use $(z_1 : \ldots : z_g)$ for the homogeneous coordinates in $\mathbb{P}^{g-1}$.
Recall that the Riemann theta function ϑ satisfies the *heat equations*

$$\frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j}\vartheta = 2\pi\sqrt{-1}(1 + \delta_{ij})\frac{\partial}{\partial \tau_{ij}}\vartheta$$

for $1 \le i, j \le g$, where δ_{ij} is the Kronecker delta. We shall abbreviate this equation as

$$\partial_i \partial_j \vartheta = 2\pi\sqrt{-1}(1 + \delta_{ij})\partial_{\tau_{ij}}\vartheta,$$

where ∂_j means the partial derivative $\partial/\partial z_j$ and $\partial_{\tau_{ij}}$ the partial derivative $\partial/\partial \tau_{ij}$. One easily checks that also all derivatives of θ verify the heat equations. We refer to [39] for an algebraic interpretation of the heat equations in terms of deformation theory.
If $\xi = (X, \Theta_X, x) \in \Theta$ corresponds to the $\mathrm{Sp}(2g, \mathbb{Z}) \ltimes \mathbb{Z}^{2g}$-orbit of a point (τ_0, z_0), then the Zariski tangent space $T_{\Theta,\xi}$ to Θ at ξ is the subspace of $T_{\mathcal{X}_g,\xi} \simeq \mathbb{C}^{g(g+1)/2+g}$ defined, with the above conventions, by the linear equation

$$\sum_{1\le i\le j\le g}\frac{1}{2\pi\sqrt{-1}(1+\delta_{ij})}a_{ij}\,\partial_i\partial_j\vartheta(\tau_0,z_0)+\sum_{1\le \ell\le g}b_\ell\,\partial_\ell\vartheta(\tau_0,z_0)=0 \tag{1}$$

in the variables (a_{ij}, b_ℓ), $1 \le i, j \le g$, $1 \le \ell \le g$. As an immediate consequence we get the result (see [36], Lemma (1.2)):

Lemma 2.1. *The point $\xi = (X, \Theta_X, x)$ is a singular point of Θ if and only if x is a point of multiplicity at least 3 for Θ_X.*

3. The locus S_g

We begin by defining a suborbifold of Θ supported on the set of points where $\pi_{|\Theta}$ fails to be of maximal rank.

Definition 3.1. The closed suborbifold S_g of Θ is defined on the universal cover $\mathbb{H}_g \times \mathbb{C}^g$ by the $g + 1$ equations

$$\vartheta(\tau,z)=0,\quad \partial_j\vartheta(\tau,z)=0,\quad j=1,\ldots,g. \tag{2}$$

Lemma 2.1 implies that the support of S_g is the union of $\mathrm{Sing}(\Theta)$ and of the set of smooth points of Θ where $\pi_{|\Theta}$ fails to be of maximal rank. Set-theoretically one has

$$S_g = \{(X, \Theta_X, x) \in \Theta : x \in \mathrm{Sing}(\Theta_X)\}$$

and $\mathrm{codim}_{\mathcal{X}_g}(S_g) \le g+1$. It turns out that every irreducible component of S_g has codimension $g+1$ in $\mathcal{X}_g$ (see [8] and an unpublished preprint by Debarre [9]). We will come back to this later in §7 and §8.

With the above identification, the Zariski tangent space to S_g at a given point (X, Θ_X, x) of $\mathcal{X}_g$, corresponding to the $\mathrm{Sp}(2g,\mathbb{Z})$-orbit of a point $(\tau_0, z_0) \in \mathbb{H}_g \times \mathbb{C}^g$, is given by the $g+1$ equations

$$\begin{aligned}\sum_{1\le i\le j\le g} a_{ij}\partial_{\tau_{ij}}\vartheta(\tau_0,z_0)&=0,\\ \sum_{1\le i\le j\le g} a_{ij}\partial_{\tau_{ij}}\partial_k\vartheta(\tau_0,z_0)+\sum_{1\le \ell\le g} b_\ell\partial_\ell\partial_k\vartheta(\tau_0,z_0)&=0,\qquad 1\le k\le g\end{aligned} \tag{3}$$

in the variables (a_{ij}, b_ℓ) with $1 \le i, j, \ell \le g$. We will use the following notation:

- (a) q is the row vector of length $g(g+1)/2$, given by $(\partial_{\tau_{ij}}\theta(\tau_0, z_0))$, with lexicographically ordered entries;
- (b) q_k is the row vector of length $g(g+1)/2$, given by $(\partial_{\tau_{ij}}\partial_k\theta(\tau_0, z_0))$, with lexicographically ordered entries;
- (c) M is the $g \times g$-matrix $(\partial_i\partial_j\vartheta(\tau_0, z_0))_{1\le i,j\le g}$.

Then we can rewrite the equations (3) as

$$a\cdot q^t=0,\quad a\cdot q_k^t+b\cdot M_k^t=0,\ (k=1,\ldots,g), \tag{4}$$

where a is the vector (a_{ij}) of length $g(g+1)/2$, with lexicographically ordered entries, b is a vector in $\mathbb{C}^g$ and M_j the j-th row of the matrix M.

In this setting, the equation (1) for the tangent space to $T_{\Theta,\xi}$ can be written as:

$$a \cdot q^t + b \cdot \partial\vartheta(\tau_0, z_0)^t = 0 \tag{5}$$

where ∂ denotes the gradient.
Suppose now the point $\xi = (X, \Theta_X, x)$ in S_g, corresponding to $(\tau_0, z_0) \in \mathbb{H}_g \times \mathbb{C}^g$ is not a point of $\mathrm{Sing}(\Theta)$. By Lemma 2.1 the matrix M is not zero and therefore we can associate to ξ a quadric Q_ξ in the projective space $\mathbb{P}(T_{X,x}) \simeq \mathbb{P}(T_{X,0}) \simeq \mathbb{P}^{g-1}$, namely the one defined by the equation

$$b \cdot M \cdot b^t = 0.$$

Recall that $b = (b_1, \dots, b_g)$ is a coordinate vector on $T_{X,0}$ and therefore $(b_1 : \dots : b_g)$ are homogeneous coordinates on $\mathbb{P}(T_{X,0})$. We will say that Q_ξ is *indeterminate* if $\xi \in \mathrm{Sing}(\Theta)$.
The vector q naturally lives in $\mathrm{Sym}^2(T_{X,0})^\vee$ and therefore, if q is not zero, the point $[q] \in \mathbb{P}(\mathrm{Sym}^2(T_{X,0})^\vee)$ determines a quadric in $\mathbb{P}^{g-1} = \mathbb{P}(T_{X,0})$. The heat equations imply that this quadric coincides with Q_ξ.
Consider the matrix defining the Zariski tangent space to S_g at a point $\xi = (X, \Theta_X, x)$. We denote by $r := r_\xi$ the corank of the quadric Q_ξ, with the convention that $r_\xi = g$ if $\xi \in \mathrm{Sing}(\Theta)$, i.e., if Q_ξ is indeterminate. If we choose coordinates on $\mathbb{C}^g$ such that the first r basis vectors generate the kernel of q then the shape of the matrix A of the system (3) is

$$A = \begin{pmatrix} q & 0_g \\ q_1 & 0_g \\ & \vdots & \\ q_r & 0_g \\ * & B \end{pmatrix}, \tag{6}$$

where q and q_k are as above and B is a $(g-r) \times g$-matrix with the first r columns equal to zero and the remaining $(g-r) \times (g-r)$ matrix symmetric of maximal rank.
Next, we characterize the smooth points $\xi = (X, \Theta_X, x)$ of S_g. Before stating the result, we need one more piece of notation. Given a non-zero vector $b = (b_1, \dots, b_g) \in T_{X,0}$, we set $\partial_b = \sum_{\ell=1}^g b_\ell \partial_\ell$. Define the matrix $\partial_b M$ as the $g \times g$-matrix $(\partial_i \partial_j \partial_b \vartheta(\tau_0, z_0))_{1 \le i,j \le g}$. Then define the quadric $\partial_b Q_\xi = Q_{\xi,b}$ of $\mathbb{P}(T_{X,0})$ by the equation

$$z \cdot \partial_b M \cdot z^t = 0.$$

If $z = e_i$ is the i-th vector of the standard basis, one writes $\partial_i Q_\xi = Q_{\xi,i}$ instead of Q_{ξ,e_i} for $i = 1, \dots, g$. We will use similar notation for higher order derivatives or even for differential operators applied to a quadric.

Definition 3.2. We let $\mathcal{Q}_\xi$ be the linear system of quadrics in $\mathbb{P}(T_{X,0})$ spanned by Q_ξ and by all quadrics $Q_{\xi,b}$ with $b \in \ker(Q_\xi)$.

Since Q_ξ has corank r, the system $\mathcal{Q}_\xi$ is spanned by $r+1$ elements and therefore $\dim(\mathcal{Q}_\xi) \leq r$. This system may happen to be empty, but then Q_ξ is indeterminate, i.e., ξ lies in $\mathrm{Sing}(\Theta)$. Sometimes we will use the lower suffix x instead of ξ to denote quadrics and linear systems, e.g. we will sometimes write Q_x instead of Q_ξ, etc. By the heat equations, the linear system $\mathcal{Q}_\xi$ is the image of the vector subspace of $\mathrm{Sym}^2(T_{X,0})^\vee$ spanned by the vectors $q, q_1, \ldots, q_r$.

Proposition 3.3. *The subscheme S_g is smooth of codimension $g+1$ in $\mathcal{X}_g$ at the point $\xi = (X, \Theta_X, x)$ of S_g if and only if the following conditions hold:*

(i) *$\xi \notin \mathrm{Sing}(\Theta)$, i.e., Q_ξ is not indeterminate and of corank $r < g$;*
(ii) *the linear system $\mathcal{Q}_\xi$ has maximal dimension r; in particular, if $b_1, \ldots, b_r$ span the kernel of Q_ξ, then the $r+1$ quadrics Q_ξ, $Q_{\xi,b_1}, \ldots, Q_{\xi,b_r}$ are linearly independent.*

Proof. The subscheme S_g is smooth of codimension $g+1$ in $\mathcal{X}_g$ at ξ if and only if the matrix A appearing in (6) has maximal rank $g+1$. Since the submatrix B of A has rank $g-r$, the assertion follows. □

Corollary 3.4. *If Q_ξ is a smooth quadric, then S_g is smooth at $\xi = (X, \Theta_X, x)$.*

4. Quadrics and Cornormal Spaces

Next we study the differential of the restriction to S_g of the map $\pi : \mathcal{X}_g \to \mathcal{A}_g$ at a point $\xi = (X, \Theta_X, x) \in S_g$. We are interested in the kernel and the image of $d\pi_{|S_g,\xi}$. We can view these spaces in terms of the geometry of $\mathbb{P}^{g-1} = \mathbb{P}(T_{X,0})$ as follows:

$$\Pi_\xi = \mathbb{P}(\ker(d\pi_{|S_g,\xi})) \subseteq \mathbb{P}(T_{X,0})$$

is a linear subspace of $\mathbb{P}(T_{X,0})$ and

$$\Sigma_\xi = \mathbb{P}(\mathrm{Im}(d\pi_{|S_g,\xi})^\perp) \subseteq \mathbb{P}(\mathrm{Sym}^2(T_{X,0})^\vee)$$

is a linear system of quadrics in $\mathbb{P}(T_{X,0})$.
The following proposition is the key to our approach; we use it to view the quadrics as elements of the conormal space to our loci in the moduli space.

Proposition 4.1. *Let $\xi = (X, \Theta_X, x)$ be a point of S_g. Then:*

(i) *Π_ξ is the vertex of the quadric Q_ξ. In particular, if ξ is a singular point of Θ, then Π_ξ is the whole space $\mathbb{P}(T_{X,0})$;*
(ii) *Σ_ξ contains the linear system $\mathcal{Q}_\xi$.*

Proof. The assertions follow from the shape of the matrix A in (6). □

This proposition tells us that, given a point $\xi = (X, \Theta_X, x) \in S_g$, the map $d\pi_{|S_g,\xi}$ is not injective if and only if the quadric Q_ξ is singular.
The orbifold S_g is stratified by the corank of the matrix $(\partial_i \partial_j \theta)$.

Definition 4.2. For $0 \leq k \leq g$ we define $S_{g,k}$ as the closed suborbifold of S_g defined by the equations on $\mathbb{H}_g \times \mathbb{C}^g$

$$
\begin{aligned}
&\vartheta(\tau, z) = 0, \quad \partial_j \vartheta(\tau, z) = 0, \quad (j = 1, \ldots, g), \\
&\operatorname{rk}\big((\partial_i \partial_j \vartheta(\tau, z))_{1 \leq i,j \leq g}\big) \leq g - k.
\end{aligned} \tag{7}
$$

Geometrically this means that $\xi \in S_{g,k}$ if and only if $\dim(\Pi_\xi) \geq k - 1$ or equivalently Q_ξ has corank at least k. We have the inclusions

$$S_g = S_{g,0} \supseteq S_{g,1} \supseteq \ldots \supseteq S_{g,g} = S_g \cap \operatorname{Sing}(\Theta)$$

and $S_{g,1}$ is the locus where the map $d\pi_{|S_g,\xi}$ is not injective. The loci $S_{g,k}$ have been considered also in [16].

We have the following dimension estimate for the $S_{g,k}$.

Proposition 4.3. *Let $1 \leq k \leq g-1$ and let Z be an irreducible component of $S_{g,k}$ not contained in $S_{g,k+1}$. Then we have*

$$\operatorname{codim}_{S_g}(Z) \leq \binom{k+1}{2}.$$

Proof. Locally, in a neighborhood U in S_g of a point z of $Z \backslash S_{g,k+1}$ we have a morphism $f : U \to \mathcal{Q}$, where $\mathcal{Q}$ is the linear system of all quadrics in $\mathbb{P}^{g-1}$. The map f sends $\xi = (X, \Theta_X, x) \in U$ to Q_ξ. The scheme $S_{g,k}$ is the pull-back of the subscheme $\mathcal{Q}_k$ of $\mathcal{Q}$ formed by all quadrics of corank k. Since $\operatorname{codim}_{\mathcal{Q}}(\mathcal{Q}_k) = \binom{k+1}{2}$, the assertion follows. □

Using the equations (7) it is possible to make a local analysis of the schemes $S_{g,k}$, e.g. it is possible to write down equations for their Zariski tangent spaces (see §6 for the case $k = g$). This is however not particularly illuminating, and we will not dwell on this here.

It is useful to give an interpretation of the points $\xi = (X, \Theta_X, x) \in S_{g,k}$ in terms of singularities of the theta divisor Θ_X. Suppose that ξ is such that $\operatorname{Sing}(\Theta_X)$ contains a subscheme isomorphic to $\operatorname{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2))$ supported at x. This subscheme of X is given by a homomorphism

$$\mathcal{O}_{X,x} \to \mathbb{C}[\epsilon]/(\epsilon^2), \quad f \mapsto f(x) + \Delta^{(1)} f(x) \cdot \epsilon,$$

where $\Delta^{(1)}$ is a non-zero differential operator of order ≤ 1, hence $\Delta^{(1)} = \partial_b$, for some non-zero vector $b \in \mathbb{C}^g$. Then the condition $\operatorname{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2)) \subset \operatorname{Sing}(\Theta_X)$ is equivalent to saying that ϑ and $\partial_b \vartheta$ satisfy the equations

$$f(\tau_0, z_0) = 0, \quad \partial_j f(\tau_0, z_0) = 0, \quad 1 \leq j \leq g, \tag{8}$$

and this, in turn, is equivalent to the fact that the quadric Q_ξ is singular at the point $[b]$.

More generally, we have the following proposition, which explains the nature of the points in $S_{g,k}$ for $k < g$.

Proposition 4.4. *Suppose that $x \in \operatorname{Sing}(\Theta_X)$ does not lie on $\operatorname{Sing}(\Theta)$. Then $\operatorname{Sing}(\Theta_X)$ contains a scheme isomorphic to $\operatorname{Spec}(\mathbb{C}[\epsilon_1, \ldots, \epsilon_k]/(\epsilon_i \epsilon_j \colon 1 \leq i, j \leq k < g))$ supported at x if and only if the quadric Q_ξ has corank $r \geq k$. Moreover, the Zariski tangent space to $\operatorname{Sing}(\Theta_X)$ at x is the kernel space of Q_ξ.*

Proof. With a suitable choice of coordinates in X, the condition that the scheme $\mathrm{Spec}(\mathbb{C}[\epsilon_1,\dots,\epsilon_k]/(\epsilon_i\epsilon_j\colon 1\le i,j\le k<g))$ is contained in $\mathrm{Sing}(\Theta_X)$ is equivalent to the fact that the functions ϑ and $\partial_i\vartheta$ for $i=1,\dots,k$ satisfy (8). But this the same as saying that $\partial_i\partial_j\vartheta(\tau_0,z_0)$ is zero for $i=1,\dots,k$, $j=1,\dots,g$, and the vectors e_i, $i=1,\dots,k$, belong to the kernel of Q_ξ. This settles the first assertion.

The scheme $\mathrm{Sing}(\Theta_X)$ is defined by the equations (2), where τ is now fixed and z is the variable. By differentiating, and using the same notation as above, we see that the equations for the Zariski tangent space to $\mathrm{Sing}(\Theta_X)$ at x are $\sum_{i=1}^g b_i\partial_i\partial_j\vartheta(\tau_0,z_0)$, $j=1,\dots,g$ i.e., $b\cdot M=0$, which proves the second assertion. $\square$

5. Curvi-linear subschemes in the singular locus of theta

A 0-dimensional curvi-linear subscheme $\mathrm{Spec}(\mathbb{C}[t]/(t^{N+1}))\subset X$ of length $N+1$ supported at x is given by a homomorphism

$$\delta:\mathcal{O}_{X,x}\to\mathbb{C}[t]/(t^{N+1}),\qquad f\mapsto\sum_{j=0}^N\Delta^{(j)}f(x)\cdot t^j, \tag{9}$$

with $\Delta^{(j)}$ a differential operator of order $\le j$, $j=1,\dots,N$, with $\Delta^{(N)}$ non-zero, and $\Delta^{(0)}(f)=f(x)$. The condition that the map δ is a homomorphism is equivalent to saying that

$$\Delta^{(k)}(fg)=\sum_{r=0}^k\Delta^{(r)}f\cdot\Delta^{(k-r)}g,\quad k=0,\dots,N \tag{10}$$

for any pair (f,g) of elements of $\mathcal{O}_{X,x}$. Two such homomorphisms δ and δ' define the same subscheme if and only if they differ by composition with a automorphism of $\mathbb{C}[t]/(t^{N+1})$.

Lemma 5.1. *The map δ defined in* (9) *is a homomorphism if and only if there exist translation invariant vector fields $D_1,\dots,D_N$ on X such that for every $k=1,\dots,N$ one has*

$$\Delta^{(k)}=\sum_{h_1+2h_2+\ldots+kh_k=k>0}\frac{1}{h_1!\cdots h_k!}D_1^{h_1}\cdots D_k^{h_k}. \tag{11}$$

Moreover, two N-tuples of vector fields $(D_1,\dots,D_N)$ and $(D'_1,\dots,D'_N)$ determine the same 0*-dimensional curvi-linear subscheme of X of length $N+1$ supported at a given point $x\in X$ if and only if there are constants $c_1,\dots,c_N$, with $c_1\neq 0$, such that*

$$D'_i=\sum_{j=1}^i c_j^{i-j+1}D_j,\quad i=1,\dots,N.$$

Proof. If the differential operators $\Delta^{(k)}$, $k=1,\dots,N$, are as in (11), one computes that (10) holds, hence δ is a homomorphism.

As for the converse, the assertion trivially holds for $k=1$. So we proceed by induction on k. Write $\Delta^{(k)}=\sum_{i=1}^{k} D_i^{(k)}$, where $D_i^{(k)}$ is the homogeneous part of degree i, and write D_k instead of $D_1^{(k)}$. Using (10) one verifies that for every $k=1,\dots,N$ and every positive $i\le k$ one has

$$iD_i^{(k)}=\sum_{j=1}^{k-i+1} D_jD_{i-1}^{(k-j)}.$$

Formula (11) follows by induction and easy combinatorics.

To prove the final assertion, use the fact that an automorphism of $\mathbb{C}[t]/(t^{N+1})$ is determined by the image $c_1t+c_2t^2+\ldots+c_Nt^N$ of t, where $c_1\neq 0$. □

In formula (11) one has $h_k\le 1$. If $\Delta^{(1)}=D_1$ then $\Delta^{(2)}=\frac{1}{2}D_1^2+D_2$, $\Delta^{(3)}=(1/3!)D_1^3+(1/2)D_1D_2+D_3$ etc.

Each non-zero summand in (11) is of the form $(1/h_{i_1}!\cdots h_{i_\ell}!)D_{i_1}^{h_{i_1}}\cdots D_{i_\ell}^{h_{i_\ell}}$, where $1\le i_1<\ldots<i_\ell\le k$, $i_1h_{i_1}+\ldots+i_\ell h_{i_\ell}=k$ and $h_{i_1},\dots,h_{i_\ell}$ are positive integers. Thus formula (11) can be written as

$$\Delta^{(k)}=\sum_{\{h_{i_1},\dots,h_{i_\ell}\}}\frac{1}{h_{i_1}!\cdots h_{i_\ell}!}D_{i_1}^{h_{i_1}}\cdots D_{i_\ell}^{h_{i_\ell}}, \tag{12}$$

where the subscript $\{h_{i_1},\dots,h_{i_\ell}\}$ means that the sum is taken over all ℓ-tuples of positive integers $(h_{i_1},\dots,h_{i_\ell})$ with $1\le i_1<\cdots<i_\ell\le k$ and $i_1h_{i_1}+\cdots+i_\ell h_{i_\ell}=k$.

Remark 5.2. Let $x\in X$ correspond to the pair (τ_0,z_0). The differential operators $\Delta^{(k)}$, $k=1,\dots,N$, defined as in (11) or (12) have the following property: if f is a regular function such that $\Delta^{(i)}f$ satisfies (8) for all $i=0,\dots,k-1$, then one has $\Delta^{(k)}f(\tau_0,z_0)=0$.

We want now to express the conditions in order that a 0-dimensional curvilinear subscheme of X of length $N+1$ supported at a given point $x\in X$ corresponding to the pair (τ_0,z_0) and determined by a given N-tuple of vector fields $(D_1,\dots,D_N)$ lies in $\mathrm{Sing}(\Theta_X)$. To do so, we keep the notation we introduced above.

Let us write $D_i=\sum_{\ell=1}^{g}\eta_{i\ell}\partial_\ell$, so that D_i corresponds to the vector $\eta_i=(\eta_{i1},\dots,\eta_{ig})$. As before we denote by M the matrix $(\partial_i\partial_j\theta(\tau_0,z_0))$.

Proposition 5.3. *The 0-dimensional curvi-linear subscheme R of X of length $N+1$, supported at the point $x\in X$ corresponding to the pair (τ_0,z_0) and determined by the N-tuple of vector fields $(D_1,\dots,D_N)$ lies in $\mathrm{Sing}(\Theta_X)$ if and only if $x\in\mathrm{Sing}(\Theta_X)$ and moreover for each $k=1,\dots,N$ one has*

$$\sum_{\{h_{i_1},\dots,h_{i_\ell}\}}\frac{1}{h_{i_1}!\cdots h_{i_\ell}!}\eta_{i_\ell}\cdot\partial_{\eta_{i_1}}^{h_{i_1}}\cdots\partial_{\eta_{i_\ell}}^{h_{i_\ell}-1}M=0, \tag{13}$$

where the sum is taken over all ℓ-tuples of positive integers $(h_{i_1},\dots,h_{i_\ell})$ with $1\le i_1<\cdots<i_\ell\le k$ and $i_1h_{i_1}+\cdots+i_\ell h_{i_\ell}=k$.

Proof. The scheme R is contained in $\mathrm{Sing}(\Theta_X)$ if and only if one has

$$\Delta^{(k)}\theta(\tau_0, z_0) = 0, \quad \partial_j \Delta^{(k)}\theta(\tau_0, z_0) = 0 \qquad k = 0, \ldots, N, \quad j = 1, \ldots, g.$$

By Remark 5.2 this is equivalent to

$$\theta(\tau_0, z_0) = 0, \quad \partial_j \Delta^{(k)}\theta(\tau_0, z_0) = 0 \qquad k = 0, \ldots, N, \quad j = 1, \ldots, g.$$

The assertion follows by the expression (12) of the operators $\Delta^{(k)}$. □

For instance, consider the scheme R_1, supported at $x \in \mathrm{Sing}(\Theta_X)$, corresponding to the vector field D_1. Then R_1 is contained in $\mathrm{Sing}(\Theta_X)$ if and only if

$$\eta_1 \cdot M = 0. \tag{14}$$

This agrees with Proposition 4.4. If R_2 is the scheme supported at x and corresponding to the pair of vector fields (D_1, D_2), then R_2 is contained in $\mathrm{Sing}(\Theta_X)$ if and only if, besides (14) one has also

$$(1/2)\eta_1 \cdot \partial_{\eta_1} M + \eta_2 \cdot M = 0. \tag{15}$$

Next, consider the scheme R_3 supported at x and corresponding to the triple of vector fields (D_1, D_2, D_3). Then R_3 is contained in $\mathrm{Sing}(\Theta_X)$ if and only if, besides (14) and (15) one has also

$$(1/3!)\eta_1 \cdot \partial^2_{\eta_1} M + (1/2)\eta_2 \cdot \partial_{\eta_1} M + \eta_3 \cdot M = 0 \tag{16}$$

and so on. Observe that (13) can be written in more than one way. For example $\eta_2 \cdot \partial_{\eta_1} M = \eta_1 \cdot \partial_{\eta_2} M$ so that (16) could also be written as

$$(1/3!)\eta_1 \cdot \partial^2_{\eta_1} M + (1/2)\eta_1 \cdot \partial_{\eta_2} M + \eta_3 \cdot M = 0.$$

So far we have been working in a fixed abelian variety X. One can remove this restriction by working on S_g and by letting the vector fields $D_1, \ldots, D_N$ vary with X, which means that we let the vectors η_i depend on the variables τ_{ij}. Then the equations (13) define a subscheme $S_g(D)$ of $\mathrm{Sing}(\Theta)$ which, as a set, is the locus of all points $\xi = (X, \Theta_X, x) \in S_g$ such that $\mathrm{Sing}(\Theta_X)$ contains a curvi-linear scheme of length $N + 1$ supported at x, corresponding to the N-tuple of vector fields $D = (D_1, \ldots, D_N)$, computed on X.
One can compute the Zariski tangent space to $S_g(D)$ at a point $\xi = (X, \Theta_X, x)$ in the same way, and with the same notation, as in §3. This gives in general a complicated set of equations. However we indicate one case in which one can draw substantial information from such a computation. Consider indeed the case in which $D_1 = \ldots = D_N \neq 0$, and call b the corresponding tangent vector to X at the origin, depending on the the variables τ_{ij}. In this case we use the notation $D_{b,N} = (D_1, \ldots, D_N)$ and we denote by $R_{x,b,N}$ the corresponding curvi-linear scheme supported at x. For a given such $D = (D_1, \ldots, D_N)$, consider the linear system of quadrics

$$\Sigma_\xi(D) = \mathbb{P}(\mathrm{Im}(d\pi_{|S_g(D),\xi})^\perp)$$

in $\mathbb{P}(T_{X,0})$. One has again an interpretation of these quadrics in terms of the normal space:

Proposition 5.4. *In the above setting, the space* $\Sigma_\xi(D_{b,N})$ *contains the quadrics* $Q_\xi, \partial_b Q_\xi, \ldots, \partial_b^N Q_\xi$.

Proof. The equations (13) take now the form

$$\theta(\tau, z) = 0,\ \partial_i\theta(\tau, z) = 0, \quad i = 1, \ldots, g$$
$$b \cdot M = b \cdot \partial_b M = \cdots = b \cdot \partial_b^{N-1} M = 0.$$

By differentiating the assertion immediately follows. □

6. Higher multiplicity points of the theta divisor

We now study the case of higher order singularities on the theta divisor. For a multi-index $I = (i_1, \ldots, i_g)$ with $i_1, \ldots, i_g$ non-negative integers we set $z^I = z_1^{i_1} \cdots z_g^{i_g}$ and denote by ∂_I the operator $\partial_1^{i_1} \cdots \partial_g^{i_g}$. Moreover, we let $|I| = \sum_{\ell=1}^g i_\ell$, which is the *length* of I and equals the *order* of the operator ∂_I.

Definition 6.1. For a positive integer r we let $S_g^{(r)}$ be the subscheme of $\mathcal{X}_g$ which is defined on $\mathbb{H}_g \times \mathbb{C}^g$ by the equations

$$\partial_I \vartheta(\tau, z) = 0, \qquad |I| = 0, \ldots, r-1. \tag{17}$$

One has the chain of subschemes

$$\ldots \subseteq S_g^{(r)} \subseteq \ldots \subseteq S_g^{(3)} \subseteq S_g^{(2)} = S_g \subset S_g^{(1)} = \Theta$$

and as a set $S_g^{(r)} = \{(X, \Theta_X, x) \in \Theta : x \text{ has multiplicity} \geq r \text{ for } \Theta_X\}$. One denotes by $\mathrm{Sing}^{(r)}(\Theta_X)$ the subscheme of $\mathrm{Sing}(\Theta_X)$ formed by all points of multiplicity at least r. One knows that $S_g^{(r)} = \emptyset$ as soon as $r > g$ (see [37]). We can compute the Zariski tangent space to $S_g^{(r)}$ at a point $\xi = (X, \Theta_X, x)$ in the same vein, and with the same notation, as in §3. Taking into account that θ and all its derivatives verify the heat equations, we find the equations by replacing in (3) the term $\theta(\tau_0, z_0)$ by $\partial_I\theta(\tau_0, z_0)$.

As in §3, we wish to give some geometrical interpretation. For instance, we have the following lemma which partially extends Lemma 2.1 or 3.3.

Lemma 6.2. *For every positive integer* r *the scheme* $S_g^{(r+2)}$ *is contained in the singular locus of* $S_g^{(r)}$.

Next we are interested in the differential of the restriction of the map $\pi : \mathcal{X}_g \to \mathcal{A}_g$ to $S_g^{(r)}$ at a point $\xi = (X, \Theta_X, x)$ which does not belong to $S_g^{(r+1)}$. This means that Θ_X has a point of multiplicity exactly r at x. If we assume, as we may, that x is the origin of X , i.e. $z_0 = 0$, then the Taylor expansion of θ has the form

$$\vartheta = \sum_{i=r}^{\infty} \vartheta_i,$$

where ϑ_i is a homogeneous polynomial of degree i in the variables $z_1, \ldots, z_g$ and

$$\theta_r = \sum_{I=(i_1,\ldots,i_g),|I|=r} \frac{1}{i_1! \cdots i_g!} \partial_I \theta(\tau_0, z_0) z^I$$

is not identically zero. The equation $\theta_r = 0$ defines a hypersurface TC_ξ of degree r in $\mathbb{P}^{g-1} = \mathbb{P}(T_{X,0})$, which is the *tangent cone* to Θ_X at x.
We will denote by $\mathrm{Vert}(TC_\xi)$ the *vertex* of TC_ξ, i.e., the subspace of $\mathbb{P}^{g-1}$ which is the locus of points of multiplicity r of TC_ξ. Note that it may be empty. In case $r = 2$, the tangent cone TC_ξ is the quadric Q_ξ introduced in §3 and $\mathrm{Vert}(TC_\xi)$ is its vertex Π_ξ.
More generally, for every $s \geq r$, one can define the subscheme $TC^{(s)}_\xi = TC^{(s)}_x$ of $\mathbb{P}^{g-1} = \mathbb{P}(T_{X,0})$ defined by the equations

$$\theta_r = \ldots = \theta_s = 0,$$

which is called the *asymptotic cone* of order s to Θ_X at x.
Fix a multi-index $J = (j_1, \ldots, j_g)$ of length $r-2$. For any pair (h,k) with $1 \leq h,k \leq g$, let $J_{(h,k)}$ be the multi-index of length r obtained from J by first increasing by 1 the index j_h and then by 1 the index j_k (that is, by 2 if they coincide). Consider then the quadric Q^J_ξ in $\mathbb{P}^{g-1} = \mathbb{P}(T_{X,0})$ defined by the equation

$$q^J_\xi(z) := \sum_{1\leq h,k\leq g} \partial_{J_{(h,k)}}\theta(\tau_0, z_0) z_h z_k = 0$$

with the usual convention that the quadric is indeterminate if the left-hand-side is identically zero. This is a *polar quadric* of TC_ξ, namely it is obtained from TC_ξ by iterated operations of polarization. Moreover all polar quadrics are in the span $\langle Q^J_\xi, |J| = r-2\rangle$. We will denote by $\mathcal{Q}^{(r)}_\xi$ the span of all quadrics Q^J_ξ, $|J| = r-2$ and $\partial_b Q^J_\xi$, $|J| = r-2$, with equation

$$\sum_{1\leq i,j\leq g} \partial_b \partial_{J_{(h,k)}}\theta(\tau_0, z_0) z_i z_j = 0,$$

for every non-zero vector $b \in \mathbb{C}^g$ such that $[b] \in \mathrm{Vert}(TC_\xi)$.
We are now interested in the kernel and the image of $d\pi_{|S^{(r)}_g,\xi}$. Equivalently we may consider the linear system of quadrics $\Sigma^{(r)}_\xi = \mathbb{P}(\mathrm{Im}(d\pi_{|S^{(r)}_g,\xi})^\perp)$, and the subspace $\Pi^{(r)}_\xi = \mathbb{P}(\ker(d\pi_{|S^{(r)}_g,\xi}))$ of $\mathbb{P}(T_{X,0})$. The following proposition partly extends Proposition 4.1 and 4.4 and its proof is similar.

Proposition 6.3. *Let $\xi = (X, \Theta_X, x)$ be a point of $S^{(r)}_g$. Then:*

(i) $\Pi^{(r)}_\xi = \mathrm{Vert}(TC_\xi)$. *In particular, if $\xi \in S^{(r+1)}_g$, then $\Pi^{(r)}_\xi$ is the whole space $\mathbb{P}(T_{X,0})$;*
(ii) $\Sigma^{(r)}_\xi$ *contains the linear system $\mathcal{Q}^{(r)}_\xi$.*

Remark 6.4. As a consequence, just like in Proposition 4.4, one sees that for $\xi = (X, \Theta_X, x)$ the Zariski tangent space to $\mathrm{Sing}^{(r)}(\Theta_X)$ at x is contained in $\mathrm{Vert}(TC_\xi)$.

As an application, we have:

Proposition 6.5. *Let* $\xi = (X, \Theta_X, x)$ *be a point of a component* Z *of* $S_g^{(3)}$ *such that* TC_ξ *is not a cone. Then* $\dim(\mathcal{Q}_\xi^{(3)}) = g-1$ *and therefore the codimension of the image of* Z *in* $\mathcal{A}_g$ *is at least* g.

Proof. Since TC_ξ is a not a cone its polar quadrics are linearly independent. $\square$

The following example shows that the above bound is sharp for $g = 5$.

Example 6.6. Consider the locus $\mathcal{C}$ of intermediate Jacobians of cubic threefolds in $\mathcal{A}_5$. Note that $\dim(\mathcal{A}_5) = 15$ and $\dim(\mathcal{C}) = 10$. These have (at least) an isolated triple point on their theta divisor whose tangent cone gives back the cubic threefold. The locus $\mathcal{C}$ is dominated by an irreducible component of $S_5^{(3)}$ for which the estimate given in Proposition 6.5 is sharp. Cf. [6] where Casalaina-Martin proves that the locus of intermediate Jacobians of cubic threefolds is an irreducible component of the locus of principally polarized abelian varieties of dimension 5 with a point of multiplicity ≥ 3.

7. The Andreotti–Mayer loci

Andreotti and Mayer consider in $\mathcal{A}_g$ the algebraic sets of principally polarized abelian varieties X with a locus of singular points on Θ_X of dimension at least k. More generally, we are interested in the locus of principally polarized abelian varieties possessing a k-dimensional locus of singular points of multiplicity r on the theta divisor. To define these loci scheme-theoretically we consider the morphism $\pi : \mathcal{X}_g \to \mathcal{A}_g$ and the quasi-coherent sheaf on $\mathcal{A}_g$

$$\mathcal{F}_k^{(r)} = \bigoplus_{i=k}^{g-2} R^i \pi_* \mathcal{O}_{S_g^{(r)}}.$$

Definition 7.1. For integers k and r with $0 \leq k \leq g-2$ and $2 \leq r \leq g$ we define $N_{g,k,r}$ as the support of $\mathcal{F}_k^{(r)}$. We also set $M_{g,k,r} = \pi^{-1}(N_{g,k,r})$, a subscheme of both $S_{g,k}$ and $S_g^{(r)}$. We write $N_{g,k}$ and $M_{g,k}$ for $N_{g,k,2}$ and $M_{g,k,2}$.

The schemes $N_{g,k}$ are the so-called *Andreotti–Mayer loci* in $\mathcal{A}_g$, which were introduced in a somewhat different way in [2].
Note that $N_{g,k,r}$ is locally defined by an annihilator ideal and so carries the structure of subscheme. Corollary 8.12 below and results by Debarre [11] (see §19) imply that the scheme structure at a general point of $N_{g,0}$ defined above coincides with the one considered by Mumford in [27].
We now want to see that as a set $N_{g,k,r}$ is the locus of points corresponding to (X, Θ_X) such that $\mathrm{Sing}(\Theta_X)$ has an irreducible component of dimension $\geq k$ of points of multiplicity $\geq r$ for Θ_X.

Lemma 7.2. *Let* X *be an abelian variety of dimension* g *and* $W \subset X$ *an irreducible reduced subvariety of dimension* n *and let* ω_W *be its dualizing sheaf. Then* $H^0(W, \omega_W) \neq (0)$.

Proof. Let $f : W' \to W$ be the normalization of W. We first claim the inequality $h^0(W', \omega_{W'}) \leq h^0(W, \omega_W)$. To see this, note that by [24], p. 48 ff (see also [17], Exerc. 6.10, p. 239, 7.2, p. 249), there exists a map $f_*\omega_{W'} \to \mathcal{H}om(f_*\mathcal{O}_{W'}, \omega_W)$, hence a map

$$H^0(W', \omega_{W'}) \to H^0(W, \mathcal{H}om(f_*\mathcal{O}_{W'}, \omega_W)).$$

Now $\mathcal{O}_W \to f_*\mathcal{O}_{W'}$ is an injection and therefore $H^0(W, \mathcal{H}om(f_*\mathcal{O}_{W'}, \omega_W)$ maps to $H^0(W, \mathcal{H}om(\mathcal{O}_W, \omega_W))$ and we thus get a map $H^0(W', \omega_{W'}) \to H^0(W, \omega_W)$ which is injective as one sees by looking at the smooth part of W.
Let $\tilde{W}$ be a desingularization of W'. According to [22] we have $h^0(\tilde{W}, \Omega^n_{\tilde{W}}) \leq h^0(W', \omega_{W'})$. Since $\tilde{W}$ maps to X we have $h^0(\tilde{W}, \Omega^1_{\tilde{W}}) \geq n$. If $h^0(\tilde{W}, \Omega^n_{\tilde{W}})$ were 0 then $\wedge^n H^0(\tilde{W}, \Omega^1_{\tilde{W}}) \to H^0(\tilde{W}, \Omega^n_{\tilde{W}})$ would be the zero map contradicting the fact that W has dimension n. $\square$

Corollary 7.3. *We have $(X, \Theta_X) \in N_{g,k,r}$ if and only if $\dim(\mathrm{Sing}^{(r)}(\Theta_X)) \geq k$.*

Proof. By the previous lemma and Serre duality for a reduced irreducible subvariety W of dimension m in X it follows that $H^m(W, \mathcal{O}_W) \neq (0)$ and we know $H^k(W, \mathcal{O}_W) = (0)$ for $k > m$. This implies the corollary. $\square$

There are the inclusions

$$N_{g,k,r} \subseteq N_{g,k,r-1}, \quad N_{g,k,r} \subseteq N_{g,k-1,r}.$$

If $p = (n_1, \ldots, n_r)$ with $1 \leq n_1 \leq \ldots \leq n_r < g$ and $n_1 + \ldots + n_r = g$ is a partition of g we write $\mathcal{A}_{g,p}$ for the suborbifold (or substack) of $\mathcal{A}_g$ corresponding to principally polarized abelian varieties that are a product of r principally polarized abelian varieties of dimensions $n_1, \ldots, n_r$. We write $r(p) = r$ for the length of the partition and write $\mathcal{A}_{g,[r]}$ for the suborbifold $\cup_{r(p)=r}\mathcal{A}_{g,p}$ of $\mathcal{A}_g$ corresponding to pairs (X, Θ_X) isomorphic as a polarized abelian variety to the product of r principally polarized abelian varieties. One has the stratification

$$\mathcal{A}_{g,[g]} \subset \mathcal{A}_{g,[g-1]} \subset \cdots \subset \mathcal{A}_{g,[2]}.$$

We will denote by:
i) $\Pi_g = \cup_{r \geq 2}\mathcal{A}_{g,r} = \mathcal{A}_{g,[2]}$ the locus of *decomposable* principally polarized abelian varieties;
ii) $\mathcal{A}_g^{(\mathrm{ns})}$ the locus of classes of *non-simple abelian varieties*, i.e., of principally polarized abelian varieties of dimension g which are isogenous to a product of abelian varieties of dimension smaller that g;
iii) $\mathcal{A}_g^{\mathrm{End} \neq \mathbb{Z}}$ the locus of classes of *singular abelian varieties*, i.e., of principally polarized abelian varieties whose endomorphism ring is larger than $\mathbb{Z}$.

Remark 7.4. Note the inclusions $\Pi_g \subset \mathcal{A}_g^{(\mathrm{ns})} \subset \mathcal{A}_g^{\mathrm{End} \neq \mathbb{Z}}$. The locus Π_g is reducible with irreducible components $\mathcal{A}_{g,p}$ with p running through the partitions $g = (i, g-i)$ of g for $1 \leq i \leq g/2$ and we have $\mathrm{codim}_{\mathcal{A}_g}(\mathcal{A}_{g,(i,g-i)}) = i(g-i)$. In contrast to this $\mathcal{A}_g^{\mathrm{End} \neq \mathbb{Z}}$ and $\mathcal{A}_g^{(\mathrm{ns})}$ are the union of infinitely countably many irreducible closed subsets of $\mathcal{A}_g$ of codimension at least $g-1$, the minimum codimension being achieved for families of abelian varieties that are isogenous to products of an elliptic curve with an abelian variety of dimension $g-1$ (compare with [7]).

We recall a result from [20] and the main result from [12].

Theorem 7.5. *For every integer r with $2 \leq r \leq g$ one has:*
 (i) *$N_{g,k,r} = \emptyset$ if $k > g-r$;*
 (ii) *$N_{g,g-r,r} = \mathcal{A}_{g,[r]}$, i.e., $(X,\Theta_X) \in N_{g,g-r,r}$ is an r-fold product.*

Hence, for every integer r such that $2 \leq r \leq g$, one has the stratification

$$N_{g,0,r} \supset N_{g,1,r} \supset \ldots \supset N_{g,k,r} \supset \ldots \supset N_{g,g-r,r} = \mathcal{A}_{g,[r]},$$

whereas for every integer k such that $0 \leq k \leq g-2$, one has the stratification

$$N_{g,k,2} \supset N_{g,k,3} \supset \ldots \supset N_{g,k,r} \supset \ldots \supset N_{g,k,g-k} = \mathcal{A}_{g,[g-k]}.$$

8. Lower bounds for the codimension of Andreotti-Mayer loci

The results in the previous sections give information about the Zariski tangent spaces to these loci and this will allow us to prove bounds on the dimension of the Andreotti–Mayer loci, which is our main objective in this paper.
We start with the results on tangent spaces. We need some notation.

Definition 8.1. Let $\zeta = (X,\Theta_X)$ represent a point in $N_{g,k,r}$. By $\mathcal{L}_{g,k,r}(\zeta)$ we denote the linear system of quadrics $\mathbb{P}(T^{\perp}_{N_{g,k,r},\zeta})$, where $T_{N_{g,k,r},\zeta}$ is the Zariski tangent space and where we view $\mathbb{P}(T_{\mathcal{A}_{g,\zeta}})$ as a space of quadrics as in Section 2. As usual, we may drop the index r if $r=2$ and write $\mathcal{L}_{g,k}(\zeta)$ for $\mathcal{L}_{g,k,2}(\zeta)$.

Notice that

$$\dim_{\zeta}(N_{g,k,r}) \leq \binom{g+1}{2} - \dim(\mathcal{L}_{g,k}(\zeta)) + 1.$$

Definition 8.2. For $\zeta = (X,\Theta_X) \in N_{g,k,r}$ we denote by $\mathrm{Sing}^{(k,r)}(\Theta_X)$ the locally closed subset

$$\mathrm{Sing}^{(k,r)}(\Theta_X) = \{x \in \mathrm{Sing}(\Theta_X) : \dim_x(\mathrm{Sing}^{(r)}(\Theta_X)) \geq k\}.$$

Moreover, we define $\mathcal{Q}^{(k,r)}_{\zeta}$ to be the linear system of quadrics in $\mathbb{P}^{g-1} = \mathbb{P}(T_{X,0})$ spanned by the union of all linear systems $\mathcal{Q}^{(r)}_{\xi}$ with $\xi = (X,\Theta_X,x)$ and $x \in \mathrm{Sing}^{(k,r)}(\Theta_X)$.

Propositions 4.1 and 6.3 imply the following basic tool for giving upper bounds on the dimension of the Andreotti–Mayer loci.

Proposition 8.3. *Let N be an irreducible component of $N_{g,k,r}$ with its reduced structure. If $\zeta = (X,\Theta_X)$ is a general point of N then the projectivized conormal space to N at ζ, viewed as a subspace of $\mathbb{P}(\mathrm{Sym}^2(T^{\vee}_{X,0}))$, contains the linear system $\mathcal{Q}^{(k,r)}_{\zeta}$.*

Proof. Let M be an irreducible component of π^*N in $S^{(r)}_{g,k}$. If ξ is smooth point of M then the image of the Zariski tangent space to M at ξ under $d\pi$ is orthogonal to $\mathcal{Q}^{(r)}_\xi$ for all $x \in \mathrm{Sing}^{k,r}(\Theta_X)$. Since we work in characteristic 0 the map $d\pi$ is surjective on the tangent spaces for general points $m \in M$ and $\pi(m) \in N$. Therefore the result follows from Propositions 4.1 and 6.3. □

We need a couple of preliminary results. First we state a well-known fact, which can be proved easily by a dimension count.

Lemma 8.4. *Every hypersurface of degree $d \leq 2n-3$ in $\mathbb{P}^n$ with $n \geq 2$ contains a line.*

Next we prove the following:

Lemma 8.5. *Let V in $\mathbb{P}^s$ be a hypersurface of degree $d \geq 3$. If all polar quadrics of V coincide, then V is a hyperplane H counted with multiplicity d, and the polar quadrics coincide with $2H$.*

Proof. If $d = 3$ the assertion follows from general properties of duality (see [41], p. 215) or from an easy calculation.
If $d > 3$, then the result, applied to the cubic polars of V, tells us that all these cubic polars are equal to $3H$, where H is a fixed hyperplane. This immediately implies the assertion. □

The next result has been announced in [8].

Theorem 8.6. *Let $g \geq 4$ and let N be an irreducible component of $N_{g,k}$ not contained in $N_{g,k+1}$. Then:*

(i) *for every positive integer $k \leq g-3$, one has $\mathrm{codim}_{\mathcal{A}_g}(N) \geq k+2$, whereas $\mathrm{codim}_{\mathcal{A}_g}(N) = g-1$ if $k = g-2$;*
(ii) *if N is contained in $N_{g,k,r}$ with $r \geq 3$, then $\mathrm{codim}_{\mathcal{A}_g}(N) \geq k+3$;*
(iii) *if $g-4 \geq k \geq g/3$, then $\mathrm{codim}_{\mathcal{A}_g}(N) \geq k+3$.*

Proof. By Theorem 7.5 and Remark 7.4, we may assume $k < g-2$. By definition, there is some irreducible component M of $\pi^{-1}_{S_g}(N)$ with $\dim(M) = \dim(N)+k$ which dominates N via π. We can take a general point $(X, \Theta_X, z) \in M$ so that $\zeta = (X, \Theta_X)$ is a general point in N. By Remark 7.4 we may assume X is simple.
Let R be the unique k-dimensional component of $\pi^{-1}_{|S_g}(\zeta)$ containing (X, Θ_X, z). Its general point is of the form $\xi = (X, \Theta_X, x)$ with x the general point of the unique k-dimensional component of $\mathrm{Sing}(\Theta_X)$ containing z, and x has multiplicity r on Θ_X. By abusing notation, we may still denote this component by R. Proposition 6.3 implies that the linear system of quadrics $\mathbb{P}(T^\perp_{N,\zeta})$ contains all polar quadrics of TC_ξ with $\xi = (X, \Theta_X, x) \in R$.
Thus we have a rational map

$$\phi : (\mathbb{P}^{g-1})^{r-2} \times R \dashrightarrow \mathcal{Q}^{(r)}_\zeta$$

which sends the general point $(b, \xi) := (b_1, \ldots, b_{r-2}, \xi)$ to the polar quadric $Q_{b,\xi}$ of TC_ξ with respect to $b_1, \ldots, b_{r-2}$. It is useful to remark that the quadric

$Q_{b,\xi}$ is a cone with vertex containing the projectivized Zariski tangent space to R at x (see Remark 6.4).

Claim 8.7 (The finiteness property). *For each $b \in (\mathbb{P}^{g-1})^{r-2}$ the map ϕ restricted to $\{b\} \times R$ has finite fibres.*

Proof of the claim. Suppose the assertion is not true. Then there is an irreducible curve $Z \subseteq R$ such that, for $\xi = (X, \Theta_X, x)$ corresponding to the general point in Z, one has $Q_{b,\xi} = Q$. Set $\Pi = \mathrm{Vert}(Q)$, which is a proper subspace of $\mathbb{P}^{g-1}$.
Consider the *Gauss map*

$$\gamma = \gamma_Z : Z \dashrightarrow \mathbb{P}^{g-1} = \mathbb{P}(T_{X,0})$$

which associates to a smooth point of Z its projectivized tangent space. Then Proposition 6.3 implies that $\gamma(\xi) \in R \subseteq \mathrm{Vert}(Q_\xi) = \Pi$. Thus $\gamma(Z)$ is degenerate in $\mathbb{P}^{g-1}$ and this yields that X is non-simple, cf. [31]. This is a contradiction which proves the claim. $\square$

Claim 8.7 implies that the image of the map ϕ has dimension at least k, hence $\mathrm{codim}_{\mathcal{A}_g}(N_{g,k}) \geq k+1$. To do better we need the following information.

Claim 8.8 (The non-degeneracy property). *The image of the map ϕ does not contain any line.*

Proof of the claim. Suppose the claim is false. Take a line L in the image of the map ϕ, and let $\mathcal{L}$ be the corresponding pencil of quadrics. By Proposition 6.3 and Remark 6.4, the general quadric in $\mathcal{L}$ has rank $\rho \leq g-k$. Then part (i) of Segre's Theorem 21.2 in §21 below implies that the Gauss image $\gamma(Z)$ of any irreducible component of the curve $Z = \phi^{-1}(L)$ is degenerate. This again leads to a contradiction. This proves the claim. $\square$

Claim 8.8 now implies that the image of ϕ spans a linear space of dimension at least $k+1$, hence (i) follows.
To prove part (ii) we now want to prove that $\dim(\mathcal{Q}_\zeta^{(r)}) > k+1$. Remember that the image of ϕ has dimension at least k by Claim 8.7. If the image has dimension at least $k+1$, then by Claim 8.8 it cannot be a projective space and therefore $\dim(\mathcal{Q}_\zeta^{(r)}) > k+1$. So we can assume that the image has dimension k. Therefore each component of the fibre F_Q over a general point Q in the image has dimension $(g-1)(r-2)$.
Consider now the projection of F_Q to R. If the image is positive-dimensional then there is a curve Z in R such that the image of the Gauss map of Z is contained in the vertex of Q. Then X is non-simple, a contradiction (see the proof of Claim 8.7).
Therefore the image of F_Q on R is constant, equal to a point ξ, hence $F_Q = (\mathbb{P}^{g-1})^{r-2} \times \{\xi\}$. By Lemma 8.5 there is a hyperplane H_ξ such that $TC_\xi = rH_\xi$, and $Q = 2H_\xi$. Therefore we have a rational map

$$\psi : R \dashrightarrow \mathbb{P}^{g-1\vee}$$

sending ξ to H_ξ. We notice that the image of ϕ is then equal to the 2-Veronese image if the image of ψ.
By Claim 8.7, the map ψ has finite fibres. By an argument as in Claim 8.8, we see that the image of ψ does not contain a line, hence is not a linear space. Thus it spans a space of dimension $s \geq k+1$. Then its 2-Veronese image, which is the image of ϕ spans a space of dimension at least $2s \geq 2k+2 > k+2$.
To prove part (iii), by part (ii), we can assume $r = 2$. It suffices to show that $\dim(\mathcal{Q}_\zeta) > k+1$, where $\mathcal{Q}_\zeta := \mathcal{Q}_\zeta^{(2)}$. Suppose instead that $\dim(\mathcal{Q}_\zeta) = k+1$ and set $\Sigma = \phi(R)$. We have to distinguish two cases:

(a) not every quadric in $\mathcal{Q}_\zeta$ is singular;
(b) the general quadric in $\mathcal{Q}_\zeta$ is singular, of rank $g - \rho < g$.

In case (a), consider the discriminant $\Delta \subset \mathcal{Q}_\zeta$, i.e. the scheme of singular quadrics in $\mathcal{Q}_\zeta$. This is a hypersurface of degree g, which, by Proposition 4.4, contains Σ with multiplicity at least k. Thus $\deg(\Sigma) \leq g/k \leq 3$ and Σ contains some line, so that we have the corresponding pencil $\mathcal{L}$ of singular quadrics. By Claim 8.8, one arrives at a contradiction.
Now we treat case (b). Let $g - h$ be the rank of the general quadric in Σ. One has $g - h \leq g - k$, hence $k \leq h$. Moreover one has $g - h \leq g - \rho$, i.e. $\rho \leq h$. Suppose first $\rho = h$, hence $\rho \geq k$. Let s be the dimension of the subspace

$$\Pi := \bigcap_{Q \in \mathcal{Q}_\zeta, \mathrm{rk}(Q) = g-\rho} \mathrm{Vert}(Q).$$

By applying part (iii) of Segre's Theorem 21.2 to a general pencil contained in $\mathcal{Q}_\zeta$, we deduce that

$$3\rho \leq g + 2s + 2. \tag{18}$$

Claim 8.9. *One has $s < r - k$.*

Proof. Suppose $s \geq \rho - k$. If Π' is a general subspace of Π of dimension $\rho - k$, then its intersection with $\mathbb{P}(T_{R,x})$, where $x \in R$ is a general point, is not empty. Since $\rho - k < g - k$ and X is simple, this is a contradiction (see [31], Lemma II.12). □

By (18) and Claim 8.9 we deduce that $\rho + 2k \leq g$ and therefore $3k \leq g$, a contradiction.
Suppose now $\rho < h$. Then part (iv) of Segre's Theorem 21.2 yields

$$\deg(\Sigma) \leq \frac{g - 2 - \rho}{h - \rho}.$$

The right-hand-side is an increasing function of ρ, thus $\deg(\Sigma) \leq g - h - 1 \leq 2k - 1$ because $g \leq 3k \leq 2k + h$. By Lemma 8.4 the locus Σ contains a line, and we can conclude as in the proof of the non-degeneracy property 8.8.

□

The following corollary was proved independently by Debarre [9] and includes a basic result by Mumford [27].

Corollary 8.10. *Let $g \geq 4$. Then:*

(i) *every irreducible component S of S_g has codimension $g+1$ in $\mathcal{X}_g$, hence S_g is locally a complete intersection in $\mathcal{X}_g$;*
(ii) *if $\xi = (X, \Theta_X, x)$ is a general point of S, then either Θ_X has isolated singularities or (X, Θ_X) is a product of an elliptic curve with a principally polarized abelian variety of dimension $g-1$;*
(iii) *every irreducible component N of $N_{g,0}$ has codimension 1 in $\mathcal{A}_g$;*
(iv) *if $(X, \Theta_X) \in N$ is a general element of an irreducible component N of $N_{g,0}$, then Θ_X has isolated singularities. Moreover for every point $x \in \mathrm{Sing}(\Theta_X)$, the quadric Q_x is smooth and independent of x.*

Proof. Take an irreducible component S of S_g and let N be its image via the map $\pi : \mathcal{X}_g \to \mathcal{A}_g$. Then there is a maximal integer $k \leq g-2$ such that N is contained in $N_{g,k}$. Suppose first that $1 \leq k \leq g-3$. Then by Theorem 8.6 the codimension of N in $\mathcal{A}_g$ is at least $k+2$. This implies that the codimension of S in $\mathcal{X}_g$ is at least $g+2$, which is impossible because S_g is locally defined in $\mathcal{X}_g$ by $g+1$ equations.
Then either $k = 0$ or $k = g-2$. In the former case S maps to an irreducible component N of $N_{g,0}$ which is a proper subvariety of codimension 1 in $\mathcal{A}_g$.
If $k = g-2$, then by Theorem 7.5 the polarized abelian variety (X, Θ_X) is a product of principally polarized abelian varieties. The resulting abelian variety has a vanishing thetanull and so N is contained in the divisor of a modular form (a product of thetanulls, cf. e.g. [27], p. 370) which is a component of $N_{g,0}$. Assertions (i)–(iii) follow by a dimension count (see Remark 7.4). Part (iv) follows by Propositions 4.1 and 8.3. □

Finally, we show a basic property of S_g.

Theorem 8.11. *The locus S_g is reduced.*

Proof. The assertion is well-known for $g \leq 3$, using the theory of curves. We may assume $g \geq 4$.
Let S be an irreducible component of S_g, let $N = \pi(S)$ and $k \leq g-2$ the maximal integer such that N is contained in $N_{g,k}$. As in the proof of Corollary 8.10, one has either $k = 0$ or $k = g-2$. Assume first $k = 0$, and let $\xi = (X, \Theta_X, x) \in S$ be a general point. We are going to prove that $\mathrm{Sing}(\Theta_X)$ is reduced of dimension 0.
First we prove that x has multiplicity 2 for Θ_X. Suppose this is not the case and x has multiplicity $r \geq 3$. Since $N_{g,0}$ has codimension 1 all polar quadrics of TC_ξ are the same quadric, say Q (see Proposition 6.3 and 8.3). By Lemma 8.5 the tangent cone TC_ξ is a hyperplane H with multiplicity r. Again by Proposition 6.3 all the derivatives of Q with respect to points $b \in H$ coincide with Q. By Proposition 5.3 the scheme $D_{b,2}$ supported at x is contained in $\mathrm{Sing}(\Theta_X)$. By taking into account Proposition 5.4 and repeating the same argument we see that this subscheme can be indefinitely extended to a 1-dimensional subscheme, containing x and contained in $\mathrm{Sing}(\Theta_X)$. This

implies that the corresponding component of $N_{g,0}$ is contained in $N_{g,1}$, which is not possible since the codimension of $N_{g,1}$ is at least 3.
If x has multiplicity 2 the same argument shows that the quadric Q_ξ is smooth. By Corollary 3.4, ξ is a smooth point of S_g and this proves the assertion.
Suppose now that $k = g-2$. Then by Theorem 7.5 N is contained in the locus of products $\mathcal{A}_{g,(1,g-1)}$, and for dimension reasons it is equal to it and then the result follows from a local analysis with theta functions. □

The following corollary is due to Debarre ([11]).

Corollary 8.12. *If (X, Θ_X) is a general point in a component of $N_{g,0}$ then Θ_X has finitely many double points with the same tangent cone which is a smooth quadric.*

9. A conjecture

As shown in [8], part (i) of Theorem 8.6 is sharp for $k = 1$ and $g = 4$ and 5. However, as indicated in [8], it is never sharp for $k = 1$ and $g \geq 6$, or for $k \geq 2$. In [8] we made the following conjecture, which is somehow a natural completion of Andreotti–Mayer's viewpoint in [2] on the Schottky problem.
Recall the *Torelli morphism* $t_g : \mathcal{M}_g \to \mathcal{A}_g$ which maps the isomorphism class of a curve C to the isomorphism class of its principally polarized Jacobian $(J(C), \Theta_C)$. As a map of orbifolds it is of degree 2 for $g \geq 3$ since the general abelian variety has an automorphism group of order 2 and the general curve one of order 1. We denote by $\mathcal{J}_g$ the *jacobian locus* in $\mathcal{A}_g$, i.e., the Zariski closure of $t_g(\mathcal{M}_g)$ in $\mathcal{A}_g$ and by $\mathcal{H}_g$ the *hyperelliptic locus* in $\mathcal{A}_g$, that is, the Zariski closure in $\mathcal{A}_g$ of $t_g(H_g)$, where H_g is the closed subset of $\mathcal{M}_g$ consisting of the isomorphism classes of the hyperelliptic curves. By Torelli's theorem we have $\dim(\mathcal{J}_g) = 3g-3$ and $\dim(\mathcal{H}_g) = 2g-1$ for $g \geq 2$.

Conjecture 9.1. *If $1 \leq k \leq g-3$ and if N is an irreducible component of $N_{g,k}$ not contained in $\mathcal{A}_g^{\mathrm{End}\neq\mathbb{Z}}$, then $\mathrm{codim}_{\mathcal{A}_g}(N) \geq \binom{k+2}{2}$. Moreover, equality holds if and only if one of the following happens:*

(i) *$g = k+3$ and $N = \mathcal{H}_g$;*
(ii) *$g = k+4$ and $N = \mathcal{J}_g$.*

By work of Beauville [4] and Debarre [11] the conjecture is true for $g = 4$ and $g = 5$. Debarre [10, 11] gave examples of components of $N_{g,k}$ for which the bound in Conjecture 9.1 for the codimension in $\mathcal{A}_g$ fails, but they are contained in $\mathcal{A}_g^{\mathrm{End}\neq\mathbb{Z}}$, since the corresponding abelian varieties are isogenous to products. Our main objective in this paper will be to prove the conjecture for $k = 1$.

Remark 9.2. The question about the dimension of the Andreotti–Mayer loci is related to the one about the loci $S_{g,k}$ introduced in §3. Note that $M_{g,k} = \pi^*(N_{g,k})$ is a subscheme of $S_{g,k}$. Let N be an irreducible component of $N_{g,k}$ not contained in $N_{g,k+1}$, let M be the irreducible component of $M_{g,k}$ dominating N and let Z be an irreducible component of $S_{g,k}$ containing M. We now give a heuristic argument. Recalling Proposition 4.3, we can consider Z to be *well-behaved* if $\operatorname{codim}_{S_g}(Z) = \binom{k+1}{2}$. Since M is contained in Z, one has also $\operatorname{codim}_{S_g}(M) \geq \binom{k+1}{2}$ and therefore $\operatorname{codim}_{\mathcal{A}_g}(N) \geq \binom{k+2}{2}$, which is the first assertion in Conjecture 9.1. In this setting, the equality holds if and only if Z is well-behaved and $M = Z$.

On the other hand, since M is got from Z by imposing further restrictions, one could expect that M is, in general, *strictly* contained in Z and therefore that $\operatorname{codim}_{\mathcal{A}_g}(N) > \binom{k+2}{2}$.

In this circle of ideas, it is natural to ask if $\mathcal{J}_g$ [resp. $\mathcal{H}_g$] is dominated by a well-behaved component of $S_{g,g-4}$ [resp. $S_{g,g-3}$]. This is clearly the case if $g = 4$.

A second, related, question is whether $\mathcal{J}_g$ [resp. $\mathcal{H}_g$] is contained in some irreducible subvariety of codimension $c < \binom{g-2}{2}$ [resp. $c < \binom{g-1}{2}$] in $\mathcal{A}_g$, whose general point corresponds to a principally polarized abelian variety (X, Θ_X) with $\operatorname{Sing}(\Theta_X)$ containing a subscheme isomorphic to $\operatorname{Spec}(\mathbb{C}[\epsilon_1, \ldots, \epsilon_k]/(\epsilon_i\epsilon_j \colon 1 \leq i, j \leq k))$ with $k = g-4$ [resp. $k = g-3$].

One might be tempted to believe that an affirmative answer to the first question implies a negative answer to the second. This is not the case. Indeed $\mathcal{H}_4$ is contained in the locus of Jacobians of curves with an effective even theta characteristic. In this case $S_{4,1}$ has two well-behaved irreducible components of dimension 8, one dominating $\mathcal{H}_4$ with fibres of dimension 1, the other one dominating $\mathcal{J}_4 \cap \theta_{4,0}$, where $\theta_{4,0}$ is the *theta-null locus*, see §19 and §10 below and [16]. These two components intersect along a 7-dimensional locus in S_4 which dominates $\mathcal{H}_4$.

Note that it is not always the case that a component of $S_{g,k}$ is well-behaved. For example, there is an irreducible component of $S_{g,g-2}$, the one dominating $\mathcal{A}_{g,(1,g-1)}$, which is also an irreducible component of S_g.

10. An Example for Genus $g = 4$

In this section we illustrate the fact that the quadrics associated to the singularities of the theta divisor may provide more information than obtained above. The locus S_4 in the universal family $\mathcal{X}_4$ consists of three irreducible components: i) one dominating $\theta_{0,4}$, the locus of abelian varieties with a vanishing theta-null; we call it $\mathcal{A}$; ii) one dominating the Jacobian locus $\mathcal{J}_4$; we call it $\mathcal{B}$; iii) one dominating $\mathcal{A}_{4,(3,1)}$, the locus of products of an elliptic curve with a principally polarized abelian variety of dimension 3.

The components $\mathcal{A}$ and $\mathcal{B}$ have codimension 5 in $\mathcal{X}_4$ and they intersect along a locus $\mathcal{C}$ of codimension 6. The image of $\mathcal{C}$ in $\mathcal{A}_4$ is the locus of Jacobians with a half-canonical g^1_3. The quadric associated to the singular point of order 2 of X in Θ_X has corank 1 at a general point. We refer to [16] for a characterization of the intersection $\theta_{0,4} \cap \mathcal{J}_4$.

Proposition 10.1. *If* $\xi = (X, \Theta_X, x)$ *is a point of* $\mathcal{C}$ *then* $\operatorname{Sing}(\Theta_X)$ *contains a scheme of length* 3 *at* x.

Proof. The scheme $\operatorname{Sing}(\Theta_X)$ contains a scheme $\operatorname{Spec}(\mathbb{C}[\epsilon])$ at x, say corresponding to the tangent vector $D_1 \in \ker q_x$, cf. Section 5. In order that ξ be a singular point of S_4, the quadrics q and q_1 associated to x and D_1 (cf. equation (6) and Proposition 3.3) must be linearly dependent. By Proposition 5.3 this implies that $\mathrm{Sing}(\Theta_X)$ contains a scheme of length 3 at x. □

Proposition 10.2. *The two components $\theta_{0,4}$ and $\mathcal{J}_4$ of $N_{4,0}$ in $\mathcal{A}_4$ are smooth at a non-empty open subset of their intersection and are tangent there.*

Proof. Let $\xi = (X, \Theta_X, x)$ be a general point of $\mathcal{C}$ so that the singular point x of Θ_X defines a quadric q_x of rank 3. Set $\eta = (X, \Theta_X) \in \mathcal{A}_4$. From the equations (3) and (6), one deduces that the tangent cone to $N_{4,0}$ at η is supported on $q_x^\perp$. Suppose, as we may by applying a suitable translation, that x is the origin 0 in X. Then $\theta_{0,4}$ is locally defined by the equation $\theta(\tau, 0)$ in $\mathcal{A}_4$ (see [16], §2). Hence $\theta_{0,4}$ is smooth at η with tangent space $q_x^\perp$. Now the jacobian locus $\mathcal{J}_4$ is also smooth at η by the injectivity of the differential of Torelli's morphism at non-hyperelliptic curves (see [15]) and the assertion follows. □

11. The Gauss map and tangencies of theta divisors

In this section we shall study the situation where a number of translates of the theta divisor of a principally polarized abelian variety are 'tangentially degenerate,' that is, are smooth but with linearly dependent tangent spaces or singular at prescribed points.

Let (B, Ξ) be a polarized abelian variety of dimension g, where Ξ is an effective divisor on B. As usual we let $B = \mathbb{C}^g/\Lambda$, with Λ a $2g$-dimensional lattice, and $p : \mathbb{C}^g \to B$ be the projection. So we have coordinates $z = (z_1, \ldots, z_g)$ in $\mathbb{C}^g$ and therefore on B, and we can keep part of the conventions and the notation used so far. Let $\xi = \xi(z)$ be the Riemann theta function whose divisor on $\mathbb{C}^g$ descends to Ξ via p.

If Ξ is reduced, then the Gauss map of (B, Ξ) is the morphism

$$\gamma = \gamma_\Xi \colon \Xi - \mathrm{Sing}(\Xi) \to \mathbb{P}(T_{B,0}^\vee), \quad x \mapsto \mathbb{P}(t_{-x}(T_{\Xi,\xi})),$$

where $t_{-x}(T_{\Xi,x})$ is the tangent space to Ξ at ξ translated to the origin. If $\Xi_b = t_b(\Xi)$ is the translate of Ξ by the point $b \in B$ defined by the equation $\xi(z - b) = 0$ then for $x \in \Xi - \mathrm{Sing}(\Xi)$ the origin is a smooth point for Ξ_{-x} and $\gamma(x) = \mathbb{P}(T_{\Xi_{-x},0})$.

As usual we have natural homogeneous coordinates $(z_1 : \ldots : z_g)$ in $\mathbb{P}(T_{B,0}) = \mathbb{P}^{g-1}$ and therefore dual coordinates in the dual projective space $\mathbb{P}(T_{B,0}^\vee) = \mathbb{P}^{g-1\vee}$. Then the expression of γ in coordinates equals $\gamma(p(z)) = (\partial_1\xi(z) : \ldots : \partial_g\xi(z))$ with $\partial_i = \partial/\partial z_i$.

The following lemma is well-known.

Lemma 11.1. *Let (B, Ξ) be a simple abelian variety of dimension g and Ξ an irreducible effective divisor on B. Then the map γ_Ξ has finite fibres. Moreover, for a smooth point $x \in \Xi$ there are only finitely many $b \in B$ such that Ξ_b is smooth at x and tangent to Ξ there.*

Proof. Suppose γ_Ξ does not have finite fibres. Then there is an irreducible curve C of positive geometric genus contained in the smooth locus of Ξ which is contracted by γ_Ξ to a point of $\mathbb{P}^{g-1\vee}$ corresponding to a hyperplane $\Pi \subset \mathbb{P}^{g-1}$. Then the image of the Gauss map γ_C of C lies in Π. This implies that C does not generate B and contradicts the fact that B is simple. As to the second statement, if it does not hold there exists an irreducible curve C such that for all $b \in C$ the divisor Ξ_b is smooth at x and tangent to Ξ. Then the curve $C' = \{x - b : b \in C\}$ is contained in Ξ and contracted by γ_Ξ. This contradicts the fact that γ_Ξ has finite fibres as we just proved. $\square$

Definition 11.2. Let h be a natural number with $1 \leq h \leq g-1$. The subscheme $T_h(B,\Xi)$ of $B \times B^h$ is defined in $\mathbb{C}^g \times (\mathbb{C}^g)^h$ with coordinates $(z, u_1, \dots, u_h)$ by the equations

$$\xi(z) = 0, \quad \xi(z-u_1) = 0, \quad \dots \quad , \xi(z-u_h) = 0,$$

$$\mathrm{rk} \begin{pmatrix} \partial_1\xi(z) & \cdots & \partial_g\xi(z) \\ \partial_1\xi(z-u_1) & \cdots & \partial_g\xi(z-u_1) \\ & \vdots & \\ \partial_1\xi(z-u_h) & \cdots & \partial_g\xi(z-u_h) \end{pmatrix} \leq h. \tag{19}$$

The projection to the first factor induces a morphism $p_1 : T_h(B,\Xi) \to \Xi$. Note that for $i = 1, \dots, h$ the variety E_i of codimension $h+1 \leq g$ in $B \times B^h$ defined by the equations

$$\xi(z) = 0, \quad \xi(z-u_j) = 0 \quad (j \neq i), \quad \text{and} \quad u_i = 0,$$

is contained in $T_h(B,\Xi)$. Moreover, the expected codimension of the irreducible components of $T_h(B,\Xi)$ is $g+1$. We will say that an irreducible component T of $T_h(B,\Xi)$ is *regular* if:

(i) $T \neq E_i$ for $i = 1, \dots, h$;

(ii) on a non-empty open subset of T all the rows of the matrix in (19) are non-zero.

In particular, if T is regular then $p_1(T) \not\subseteq \mathrm{Sing}(\Xi)$.

Proposition 11.3. *If B is simple each regular component of $T_h(B,\Xi)$ has dimension $hg-1$.*

Proof. Let us first assume $h = 1$ and denote a component of $T_h(B,\Xi)$ by T. By composing p_1 with the Gauss map $\gamma = \gamma_\Xi$, we obtain a rational map $\phi : T \dashrightarrow \mathbb{P}(T^\vee_{B,0})$. We shall prove that ϕ has finite fibres. Let v be a point in the image of ϕ coming from a point in the open subset as in Definition 11.2, (ii). By Lemma 11.1 there are only finitely many smooth points $z_1, \dots, z_a \in \Xi$ such that $\gamma(z_i) = v$ for $i = 1, \dots, a$. For each $1 \leq i \leq a$ we consider the theta divisor defined by $\xi(z_i - u) = 0$. Again by Lemma 11.1 there are only finitely many points $u_{i1}, \dots, u_{i\ell_i}$ in it such that (z_i, u_{ij}) may give rise to a point on T. So ϕ has finite fibres. Thus it is dominant and T has dimension $g-1$.

Now we prove the assertion by induction on h. Consider the projection $q : T \to B \times B^{h-1}$ by forgetting the last factor B. If the image T' of T is contained in $T_{h-1}(B,\Xi)$, then it is contained in a regular component of $T_{h-1}(B,\Xi)$, hence

by induction the codimension of T is at least $g+1$, while we know from the equations that it is at most $g+1$, and the assertion follows.
Suppose T' is not contained in $T_{h-1}(B,\Xi)$. Let $\mathbb{U}(h-1,g-1) \to \mathbb{G}(h-1,g-1)$ be the universal bundle over the Grassmannian of $(h-1)$-planes in $\mathbb{P}^{g-1}$. Then we define a rational map $\psi : T \dashrightarrow \mathbb{U}(h-1,g-1)$ in the following way. If $(z,u_1,\ldots,u_h)$ is a general point in T, then by the assumption that $T' \not\subseteq T_{h-1}(B,\Xi)$ the hyperplanes $\gamma(z),\gamma(z-u_1),\ldots,\gamma(z-u_{h-1})$ are linearly independent and we define

$$\psi(z,u_1,\ldots,u_h) = (\langle\gamma(z),\gamma(z-u_1),\ldots,\gamma(z-u_{h-1})\rangle,\gamma(z-u_h)).$$

We claim that Lemma 11.1 implies that the general fibre of ψ has dimension $\leq h(h-1)$. Indeed, in a fibre of ψ the points $\gamma(z),\gamma(z-u_1),\ldots,\gamma(z-u_{h-1})$ vary in a $(h-1)$-dimensional space giving at most $h(h-1)$ parameters and if $\gamma(z),\gamma(z-u_1),\ldots,\gamma(z-u_h)$ are fixed there are for $(z,u_1,\ldots,u_h)$ only finitely many possibilities. Thus $\dim(T)$ is bounded from above by

$$\dim(\mathbb{U}(h-1,g-1)) + h(h-1) = h(g-h) + (h-1) + h(h-1) = hg-1,$$

and this proves the assertion. □

We will also consider the closed subscheme $T_h^0(B,\Xi)$ of $T_h(B,\Xi)$ which is defined in $\mathbb{C}^g \times (\mathbb{C}^g)^h$ by the equations

$$\xi(z)=0, \quad \xi(z-u_i)=0, \quad i=1,\ldots,h$$

and

$$\mathrm{rk}\begin{pmatrix} \partial_1\xi(z) & \cdots & \partial_g\xi(z) \\ \partial_1\xi(z-u_1) & \cdots & \partial_g\xi(z-u_1) \\ & \vdots & \\ \partial_1\xi(z-u_h) & \cdots & \partial_g\xi(z-u_h) \end{pmatrix} = 0.$$

Finally we will consider the closed subset $\mathcal{T}_h(X,B)$ which is the union of $T_h^0(X,B)$ and of all regular components of $T_h(B,\Xi)$. Look at the projection

$$p = p_2 : \mathcal{T}(X,B) \to B^h.$$

Definition 11.4. We define $N_{0,h}(B,\Xi)$ to be the image of $\mathcal{T}_h(X,B)$ under the map p. More generally, for each integer k we define

$$N_{k,h}(B,\Xi) := \overline{\{u=(u_1,\ldots,u_h)\in B^h\colon u_1,\ldots,u_h\neq 0, \quad \dim(p_2^{-1}(u))\geq k\}}.$$

Roughly speaking, $N_{k,h}(B,\Xi)$ is the closure of the set of all $(u_1,\ldots,u_h)\in B^h$ such that Ξ contains an irreducible subvariety V of dimension $n\geq k$ and for all $z\in V$ either:

(a) $z, z-u_1,\ldots,z-u_h$ are smooth points of Ξ and $\gamma(z),\gamma(z-u_1),\ldots,\gamma(z-u_h)$ are linearly dependent, or
(b) all the points z, $z-u_1,\ldots,z-u_h$ are contained in $\mathrm{Sing}(\Xi)$.

In case (a) we say that the divisors $\Xi,\Xi_{u_1},\ldots,\Xi_{u_h}$, all passing through z with multiplicity one, are *tangentially degenerate* at z.

In case $h = 1$ we have only two divisors which are just *tangent* at z. We will drop the index h if $h = 1$. Thus, $N_0(B, \Xi)$ is the set of all u such that Ξ and Ξ_u are either tangent, or both singular, somewhere. Note that, if Ξ is symmetric, i.e., if $\xi(z)$ is even, as happens for the Riemann theta function, then $N_0(B, \Xi)$ contains the divisorial component $2\Xi := \{2\xi \colon \xi \in \Xi\}$, since $\gamma(\xi) = \gamma(-\xi)$ for all smooth points $\xi \in \Xi$.
One has the following result by Mumford (see [27], Proposition 3.2).

Theorem 11.5. *If (B, Ξ) is a principally polarized abelian variety of dimension g with Ξ smooth, then $N_0(B, \Xi)$ is a divisor on B algebraically equivalent to $\frac{(g+2)!}{6}\,\Xi$.*

As we will see later, the following result is related to Conjecture 9.1.

Proposition 11.6. *Suppose that (B, Ξ) is a simple principally polarized abelian variety of dimension g. Assume that $(B, \Xi) \notin N_{g,k}$. Then for $k \geq 0$ and $1 \leq h \leq g-1$ and for every irreducible component Z of $N_{k,h}(B, \Xi)$ one has* $\operatorname{codim}_{B^h}(Z) \geq k+1$.

Proof. By the definition of $N_{k,h}(B, \Xi)$ and by the fact that $\mathrm{Sing}(\Xi)$ has dimension $< k$ an irreducible component of $N_{k,h}(B, \Xi)$ can only be contained in the image of a regular component of $T_h(B, \Xi)$. The assertion follows by Proposition 11.3 and the fact that the fibres of p_2 on a regular component have dimension $\geq k$. □

12. Properties of the loci $N_{k,h}(B, \Xi)$

We will prove now a more precise result in the spirit of Proposition 11.6.

Proposition 12.1. *Suppose that (B, Ξ) is a simple, principally polarized abelian variety of dimension g. Assume that $N_{k,h}(B, \Xi)$ is positive-dimensional for some $h \geq 1$ and $k \geq 1$. Then $(B, \Xi) \in N_{g,k-1}$.*

Proof. We may assume $(B, \Xi) \notin N_{g,k}$, otherwise there is nothing to prove. We may also assume $h \geq 1$ is minimal under the hypothesis that $N_{k,h}(B, \Xi)$ has positive dimension.
Let $(u_1, \ldots, u_h)$ be a point in $N_{k,h}(B, \Xi)$, so that $\Xi_0 := \Xi$ and $\Xi_i := \Xi_{u_i}, i = 1, \ldots, h$, are tangentially degenerate along a non-empty subset V^0 of an irreducible subvariety V of B of dimension k such that Ξ_i for $i = 0, \ldots, h$ are smooth at all points $v \in V^0$.
For $j = 0, \ldots, h$ the element $s_j^{(i)} := \partial_i \xi(z - u_j)$ (with the convention that $u_0 = 0$) is a section of $\mathcal{O}(\Xi_j)$ when restricted to V since Ξ_j contains V. We know that for given j not all $s_j^{(i)}$ are identically zero on V^0. Our assumptions on the tangential degenerateness and minimality tell us that there exist non-zero rational functions a_j such that we have a non-trivial relation

$$\sum_{j=0}^{h} a_j s_j^{(i)} = 0 \quad \text{for} \quad i = 1, \ldots, g. \tag{20}$$

Suppose that the a_j are regular on all of a desingularization $f: W \to V$ of V. Then they are constant and the relation $\sum a_j s_j^{(i)} = 0$ holds on the whole of W. By writing relation (20) in different patches which trivialize the involved line bundles, and comparing them, we see that, if the transition functions are not all proportional, then we can shorten the original relation by subtracting two of them. This would contradict the minimality assumption. Therefore, we have that all of the divisors $f^*(\Xi - \Xi_{u_j})_{|V}$ with $j = 1, \ldots h$ are linearly equivalent. Since $u = (u_1, \ldots, u_h)$ varies in a subvariety of positive dimension this implies that the map $\mathrm{Pic}^0(B) \to \mathrm{Pic}(W)$ has a positive-dimensional kernel, and this is impossible since B is simple. So there exists an index j such that the function a_j has poles on a divisor Z_j of W. Note now that a point which is non-singular for all the divisors Ξ_j, $j = 1, \ldots, h$, is certainly not a pole for the functions a_j. Therefore, for each $j = 1, \ldots, h$, there an ℓ depending of j such that Z_j is contained in the divisorial part of the scheme $f^*(\mathrm{Sing}(\Xi_\ell))$. Moreover, since Z_j moves in a linear system on W, it cannot be contracted by the birational morphism f. This proves that Ξ_ℓ is singular along a variety of dimension $k-1$ contained in V, which proves the assertion. $\square$

Corollary 12.2. *Suppose that (B, Ξ) is a simple, principally polarized abelian variety of dimension g. Assume that $(B, \Xi) \notin N_{g,0}$, that is, Ξ is smooth. Then every irreducible component Z of $N_{0,h}(B, \Xi)$ is a divisor of B^h.*

Proof. Each irreducible component Z of $N_{0,h}(B, \Xi)$ is dominated by a regular component T of $T_h(B, \Xi)$, which has dimension $hg - 1$ by Proposition 11.3. The map $p: T \to Z$ is finite by Corollary 11.1. The assertion follows. $\square$

Remark 12.3. If we have two divisors Ξ_0, Ξ_1 which are tangentially degenerate along an irreducible k-dimensional variety V whose general point is smooth for both Ξ_0, Ξ_1, then Ξ_0, Ξ_1 are *both* singular along some $(k-1)$-dimensional variety contained in V. This can be easily proved by looking at the relation (20) in this case, and noting that the polar divisor Z_j is contained in $f^*(\mathrm{Sing}(\Xi_j))$, $j = 0, 1$.

Remark 12.4. Suppose $(B, \Xi) \notin N_{g,0}$. Then $N_0(B, \Xi)$ is described by all differences of pairs of points of Ξ having the same Gauss image.
Suppose $(B, \Xi) \in N_{g,0} - N_{g,1}$ and assume $\{x, -x\} = \mathrm{Sing}(\Xi)$ have multiplicity 2 and the quadric $Q_x = Q_{-x}$ is smooth. It may or may not be the case that $b = -2x \in N_1(B, \Xi)$. In any case, we claim that $N_1(B, \Xi) - \{-2x\}$ is contained in the set of all differences of points in $\gamma_\Xi^{-1}(Q_x^*)$ with x. Let us give a sketch of this assertion, which provides, in this case, a different argument for the proof of Proposition 11.6.
If $b \in N_1(B, \Xi) - \{-2x\}$, there is a curve $C \subset \Xi$ such that for $t \in C$ general $\gamma_\Xi(t) = \gamma_\Xi(t + b)$. Along C the divisors Ξ and Ξ_b are tangent, hence the curve contains x (see Remark 12.3). Note that the curve C is smooth at x. Indeed, locally at x, the divisor Ξ is a quadric cone of rank $g - 1$ in $\mathbb{C}^g$ with vertex x, whereas Ξ_b is a hyperplane through x, and they can only be tangent along a line.

Thus it makes sense to consider the image of x for γ_C, which is a point on Q_x. The point $x+b \in C+b \subset \Xi$ is smooth for Ξ and $\gamma(x+b)$ is clearly tangent to Q_x at $\gamma_C(x)$.

13. On N_{g-2} and N_{g-3} for Jacobians

The following result shows that the bound in Proposition 11.6 is sharp in the case $h=1$. Recall that a curve C is called *bielliptic* if it is a double cover of an elliptic curve.

Proposition 13.1. *Let C be a smooth, irreducible projective curve of genus g and let $(J=J(C),\Theta_C)$ be its principally polarized Jacobian. Then one has:*

(i) $\{\mathcal{O}_C(p-q)\in J : p,q\in C\}\subseteq N_{g-3}(J,\Theta_C)$*;*
(ii) *either C is bielliptic or the equality holds in (i);*
(iii) *if C is hyperelliptic, then*

$$\{\mathcal{O}_C(p-q)\in J : p,q\in C, h^0(C,\mathcal{O}_C(p+q))=2\}=N_{g-2}(J,\Theta_C).$$

Proof. We begin with (i). We assume that C is not hyperelliptic; the hyperelliptic case is similar and can be left to the reader. We may identify C with its canonical image in $\mathbb{P}^{g-1}$. Moreover, we identify J with $\mathrm{Pic}^{g-1}(C)$ and $\Theta_C\subset \mathrm{Pic}^{g-1}(C)$ with the set of effective divisor classes of degree $g-1$. Then the Gauss map $\gamma_C:=\gamma_{\Theta_C}$ can be geometrically described as the map which sends the class of an effective divisor D of degree $g-1$ such that $h^0(C,\mathcal{O}_C(D))=1$ to the hyperplane in $\mathbb{P}^{g-1}$ spanned by D (see [15], p. 360).

Take two distinct points p_1,p_2 on C. Then $|\omega_C(-p_1-p_2)|$ is a linear series of degree $2g-4$ and dimension $g-3$. For $i=1,2$ we let V_i be the subvariety of Θ_C which is the Zariski closure of the set of all divisor classes of the type $E+p_i$, where E is a divisor of degree $g-2$ contained in some divisor of $|\omega_C(-p_1-p_2)|$. Clearly $\dim(V_i)=g-3$, hence V_i is not contained in $\mathrm{Sing}(\Theta_C)$ which is of dimension $g-4$. If u denotes the divisor class p_2-p_1 then $x\mapsto x+u$ defines an isomorphism $V_1\cong V_2$. Moreover, for x in an non-empty open subset of V_1 we have $\gamma_C(x)=\gamma_C(x+u)$. This proves (i).

Conversely, assume there is a point $u\in \mathrm{Pic}^0(C)-\{0\}$, and a pair of irreducible subvarieties V_1,V_2 of Θ_C of dimension $g-3$ such that $x\mapsto x+u$ gives a birational map from V_1 to V_2. For x in a non-empty open subset $U\subset V_1$ we have $\gamma_C(x)=\gamma_C(x+u)$. If D and D' are the effective divisors of degree $g-1$ on C corresponding to x and $x+u$ and if E is the greatest common divisor of D and D' then by the geometric interpretation of the Gauss map γ_C there is an effective divisor F with $\deg(E)=\deg(F)$ such that $D+D'-E+F\equiv K_C$. Thus $(2D-E)+F\equiv K_C-u$.

Consider the linear series $|K_C-u|$, which is a g^{g-2}_{2g-2}. If this linear series has a base point q, then there is a point p such that $K_C-u-q\equiv K_C-p$, i.e. $D'-D\equiv u\equiv p-q$, proving (ii). So we may assume that $|K_C-u|$ has no base point. If C is not bielliptic, then $|K_C-u|$ determines a birational map of C to a curve in $\mathbb{P}^{g-2}$. On the other hand it contains the $(g-3)$-dimensional family of divisors of the form $(2D-E)+F$, which are singular along the divisor

$D-E$. This is only possible if $\deg(D-E)=1$, i.e., $u \equiv D'-D \equiv p-q$, with $p,q \in C$. But in this case $|K_C-u|$ has the base point q, a contradiction. This proves (ii).
Assume now C is hyperelliptic. Let p_1+p_2 be an effective divisor of the g^1_2 on C with $p_1 \neq p_2$. Then $|\omega_C(-p_1-p_2)|$ is a linear series of degree $2g-4$ and dimension $g-2$. For $i=1,2$ we let V_i be the subvariety of Θ_C which is the Zariski closure of the set of all divisors classes of the type $E+p_i$, where E is a divisor of degree $g-2$ contained in a divisor of $|\omega_C(-p_1-p_2)|$. The variety V_i has dimension $g-2$ and is not contained in $\mathrm{Sing}(\Theta_C)$, which is of dimension $g-3$. The translation over u induces an isomorphism $V_1 \cong V_2$ and for x in a non-empty subset U of V_1 we have $\gamma_C(x)=\gamma_C(x+u)$. Hence the left-hand-side in (iii) is contained in $N_{g-2}(J,\Theta_C)$.
Finally, assume there is a point $u \in \mathrm{Pic}^0(C)-\{0\}$, and a pair of irreducible subvarieties V, V' of Θ_C of dimension $g-2$ such that translation by u gives a rational map $V \dashrightarrow V'$ with $\gamma_C(x)=\gamma_C(x+u)$ on a non-empty open subset of V. Let D, D' be the effective divisors of degree $g-1$ on C corresponding to x and $x+u$. As in the proof of part (ii), let E be the greatest common divisor of D and D'. In the present situation the linear series $|K_C-u|$ of dimension $g-2$ contains the $(g-2)$-dimensional family of divisors of the form $2D-E$, which are singular along the divisor $D-E$. This means that $2(D-E)$ is in the base locus of $|K_C-u|$. This is only possible if $D=E+q$ for some point $q \in C$, and $K_C-u-2q \equiv (g-2)(p+q)$, where p is conjugated to q under the hyperelliptic involution. In conclusion, we have $u \equiv p-q$ and the equality in (iii) follows. □

Remark 13.2. The hypothesis that C is not bielliptic is essential in (ii) of Proposition 13.1. Let in fact C be a non-hyperelliptic bielliptic curve which is canonically embedded in $\mathbb{P}^{g-1}$. Let $f: C \to E$ be the bielliptic covering. One has $f_*\mathcal{O}_C \simeq \mathcal{O}_E \oplus \xi^\vee$, with $\xi^{\oplus 2} \simeq \mathcal{O}_E(B)$, where B is the branch divisor of f. Let $u \in \mathrm{Pic}^0(E)-\{0\}$ be a general point, which we can consider as a non-trivial element in $\mathrm{Pic}^0(C)$ via the inclusion $f^*: \mathrm{Pic}^0(E) \to \mathrm{Pic}^0(C)$ Note that f is ramified, hence f^* is injective. We want to show that $u \in N_{g-3}(J,\Theta_C)$, proving that equality does not hold in (i) in this case.
The canonical image of C is contained in a cone X with vertex a point $v \in \mathbb{P}^{g-1}$ over the curve E embedded in $\mathbb{P}^{g-2}$ as a curve of degree $g-1$ via the linear system $|\xi|$. Let us consider the subvariety W of $|\xi|$ consisting of all divisors $M \in |\xi|$ such that there is a subdivisor $p+q$ of M with $p,q \in E$ and $p-q \equiv u$. It is easily seen that W is irreducible of dimension $g-3$.
Notice that for M general in W one has $M=p+q+N$, with N effective of degree $g-3$. Therefore we may write $K_C \equiv f^*(M) \equiv f^*(p)+f^*(q)+F+F'$, where F,F' are disjoint, effective divisors of degree $g-3$ which are exchanged by the bielliptic involution.
We let V be the $(g-3)$-dimensional subvariety of Θ_C described by all classes of divisors D of degree $g-1$ on C of the form $D=f^*(p)+F$, as M varies in W. Then $D+u \equiv D' := f^*(q)+F$ and D and D' span the same hyperplane

through v in $\mathbb{P}^{g-1}$. Therefore, if $x \in V$ is the point corresponding to D, one has $\gamma_C(x) = \gamma_C(x+u)$. This proves that $u \in N_{g-3}(J, \Theta_C)$.

14. A boundary version of the Conjecture

We will now formulate a conjecture. As we will see later, it can be considered as a *boundary version* of Conjecture 9.1 (see also Proposition 11.6).

Conjecture 14.1. *Suppose that* (B, Ξ) *is a simple principally polarized abelian variety of dimension* g. *Assume that* $(B, \Xi) \notin N_{g,i}$ *for all* $i \geq k \geq 1$. *Then there is an irreducible component* Z *of* $N_k(B, \Xi)$ *with* $\operatorname{codim}_B(Z) = k+1$ *if and only if one of the following happens:*

(i) *either* $g \geq 2$, $k = g-2$ *and* B *is a hyperelliptic Jacobian,*
(ii) *or* $g \geq 3$, $k = g-3$ *and* B *is a Jacobian.*

One implication in this conjecture holds by Proposition 13.1. Note that the conjecture would give an answer for simple abelian varieties to the Schottky problem that asks for a characterization of Jacobian varieties among all principally polarized abelian varieties. For related interesting questions, see [30].

15. Semi-abelian Varieties of Torus Rank One

Let (B, Ξ) be a principally polarized abelian variety of dimension $g-1$. The polarization Ξ gives rise to the isomorphism

$$\phi_\Xi : B \to \hat{B} = \operatorname{Pic}^0(B),\, b \to \mathcal{O}_B(\Xi - \Xi_b).$$

and we shall identify B and $\hat{B}$ via this isomorphism. Thus an element $b \in \hat{B} \cong B$ determines a line bundle $L = L_b = \mathcal{O}_B(\Xi - \Xi_b)$ with trivial first Chern class. We can associate to L a *semi-abelian variety* $X = X_{B,b}$, namely the $\mathbb{G}_m$-bundle over B defined by L which is an algebraic group since it coincides with the theta group $\mathcal{G}_b := \mathcal{G}(L)$ of L (cf. [25], p. 221). This gives the well-known equivalence between $\hat{B}$ and $\operatorname{Ext}(B, \mathbb{G}_m)$, the group of extension classes of B with $\mathbb{G}_m$ in the category of algebraic groups (see [35], p. 184).
Both the line bundle L and the $\mathbb{G}_m$-bundle X determine a $\mathbb{P}^1$-bundle $\mathbb{P} = \mathbb{P}(L \oplus \mathcal{O}_B)$ over B with projection $\pi : \mathbb{P} \to B$ and two sections over B, say s_0 and s_∞ given by the projections $L \oplus \mathcal{O}_B \to L$ and $L \oplus \mathcal{O}_B \to \mathcal{O}_B$. If we set $P_0 = s_0(B)$ and $P_\infty = s_\infty(B)$ then we can identify $\mathbb{P} - P_0 - P_\infty$ with X. By [17], Proposition 2.6 on page 371 we have $\mathcal{O}_\mathbb{P}(1) \cong \mathcal{O}_\mathbb{P}(P_0)$. We can complete X by considering the non-normal variety $\overline{X} = \overline{X}_{B,b}$ obtained by glueing P_0 and P_∞ by a translation over $b \in B \cong \hat{B}$. On $\mathbb{P}$ we have the linear equivalence $P_0 - P_\infty \equiv \pi^{-1}(\Xi - \Xi_b)$. We set $E := P_\infty + \pi^{-1}(\Xi)$ and put $M_\mathbb{P} = \mathcal{O}_\mathbb{P}(E)$. This line bundle restricts to $\mathcal{O}_B(\Xi)$ on P_0 and to $\mathcal{O}_B(\Xi_b)$ on P_∞, and thus descends to a line bundle $M = M_{\overline{X}}$ on $\overline{X}$. We have $\pi_*(\mathcal{O}_\mathbb{P}(E)) = \mathcal{O}_B(\Xi) \oplus \mathcal{O}_B(\Xi_b)$ and $H^0(\mathbb{P}, M_\mathbb{P})$ is generated by two sections with divisors $P_\infty + \pi^{-1}(\Xi)$ and $P_0 + \pi^{-1}(\Xi_b)$. One concludes that $H^0(\overline{X}, M)$ corresponds to the sections of $M_\mathbb{P}$ such that translation over b carries its restriction to P_0 to the restriction

to P_∞. It follows that $h^0(\overline{X}, M) = 1$ with effective divisor $\overline{\Xi}$, which is called the *generalized theta divisor* on $\overline{X}$.

Analytically we can describe a section of $\mathcal{O}_{\bar{X}}(\overline{\Xi})$ on the universal cover $\mathbb{C}\times\mathbb{C}^{g-1}$ by a function

$$\xi(\tau, z) + u\,\xi(\tau, z-\omega),$$

where $\omega \in \mathbb{C}^{g-1}$ represents $b \in B = \mathbb{C}^{g-1}/\mathbb{Z}^{g-1} + \tau\mathbb{Z}^{g-1}$, $\xi(\tau, z)$ is Riemann's theta function for B and $u = \exp(2\pi i\zeta)$ is the coordinate on $\mathbb{C}^*$. This is called the *generalized theta function* of $\overline{X}$.

Let D be the Weil divisor on $\overline{X}$ that is the image of P_0 (or, what is the same, of P_∞). We consider a locally free subsheaf $T_{\rm vert}$ of the tangent sheaf to $\overline{X}$, namely the dual of the sheaf $\Omega^1(\log D)$ of rank g. If $d \in D$ is a point such that on the normalization $\mathbb{P}$ near the two preimages $z_1, \ldots, z_{g-1}, u$ and $z_1+b, \ldots, z_{g-1}+b, v$ are local coordinates such that $u = 0$ defines P_0 (resp. $v = 0$ defines P_∞) with $uv = 1$ then $T_{\rm vert}$ is generated by $\partial/\partial z_1, \ldots, \partial/\partial z_{g-1}, u\partial/\partial u - v\partial/\partial v$. Here $z_1, \ldots, z_{g-1}$ are coordinates on B. We interpret local sections of $T_{\rm vert}$ as derivations. In particular, if an effective Cartier divisor Y of $\overline{X}$ has local equation $f = 0$, then for each local section ∂ of $T_{\rm vert}$, the restriction to Y of ∂f is a local section of $\mathcal{O}_{\overline{X}}(Y)/\mathcal{O}_{\overline{X}}$. Then the subscheme $\mathrm{Sing}_{\rm vert}(\overline{\Xi})$ of $\overline{\Xi}$ is locally defined by the g equations

$$\partial_i f = 0 \quad \text{modulo } f \text{ in } \mathcal{O}_{\overline{X}}(\overline{\Xi})/\mathcal{O}_{\overline{X}} \tag{21}$$

with $f = 0$ a local equation of $\overline{\Xi}$ and ∂_i local generators of $T_{\rm vert}$.

The equations for $\mathrm{Sing}_{\rm vert}(\overline{\Xi})$ on X are thus given by

$$\begin{aligned} \xi(\tau, z) + u\,\xi(\tau, z-\omega) &= 0, \\ \xi(\tau, z-\omega) &= 0, \\ \partial_i\xi(\tau, z) + u\partial_i\xi(\tau, z-\omega) &= 0, \quad (1 \leq i \leq g-1). \end{aligned}$$

The points in $\mathrm{Sing}_{\rm vert}(\overline{\Xi})$ are of two sorts depending on whether they lie on the double locus D of $\overline{X}$ or not. The singular points of $\mathrm{Sing}(\overline{\Xi})$ on $X = \overline{X} - D$ are the points (z, u), with $u \neq 0$, which are zeros of $\xi(\tau, z)$ and $\xi(\tau, z-\omega)$ and such that $\gamma(\tau, z) = -u\gamma(\tau, z-\omega)$. That is, geometrically, these correspond under the projection on B to the points on B where Ξ and Ξ_b are tangent to each other. To describe the singular points of $\mathrm{Sing}(\overline{\Xi})$ on D, we consider the composition

$$\phi\colon B \cong P_0 \to \mathbb{P} \xrightarrow{\nu} \overline{X},$$

where ν is the normalisation. Then we have $\phi^{-1}(\mathrm{Sing}_{\rm vert}(\overline{\Xi})) = \mathrm{Sing}(\Xi)$ and the same if we identify B with P_∞.

Points of $\mathrm{Sing}_{\rm vert}(\overline{\Xi})$ determine again quadrics in $\mathbb{P}^{g-1}$ as follows. Note that the projective space $\mathbb{P}(T_{X,0})$ contains a point P_b corresponding to the tangent space $T_{\mathbb{G}_m} \subset T_X$ of the algebraic torus $\mathbb{G}_m$ at the origin. Recall that we write $\gamma(\tau, z)$ for the row vector

$$\gamma(\tau, z) = (\partial_1\xi, \ldots, \partial_{g-1}\xi)(\tau, z).$$

Then a singular point determines a, possibly indeterminate, quadric defined by the matrix

$$\begin{pmatrix} 0 & \gamma(\tau, z-\omega) \\ \gamma(\tau, z-\omega)^t & M \end{pmatrix} \tag{22}$$

with M the $(g-1)\times(g-1)$ matrix $(\partial/\partial\tau_{ij}\xi(\tau,z) + u\partial/\partial\tau_{ij}\xi(\tau,z-\omega))$. Note that we have $\gamma(\tau,z) = -u\gamma(\tau,z-\omega)$. The quadric passes through the point P_b. For a point on D the quadric is a cone with vertex P_b over a quadric in $\mathbb{P}^{g-2}$ given by M.

Remark 15.1. The above considerations show that a point in $\mathrm{Sing}_{\mathrm{vert}}(\overline{\Xi})$ has to be regarded as a point of multiplicity larger than 2 if the matrix (22) vanishes identically. This can happen only if z and $z+\omega$ are both singular for Ξ.

16. Standard Compactifications of Semi-Abelian Varieties

Let (B,Ξ) be a principally polarized abelian variety. We assume now that $\dim(B) = g-r$ with $r \geq 1$ and extend the considerations of the previous section.

The extensions of B by $\mathbb{G}_m^r$ are parametrized by $\mathrm{Ext}^1(B,\mathbb{G}_m^r) \cong \hat{B}^r$. To a point $(b_1,\ldots,b_r) \in \hat{B}^r$ one associates the $\mathbb{G}_m^r$-extension $X = X_b$ obtained as the fibre product of theta groups $\mathcal{G}_{b_1} \times_B \cdots \times_B \mathcal{G}_{b_r}$.

One of the type of degenerations of abelian varieties that we shall encounter are special compactifications of semi-abelian varieties. We shall call them *standard compactifications of torus rank r*. Let $b = (b_1,\ldots,b_r) \in \hat{B}^r$. The algebraic group $X = X_b$ sits in a $\mathbb{P}^1 \times_B \cdots \times_B \mathbb{P}^1$-bundle $\pi : \mathbb{P} \to B$ that is obtained as the fibre product over B of the $\mathbb{P}^1$-bundles $P_{b_i} = \mathbb{P}(L_{b_i} \oplus \mathcal{O}_B)$. The complement $\mathbb{P} - X$ is a union of $2r$ divisors $\sum_{i=1}^r \Pi_0^{(i)} + \Pi_\infty^{(i)}$, where $\Pi_0^{(i)}$ (resp. $\Pi_\infty^{(i)}$) is given by taking 0 (resp. ∞) in the i-th fibre coordinate, with projections $\pi_{i,0}, \pi_{i,\infty}$ to B.

We now define a non-normal variety obtained from $\mathbb{P}$ by glueing $\Pi_0^{(i)}$ with $\Pi_\infty^{(i)}$ for $i = 1,\ldots,r$. This identification depends on a $r\times r$-matrix $T = (t_{ij})$ with entries from $\mathbb{G}_m$ such that $t_{ii} = 1$ and $t_{ij} = t_{ji}^{-1}$. Let $s_0^{(i)} : B \to P_{b_i}$ (resp. $s_\infty^{(i)}$) be the zero-section (infinity section) of P_{b_i}. We glue the point

$$(\beta, x_1, \ldots, x_{i-1}, s_0^{(i)}(\beta), x_{i+1}, \ldots, x_r)$$

on $\Pi_0^{(i)}$ with the point

$$(\beta + b_i, t_{i,1}x_1, \ldots, t_{i,i-1}x_{i-1}, s_\infty^{(i)}(\beta), t_{i,i+1}x_{i+1}, \ldots, t_{i,r}x_r)$$

on $\Pi_\infty^{(i)}$. We denote the resulting variety by $\overline{X}$. It depends on the parameters $b \in \hat{B}^r$ and $t \in \mathrm{Mat}(r\times r, \mathbb{G}_m)$.

We have the linear equivalences $\Pi_0^{(i)} - \Pi_\infty^{(i)} \equiv \pi^*(\Xi - \Xi_{b_i})$. We set $E = \Pi_\infty + \pi^*(\Xi) = \sum \Pi_\infty^{(i)} + \pi^*(\Xi)$ and $E_i = \Pi_\infty - \Pi_\infty^{(i)}$ and $M := M_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}}(E)$. This line bundle restricts to $\mathcal{O}_{\Pi_0^{(i)}}(E_i + \pi_{i,0}^*(\Xi))$ on $\Pi_0^{(i)}$ and to $\mathcal{O}_{\Pi_\infty^{(i)}}(E_i + \pi_{i,\infty}^*(\Xi_{b_i}))$

on $\Pi_\infty^{(i)}$. Thus, by the definition of the glueing, M descends to a line bundle $\overline{M} := M_{\overline{X}}$ on $\overline{X}$. We have

$$\begin{aligned}\pi_*(M) =& \big(\otimes_{i=1}^r (\mathcal{O}_B \oplus L_i^{-1})\big) \otimes \mathcal{O}_B(\Xi) \\ &\cong \oplus_{k=1}^r \big(\oplus_{0\leq i_1<\ldots<i_k\leq r} \mathcal{O}_B(\Xi_{b_{i_1}+\ldots+b_{i_k}})\big).\end{aligned}$$

Hence we have $h^0(\mathbb{P}, M) = 2^r$. As in the preceding section one sees that only a 1-dimensional space of sections descends to sections of $\overline{M}$ on $\overline{X}$. In terms of coordinates $(\zeta_1,\ldots,\zeta_r,z_1,\ldots,z_{g-r})$ on the universal cover of X, where $(z_1,\ldots,z_{g-r}) \in \mathbb{C}^{g-r}$ are coordinates on the universal cover of B, a non-zero section of $\overline{M}$ is given by

$$\sum_{I\subseteq\{1,\ldots,r\}} u_I\, t_I\, \xi(\tau, z-\omega_I),$$

where I runs through the subsets of $\{1,\ldots,r\}$, $u_I = \prod_{i\in I} u_i$ with $u_i = \exp(2\pi\zeta_i)$, $t_I = \prod_{i,j\in I, i<j} t_{ij}$, $b_I = \sum_{i\in I} b_i$ and $\omega_I \in \mathbb{C}^{g-r}$ represents $b_I \in B$. This is the *generalized theta function* of $\overline{X}$, whose zero locus is the *generalized theta divisor* $\overline{\Xi}$ of $\overline{X}$.

Next we look at the singular points of $\overline{\Xi}$. All points in $\overline{\Xi}\cap D$ are singular points of $\overline{\Xi}$. However, just as in the rank one case in the preceding section we will in general disregard these singularities of $\overline{\Xi}$, and we will only look at the so-called *vertical singularities*, which we are going to define now (cf. [27], §2).

The locally free subsheaf T_{vert} of rank g of the tangent sheaf $T_{\overline{X}}$ is the dual of $\Omega^1(\log D)$. Its pull back to $\mathbb{P}$ is generated, in the (u,z)-coordinates, by the differential operators $u_i\partial/\partial u_i - v_i\partial/\partial v_i$ with $u_iv_i = 1$ for $i = 1,\ldots,r$ and $\partial_j = \partial/\partial z_j$ with $j = 1,\ldots,g-r$. We interpret local sections of T_{vert} as derivations as above and define the scheme $\mathrm{Sing}_{\mathrm{vert}}(\overline{\Xi})$ of *vertical singular points* of $\overline{\Xi}$ as the subscheme of $\overline{\Xi}$ defined by the equations (21) with $f = 0$ a local equation of $\overline{\Xi}$ for all local sections $\partial \in T_{\mathrm{vert}}$. This is independent of the choice of a local equation.

Lemma 16.1. *Let $(\overline{X},\overline{\Xi})$ be a standard compactification of a semi-abelian variety X of torus rank r with abelian part (B,Ξ). If $\dim(\mathrm{Sing}_{\mathrm{vert}}(\overline{\Xi})) \geq 1$ then $(B,\Xi) \in N_{g-r,0}$.*

Proof. The compactification $\overline{X}$ is a stratified space and the (closed) strata are (standard) compactifications of semi-abelian extensions of (B,Ξ) of torus rank s with $0 \leq s \leq r$. In view of the relations $\sum_I u_I t_I \partial_i \xi(z-\omega_I) = 0$ the vertical singularities of $\overline{\Xi}$ correspond to points where 2^s translates of Ξ are tangentially degenerate. Therefore we have $\dim(N_{1,h}(B,\Xi)) \geq 1$ with $h = 2^s$. By Lemma 12.1 it follows that $(B,\Xi) \in N_{g-r,0}$. □

17. Semi-abelian varieties of torus rank two

In the compactification of the moduli space of principally polarized abelian varieties of dimension g we shall encounter two types of degenerations of torus rank 2. The first of these is a standard compactification introduced above and its normalization is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over a principally polarized abelian variety of dimension $g-2$. For such a standard compactification the equations for $\mathrm{Sing}_{\mathrm{vert}}(\Xi)$ are given in terms of the (u,z)-coordinates by the system (we write t instead of $t_{1,2}$; note that $t \neq 0$)

$$\begin{aligned} \xi(z) - tu_1u_2\xi(z-\omega_1-\omega_2) =& 0, \\ u_1\xi(z-\omega_1) + tu_1u_2\xi(z-\omega_1-\omega_2) =& 0, \\ u_2\xi(z-\omega_2) + tu_1u_2\xi(z-\omega_1-\omega_2) =& 0, \\ \partial_i\xi(z) + u_1\partial_i\xi(z-\omega_1) + u_2\partial_i\xi(z-\omega_2) + tu_1u_2\partial_i\xi(z-\omega_1-\omega_2) =& 0, \\ i = 1, \ldots, g-2.& \end{aligned}$$

From this and the analogous equations in the v-coordinates (with $u_iv_i = 1$) we see that the vertical singular points of $\overline{\Xi}$ are essentially of three types:
(i) A point $x \in D$ that is the image via $\varphi : X \to \overline{X}$ of a point in $\Pi_{1,0} \cap \Pi_{2,0} \cong B$, (i.e. in the (u,z) coordinates one has $u_1 = u_2 = 0$) is a vertical singular point of $\overline{\Xi}$ if and only if it corresponds to a singular point of Ξ on $\Pi_{1,0} \cap \Pi_{2,0} \cong B$ and to a singular point of $\Xi_{b_1+b_2}$ on $\Pi_{1,\infty} \cap \Pi_{2,\infty} \cong B$.
(ii) A point $x \in D$ which is the image via φ of a point in $\Pi_{j,0}$ but not of a point in $\Pi_{3-j,0}$, (i.e., in the (u,z) coordinates one has $u_i = 0, u_{3-j} \neq 0$, for a $j = 1,2$) is a vertical singular point of $\overline{\Xi}$ if and only if

$$\begin{aligned} \xi(\tau,z) = 0, \quad \xi(\tau, z-\omega_{3-j}) =& 0, \\ \partial_i\xi(\tau,z) + u_{3-j}\partial_i\xi(\tau,z-\omega_{3-j}) =& 0, \quad i = 1, \ldots, g-2. \end{aligned}$$

i.e. if and only if z and $z-b$ are in Ξ and $\gamma_\Xi(z) = \gamma_\Xi(z-b)$.
(iii) A point $x \notin D$, (i.e. in the (u,z) coordinates one has $u_1 \neq 0 \neq u_2$) is a vertical singularity if and only if z is a singular point of the divisor $H \in |2\Xi_{b_1+b_2}|$ defined by the equation

$$\xi(\tau, z-\omega_1)\xi(\tau,z-\omega_2) = t\,\xi(\tau,z)\xi(\tau,z-\omega_1-\omega_2).$$

By the way, this occurs even in case (ii) above. Note also that, by the above equations, the existence of a vertical singularity implies that the theta divisors Ξ, Ξ_{b_1}, Ξ_{b_2} and $\Xi_{b_1+b_2}$ are tangentially degenerate at some point x of B, i.e., $z \in N_{0,3}(B,\Xi)$.
We call *of type* (i), (ii) or (iii) the singular points of $\overline{\Xi}$ according to whether cases (i), (ii) or case (iii) occur.

Remark 17.1. A point x in $\mathrm{Sing}_{\mathrm{vert}}(\overline{\Xi})$ again determines a quadric Q_x in $\mathbb{P}^{g-1}$. It is useful to remark that:
(a) in case (i) the quadric Q_x is a cone with vertex the line $L_{\mathbf{b}} := \langle P_{b_1}, P_{b_2}\rangle$ given by the tangent space to the $\mathbb{G}_m$-part over the quadric Q_z in $\mathbb{P}^{g-3}$ which corresponds to the singular point z of Ξ;
(b) in case (ii), say we are at a point with $u_1 = 0, u_2 \neq 0$. Then Q_x is a cone with vertex P_{b_1} over the quadric in the hyperplane $u_1 = 0$ with matrix

$$\begin{pmatrix} 0 & -\gamma(\tau,z)^t \\ -\gamma(\tau,z) & M \end{pmatrix}$$

with $\gamma(\tau, z) = (\partial_1\xi, \ldots, \partial_{g-2}\xi)(\tau, z)$ and the matrix M is given by

$$M = (\partial/\partial\tau_{ij}\xi(\tau, z) + u\partial/\partial\tau_{ij}\xi(\tau, z-\omega))_{1\leq i,j\leq g-2}.$$

In §15 we saw that all rank 1 compactifications of $\mathbb{G}_m$-extensions of a principally polarized abelian variety B form a compact family $\hat{B}$. This is no longer the case in the higher rank case. This is where semi-abelian varieties of *non-standard type* come into the picture. This will depend on choices. It is good to see this in some detail in the rank 2 case.

Given a principally polarized abelian variety (B, Ξ) of dimension $g-2$, all standard rank 2 compactifications of (B, Ξ) are of the form $(\overline{X}, \overline{\Xi})$ with $\overline{X} = \overline{X}_{B,b,t}$ with $b = (b_1, b_2) \in B \times B$ and $t \in \mathbb{C}^*$. Thus the parameter space may be identified (up to dividing by automorphisms) with the total space of the Poincaré bundle $\mathcal{P} \to B \times B$ minus the 0-section P_0. It is then natural to compactify this by looking at the associated $\mathbb{P}^1$-bundle and by glueing on it the 0-section P_0 with the infinity section P_∞. This in fact works and it is explained in [26], §7, and in [29]. We describe next the new objects that arise. We denote by L_i the line bundle associated to b_i, for $i = 1, 2$. We consider again the $\mathbb{P}^1 \times \mathbb{P}^1$-bundle P on B as in §16 and in the glueing operation described in §16, we let $t = t_{12}$ tend to 0 (or equivalently to ∞). Letting $t \to \infty$, one contracts $\Pi_{1,0}$ and $\Pi_{2,0}$ to the section $A = \Pi_{1,\infty} \cap \Pi_{2,\infty} \cong B$, and $\Pi_{1,\infty}$ and $\Pi_{2,\infty}$ to the section $\Delta = \Pi_{1,0} \cap \Pi_{2,0} \cong B$. In order to properly describe the glueing process, we have first to blow up the two sections A and Δ in P. Let us do that. Let $w : \tilde{P} \to P$ be the blow-up, on which we have the following divisors: α is the exceptional divisor over A and δ is the exceptional divisor over Δ; $\beta, \gamma, \epsilon, \zeta$ are the proper transforms on $\tilde{P}$ of $\Pi_{1,\infty}, \Pi_{2,0}, \Pi_{1,0}, \Pi_{2,\infty}$, respectively. We will abuse notation and denote by the same letters the restrictions of these divisors on the general fibre Φ of $\tilde{P}$ over B, which is a $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at two points, hence a $\mathbb{P}^2$ blown up at three points. Note that $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ are $\mathbb{P}^1$-bundles over B and one has

$$\begin{array}{llll} (23) & \alpha \cong \mathbb{P}(L_1^\vee \oplus L_2^\vee), & \gamma \cong \mathbb{P}(L_2 \oplus \mathcal{O}_B), & \epsilon \cong \mathbb{P}(L_1 \oplus \mathcal{O}_B), \\ & \delta \cong \mathbb{P}(L_1 \oplus L_2), & \zeta \cong \mathbb{P}((L_1 \otimes L_2) \oplus L_2), & \beta \cong \mathbb{P}((L_1 \otimes L_2) \oplus L_1). \end{array}$$

At this point one could be tempted to suitably glue α with γ and ϵ and δ with β and ζ. This however, as (23) shows, does not work. The right construction is instead the following.

One considers two $\mathbb{P}^2$-bundles $\phi_i : P_i^\sharp \to B$, $i = 1, 2$, associated to the vector bundles $L_1 \oplus L_2$ and $L_1^\vee \oplus L_2^\vee$ on B, i.e.,

$$P_1^\sharp = \mathbb{P}(L_1^\vee \oplus L_2^\vee \oplus \mathcal{O}_B), \quad P_2^\sharp = \mathbb{P}(L_1 \oplus L_2 \oplus \mathcal{O}_B).$$

There are three relevant $\mathbb{P}^1$-subbundles of the bundles $P_i^\sharp$, $i = 1, 2$, namely

$$\begin{array}{lllll} (24) & \bar{\alpha} = \mathbb{P}(L_1^\vee \oplus L_2^\vee), & \bar{\gamma} = \mathbb{P}(L_2^\vee \oplus \mathcal{O}_B), & \bar{\epsilon} = \mathbb{P}(L_1^\vee \oplus \mathcal{O}_B) & \text{in} \quad P_1^\sharp \\ & \bar{\delta} = \mathbb{P}(L_1 \oplus L_2), & \bar{\zeta} = \mathbb{P}(L_1 \oplus \mathcal{O}_B), & \bar{\beta} = \mathbb{P}(L_2 \oplus \mathcal{O}_B) & \text{in} \quad P_2^\sharp. \end{array}$$

As (23) and (24) show, we can glue P with $P_1^\sharp$ and $P_2^\sharp$ in such a way that α and δ are respectively glued to $\bar{\alpha}$ and $\bar{\delta}$; ϵ is glued to $\bar{\epsilon}$ and β to $\bar{\beta}$ with a shift by $-b_1$, and ζ to $\bar{\zeta}$ and γ to $\bar{\gamma}$ with a shift by $-b_2$. The resulting variety is denoted by $\overline{X} = \overline{X}_{B,b}$. As usual, we will denote by D its singular locus. On $\tilde{P}$ we have the line bundle

$$\tilde{M} = w^*\mathcal{O}_P(\Pi_{1,\infty} + \Pi_{2,\infty}) \otimes \mathcal{O}_{\tilde{P}}(\Xi - \alpha),$$

where we write L instead of $w^*(\pi^*(L))$ for a line bundle, or divisor, L on B. With similar notation, one has

$$\begin{aligned}\tilde{M} \cong \mathcal{O}_{\tilde{P}}(\alpha + \beta + \zeta + \Xi) &\cong \mathcal{O}_{\tilde{P}}(\delta + \epsilon + \zeta + \Xi_{b_1}) \\ &\cong \mathcal{O}_{\tilde{P}}(\delta + \gamma + \beta + \Xi_{b_2}).\end{aligned} \tag{25}$$

One has $\mathcal{O}_{P_1^\sharp}(1) = \mathcal{O}_{P_1^\sharp}(\bar{\alpha})$ and the following linear equivalences

$$\bar{\alpha} - \bar{\gamma} \equiv L_2^\vee, \quad \bar{\alpha} - \bar{\epsilon} \equiv L_1^\vee \tag{26}$$

where again we write L instead of $\phi_i^*(L)$, $i = 1, 2$, for a line bundle, or divisor, L on B (see again [17], Proposition 2.6, p. 371). From (25) and (26) one deduces that $\tilde{M}$ glues with the line bundle $M_1^\sharp = \mathcal{O}_{P_1^\sharp}(\bar{\alpha} + \Xi)$ and with the line bundle $M_2^\sharp = \mathcal{O}_{P_2^\sharp}(\Xi)$, to give a line bundle $\overline{M}$ on $\overline{X}$. In the obvious coordinates $(u, z) = ((u_1, u_2), (z_1, \ldots, z_{g-2}))$, which can be considered as coordinates on $\tilde{P} - (\alpha \cup \cdots \cup \zeta)$, the sections of $\tilde{M}$ can be expressed as

$$a\,\xi(\tau, z) + a_1 u_1\,\xi(\tau, z - \omega_1) + a_2 u_2\,\xi(\tau, z - \omega_2)$$

with a, a_1, a_2 complex numbers. By taking into account the glueing conditions, one sees that only a 1-dimensional subspace V of $H^0(\tilde{P}, \tilde{M})$ gives rise to a space of sections of $H^0(\overline{X}, \overline{M})$; V is generated by

$$\xi(\tau, z) + u_1\xi(\tau, z - \omega_1) + u_2\xi(\tau, z - \omega_2). \tag{27}$$

in the (u, z)-coordinates. Note that (27) is just obtained from the generalized theta function in 16 by letting $t = t_{12}$ tend to 0.
In conclusion, one has $h^0(\overline{X}, \overline{M}) = 1$, hence there is a unique effective divisor $\overline{\Xi} = \overline{\Xi}$ which is the zero locus of a non-zero section of $H^0(\overline{X}, \overline{M})$.
As in the standard case, we parametrize an open subset of $\mathrm{Pic}^0(\overline{\Xi})$ with points in the union of $P - \bigcup_{i=1,2;h=1\infty} \Pi_{i,h}$ with $P_1^\sharp - (\bar{\alpha} \cup \bar{\gamma} \cup \bar{\epsilon})$ and $P_1^\sharp - (\bar{\delta} \cup \bar{\zeta} \cup \bar{\beta})$.
We can define the *vertical singularities* of the divisor $\overline{\Xi}$, whose equations, in the (u, z) coordinates, take the form
(28)

$$\begin{aligned}\xi(\tau, z) = 0, \quad u_1\xi(\tau, z - \omega_1) = 0, \quad u_2\xi(\tau, z - \omega_2) &= 0, \\ \partial_i\xi(\tau, z) + u_1\partial_i\xi(\tau, z - \omega_1) + u_2\partial_i\xi(\tau, z - \omega_2) &= 0, \quad i = 1, \ldots, g - 2.\end{aligned}$$

Again the vertical singular points of $\overline{\Xi}$ are essentially of three types:
(i) Consider a point $x \in D$ which is the image via $\varphi : X \to \overline{X}$ of a point in $\Pi_{1,0} \cap \Pi_{2,0} \cong B$, i.e. in the (u, z) coordinates one has $u_1 = u_2 = 0$. Then this is a vertical singular point of $\overline{\Xi}$ if and only if it corresponds to a singular point of Ξ on $\Pi_{1,0} \cap \Pi_{2,0} \cong B$ and to a singular point of $\Xi_{b_1+b_2}$ on $\Pi_{1,\infty} \cap \Pi_{2,\infty} \cong B$.

(ii) Consider a point $x \in D$ which is the image via φ of a point in $\Pi_{i,0}$ but not of a point in $\Pi_{3-i,0}$, i.e. in the (u,z) coordinates one has $u_i = 0, u_{3-i} \neq 0$, for an $i = 1,2$. If $i = 1$, this is a vertical singular point of $\overline{\Xi}$ if and only if

$$\begin{aligned} \xi(\tau, z) = 0, \quad \xi(\tau, z - \omega_2) &= 0, \\ \partial_i \xi(\tau, z) + u_2\, \partial_i \xi(\tau, z - \omega_2) &= 0, \quad i = 1, \dots, g-2. \end{aligned}$$

i.e., if and only if z and $z - b_2$ are in Ξ and $\gamma_\Xi(z) = \gamma_\Xi(z - b_2)$. Thus points of this type correspond to points in $N_0(B, \Xi)$.
(iii) Consider a point $x \notin D$, i.e., in the (u,z) coordinates one has $u_1 \neq 0 \neq u_2$. Then equations (28) mean that z corresponds to a point in $\Xi \cap \Xi_{b_1} \cap \Xi_{b_2}$ where $\Xi, \Xi_{b_1}, \Xi_{b_2}$ are tangentially degenerate. In other words points of this type correspond to points in $N_{0,2}(B, \Xi)$.

Remark 17.2. A point x in $\mathrm{Sing}_{\mathrm{vert}}(\overline{\Xi})$ determines a quadric Q_x in $\mathbb{P}^{g-1}$. Remark 17.1 is still valid here.

A variant of this second type of rank-2 degeneration is obtained as follows. Given a $\mathbb{G}_m^2$-extension $X = X_b$ of B determined by a point $b = (b_1, b_2) \in \hat{B}^2$ we consider two $\mathbb{P}^2$-bundles $\mathbb{P}$ and $\mathbb{P}'$ over B:

$$\mathbb{P} = \mathbb{P}(\mathcal{O}_B \oplus L_1 \oplus L_2) \qquad \text{and} \qquad \mathbb{P}' = \mathbb{P}(L_2 \oplus L_1 \oplus (L_1 \otimes L_2)),$$

where we write as before L_i for L_{b_i}. We can glue these over the common $\mathbb{P}^1$-subbundle $\mathbb{P}(L_1 \oplus L_2)$. Then we glue the $\mathbb{P}^1$-subbundle $\mathbb{P}(\mathcal{O}_B \oplus L_1)$ of $\mathbb{P}$ with the $\mathbb{P}^1$-subbundle $\mathbb{P}(L_2 \oplus (L_1 \otimes L_2))$ of $\mathbb{P}'$ via a shift over b_2. Similarly, we glue the $\mathbb{P}^1$-subbundle $\mathbb{P}(\mathcal{O}_B \oplus L_2)$ of $\mathbb{P}$ with the $\mathbb{P}^1$-subbundle $\mathbb{P}(L_1 \oplus (L_1 \otimes L_2))$ of $\mathbb{P}'$ via a shift over b_1. In this way we obtain a non-normal variety over B. Both $\mathbb{P}$ and $\mathbb{P}'$ come with a relatively ample bundle $\mathcal{O}_{\mathbb{P}}(1)$ and $\mathcal{O}_{\mathbb{P}'}(1)$. On $\mathbb{P}$ we have the linear equivalences

$$\Pi_1 + \pi^*(\Xi_{b_1}) \equiv \Pi_2 + \pi^*(\Xi_{b_2}) \equiv \Pi_3 + \pi^*(\Xi_{b_1+b_2}).$$

with $\Pi_i = \mathbb{P}(\mathcal{O}_B \oplus L_i)$ for $i = 1,2$ and $\Pi_3 = \mathbb{P}(L_1 \oplus L_2)$. We let M be the line bundle $\mathcal{O}(\Pi_3 + \pi^*(\Xi_{b_1+b_2}))$ on $\mathbb{P}$ and M' the line bundle $\mathcal{O}(\Pi_3' + \pi^*(\Xi_{b_1+b_2}))$ on $\mathbb{P}'$, where Π_3' is the bundle $\mathbb{P}(L_1 \oplus L_2)$. This descends to a line bundle $\overline{M}$ on $\overline{X}$. This line bundle has a 1-dimensional space of sections generated by

$$\theta(\tau, z) = \xi(\tau, z) + u_1\, \xi(\tau, z_1 - \omega_1) + u_2\, \xi(\tau, z - \omega_2)$$

in suitable affine coordinates (u_1, u_2) on $\mathbb{P}^2$. Again the vertical singular points of $\overline{\Xi}$ are essentially of three types:
(i) A point $x \in D$ which is the image via $\varphi : X \to \overline{X}$ of a point in $\Pi_1 \cap \Pi_2 = \mathbb{P}(\mathcal{O}_B) \cong B$ is a vertical singularity if it corresponds to a singularity on Ξ, to a singularity on Ξ_{b_1} on $\Pi_1 \cap \Pi_3$ and a singularity on Ξ_{b_2} on $\Pi_2 \cap \Pi_3$;
(ii) A point $x \in D$ which is the image via φ of a point on one Π_3 but not of a point in Π_1 or Π_2 is a vertical singular point of $\overline{\Xi}$ if and only if $x \in \Xi_{b_1} \cap \Xi_{b_2}$ and $\gamma_{\Xi_{b_1}}(x) = \gamma_{\Xi_{b_2}}(x)$. Thus points of this type correspond to points in $N_0(B, \Xi)$. Something similar happens for the points on exactly one of Π_1 or Π_2.

(iii) A point $x \notin D$ is a vertical singularity if $x \in \Xi \cap \Xi_{b_1} \cap \Xi_{b_2}$ and $\Xi, \Xi_{b_1}, \Xi_{b_2}$ are tangentially degenerate at x. In other words points of this type correspond to points in $N_{0,2}(B, \Xi)$.
We see that the compactification depends on a choice, but in both cases we can deal explicitly with the singularities of the theta divisors.

REMARK 17.3. The variant just given corresponds to a tesselation of $\mathbb{R}^2$ given by the lines $x = a$, $y = b$ and $x + y = c$ with $a, b, c \in \mathbb{Z}$. To a triangle with integral vertices $(n, m), (n+1, m)$ and $(n, m+1)$ (resp. $(n-1, m)$, $(n, m-1)$ and (n, m)) we associate the $\mathbb{P}^2$-bundle $\mathbb{P}(L_{nb_1+mb_2} \oplus L_{(n+1)b_1+mb_2} \oplus L_{nb_1+(m+1)b_2})$ (resp. $\mathbb{P}(L_{(n-1)b_1+mb_2} \oplus L_{nb_1+(m-1)b_2} \oplus L_{nb_1+mb_2})$ and we glue the bundles belonging to adjacent triangles over the $\mathbb{P}^1$-bundle defined by the common edge. Then the generator τ_1 (resp. τ_2) of $\mathbb{Z}^2$ acts by glueing the $\mathbb{P}^1$-bundle associated to an edge to the $\mathbb{P}^1$-bundle associated to the translate by $x \mapsto x+1$ (resp. $y \mapsto y + 1$) of this edge using a translation over b_2 (resp. b_1). The quotient under $\mathbb{Z}^2$ is the non-normal variety we just constructed. Also the earlier compactifications thus correspond to tesselations (see [26], §7).

18. COMPACTIFICATION OF $\mathcal{A}_g$

In order to study the Andreotti-Mayer loci we need to compactify $\mathcal{A}_g$. The moduli space $\mathcal{A}_g$ admits a 'minimal' compactification, the *Satake compactification* constructed by Satake and Baily-Borel in characteristic 0 and by Faltings-Chai over the integers (see [32], [13]). This compactification $\mathcal{A}_g^*$ is an orbifold or stack which admits a stratification

$$\mathcal{A}_g^* = \sqcup_{i=0}^{g} \mathcal{A}_i$$

and the closure of $\mathcal{A}_m$ in $\mathcal{A}_g^*$ is $\mathcal{A}_m^* = \sqcup_{i=0}^{m} \mathcal{A}_i$. This compactification is highly singular for $g \geq 2$. Smooth compactifications can be constructed by the theory developed by Mumford and his co-workers in characteristic 0 and by Faltings-Chai in general. These compactifications depend on combinatorial data. We shall use the *Voronoi compactification* $\tilde{\mathcal{A}}_g = \tilde{\mathcal{A}}_g^{\mathrm{Vor}}$ as described by Namikawa, Nakamura and Alexeev (see [29, 28, 5]). This compactification is a smooth orbifold with a natural map $q : \tilde{\mathcal{A}}_g \to \mathcal{A}_g^*$. It has the stratification induced by that of $\mathcal{A}_g^*$: the stratum

$$\mathcal{A}_g^{(r)} = q^{-1}(\mathcal{A}_{g-r})$$

is called the stratum of *quasi-abelian varieties of torus rank* r.
In §20 we also shall use another compactification, the *perfect cone compactification*, cf. [29, 34]. It also has a map to the Satake compactification denoted by $q : \tilde{\mathcal{A}}_g^{\mathrm{pc}} \to \mathcal{A}_g^*$; sometimes we shall denote $\tilde{\mathcal{A}}_g^{\mathrm{pc}}$ simply by $\tilde{\mathcal{A}}_g$ in order to avoid introducing more notation. It has the following properties: i) the boundary is an irreducible $\mathbb{Q}$-Cartier divisor, ii) the general point of the boundary is smooth, iii) the codimension of $\mathcal{A}_g^{(r)} = q^{-1}(\mathcal{A}_{g-r})$ equals r, iv) there is a dense open subset U of $\mathcal{A}_g^{(\leq 4)}$ and a family of compactified semi-abelian varieties $\mathcal{X} \to U$ extending the universal compactified semi-abelian variety over $\mathcal{A}_g^{\leq 1}$ such that

the standard compactifications of §16 form a dense open subset of $\mathcal{A}_g^{(r)}$. We point out that in this case for $r \leq 4$ the general fibre of $q : \mathcal{A}_g^{(r)} \to \mathcal{A}_{g-r}$ has dimension $gr - r(r+1)/2$, thus $\dim(\mathcal{A}_g^{(r))}) = g(g+1)/2 - r$.
We define for our chosen compactification $\tilde{\mathcal{A}}_g$ the boundary as $\partial\tilde{\mathcal{A}}_g := \mathcal{A}_g^{(\geq 1)}$. Moreover we set $\mathcal{A}_g^{(\leq r)} := \tilde{\mathcal{A}}_g - \mathcal{A}_g^{(\geq r+1)}$.
For the Voronoi compactification the fibres of the map $q : \mathcal{A}_g^{(\leq r)} \to \mathcal{A}_g^*$ are well-behaved if $r \leq 4$. Indeed, the points of $\mathcal{A}_g' := \mathcal{A}_g^{(\leq 4)}$ correspond to so-called *stable* quasi-abelian varieties which are compactifications of semi-abelian varieties, which can be explicitely described (see [29, 28, 5] and §§15, 17 for torus ranks 1 and 2). Thus one can define the vertical singular locus and the Andreotti-Mayer loci on the partial compactification $\mathcal{A}_g'$ (see Remark 18.1 below). For higher torus rank the situation is more complicated. For instance the fibres of $q : \mathcal{A}_g^{(\geq r)} \to \mathcal{A}_{g-r}^*$ might be non-reduced. However we will not need $r > 4$ here.
An alternative approach might be to use the idea of Alexeev and Nakamura (cf. [1], [28], [5]) who have constructed canonical limits for 1-dimensional families of abelian varieties equipped with principal theta divisors.
The stable quasi-abelian varieties that occur in $\mathcal{A}_g'$ for torus rank 1 and 2 are exactly those described in Section 15 and 17. For torus rank 3 these are described by the tesselations of $\mathbb{R}^3$ occuring on p. 188 of [28], cf. also Remark 17.3. The open stratum of $\mathcal{A}_g^{(3)}$ over $\mathcal{A}_{g-3}$ corresponds to the standard compactifications (see §16), obtained by glueing six $\mathbb{P}^3$-bundles over a $(g-3)$-dimensional abelian variety B generalizing the construction for torus rank 2 where two $\mathbb{P}^2$-bundles were glued. These closed strata correspond to degenerations of the matrix T of the glueing data on which the standard compactifications depend (see §16). For instance the codimension 3 stratum corresponds to the fact that in T two of the three elements above the main diagonal tend to zero (or to ∞).

Remark 18.1. As we remarked before, for stable quasi-abelian varieties corresponding to points $(\overline{X}, \overline{\Xi})$ of $\mathcal{A}_g'$ one can define the vertical singularities $\mathrm{Sing}_{\mathrm{vert}}(\overline{\Xi})$ using $\Omega^1(\log D)$ as in the previous sections. One checks that for these compactifications the analogue of Lemma 16.1 still holds.

We will need the following result from [14].

Theorem 18.2. *Let Z be an irreducible, closed subvariety of $\tilde{\mathcal{A}}_g$. Then $Z \cap \partial\tilde{\mathcal{A}}_g$ is not empty as soon as* $\mathrm{codim}_{\tilde{\mathcal{A}}_g}(Z) < g$.

19. The Andreotti–Mayer loci and the boundary

We are working with a fixed compactification $\tilde{\mathcal{A}}_g = \tilde{\mathcal{A}}_g^{\mathrm{Vor}}$ and, as indicated in §18 above, we can define the Andreotti–Mayer loci over the part $\mathcal{A}_g' = \mathcal{A}_g^{(\leq 4)}$ of $\tilde{\mathcal{A}}_g$. We have $N_{g,k}$ as a subscheme of $\mathcal{A}_g$ and we define $\tilde{N}_{g,k}$ as the schematic closure. The support of $\tilde{N}_{g,k}$ contains the set of points corresponding to pairs $(\overline{X}, \overline{\Xi})$ such that $\mathrm{Sing}_{\mathrm{vert}}(\overline{X})$ has a component of dimension at least k (see [27]).

It is interesting to look at the case $k = 0$, which has been worked out by Mumford [27] and Debarre [11]. In this case $\tilde{N}_{g,0}$ is a divisor and by Theorem 18.2, every irreducible component N of this divisor intersects $\partial\tilde{\mathcal{A}}_g$. Let M be an irreducible component of $N \cap \partial\tilde{\mathcal{A}}_g$, which has dimension $\binom{g+1}{2} - 2$.

First of all, notice that M cannot be equal to $\mathcal{A}_g^{(\geq 2)}$. This follows by the results in §17 and by Propositions 11.6 and 12.1. More generally, in the same way, one proves that M cannot contain $\mathcal{A}_g^{(r)}$ for any $r = 2, 3$ and 4.

Hence M intersects $\mathcal{A}_g^{(1)}$ in a non-empty open set of M, i.e., the intersection with the boundary has points corresponding to semi-abelian varieties of torus rank 1. If M does not dominate $\mathcal{A}_{g-1}$ via the map q, then each fibre must have dimension $g - 1$. By Proposition 11.6 this implies that M dominates $N_{0,g-1}$.

If M dominates $\mathcal{A}_{g-1}$ via q, the fibre of $q_{|M}$ over a general point $(B, \Xi) \in \mathcal{A}_{g-1}$ is $N_0(B, \Xi)$.

Recall now that Debarre proves in [11] that $N_{0,g}$ has two irreducible components, one of which is the so-called *theta-null* component $\theta_{0,g}$: the general abelian variety (X, Θ_X) in $\theta_{0,g}$, with Θ_X symmetric, is such that Θ_X has a unique double point which is a 2-torsion point of X lying on Θ_X.

Let $M_{0,g}$ be the other component. The general abelian variety (X, Θ_X) in $M_{0,g}$, with Θ_X symmetric, is such that Θ_X has exactly two double points x and $-x$. It is useful to recall that, by Corollary 8.12, at a general point of either one of these component of $N_{g,0}$, the tangent cone to the theta divisor at the singular points is a smooth quadric.

The component $\theta_{0,g}$ intersects the boundary in two components, $\theta'_{0,g}, \theta''_{0,g}$, one dominating $\theta_{0,g-1}$, the other dominating $\mathcal{A}_{g-1}$ with fibre over the general point $(B, \Xi) \in \mathcal{A}_{g-1}$ given by the component 2Ξ of $N_0(B, \Xi)$ (see §11). Also $M_{0,g}$ intersects the boundary in two irreducible divisors $M'_{0,g}, M''_{0,g}$. The former is irreducible and dominates $M_{0,g-1}$, the latter dominates $\mathcal{A}_{g-1}$ with fibre over the general point $(B, \Xi) \in \mathcal{A}_{g-1}$ given by the components of $N_0(B, \Xi)$ different from 2Ξ.

The main ingredient for Debarre's proof of the irreducibility of $M_{0,g}$ is a monodromy argument which implies that, if (B, Ξ) is a general principally polarized abelian variety of dimension g, then $N_0(B, \Xi)$ consists of only two irreducible components.

Remark 19.1. Let (B, Θ_B) be a general element in $\theta_{0,g}$ and let $(\overline{X}, \overline{\Xi})$ be a semi-abelian variety of torus rank one with abelian part B. Then there are no points in $\mathrm{Sing}_{\mathrm{vert}}(\overline{\Xi})$ with multiplicity larger than 2. This follows from the fact that Θ_B has a unique singular point and by Remark 15.1.

We finish this section with the following result which will be useful later on. It uses the notion of asymptotic cone given in §6.

Proposition 19.2. *One has:*

(i) *let $g \geq 3$, let (B, Ξ) be a general point of $\theta_{0,g}$ and let x be the singular point of Ξ. Then the asymptotic cone $TC_\xi^{(4)}$ is strictly contained in the quadric tangent cone Q_x;*

(ii) *let $g \geq 4$, let (B, Ξ) be a general point of $M_{0,g}$ and let $x, -x$ be the singular points of Ξ. Then the asymptotic cone $TC_{\xi}^{(3)}$ is strictly contained in the quadric tangent cone $Q_x = Q_{-x}$.*

Proof. Degenerate to the jacobian locus and apply the results from [18, 19]. □

20. The Conjecture for $N_{1,g}$

In this section we prove Conjecture 9.1 for $k = 1$. We consider an irreducible component N of $\tilde{N}_{g,1}$ which is of codimension 3. The first observation is that the assumption about the codimension of N implies that the generic principally polarized abelian variety is simple since by Remark 7.4 every irreducible component of $\mathcal{A}_g^{(\mathrm{ns})}$ has codimension $\geq g - 1 > 3$ if we assume $g \geq 5$.

Proposition 20.1. *Let $g \geq 6$ and let N be an irreducible component of $\tilde{N}_{g,1}$ which is of codimension 3 in $\tilde{\mathcal{A}}_g$. Then N intersects the stratum $\mathcal{A}_g^{(1)}$.*

Proof. We begin by remarking that N cannot be complete in $\mathcal{A}_g$ in view of Theorem 18.2. Therefore N intersects $\partial\tilde{\mathcal{A}}_g$. Here we shall use the perfect cone compactification, see §18. Since $\partial\tilde{\mathcal{A}}_g$ is a divisor in $\tilde{\mathcal{A}}_g$ it intersects N in codimension one. Let M be an irreducible component of $N \cap \partial\tilde{\mathcal{A}}_g$. It is our intention to prove that M has a non-empty intersection with $\mathcal{A}_g^{(1)}$.
Suppose that $M \subseteq \mathcal{A}_g^{(\geq 4)}$. For dimension reasons we have $M = \mathcal{A}_g^{(\geq 4)}$. Since we are using a compactification $\tilde{\mathcal{A}}_g$ such that the general point of $\mathcal{A}_g^{(4)}$ corresponds to a standard compactification $(\overline{X}, \overline{\Xi})$ of torus rank 4 with abelian part $(B, \Xi) \in \mathcal{A}_{g-4}$ we deduce from Lemma 16.1 and Remark 18.1 that if $\dim(\mathrm{Sing}_{\mathrm{vert}}(\overline{\Xi})) \geq 1$ then $(B, \Xi) \in N_{g-4,0}$. But for $g \geq 5$ the locus $N_{g-4,0}$ is a divisor in $\mathcal{A}_{g-4}$ and we obtain the inequality $\mathrm{codim}_{\partial\tilde{\mathcal{A}}_g}(M) \geq \mathrm{codim}_{\partial\tilde{\mathcal{A}}_g}(\mathcal{A}_g^{(4)}) + 1 \geq 5$, a contradiction. Therefore we can assume that $M \cap \mathcal{A}_g^{(\leq 3)} \neq \emptyset$.
Suppose that $M \cap \mathcal{A}_g^{(3)}$ has codimension 1 in $\mathcal{A}_g^{(3)}$. Then either M maps dominantly to $\tilde{\mathcal{A}}_{g-3}$ via the map $q : \mathcal{A}_g^{(3)} \dashrightarrow \tilde{\mathcal{A}}_{g-3}$ and M intersects the general fibre F of q in a divisor, or M maps to a divisor in $\mathcal{A}_{g-3}$ under q with full fibres F contained in M.
The former case is impossible by Proposition 12.1. In the latter case for a general (B, Ξ) in $q(M)$ all the quasi-abelian varieties $(\overline{X}, \overline{\Xi})$ in the fibre F over (B, Ξ) must have a 1-dimensional vertical singular locus of $\overline{\Xi}$. Note that $(\overline{X}, \overline{\Xi})$ corresponds to a standard compactification as considered in §16. By the discussion given in §16 and by Proposition 11.6, we see that $q(M)$ has to be contained in $N_{g-3,1}$, against Theorem 8.6. We thus conclude that $M \cap \mathcal{A}_g^{(\leq 2)} \neq \emptyset$.
Suppose that $M \cap \mathcal{A}_g^{(\geq 2)} \neq \emptyset$. Then $M \cap \mathcal{A}_g^{(2)}$ has codimension 2 in $\mathcal{A}_g^{(2)}$. As above we have that $q(M)$ is contained in $N_{g-2,0}$.
Suppose first $q(M)$ is dense in a component of $N_{g-2,0}$. If (B, Ξ) is a general element of $q(M)$, then M intersects the fibre of $q : \mathcal{A}_g^{(2)} \dashrightarrow \tilde{\mathcal{A}}_{g-2}$ over (B, Ξ) in

codimension one. This gives a contradiction by the analysis §17 and Proposition 11.6.

Suppose that $q(M)$ is not dense in a component of $N_{g-2,0}$. If (B,Ξ) is a general element of $q(M)$, then M contains the full fibre of $q : \mathcal{A}_g^{(2)} \to \mathcal{A}_{g-2}$ over (B,Ξ). By taking into account the analysis §17 and Proposition 11.6, this implies $q(M)$ contained in $N_{g-2,1}$, giving again a contradiction.

This proves that $M \cap \mathcal{A}_g^{(\leq 1)} \neq \emptyset$. □

From now on we use again the Voronoi compactification. Let $g \geq 4$ and let N be an irreducible component of $N_{g,1}$ of codimension 3 in $\mathcal{A}_g$. As in the proof above we denote by M an irreducible component of the intersection of the closure of N in $\tilde{\mathcal{A}}_g$ with the boundary $\partial\tilde{\mathcal{A}}_g$. According to Lemma 20.1 the morphism $q : \tilde{\mathcal{A}}_g \to \mathcal{A}_g^*$ induces a rational map $\alpha : M \dashrightarrow \tilde{\mathcal{A}}_{g-1}$, whose image is not contained in $\partial\tilde{\mathcal{A}}_{g-1}$.

Lemma 20.2. *In the above setting, the Zariski closure of $q(M)$ in $\tilde{\mathcal{A}}_{g-1}$ is:*

(i) *either an irreducible component N_1 of $\tilde{N}_{g-1,1}$ of codimension 3 in $\tilde{\mathcal{A}}_{g-1}$;*
(ii) *or an irreducible component N_0 of $\tilde{N}_{g-1,0}$ and in this case:*
  (a) *if $\eta = (B,\Xi) \in N_0$ is a general point, then the closure of $q^{-1}(\eta)$ in B is an irreducible component of $N_1(B,\Xi)$ of codimension 2 in B;*
  (b) *if $\xi = (\overline{X},\overline{\Xi}) \in M$ is a general point, then $\mathrm{Sing}_{\mathrm{vert}}(\overline{\Xi})$ meets the singular locus D of $\overline{X}$ in one or two points, whose associated quadric has corank 1.*

Proof. If $q(M) \subseteq N_{g-1,1}$, then Theorem 8.6 implies $3 \leq \mathrm{codim}_{\mathcal{A}_{g-1}}(q(M)) \leq \mathrm{codim}_{\partial\tilde{\mathcal{A}}_g}(M) = 3$ and the closure of $q(M)$ must an irreducible component of $\tilde{N}_{g-1,1}$. If $q(M) \not\subseteq N_{g-1,1}$ then by Proposition 12.1 we have that $q(M) \subseteq N_{g-1,0}$ and the fibre $q^{-1}(B,\Xi) \subseteq N_1(B,\Xi)$. By Proposition 11.6 we have $\mathrm{codim}_B(N_1(B,\Xi)) \geq 2$ and since $N_{g-1,0}$ is a divisor in $\mathcal{A}_{g-1}$ we see that (iia) follows. The last statement (iib) follows from Remark 12.3, the analysis in Section 17, the description of $N_{g,0}$ by Mumford and Debarre (see [27], [11], and §19) and Corollary 8.12. □

We are now ready for the proof of the conjecture for $N_{g,1}$.

Theorem 20.3. *Let $g \geq 4$. Then the codimension of an irreducible component N of $N_{g,1}$ in $\mathcal{A}_g$ is at least 3 with equality if and only if:*

(i) *$g = 4$ and either $N = \mathcal{H}_4$ is the hyperelliptic locus or $N = \mathcal{A}_{4,(1,3)}$;*
(ii) *$g = 5$ and $N = \mathcal{J}_5$ is the jacobian locus.*

Proof. By Theorem 8.6, the codimension of N is at least 3. Suppose that N has codimension 3. It is well-known that the assertion holds true for $g = 4$ and 5 (see [4], [10], [8]). We may thus assume $g \geq 6$ and proceed by induction.

Let $\zeta = (X,\Theta_X)$ be a general point of N, so that X is simple (see Remark 7.4). Let S be a 1-dimensional component of $\mathrm{Sing}(\Theta_X)$. We can assume that the

class of S in X is a multiple $m\gamma_X$ of the *minimal class* $\gamma_X = \Theta_X^{g-1}/(g-1)! \in H^2(X,\mathbb{Z})$. If not so, then $\mathrm{End}(X) \neq \mathbb{Z}$ and this implies that $\mathrm{codim}_{\mathcal{A}_g}(N) \geq g-1$ (see Remark 7.4).
By Theorem 8.6, the general point in S is a double point for Θ_X. We let R be the curve in S_g whose general point is $\xi = (X, \Theta_X, x)$, with $x \in S$ a general point. Note that R is birationally equivalent to S. Let $\mathcal{Q}$ be the linear system of all quadrics in $\mathbb{P}(T_{X,0})$. One has the map

$$\phi : \xi \in R \dashrightarrow Q_\xi \in \mathcal{Q}.$$

As in the proof of Theorem 8.6, the map ϕ is not constant. Let $\mathcal{Q}_R$ be the span of the image of ϕ. As in the proof of Theorem 8.6, one has $\dim(\mathcal{Q}_R) \geq 2$. By Proposition 8.3, $\mathcal{Q}_R$ is contained in the linear system $\mathcal{N}_{g,1}(\zeta)$ (see §7 for the definition), which has dimension at most 2 since $\mathrm{codim}_{\mathcal{A}_g}(N) = 3$. Thus $\mathcal{Q}_R = \mathcal{N}_{g,1}(\zeta)$ has dimension 2.
By Lemma 20.1, the closure of N in $\mathcal{A}_g^{(\leq 1)}$ has non-empty intersection with the boundary. As in the proof of Lemma 20.1, we let M be an irreducible component of the intersection of the closure of N in $\mathcal{A}_g^{(\leq 1)}$ with the boundary. Consider the rational map $\alpha : M \dashrightarrow \mathcal{A}_{g-1}$ and the closure of the image $\alpha(M)$, for which we have the possibilities described in Lemma 20.2.

Claim 20.4. *Possibility (i) in Lemma 20.2 does not occur.*

Proof of the claim. By induction, one reduces to the case $g = 6$ and $\alpha(M) = \mathcal{J}_5$. Let $(\overline{X}, \overline{\Xi}) \in M$ be a general point. Then $(\overline{X}, \overline{\Xi})$ is a general rank one extension of the Jacobian $(J(C), \Theta_C)$ of a general curve C of genus 5. Note that if $x \in J(C)$ corresponds to the extension, then Θ_C and $x + \Theta_C$ are not tangentially degenerate (see [17], Thm. 10.8, p. 273). Then the analysis of §15 implies that the vertical singular locus S_0 of $\overline{\Xi}$ sits on the singular locus $D \cong J(C)$ of $\overline{X}$ and it is isomorphic to $S_C = \mathrm{Sing}(\Theta_C)$ with cohomology class $\Theta^4/12$ (see [3]). Thus $\overline{\Xi} \cdot S_0 = \Theta_C \cdot S_C = 10$. Hence, if $\zeta = (X, \Theta_X)$ is a general point of N, then $\mathrm{Sing}(\Theta_X)$ is a curve S such that $\Theta_X \cdot S = 10$. On the other hand S is homologous to $m\gamma_X$ and one has $10 = m\Theta_X \cdot \gamma_X = 6m$, a contradiction. $\square$

Claim 20.4 shows that only possibility (ii) in Lemma 20.2 can occur. In particular, by (iib) of Lemma 20.2, for $\xi = (X, \Theta_X, x)$ general in R, the quadric Q_ξ has corank 1. Let $v_\xi \in \mathbb{P}^{g-1}$ be the vertex of Q_ξ. Remember that R is birational to S. Hence, by Proposition 4.4, the map

$$\gamma : \xi \in R \dashrightarrow v_\xi \in \mathbb{P}^{g-1}$$

can be regarded as the Gauss map γ_S of S.

Claim 20.5. *If the general quadric in the linear system $\mathcal{Q}_R$ is singular, then for $\xi = (X, \Theta_X, x)$ general in R, the vertex v_ξ of Q_ξ is contained in the asymptotic cone $TC_\xi^{(4)}$.*

Proof of the claim. Suppose the general quadric in $\mathcal{Q}_R$ is singular. Then the general quadric in $\mathcal{Q}_R$ has corank 1 (see Lemma 20.2, (iib)) and, by Bertini's theorem, its vertex lies in the base locus of $\mathcal{Q}_R$. In particular, for $\xi = (X, \Theta_X, x)$ general in R, the vertex v_ξ of the quadric Q_ξ lies in all the quadrics of $\mathcal{Q}_R$. Choose a local parametrization $x = x(t)$ of S around a general point of it, with t varying in a disc Δ. Then $\xi(t) = (X, \Theta_X, x(t)) \in R$ and we set $Q_{\xi(t)} := Q_t$, its equation being

$$\sum_{ij} \partial_i \partial_j \theta(x(t)) z_i z_j = 0,$$

where we set $\theta(z) := \theta(\tau, z)$ for the theta function of X. The main remark is that all the subsequent derivatives of Q_t with respect to t lie in $\mathcal{Q}_R$ and actually Q_t and its first two derivatives Q_t' and Q_t'' span $\mathcal{Q}_R$, because $\dim(\mathcal{Q}_R) = 2$. Hence Bertini's theorem implies that $x' := x'(s)$ sits on all these quadrics for t and s general in Δ. The equations of Q_t' and Q_t'' are respectively

$$\sum_{ijh} \partial_i \partial_j \partial_h \theta(x(t)) x_h'(t) z_i z_j = 0$$

$$\sum_{ijhk} \partial_i \partial_j \partial_h \partial_k \theta(x(t)) x_h'(t) x_k'(t) z_i z_j + \sum_{ijh} \partial_i \partial_j \partial_h \theta(x(t)) x_h''(t) z_i z_j = 0$$

Thus we have the relations

(29)

$$\sum_{ij} \partial_i \partial_j \theta(x(t)) x_i'(s) x_j'(t) = 0, \qquad \sum_{ijh} \partial_i \partial_j \partial_h \theta(x(t)) x_h'(t) x_i'(s) x_j'(s) = 0$$

$$\sum_{ijhk} \partial_i \partial_j \partial_h \partial_k \theta(x(t)) x_h'(t) x_k'(t) x_i'(s) x_j'(s) + \sum_{ijh} \partial_i \partial_j \partial_h \theta(x(t)) x_h''(t) x_i'(s) x_j'(s) = 0$$

identically in $s, t \in \Delta$. The first of these relations says that the tangent hyperplane to Q_t at $x'(t)$ contains the vertex $x'(s)$. From the second relation we have

$$\sum_{ijh} \partial_i \partial_j \partial_h \theta(x(t)) x_h'(t) x_i'(t) x_j'(t) = 0 \tag{30}$$

which shows that v_ξ is contained in the asymptotic cone $TC_\xi^{(3)}$. By differentiating (30), one finds

$$\sum_{ijhk} \partial_i \partial_j \partial_h \partial_k \theta(x(t)) x_h'(t) x_k'(t) x_i'(t) x_j'(t) + 3 \sum_{ijh} \partial_i \partial_j \partial_h \theta(x(t)) x_h''(t) x_i'(t) x_j'(t) = 0.$$

By comparing with (29) for $s = t$, we deduce that

$$\sum_{ijhk} \partial_i \partial_j \partial_h \partial_k \theta(x(t)) x_h'(t) x_k'(t) x_i'(t) x_j'(t) = 0$$

which proves that $v_\xi \in TC_\xi^{(4)}$. □

The crucial step in our proof is the following claim.

Claim 20.6. *The general quadric in the linear system $\mathcal{Q}_R$ is non singular.*

Proof of the claim. Suppose this is not the case. Again we consider an irreducible component M of $\bar{N} \cap \partial\tilde{\mathcal{A}}'_g$ and let $(\overline{X}, \overline{\Xi})$ be a general point of M. By Claim 20.4 and Lemma 20.2, $(\overline{X}, \overline{\Xi})$ is a rank 1 extension of (B, Ξ) corresponding to a general point in a component of $N_{g-1,0}$, with extension datum b varying in a codimension 2 component of $N_1(B, \Xi)$. We let S_0 be the vertical singular locus of $\overline{\Xi}$. By the analysis of §15, this corresponds to a contact curve $C := C_b$ of Ξ with Ξ_b, which contains the singular points of both Ξ and Ξ_b (see Remark 12.3). This means that we have a point on S_0, corresponding to a singular point x of Ξ, where the tangent cone is the cone over Q_x with vertex the point of $\mathbb{P}^{g-1}$ corresponding to b (see S 15).
Now we note that C is smooth at x. Indeed locally around x, the divisor Ξ looks like a quadric cone of corank 1 in $\mathbb{P}^{g-2}$ and Ξ_b looks like a hyperplane, which touches it along a curve. This implies that C locally at x looks like a line along which a hyperplane touches a quadric cone of corank 1 (see Remark 12.4).
Hence, the Gauss image $x_b := \gamma_{C_b}(x)$ lies in Q_x and actually, by Claim 20.5, the point x_b lies in the asymptotic cone $TC_x^{(4)}$.
The Gauss map $\gamma_\Xi : \Xi \dashrightarrow \mathbb{P}^{g-2}$ of Ξ has an indeterminacy point at x and $-x$, which can be resolved by blowing up x and $-x$, since we may assume Ξ to be symmetric. Let $p : \tilde{\Xi} \to \Xi$ be the blow-up and let $\tilde{\gamma}_\Xi$ be the morphism which coincides with $\gamma_\Xi \circ p : \tilde{\Xi} \dashrightarrow \mathbb{P}^{g-2}$ on an open subset. The exceptional divisor at x and $-x$ is isomorphic to Q_x and $\tilde{\gamma}_\Xi(x_b)$ is the tangent hyperplane to Q_x at x_b. The tangency property of Ξ and Ξ_b along C implies that the tangent hyperplane to Q_x at x_b coincides with $\gamma_{\Xi_b}(x)$.
Now we let b vary in a component Z of $N_1(B, \Xi)$ of dimension $g-3$, so that we have a rational map

$$f : Z \dashrightarrow Q_x, \quad b \mapsto x_b.$$

Note that Q_x also has dimension $g-3$ and we claim that f has finite fibres, hence it is dominant. If not, we would have an irreducible curve Γ in Z such that for all $b \in \Gamma$, x_b stays fixed. But then for the general $b \in \Gamma$, the divisor Ξ_b has a fixed tangent hyperplane at x, a contradiction by Lemma 11.1.
On the other hand, by Claim 20.5 and by part (i) of Proposition 19.2, one has that f cannot be dominant, a contradiction. □

By Lemma 20.6, the curve $\phi(R) := \Sigma$ is an irreducible component of the discriminant $\Delta \subset \mathcal{Q}_R$ of singular quadrics in $\mathcal{Q}_R \cong \mathbb{P}^2$. Note that Δ has degree g in $\mathbb{P}^2$. The map ϕ has degree at least 2 since we may assume Θ_X to be symmetric, hence it factors through the multiplication by -1_X on X.

Claim 20.7. *The map $\phi : R \to \Sigma$ has degree 2.*

Proof. Let $d \geq 2$ be the degree of the map. Then for $\xi = (X, \Theta_X, x)$ general in R, we have distinct points $\xi_i = (X, \Theta_X, x_i) \in R$, with $\xi = \xi_1$, such that all the quadrics Q_{ξ_i}, $i = 1, \ldots, d$, coincide with the quadric $Q = Q_\xi$. If for $i = 1, \ldots, d$ we let η_i be the tangent vector to S at x_i, we have that $\eta_1 = \cdots = \eta_d$ and

$\partial_{\eta_i} Q_{\xi_i}$, for $i = 1, \ldots, d$, coincide with the quadric $Q_1 = \partial_{\eta_1} Q$, which is linearly independent from Q. The analysis we made in §3 shows that S_g is smooth at each of the points ξ_i, $i = 1, \ldots, d$, and the tangent space there is determined by Q and Q_{η_1}. This implies that a general deformation of $\xi = (X, \Theta_X, x)$ inside S_g carries with it a deformation of each of the points ξ_i $(i = 1, \ldots, d)$ in S_g, because the involved quadrics are the same at these points. This yields that a general element of an irreducible component of $N_{g,0}$ containing N has at least d singular points. By Debarre's result in [11] (see §19), one has $d = 2$, proving our claim. $\square$

Claim 20.8. *The map ϕ is a morphism.*

Proof. To prove the claim, it suffices to show that N is not contained in $N_{g,0,r}$, with $r \geq 3$. In order to prove this, one verifies that, for a general point $(\overline{X}, \overline{\Xi})$ in a component M of the intersection of N with the boundary, there are no points of multiplicity $r \geq 3$ in $\mathrm{Sing}_{\mathrm{vert}}(\overline{\Xi})$. Recall that, by Proposition 20.1, we may assume that $(\overline{X}, \overline{\Xi})$ is a semi-abelian variety of torus rank 1, with abelian part $\eta = (B, \Xi)$. Moreover, by Claim 20.4, only case (ii) of Lemma 20.2 can occur. Therefore we may assume that η is either a general point of $\theta_{0,g-1}$ or a general point of $M_{0,g-1}$ and the extension corresponds to a general point in an irreducible component of $N_1(B, \Xi)$ which has dimension $g - 3 > 0$. By Remark 15.1 we see that no triple points can occur on $\mathrm{Sing}_{\mathrm{vert}}(\overline{\Xi})$. $\square$

Note now that the morphism ϕ is defined on $R \cong S$ by sections of $\mathcal{O}_S(\Theta_X)$, since the points of S verify the equations (2) and, if $\xi = (X, \Theta_X, x) \in R$, the entries of the matrix of Q_ξ are $\partial_i \partial_j \theta(\tau, z)$, where z corresponds to x. We deduce from $\deg(\Delta) = g$ and from Claim 20.7, that

$$S \cdot \Theta_X \leq 2g. \tag{31}$$

As we assumed at the beginning of the proof, the class of S in X is a multiple $m\gamma_X$ of the minimal class. In view of (31), we find $m \leq 2$. The Matsusaka-Ran criterion [31] and a result of Welters [40] imply that (X, Θ_X) is either a Jacobian or a Prym variety or depends on less than $3g$ parameters. Since $g \geq 6$ this is not possible in view of the dimensions. This ends the proof. $\square$

Remark 20.9. A. Verra communicated to us an interesting example of an irreducible component M of codimension 6 of $N_{6,1}$ contained in the Prym locus. We briefly sketch, without entering in any detail, its construction and properties. Let C be the normalization of a general curve of type $(4, 4)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ with two nodes on a line of type $(0, 1)$, so that C has genus $g = 7$. Let d, t be the linear series formed by pull-back divisors on C of the rulings of type $(1, 0), (0, 1)$ respectively. Consider a non trivial, unramified double cover $f : \tilde{C} \to C$ of C and let (P, Ξ) be the corresponding Prym variety. Then Ξ has a 1-dimensional unstable singular locus R (see [23]), homologically equivalent to twice the minimal class, described by all classes in $\mathrm{Pic}^{12}(\tilde{C})$ of divisors of the form $f^*(d) + M$, with $f_*(M) \in t$. One proves that the map ϕ described in the proof of Theorem 20.3 sends R to a plane sextic of genus 7 which is tetragonally associated to C (see [23] for the tetragonal construction). The divisor Ξ has 24 further isolated singular points, which are pairwise exchanged by the multiplication by -1. One shows that the corresponding 12 tangent cones span the same linear system $\mathcal{Q}$ of dimension 2 spanned by $\phi(R)$. The linear system $\mathcal{Q}$ is the tangent space to the Prym locus $\mathcal{P}_6$ at (P, Ξ), which is therefore a smooth point for $\mathcal{P}_6$. By contrast, M is a non-reduced component of $N_{6,1}$ of codimension 6 such that the projectivized normal space $\mathcal{Q}$ at a general point has dimension 2 rather than 5. This shows that the hypotheses of Theorem 20.3 cannot be relaxed by assuming only that the projectivized normal space to M at a general point has dimension 2.

21. Appendix: A Result on Pencils of Quadrics

One of the ingredients of the proof Theorem 8.6 is a classical result of Corrado Segre from [33] on pencils of quadrics.
First we recall the following:

Proposition 21.1. *Let $\mathcal{L}$ be a pencil of quadrics in $\mathbb{P}^n$ with $n \geq 1$ whose general member is smooth. Then:*

(i) *the number of singular quadrics $Q \in \mathcal{L}$ is $n+1$, where each such quadric Q has to be counted with a suitable multiplicity $\mu(Q) \geq n+1-\mathrm{rk}(Q)$;*
(ii) *for a singular quadric $Q \in \mathcal{L}$ one has $\mu(Q) \geq 2$ if and only if either $\mathrm{rk}(Q) < n$ or the singular point of Q is also a base point of $\mathcal{L}$;*
(iii) *for a singular quadric $Q \in \mathcal{L}$ with rank n one has $\mu(Q) = 2$ if and only if any other quadric $Q' \in \mathcal{L}$ is smooth at p and the tangent hyperplane to Q' at p is not tangent to Q along a line.*

Proof. Consider the linear system $\mathcal{Q}_n$ of dimension $n(n+3)/2$ all quadrics in $\mathbb{P}^n$. Inside $\mathcal{Q}_n$ we have the discriminant locus Δ_n of singular quadrics, which is a hypersurface of degree $n+1$, defined by setting the determinant of a general quadric equal to zero. The differentiation rule for determinants implies that the locus $\Delta_{n,r}$ of quadrics of rank $r < n+1$ has multiplicity $n+1-r$ for Δ_n. By intersecting Δ_n with the line corresponding to $\mathcal{L}$ we have (i).
As for assertion (ii), we may assume $\mathrm{rk}(Q) = n$, so that Q has a unique double point p, which we may suppose to be the point $(1, 0, \ldots, 0)$. Thus the matrix of Q is of the form

$$\begin{pmatrix} 0 & 0_n \\ 0_n^t & A \end{pmatrix}$$

where $0_n \in \mathbb{C}^n$ is the zero vector and A is a symmetric matrix of order n and maximal rank. Let Q' be another quadric in $\mathcal{L}$, with matrix

$$\begin{pmatrix} \beta & b \\ b^t & B \end{pmatrix}$$

with $\beta \in \mathbb{C}$, $b = (b_1, \ldots, b_n) \in \mathbb{C}^n$ and B is a symmetrix matrix of order n. By intersecting $\mathcal{L}$ with Δ_n, we find the equation

$$\det\begin{pmatrix} t\beta & tb \\ tb^t & A+tB \end{pmatrix} = 0. \tag{32}$$

The constant term in the left-hand-side is 0. The coefficient of the linear term is

$$\det\begin{pmatrix} \beta & b \\ 0_n^t & A \end{pmatrix} = \beta\det(A)$$

which proves (ii).

Let us prove (iii). Suppose $\mathrm{rk}(Q) = n$, so that Q has a unique double point p, which is a base point of $\mathcal{L}$. Again we may suppose p is the point $(1 : 0 : \ldots : 0)$ and we can keep the above notation and continue the above analysis. The left-hand-side in (32) is

$$t^2\det\begin{pmatrix} 0 & b \\ b^t & A+tB \end{pmatrix} = 0$$

hence the coefficient of the third order term is

$$\det\begin{pmatrix} 0 & b \\ b^t & A \end{pmatrix}. \tag{33}$$

One has $\mu(Q) = 2$ if and only if this determinant is not zero, hence $b \neq 0$, which is equivalent to saying that all quadrics in the pencil different from Q are smooth at p. Note that there is a vector $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$ such that $b = c \cdot A$. Now the determinant in (33) vanishes if and only if $c \cdot b^t = 0$, i.e. $c \cdot A \cdot c^t = 0$. This means that the line L joining p with $(0 : c_1 : \ldots : c_n)$ sits on Q and that the tangent hyperplane to Q' at p, which has equation $b_1x_1 + \cdot + b_nx_n = 0$, is tangent to Q along r. □

Next we prove Segre's theorem.

Theorem 21.2. *Let $\mathcal{L}$ be a linear pencil of singular quadrics in $\mathbb{P}^n$ with $n \geq 2$ whose general member Q has rank $n+1-r$, i.e. $\mathrm{Vert}(Q) \cong \mathbb{P}^{r-1}$. We assume that $\mathrm{Vert}(Q)$ is not constant when Q varies in $\mathcal{L}$ with rank $n+1-r$. Then:*

(i) *the Zariski closure*

$$V_{\mathcal{L}} = \overline{\Big(\bigcup_{Q\in\mathcal{L},\,\mathrm{rk}(Q)=n+1-r} \mathrm{Vert}(Q)\Big)}$$

is a variety of dimension r spanning a linear subspace Π of dimension m in $\mathbb{P}^n$ with $r \leq m \leq (n+r-1)/2$;

(ii) *$V_{\mathcal{L}}$ is a variety of minimal degree $m-r+1$ in Π;*

(iii) *if*

$$\dim\Big(\bigcap_{Q\in\mathcal{L},\,\mathrm{rk}(Q)=n+1-r} \mathrm{Vert}(Q)\Big) = s$$

then $r \leq (n+2s+3)/3$;

(iv) *the number of quadrics $Q \in \mathcal{L}$ of rank $\mathrm{rk}(Q) < n+1-r$ is $n+r-2m-1 \leq n-r-1$, where each such quadric Q has to be counted with a suitable multiplicity $\nu(Q) \geq n+1-r-\mathrm{rk}(Q)$.*

Proof. We start with the proof of part (i). Notice that, by iteratedly restricting to a general hyperplane, we can reduce to the case $r = 1$.

In this case $V_{\mathcal{L}}$ is a rational curve which, by Bertini's theorem, is contained in the base locus B of $\mathcal{L}$. Let p, q be general points on it and let L be the line joining them. There is a quadric $Q_p \in \mathcal{L}$ with vertex at p. Hence Q_p contains L. Similarly there is a different quadric Q_q with vertex at q, and it also contains L. Since Q_p and Q_q span $\mathcal{L}$, we see that L is contained in B, i.e., the secant variety to $V_{\mathcal{L}}$ is contained in B. Take now three general points p, q, r on $V_{\mathcal{L}}$. Since the lines pq, pr, qr are contained in B also the plane spanned by p, q, r is contained in B. Continuing this way, we see that $\Pi = \langle V_{\mathcal{L}} \rangle$ is contained in B. Since the general quadric in $\mathcal{L}$ has rank n (recall we are assuming $r = 1$ now), the maximal dimension of subspaces on it is $n/2$. Thus $\dim(\Pi) \leq n/2$ which proves part (i).

Also for (ii) we can reduce ourselves to the case $r = 1$, in which we have to prove that $V_{\mathcal{L}}$ is a rational normal curve in $\Pi = \langle V_{\mathcal{L}} \rangle$. Set $\dim(\Pi) = m$.

Let $p \in V_{\mathcal{L}}$ be a general point. The polar hyperplane π_p of p with respect to $Q \in \mathcal{L}$ does not depend on Q, since there is a quadric in $\mathcal{L}$ which is singular at p (see the proof of Proposition 21.1). Note that π_p has to contain all vertices of the quadrics in $\mathcal{L}$, hence it contains Π. By the linearity of polarity, we have that polarity with respect to all quadrics in $\mathcal{L}$ is constant along Π and for a general point $x \in \Pi$, the polar hyperplane π_x with respect to all quadrics in $\mathcal{L}$ contains Π. Furthermore the linear system of hyperplanes $\mathcal{P} = \{\pi_x\}_{x \in \Pi}$ has dimension $m - 1$.

Now, let $p \in V_{\mathcal{L}}$ be a general point and let Q_p be the unique quadric in $\mathcal{L}$ with a double point at p. We denote by $\mathrm{Star}(p)$ the $\mathbb{P}^{m-1}$ of all lines in Π containing p. Let $\pi \in \mathcal{P}$ be a general hyperplane, which is tangent to Q_p along a line L containing p. Moreover L sits in Π, because this is the case if $\pi = \pi_q$ with q another general point on $V_{\mathcal{L}}$, in which case L is the line $\langle p, q \rangle$. Thus we have a linear map $\phi_p : \mathcal{P} \to \mathrm{Star}(p)$, which is clearly injective and therefore an isomorphism.

Fix now another general point q on $V_{\mathcal{L}}$. The two maps ϕ_p and ϕ_q determine a linear isomorphism $\phi : \mathrm{Star}(p) \to \mathrm{Star}(q)$. Note that L meets $\phi(L)$ if and only if $L \cap \phi(L)$ is a point of $V_{\mathcal{L}}$. This implies that $V_{\mathcal{L}}$ is a rational normal curve in Π, proving part (ii).

Let us prove part (iii). It suffices to prove the assertion if $s = -1$. The variety $V_{\mathcal{L}}$ is swept out by a 1-dimensional family of projective spaces of dimension $r - 1$, i.e., the vertices of the quadrics in $\mathcal{L}$. Under the assumption $s = -1$ no two of these vertices can intersect. Thus we must have $2(r - 1) < m$. Using part (i), the assertion follows.

Finally we come to part (iv). Let us restrict $\mathcal{L}$ to a general subspace Λ of dimension $n - r$. We get a pencil $\bar{\mathcal{L}}$ of quadrics in Λ whose general member is smooth.

We get a singular quadric in $\bar{\mathcal{L}}$ when we intersect Λ with a quadric in $\mathcal{L}$ whose vertex intersects Λ. We claim that this is the only possibility for getting a singular quadric in $\bar{\mathcal{L}}$. Indeed, let $Q \in \mathcal{L}$ and suppose that its intersection

$\bar{Q} \in \bar{\mathcal{L}}$ with Λ is singular at $p \in \Lambda$, but Q is not singular at p. Then Λ is tangent to Q at p and therefore also intersects the vertex of Q.
In conclusion we have only two possibilities for getting singular quadrics in $\bar{\mathcal{L}}$:

(a) there is quadric of rank $n+1-r$ in $\mathcal{L}$ whose vertex intersects Λ;
(b) there is quadric of rank $n+1-h < n+1-r$ in $\mathcal{L}$ giving rise to a quadric of rank $n-h$ in $\bar{\mathcal{L}}$.

Case (a) occurs as many times as the degree of $V_{\mathcal{L}}$, that is, $m-r+1$ times. According to part (ii) of Proposition 21.1, each quadric Q in case (a) contributes with multiplicity at least 2 in the counting of singular quadrics in $\bar{\mathcal{L}}$. We claim that, because of the generality of Λ, this multiplicity is exactly 2. To prove this, by part (iii) of Proposition 21.1, we will prove that for each quadric Q in case (a), with vertex $p \in \Lambda$ and for any other quadric $Q' \in \mathcal{L}$, the tangent hyperplane π to Q' at p is not tangent to Q along a line contained in Λ. To see this we can, by first cutting with a general subspace of dimension $n-r+1$ through Λ, reduce ourselves to the case $r=1$, in which $V_{\mathcal{L}}$ is a rational normal curve. Choose then a general point $q \in V_{\mathcal{L}}$ and let $Q' = Q_q$ be the unique quadric in $\mathcal{L}$ with a double point at q. The hyperplane π is tangent to Q_q along the line $\langle p, q\rangle$. This implies that π is tangent to Q only along the tangent line L_p to $V_{\mathcal{L}}$ at p (see the proof of part ii)). By the generality assumption, Λ is not tangent to $V_{\mathcal{L}}$ at p. Thus the assertion follows.
As for quadrics in case (b), again by part (i) of Proposition 21.1, each such quadric contributes to the same count with multiplicity $h-r$. Since, by part (i) of Proposition 21.1, the number of singular quadrics in $\bar{\mathcal{L}}$, counted with appropriate multiplicity, is $n-r+1$, the assertion follows. □

One has the following consequence:

Corollary 21.3. *Let $\mathcal{L}$ be a linear pencil of quadrics in $\mathbb{P}^n$ with $n \geq 2$. Then the general member $Q \in \mathcal{L}$ has rank $n+1-r$ (i.e., $\mathrm{Vert}(Q) \cong \mathbb{P}^{r-1}$) if and only if the base locus of $\mathcal{L}$ contains a linear subspace Π of dimension m with $r \leq m \leq (n+r-1)/2$, along which all the quadrics in $\mathcal{L}$ have a common tangent subspace of dimension $n+r-m-1$. In this case Π is the span of the variety $V_{\mathcal{L}}$.*

Proof. As usual it suffices to prove the assertion for $r=1$. If the general quadric $Q \in \mathcal{L}$ has rank $\mathrm{rk}(Q) = n$, the assertion follows from the proof of Theorem 21.2. The converse is trivial, since a smooth quadric in $\mathbb{P}^n$ has a tangent subspace of dimension $n-m-1$, and not larger, along a subspace of dimension m. □

These results imply the existence of *canonical forms* for pencils of singular quadrics, originally due to Weierstrass [38] and Kronecker [21]. This is explained in some detail in [33], §§20-25, and we will not dwell on this here.
It would be desirable to have an extension of the results in this Appendix to higher-dimensional linear families of quadrics.

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K-Theory of Log-Schemes I

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Abstract. We set down some basic facts about the algebraic and topological K-theory of log-schemes. In particular, we show that the l-adic topological log-étale K-theory of log-regular schemes computes the l-adic étale K-theory of the largest open sets where the log-structure is trivial.

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Contents

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1. Introduction

The purpose of this paper is to set down some basic facts about the algebraic and topological K-theory of log-schemes. Log-schemes come equipped with several natural topologies. The main two are the Kummer log-étale topology, well suited to study l-adic phenomena, and the Kummer log-flat topology (together with its derivative – the Kummer log-syntomic topology) reasonably well-suited to study p-adic phenomena. These topologies are often enhanced by adding log-blow-ups as coverings, a procedure that yields better behaved topoi.

The investigation of coherent and locally free sheaves in these topologies as well as of the related descent questions was initiated by Kato in [24]. In particular, Kato was able to compute the Picard groups of strictly local rings. A foundational study of the algebraic K-theory of the Kummer log-étale topos (i.e., the Quillen K-theory of locally free sheaves in that topos) was done by Hagihara in [14]. He has shown that over a separably closed field Kummer log-étale K-theory satisfies devissage, localization as well as Poincaré duality for log-regular regular schemes. Using these facts and an equivariant K-theory computation of the Kummer log-étale K'-theory of log-points (fields equipped with log-structure) he obtained a structure theorem (see Theorem 4.13 below) for Kummer log-étale K-theory of a certain class of log-schemes including those coming from a smooth variety with a divisor with strict normal crossings.

This paper builts on the results of Kato and Hagihara. In section 2 we focus on some basic properties of the topologies we will use. In section 3 we study coherent and locally free sheaves in these topologies. Since Kato's paper remains unfinished and unpublished, for the convenience of the reader (and the author), this section contains some of Kato's proofs as well as supplies proofs of the results only announced in [24]. In section 4 we study algebraic K-theory. We generalize Hagihara's work to schemes over fields with Kummer log-étale topology and to arbitrary schemes with Kummer log-flat topology. This is rather straightforward and is done by studying equivariant K-theory of finite flat group schemes instead of just finite groups as in Hagihara. The following structure theorem follows. Let X be a regular, log-regular scheme with the log-structure associated to a divisor D with strict normal crossing. Let $\{D_i | i \in I\}$ be the set of the irreducible (regular) components of D. For an index set $J \subset I$ denote by D_J the intersection of irreducible components indexed by J and by $\Lambda_{|J|}$ (resp. $\Lambda'_{|J|}$) the free abelian groups generated by the set $\{(a_1,\ldots,a_{|J|}) | a_i \in \mathbf{Q}/\mathbf{Z}\setminus\{0\}\}$ (resp. the set $\{(a_1,\ldots,a_{|J|}) | a_i \in (\mathbf{Q}/\mathbf{Z})'\setminus\{0\}\}$).

Theorem 1.1. *For any $q \geq 0$ we have the canonical isomorphism*

$$K_q(X_{\mathrm{kfl}}) \simeq \bigoplus_{J\subset I} K_q(D_J)\otimes \Lambda_{|J|}$$

Moreover, if D is equicharacteristic then canonically

$$K_q(X_{\mathrm{k\acute{e}t}}) \simeq \bigoplus_{J\subset I} K_q(D_J)\otimes \Lambda'_{|J|}$$

Section 5 is devoted to topological K-theory. By definition this is K-cohomology of the various sites considered in this paper. The main theorem (Theorem 5.14 and Corollary 5.17) states that l-adic log-étale K-theory of a log-regular scheme computes the étale K-theory of the largest open set on which the log-structure is trivial.

Theorem 1.2. *Let X be a log-regular scheme satisfying condition (*) from section 5. Let n be a natural number invertible on X. Then the open immersion $j : U \hookrightarrow X$, where $U = X_{\mathrm{tr}}$ is the maximal open set of X on which the log-structure is trivial, induces an isomorphism*

$$j^* : K_m^{\mathrm{vk\acute{e}t}}(X, \mathbf{Z}/n) \xrightarrow{\sim} K_m^{\mathrm{\acute{e}t}}(U, \mathbf{Z}/n), \quad m \geq 0.$$

This follows from the fact that we can resolve singularities of log-regular schemes by log-blow-ups and that the étale sheaves of nearby cycles can be killed by coverings that are étale where the log-structure is trivial and tamely ramified at infinity.

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For a log-scheme X, M_X will always denote the log-structure of X. Unless otherwise stated all the log-structures on schemes are fine and saturated (in short: fs) and come from the étale topology, and all the operations on monoids are performed in the fine and saturated category.

2. Topologies on log-schemes

In this section we collect some very basic facts about topologies on log-schemes.

2.1. The Kummer log-flat and the Kummer log-syntomic topology.

2.1.1. *The log-étale, log-syntomic, and log-flat morphisms.* The notion of the log-étale and the log-flat morphism recalled below is the one of Kato [23, 3.1.2]. The notion of log-syntomic morphism we introduce is modeled on that. Our main reason for introducing it is the local lifting property it satisfies (see Lemma 2.9).

Definition 2.1. Let $f : Y \to X$ be a morphism of log-schemes. We say that f is log-étale (resp. log-flat, resp. log-syntomic) if locally on X and Y for the (classical) étale (resp. fppf, resp. syntomic) topology, there exists a chart $(P \to M_X, Q \to M_Y, P \to Q)$ of f such that the induced morphisms of schemes

- $Y \to X \times_{\mathrm{Spec}(\mathbf{Z}[P])} \mathrm{Spec}(\mathbf{Z}[Q])$,
- $\mathrm{Spec}(\mathcal{O}_Y[Q^{gp}]) \to \mathrm{Spec}(\mathcal{O}_Y[P^{gp}])$

are classically étale (resp. flat, resp. syntomic).

Recall the definition of (classical) syntomic morphism.

Definition 2.2. Let $f : Y \to X$ be a morphism of schemes. We say that f is syntomic if locally on X and Y for the classical étale topology f can be written as $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$, with $B = A[X_1, \ldots, X_r]/(f_1, \ldots, f_s)$, where the sequence $(f_1, \ldots, f_s)$ is regular in $A[X_1, \ldots, X_r]$ and the algebras $A[X_1, \ldots, X_r]/(f_1, \ldots, f_i)$ are flat over A, for all i.

Syntomic morphisms are stable under composition and base change.

Remark 2.3. We should mention that, a priori, in the Definition 2.1 we have used (after Kato [23, 3.1]) the following meaning of property being local on X and Y: there exist coverings $(X_i \to X)_i$ and $(Y_{ij} \to X_i \times_X Y)_j$, for each i, for the corresponding topology such that each morphism $Y_{ij} \to X_i$ has the required property. By Lemma 2.8 below, this is equivalent for log-étale, log-flat, and log-syntomic morphisms, to the more usual meaning: for every point $y \in Y$ and its image $x \in X$, there exist neighbourhoods U and V of y and x respectively (for the corresponding topology) such that U maps to V and the morphism $U \to V$ has the required property. In particular, in the Definition 2.1 we may use the second meaning of "locally" and change the second condition to "$\mathrm{Spec}(\mathcal{O}_X[Q^{gp}]) \to \mathrm{Spec}(\mathcal{O}_X[P^{gp}])$ is classically étale (resp. flat, resp. syntomic)."

Remark 2.4. The notion of log-syntomic morphism presented here is not the same as the one used by Kato [21, 2.5]. Recall that Kato defines an integral morphism $f : Y \to Z$ of fine log-schemes to be log-syntomic if étale locally Y (over Z) embeds into a log-smooth Z-scheme via an exact classically regular embedding over Z. In particular, Kato's log-syntomic morphisms are classically flat while ours are not necessarily so.

Lemma 2.5. *Let S be a nonempty scheme and let $h : G \to H$ be a homomorphism of finitely generated abelian groups. Then the morphism $\mathcal{O}_S[G] \to \mathcal{O}_S[H]$ is étale (resp. flat or syntomic) if and only if the kernel and the cokernel of h are finite groups whose orders are invertible on S (resp. if the kernel of h is a finite group whose order is invertible on S).*

Proof. The étale case follow from [22, 3.4]. The "if" part of the flat case follows from [22, 4.1]. We will now show that if the induced morphism $f : k[G] \to k[H]$, where k is a field is flat, then the kernel N of h is torsion of order invertible in k. Take an element g from N. It is easy to see that the kernel of the multiplication by $g - 1$ on $k[G]$ is generated, as an ideal, by elements $1 + g + \ldots + g^{n-1}$, such that $g^n = 1$. By the flatness of f, the images of these elements in $k[H]$ generate as an ideal the whole of $k[H]$. In particular, the element g has to be of finite order d and the ideal of the multiplication by $g - 1$ on $k[G]$ is generated by the element $1 + g + \ldots + g^{d-1}$. But $f(1 + g + \ldots + g^{d-1}) = d$. Hence d is invertible in k, as wanted. The syntomic case follows from Lemma 2.6 below. □

Lemma 2.6. *With the notation as in the above lemma, the morphism $\mathcal{O}_S[G] \to \mathcal{O}_S[H]$ is flat if and only if it is syntomic.*

Proof. Since syntomic morphism is flat, we have to show that if the morphism $\mathcal{O}_S[G] \to \mathcal{O}_S[H]$ is flat it is already syntomic. Let $N = \ker(G \to H)$. Our morphism $\mathcal{O}_S[G] \to \mathcal{O}_S[H]$ factors as $\mathcal{O}_S[G] \to \mathcal{O}_S[G/N] \to \mathcal{O}_S[H]$. Since the morphism $\mathcal{O}_S[G] \to \mathcal{O}_S[H]$ is flat, the group N is torsion of order invertible on S (see the previous lemma). This yields that the first morphism in our factorization is étale hence syntomic. This allows us to reduce the question to proving that if the morphism $h : G \to H$ is injective then the induced morphism $\mathbf{Z}[G] \to \mathbf{Z}[H]$ is syntomic.

Write $H = H_1/G_1$, for $H_1 = G \oplus \mathbf{Z}r_1 \oplus \ldots \oplus \mathbf{Z}r_n$ and a subgroup G_1 of H_1. Since $H_{1,tor} = G_{tor}$ and the map $G \to H$ is injective, the group G_1 is finitely generated and torsion-free. Write $G_1 = \mathbf{Z}a_1 \oplus \ldots \oplus \mathbf{Z}a_k$. We claim that $\mathbf{Z}[H] \simeq \mathbf{Z}[H_1]/(a_1-1, \ldots, a_k-1)$ and the sequence $\{a_1-1, \ldots, a_k-1\}$ is regular. Set $H_{1,l} := H_1/\mathbf{Z}a_1 \oplus \ldots \oplus \mathbf{Z}a_l$. Note that, since the group $\mathbf{Z}a_1 \oplus \ldots \oplus \mathbf{Z}a_l \oplus \mathbf{Z}a_{l+1}$ is torsion free, the element a_{l+1} is not torsion in $H_{1,l}$. This easily implies (cf. [5, 2.1.6]) that $a_{l+1}-1$ is not a zero-divisor in $\mathbf{Z}[H_{1,l}]$. To finish, it suffices to check that the natural map $\mathbf{Z}[H_{1,l}]/(a_{l+1}-1) \to \mathbf{Z}[H_{1,l}/\mathbf{Z}a_{l+1}]$ is an isomorphism. But this is clear since we have the inverse induced by $\overline{x} \mapsto x$, for $x \in H_{1,l}$. ☐

Lemma 2.7.
(1) *Log-étale, log-flat, log-syntomic morphisms are stable under compositions and under base changes.*
(2) *Let $f : Y \to X$ be a strict morphism of log-schemes, i.e., a morphism such that $f^*M_X \overset{\simeq}{\to} M_Y$. Then f is log-étale (resp. log-flat, resp. log- syntomic) if and only if the underlying morphism of schemes is (classically) étale (resp. flat, resp. syntomic).*
(3) *Let S be a scheme and let $P \to Q$ be a morphism of monoids. Then the induced morphism of log-schemes* $\mathrm{Spec}(\mathcal{O}_S[Q]) \to \mathrm{Spec}(\mathcal{O}_S[P])$ *is log-étale (resp. log-flat, resp. log-syntomic) if and only if the morphism of schemes* $\mathrm{Spec}(\mathcal{O}_S[Q^{gp}]) \to \mathrm{Spec}(\mathcal{O}_S[P^{gp}])$ *is (classically) étale (resp. flat, resp. syntomic).*

Proof. The only nonobvious statement is the one concerning compositions, which follow easily from Lemma 2.8 below. ☐

Lemma 2.8. *Let $f : Y \to X$ be a morphism of log-schemes and let $\beta : P \to M_X$ be a chart. Assume that f is log-étale (resp. log-flat, resp. log-syntomic). Then, étale (resp. flat, resp. syntomic) locally on X and on Y in the classical sense, there exists a chart $(P \to M_X, Q \to M_Y, P \to Q)$ including β satisfying the conditions in Definition 2.1. We can require further $P^{gp} \to Q^{gp}$ to be injective.*

Proof. For the log-étale and the log-flat topology this is Lemma 3.1.6 from [23]. We will argue in a similar fashion for the log-syntomic case taking into account that (unlike in [23]) our monoids are always saturated.

Let $(P' \to M_X, Q' \to M_Y, P' \to Q')$ be a chart satisfying the conditions in Definition 2.1. Fix $y \in Y, x = f(y) \in X$. By replacing P' with the inverse image P_1 (which is always saturated) of $M_{X,x}$ under the map

$$P^{gp} \oplus (P')^{gp} \to M_{X,x}^{gp}; \qquad (a, b) \mapsto ab,$$

and by replacing Q' with the pushout $P_1 \leftarrow P' \rightarrow Q'$, we may assume that $\beta : P \rightarrow M_X$ factors as $P \rightarrow P' \rightarrow M_X$. By (Zariski) localization we may also assume that $P'/(P')^* \simeq M_{X,x}/\mathcal{O}^*_{X,x}$ and $Q'/(Q')^* \simeq M_{Y,y}/\mathcal{O}^*_{Y,y}$.
Assume for the moment that the morphism $(P')^{gp} \rightarrow (Q')^{gp}$ is injective. Consider the pushout diagrams with exact rows

$$\begin{tikzcd}
0 \arrow[r] & P^{gp} \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (P')^{gp} \arrow[r] \arrow[d] & (Q')^{gp} \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G \arrow[r] & T \arrow[r] & W \arrow[r] & 0,
\end{tikzcd}$$

where the group G is the cokernel of the map $P^{gp} \rightarrow (P')^{gp}$. We want to construct the group H. For that, it suffices to show that the map $T \rightarrow W$ has a section. Consider a direct summond $\mathbf{Z}/n\mathbf{Z}$ of W. Let $t \in T$ be a preimage of a generator of $\mathbf{Z}/n\mathbf{Z}$. Then $t^n = b, b \in G$. Take $b' \in (P')^*$ in the preimage of b. Since $P'/(P')^* \simeq M_{X,x}/\mathcal{O}^*_{X,x}$ and P^{gp} maps onto $M^{gp}_{X,x}/\mathcal{O}^*_{X,x}$ such a b' exists. Define the group G_1 by adjoining the n'th root of b' to $(P')^*$. By localizing in the classical syntomic topology, we can now change P' and Q' into the pushouts $P' \leftarrow (P')^* \rightarrow G_1$ and $Q' \leftarrow (P')^* \rightarrow G_1$. Note that we can do that since the morphism $(P')^* \rightarrow G_1$ is injective with finite cokernel, hence the induced morphism $\mathrm{Spec}(\mathbf{Z}[G_1]) \rightarrow \mathrm{Spec}(\mathbf{Z}[(P')^*])$ is syntomic (Lemma 2.5) and surjective. Moreover, the above pushouts taken in the category of monoids are already fine and saturated. Now, $b = a^n$ for some $a \in G$. Changing t to t/a gives us an element in the preimage of our generator of $\mathbf{Z}/n\mathbf{Z}$ whose n'th power is one, hence the section we wanted.
Let now Q be the inverse image of $M_{Y,y}$ under the map $H \rightarrow M^{gp}_{Y,y}$ (it is saturated). Since $P'/(P')^* \simeq M_{X,x}/\mathcal{O}^*_{X,x}$ and $Q'/(Q')^* \simeq M_{Y,y}/\mathcal{O}^*_{Y,y}$ this gives a local chart at y. We claim that the natural morphism $P \rightarrow Q$ gives us the chart we wanted. The map $P^{gp} \rightarrow Q^{gp}$ is clearly injective. Let Q_1 be the pushout $P' \leftarrow P \rightarrow Q$. There is a natural morphism $Q_1 \rightarrow Q'$. By Zariski localizing on Y, we may assume that $Q_1/Q_1^* \xrightarrow{\sim} Q'/(Q')^*$. Since the map $Q_1^{gp} \rightarrow (Q')^{gp}$ is an isomorphism, this yields that $Q_1 \xrightarrow{\sim} Q'$. Hence the morphism $P \rightarrow Q$ is indeed the chart we wanted.
Let now $(P \rightarrow M_X, Q \rightarrow M_Y, P \rightarrow Q)$ be a chart satisfying the conditions in Definition 2.1. It remains to show that we may assume $P^{gp} \rightarrow Q^{gp}$ to be injective. Indeed, let N be the kernel of $P^{gp} \rightarrow Q^{gp}$. Consider the pushout diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & P^{gp} \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P^{gp}/N \arrow[r] & Q^{gp} \arrow[r] & W \arrow[r] & 0.
\end{tikzcd}$$

It is easy to construct the group H. Let now Q' be the inverse image of $M_{Y,y}$ under the map $H \to M^{gp}_{Y,y}$ (it is saturated). Q' gives a local chart at y. Since N is the kernel of the map $(Q')^{gp} \to Q^{gp}$ and N is a finite group of order invertible on Y, there exists an open set $U \subset X \times_{\mathrm{Spec}(\mathbf{Z}[P])} \mathrm{Spec}(\mathbf{Z}[Q'])$ such that the order of N is invertible on U. Then the morphism $U \to X \times_{\mathrm{Spec}(\mathbf{Z}[P])} \mathrm{Spec}(\mathbf{Z}[Q'])$ is log-étale. On the other hand this morphism is (perhaps after Zariski localization) strict. Hence it is étale. This shows that $(P \to M_X, Q' \to M_Y, P \to Q')$ is a chart satisfying the conditions in Definition 2.1 such that the map $P^{gp} \to (Q')^{gp}$ is injective. $\square$

Lemma 2.9. *Let $Y \hookrightarrow X$ be a exact closed immersion defined by a nilideal. Étale locally log-syntomic morphisms over Y can be lifted to log-syntomic morphisms over X.*

Proof. Immediate from Lemma 2.8 (note that it suffices to localize in the étale topology on Y) and the well-known lifting property for classical syntomic morphisms that we recall below. $\square$

Lemma 2.10. *Let A be a commutative ring, $B \to A$ a closed immersion defined by a nilideal, and $C = A[X_1, \dots, X_n]/(G_1, \dots, G_r)$ an A-algebra such that the sequence $(G_1, \dots, G_r)$ is regular and each $A[X1, \dots, X_n]/(G_1, \dots, G_i)$, $i \leq r$, is flat over A. Let $(\check{G}_1, \dots, \check{G}_r)$ be liftings of $(G_1, \dots, G_r)$ to $B[X_1, \dots, X_n]$. Then the sequence $(\check{G}_1, \dots, \check{G}_r)$ is regular and each $B[X_1, \dots, X_n]/(\check{G}_1, \dots, \check{G}_i)$, $i \leq r$, is flat over B.*

2.1.2. *Kummer topologies.* Recall first the definition of Kummer morphisms.

Definition 2.11.
1. A homomorphism of monoids $h : P \to Q$ is said to be of Kummer type if it is injective and, for any $a \in Q$, there exists $n \geq 1$ such that $a^n \in h(P)$.
2. A morphism $f : X \to Y$ of log-schemes is of Kummer type if for any $x \in X$, the induced homomorphism of monoids $(M/\mathcal{O}^*)_{Y,\overline{f(x)}} \to (M/\mathcal{O}^*)_{X,\overline{x}}$ is of Kummer type in the sense of (1).

One checks [27, 2.1.2] that Kummer morphisms are stable under base changes and compositions.

Remark 2.12. Note that if the morphism $P \to Q$ is Kummer, then by Lemma 2.6 the associated morphism $\mathrm{Spec}(\mathbf{Z}[Q]) \to \mathrm{Spec}(\mathbf{Z}[P])$ is both log-flat and log-syntomic.

Definition 2.13. Let X be a log-scheme. A morphism $Y \to X$ is called Kummer log-étale (resp. log-flat, log-syntomic) if it is log-étale (resp. log-flat, log-syntomic) and of Kummer type and the underlying morphism of schemes is locally of finite presentation. The log-étale (resp. log-flat, log-syntomic) topology on the category of Kummer log-étale (resp. log-flat, log-syntomic) morphisms over X is defined by taking as coverings families of morphisms $\{f_i : U_i \to T\}_i$ such that each f_i is log-étale (resp. log-flat, log-syntomic) and $T = \bigcup_i f_i(U_i)$ (set theoretically).

This defines a Grothendieck topology by the following result of Nakayama [27, 2.2.2].

Lemma 2.14. *Let $f : Y \to X$ be a morphism of log-schemes that is Kummer and surjective. Then, for any log-scheme $X' \to X$, the morphism $Y \times_X X' \to X'$ is surjective. In fact, for any $y \in Y$ and $x \in X'$ having the same image in X, there exists $z \in Y \times_X X'$ mapping to x and to y.*

The following proposition describes a very useful cofinal system of coverings for the Kummer log- étale, log-flat and log-syntomic sites.

Proposition 2.15. *Let $f : Y \to T$ be a Kummer log-étale (resp. log-flat, log-syntomic) morphism. Let $y \in Y, t = f(y)$, and $P \to M_T$ be a chart such that $P^* \simeq \{1\}$. Then there exists a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{h} & T_n \\ {\scriptstyle g}\downarrow & & \downarrow \\ Y & \xrightarrow{f} & T, \end{array}$$

where y is in the image of g, h is classically étale (resp. flat, syntomic), g is Kummer log-étale (resp. log-flat, log-syntomic), and n is invertible on X (resp. any, any).

Proof. We will argue the case of Kummer log-flat topology. The other cases are similar. By Lemma 2.8 localizing on Y (but keeping $y \in Y$) for the flat topology, we get a chart $(P \to M_T, Q \to M_Y, P \to Q)$ as in Definition 2.1 such that $P^{gp} \to Q^{gp}$ is injective. Note that localizing on T is not necessary. Arguing further as in the proof of Proposition A.2 in [28] we may assume that Q is torsion free. Hence $P^{gp} \simeq Q^{gp}$ as abelian groups. Write $n : P \to Q \to P^{1/n}$ for some n, where $P^{1/n}$ is a P-monoid such that $P \to P^{1/n}$ is isomorphic to $n : P \to P$. Set $X = Y \times_{T_Q} T_n$, where $T_n = T \otimes_{\mathbf{Z}[P]} \mathbf{Z}[P^{1/n}]$, $T_Q = T \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q]$. By definition the map $h : X \to T_n$ is classically flat and, since $Q^{gp} \hookrightarrow P^{1/n,gp}$, the induced map $g : X \to Y$ is surjective and Kummer log-flat. □

Corollary 2.16. *Let $f : Y \to T$ be a Kummer log-flat (resp. log-syntomic) morphism. Let $P \to M_T$ be a chart such that $P^* \simeq \{1\}$. Then there exists a Kummer log-flat (resp. log-syntomic) covering $V \to Y$ such that for some n the map $V \times_T T_n \to T_n$ is classically flat (resp.syntomic).*

Proof. We will treat the flat case. The syntomic case is similar. By Proposition 2.15, there exists n such that for a flat covering $V \to Y$ the induced map $V \to T$ factors as $V \to T_n \to T$, where the map $V \to T_n$ is classically flat. We have the following cartesian diagram

$$\begin{array}{ccccc} V_n = V \times_{T_n} T_n \times_T T_n & \longrightarrow & T_n \times_T T_n & \xrightarrow{p_2} & T_n \\ \downarrow & & {\scriptstyle p_1}\downarrow & & \downarrow \\ V & \longrightarrow & T_n & \longrightarrow & T, \end{array}$$

Since p_1, p_2 are classicaly flat, so is the map $V_n \to T_n$, as wanted. $\square$

Similarly one proves the following

Corollary 2.17. *Let $f : Y \to T$ be a Kummer log-étale covering. Let $P \to M_T$ be a chart such that $P^* \simeq \{1\}$. Then, Zariski locally on T, there exists a Kummer log-étale covering $V \to T$ refining f such that, for some n invertible on T, the map $V \times_T T_n \to T_n$ is classically étale.*

For a log-scheme X, we will denote by $X_{\text{két}}$ (resp. X_{kfl}, X_{ksyn}) the site defined above. In what follows, I will denote sites and the associated topoi in the same way. I hope that this does not lead to a confusion.
We will need to know that certain presheaves are sheaves for the Kummer topologies.

Proposition 2.18. *Let X be a log-scheme. Then the presheaf $(Y \to X) \mapsto \Gamma(Y, \mathcal{O}_Y)$ is a sheaf on all Kummer sites.*

Proof. It is clearly enough to show this for the Kummer log-flat site. In that case it follows from a Kummer descent argument (see Lemma 3.28 below). $\square$

More generally

Proposition 2.19. *Let X be a log-scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X_{Zar}. Then the presheaf*

$$(f : T \to X) \mapsto \Gamma(T, f^*\mathcal{F})$$

is a sheaf on all Kummer sites.

Proof. It is clearly enough to show this for the Kummer log-flat site. In that case it follows from the proof of the Kummer descent argument below via exhibiting an explicite contracting homotopy (see Lemma 3.28). $\square$

And in a different direction, we have the following theorem. The proof presented here is that of Kato [24, 3.1].

Theorem 2.20. *Let X be a log-scheme, and let Y be a log-scheme over X. Then the functor*

$$\mathrm{Mor}_X(\ ,Y) : T \mapsto \mathrm{Mor}_X(T, Y)$$

on (fs/X) is a sheaf for all the Kummer topologies.

Proof. We claim that it suffices to show that the functors

$$T \mapsto \Gamma(T, \mathcal{O}_T), \quad T \mapsto \Gamma(T, M_T). \tag{2.1}$$

are sheaves for the Kummer log-flat topology. To see that assume that $X = \mathrm{Spec}(\mathbf{Z})$ with the trivial log-structure, Y is an affine scheme with a chart $P \to \Gamma(Y, M_Y)$. Let F, G, H be the following functors from $(fs)/\,\mathrm{Spec}(\mathbf{Z})$ to $(Sets)$

$$F(T) = \{\text{ring homomorphisms } \Gamma(Y, \mathcal{O}_Y) \to \Gamma(T, \mathcal{O}_T)\},$$
$$G(T) = \{\text{monoid homomorphisms } P \to \Gamma(T, M_T)\},$$
$$H(T) = \{\text{monoid homomorphisms } P \to \Gamma(T, \mathcal{O}_T)\}.$$

The functor $\mathrm{Mor}_X(\ ,Y) : T \mapsto \mathrm{Mor}_X(T,Y)$ is the fiber product $F \to H \leftarrow G$, where the first arrow is induced by $P \to \Gamma(Y,\mathcal{O}_Y)$ and the second one by $\Gamma(T,M_T) \to \Gamma(T,\mathcal{O}_T)$. It follows that it suffices to show that the functors F,G,H are sheaves.
Take now a presentation

$$\Gamma(Y,\mathcal{O}_Y) = \mathbf{Z}[T_i; i\in I]/(f_j; j\in J),$$
$$\mathbf{N}^r \rightrightarrows \mathbf{N}^s \to P.$$

We get that $F(T)$ is the kernel of $\Gamma(T,\mathcal{O}_T)^I \to \Gamma(T,\mathcal{O}_T)^J$ and $G(T)$ and $H(T)$ are the equalizers of $\Gamma(T,M_T)^s \rightrightarrows \Gamma(T,M_T)^t$ and $\Gamma(T,\mathcal{O}_T)^s \rightrightarrows \Gamma(T,\mathcal{O}_T)^t$, respectively. Thus it suffices to show that the functors in (2.1) are sheaves.
For the functor $T \mapsto \Gamma(T,\mathcal{O}_T)$ this follows from Lemma 3.28. For the functor $T \mapsto \Gamma(T,M_T)$ we first show that it is a sheaf for the classical flat topology. If $T' \to T$ is a fppf covering, then we know that the sequence

$$\Gamma(T,\mathcal{O}_T^*) \to \Gamma(T',\mathcal{O}_{T'}^*) \rightrightarrows \Gamma(T'',\mathcal{O}_{T''}^*)$$

where $T' = T'\times_T T'$, is exact. Since $M_{T'}/\mathcal{O}_{T'}^*$ and $M_{T''}/\mathcal{O}_{T''}^*$ are pulbacks of $M_T/\mathcal{O}_T^*$, the sequence

$$\Gamma(T,M_T/\mathcal{O}_T^*) \to \Gamma(T',M_{T'}/\mathcal{O}_{T'}^*) \rightrightarrows \Gamma(T'',M_{T''}/\mathcal{O}_{T''}^*)$$

is exact as well. Next we treat Kummer coverings

Lemma 2.21. *Take $T = \mathrm{Spec}(A)$ for a local ring A equipped with a chart $P \to \Gamma(T,M_T)$, $P \simeq (M_T/\mathcal{O}_T^*)_t$, where t is the closed point of T. Let Q be a monoid with no torsion. Let $P \to Q$ be a homomorphism of Kummer type. Let $T' = T\otimes_{\mathbf{Z}[P]}\mathbf{Z}[Q]$ endowed with the log-structure associated to Q. Let $T'' = T'\times_T T'$. Then*

$$\Gamma(T,M_T) \to \Gamma(T',M_{T'}) \rightrightarrows \Gamma(T'',M_{T''})$$

is exact.

Proof. Set $A' = \Gamma(T',\mathcal{O}_{T'}) = A\otimes_{\mathbf{Z}[P]}\mathbf{Z}[Q]$, $A'' = \Gamma(T'',\mathcal{O}_{T''}) = A\otimes_{\mathbf{Z}[P]}\mathbf{Z}[Q\oplus(Q^{gp}/P^{gp})]$. By Lemma 3.28 the sequence $A \to A' \rightrightarrows A''$ is exact. Let I,I',I'' be the ideals of A,A',A'', respectively, generated by the images of $P\setminus\{1\}, Q\setminus\{1\}, Q\setminus\{1\}$, respectively. Let V,V',V'' be the subgroups of $A^*,(A')^*,(A'')^*$, respectively, consisting of elements that are congruent to 1 modulo I,I',I'', respectively. Since $A/I \simeq A'/I'$ the sequence $V \to V' \rightrightarrows V''$ is exact. It remains to show that the sequence

$$\Gamma(T,M_T)/V \to \Gamma(T',M_{T'})/V' \rightrightarrows \Gamma(T'',M_{T''})/V''$$

is exact. This sequence is isomorphic to

$$P\oplus(A/I)^* \to Q\oplus(A/I)^* \rightrightarrows Q\oplus\{(A/I)[Q^{gp}/P^{gp}]\}^*,$$

where the two arrows in the middle are $\beta_1 : (q,u)\mapsto(q,u)$, $\beta_2 : (q,u)\mapsto(q,qu)$. The exactness of the last sequence follows from the exactness of $P \to Q \to Q\oplus(Q^{gp}/P^{gp})$. □

□

Denote by $\mathbf{G}_m^\times$ the functor $T \mapsto \Gamma(T, M_T^{\mathrm{gp}})$ on (fs/X). The above theorem yields

Corollary 2.22. *([24, 3.6]) The functor $\mathbf{G}_m^\times$ is a sheaf for all the Kummer topologies.*

Proof. This argument is also due to Kato [24, 3.6]. It suffices to show that $\mathbf{G}_m^\times$ is a sheaf for the Kummer log-flat topology. Let $T' \to T$ be a Kummer log-flat covering equipped with a chart $P \to \Gamma(T, M_T)$. Set $T'' = T' \times_T T'$. We have $\Gamma(T, M_T^{gp}) = \operatorname{inj\,lim}_a \Gamma(T, a^{-1}M_T)$, where a ranges over all elements of P. Since both T' and T'' are of Kummer type over T, we also have $\Gamma(T', M_{T'}^{gp}) = \operatorname{inj\,lim}_a \Gamma(T', a^{-1}M_{T'})$ and $\Gamma(T'', M_{T''}^{gp}) = \operatorname{inj\,lim}_a \Gamma(T'', a^{-1}M_{T''})$. It follows that the exactness of the sequence

$$\Gamma(T, M_T^{gp}) \to \Gamma(T', M_{T'}^{gp}) \rightrightarrows \Gamma(T'', M_{T''}^{gp})$$

is reduced to the exactness of the sequence

$$\Gamma(T, M_T) \to \Gamma(T', M_{T'}) \rightrightarrows \Gamma(T'', M_{T''})$$

that was proved above. □

2.2. The valuative topologies. The valuative topologies refine Kummer topologies with log-blow-up coverings. That makes them slightly pathological (blow-ups do not change the global sections of sheaves) but also allows for better functorial properties [15].

Definition 2.23. Let X be a log-scheme. A morphism $Y \to X$ is called Zariski (resp. étale, log-étale, log-flat, log-syntomic) valuative if it is a composition of Zariski open (resp. étale, Kummer log-étale, Kummer log-flat, Kummer log-syntomic) morphisms and log-blow-ups. The Zariski (resp. étale, log-étale, log-flat, log-syntomic) valuative topology on this category of morphisms over X is defined by taking as coverings families of morphisms $\{f_i : U_i \to T\}_i$ such that each f_i is Zariski (resp. étale, log-étale, log-flat, log-syntomic) valuative and $T = \bigcup_i f_i(U_i)$ universally (i.e., this equality is valid after any base change by a map $S \to T$ of log-schemes). We will denote the corresponding site by X_{val} (resp. $X_{\mathrm{v\acute{e}t}}$, $X_{\mathrm{vk\acute{e}t}}$, X_{vkfl}, X_{vksyn}).

Note that, since any base change of a log-blow-up is a log-blow-up [30, Cor.4.8], the above definition makes sense. We have the following commutative diagram of continuous maps of sites

$$\begin{array}{ccccccccc}
X_{\mathrm{vkfl}} & \longrightarrow & X_{\mathrm{vksyn}} & \longrightarrow & X_{\mathrm{vk\acute{e}t}} & \longrightarrow & X_{\mathrm{v\acute{e}t}} & \longrightarrow & X_{\mathrm{val}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
X_{\mathrm{kfl}} & \longrightarrow & X_{\mathrm{ksyn}} & \longrightarrow & X_{\mathrm{k\acute{e}t}} & \longrightarrow & X_{\mathrm{\acute{e}t}} & \longrightarrow & X_{\mathrm{Zar}}.
\end{array}$$

Remark 2.24. Note that the site $X_{\mathrm{vk\acute{e}t}}$ is the same as the full log-étale site [15].

Denote by $\mathcal{O}_{X^*}$ (or by $\mathcal{O}_X$ if there is no risk of confusion) the structure sheaf of the topos on X induced by one of the above topologies, i.e., the sheaf associated to the presheaf $(Y \to X) \mapsto \Gamma(Y, \mathcal{O}_Y)$.

We will now describe points of the topoi associated to some of the above sites. Recall [27, 2.4] that a log-geometric point is a scheme $\operatorname{Spec}(k)$, for a separably closed field k, equipped with a saturated monoid M such that the map $a \mapsto a^n$ on $P = M/k^*$ is bijective for any integer n prime to the characteristic of k. Log-geometric points form a conservative system for the Kummer log-étale topos [27, 2.5]. We get enough points of the full log-étale topos by taking (valuative) log-geometric points, i.e., log-geometric points with M/k^* valuative (recall that a saturated monoid P is called valuative if for any $a \in P^{gp}$, either a or a^{-1} is in P). There is an alternative way of describing a conservative family of points for the log-étale topos. For $x \in X$, choose a chart $x \in U$, $U \to \operatorname{Spec} \mathbf{Z}[P]$. For each finitely generated and nonempty ideal $J \subset P$, let U_J be the log-blow-up of U along J. These U_J's form an inverse system indexed by the set of finitely generated and nonempty ideals J partially ordered by divisibility. Take now a compatible system of log-geometric points of the $U_J's$ lying above x.

A conservative family of points of X_{val} (resp. $X_{\text{vét}}$) can be described in a similar fashion by taking compatible systems of Zariski (resp. geometric) points. Recall [23, 1.3.5] that in the case of X_{val} and a chart $X \to \operatorname{Spec} \mathbf{Z}[P]$ there is a canonical bijection between this set of points and all pairs $(V, \mathfrak{p})$ such that V is a valuative submonoid of P^{gp} containing P and $\mathfrak{p}$ is a point of $X_V = X \otimes_{\mathbf{Z}[P]} \mathbf{Z}[V]$ satisfying the following condition: If $a \in V$ and the image of a in $\mathcal{O}_{X_V,\mathfrak{p}}$ is invertible, then $a \in V^*$. We have then the following description of stalks of the structure sheaf: $\mathcal{O}_{X_{\text{val}},(V,\mathfrak{p})} \simeq \mathcal{O}_{X_V,\mathfrak{p}}$.

Lemma 2.25. *Let $Y \to X$ be a log-flat valuative morphism. Then there is a log-blow-up $Y' \to Y$ (hence necessarily a covering) such that the morphism $Y' \to X$ can be written as a composition $Y' \to T \to X$, where $Y' \to T$ is Kummer log-flat and $T \to X$ is a log-blow-up.*

Proof. Since composition of log-blow-ups is a log-blow-up [30, Cor.4.11], it is enough to show this for a composition $Y \to Z \to X$ of a log-blow-up $Y \to Z$ with a Kummer log-flat morphism $Z \to X$. Recall that by [20, 3.13] we can find a log-blow-up $B \to X$ such that the base change $Y' := Y \times_X B \to B$ is exact. Here a morphism of log-schemes $f : T \to S$ is called exact if, for every $t \in T$, the morphism $f : M_{S,\overline{s}}/\mathcal{O}^*_{S,\overline{s}} \to M_{T,\overline{t}}/\mathcal{O}^*_{T,\overline{t}}$, $s = f(t)$, is exact, i.e., $(f^{gp})^{-1}(M_{T,\overline{t}}/\mathcal{O}^*_{T,\overline{t}}) = M_{S,\overline{s}}/\mathcal{O}^*_{S,\overline{s}}$. Consider now the following commutative diagram

$$\begin{tikzcd} Y \arrow[r] & Z \arrow[r] & X \\ Y' \arrow[u] \arrow[r, "\sim"] & Z \times_X B \arrow[u] \arrow[r] & B. \arrow[u] \end{tikzcd}$$

Since base change of a log-blow-up is a log-blow-up [30, Cor.4.8] the morphisms $Y' \to Y$, $Z \times_X B \to Z$ and $Y' \to Z \times_X B$ are log-blow-ups. But because the composition $Y' \to B$ is exact, the morphism $Z \times_X B \to B$ is Kummer, and the log-schemes are saturated, the morphism $Y' \to Z \times_X B$ is actually an isomorphism. Hence $Y' \to B$ is Kummer log-flat as wanted. $\square$

For a general scheme X, the presheaf $(Y \to X) \mapsto \Gamma(Y, \mathcal{O}_Y)$ on X_{val} is not always a sheaf (see [12, 2.5]). Let, for example, $X = \mathrm{Spec}(k[T_1, T_2]/(T_1^2, T_2^2))$ with the log-structure $\mathbf{N}^2 \to \mathcal{O}_X$; $e_i \mapsto T_i$, and let $Y \to X$ be the log-blow-up of the ideal generated by e_1 and e_2. Then the map $\Gamma(X, \mathcal{O}_X) \to \Gamma(Y, \mathcal{O}_Y)$ is not injective. On the other hand, since Y covers X and $Y \times_X Y \simeq Y$, the map $\Gamma(X, \mathcal{O}_{X_{\mathrm{val}}}) \to \Gamma(Y, \mathcal{O}_{Y_{\mathrm{val}}})$ is necessarily an isomorphism. We have however the proposition below. But first we need to recall the notion of a log-regular scheme.

Definition 2.26. A log-scheme X is called log-regular at $x \in X$ if $\mathcal{O}_{X,\overline{x}}/I_{\overline{x}}\mathcal{O}_{X,\overline{x}}$ is regular and $\dim(\mathcal{O}_{X,\overline{x}}) = \dim(\mathcal{O}_{X,\overline{x}}/I_{\overline{x}}\mathcal{O}_{X,\overline{x}}) + \mathrm{rank}_{\mathbf{Z}}((M_X^{gp}/\mathcal{O}_X^*)_{\overline{x}})$, where $I_{\overline{x}} = M_{X,\overline{x}} \setminus \mathcal{O}^*_{X,\overline{x}}$. We say that X is log-regular if X is log-regular at every point $x \in X$.

Proposition 2.27. *Let X be a log-regular log-scheme. Then the presheaf $(Y \to X) \mapsto \Gamma(Y, \mathcal{O}_Y)$ is a sheaf on all valuative sites.*

Proof. It is clearly enough to show this for the log-flat valuative site. Since this presheaf is a sheaf on the Kummer log-flat site, by Lemma 2.25, it suffices to show that if $\pi : B \to T$ is a log-blow-up of a log-scheme $T \to X$, log-flat valuative over X, then $\Gamma(T, \mathcal{O}_T) \to \Gamma(B, \mathcal{O}_B)$ is an isomorphism. We will show that $\mathcal{O}_T \stackrel{\sim}{\to} R\pi_*\mathcal{O}_B$. Assume for the moment that T is log-regular. Then T behaves like a toric variety, and this is a well-known result. As the argument in [30] shows the key-point is that (flat) locally there is a chart $P \to \mathcal{O}_T$, with a torsion free monoid P, such that

$$\text{(2.2)} \qquad \text{for injective morphism } P \to Q, \qquad \mathrm{Tor}_i^{\mathbf{Z}[P]}(\mathcal{O}_T, \mathbf{Z}[Q]) = 0, \quad i \geq 1.$$

We will show that this is also the case for our (general now) T. By induction, assume that a log-scheme $Z \to X$, log-flat valuative over X satisfies the condition (2.2). We have to show that any log-scheme $T \to Z$, Kummer log-flat or log-blow-up over Z, also satisfies this condition. We will show the argument in the case when $T \to Z$ is Kummer log-flat. The argument for log-blow-up is similar but simpler.
Consider a "good" chart

$$\begin{array}{ccc} T & \longrightarrow & \mathrm{Spec}(\mathbf{Z}[Q]) \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathrm{Spec}(\mathbf{Z}[P]), \end{array}$$

where the monoid P has no torsion, the morphism $P \to Q$ is injective, and the morphism $T \to Y_1$, $Y_1 := Y \times_{\mathrm{Spec}(\mathbf{Z}[P])} \mathrm{Spec}(\mathbf{Z}[Q])$ is flat. A slight modification of an argument of Nakayama in [28, A.2.], yields that, modulo a flat localization, we may assume Q to be torsion free as well. Since the morphism $T \to Y_1$ is flat, we just need to show that $\mathrm{Tor}_i^{\mathbf{Z}[Q]}(\mathcal{O}_{Y_1}, \mathbf{Z}[Q_1]) = 0$, $i \geq 1$, for any injection

$Q \to Q_1$. But this follows from the fact that $\mathrm{Tor}_i^{\mathbf{Z}[P]}(\mathcal{O}_Y, \mathbf{Z}[P_1]) = 0$, $i \geq 1$, for any injection $P \to P_1$. □

3. Coherent and locally free sheaves on log-schemes

Let us first collect some basic facts about coherent and locally free sheaves in the various topologies on log-schemes discussed above. Let $\mathcal{F}(X)_*$ be the category of $\mathcal{O}_X$-modules, where $*$ stands for one of the considered here topologies. It is an abelian category. Let $\mathcal{P}(X)_*$ denote the category of $\mathcal{O}_X$-modules that are locally a direct factor of a free module of finite type. By [3, I.2.15.1.ii] this is the same as the category of locally free sheaves of finite type. Let $\mathcal{M}(X)_*$ denote the category of coherent $\mathcal{O}_X$-modules, i.e., $\mathcal{O}_X$-modules that are of finite type and precoherent. Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is called precoherent [3, I.3.1] if for every object $Y \to X$ in X_* and for every map $\mathcal{E} \xrightarrow{f} \mathcal{F}|Y_*$ from a locally free finite type $\mathcal{O}_Y$-module $\mathcal{E}$, the kernel of f is of finite type.

Lemma 3.1. (1) *The category $\mathcal{M}(X)_*$ is abelian and closed under extensions.*
(2) *The category $\mathcal{P}(X)_*$ is additive and when embedded in $\mathcal{F}(X)_*$ with the induced notion of a short exact sequence, it is exact.*

Proof. The first statement follows from [3, I.3.3]. For the second one it suffices to check that $\mathcal{P}(X)_*$ is closed under extensions in $\mathcal{F}(X)_*$. That follows from the fact that $\mathcal{O}_{X_*}$-modules of finite type are closed under extensions [3, I.3.3] and that all epimorphisms $\mathcal{M}_1 \to \mathcal{M}_2$, $\mathcal{M}_2 \in \mathcal{P}(X)_*$, locally admit a section [3, I.1.3.1]. □

The simplest coherent sheaves come from the Zariski topology. Let X_* denote one of the Kummer topologies and let $\varepsilon_X : X_* \to X_{\mathrm{Zar}}$ be the natural projection. We have

Lemma 3.2. *The pullback functor $\varepsilon_X^* : \mathrm{Q}\mathcal{M}(X_{\mathrm{Zar}}) \to \mathcal{F}(X_*)$ (from the category of quasicoherent Zariski sheaves) is fully faithful.*

Proof. Immediate from Proposition 2.19. □

Proposition 3.3. *Let X_* satisfy the following property*

$$\varepsilon^* \text{ is exact for a cofinal system of coverings in } X_* \tag{3.1}$$

Then the structure sheaf $\mathcal{O}_{X_*}$ is coherent on all Kummer sites.

Proof. We need to check that for any object $Y \to X$ in the Kummer site X_* the kernel of any morphism $f : \mathcal{O}_{Y_*}^m \to \mathcal{O}_{Y_*}$ is of finite type. But f comes from a Zariski morphism $f' : \mathcal{O}_{Y_{\mathrm{Zar}}}^m \to \mathcal{O}_{Y_{\mathrm{Zar}}}$ and by exactness $\varepsilon_Y^* \ker f' = \ker f$. Since $\ker f'$ is of finite type so is $\ker f$. □

Corollary 3.4. *If X has property (3.1) then the $\mathcal{F}$ is coherent if and only if there exists a covering $X_i \to X$ of X such that $\mathcal{F}|X_i$ is isomorphic to $\varepsilon_{X_i}^* \mathcal{F}_i'$ for some coherent sheaf $\mathcal{F}_i'$ on $X_{i,\mathrm{Zar}}$.*

Example 3.5. A log-scheme X such that $(M_X/\mathcal{O}_X^*)_{\overline{x}} \simeq \mathbf{N}^{r(x)}$ has property (3.1). In particular, X can be a strict closed subscheme of a regular, log-regular scheme.

Definition 3.6. The coherent sheaves or locally free sheaves in the (essential) image of the functor ε^* are called *classical*.

Lemma 3.7. *Let X be a log-regular log-scheme. Let $Y \to X$ be a log-blow-up. Then the restrictions*

$$r : \mathcal{F}(X)_* \to \mathcal{F}(Y)_*, \quad r : \mathcal{M}(X)_* \to \mathcal{M}(Y)_*, \quad r : \mathcal{P}(X)_*^{\mathrm{val}} \to \mathcal{P}(Y)_*$$

are equivalences of categories for $*$ any of the valuative topologies.

Proof. Let $\mathcal{M} \in \mathcal{F}(Y)_*$. Consider the functor $\pi : \mathcal{F}(X)_* \to \mathcal{F}(Y)_*$ given by $\pi(\mathcal{M}) : (T \to X) \mapsto \Gamma(T \times_X Y, \mathcal{M})$. Since $Y \times_X Y \simeq Y$, the compositions $r\pi$ and πr are naturally equivalent to the identity. Hence the restriction induces an equivalence of categories $\mathcal{F}$. The remaining equivalences follow since the map $Y \to X$ is covering. $\square$

Lemma 3.8. *Let X be a log-regular quasi-compact log-scheme. Let $\mathcal{F} \in \mathcal{P}(X)_{\mathrm{vkfl}}$ be a locally free sheaf of rank n. Then, for some log-blow-up $T \to X$, $\mathcal{F}|T_{\mathrm{vkfl}}$ is isomorphic to a pullback of a locally free sheaf of rank n from T_{kfl}.*

Proof. By Lemma 2.25 we can restrict our attention to trivializing coverings of the form $Y \to T \to X$, where $Y \to T$ is a Kummer log-flat covering and $T \to X$ is a log-blow-up. Since the isomorphism classes of locally free sheaves of rank n are classified by the first Čech cohomology groups of the sheaf $\mathbf{GL}_n$, the statement of the lemma follows now easily from the following commutative diagram

$$\begin{array}{ccccc}
\mathbf{GL}_n(Y) & \longrightarrow & \mathbf{GL}_n(Y \times_X Y) & \longrightarrow & \mathbf{GL}_n(Y \times_X Y \times_X Y) \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
\mathbf{GL}_n(Y) & \longrightarrow & \mathbf{GL}_n(Y \times_T Y) & \longrightarrow & \mathbf{GL}_n(Y \times_T Y \times_T Y),
\end{array}$$

where the equalities hold already on the level of schemes (since $T \times_X T \simeq T$). $\square$

Corollary 3.9. *Let X be a log-regular quasi-compact log-scheme. Then the pullback functor*

$$\operatorname{inj}\lim_{Y} \mathcal{P}(Y_{\mathrm{kfl}}) \to \mathcal{P}(X_{\mathrm{vkfl}})$$

is an equivalence of categories, where the limit is over log-blow-ups $Y \to X$.

Lemma 3.10. *Let X be a log-regular log-scheme. The the pullback functor $\mathcal{P}(X_{\mathrm{vksyn}}) \to \mathcal{P}(X_{\mathrm{vkfl}})$ is an equivalence of categories.*

Proof. Let $\mathcal{E}$ be a locally free sheaf on X_{vkfl}. Denote by $\mathcal{E}'$ its restriction to X_{vksyn}. It is a sheaf. We claim that $\mathcal{E}'$ is actually a locally free sheaf and that $\varepsilon^*\mathcal{E}' \xrightarrow{\sim} \mathcal{E}$, where $\varepsilon : X_{\mathrm{vkfl}} \to X_{\mathrm{vksyn}}$ is the natural map. By Corollary 2.16 and Lemma 2.25, $\mathcal{E}$ can be trivialized by a covering of the form $U \to T \to Y \to X$, where $U \to T$ is a (classical) flat covering, $T \to X$ is a Kummer log-syntomic covering, and $Y \to X$ is a log-blow-up. The restriction of $\mathcal{E}$ to T, $\mathcal{E}|T$, comes from flat topology hence by faithfully flat descent from a Zariski locally free sheaf. This allows us to show that $(\varepsilon^*\mathcal{E}')|T \xrightarrow{\sim} \mathcal{E}|T$, as wanted. $\square$

Basically the same argument gives the following

Lemma 3.11. *For any Noetherian log-scheme X, the pullback functors*

$$\mathcal{P}(X_{\mathrm{ksyn}}) \to \mathcal{P}(X_{\mathrm{kfl}}), \qquad \mathcal{M}(X_{\mathrm{ksyn}}) \to \mathcal{M}(X_{\mathrm{kfl}})$$

are equivalences of categories.

3.1. Invertible sheaves. We will compute now the groups $H^1(X_*, \mathbf{G}_m)$ of isomorphism classes of invertible sheaves for X local and equipped with one of the Kummer topologies. The main ideas here are due to Kato [24]. Let X be a log-scheme. We have the following Kummer exact sequences on X_{kfl}, respectively $X_{\mathrm{k\acute{e}t}}$,

$$0 \to \mathbf{Z}/n(1) \to \mathbf{G}_m^{\times} \xrightarrow{n} \mathbf{G}_m^{\times} \to 0,$$
$$0 \to \mathbf{Z}/n(1) \to \mathbf{G}_m^{\times} \xrightarrow{n} \mathbf{G}_m^{\times} \to 0,$$

for any nonzero integer n, respectively for any integer n which is invertible on X. Here $\mathbf{Z}/n(1)$ is by definition the kernel of the multiplication by n on the multiplicative group $\mathbf{G}_m$.

The following theorem was basically proved by Kato in [24, Theorem 4.1]. We supplied the missing arguments.

Theorem 3.12. *Let X be a log-scheme and assume X to be locally Noetherian. Let $\varepsilon : X_{\mathrm{kfl}} \to X_{\mathrm{fl}}$ be the canonical map. Let G be a commutative group scheme over the underlying scheme of X satisfying one of the following two conditions*

(1) *G is finite flat over the underlying scheme of X;*
(2) *G is smooth and affine over the underlying scheme of X.*

We endow G with the inverse image of the log-structure of X. Then we have a canonical isomorphism

$$R^1\varepsilon_*G \simeq \operatorname*{inj\,lim}_{n\neq 0} \mathcal{H}\mathrm{om}(\mathbf{Z}/n(1), G) \otimes_{\mathbf{Z}} (\mathbf{G}_m^{\times}/\mathbf{G}_m).$$

Proof. Let X be a log-scheme and let G be a sheaf of abelian groups on X_{kfl}. Define a canonical homomorphism of sheaves on X_{kfl}

$$\mu : \qquad \operatorname*{inj\,lim}_{n\neq 0} \mathcal{H}\mathrm{om}(\mathbf{Z}/n(1), G) \otimes_{\mathbf{Z}} (\mathbf{G}_m^{\times}/\mathbf{G}_m) \to R^1\varepsilon_*G$$

as follows. Let h be a local section of $\mathcal{H}\mathrm{om}(\mathbf{Z}/n(1), G)$. The Kummer exact sequence on X_{kfl}

$$0 \to \mathbf{Z}/n(1) \to \mathbf{G}_m^{\times} \xrightarrow{n} \mathbf{G}_m^{\times} \to 0$$

yields the composition

$$\mathbf{G}_m^{\times} = \varepsilon_*\mathbf{G}_m^{\times} \xrightarrow{\partial} R^1\varepsilon_*(\mathbf{Z}/n(1)) \xrightarrow{h} R^1\varepsilon_*G,$$

where ∂ is the connecting morphism. Since multiplication by n on $\mathbf{G}_m$ on the site X_{fl} is surjective, the map ∂ kills $\mathbf{G}_m$. That gives us the definition of the map μ.

It is easy to see now that the first case of the theorem follows from the second. Indeed, if G is a finite flat commutative group scheme on X we can take its (see [26, A.5]) smooth resolution, i.e., an exact sequence of sheaves on X_{fl}

$$0 \to G \to L \to L' \to 0,$$

where both L and L' are smooth and affine group schemes over the underlying scheme of X. We endow both L and L' with the inverse image log-structure. By applying the pushforward ε_* to the above exact sequence and using the fact that $L = \varepsilon_* L \to L' = \varepsilon_* L'$ is surjective on X_{fl} we get an exact sequence

$$0 \to R^1\varepsilon_* G \to R^1\varepsilon_* L \to R^1\varepsilon_* L'.$$

Hence bijectivity of the map μ for G is reduced to the bijectivity of this map for L and L'.

It suffices now to prove the following proposition

Proposition 3.13. *Assume $X = \operatorname{Spec}(A)$, where A is strictly local, and assume that G is represented by a smooth commutative group scheme over X endowed with the induced log-structure. Assume that $P \xrightarrow{\sim} (M_X/\mathcal{O}_X^*)_x$, where x is the closed point of X. Then the map*

$$\operatorname{inj}\lim_n \operatorname{Hom}(\mathbf{Z}/n(1), G) \otimes_{\mathbf{Z}} P^{\mathrm{gp}} \xrightarrow{\mu} H^1(X_{\mathrm{kfl}}, G),$$

is an isomorphism.

Proof. For $n \geq 1$, consider $X_n = X \otimes_{\mathbf{Z}[P]} \mathbf{Z}[P^{1/n}]$, with the induced log-structure. Here $P^{1/n}$ is a P-monoid such that $P \to P^{1/n}$ is isomorphic to $n : P \to P$. The map $X_n \to X$ is a covering in X_{kfl}. Denote by $X_{n,i}$ the fiber product of $i+1$ copies of X_n over X. For any sheaf of abelian groups G on X_{kfl}, we have a Čech complex

$$C^{\cdot}_{G,n}: \qquad \Gamma(X_{n,0}, G) \to \Gamma(X_{n,1}, G) \to \Gamma(X_{n,2}, G) \to \dots$$

Assume that A is Noetherian and complete. Then our proposition is proved in two steps via the following two lemmas

Lemma 3.14. *Assume $X = \operatorname{Spec}(A)$, where A is strictly local, and assume that G is represented by a smooth commutative group scheme over X endowed with the induced log-structure. Then*

$$\operatorname{inj}\lim_n H^1(C^{\cdot}_{G,n}) \xrightarrow{\sim} H^1(X_{\mathrm{kfl}}, G).$$

Proof. From Čech cohomology we know that the map $\operatorname{inj}\lim_n H^1(C_{G,n}) \to H^1(X_{\mathrm{kfl}}, G)$ is injective and its cokernel injects into $\operatorname{inj}\lim_n H^1((X_n)_{\mathrm{kfl}}, G)$. Hence it suffices to show that $\operatorname{inj}\lim_n H^1((X_n)_{\mathrm{kfl}}, G) = 0$. Take an element α of $H^1((X_n)_{\mathrm{kfl}}, G)$. Let $T \to X_n$ be a log-flat Kummer covering such that α dies in $H^1(T_{\mathrm{kfl}}, G)$. By Corollary 2.16, we may assume that for some m, we have a factorization $T \to X_{mn} \to X_n$, where $T \to X_{mn}$ is a classically flat covering. It follows that the class α on X_{mn} is trivialised by a classically flat cover. Thus the class of α in $H^1((X_{mn})_{\mathrm{kfl}}, G)$ comes from $H^1((X_{mn})_{\mathrm{fl}}, G)$. But the group scheme G being smooth, $H^1((X_{mn})_{\mathrm{fl}}, G) \simeq H^1((X_{mn})_{\mathrm{\acute{e}t}}, G)$. Finally, since X_{mn} is a disjoint union of a finite number of Spec of strictly local rings, we have $H^1((X_{mn})_{\mathrm{\acute{e}t}}, G) = 0$, as wanted. □

Before stating the second lemma, we would like to show that the composition of the structure map $P^{\mathrm{gp}} \to H^0(X_{\mathrm{kfl}}, \mathbf{G}_m^{\times})$ with the map μ_h : $H^0(X_{\mathrm{kfl}}, \mathbf{G}_m^{\times}) \to H^1(X_{\mathrm{kfl}}, G)$ induced by a section $h \in \mathrm{Hom}(\mathbf{Z}/n(1), G)$ factors through $H^1(C^{\cdot}_{G,n})$. For that, consider the classical commutative group scheme $H_n = \mathrm{Spec}(\mathbf{Z}[P^{\mathrm{gp}}/(P^{\mathrm{gp}})^n])$ over $\mathrm{Spec}(\mathbf{Z})$. It defines the sheaf $T \mapsto \mathrm{Hom}(P^{\mathrm{gp}}/(P^{\mathrm{gp}})^n, \Gamma(T, \mathbf{Z}/n(1)))$ on X_{kfl}. The group scheme H_n acts on X_n over X, and we have $H_n \times_{\mathbf{Z}} X_n \simeq X_n \times_X X_n$. Hence, $X_{n,i} \simeq (H_n)^{\times i} \times_{\mathbf{Z}} X_n$. For a sheaf of abelian groups G on X_{kfl}, let G_n be the sheaf of abelian groups on X_{kfl} defined by $G_n(T) = \Gamma(T \times_X X_n, G)$. The sheaf H_n acts on G_n. The Čech complex $C^{\cdot}_{G,n}$ can now be written as

$$C^{\cdot}_{G,n}: \quad \mathrm{Mor}(1, G_n) \xrightarrow{\partial_0} \mathrm{Mor}(H_n, G_n) \xrightarrow{\partial_1} \mathrm{Mor}(H_n^{\times 2}, G_n) \xrightarrow{\partial_2} \ldots,$$

where Mor refers to morphisms of sheaves of sets, and

$$\partial_0(x) = (\sigma \mapsto \sigma x - x), \quad \partial_1(x) = ((\sigma, \tau) \mapsto \sigma x(\tau) - x(\sigma\tau) + x(\sigma)), \ldots$$

Note that the above complex is the standard complex that computes the cohomology of the H_n-module G_n (see [6, II.3]).

Consider now G with the trivial action of H_n. Note that

$$H^1(H_n, G) = \mathrm{Hom}(H_n, G) = \mathrm{Hom}(\mathbf{Z}/n(1), G) \otimes_{\mathbf{Z}} P^{\mathrm{gp}}.$$

It can be easily checked that the map

$$\mathrm{Hom}(\mathbf{Z}/n(1), G) \otimes_{\mathbf{Z}} P^{\mathrm{gp}} \simeq H^1(H_n, G) \to H^1(H_n, G_n) \simeq H^1(C^{\cdot}_{G,n}) \to H^1(X_{\mathrm{kfl}}, G)$$

maps $h \otimes a$ to the image of a under the above composition. We can now state the second lemma.

Lemma 3.15. *Assume $X = \mathrm{Spec}(A)$, where A is a Noetherian complete local ring with separably closed residue field, and assume that G is represented by a smooth commutative group scheme over X endowed with the induced log-structure. Assume that $P \xrightarrow{\sim} (M_X/\mathcal{O}_X^*)_x$, where x is the closed point of X. Then, for any $n \neq 0$,*

$$\mu: \quad \mathrm{Hom}(\mathbf{Z}/n(1), G) \otimes_{\mathbf{Z}} P^{\mathrm{gp}} \xrightarrow{\sim} H^1(C^{\cdot}_{G,n}).$$

Proof. Let's treat first the case when A is Artinian. Let I (resp. J) be the ideal of A (resp. $\mathcal{O}_{X_n}$) generated by the image of $P \setminus \{1\}$ (resp. $P^{1/n} \setminus \{1\}$). Then I (resp. J) is a nilpotent ideal. Define a descending filtration G^i on the H_n-module G and G_n^i on the H_n-module G_n by

$$G^i(T) = \ker(G(T) \to G(T \times_X \mathrm{Spec}(\mathcal{O}_X/I^i)));$$

$$G_n^i(T) = \ker(G_n(T) \to G(T \times_X \mathrm{Spec}(\mathcal{O}_{X_n}/J^i))).$$

Since I and J are nilpotent, we have that $G^i(T) = G^i_n(T) = 0$ for a large enough i. Since the group scheme G is smooth, for $i \geq 1$ we get

$$\mathrm{gr}^i(G)(T) \simeq Lie(G) \otimes_A \Gamma(T, I^i\mathcal{O}_T/I^{i+1}\mathcal{O}_T),$$
$$\mathrm{gr}^i(G_n)(T) \simeq Lie(G) \otimes_A \Gamma(T, J^i\mathcal{O}_T/J^{i+1}\mathcal{O}_T).$$

Also, since $\mathcal{O}_X/I \xrightarrow{\sim} \mathcal{O}_{X_n}/J$, we have that $\mathrm{gr}^0(G)(T) \xrightarrow{\sim} \mathrm{gr}^0(G_n)(T)$. We will prove now the following lemma

Lemma 3.16. *For any $i \geq 1$ and any $m \geq 1$, the groups $H^m(H_n, \mathrm{gr}^i(G))$ and $H^m(H_n, \mathrm{gr}^i(G_n))$ are zero.*

Proof. Let $i \geq 1$ and consider the standard complex $C^\cdot(H_n, \mathrm{gr}^i(G))$ that computes the cohomology of the H_n-module $\mathrm{gr}^i(G)$. Then for $m \geq 0$, since H_n is flat over $\mathbf{Z}$ and G is smooth over X, for a certain number k we have

$$\begin{aligned} C^m(H_n, \mathrm{gr}^i(G)) &= \mathrm{Mor}(H_n^{\times m}, \mathrm{gr}^i(G)) \\ &= \mathrm{gr}^i(G)(H_n^{\times m} \times_{\mathbf{Z}} X) \\ &= Lie(G) \otimes_A \Gamma(H_n^{\times m} \times_{\mathbf{Z}} X, I^i\mathcal{O}/I^{i+1}\mathcal{O}) \\ &= Lie(G) \otimes_A \mathcal{O}_{H_n^{\times m} \times_{\mathbf{Z}} X} \otimes_A I^i/I^{i+1} \\ &= \mathbf{G}_a^k(H_n^{\times m} \times_{\mathbf{Z}} X) \otimes_A I^i/I^{i+1} \\ &= \mathrm{Mor}(H_n^{\times m}, \mathbf{G}_a^k) \otimes_A I^i/I^{i+1} = C^m(H_n, \mathbf{G}_a^k) \otimes_A I^i/I^{i+1}. \end{aligned}$$

Similarly, for the standard complex $C^\cdot(H_n, \mathrm{gr}^i(G_n))$ that computes the cohomology of the H_n-module $\mathrm{gr}^i(G_n)$, we get

$$\begin{aligned} C^m(H_n, \mathrm{gr}^i(G_n)) &= \mathrm{Mor}(H_n^{\times m}, \mathrm{gr}^i(G_n)) = \mathrm{Mor}(H_n^{\times m}, \mathbf{G}_a^k) \otimes_A J^i/J^{i+1} \\ &= C^m(H_n, \mathbf{G}_a^k) \otimes_A J^i/J^{i+1}, \end{aligned}$$

Since H_n is diagonalizable and it acts trivially on $\mathbf{G}_a^k$, we know that $H^m(H_n, \mathbf{G}_a^k) = 0$ for $m \geq 1$ [33, Exp.I, Theorem 5.3.3]. Moreover, $\mathbf{G}_a^k$ embeds into $\mathcal{M}\mathrm{or}(H_n, \mathbf{G}_a^k)$ with an H_n-equivariant section [33, Exp.I, Prop. 4.7.4]. Hence

$$\mathcal{M}\mathrm{or}(H_n, \mathbf{G}_a^k) \simeq \mathbf{G}_a^k \oplus \mathcal{M}\mathrm{or}(H_n, \mathbf{G}_a^k)/\mathbf{G}_a^k$$

as H_n-modules. That gives us that

$$C^\cdot(H_n, \mathcal{M}\mathrm{or}(H_n, \mathbf{G}_a^k)) \simeq C^\cdot(H_n, \mathbf{G}_a^k) \oplus C^\cdot(H_n, \mathcal{M}\mathrm{or}(H_n, \mathbf{G}_a^k)/\mathbf{G}_a^k).$$

Now, $C^\cdot(H_n, \mathcal{M}\mathrm{or}(H_n, \mathbf{G}_a^k))$ has an A-linear contracting homotopy [33, Exp.I, Lemma 5.2.]. It follows that $C^\cdot(H_n, \mathcal{M}\mathrm{or}(H_n, \mathbf{G}_a^k)) \otimes_A I^i/I^{i+1}$ also has a contracting homotopy. Hence $H^m(C^\cdot(H_n, \mathcal{M}\mathrm{or}(H_n, \mathbf{G}_a^k)) \otimes_A I^i/I^{i+1}) = 0$, for $m \geq 1$, and by the above splitting $H^m(C^\cdot(H_n, \mathbf{G}_a^k) \otimes_A I^i/I^{i+1}) = 0$, as wanted. Similarly, $H^m(H_n, \mathrm{gr}^i(G_n))) = H^m(C^\cdot(H_n, \mathbf{G}_a^k) \otimes_A J^i/J^{i+1}) = 0$, for $m \geq 1$. □

Using the above lemma, we get

$$\begin{aligned} &\mathrm{Hom}(\mathbf{Z}/n(1), G) \otimes_{\mathbf{Z}} P^{\mathrm{gp}} = H^1(H_n, G) \\ &\quad \xrightarrow{\sim} H^1(H_n, \mathrm{gr}^0(G)) \xrightarrow{\sim} H^1(H_n, \mathrm{gr}^0(G_n)) \xleftarrow{\sim} H^1(H_n, G_n) = H^1(C^\cdot_{G,n}), \end{aligned}$$

as wanted.
Let's turn now to the general case of A complete. We will basically "go to the limit over the argument for A Artinian". Denote the maximal ideal of A by m_A. Note that $G(A) \xrightarrow{\sim} \operatorname{proj\,lim}_i G(A/\mathrm{m}_A^i)$ and $G(X_{n,k}) \xrightarrow{\sim} \operatorname{proj\,lim}_i G(X_{n,k} \otimes_A A/\mathrm{m}_A^i)$. Moreover, since G is smooth, we have that the maps

$$G(A/\mathrm{m}_A^{i+1}) \to G(A/\mathrm{m}_A^i), \qquad G(X_{n,k} \otimes_A A/\mathrm{m}_A^{i+1}) \to G(X_{n,k} \otimes_A A/\mathrm{m}_A^i)$$

are surjective. Hence we get the following exact sequences

$$\begin{aligned} 0 \to & G(A) \to G(X_{n,0}) \to D \to 0, \\ 0 \to & E \to G(X_{n,1}) \to G(X_{n,2}), \end{aligned}$$

where $E = \operatorname{proj\,lim}_i E_i$ and $D = \operatorname{proj\,lim}_i D_i$, and E_i and D_i are defined by the following exact sequences

$$\begin{aligned} 0 \to & G(A/\mathrm{m}_A^i) \to G(X_{n,0} \otimes_A A/\mathrm{m}_A^i) \to D_i \to 0, \\ 0 \to & E_i \to G(X_{n,1} \otimes_A A/\mathrm{m}_A^i) \to G(X_{n,2} \otimes_A A/\mathrm{m}_A^i). \end{aligned}$$

We have $D_i \subset E_i$ and E_i/D_i is H^1 of the complex $C^{\cdot}_{G,n}$ for $\operatorname{Spec}(A/\mathrm{m}_A^i)$. Also $E/D \simeq H^1(C^{\cdot}_{G,n})$ and, since the maps $D_{i+1} \to D_i$ are surjective, $E/D \simeq \operatorname{proj\,lim}_i(E_i/D_i)$. On the other hand, let $\operatorname{Hom}(\mathbf{Z}/n(1), G)_i$ denote the group $\operatorname{Hom}(\mathbf{Z}/n(1), G)$ over $\operatorname{Spec}(A/\mathrm{m}_A^i)$. Since $\mathcal{H}\mathrm{om}(\mathbf{Z}/n(1), G)$ is representable by an étale scheme [1, Exp.XI, Prop. 3.12], [2, Exp.XV, Prop.16], we have $\operatorname{Hom}(\mathbf{Z}/n(1), G) \simeq \operatorname{Hom}(\mathbf{Z}/n(1), G)_i$, $i \geq 1$. The proof of our lemma for A Artinian gives that

$$\operatorname{Hom}(\mathbf{Z}/n(1), G)_i \otimes_{\mathbf{Z}} P^{\mathrm{gp}} \xrightarrow{\sim} E_i/D_i.$$

Hence taking limits

$$\operatorname{Hom}(\mathbf{Z}/n(1), G) \otimes_{\mathbf{Z}} P^{\mathrm{gp}} \xrightarrow{\sim} E/D \simeq H^1(C^{\cdot}_{G,n}),$$

as wanted. □

In the general case we have to argue a little bit more. Let $\widehat{X} = \operatorname{Spec}(\widehat{A})$, where $\widehat{A}$ is the completion of A. Endow $\widehat{X}$ with the inverse image log-structure. Since $\mathcal{H}\mathrm{om}(\mathbf{Z}/n(1), G)$ is represented by an étale scheme and the morphism $A \to \widehat{A}$ is a covering for the fpqc topology, $\operatorname{Hom}(\mathbf{Z}/n(1), G)$ does not change when we pass to the completion. It suffice thus to show that $H^1(X_{\mathrm{kfl}}, G) \to H^1(\widehat{X}_{\mathrm{kfl}}, G)$ is injective. Let $\alpha \in H^1(X_{\mathrm{kfl}}, G)$ be a class that dies in $H^1(\widehat{X}_{\mathrm{kfl}}, G)$. By fpqc descent, α is a class of a representable smooth affine G-torsor Y over X (equipped with the inverse image log-structure). Since X is strictly local, Y has an X-rational point. Hence $\alpha = 0$. □

□

Corollary 3.17. *Let $X = \operatorname{Spec}(A)$ be a log-scheme such that A is Noetherian and strictly local. We have the following isomorphisms*

$$\begin{aligned} H^1(X_{\mathrm{kfl}}, \mathbf{G}_m) &\simeq (M_X^{gp}/\mathcal{O}_X^*)_x \otimes (\mathbf{Q}/\mathbf{Z}), \\ H^1(X_{\mathrm{k\acute{e}t}}, \mathbf{G}_m) &\simeq (M_X^{gp}/\mathcal{O}_X^*)_x \otimes (\mathbf{Q}/\mathbf{Z})', \end{aligned}$$

where x denotes the closed point of X and $(\mathbf{Q}/\mathbf{Z})' = \oplus_{l\neq char(x)}\mathbf{Q}_l/\mathbf{Z}_l$.

Proof. The case of X_{kfl} follows from Proposition 3.13. Inspecting its proof we see that together with Corollary 2.17 it actually proves the statement for $H^1(X_{\mathrm{k\acute{e}t}}, \mathbf{G}_m)$ as well. □

Corollary 3.18. *Let $X = \operatorname{Spec}(A)$ be a log-scheme equipped with a Zariski log-structure such that A is Noetherian and local. We have the following isomorphisms*

$$H^1(X_{\mathrm{kfl}}, \mathbf{G}_m) \simeq (M_X^{gp}/\mathcal{O}_X^*)_x \otimes (\mathbf{Q}/\mathbf{Z}),$$
$$H^1(X_{\mathrm{k\acute{e}t}}, \mathbf{G}_m) \simeq (M_X^{gp}/\mathcal{O}_X^*)_x \otimes (\mathbf{Q}/\mathbf{Z})'.$$

Proof. The proof of Proposition 3.13 goes through with few small changes. In Lemma 3.14 we have to use the fact that X_{mn} is a product of a finite number of Spec of local rings and we have $H^1((X_{mn})_{\mathrm{\acute{e}t}}, \mathbf{G}_m) = H^1((X_{mn})_{\mathrm{Zar}}, \mathbf{G}_m) = 0$. Similarly, at the very end of the proof of the proposition we get that, since X is local, and Y is a $\mathbf{G}_m$-torsor, it has a rational point. □

Example 3.19. We can obtain invertible sheaves on the Kummer log-flat site in the following way. Take a log-scheme X with a chart $P \to M_X$. Consider the covering $Y = X \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q]$ associated to a Kummer map $P \to Q$. The $\mathcal{O}_X$-module $f_*\mathcal{O}_Y$ on X_{kfl}, $f : Y \to X$, has an action of the group scheme $H = \operatorname{Spec}(\mathbf{Z}[Q^{\mathrm{gp}}/P^{\mathrm{gp}}])$. It decomposes under this action into a direct sum of invertible sheaves $f_*\mathcal{O}_Y \simeq \oplus_a \mathcal{O}_X(a)$, $a \in Q^{\mathrm{gp}}/P^{\mathrm{gp}}$. Here $\mathcal{O}_X(a)$ is the part of $f_*\mathcal{O}_Y$ on which H acts via the character $H \to \mathbf{G}_m$ corresponding to a. More specifically,

$$f_*\mathcal{O}_Y(Y) \simeq \mathcal{O}_{Y\times_X Y} \simeq \mathcal{O}_X \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q \oplus Q^{\mathrm{gp}}/P^{\mathrm{gp}}] = \oplus_{a\in Q^{\mathrm{gp}}/P^{\mathrm{gp}}} a\mathcal{O}_Y$$

and $\mathcal{O}_X(a)|Y_{\mathrm{kfl}} = \varepsilon^* a\mathcal{O}_Y$. The element of $H^1(X_{\mathrm{kfl}}, \mathbf{G}_m)$ corresponding to the invertible sheaf $\mathcal{O}_X(a)$ is given by the image of a^m under

$$H^0(X_{\mathrm{kfl}}, \mathbf{G}_m^{\times}) \to H^1(X_{\mathrm{kfl}}, \mathbf{Z}/m(1)) \to H^1(X_{\mathrm{kfl}}, \mathbf{G}_m),$$

where the first arrow is the connecting map of the Kummer sequence

$$0 \to \mathbf{Z}/m(1) \to \mathbf{G}_m^{\times} \xrightarrow{m} \mathbf{G}_m^{\times} \to 0$$

Here m is a number such that $a^m \in P^{\mathrm{gp}}$ and the above image is independent of m chosen. If X and x are as in the above corollary then this element corresponds to $a \otimes m^{-1}$ of $(M_X^{gp}/\mathcal{O}_X^*)_x \otimes (\mathbf{Q}/\mathbf{Z})$.
To get nontrivial Kummer log-flat coherent sheaves note that, for $a \in Q$ and for the natural map $\alpha : Q \to M_Y$, the element $a \otimes \alpha(a) \in f_*\mathcal{O}_Y(Y)$ is a global section of $\mathcal{O}_X(a)$. Define $\mathcal{O}_X\{a\}$ to be the image of the map $\alpha(a) : \mathcal{O}_X \to \mathcal{O}_X(a)$.

As the next corollary we get the following log-version of Hilbert 90.

Theorem 3.20. *(Hilbert 90) Let X be a log-scheme whose underlying scheme is locally Noetherian. Then the canonical maps*

$$H^1(X_{\mathrm{fl}}, \mathbf{G}_m^{\times}) \xrightarrow{\sim} H^1(X_{\mathrm{kfl}}, \mathbf{G}_m^{\times}), \qquad H^1(X_{\mathrm{\acute{e}t}}, \mathbf{G}_m^{\times}) \xrightarrow{\sim} H^1(X_{\mathrm{k\acute{e}t}}, \mathbf{G}_m^{\times})$$

are isomorphisms.

Proof. We have the short exact sequence

$$0 \to \mathbf{G}_m \to \mathbf{G}_m^{\times} \to \mathbf{G}_m^{\times}/\mathbf{G}_m \to 0$$

of sheaves on X_{fl}. For $X = \mathrm{Spec}(A)$, where A is a Noetherian strictly local ring, this yields $H^1(X_{\mathrm{fl}}, \mathbf{G}_m^{\times}) = 0$. Indeed, we have $H^1(X_{\mathrm{fl}}, \mathbf{G}_m) = 0$. And, since $\mathbf{G}_m^{\times}/\mathbf{G}_m = \varepsilon^*(\mathbf{G}_m^{\times}/\mathbf{G}_m)$, where $\varepsilon : X_{\mathrm{fl}} \to X_{\text{ét}}$ is the natural map, $H^1(X_{\mathrm{fl}}, \mathbf{G}_m^{\times}/\mathbf{G}_m) = H^1(X_{\text{ét}}, \mathbf{G}_m^{\times}/\mathbf{G}_m) = 0$ (cf., [25, II.3]).
The above implies that this theorem is equivalent to the following local form. □

Corollary 3.21. *Let X be a log-scheme whose underlying scheme is* Spec *of a Noetherian strictly local ring. Then the groups $H^1(X_{\mathrm{kfl}}, \mathbf{G}_m^{\times})$, $H^1(X_{\text{két}}, \mathbf{G}_m^{\times})$ and $H^1(X_{\text{ét}}, \mathbf{G}_m^{\times})$ are zero.*

Proof. Let $X = \mathrm{Spec}(A)$, where A is a Noetherian strictly local ring. Assume that $P \xrightarrow{\sim} (M_X/\mathcal{O}_X^*)_x$, where x is the closed point of X. We will show that $H^1(X_{\mathrm{kfl}}, \mathbf{G}_m^{\times}) = 0$ (the proof for the Kummer log-étale site is almost the same and the case of the étale site is obvious). From Čech cohomology we know that the map $\mathrm{inj\,lim}_n \check{H}^1(X_n/X, \mathbf{G}_m^{\times}) \to H^1(X_{\mathrm{kfl}}, \mathbf{G}_m^{\times})$ is injective and its cokernel injects into $\mathrm{inj\,lim}_n H^1((X_n)_{\mathrm{kfl}}, \mathbf{G}_m^{\times})$. Hence it suffices to show that $\mathrm{inj\,lim}_n H^1((X_n)_{\mathrm{kfl}}, \mathbf{G}_m^{\times}) = 0$ and $\check{H}^1(X_n/X, \mathbf{G}_m^{\times}) = 0$. Here the covering $X_n = X \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q]$, $Q = P^{1/n}$.
First, let's show that $\mathrm{inj\,lim}_n H^1((X_n)_{\mathrm{kfl}}, \mathbf{G}_m^{\times}) = 0$. Take an element α of $H^1((X_n)_{\mathrm{kfl}}, \mathbf{G}_m^{\times})$. Let $T \to X_n$ be a log-flat Kummer covering such that α comes from $\check{H}^1(T/X_n, \mathbf{G}_m^{\times})$. By Corollary 2.16, for some m, we may assume that we have a factorization $T \to X_{mn} \to X_n$, where $T \to X_{mn}$ is classically flat and surjective. It follows that the class α on X_{mn} comes from $\check{H}^1(T \times_{X_n} X_{mn}/X_{mn}, \mathbf{G}_m^{\times})$. Thus the class of α in $H^1((X_{mn})_{\mathrm{kfl}}, \mathbf{G}_m^{\times})$ comes from $H^1((X_{mn})_{\mathrm{fl}}, \mathbf{G}_m^{\times})$. Since X_{mn} is a disjoint union of a finite number of Spec of strictly local rings, the last group is trivial as we have shown above.
Now, let's show that $\check{H}^1(X_n/X, \mathbf{G}_m^{\times}) = 0$. Consider the exact sequence of presheaves (!) on X_{kfl}

$$0 \to \mathbf{G}_m \to \mathbf{G}_m^{\times} \to \mathbf{G}_m^{\times}/\mathbf{G}_m \to 0.$$

It gives us the exact sequence of Čech cohomology groups

$$\to \check{H}^0(X_n/X, \mathbf{G}_m^{\times}/\mathbf{G}_m) \xrightarrow{\partial} \check{H}^1(X_n/X, \mathbf{G}_m) \to \check{H}^1(X_n/X, \mathbf{G}_m^{\times}) \to \check{H}^1(X_n/X, \mathbf{G}_m^{\times}/\mathbf{G}_m) \to$$

By Proposition 3.13 the connecting morphism ∂ is surjective. Indeed, consider an element $a \otimes n^{-1} \in \check{H}^1(X_n/X, \mathbf{G}_m) \simeq P^{\mathrm{gp}} \otimes \mathbf{Z}/n$, $a \in P^{\mathrm{gp}}$. Choose an element $b \in Q^{gp}$ such that $b^n = a$. It belongs to $\check{H}^0(X_n/X, \mathbf{G}_m^{\times}/\mathbf{G}_m)$. To see that recall that the exact sequence of the covering X_n/X

$$0 \to \Gamma(X, \mathcal{O}_X) \to \Gamma(X_n, \mathcal{O}_{X_n}) \to \Gamma(X_n \times_X X_n, \mathcal{O}_{X_n \times_X X_n})$$

is isomorphic to

$$0 \to A \to A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q] \stackrel{\beta_1-\beta_2}{\longrightarrow} A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[Q \oplus Q^{\mathrm{gp}}/P^{\mathrm{gp}}],$$

where $\beta_1(x) = 1 \otimes x$, $x \in Q$, $\beta_2(x) = 1 \otimes (x, x \mod P^{\mathrm{gp}})$. Hence

$$(\beta_1^* - \beta_2^*)(b) = 1 \otimes b - 1 \otimes (b, b \mod P^{\mathrm{gp}}) = 1 \otimes b - 1 \otimes b = 0$$

and $b \in \check{H}^0(X_n/X, \mathbf{G}_m^{\times}/\mathbf{G}_m)$, as wanted. One easily now checks that $\partial(b) = a \otimes n^{-1}$.

It remains to show that $\check{H}^1(X_n/X, \mathbf{G}_m^{\times}/\mathbf{G}_m) = 0$. Or that the sequence

$$\mathbf{G}_m^{\times}/\mathbf{G}_m(X_n) \stackrel{d_0}{\to} \mathbf{G}_m^{\times}/\mathbf{G}_m(X_n \times_X X_n) \stackrel{d_1}{\to} \mathbf{G}_m^{\times}/\mathbf{G}_m(X_n \times_X X_n \times_X X_n)$$

is exact. By Lemma 3.28 this sequence is isomorphic to

$$Q^{gp} \stackrel{d_0}{\to} Q^{gp} \stackrel{d_1}{\to} Q^{gp},$$

where $d_0 = 0$ and $d_1 = 1$. Hence it is exact, as wanted. □

3.2. Locally free sheaves of higher rank. For isomorphism classes of locally free sheaves of arbitrary rank we have the following theorem stated already by Kato [24, Cor. 6.4].

Theorem 3.22. *Let $X = \mathrm{Spec}(A)$ be a log-scheme such that A is Noetherian and strictly local. The map*

$$\prod^n \check{H}^1(X_{\mathrm{kfl}}, \mathbf{G}_m) \to \check{H}^1(X_{\mathrm{kfl}}, \mathbf{GL}_n)$$

given by the diagonal embedding $\prod^n \mathbf{G}_m \hookrightarrow \mathbf{GL}_n$ induces an isomorphism

$$\check{H}^1(X_{\mathrm{kfl}}, \mathbf{GL}_n) \simeq S_n \backslash (\prod^n (M_X^{gp}/\mathcal{O}_X^*)_x \otimes (\mathbf{Q}/\mathbf{Z})),$$

where $S_n\backslash$ denotes the quotient by the action of the symmetric group of degree n on the product of n copies. Similarly, we have an isomorphism

$$\check{H}^1(X_{\mathrm{k\acute{e}t}}, \mathbf{GL}_n) \simeq S_n \backslash (\prod^n (M_X^{gp}/\mathcal{O}_X^*)_x \otimes (\mathbf{Q}/\mathbf{Z})').$$

Proof. Assume that $P \stackrel{\sim}{\to} (M_X/\mathcal{O}_X^*)_x$. For $m \geq 1$, let $X_m = X \otimes_{\mathbf{Z}[P]} \mathbf{Z}[P^{1/m}]$, with the induced log-structure.

Lemma 3.23. *We have*

$$\operatorname{inj\,lim}_m \check{H}^1(X_m/X, \mathbf{GL}_n) \stackrel{\sim}{\to} \check{H}^1(X_{\mathrm{kfl}}, \mathbf{GL}_n).$$

Proof. The injectivity is obvious. For the surjectivity, consider a class $\alpha \in \check{H}^1(X_{\mathrm{kfl}}, \mathbf{GL}_n)$. Let $T \to X$ be a log-flat Kummer covering such that $\alpha \in \check{H}^1(T/X, \mathbf{GL}_n)$. By Corollary 2.16, we may assume that for some m, we have a factorization $T \to X_m \to X$, where $T \to X_m$ is classically flat and surjective. Since X_m is a disjoint union of a finite number of Spec of strictly local rings we have $\check{H}^1(X_{m,\mathrm{fl}}, \mathbf{GL}_n) = 0$. It follows that α is trivialised on X_m hence $\alpha \in \check{H}^1(X_m/X, \mathbf{GL}_n)$, as wanted. □

Lemma 3.24. *Assume $X = \mathrm{Spec}(A)$, where A is a Noetherian complete local ring with separably closed residue field. Then, for any $n \neq 0$,*

$$(\mathrm{Hom}(H_m, \mathbf{GL}_n)/\equiv) \xrightarrow{\sim} \check{H}^1(X_m/X, \mathbf{GL}_n),$$

where H_m is the group scheme $\mathrm{Spec}(\mathbf{Z}[P^{\mathrm{gp}}/(P^{\mathrm{gp}})^m])$ and $/\equiv$ means the quotient set by the inner conjugation by elements of $\mathbf{GL}_n(A)$.

Proof. We proceed as in the proof of Lemma 3.15 and keep its notation. Note that

$$(\mathrm{Hom}(H_m, \mathbf{GL}_n)/\equiv) = H^1(H_m, \mathbf{GL}_n)$$

Let's treat first the case when A is Artinian. Consider the corresponding filtrations $\mathbf{GL}_n^i$, $\mathbf{GL}_{n,m}^i$ of $\mathbf{GL}_n$ and $\mathbf{GL}_{n,m}$. The computation of the graded pieces goes through and, since $Lie(\mathbf{GL}_n) \simeq \mathbf{G}_a^{n^2}$, so does the proof of Lemma 3.16. Hence

$$H^k(H_n, \mathrm{gr}^i(\mathbf{GL}_n)) = H^k(H_n, \mathrm{gr}^i(\mathbf{GL}_{n,m})) = 0, \quad i \geq 1, k \geq 1.$$

Using now the exact sequences

$$0 \to \mathbf{GL}_n^{i-1}/\mathbf{GL}_n^i \to \mathbf{GL}_n/\mathbf{GL}_n^i \to \mathbf{GL}_n/\mathbf{GL}_n^{i-1} \to 0$$

(starting from i such that $\mathbf{GL}_n^i = 0$) we get that $H^1(H_n, \mathbf{GL}_n) \xrightarrow{\sim} H^1(H_n, \mathrm{gr}^0(\mathbf{GL}_n))$. Similarly, $H^1(H_n, \mathbf{GL}_{n,m}) \xrightarrow{\sim} H^1(H_n, \mathrm{gr}^0(\mathbf{GL}_{n,m}))$. Since $\mathrm{gr}^0(\mathbf{GL}_n) \xrightarrow{\sim} \mathrm{gr}^0(\mathbf{GL}_{n,m})$, we are done.

Let's turn now to the general case of A complete. We compute

$$\begin{aligned}(\mathrm{Hom}(H_m, \mathbf{GL}_n)/\equiv) &= S_n \backslash \mathrm{Hom}(H_m, \prod^n \mathbf{G}_m) = S_n \backslash \prod^n \mathrm{Hom}(H_m, \mathbf{G}_m) \\ &= S_n \backslash \prod^n \mathrm{Hom}(\mathbf{Z}/m(1), \mathbf{G}_m) \otimes P^{gp} = S_n \backslash \prod^n \mathbf{Z}/m \otimes P^{\mathrm{gp}}.\end{aligned}$$

The same computation works over each A/m_A^i. Passing now to the limit over i it suffices to show that the natural map

$$\check{H}^1(X_m, \mathbf{GL}_n) \to \operatorname{proj}\lim_i \check{H}^1(X_{m,i}, \mathbf{GL}_n),$$

where $X_{m,i}$ is the base change of X_m to A/m_A^i, is injective. By straightforward computation this follows from the fact that $\mathbf{GL}_n$ defines a sheaf for the Kummer log-flat topology. □

In the general case we have to argue a little bit more. Let $\widehat{X} = \mathrm{Spec}(\widehat{A})$, where $\widehat{A}$ is the completion of A. Endow $\widehat{X}$ with the inverse image log-structure. Since $\mathrm{Hom}(H_m, \mathbf{GL}_n)/\equiv$ does not change when we pass to the completion (see above), it suffice to show that $\check{H}^1(X_{\mathrm{kfl}}, \mathbf{GL}_n) \to \check{H}^1(\widehat{X}_{\mathrm{kfl}}, \mathbf{GL}_n)$ is injective. This is proved exactly like the corresponding fact in the proof of Lemma 3.15. The proof for $X_{\mathrm{k\acute{e}t}}$ is analogous (using Corollary 2.17). □

Corollary 3.25. *In the above theorem we may take $X = \mathrm{Spec}(A)$ to be a log-scheme equipped with a Zariski log-structure such that A is Noetherian and local.*

Proof. The proof of Theorem 3.22 goes through with few small changes. In Lemma 3.23 we have to use the fact that X_m is a product of a finite number of Spec of local rings and we have $\check{H}^1((X_m)_{\mathrm{fl}}, \mathbf{GL}_n) = 0$. Similarly, at the very end of the proof of the theorem we get that, since X is local, and Y is a $\mathbf{GL}_n$-torsor, it has a rational point. □

Corollary 3.26. *Let $X = \mathrm{Spec}(A)$ be a log-scheme such that A is Noetherian and strictly local. Let $\mathcal{F}$ be a locally free finite type $\mathcal{O}_X$-module on X_{kfl} (resp. $X_{\mathrm{k\acute{e}t}}$). Then $\mathcal{F}$ is a direct sum of invertible sheaves on X_{kfl} (resp. $X_{\mathrm{k\acute{e}t}}$). Similarly for A local and equipped with a Zariski log-structure.*

The following proposition will be useful in computing K-theory groups. It was originally stated by Kato [24, Prop. 6.5].

Proposition 3.27. *Let X be an affine log-scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module on X_{kfl} such that for some Kummer log-flat covering $Y \to X$ the restriction $\mathcal{F}|Y$ is isomorphic to the inverse image of a quasi-coherent sheaf on the small Zariski site of Y. Then $H^n(X_{\mathrm{kfl}}, \mathcal{F}) = 0$ for any $n \geq 1$. Similar statement holds for $X_{\mathrm{k\acute{e}t}}$.*

Proof. Consider the case of X_{kfl}. Assume first that $\mathcal{F}$ is isomorphic to the inverse image of a quasi-coherent sheaf on the small Zariski site of X. Since $H^n(X_{\mathrm{Zar}}, \mathcal{F}) = 0$, $n \geq 1$, we may assume that X is equipped with a chart $P \to M_X$, $P^* = \{1\}$. We may work on the small site of the Kummer log-flat site built from affine maps. It suffices now to show that our sheaf $\mathcal{F}$ is flasque. We will show that for every covering $Y \to X$ from some cofinal system of coverings the Čech cohomology groups $\check{H}^n(Y/X, \mathcal{F}) = H^n(C^{\cdot}(Y/X))$, $n \geq 1$, are trivial. Since our coverings are log-flat and of Kummer type, by Corollary 2.16 we may assume that there exists a factorization of $Y \to X$ into $f : Y \to Y_1$ and $g : Y_1 \to X$, where f is affine, strictly flat and a covering and $Y_1 = Y \times_{\mathrm{Spec}(\mathbf{Z}[P])} \mathrm{Spec}(\mathbf{Z}[Q])$, for a Kummer morphism $u : P \to Q$.
We will show now that the complex $C^{\cdot}(Y/X)$ has trivial cohomology in degrees higher than 0. Assume first that the augmentation $\Gamma(X, \mathcal{F}) \stackrel{g^*}{\to} C^{\cdot}(Y_1/X)$ is a quasi-isomorphism. We will check that this implies that the augmentation $\Gamma(X, \mathcal{F}) \stackrel{(gf)^*}{\to} C^{\cdot}(Y/X)$ is a quasi–isomorphism as well. The reader will note that because the schemes Y, Y_1, and X are assumed to be affine, all the schemes appearing in the argument below are affine as well. Consider the double complex

$$C^{\cdot}(Y, Y_1, X) : (i, j) \mapsto \Gamma(Y^{(i+1)} \times_X Y_1^{(j+1)}, \mathcal{F}),$$

where, for any $n \geq 1$, $Y^n = (Y/X)^{\times n}$, and $Y_1^n = (Y_1/X)^{\times n}$. Consider the natural maps $C^{\cdot}(Y/X) \stackrel{f_1^*}{\to} C^{\cdot}(Y, Y_1, X)$ and $C^{\cdot}(Y_1/X) \stackrel{g_1^*}{\to} C^{\cdot}(Y, Y_1, X)$. First, we claim that f_1^* is a quasi-isomorphism. For that, it suffices to show that, for any $n \geq 1$, the map $\Gamma(Y^n, \mathcal{F}) \stackrel{f_1^*}{\to} C^{\cdot}(Y^n \times_X Y_1/Y^n)$ is a quasi-isomorphism. Since the projection $Y^n \times_X Y_1 \to Y^n$ admits a section $Y^n \stackrel{s_n}{\to} Y^n \times_X Y_1$, this is clear. Next, we will show that g_1^* is a quasi-isomorphism. It suffices to show

that the augmentation $\Gamma(Y_1^n,\mathcal{F})\stackrel{g_1^*}{\to} C^{\cdot}(Y_1^n\times_X Y/Y_1^n)$ is a quasi-isomorphism. Consider the composition $Y^n \stackrel{f^{n-1}\times s_1}{\to} Y_1^n\times_X Y\to Y_1^n$. It is equal to the map f^n, which is faithfully flat. By faithfully flat descent, since the base-change of the augmentation g_1^* by f^n is a quasi-isomorphism $\Gamma(Y^n,\mathcal{F})\to C^{\cdot}(Y^{(n+1)}/Y^n)$ (the morphism $Y^{(n+1)}=Y^n\times_{Y_1^n}Y_1^n\times_X Y\to Y^n$ admitting a section), so is the augmentation g_1^*.
Finally, we have that

$$f_1^*(gf)^*=g_1^*g^*:\Gamma(X,\mathcal{F})\to C^{\cdot}(Y,Y_1,X).$$

Since f_1^*, g_1^*, and g^* are quasi-isomorphisms, so is $(gf)^*$.

Lemma 3.28. *Let $A=\Gamma(X,\mathcal{O}_X)$. Then the augmentation*

$$A\stackrel{e^*}{\to} C^{\cdot}(Y_1/X) \tag{3.2}$$

is a quasi-isomorphism.

Proof. The essential point is that the morphism of monoids $u:P\to Q$ is exact, i.e., $P=(u^{gp})^{-1}(Q)$ in P^{gp}, where $u^{gp}:P^{gp}\to Q^{gp}$. Set $G=Q^{gp}/P^{gp}$. The augmentation e^* is isomorphic to

$$A\stackrel{e^*}{\to} A\otimes_{\mathbf{Z}[P]}\mathbf{Z}[Q]\stackrel{d_0}{\to} A\otimes_{\mathbf{Z}[P]}\mathbf{Z}[Q\oplus G]\stackrel{d_1}{\to} A\otimes_{\mathbf{Z}[P]}\mathbf{Z}[Q\oplus G^{\oplus 2}]\stackrel{d_2}{\to}\dots$$

Here the A-linear morphism $d_n:A\otimes_{\mathbf{Z}[P]}\mathbf{Z}[Q\oplus G^{\oplus n}]\to A\otimes_{\mathbf{Z}[P]}\mathbf{Z}[Q\oplus G^{\oplus n+1}]$ is equal to the alternating sum of maps $\beta_1,\beta_2,\dots,\beta_{n+2}$, where

$$\beta_k(b_1,b_2,\dots,b_{n+1})=\begin{cases}(b_1,b_1b_2^{-1}\cdots b_{n+1}^{-1},b_2,\dots,b_{n+1}) & \text{if } k=1\\ (b_1,b_2,\dots,b_{k-1},1,b_{k+1},\dots,b_{n+1}) & \text{if } k\neq 1,\end{cases}$$

for $b_1\in Q$, $b_2,\dots,b_{n+1}\in G$. Consider now the following A-module homomorphisms $h_{n+1}:A\otimes_{\mathbf{Z}[P]}\mathbf{Z}[Q\oplus G^{\oplus n}]\to A\otimes_{\mathbf{Z}[P]}\mathbf{Z}[Q\oplus G^{\oplus n-1}]$ for $n\geq 1$,

$$h_{n+1}(b_1,b_2,\dots,b_{n+1})=\begin{cases}(-1)^n(b_1,b_2,\dots,b_n) & \text{if } b_{n+1}=1\\ 0 & \text{if } b_{n+1}\neq 1\end{cases}$$

We claim that h_n's together with the morphism $h_1:A\otimes_{\mathbf{Z}[P]}\mathbf{Z}[Q]\to A$ sending $1\otimes b$ to b if $b\in P$ and to 0 if $b\notin P$ (h_1 is well-defined since u is exact), form a contracting homotopy, i.e., that $h_1e^*=\mathrm{Id}$, $h_2d_0+e^*h_1=\mathrm{Id}$, and $h_{n+2}d_n+d_{n-1}h_{n+1}=\mathrm{Id}$, $n\geq 1$. We compute that

$$\begin{aligned}(h_2d_0+e^*h_1)(b_1) &= \begin{cases}h_2((b_1,1)-(b_1,1))+1\otimes b_1 & \text{if } b_1\in P\\ h_2((b_1,b_1)-(b_1,1)) & \text{if } b_1\notin P\end{cases}\\ &= 1\otimes b_1\end{aligned}$$

(use that u is exact), and that, for $n \geq 1$,

$$\begin{aligned}
&h_{n+2}d_n(b_1, b_2, \ldots, b_{n+1}) \\
&\quad = h_{n+2}[(b_1, b_1b_2^{-1}\cdots b_{n+1}^{-1}, b_2, \ldots, b_{n+1}) - (b_1, 1, b_3, \ldots, b_{n+1}) + \ldots \\
&\qquad + (-1)^{n+1}(b_1, b_2, b_3, \ldots, b_{n+1}, 1)] \\
&\quad = \begin{cases} (-1)^{n+1}(b_1, b_1b_2^{-1}\cdots b_{n+1}^{-1}, b_2, \ldots, b_n) & \\ \quad -(-1)^{n+1}(b_1, 1, b_3, \ldots, b_n) + \ldots & \\ \quad -(b_1, b_2, b_3, \ldots, b_n, 1) + (b_1, b_2, b_3, \ldots, b_{n+1}) & \text{if } b_{n+1} = 1 \\ (b_1, b_2, b_3, \ldots, b_{n+1}) & \text{if } b_{n+1} \neq 1 \end{cases}
\end{aligned}$$

$$\begin{aligned}
&d_{n-1}h_{n+1}(b_1, b_2, \ldots, b_{n+1}) \\
&\quad = \begin{cases} (-1)^n d_{n-1}(b_1, b_2, \ldots, b_n) & \text{if } b_{n+1} = 1 \\ 0 & \text{if } b_{n+1} \neq 1 \end{cases} \\
&\quad = \begin{cases} (-1)^n(b_1, b_1b_2^{-1}\cdots b_n^{-1}, b_2, \ldots, b_n) & \\ \quad -(-1)^n(b_1, 1, b_3, \ldots, b_n) + \ldots & \\ \quad +(b_1, b_2, b_3, \ldots, b_n, 1) & \text{if } b_{n+1} = 1 \\ 0 & \text{if } b_{n+1} \neq 1 \end{cases}
\end{aligned}$$

Hence we get that

$$(h_{n+2}d_n + d_{n-1}h_{n+1})(b_1, b_2, \ldots, b_{n+1}) = \begin{cases} (b_1, b_2, \ldots, b_n, 1) & \text{if } b_{n+1} = 1 \\ (b_1, b_2, b_3, \ldots, b_{n+1}) & \text{if } b_{n+1} \neq 1, \end{cases}$$

as wanted. □

This proves the vanishing of cohomology for $\mathcal{F} = \mathcal{O}_X$. For general $\mathcal{F}$, the complex $\Gamma(X, \mathcal{F}) \xrightarrow{e^*} C^{\cdot}(Y_1/X)$ is isomorphic to the tensor product (over A) of the complex (3.2) with $\Gamma(X, \mathcal{F})$. Since the contracting homotopy we have constructed above is A-linear, this complex is clearly exact.
Let us turn now to the case of general Y. By Corollary 2.16 and faithfully flat descent we may assume that $Y = \operatorname{Spec}(A \otimes_{\mathbf{Z}[P]} \mathbf{Z}[P^{1/m}])$ for some m. Then (see the proof of Proposition 3.13)

$$\check{H}^n(Y/X, \mathcal{F}) = H^n(H_m, f_*\mathcal{F})$$

where H_m is the group scheme $\operatorname{Spec}(\mathbf{Z}[P^{gp}/(P^{gp})^m])$. Since H_m is diagonalizable, we know that $H^n(H_m, f_*\mathcal{F}) = 0$ for $n \geq 1$ [33, Exp. I, Thm. 5.3.3]. This finishes our proof for the Kummer log-flat topology. The proof for the Kummer log-étale topology is analogous (replace Corollary 2.16 with Corollary 2.17). □

The above proposition implies the following

Proposition 3.29. *Let X be a log-scheme and let*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

be an exact sequence of locally free finite rank $\mathcal{O}_X$-sheaves on X_{kfl} or $X_{\mathrm{két}}$. Then

(1) *if X is affine, this exact sequence splits;*
(2) *$\mathcal{F}$ is classical if and only if so is $\mathcal{F}'$ and $\mathcal{F}''$.*

Proof. Consider the following exact sequence of sheaves on X_{kfl}

$$0 \to \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{F}') \to \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{F}) \to \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{F}'') \to 0.$$

Since, by Proposition 3.27, $H^1(X_{\mathrm{kfl}}, \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}'', \mathcal{F})) = 0$, (1) follows. To prove (2) reduce to the case of X affine and use (1). Treat the case of $X_{\mathrm{két}}$ similarly. □

4. Algebraic K-theory of log-schemes

We present in this section basic properties and some examples of calculations of algebraic (Quillen) K-theory of log-schemes for the topologies discussed earlier. Hagihara [14] was the first one to study algebraic K-theory of Kummer log-étale topos. Most of his results hold for log-schemes over (separably) closed fields. Working with equivariant K-theory for finite flat group schemes instead of finite groups and using some of the results from earlier sections we show that they hold in greater generality. In particular, for the Kummer log-flat site.
Let X be a Noetherian log-scheme. Let $K(X_*) = K(\mathcal{P}(X)_*)$ denote the higher K-theory groups of the exact category $\mathcal{P}(X)_*$ as defined by Quillen [32]. Similarly, let $K'(X_*) = K(\mathcal{M}(X)_*)$ be the Quillen's K-theory of the abelian category $\mathcal{M}(X)_*$. Denote by $\mathbf{K}(X_*), \mathbf{K}'(X_*)$ the Waldhausen spectra [35, 1.5.3] corresponding respectively to the categories $\mathcal{P}(X)_*, \mathcal{M}(X)_*$. Recall that they are functorial with respect to exact functors. We have

$$\pi_i(\mathbf{K}(X_*)) = K_i(X_*), \quad \pi_i(\mathbf{K}'(X_*)) = K'_i(X_*)$$

Let $\mathbf{K}/n(X_*), \mathbf{K}'/n(X_*)$ be the associated mod-n spectra. Set

$$K_i(X_*, \mathbf{Z}/n) = \pi_i(\mathbf{K}/n(X_*)), \quad K'_i(X_*, \mathbf{Z}/n) = \pi_i(\mathbf{K}'/n(X_*)).$$

4.1. Basic properties. We easily check that we have the following morphisms

- $K(X_*) \to K'(X_*)$ if $\mathcal{O}_{X_*}$ is a coherent sheaf;
- $f^* : K(X_*) \to K(Y_*)$, for any morphism $f : Y \to X$;
- $f^* : K'(X_*) \to K'(Y_*)$ for any object $f : Y \to X$ in X_* or f classically flat and $*$ any Kummer site.

Less obvious is the existence of pushforward for exact closed immersions.

Lemma 4.1. *The pushforward functor $i_* : K'(Y_*) \to K'(X_*)$ exists for an exact closed immersion $i : Y \hookrightarrow X$, X such that $(M_X/\mathcal{O}_X^*)_{\overline{x}} \simeq \mathbf{N}^{r(x)}$ for every point $x \in X$, and $*$ any Kummer topology.*

Proof. This follows easily for the Kummer étale topology from the exactness of i_* on all abelian sheaves (check on stalks at log-geometric points of X). We present here the argument for the Kummer log-flat topology (the log-syntomic case is analogous). In that case it can be reduced to the exactness of i_* for the Zariski topology. Let $f : \mathcal{F}_1 \to \mathcal{F}_2$ be a surjective morphism of Kummer log-flat coherent sheaves on Y. Cover X with étale open sets U that are affine and equipped with charts $P \to M_U$, $P \simeq \mathbf{N}^r$. For each U, by Corollary 2.16 and faithfully flat descent, there exists an n such that the map $f|U$ comes from a Zariski map $f_{\mathrm{Zar}} : \mathcal{F}_{1,\mathrm{Zar}} \to \mathcal{F}_{2,\mathrm{Zar}}$ on $U_{Y,n} = Y_U \times_U U_n$. Since $\varepsilon^* : U_{Y,n,\mathrm{kfl}} \to U_{Y,n,\mathrm{Zar}}$ is exact and faithful, the map f_{Zar} is surjective as well. It follows that the pushforward $i_* f_{\mathrm{Zar}} : i_*\mathcal{F}_{1,\mathrm{Zar}} \to i_*\mathcal{F}_{2,\mathrm{Zar}}$ is a surjection on X_n. Since $\varepsilon^* i_* \simeq i_* \varepsilon^*$ (easy to check), we are done. □

The following two propositions follow from Corollary 3.9, Lemma 3.10, and Lemma 3.11.

Proposition 4.2. *Let X be a log-regular quasi-compact log-scheme. Then*

(1) $\operatorname{inj\,lim}_Y K_*(Y_{\mathrm{kfl}}) \xrightarrow{\sim} K_*(X_{\mathrm{vkfl}})$, *where the limit is over log-blow-ups $Y \to X$;*
(2) $K_*(X_{\mathrm{vksyn}}) \xrightarrow{\sim} K_*(X_{\mathrm{vkfl}})$.

Proposition 4.3. *For any Noetherian log-scheme X, the pullback functors induce isomorphisms*

$$K_*(X_{\mathrm{ksyn}}) \xrightarrow{\sim} K_*(X_{\mathrm{kfl}}), \quad K'_*(X_{\mathrm{ksyn}}) \xrightarrow{\sim} K'_*(X_{\mathrm{kfl}}).$$

The following two propositions are proved in a similar way to their classical versions.

Proposition 4.4. *Let X be a Noetherian, log-scheme satisfying property (3.1). Then the natural immersion $i : X_{\mathrm{red}} \hookrightarrow X$ induces an isomorphism $i_* : K'_q(X_{\mathrm{red},*}) \xrightarrow{\sim} K'_q(X_*)$, for any Kummer topology.*

Proposition 4.5. *Let $\{X_i\}$ be a filtered system of Noetherian log-schemes. Assume that all the schemes X_i satisfy property (3.1) and the transition maps $\alpha_{ij} : X_j \to X_i$ are affine and classically flat. Then, for any Kummer site $*$,*

$$\operatorname*{inj\,lim}_i K'_q(X_{i,*}) \simeq K'_q((\operatorname*{proj\,lim}_i X_i)_*).$$

We have the following versions of the localization exact sequence. Their proofs are analogous to the proof of their classical version and the interested reader will find the details of the Kummer log-étale case in Hagihara [14, Theorem 4.5].

Proposition 4.6. *Let X be a Noetherian, equicharacteristic log-scheme, Y a strictly closed subscheme and U its complement. Assume that $(M_X/\mathcal{O}^*_X)_{\overline{x}} \simeq \mathbf{N}^{r(x)}$ for every point $x \in X$. Then we have the canonical long exact sequence*

$$\to K'_i(Y_{\mathrm{k\acute{e}t}}) \to K'_i(X_{\mathrm{k\acute{e}t}}) \to K'_i(U_{\mathrm{k\acute{e}t}}) \to K'_{i-1}(Y_{\mathrm{k\acute{e}t}}) \to$$

Proposition 4.7. *Let X be a Noetherian log-scheme, Y a strictly closed subscheme and U its complement. Assume that $(M_X/\mathcal{O}^*_X)_{\overline{x}} \simeq \mathbf{N}^{r(x)}$ for every point $x \in X$. Then we have the canonical long exact sequence*

$$\to K'_i(Y_{\mathrm{kfl}}) \to K'_i(X_{\mathrm{kfl}}) \to K'_i(U_{\mathrm{kfl}}) \to K'_{i-1}(Y_{\mathrm{kfl}}) \to$$

Recall Hagihara's notion of an M-framed log-scheme. Let $M \simeq \mathbf{N}^r$ be a monoid. An M-framed log-scheme is a pair (X, θ), where $\theta : M \to \Gamma(X, M_X/\mathcal{O}_X^*)$ is a frame such that for all points $x \in X$ the composite $M \to \Gamma(X, M_X/\mathcal{O}_X^*) \to (M_X/\mathcal{O}_X^*)_x$ is isomorphic to a projection $\mathbf{N}^r \to \mathbf{N}^m$, $r \geq m$. Note that the log-structure on X is Zariski. A standard example is given by a regular scheme with the log-structure coming from a strict normal crossing divisor (generate M from the irreducible components of the divisor at infinity).

Proposition 4.8. *(Poincaré isomorphism) Let X be a log-regular, regular quasi-compact log-scheme with a frame M. Then the natural morphism $K_i(X_*) \to K_i'(X_*)$ is an isomorphism for all i and any Kummer topology.*

Proof. We will argue the case of the Kummer log-flat topology. Assume that X has dimension n. Let $\mathcal{F}$ be a log-flat coherent sheaf. By the lemma below we can find a resolution

$$0 \to \mathcal{P} \to \mathcal{E}_{n-1} \to \dots \mathcal{E}_0 \to \mathcal{F} \to 0,$$

where each $\mathcal{E}_i$ is a locally free sheaf and $\mathcal{P}$ is coherent. Zariski localize now on X and take $Y = \operatorname{Spec}(\mathcal{O}_{X,x})$ for a point $x \in X$ with a chart $P \to M_Y$, $P \simeq (M_X/\mathcal{O}_X^*)_x \simeq \mathbf{N}^r$. We may assume that the above long exact sequence pullbacked to Y_{kfl} comes from a Zariski long exact sequence on $B = \mathcal{O}_{X,x} \otimes_{\mathbf{Z}[P]} \mathbf{Z}[P^{1/m}]$, for some m. Note that B is log-regular and regular. Hence $\mathcal{P}|\operatorname{Spec}(B)_{\mathrm{kfl}}$ is locally free. This suffices to exhibit a covering $U \to X$ for the Kummer log-flat topology such that $\mathcal{P}|U_{\mathrm{kfl}}$ is locally free, as wanted. □

Lemma 4.9. *For any Kummer log-flat or log-étale coherent sheaf $\mathcal{F}$ there exists a locally free sheaf $\mathcal{E}$ that surjects onto $\mathcal{F}$.*

Proof. Take a point $x \in X$. By interpreting kfl-modules as equivariant modules, we can construct a surjection: $f_x : \mathcal{E}_x \to \mathcal{F}_x$ on $\operatorname{Spec}(\mathcal{O}_{X,x})_{\mathrm{kfl}}$. Note that, by Corollary 3.25, $\mathcal{E}_x$ is a sum of invertible sheaves.
Consider now the following commutative diagram

$$\begin{array}{ccccc}
M^{\mathrm{div}} & \xrightarrow{\theta} & \Gamma(X_{\mathrm{kfl}}, M_X/\mathcal{O}_X^*) & \xrightarrow{\partial} & \operatorname{Pic}(X_{\mathrm{kfl}}) \\
\| & & \downarrow & & \downarrow \\
M^{\mathrm{div}} & \xrightarrow{\theta_x} & \Gamma(\mathcal{O}_{X,x,\mathrm{kfl}}, M_X/\mathcal{O}_X^*) & \xrightarrow{\partial} & \operatorname{Pic}(\mathcal{O}_{X,x,\mathrm{kfl}}),
\end{array}$$

where $M^{\mathrm{div}} = \operatorname{inj\,lim}_n M^{1/n}$ or $M^{\mathrm{div}} = \operatorname{inj\,lim}_{(n,p)=1} M^{1/n}$, $p = char(x)$. By Corollary 3.18, the map $\partial\theta_x$ is surjective. Hence there exists a locally free sheaf $\mathcal{E}$ on X_{kfl} that restricts to $\mathcal{E}_x$. By [14, Lemma 4.11] , there exists an invertible sheaf $\mathcal{L}$ on X_{kfl} such that the map $\mathcal{E}_x \to \mathcal{F}_x$ extends to a map $f : \mathcal{E} \otimes \mathcal{L} \to \mathcal{F}$. The map f is surjective in a neighbourhood U_x of x. We finish by covering X with a finite number of such U_x's and taking a direct sum of the corresponding maps f. □

Remark 4.10. It is easy to see that all of the above holds for the K-theory groups with coefficients: $K_*(X_*, \mathbf{Z}/n)$ and $K_*'(X_*, \mathbf{Z}/n)$.

4.2. Calculations.

Proposition 4.11. *Let $X = \operatorname{Spec}(A)$ be a log-scheme such that A is Noetherian and strictly local. We have the following isomorphisms*

$$\operatorname{Pic}(X_{\mathrm{kfl}}) \simeq (M_X^{gp}/\mathcal{O}_X^*)_x \otimes (\mathbf{Q}/\mathbf{Z}), \quad \operatorname{Pic}(X_{\mathrm{k\acute{e}t}}) \simeq (M_X^{gp}/\mathcal{O}_X^*)_x \otimes (\mathbf{Q}/\mathbf{Z})',$$
$$K_0(X_{\mathrm{kfl}}) \simeq \mathbf{Z}[(M_X^{gp}/\mathcal{O}_X^*)_x \otimes \mathbf{Q}/\mathbf{Z}], \quad K_0(X_{\mathrm{k\acute{e}t}}) \simeq \mathbf{Z}[(M_X^{gp}/\mathcal{O}_X^*)_x \otimes (\mathbf{Q}/\mathbf{Z})']$$

where x denotes the closed point of X.

Proof. The statement about the Picard groups is simply a reformulation of Corollary 3.17. Since, by Theorem 3.22, every locally free sheaf is a sum of invertible sheaves and, by Proposition 3.29, there are no nontrivial relations we get the statement about K_0-groups. □

Proposition 4.12. *Let $X = \operatorname{Spec}(K)$, for a field K, be a log-scheme with a chart $P \to M_X$, $P \simeq M_X/\mathcal{O}_X^* \simeq \mathbf{N}^r$. Then*

$$K'_*(K_{\mathrm{kfl}}) \simeq K'_*(K_{\mathrm{Zar}}) \otimes_{\mathbf{Z}} \mathbf{Z}[P^{gp} \otimes \mathbf{Q}/\mathbf{Z}],$$
$$K'_*(K_{\mathrm{k\acute{e}t}}) \simeq K'_*(K_{\mathrm{Zar}}) \otimes_{\mathbf{Z}} \mathbf{Z}[P^{gp} \otimes (\mathbf{Q}/\mathbf{Z})'].$$

Proof. For any m, denote by $F^m\mathcal{M}(X_{\mathrm{kfl}})$ the full subcategory of the category of Kummer log-flat coherent sheaves that become classical on the covering X_m of X. We have $F^m\mathcal{M}(X_{\mathrm{kfl}}) \simeq \mathcal{M}(X_{m,\mathrm{Zar}}, H_m)$, where the group scheme $H_m = \operatorname{Spec}(\mathbf{Z}[P^{1/m,gp}/P^{gp}])$. Here the right hand side denotes the category of H_m-equivariant Zariski coherent sheaves on X_m. By devissage, the natural functor $\mathcal{M}(X_{\mathrm{Zar}}, H_m) \to \mathcal{M}(X_{m,\mathrm{Zar}}, H_m)$ induces an isomorphism on K'-theory groups. Here H_m acts trivially on K.
Consider now the functor

$$\bigoplus_{\xi \in P^{1/m,gp}/P^{gp}} \mathcal{M}(X_{\mathrm{Zar}}) \to \mathcal{M}(X_{\mathrm{Zar}}, H_m); \quad \{\mathcal{F}_\xi\} \mapsto \oplus \mathcal{F}_\xi \otimes \mathcal{L}_\xi,$$

where $\mathcal{L}_\xi$ is the invertible sheaf corresponding to the map $K \to K[P^{1/n,gp}/P^{gp}]$, $a \mapsto a\xi$. Since H_m is diagonalizable this is an equivalence of categories (cf. [33, Exp.I,Prop.4.7.3]). This yields the isomorphism $\bigoplus_{\xi \in P^{1/m,gp}/P^{gp}} K'(K_{\mathrm{Zar}}) \xrightarrow{\sim} K(F^m\mathcal{M}(X_{\mathrm{kfl}}))$ and, by passing to the limit with respect to m, our proposition. □

For a framed log-scheme (X, M) and a prime ideal $\mathfrak{p}$ of M, we write $V(\mathfrak{p}) = \{x \in X | \mathfrak{p} \subset \theta_x((M_X/\mathcal{O}_X^*)_x \setminus \{1\})\}$, where $\theta_x : M \xrightarrow{\theta} \Gamma(X, M_X) \to (M_X/\mathcal{O}_X^*)_x$. $V(\mathfrak{p})$ is a closed subset of X and we equip it with the reduced subscheme structure. We write $M(\mathfrak{p})$ for the unique face of M such that $M(\mathfrak{p}) \oplus (M \setminus \mathfrak{p}) = M$, and set

$$\Lambda[\mathfrak{p}] = \mathbf{Z}[(M(\mathfrak{p})^{gp} \otimes \mathbf{Q}/\mathbf{Z}) \setminus \cup_{\mathfrak{q} \subsetneq \mathfrak{p}} (M(\mathfrak{q})^{gp} \otimes \mathbf{Q}/\mathbf{Z})].$$

We will denote by $\Lambda'[\mathfrak{p}]$ the same group as $\Lambda[\mathfrak{p}]$ but defined using $(\mathbf{Q}/\mathbf{Z})'$ instead of $\mathbf{Q}/\mathbf{Z}$.

Theorem 4.13. *Let X be a Noetherian M-framed log-scheme. Then*

(1) *if X is equicharacteristic then there is a natural isomorphism*

$$\beta : \bigoplus_{\mathfrak{p},\ \textit{prime of } M} K'_*(V(\mathfrak{p})_{\mathrm{Zar}}) \otimes \Lambda'[\mathfrak{p}] \to K'_*(X_{\text{két}});$$

(2) *there is a natural isomorphism*

$$\beta : \bigoplus_{\mathfrak{p},\ \textit{prime of } M} K'_*(V(\mathfrak{p})_{\mathrm{Zar}}) \otimes \Lambda[\mathfrak{p}] \to K'_*(X_{\mathrm{kfl}}).$$

Proof. Let us define the map β (in the second case). We fix $\xi \in M(\mathfrak{p})^{gp} \otimes \mathbf{Q}/\mathbf{Z}$. The corresponding map $\beta_\xi : K'_*(V(\mathfrak{p})_{\mathrm{Zar}}) \to K'_*(X_{\mathrm{kfl}})$ is induced by the functor

$$\beta_\xi : \mathcal{M}(V(\mathfrak{p})_{\mathrm{Zar}}) \to \mathcal{M}(X_{\mathrm{kfl}}), \quad \mathcal{F} \mapsto i_*(\varepsilon^*\mathcal{F} \otimes \mathcal{O}_{V(\mathfrak{p})}\{\xi\}),$$

where $i : V(\mathfrak{p}) \hookrightarrow X$ is the natural closed immersion and $\mathcal{O}_{V(\mathfrak{p})}\{\xi\}$ is the coherent sheaf on $V(\mathfrak{p})_{\mathrm{kfl}}$ (see Example 3.19) associated to the locally free sheaf $\mathcal{O}_{V(\mathfrak{p})}(\xi)$ on $V(\mathfrak{p})_{\mathrm{kfl}}$ obtained as the image of ξ (or rather of the minimal lifting of ξ) by the following map

$$M(\mathfrak{p})^{\mathrm{div}} \to M^{\mathrm{div}} \to \Gamma(X_{\mathrm{Zar}}, (M_X/\mathcal{O}_X^*)^{\mathrm{div}}) \to \Gamma(V(\mathfrak{p})_{\mathrm{Zar}}, (M/\mathcal{O}^*)^{\mathrm{div}}) \xrightarrow{\partial} \mathrm{Pic}(V(\mathfrak{p})_{\mathrm{kfl}}).$$

Note here that using $\mathcal{O}_{V(\mathfrak{p})}(\xi)$ instead of $\mathcal{O}_{V(\mathfrak{p})}\{\xi\}$ would tend to give a zero map.

The functor β_ξ is exact (follow [14, 6.2] replacing $\mathrm{Spec}(k)$ by $\mathrm{Spec}(\mathbf{Z})$). The rest of the argument goes as follows. One proves that the map β is compatible with localization sequences and by a limit argument reduces the proof to the case of a field. Then it suffices to evoke Proposition 4.12, and we are done.

Compatibility with localization sequences requires the following lemma (Lemma 9.4 in [14]) that we have to reprove in our setting.

Lemma 4.14. *Let N be a face of M and U an M-framed log-scheme. Assume that the frame of U comes from a chart $M \to M_U$ that maps $N \setminus \{1\}$ to zero in $\Gamma(U, \mathcal{O}_U)$. Then for any exact closed immersion $i : V \hookrightarrow U$ with the induced M-frame and $\xi \in N^{\mathrm{div}}$ we have $i^*\mathcal{O}_U\{\xi\} \simeq \mathcal{O}_V\{\xi\}$.*

Proof. Write $M \simeq \mathbf{N}^m, N \simeq \mathbf{N}^k, M = N \oplus Q$ for a face Q. Let $\xi \in N^{1/n}$. Set $M' = N^{1/n} \oplus Q$. We have $U' = U \otimes_{\mathbf{Z}[M]} \mathbf{Z}[M'] = U \times_{\mathrm{Spec}(\mathbf{Z})} S$, where $S = \mathrm{Spec}(\mathbf{Z} \otimes_{\mathbf{Z}[N]} \mathbf{Z}[N^{1/n}])$. Similarly, $V' = V \otimes_{\mathbf{Z}[M]} \mathbf{Z}[M'] = V \times_{\mathrm{Spec}(\mathbf{Z})} S$. One easily computes

$$S = \mathrm{Spec}(\mathbf{Z}[x_1, \ldots, x_k]/(x_1^n, \ldots, x_k^n));$$

$$\mathcal{O}_S(x_I)(S) = x_I^{-1}(\oplus_J x_J \mathbf{Z}), \quad x_I = x_1^{i_1} \ldots x_k^{i_k}, x_J = x_1^{j_1} \ldots x_k^{j_k}, 0 \le j_l \le k-1;$$

$$\mathcal{O}_S\{x_I\}(S) = x_I^{-1}(\oplus_J x_J \mathbf{Z}), \quad x_I = x_1^{i_1} \ldots x_k^{i_k}, x_J = x_1^{j_1} \ldots x_k^{j_k}, i_l \le j_l \le k-1.$$

Hence, if we write $\xi = x_I$, then $\mathcal{O}_S\{x_I\}$ is a direct factor of $\mathcal{O}_S(x_I)$ and the cokernel is a free $\mathbf{Z}$-module. It follows that

$$\mathcal{O}_{U'}\{x_I\} = \mathcal{O}_S\{x_I\} \otimes_{\mathbf{Z}} \mathcal{O}_U, \quad \mathcal{O}_{V'}\{x_I\} = \mathcal{O}_S\{x_I\} \otimes_{\mathbf{Z}} \mathcal{O}_V.$$

Thus $\mathcal{O}_{V'}\{x_I\} = i^*\mathcal{O}_{U'}\{x_I\}$, as wanted. □

□

Example 4.15. Let A be a complete discrete valuation ring with residue field k and the log-structure coming from the closed point. Then, by Theorem 4.13 (see the argument below), we have

$$K_*(A_{\mathrm{kfl}}) \simeq K_*(A) \oplus K_*(k) \otimes \mathbf{Z}[\mathbf{Q}/\mathbf{Z} \setminus \{0\}],$$
$$K_*(A_{\mathrm{k\acute{e}t}}) \simeq K_*(A) \oplus K_*(k) \otimes \mathbf{Z}[(\mathbf{Q}/\mathbf{Z})' \setminus \{0\}].$$

When comparing this with Proposition 4.11 we get that $[A(a)] = [k\{a\}] + [A]$ in $K_*(A) \oplus K_*(k) \otimes \mathbf{Z}[\mathbf{Q}/\mathbf{Z} \setminus \{0\}]$.

Example 4.16. More generally, let X be a regular, log-regular scheme with the log-structure associated to a divisor D with strict normal crossing. Let $\{D_i | i \in I\}$ be the set of the irreducible (regular) components of D. For an index set $J \subset I$ denote by D_J the intersection of irreducible components indexed by J and by $\Lambda_{|J|}$ (resp. $\Lambda'_{|J|}$) the free abelian groups generated by the set $\{(a_1, \ldots, a_{|J|}) | a_i \in \mathbf{Q}/\mathbf{Z} \setminus \{0\}\}$ (resp. the set $\{(a_1, \ldots, a_{|J|}) | a_i \in (\mathbf{Q}/\mathbf{Z})' \setminus \{0\}\}$).

Corollary 4.17. *For any $q \geq 0$ we have the canonical isomorphism*

$$K_q(X_{\mathrm{kfl}}) \simeq \bigoplus_{J \subset I} K_q(D_J) \otimes \Lambda_{|J|}$$

Moreover, if D is equicharacteristic then canonically

$$K_q(X_{\mathrm{k\acute{e}t}}) \simeq \bigoplus_{J \subset I} K_q(D_J) \otimes \Lambda'_{|J|}$$

Proof. The Kummer log-flat statement follows from Theorem 4.13. For the Kummer log-étale note that we do have a localization sequence

$$\to K'_q(D_{\mathrm{k\acute{e}t}}) \to K'_q(X_{\mathrm{k\acute{e}t}}) \to K'_q(U_{\mathrm{k\acute{e}t}}) \to K'_{q-1}(D_{\mathrm{k\acute{e}t}}) \to$$

where $U = X_{\mathrm{tr}}$. This follows just like in the classical situation using the fact that Kummer log-étale coherent sheaves on U are simply the Zariski coherent sheaves and those can be extended to the whole of X. Now the proof of Theorem 4.13 goes through. □

Example 4.18. Again, all of the above holds for the K-theory groups with coefficients. For example, let A be a complete discrete valuation ring of mixed characteristic $(0, p)$. Let X be a smooth A-scheme equipped with the log-structure coming from the special fiber X_0. Then

$$K_*(X_{\mathrm{kfl}}, \mathbf{Z}/p^k) \simeq K_*(X_0, \mathbf{Z}/p^k) \otimes \mathbf{Z}[\mathbf{N} \setminus \{0\}] \oplus K_*(X, \mathbf{Z}/p^k)$$

Since, by Geisser-Levine [13], $K_i(X_0, \mathbf{Z}/p^k) = 0$, for $i \geq \dim X_0$, we get

$$K_i(X_{\mathrm{kfl}}, \mathbf{Z}/p^k) \simeq K_i(X, \mathbf{Z}/p^k) \simeq K_i(X[1/p], \mathbf{Z}/p^k), \quad i \geq \dim X_0.$$

5. Topological K-theory of log-schemes

In this section we initiate the study of topological K-theory of log-schemes.

5.1. Homotopy theory of simplicial presheaves and sheaves. The formalism of cohomologies of simplicial presheaves we use here is based on the closed model structures for the category of simplicial presheaves and sheaves on an arbitrary Grothendieck site developed by Jardine [16], [17], [18], [19].
We begin by recalling basic facts about cohomology of simplicial presheaves. Let us start with some definitions. A *closed model category* is a category $\mathcal{M}$ equipped with three classes of maps called cofibrations, fibrations and weak equivalences, such that the following axioms are satisfied:

(1) $\mathcal{M}$ is closed under all finite limits and colimits.
(2) Given $f : X \to Y$ and $g : Y \to Z$, if any of the two of f, g or gf are weak equivalences, then so is the third.
(3) If f is a retract of g and g is a weak equivalence, fibration or cofibration, then so is f.
(4) Given any commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r] & Y \end{tikzcd}$$

in $\mathcal{M}$, where i is a cofibration and p is a fibration, then an arrow $V \to X$ exists making this diagram commute assuming that either i or p is a weak equivalence.
(5) Any map $f : X \to Y$ may be factored
   - $f = pi$, where p is a fibration and i is a trivial cofibration, and
   - $f = qj$, where q is a trivial fibration and j is a cofibration.

A *trivial fibration* is a map that is a fibration and a weak equivalence and a *trivial cofibration* is a map that is a cofibration and a weak equivalence. A basic example of a closed model category is the category **S** of simplicial sets: the cofibrations of **S** are the monomorphisms, the weak equivalences are the maps which induce isomorphisms on all possible homotopy groups of associated realizations, and the fibrations are the Kan fibrations.
A *closed simplicial model category* is a closed model category $\mathcal{M}$ which has a natural function complex $\mathrm{Hom}(U, X)$ in the category **S** of simplicial sets for each pair of objects U, X in $\mathcal{M}$. This simplicial set is supposed to satisfy some adjointness properties as well as the following axiom:

- If $i : A \to B$ is a cofibration and $p : X \to Y$ is a fibration, then the induced map of simplicial sets

$$\mathrm{Hom}(B, X) \xrightarrow{(i^*, p_*)} \mathrm{Hom}(A, X) \times_{\mathrm{Hom}(A,Y)} \mathrm{Hom}(B, X)$$

is a Kan fibration, which is trivial if either i or p is trivial.

A closed model category $\mathcal{M}$ is called *proper* if it satisifes the following additional axiom:

- Given a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ B & \xrightarrow{g} & D \end{array}$$

(1) if the square is a pullback, j is a fibration and g is a weak equivalence, then f is a weak equivalence.
(2) if the square is a push out, i is a cofibration and f is a weak equivalence, then g is a weak equivalence.

The category $\mathbf{S}$ of simplicial sets is a proper closed simplicial model category. Let C be a site and let T be the Grothendieck topos of sheaves on C. Denote by pT (resp. sT) the category of presheaves (resp. sheaves) of simplicial sets on C. When X is a presheaf, we denote by $\pi_0(X)$ the sheaf on T associated to the presheaf

$$U \mapsto \pi_0(X(U)).$$

For an object U in C, we let $X|U$ be the image of X in the site $C|U$. When $n > 0$ is an integer and $x \in X_0(U)$, we denote by $\pi_n(X|U, x)$ the sheaf on $C|U$ associated to the preasheaf

$$V \mapsto \pi_n(X(V), x).$$

Here, for a simplicial set S, we take $\pi_n(S) = \pi_n(|S|)$, where $|S|$ is the geometric realization of S.

Definition 5.1. Let $f : X \to Y$ be a map of presheaves. Then

- f is called a weak equivalence if the induced map $f_* : \pi_0(X) \to \pi_0(Y)$ is an isomorphism, and for all $n > 0$, all objects U in C, and all $x \in X_0(U)$, the natural maps

$$f_* : \pi_n(X|U, x) \to \pi_n(Y|U, f(x))$$

are isomorphisms;

- f is called a cofibration if, for any object U from C, the induced map $f(U) : X(U) \to Y(U)$ is injective;

- f is called a fibration if it satisfies the following lifting property: for any commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & Y, \end{array}$$

where i is a trivial cofibration, there exists a map $B \to X$ such that the resulting diagram commutes.

For two simplicial presheaves X and Y, the simplicial set $\mathrm{Hom}(X,Y)$ is defined by

$$n \mapsto \mathrm{Hom}_{pT}(X \times \Delta^n, Y),$$

where Δ^n is the standard n-simplex. We also have the simplicial presheaf $\mathcal{H}\mathrm{om}(X,Y)$ defined by

$$U \mapsto (n \mapsto \mathrm{Hom}_{p(T|U)}(X|U \times \Delta^n, Y|U)).$$

Jardine proves (see Prop. 1.4 in [17]) the following

Theorem 5.2. *With the above definitions the categories sT and pT are proper closed simplicial model categories.*

We can associate to sT and pT the homotopy categories $\mathcal{H}\mathrm{o}(sT)$ and $\mathcal{H}\mathrm{o}(pT)$ by formally inverting all weak equvalences. We have (Prop. 2.8 from [16])

Theorem 5.3. *The associated sheaf functor induces an equivalence*

$$\mathcal{H}\mathrm{o}(pT) \simeq \mathcal{H}\mathrm{o}(sT)$$

between the associated homotopy categories.

For simplicial presheaves X and Y, we denote by $[X,Y]$ the set of morphisms from X to Y in the homotopy category. A simplicial presheaf X is called fibrant if the unique map $X \to *_C$ is a fibration. Here $*_C$ is the final object of the category of presheaves on C. For any simplicial presheaf X the canonical map $X \to *_C$ admits a factorization $X \to X^f \to *_C$, where $X \to X^f$ is a trivial cofibration and X^f is fibrant. Such a map $X \to X^f$ is called a fibrant replacement of X. For two simplicial presheaves X and Y, we have

$$[X,Y] = [X,Y^f] = \pi_0 \mathrm{Hom}(X,Y^f),$$

where $Y \to Y^f$ is a fibrant replacement of Y. That is, the set $[X,Y^f]$ is given by morphisms $X \to Y^f$ modulo simplicial homotopy.

5.1.1. *Cohomology of simplicial presheaves.* Let F be a pointed simplicial presheaf . Define cohomology of C with coefficients in F (see [16, 3]) by

$$H^{-m}(C,F) = [*_C, \Omega^m F] \quad \text{for} \quad m \geq 0.$$

In the case the site C has a final object X we will write $H^{-m}(X,F)$ for $H^{-m}(C,F)$. Note that $H^{-m}(C,F) \simeq [S^m,F]_*$, where the subscript $*$ refers to morphisms in the pointed homotopy category. Here S^m is the simplicial m-sphere $\Delta^m/\partial\Delta^m$. $H^{-m}(C,F)$ is a pointed set for $m=0$, a group for $m>0$, and an abelian group for $m>1$.

5.1.2. *Change of sites.* This section is based on [19]. Let $f : C \to D$ be a morphism of sites given by a functor $f : D \to C$ that preserves finite limits and sends covers to covers. We have the associated presheaf functors

$$f_* : C^\wedge \to D^\wedge, \qquad f^p : D^\wedge \to C^\wedge,$$

where $C^\wedge$ denotes the category of presheaves on C. The functor f^p is left adjoint to f_*. Both functors are exact and f_* maps sheaves to sheaves. Both f^p

and f_* preserve cofibrations and f_* preserves fibrations. In particular, the functor $F \mapsto F(U)$ preserves fibrations. Thus a fibration is a pointwise fibration. The functor f^p also preserves weak equivalences.
Jardine proves the following

Theorem 5.4. *Let $f : C \to D$ be a morphism of sites. Let F be a pointed simplicial presheaf on the site C. Take a fibrant replacement $F \to F^f$ of F. Then we have an isomorphism*

$$H^m(C, F) \simeq H^m(D, f_*F^f),$$

for all $m \leq 0$.

Proof. We start with the following lemma.

Lemma 5.5. *Suppose that F is a fibrant simplicial presheaf on C. Then there is an adjointness isomorphism*

$$[*_D, f_*F] \simeq [*_C, F].$$

Proof. We know that f_*F is also fibrant. Hence we have the following sequence of isomorphisms

$$[*_D, f_*F] \simeq \pi_0 \operatorname{Hom}(*_D, f_*F) \simeq \pi_0 \operatorname{Hom}(f^p*_D, F) \simeq \pi_0 \operatorname{Hom}(*_C, F) \simeq [*_C, F],$$

as wanted. $\square$

Since fibrant objects are preserved by the loop functor (Corollary 3.2 from [16]), the above lemma gives us the following isomorphisms

$$H^m(C, F) \simeq [*_C, \Omega^m F] \simeq [*_C, \Omega^m F^f] \simeq [*_D, f_*\Omega^m F^f].$$

Since the loop functor commutes with the direct image functor, we also get

$$[*_D, f_*\Omega^m F^f] \simeq [*_D, \Omega^m f_*F^f].$$

This proves our theorem. $\square$

It will be useful for us to identify the homotopy group presheaves of the presheaf f_*F^f from the above theorem.

Proposition 5.6. *We have*

$$\pi_k f_*F^f(V) \simeq H^{-k}(f(V), F|f(V)).$$

Proof. This follows from the following sequence of isomorphisms

$$\pi_k f_*F^f(V) = \pi_k F^f(f(V)) \simeq [*_{f(V)}, \Omega^k F^f|f(V)] \simeq [*_{f(V)}, \Omega^k F|f(V)].$$

$\square$

5.2. Topological K-theory. We base this section on Gillet and Soulé [9, 3.1]. Let $(C, \mathcal{O}_C)$ be a ringed site with enough points. We assume that $\mathcal{O}_C$ is unitary and commutative. For any $n \geq 1$, we consider the following presheaves

$$GL_n : U \mapsto GL_n(\Gamma(U, \mathcal{O}_U)), \qquad BGL_n : U \mapsto BGL_n(\Gamma(U, \mathcal{O}_U)).$$

Here $BGL_n(\Gamma(U, \mathcal{O}_U))$ is the classifying space of $GL_n(\Gamma(U, \mathcal{O}_U))$.
Let F be a simplicial presheaf such that $\pi_0(F) = *$. We define its Bousfield-Kan integral completion $\mathbf{Z}_\infty F$ to be the simplicial presheaf $U \mapsto \mathbf{Z}_\infty F(U)$. The functor $\mathbf{Z}_\infty$ for simplicial sets is defined in [4]. Its basic property gives us that if a map of simplicial presheaves $f : F \to G$ induces an isomorphism of presheaves of integral homology groups $f : H_n(F, \mathbf{Z}) \to H_n(G, \mathbf{Z})$, then the map $\mathbf{Z}_\infty f : \mathbf{Z}_\infty F \to \mathbf{Z}_\infty G$ is a weak equivalence. We set $BGL = \operatorname{inj}\lim_n BGL_n$ and

$$K = \mathbf{Z} \times \mathbf{Z}_\infty BGL,$$

where the constant presheaf $\mathbf{Z}$ is concentrated in degree zero and pointed by zero.
To compare the above definition with Quillen's K-theory, take, for any ringed site $(C, \mathcal{O}_C)$, the functor $U \mapsto P_C(U)$, where $P_C(U)$ is the category of locally free $\mathcal{O}_C|U$-modules of finite rank. Consider the simplicial presheaf $\Omega BQP_C : U \mapsto \Omega BQP_C(U)$. Here Q is the Quillen Q-construction. Consider also a related simplicial presheaf $\Omega BQP : U \mapsto \Omega BQP(\mathcal{O}_C(U))$, where $P(\mathcal{O}_C(U))$ is the category of finitely generated projective $\mathcal{O}_C(U)$-modules. There is a natural map $\Omega BQP \to \Omega BQP_C$ and, by [8, 2.15], a natural map (in the homotopy category) $\mathbf{Z} \times \mathbf{Z}_\infty BGL \to \Omega BQP_C$. Gillet and Soulé [9, 3.2.1] prove the following

Lemma 5.7. *If C is locally ringed, then the natural maps of pointed simplicial presheaves*

$$\mathbf{Z} \times \mathbf{Z}_\infty BGL \to \Omega BQP \to \Omega BQP_C$$

are weak equivalences.

Let C be now the Zariski site of some scheme X. Choose a fibrant replacement K^f of $\Omega BQP_{\mathrm{Zar}}$. It defines a map $K_m(X) = \pi_m(\Omega BQP_{\mathrm{Zar}}(X)) \to H^{-m}(X_{\mathrm{Zar}}, K)$. Gillet and Soulé show [9, 3.2.2] that the Mayer-Vietoris property implies the following

Proposition 5.8. *Suppose that X is a Noetherian regular scheme of finite Krull dimension. Then the above map gives an isomorphism*

$$K_m(X) \xrightarrow{\sim} H^{-m}(X_{\mathrm{Zar}}, K), \qquad m \geq 0$$

5.2.1. *Topological K/n-theory.* For a scheme X, write

$$\mathbf{K}(X) = \mathbf{K}(X_{\mathrm{Zar}}) = \{K^0(X), K^1(X), \ldots,\}$$

for the Waldhausen spectrum associated to the category of Zariski locally free sheaves (cf. [35, 1.5.2]). Write

$$\mathbf{K}/n(X) = \mathbf{K}/n(X_{\mathrm{Zar}}) = \{K^0/n(X), K^1/n(X), \ldots\}$$

for the corresponding mod-n spectrum. Both spectra are connective and contravariant in X. For a site C built from schemes, denote by K and K/n the pointed simplicial presheaves $K : X \mapsto K^0(X)$ and $K/n : X \mapsto K^0/n(X)$. Since, by the $+ = Q$ theorem, the map (of simplicial presheaves) $\mathbf{Z} \times \mathbf{Z}_\infty BGL \to \Omega BQP$ is a weak equivalence and there exists a (local) weak equivalence $\Omega BQP \to (U \mapsto K^0(U))$ [35, 1.11.2] this notation is compatible with the one used above.
Set

$$K_m^C(X) := H^{-m}(X_C, K), \quad K_m^C(X, \mathbf{Z}/n) := H^{-m}(X_C, K/n), \quad m \geq 0.$$

Corollary 5.9. *Suppose that X is a Noetherian regular scheme of finite Krull dimension. Then we have a natural isomorphism*

$$K_m(X, \mathbf{Z}/n) \xrightarrow{\sim} K_m^{\mathrm{Zar}}(X, \mathbf{Z}/n) = H^{-m}(X_{\mathrm{Zar}}, K/n), \qquad m \geq 0$$

Proof. The fibration sequence

$$K^0/n \to K^1 \xrightarrow{n} K^1$$

gives compatible long exact sequences

$$\begin{tikzcd}
H^{-m}(X_{\mathrm{Zar}},K) \arrow[r, "n"] & H^{-m}(X_{\mathrm{Zar}},K) \arrow[r] & H^{-m}(X_{\mathrm{Zar}},K/n) \arrow[r] & H^{-m+1}(X_{\mathrm{Zar}},K) \\
K_m(X) \arrow[r, "n"] \arrow[u, "\wr"'] & K_m(X) \arrow[r] \arrow[u, "\wr"'] & K_m(X, \mathbf{Z}/n) \arrow[r] \arrow[u] & K_{m-1}(X) \arrow[u, "\wr"']
\end{tikzcd}$$

Our corollary easily follows. □

5.3. Topological log-étale K/n-theory. We show in this section that l-adic topological log-étale K-theory of a log-regular scheme computes étale K-theory of the largest open set on which the log-structure is trivial. As the reader will see the log-étale story presented here is very similar to the story of étale K-theory. We will mainly work with schemes S such that

(*) S is separated, Noetherian and regular. The natural number n is invertible on S and $\sqrt{(-1)} \in \mathcal{O}_X$ if n is even. S has finite Krull dimension and a uniform bound on n-torsion étale cohomological dimension of all residue fields. Each residue field of S has a Tate-Tsen filtration.

We quote from Jardine (Theorem 3.9 in [16])

Theorem 5.10. *Suppose that X satisfies the above condition. Then, for $n \geq 0$, we have an isomorphism*

$$[*_{X_{\text{ét}}}, \Omega^m K^1/n] \simeq K_{m-1}^{DF}(X, \mathbf{Z}/n), \quad m \geq 0,$$

where $K_*^{DF}(X, \mathbf{Z}/n)$ is the étale K-theory.

This yields the following

Corollary 5.11. *Suppose that X satisfies the above condition. Then there is an isomorphism*

$$K_m^{\text{ét}}(X,\mathbf{Z}/n)\simeq K_m^{DF}(X,\mathbf{Z}/n),\quad m\geq 0.$$

Proof. The above theorem and the weak equivalence $K^0/n\simeq\Omega K^1/n$ give the following isomorphisms

$$H^{-m}(X_{\text{ét}},K/n)=[*_{X_{\text{ét}}},\Omega^m K^0/n]\simeq[*_{X_{\text{ét}}},\Omega^{m+1}K^1/n]\simeq K_m^{DF}(X,\mathbf{Z}/n),$$

as wanted. □

We will now compute the homotopy groups of K-presheaves. Recall [36, 2.7, 2.7.2] that, for a scheme Y satisfying condition (*) such that $\Gamma(Y,\mathcal{O}_Y)$ contains a primitive n'th root of unity, there are compatible functorial Bott element homomorphisms

$$\beta_n:\mu_n(Y)\to K_2(Y,\mathbf{Z}/n),$$

where $\mu_n(Y)$ denotes the group of n'th roots of unity in $\Gamma(Y,\mathcal{O}_Y)$.

Proposition 5.12. *Suppose that X satisfies condition (*) and that for all $x\in X$, $M_{X,\overline{x}}/\mathcal{O}^*_{X,\overline{x}}$ is isomorphic to a direct sum of $\mathbf{N}$. Let n be invertible on X. Then the sheaves of homotopy groups of K/n in the Kummer log-étale topology are given by*

$$\tilde{\pi}_q(K/n)\simeq\begin{cases}\mathbf{Z}/n(i) & \textit{for } q=2i\geq 0\\ 0 & \textit{for } q\geq 0,\ \textit{odd.}\end{cases}$$

Proof. We have a map of sheaves

$$\mathbf{Z}/n(i)\to\tilde{\pi}_{2i}(K/n)$$

induced locally by taking the product of the map $\beta_n\to\pi_2(K/n(Y))$.
It suffices to show that this map is an isomorphism and that, for q odd, the sheaf $\tilde{\pi}_q(K/n)$ is trivial. For that we need to compute the stalks of the presheaves K/n. For any point $x\in X$, consider the natural chart $P\to\mathcal{O}_{X,\overline{x}}$, where $P=M_{X,\overline{x}}/\mathcal{O}^*_{X,\overline{x}}$. By assumption $P\simeq\mathbf{N}^r$, for some r. We have

$$K/n_{x(log)}=\operatorname{inj\,lim}_U K/n(U)=\operatorname{inj\,lim}_k K/n(\mathcal{O}_{X,\overline{x},k}),$$

where the first limit is over the Kummer log-étale neighbourhoods U of the log geometric point $x(log)$ in X, and the second limit is over the base changes $\mathcal{O}_{X,\overline{x},k}=\mathcal{O}_{X,\overline{x}}\otimes_{\mathbf{Z}[P]}\mathbf{Z}[P]$ of $\mathcal{O}_{X,\overline{x}}$ by the k-power map $k:P\to P$, k being invertible in $\mathcal{O}_{X,\overline{x}}$. Since $P\simeq\mathbf{N}^r$, the ring $\mathcal{O}_{X,\overline{x},k}$ is local.
By Gabber's rigidity [7] we have the following commutative diagram

$$\begin{tikzcd}\pi_q(K/n(\mathcal{O}_{X,\overline{x}})) \arrow[r, "\sim"] \arrow[d] & \pi_q(K/n(\overline{k})) \arrow[d, "\wr"] \\ \pi_q(K/n(\mathcal{O}_{X,\overline{x},k})) \arrow[r, "\sim"] & \pi_q(K/n(\overline{k})).\end{tikzcd}$$

Hence, $\operatorname{inj\,lim}_k\pi_q(K/n(\mathcal{O}_{X,\overline{x},k}))\xrightarrow{\sim}\pi_q(K/n(\overline{k}))$. The proposition now follows from the computations of K-theory of separably closed fields. □

The above computation yields the following

Proposition 5.13. *Suppose that X satisfies condition (*) and that, for all $x \in X$, $M_{X,\overline{x}}/\mathcal{O}^*_{X,\overline{x}}$ is isomorphic to a direct sum of $\mathbf{N}$. Let n be invertible on X. Then there exists a cohomological spectral sequence $E_r^{p,q}$, $r \geq 2$, such that*

$$E_2^{p,q} = \begin{cases} H^p(X_{\text{két}}, \mathbf{Z}/n(q/2)) & \text{for } q-p \geq 0 \text{ and } q \text{ even} \\ 0 & \text{for } q-p \geq 0 \text{ and } q \text{ odd.} \end{cases}$$

This spectral sequence converges strongly to $K^{\text{két}}_{q-p}(X, \mathbf{Z}/n)$ for $q-p \geq 1$. The differential d_r in the above spectral sequence maps $E_r^{p,q}$ to $E_r^{p+r,q+r-1}$.

Theorem 5.14. *Let X be a log-regular, regular scheme satisfying condition (*). Let n be a natural number invertible on X. Then the open immersion $j : U \hookrightarrow X$, where $U = X_{\text{tr}}$ is the maximal open set of X on which the log-structure is trivial, induces an isomorphism*

$$j^* : K^{\text{két}}_m(X, \mathbf{Z}/n) \xrightarrow{\sim} K^{\text{ét}}_m(U, \mathbf{Z}/n), \quad m \geq 0.$$

Proof. Let $K/n \to K^f/n$ be a fibrant replacement. By Theorem 5.4,

$$H^{-m}(U_{\text{ét}}, \mathbf{Z}/n) \simeq H^{-m}(U_{\text{két}}, \mathbf{Z}/n) \simeq H^{-m}(X_{\text{két}}, j_*K^f/n).$$

It suffices to show that the natural map of presheaves on $X_{\text{két}}$,

$$K/n \to j_*(K^f/n)$$

is a weak equivalence. Or that the induced map on all the log-geometric stalks is a weak equivalence. By Proposition 5.13, $\pi_q(K/n_{x(log)})$ is trivial for q odd and isomorphic to $\mathbf{Z}/n(i)$ for $q = 2i$. From Proposition 5.6,

$$\pi_q((j_*(K/n))_{x(log)}) \simeq \operatorname*{inj\,lim}_Y K^{\text{ét}}_q(Y_U, \mathbf{Z}/n),$$

where the limit is over the Kummer log-étale neighbourhoods Y of $x(log)$ in X. Consider now the composition

$$\begin{aligned} \pi_q((K^f/n)_{x(log)}) &= \operatorname*{inj\,lim}_Y K_q(Y, \mathbf{Z}/n) \\ &\overset{j^*}{\simeq} \operatorname*{inj\,lim}_Y K_q(Y_U, \mathbf{Z}/n) \\ &\xrightarrow{\rho} \operatorname*{inj\,lim}_Y K^{\text{ét}}_q(Y_U, \mathbf{Z}/n) \simeq \pi_q((j_*(K^f/n))_{x(log)}). \end{aligned}$$

By Proposition 5.15 below, the map j^* is an isomorphism. By Thomason [34, 11.5], the map ρ is an isomorphism after inverting the Bott element. This yields the isomorphism

$$\pi_*((K/n)_{x(log)})[\beta_n^{-1}] \xrightarrow{\sim} \pi_*((j_*(K^f/n))_{x(log)})[\beta_n^{-1}].$$

Since the Bott element is invertible on both sides, we get the isomorphism

$$\pi_q((K/n)_{x(log)}) \xrightarrow{\sim} \pi_q((j_*(K^f/n))_{x(log)}),$$

as wanted. □

Proposition 5.15. *Let X be a log-regular, regular scheme. Let n be a natural number invertible on X. For any point $x \in X$, the natural map*

$$\operatorname*{inj\,lim}_{Y} K_q(Y, \mathbf{Z}/n) \to \operatorname*{inj\,lim}_{Y} K_q(Y_U, \mathbf{Z}/n)$$

is an isomorphism. Here, the limit is taken over the Kummer log-étale neighbourhoods of $x(log)$ in X.

Proof. Looking étale locally, we may assume that $X = \operatorname{Spec}(\mathcal{O}_{X,\overline{x}})$ (abusing notation a bit), and we have a chart $P \to \mathcal{O}_{X,\overline{x}}$, for $P = M_{X,\overline{x}}/\mathcal{O}^*_{X,\overline{x}} \simeq \mathbf{N}^k$. Consider the closed subscheme of X

$$Z = X \otimes_{\mathbf{Z}[X_1,\dots,X_k]} \mathbf{Z}[X_1,\dots,X_k]/(X_1 \dots X_k).$$

Up to reindexing, Z can be covered by closed subschemes

$$Z_i = X \otimes_{\mathbf{Z}[X_1,\dots,X_k]} \mathbf{Z}[X_1,\dots,X_k]/(X_1,\dots,X_i).$$

We will need the following lemma

Lemma 5.16. *Consider the cartesian diagram*

$$\begin{tikzcd} Z_i' \arrow[r] \arrow[d, "m_r"] & \operatorname{Spec}(\mathbf{Z}[X_1,\dots,X_k]/(X_1^r,\dots,X_i^r)) \arrow[d, "m_r"] \\ Z_i \arrow[r] & \operatorname{Spec}(\mathbf{Z}[X_1,\dots,X_k]/(X_1,\dots,X_i)), \end{tikzcd}$$

where the map m_r is defined by sending X_l to X_l^r. The pullback map m_r^* : $K'_*(Z_i, \mathbf{Z}/n) \to K'_*(Z_i', \mathbf{Z}/n)$ is trivial for r large enough and invertible on X.

Proof. We can filter the ring $\mathbf{Z}[X_1,\dots,X_k]/(X_1^r,\dots,X_i^r))$ (as an $\mathbf{Z}[X_1,\dots,X_k]/(X_1,\dots,X_i)$ module) with r^i graded pieces isomorphic to $\mathbf{Z}[X_1,\dots,X_k]/(X_1,\dots,X_i) \simeq \mathbf{Z}[X_{i+1},\dots,X_k]$. Now, we can do the same for the ring $\mathcal{O}_{Z_i'}$ assuming that there is enough flatness, i.e., that

$$\operatorname{Tor}_j^{\mathbf{Z}[X_1,\dots,X_k]/(X_1,\dots,X_i)}(\mathcal{O}_{Z_i}, \mathbf{Z}[X_1,\dots,X_k]/(X_1^{a_1},\dots,X_i^{a_i})) = 0, \qquad j > 0, \quad a_1,\dots,a_i \geq 1.$$

But that follows from the results of Kato [22, 6.1] in the following way

$$\begin{aligned} &\operatorname{Tor}_j^{\mathbf{Z}[X_1,\dots,X_k]/(X_1,\dots,X_i)}(\mathcal{O}_{Z_i}, \mathbf{Z}[X_1,\dots,X_k]/(X_1^{a_1},\dots,X_i^{a_i})) \\ &= \operatorname{Tor}_j^{\mathbf{Z}[X_1,\dots,X_k]/(X_1,\dots,X_i)}(\mathcal{O}_X \\ &\qquad \otimes_{\mathbf{Z}[X_1,\dots,X_k]} \mathbf{Z}[X_1,\dots,X_k]/(X_1,\dots,X_i), \mathbf{Z}[X_1,\dots,X_k]/(X_1^{a_1},\dots,X_i^{a_i})) \\ &\xrightarrow{\sim} \operatorname{Tor}_j^{\mathbf{Z}[X_1,\dots,X_k]}(\mathcal{O}_X, \mathbf{Z}[X_1,\dots,X_k]/(X_1^{a_1},\dots,X_i^{a_i})) = 0. \end{aligned}$$

Hence we have a filtration of $\mathcal{O}_{Z_i'}$ by $\mathcal{O}_{Z_i}$ modules. This filtration has length r^i and the graded pieces are isomorphic to $\mathcal{O}_{Z_i} \otimes_{\mathbf{Z}[X_{i+1},\dots,X_k]} \mathbf{Z}[X_{i+1},\dots,X_k]$, where the tensor product is over the map m_r (sending X_l to X_l^r). Since the map m_r is flat, this yields (by devissage) that the map $K'_*(Z_i, \mathbf{Z}/n) \to K'_*(Z_i', \mathbf{Z}/n)$ is zero for $r^i \geq n$. Clearly $r = n$ will do. $\square$

Mayer-Vietoris for K'-theory and the above lemma yield that the map $m_r : X \to X$ defined by $X_l \mapsto X_l^r$ kills $K'_*(Z, \mathbf{Z}/n)$ for some $r = n^j$. Since m_r is Kummer log-étale, this gives the isomorphism in our proposition (note that we can assume all the schemes Y in the limits to be regular). □

Corollary 5.17. *Let X be a log-regular scheme satisfying condition (*). Let n be a natural number invertible on X. Then the open immersion $j : U \hookrightarrow X$, where $U = X_{\mathrm{tr}}$ is the maximal open set of X on which the log-structure is trivial, induces an isomorphism*

$$j^* : K_m^{\mathrm{vk\acute{e}t}}(X, \mathbf{Z}/n) \xrightarrow{\sim} K_m^{\mathrm{\acute{e}t}}(U, \mathbf{Z}/n), \quad m \geq 0.$$

Proof. By Theorem 5.4,

$$H^{-m}(U_{\mathrm{\acute{e}t}}, \mathbf{Z}/n) \simeq H^{-m}(U_{\mathrm{vk\acute{e}t}}, \mathbf{Z}/n) \simeq H^{-m}(X_{\mathrm{vk\acute{e}t}}, j_* K^f/n).$$

It suffices to show that the natural map of presheaves on $X_{\mathrm{vk\acute{e}t}}$,

$$K/n \to j_*(K^f/n)$$

induces a weak equivalence on the stalks at a conservative family of valuative log-geometric points. Recall (section 2.2) that, for $x \in U$, $U \to \mathrm{Spec}(\mathbf{Z}[P])$, a valuative log-geometric point over x can be described as a compatible system of log-geometric points of certain log-blow-ups U_J of U. Since X is log-regular, all the log-blow-ups U_J can be assumed to be regular (see [30, Thm 5.5]). On each U_J, the computations in the proof of Theorem 5.14, show that the map

$$\pi_q(K/n_{x(log)}) \to \pi_q((j_*(K^f/n))_{x(log)})$$

is an isomorphism. This finishes our proof. □

Similarly, Proposition 5.12 implies the following two corollaries.

Corollary 5.18. *Suppose that X is log-regular and satisfies condition (*). Let n be invertible on X. Then the sheaves of homotopy groups of K/n in the log-étale topology are given by*

$$\tilde{\pi}_q(K/n) \simeq \begin{cases} \mathbf{Z}/n(i) & \text{for } q = 2i \geq 0 \\ 0 & \text{for } q \geq 0, \text{ odd.} \end{cases}$$

Corollary 5.19. *Suppose that X is log-regular and satisfies condition (*). Let n be invertible on X. Then there exists a cohomological spectral sequence $E_r^{p,q}$, $r \geq 2$, such that*

$$E_2^{p,q} = \begin{cases} H^p(X_{\mathrm{vk\acute{e}t}}, \mathbf{Z}/n(q/2)) & \text{for } q - p \geq 0 \text{ and } q \text{ even} \\ 0 & \text{for } q - p \geq 0 \text{ and } q \text{ odd.} \end{cases}$$

This spectral sequence converges strongly to $K_{q-p}^{\mathrm{vk\acute{e}t}}(X, \mathbf{Z}/n)$ for $q - p \geq 1$.

Remark 5.20. Let X_* be one of the Kummer sites studied in this paper. Consider the presheaves $K_*^0 : X \mapsto K^0(X_*)$ and $K_*^0/n : X \mapsto K^0/n(X_*)$. They are weakly equivalent to the presheaves K and K/n. Choose their fibrant resolutions $K^f, K^f/n$. For $m \geq 0$ they define functorial maps from the algebraic K-theory to topological K-theory

$$\rho_m : \quad K_m(X_*) = \pi_m(K^0_*(X)) \to \pi_m(K^f(X)) = K^*_m(X),$$

$$\rho_m : \quad K_m(X_*, \mathbf{Z}/n) = \pi_m(K^0_*/n(X)) \to \pi_m(K^f/n(X)) = K^*_m(X, \mathbf{Z}/n)$$

The above yields that for a log-regular regular scheme X satisfying condition (*), a number n invertible on X, and $m \geq 0$, the map

$$\rho_m : \quad K_m(X_{\text{két}}, \mathbf{Z}/n) \to K^{\text{két}}_m(X, \mathbf{Z}/n)$$

factors through the projection

$$\pi : \quad K_m(X_{\text{két}}, \mathbf{Z}/n) \to K_m(X_{\text{két}}, \mathbf{Z}/n)/K'_m(Z_{\text{két}}, \mathbf{Z}/n),$$

where Z is the divisor at infinity. Indeed, we have the following commutative diagram

$$\begin{array}{ccc} K_m(X_{\text{két}}, \mathbf{Z}/n) & \xrightarrow{j^*} & K_m(U_{\text{ét}}, \mathbf{Z}/n) = K_m(U, \mathbf{Z}/n) \\ \downarrow{\scriptstyle \rho_m} & & \downarrow{\scriptstyle \rho_m} \\ K^{\text{két}}_m(X, \mathbf{Z}/n) & \underset{\sim}{\xrightarrow{j^*}} & K^{\text{két}}_m(U, \mathbf{Z}/n), \end{array}$$

where $j : U = X_{\text{tr}} \hookrightarrow X$ is the natural immersion. And our claim follows now from the localization sequence and Theorem 5.14.

Remark 5.21. Corollary 5.17 is closely related to the following absolute log-purity conjecture (see [15, 3.4.2]).

Conjecture 5.22. *Let X be a log-scheme, locally Noetherian. Assume that X is log-regular and let $j : U \hookrightarrow X$ be the open set of triviality of the log-structure of X. Assume that n is invertible on X. Then the adjunction map*

$$\mathbf{Z}/n(q) \to Rj_*j^*\mathbf{Z}/n(q)$$

is an isomorphism for any q.

Indeed, the log-purity conjecture coupled with the spectral sequences 5.19 for X and U implies Corollary 5.17. On the other hand, the usual computation with Adams operations on the spectral sequences 5.19 for X and U should imply their degeneration up to small torsion. Hence the absolute log-purity conjecture (up to small torsion).

Since log-regular schemes can be desingularized by a log-blow-up, the absolute log-purity conjecture follows easily from the following absolute purity conjecture in étale cohomology.

Conjecture 5.23. *Let $i : Y \hookrightarrow X$ be a closed immersion of Noetherian, regular schemes of pure codimension d. Let n be an integer invertible on X. Then*

$$\mathcal{H}^q_Y(X_{\text{ét}}, \mathbf{Z}/n) \simeq \begin{cases} 0 & \text{for } q \neq 2d \\ \mathbf{Z}/n(-d) & \text{for } q = 2d \end{cases}$$

This conjecture was proved by Gabber [11]. Thus to prove Corollary 5.17, we could have used spectral sequences 5.19 and evoke the purity conjecture in étale cohomology.

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Brascamp-Lieb Inequalities for Non-Commutative Integration

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Abstract. We formulate a non-commutative analog of the Brascamp-Lieb inequality, and prove it in several concrete settings.

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1 Introduction

1.1 Young's inequality in the context of ordinary Lebesgue integration

In this paper, we shall extend the class of generalized Young's inequalities known as Brascamp-Lieb inequalities (B-L inequalities) to an operator algebra setting entailing non-commutative integration.

The original Young's inequality [40] states that for non negative measurable functions f_1, f_2 and f_3 on $\mathbb{R}$, and $1 \le p_1, p_2, p_3 \le \infty$, with $1/p_1 + 1/p_2 + 1/p_3 = 2$,

$$\int_{\mathbb{R}^2} f_1(x) f_2(x-y) f_3(y) \mathrm{d}x\mathrm{d}y \le \left(\int_{\mathbb{R}} f_1^{p_1}(t)\mathrm{d}t\right)^{1/p_1} \left(\int_{\mathbb{R}} f_2^{p_2}(t)\mathrm{d}t\right)^{1/p_2} \left(\int_{\mathbb{R}} f_3^{p_3}(t)\mathrm{d}t\right)^{1/p_3} . \quad (1.1)$$

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Thus, it provides an estimate of the integral of a product of functions in terms of a product of L^p norms of these functions. The crucial difference with a Hölder type inequality is that the integrals on the right are integrals over only $\mathbb{R}$, while the integral on the left is an integral over $\mathbb{R}^2$, and none of the three factors in the product on the left – $f(x)$, $g(x-y)$ or $h(y)$ – are integrable (to any power) on $\mathbb{R}^2$.
To frame the inequality in terms that are more amenable to the generalizations considered here, define the maps $\phi_j : \mathbb{R}^2 \to \mathbb{R}$, $j = 1, 2, 3$, by

$$\phi_1(x, y) = x \qquad \phi_2(x, y) = x - y \qquad \text{and} \qquad \phi_3(x, y) = y \ .$$

Then (1.1) can be rewritten as

$$\int_{\mathbb{R}^2} \left(\prod_{j=1}^{3} f_j \circ \phi_j \right) \mathrm{d}^2 x \leq \prod_{j=1}^{3} \left(\int_{\mathbb{R}} f_j^{p_j}(t) \mathrm{d}t \right)^{1/p_j} \ . \tag{1.2}$$

There is now no particular reason to limit ourselves to products of only three functions, or to integrals over $\mathbb{R}^2$ and $\mathbb{R}$, or even any Euclidean space for that matter:

1.1 DEFINITION. Given measure spaces $(\Omega, \mathcal{S}, \mu)$ and $(M_j, \mathcal{M}_j, \nu_j)$, $j = 1, \dots, N$, not necessarily distinct, together with measurable functions $\phi_j : \Omega \to M_j$ and numbers $p_1, \dots, p_N$ with $1 \leq p_j \leq \infty$, $1 \leq j \leq N$, we say that *a B-L inequality holds for* $\{\phi_1, \dots, \phi_N\}$ *and* $\{p_1, \dots, p_N\}$ in case there is a finite constant C such that

$$\int_\Omega \prod_{j=1}^{N} f_j \circ \phi_j \mathrm{d}\mu \leq C \prod_{j=1}^{N} \|f_j\|_{L^{p_j}(\nu_j)} \tag{1.3}$$

holds whenever each f_j is non-negative and measurable on M_j, $j = 1, \dots, N$.
There are by now many examples. One of the oldest is the original *discrete Young's inequality.* In the current notation, this concerns the case in which $\Omega = \mathbb{Z}^2$ equipped with counting measure, $N = 3$, and each M_j is $\mathbb{Z}$, equipped with counting measure. Then with

$$\phi_1(m, n) = m \qquad \phi_2(m, n) = m - n \qquad \text{and} \qquad \phi_3(m, n) = n \ ,$$

(1.2) holds for any three non-negative functions $f_j : \mathbb{Z} \to \mathbb{R}_+$ under the same conditions on the p_j as in the continuous case; i.e., $1/p_1 + 1/p_2 + 1/p_3 = 2$. There is a significant difference: In the discrete case, the constant $C = 1$ is sharp, and there is equality if and only if one of the f_j is identically zero, or else f_1 vanishes except at some m_0, f_3 vanishes except at some n_0, and f_2 vanishes except at $m_0 - n_0$. This discrete inequality is also due to Young [40], while the statement about cases of equality is proved in [20], where the authors also consider extensions to more than three functions.

In the continuous case, a much wider generalization to more than three functions was made by B-L in [5], where the sharp constant in Young's inequality – which is strictly less than 1 unless $p_1 = p_2 = 1$ – was obtained, with a proof that the only non-negative functions yielding equality are certain Gaussian functions. (This best constant was also obtained at the same time by Beckner [4], for three functions.)
These inequalities generalize from $\mathbb{R}$ to $\mathbb{R}^n$. The complete generalization to the case in which the M_j are all Euclidean spaces, but of different dimension, and the ϕ_j are linear transformations from $\mathbb{R}^n$ to M_j, was proved by Lieb [23]. Again, the maximizers are Gaussians. Another proof of this generalized version, together with a reverse form, was obtained by Barthe [1], who also provided a detailed analysis of the cases of equality in the original B-L inequality from [5]. The cases of equality in the higher dimensional generalization from [23] were analyzed in detail in [7, 8].
Examples in which Ω is the sphere S^{N-1} or the permutation group $\mathcal{S}^N$ were proved in [13, 14], and the above definition of B-L inequalities in arbitrary measure spaces is taken from [10], where a duality between B-L inequalities and subadditivity of entropy inequalities is proved.

1.2 A generalized Young's inequality in the context of non-commutative integration

In non-commutative integration theory, as developed by Irving Segal [31, 32, 33], the basic framework is a triple $(\mathcal{H}, \mathfrak{A}, \lambda)$ where $\mathcal{H}$ is a Hilbert space, $\mathfrak{A}$ is a W^* algebra (a von Neumann algebra) of operators on $\mathcal{H}$, and λ is a positive linear functional on the finite rank operators in $\mathfrak{A}$. In Segal's picture, the algebra $\mathfrak{A}$ corresponds to the algebra of bounded measurable functions, and applying the positive linear functional λ to a positive operator corresponds to taking the integral of a positive function. That is,

$$A \mapsto \lambda(A) \qquad \text{corresponds to} \qquad f \mapsto \int_M f \mathrm{d}\nu \ .$$

Such a triple $(\mathcal{H}, \mathfrak{A}, \lambda)$ is called a *non-commutative integration space*. Certain natural regularity properties must be imposed on λ if one is to get a well-behaved non-commutative integration theory, but we shall not go into these here as the examples that we consider are all based on the case in which λ is the *trace* on operators on $\mathcal{H}$, or some closely related functional, for which discussion of these extra conditions would be a digression.
In this operator algebra setting there are natural non-commutative analogs of the usual L^p spaces: If A is a finite rank operator in $\mathfrak{A}$, and $1 \le q < \infty$, define

$$\|A\|_{q,\lambda} = \left(\lambda(A^*A)^{q/2}\right)^{1/q} \ .$$

This defines a norm (under appropriate conditions on λ that are obvious for the trace), and the completion of the space of finite rank operators in $\mathfrak{A}$ under this norm defines a non-commutative L^p space. (The completion may contain unbounded operators "affiliated" to $\mathfrak{A}$.) For more on the general theory of non-commutative integration, see the early papers [15, 31, 33, 34] and the more recent work in [16, 19, 21, 25].

To frame an analog of (1.3) in an operator algebra setting, we replace the measure spaces by non-commutative integration spaces:

$$(\Omega, \mathcal{S}, \mu) \longrightarrow (\mathcal{H}, \mathfrak{A}, \lambda) \quad \text{and} \quad (M_j, \mathcal{M}_j, \nu_j) \longrightarrow (\mathcal{H}_j, \mathfrak{A}_j, \lambda_j) \quad j = 1, \ldots, N \ .$$

The right hand side of (1.3) has an obvious generalization to the operator algebra setting in terms of the non-commutative L_p norms introduced above. As for the left hand side of (1.3), regard $f_j \mapsto f_j \circ \phi_j$ as a W^* algebra homomorphism (which, restricted to the W^* algebra $L^\infty(M_j)$, it is), and suppose we are given W^* homomorphisms

$$\phi_j : \mathfrak{A}_j \to \mathfrak{A} \ .$$

Then each $\phi_j(A_j)$ belongs to $\mathfrak{A}$. However in the non-commutative case, the product of the $\phi_j(A_j)$ depends on their order in the product, and need not be self-adjoint even – let alone positive – even if each of the A_j are positive.

Therefore, let us return to the left side of (1.3) and suppose that each f_j is strictly positive. Then defining

$$h_j = \ln(f_j) \qquad \text{so that} \qquad f_j \circ \phi_j = e^{h \circ \phi_j} \ ,$$

we can then rewrite (1.3) as

$$\int_\Omega \exp\left(\sum_{j=1}^N h_j \circ \phi_j\right) \mathrm{d}\mu \le C \prod_{j=1}^N \|e^{h_j}\|_{L^{p_j}(\nu_j)} \ , \tag{1.4}$$

We can now formulate our operator algebra analog of (1.3):

1.2 DEFINITION. Given non-commutative integration spaces $(\mathcal{H}, \mathfrak{A}, \lambda)$ and $(\mathcal{H}_j, \mathfrak{A}_j, \lambda_j)$, $j = 1, \ldots, N$, together with W^* algebra homomorphisms $\phi_j : \mathfrak{A}_j \to \mathfrak{A}$, $j = 1, \ldots, N$, and indices $1 \le p_j \le \infty$, $j = 1, \ldots, N$, a *non-commutative B-L inequality holds for* $\{\phi_1, \ldots, \phi_N\}$ *and* $\{p_1, \ldots, p_N\}$ if there is a finite constant C so that

$$\boxed{\lambda\left(\exp\left[\sum_{j=1}^N \phi_j(H_j)\right]\right) \le C \prod_{j=1}^N \left(\lambda_j \exp\left[p_j H_j\right]\right)^{1/p_j}} \tag{1.5}$$

whenever H_j is self-adjoint in $\mathfrak{A}_j$, $j = 1, \ldots, N$.

In this paper, we are concerned with determining the indices and the best constant C for which such an inequality holds, and shall focus on two examples: The first concerns *operators on tensor products of Hilbert spaces*, and the second concerns *Clifford algebras*.

1.3 A generalized Young's inequality for tensor products

1.3 EXAMPLE. Let $\mathcal{H}_j$, $j = 1, \ldots, n$ be separable Hilbert spaces, and let Let $\mathcal{K}$ denote the tensor product

$$\mathcal{K} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \ .$$

Define $\mathfrak{A}$ to be $\mathfrak{B}(\mathcal{K})$, the algebra of bounded operators on $\mathcal{K}$, and define λ to be Tr, the trace Tr on $\mathcal{K}$, so that $(\mathcal{H}, \mathfrak{A}, \lambda) = (\mathcal{K}, \mathfrak{B}(\mathcal{K}), \mathrm{Tr})$.
For any non-empty subset J of $\{1, \ldots, n\}$, let $\mathcal{K}_J$ denote the tensor product

$$\mathcal{K}_J = \bigotimes_{j \in J} \mathcal{H}_j \ .$$

Define $\mathfrak{A}_J$ to be $\mathfrak{B}(\mathcal{K}_J)$, the algebra of bounded operators on $\mathcal{K}_J$, and define λ_J to be Tr_J, the trace on $\mathcal{K}_J$, so that $(\mathcal{H}_J, \mathfrak{A}_J, \lambda_J) = (\mathcal{K}_J, \mathfrak{B}(\mathcal{K}_J), \mathrm{Tr}_J)$.
There are natural homomorphisms ϕ_J embedding the $2^n - 1$ algebras $\mathfrak{A}_J$ into $\mathfrak{A}$. For instance, if $J = \{1, 2\}$,

$$\phi_{\{1,2\}}(A_1 \otimes A_2) = A_1 \otimes A_2 \otimes I_{\mathcal{H}_3} \otimes \cdots \otimes I_{\mathcal{H}_N} \ , \tag{1.6}$$

and $\phi_{\{1,2\}}$ is extended linearly.
It is obvious that in case $J \cap K = \emptyset$ and $J \cup K = \{1, \ldots, n\}$, then for all $H_J \in \mathfrak{A}_J$ and $H_K \in \mathfrak{A}_K$,

$$\mathrm{Tr}\left(e^{H_J + H_K}\right) = \mathrm{Tr}_J\left(e^{H_J}\right) \mathrm{Tr}_K\left(e^{H_K}\right) \ , \tag{1.7}$$

but things are more interesting when $J \cap K \neq \emptyset$ and J and K are both proper subsets of $\{1, \ldots, n\}$. If H_J and H_K do not commute, which is the typical situation for $J \cap K \neq \emptyset$, one can estimate the left hand side of (1.7) by first applying the Golden–Thompson inequality [17, 36], which says that for self-adjoint operators H_J and H_K,

$$\mathrm{Tr}\left(e^{H_J + H_K}\right) \leq \mathrm{Tr}\left(e^{H_J} e^{H_K}\right) \ .$$

One might then apply Hölder's inequality – but if J and K are proper subsets of $\{1, \ldots, n\}$, this will yield a finite bound if and only if all of the Hilbert spaces whose indices are not included in *both* J and K are finite dimensional. Even then, the bound depends on the dimension in an unpleasant way. The non-commutative B-L Inequalities provided by the next theorem do not have this defect.

1.4 THEOREM. *Let $J_1, \ldots, J_N$ be N non-empty subsets of $\{1, \ldots, n\}$ For each $i \in \{1, \ldots, n\}$, let $p(i)$ denote the number of the sets $J_1, \ldots, J_N$ that contain i, and let p denote the minimum of the $p(i)$. Then, for self-adjoint operators H_j on $\mathcal{K}_{J_j}$, $j = 1, \ldots, N$,*

$$\boxed{\mathrm{Tr}\left(\exp\left[\sum_{j=1}^{N} \phi_{J_j}(H_j)\right]\right) \leq \prod_{j=1}^{N} \left(\mathrm{Tr}_{J_j}\ e^{qH_j}\right)^{1/q}} \tag{1.8}$$

for $q = p$ (and hence all $1 \leq q \leq p$), while for all $q > p$, it is possible for the left hand side to be infinite, while the right hand side is finite.

Note that in Theorem 1.4, the constant C in Definition (1.2) is 1. The fact that the constant $C = 1$ is best possible, and that the inequality cannot hold for $q > p = \min\{p(1), \ldots, p(N)\}$ is easy to see by considering the case that each $\mathcal{H}_j$ has finite dimension d_j, and $H_j = 0$ for each j. Then

$$\mathrm{Tr}\left(\exp\left[\sum_{j=1}^{N} \phi_{J_j}(H_j)\right]\right) = \prod_{j=1}^{n} d_j$$

and

$$\prod_{j=1}^{N} \left(\mathrm{Tr}_{J_j} e^{qH_j}\right)^{1/q} = \prod_{j=1}^{N} \prod_{k\in J_j} d_k^{1/q} = \prod_{j=1}^{n} d_j^{p(j)/q} \ .$$

We will prove inequality (1.8) for $q = p$ in Section 3.

As an example, consider the case in which $n = 6$, $N = 3$ and

$$J_1 = \{1,2,3\} \quad J_2 = \{3,4,5\} \qquad \text{and} \quad J_3 = \{5,6,1\} \ .$$

Here, $p = 1$, and hence

$$\mathrm{Tr}\left(\exp\left[\sum_{j=1}^{3} \phi_{J_j}(H_j)\right]\right) \leq \prod_{j=1}^{3} \left(\mathrm{Tr}_{J_j}\ e^{H_j}\right) \ . \tag{1.9}$$

Theorem 1.4 says that the inequality (1.9) can be extended to larger tensor products, and this obviously has an statistical mechanical interpretation as a bound on the *partition function* of a collection of interacting spins in terms of a product of partition functions of simple constituent sub-systems. For more background on this, see Thirring's book [35].

To estimate the left side of (1.9) without using Theorem 1.4, one might use the Golden-Thompson inequality and then Schwarz's inequality to write

$$\begin{aligned}\mathrm{Tr}\left(\exp\left[\sum_{j=1}^{3} \phi_{J_j}(H_j)\right]\right) &\leq \mathrm{Tr}\left(e^{\phi_1(H_1)+\phi_3(H_3)}e^{\phi_2(H_2)}\right)\\ &\leq \left(\mathrm{Tr}\ e^{2[\phi_1(H_1)+\phi_3(H_3)]}\right)^{1/2}\left(\mathrm{Tr}\ e^{2\phi_2(H_2)}\right)^{1/2} \ .\end{aligned}$$

While the L^2 norms are an improvement over the L^1 norms in (1.9), the traces are now over the entire tensor product space. Thus, for example,

$$\left(\mathrm{Tr}\ e^{2\phi_2(H_2)}\right)^{1/2} = (d_1 d_2 d_6)^{1/2}\left(\mathrm{Tr}_{J_2}\ e^{2H_2}\right)^{1/2}$$

where d_j is the dimension of Hilbert space $\mathcal{H}_j$. This dimension dependence may be unfavorable if any of the dimensions is large.

1.4 A generalized Young's inequality in Clifford algebras

Our next example concerns Clifford algebras, which, as Segal emphasized [32], allow one to represent fermion Fock space as an L^2 space – albeit a non-commutative L^2 space, but still with many of the advantages of having a Hilbert space represented as a function space, as in the usual Schrödinger representation in quantum mechanics.

In the finite dimensional setting, with n degrees of freedom, one starts with n operators $Q_1, \ldots, Q_n$ on some Hilbert space $\mathcal{H}$ that satisfy the *canonical anticommutation relations*

$$Q_i Q_j + Q_j Q_i = 2\delta_{i,j} I \ .$$

One can concretely construct such operators acting on $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$, the n–fold tensor product of $\mathbb{C}^2$ with itself; see [6]. The Clifford algebra $\mathfrak{C}$ is the operator algebra on $\mathcal{H}$ that is generated by $Q_1, \ldots, Q_n$.

The Clifford algebra $\mathfrak{C}$ itself is 2^n dimensional. In fact, let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a *fermionic multi-index*, which means that each α_j is either 0 or 1. Then define

$$Q^\alpha = Q_1^{\alpha_1} Q_2^{\alpha_2} \cdots Q_n^{\alpha_n} \ . \tag{1.10}$$

it is easy to see that the 2^n operators Q^α are a basis for the Clifford algebra, so that any operator A in $\mathfrak{C}$ has a unique expression

$$A = \sum_\alpha x_\alpha Q^\alpha \ .$$

The linear functional τ on $\mathfrak{C}$ is defined by

$$\tau\left(\sum_\alpha x_\alpha Q^\alpha\right) = x_{(0,\ldots,0)} \ . \tag{1.11}$$

That is, τ acting on A picks out the coefficient of the identity in $A = \sum_\alpha x_\alpha Q^\alpha$. It turns out that when the Clifford algebra is constructed in the way described here, as an algebra operators on the 2^n dimensional space $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$, τ is nothing other than the normalized trace:

$$\tau(A) = \frac{1}{2^n} \mathrm{Tr}_{\mathcal{H}}(A) \ .$$

Hence τ is a positive linear functional, and $((\mathbb{C}^2)^{\otimes n}, \mathfrak{C}, \tau)$ is a non-commutative integration space in the sense of Segal.

Clifford algebras have infinitely many subalgebras that are also Clifford algebras of lower dimension. This is in contrast to the setting described in Example 1.3, in which the only natural subalgebras are the $2^n - 1$ subalgebras corresponding to the $2^n - 1$ non-empty subsets of the index set $\{1, \ldots, n\}$.

To describe these subalgebras, let $\mathcal{J}$ be the *canonical injection* of $\mathbb{R}^n$ into $\mathfrak{C}$, which is given by

$$\mathcal{J}((x_1, \dots, x_n)) = \sum_{j=1}^{n} x_j Q_j \ . \tag{1.12}$$

If x and y are any two vectors in $\mathbb{R}^n$, it is easy to see from the canonical anticommutation relations that

$$(\mathcal{J}(x))(\mathcal{J}(y)) = 2(x \cdot y)I \ .$$

Hence if V is *any* m dimensional subspace of $\mathbb{R}^n$, and $\{u_1, \dots, u_m\}$ is *any* orthonormal basis for V, the m operators

$$\mathcal{J}(u_1), \dots, \mathcal{J}(u_m)$$

again satisfy the canonical anticommutation relations, and generate a subalgebra of $\mathfrak{C}$ that is denoted by $\mathfrak{C}(V)$, and referred to as *the Clifford algebra over* V. In the same vein, it is convenient to refer to $\mathfrak{C}$ itself as the *Clifford algebra over* $\mathbb{R}^n$. Obviously, $\mathfrak{C}(V)$ is naturally isomorphic to $\mathfrak{C}(\mathbb{R}^m)$, and for $A \in \mathfrak{C}(V)$ one may compute $\tau(A)$ using either the normalized trace τ inherited from $\mathfrak{C}$, or the normalized trace τ_V induced by the identification of $\mathfrak{C}(V)$ with $\mathfrak{C}(\mathbb{R}^m)$. As Segal emphasized, $((\mathbb{C}^2)^{\otimes n}, \mathfrak{C}, \tau)$ is, in many ways, a non-commutative analog of the Gaussian measure space $(\mathbb{R}^n, \gamma(x)\mathrm{d}x)$ where

$$\gamma(x) = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2} \ . \tag{1.13}$$

In fact, just as orthogonality implies independence in $(\mathbb{R}^n, \gamma(x)\mathrm{d}x)$, if V and W are two orthogonal subspaces of $\mathbb{R}^n$, and if $A \in \mathfrak{C}(V)$ and $B \in \mathfrak{C}(W)$, then

$$\tau(AB) = \tau(A)\tau(B) \ .$$

The results we prove here reinforce this analogy. We are now ready to introduce our next example:

1.5 EXAMPLE. For some $n > 1$, let $\mathfrak{A}$ be the Clifford algebra over $\mathbb{R}^n$ with its usual inner product, and let $\mathfrak{A}$ be equipped with its unique tracial state τ, which is simply the normalized trace.
For each $j = 1, \dots, N$, let V_j be a subspace of $\mathbb{R}^n$, and let $\mathfrak{A}_j$ be $\mathfrak{C}(V_j)$, the Clifford algebra over V_j with the inner product that V_j inherits from $\mathbb{R}^n$. Let $\mathfrak{A}_j$ be equipped with its unique tracial state τ_j. The natural embedding of V_j into $\mathbb{R}^n$ induces a homomorphism of $\mathfrak{A}_j$ into $\mathfrak{A}$, and we define this to be ϕ_j. In this setting, we shall prove

1.6 THEOREM. *Let $V_1, \dots, V_N$ be N subspaces of $\mathbb{R}^n$, and let $\mathfrak{A}_j$ be the Clifford algebra over V_j with the inner product that V_j inherits from $\mathbb{R}^n$, and let $\mathfrak{A}_j$*

be equipped with its unique tracial state τ_j. Let ϕ_j be the natural homomorphism of $\mathfrak{A}_j$ into $\mathfrak{A}$ induced by the natural embedding of V_j into $\mathbb{R}^n$. Then

$$\boxed{\tau\left(\exp\left[\sum_{j=1}^{N}\phi_j(H_j)\right]\right) \le \prod_{j=1}^{N}\left(\tau_j\ e^{p_j H_j}\right)^{1/p_j}} \tag{1.14}$$

for all self-adjoint operators $H_j \in \mathfrak{A}_j$ if and only if

$$\sum_{j=1}^{N}\frac{1}{p_j}P_j \le I_{\mathbb{R}^n}\ . \tag{1.15}$$

where P_j is the orthogonal projection onto V_j in $\mathbb{R}^n$.

In the special case in which $\dim(V_j) = 1$ for each j, (1.14) reduces to an interesting inequality for the hyperbolic cosine. Indeed, let u_j be one of the two unit vectors in V_j.
Then, with $u_j \otimes u_j$ denoting the orthogonal projection onto the span of u_j, (1.15) reduces to

$$\sum_{j=1}^{N}\frac{1}{p_j}u_j \otimes u_j \le I_{\mathbb{R}^n}\ . \tag{1.16}$$

The greater simplification, however, is that in this case, the space of self-adjoint operators in each $\mathfrak{A}_j$ is just two dimensional, and, with $\mathcal{J}$ denoting the canonical injection defined in (1.12),

$$H_j = a_j I + b_j \mathcal{J}(u_j)$$

for some real numbers a_j and b_j. Then

$$\sum_{j=1}^{N} H_j = \left(\sum_{j=1}^{N} a_j\right) I + \mathcal{J}\left(\sum_{j=1}^{N} b_j u_j\right)\ .$$

This operator has exactly two eigenvalues,

$$\left(\sum_{j=1}^{N} a_j\right) \pm \left|\sum_{j=1}^{N} b_j u_j\right|$$

with equal multiplicities.
Likewise, $p_j H_j$ has exactly two eigenvalues $p_j a_j \pm p_j b_j$ with equal multiplicities.
Hence, in this case, (1.14) reduces to

$$\cosh\left(\left|\sum_{j=1}^{N} b_j u_j\right|\right) \le \prod_{j=1}^{N}\left(\cosh(p_j b_j)\right)^{1/p_j} \quad \text{for all} \quad (b_1,\dots,b_N) \in \mathbb{R}^N\ , \tag{1.17}$$

which, according to Theorem 1.6, must hold whenever (1.16) is satisfied. (The a_j's make the same contribution to both sides, and may be cancelled away.) Taking the logarithm of both sides, this can be rewritten as

$$\ln\cosh\left(\left|\sum_{j=1}^N b_j u_j\right|\right) \le \sum_{j=1}^N \frac{1}{p_j}\ln\cosh(p_j b_j) \quad \text{for all} \quad (b_1,\dots,b_N)\in\mathbb{R}^N\ , \tag{1.18}$$

and this inequality must hold whenever the unit vectors $\{u_1,\dots,u_N\}$ and the positive numbers $\{p_1,\dots,p_N\}$ satisfy (1.16).
Later on, we shall give an elementary proof of this inequality, and hence an elementary proof of Theorem 1.6 when each V_j is one dimensional. Our proof of the other cases is less than elementary, and even our elementary proof of (1.18) is less than direct.

2 Subadditivty of Entropy and Generalized Young's Inequalities

In the examples we have introduced in the previous section, the positive linear functionals λ under consideration are either traces or normalized traces. Throughout this section, we assume that our non-commutative integration spaces $(\mathcal{H},\mathfrak{A},\lambda)$ are based on *tracial* positive linear functionals λ. That is, we require that for all $A,B\in\mathfrak{A}$,

$$\lambda(AB)=\lambda(BA)\ .$$

In such a non-commutative integration space $(\mathcal{H},\mathfrak{A},\lambda)$, a *probability density* is a non-negative element ρ of $\mathfrak{A}$ such that $\lambda(\rho)=1$. Indeed, the tracial property of λ ensures that

$$\lambda(\rho A)=\lambda(A\rho)=\lambda(\rho^{1/2}A\rho^{1/2})$$

so that $A\mapsto\lambda(\rho A)$ is a positive linear functional that equals 1 on the identity.
Now suppose we have N non-commutative integration spaces $(\mathcal{H}_j,\mathfrak{A}_j,\lambda_j)$ and W^* homomorphisms $\phi_j:\mathfrak{A}_j\to\mathfrak{A}$. Then these homomorphisms induce maps from the space of probability densities on $\mathfrak{A}$ to the spaces of probability densities on the $\mathfrak{A}_j$:
For any probability density ρ on $(\mathfrak{A},\lambda)$, let ρ_j be the probability density on $(\mathfrak{A}_j,\lambda_j)$ defined by

$$\lambda_j(\rho_j A)=\lambda(\rho\phi_j(A))$$

for all $A\in\mathfrak{A}_j$.
For example, in the setting of Example 1.3, ρ_{J_j} is just the partial trace of ρ over $\bigotimes_{k\in J_j^c}\mathcal{H}_k$, leaving an operator on $\bigotimes_{k\in J_j}\mathcal{H}_k$. In the Clifford algebra setting of Example 1.5, ρ_j is simply the orthogonal projection of ρ in $L^2(\mathfrak{C},\tau)$ onto $\mathfrak{C}(V_j)$, which is also known as the conditional expectation [38] of ρ given $\mathfrak{C}(V_j)$.
In this section, we are concerned with the relations between the *entropies* of ρ and the $\rho_1,\dots,\rho_N$. The entropy of a probability density ρ, $S(\rho)$, is defined by

$$S(\rho)=-\lambda(\rho\ln\rho)\ .$$

Evidently, the entropy functional is concave on the set of probability densities.

2.1 DEFINITION. Given tracial non-commutative integration spaces $(\mathcal{H}, \mathfrak{A}, \lambda)$ and $(\mathcal{H}_j, \mathfrak{A}_j, \lambda_j)$, $j = 1, \ldots, N$, together with W^* algebra homomorphisms $\phi_j : \mathfrak{A}_j \to \mathfrak{A}$, $j = 1, \ldots, N$, and numbers $1 \leq p_j \leq \infty$, $j = 1, \ldots, N$, a *generalized subadditivity of entropy inequality* holds if there is a finite constant C so that

$$\boxed{\sum_{j=1}^{N} \frac{1}{p_j} S(\rho_j) \geq S(\rho) - \ln C} \tag{2.1}$$

for all probability densities ρ in $\mathfrak{A}$.

It turns out that for tracial non-commutative integration spaces, generalized subadditivity of entropy inequalities and B-L inequalities are dual to one another, just as they are in the commutative case [10], so that if one holds, so does the other, with the same values of $p_1, \ldots, p_N$ and C. The following is, in fact, a direct non-commutative analog of the main theorem of [10].

2.2 THEOREM. *Let $(\mathcal{H}, \mathfrak{A}, \lambda)$ and $(\mathcal{H}_j \mathfrak{A}_j, \lambda_j)$, $j = 1, \ldots, N$, be tracial non-commutative integration spaces. Let $\phi_j : \mathfrak{A}_j \to \mathfrak{A}$, $j = 1, \ldots, N$ be W^* algebra homomorphisms. Then for any numbers $1 \leq p_j \leq \infty$, $j = 1, \ldots, N$, and any finite constant C, the generalized subadditivity of entropy inequality (2.1) is true for all probability densities ρ on $\mathfrak{A}$ if and only if the non-commutative B-L inequality (1.5) is true for all self-adjoint $H_j \in \mathfrak{A}_j$, $j = 1, \ldots, N$, with the same $p_1, \ldots, p_N$ and the same C.*

As a consequence of Theorem 2.2, one strategy for proving a non-commutative B-L inequality is to prove the corresponding generalized subadditivity of entropy inequality. We shall see in our examples that this is an effective strategy; indeed, this is how we prove Theorems 1.4 and 1.6.

In the current tracial context, the proof of Theorem 2.2 is a direct adaptation of the proof of the corresponding result in the context of Lebesgue integration given in [10]. It turns on a well–known formula for the Legendre transform of the entropy. For completeness, we give this formula in Lemma 2.3 below. Except for the $S(A) = -\infty$ case in (2.2) below, it is the same as [26, Prop. 1.10]. The Legendre transform is usually defined for functions on a dual pair of real linear spaces. Therefore, it is convenient to extend the definition of S to all of $\mathfrak{A}_{\mathrm{h},1}$, the (real) subspace of self-adjoint elements A of $\mathfrak{A}$ with $\lambda(|A|) < \infty$. as follows:

$$S(A) = \begin{cases} -\lambda(A \ln A) & \text{if } A \geq 0 \text{ and } \lambda(A) = 1, \\ -\infty & \text{otherwise.} \end{cases} \tag{2.2}$$

2.3 LEMMA. *Let $\mathfrak{A}$ be $\mathfrak{B}(\mathcal{H})$, the algebra of bounded operators on a separable Hilbert space $\mathcal{H}$. Let λ denote either the trace* Tr *on $\mathcal{H}$, or, if $\mathcal{H}$ is finite* *dimensional, the normalized trace τ. Then for all $A \in \mathfrak{A}_{\mathrm{h},1}$,*

$$-S(A) = \sup_{H \text{ self adjoint, semibounded above}} \left\{\lambda(AH) - \ln\left(\lambda\left(e^H\right)\right)\right\} \ . \tag{2.3}$$

The supremum is an attained maximum if and only if A is a strictly positive probability density, in which case it is attained at H if and only if $H = \ln A + cI$ for some $c \in \mathbb{R}$. Consequently, for all self adjoint H with H semibounded above,

$$\ln\left(\lambda\left(e^H\right)\right) = \sup_{A \in \mathfrak{A}_{\mathrm{h},1}} \left\{\lambda(AH) + S(A)\right\} \ . \tag{2.4}$$

The supremum is a maximum at all points of the domain of $\ln\left(\lambda\left(e^H\right)\right)$, in which case it is attained only at the single point $A = e^H/(\lambda(e^H))$.

Proof: We consider first the case that $\lambda = \mathrm{Tr}$, and $\mathcal{H}$ has finite dimension d. To see that the supremum is ∞ unless $0 \leq A \leq I$, let c be any real number, and let u be any unit vector. Then let H be c times the orthogonal projection onto u. For this choice of H,

$$\lambda(AH) - \ln\left(\lambda\left(e^H\right)\right) = c\langle u, Au\rangle - \ln(e^c + (d-1)) \ .$$

If $\langle u, Au\rangle < 0$, this tends to ∞ as c tends to $-\infty$. If $\langle u, Au\rangle > 1$, this tends to ∞ as c tends to ∞. Hence we need only consider $0 \leq A \leq I$. Next, taking $H = cI$, $c \in \mathbb{R}$,

$$\lambda(AH) - \ln\left(\lambda\left(e^H\right)\right) = c\lambda(A) - c - \ln(d) \ .$$

Unless $\lambda(A) = 1$, this tends to ∞ as c tends to ∞. Hence we need only consider the case that A is a density matrix ρ.
Let ρ be any density matrix on $\mathcal{H}$ and let H be any self-adjoint operator such that $\mathrm{Tr}(e^H) < \infty$, and define the density matrix σ by

$$\sigma = \frac{e^H}{\mathrm{Tr}(e^H)} \ .$$

Then, by the positivity of the relative entropy,

$$\mathrm{Tr}(\rho \ln \rho - \rho \ln \sigma) \geq 0 \ ,$$

with equality if and only if $\sigma = \rho$. But by the definition of σ, this reduces to

$$\mathrm{Tr}(\rho \ln \rho) \geq \mathrm{Tr}(\rho H) - \ln\left(\mathrm{Tr}\left(e^H\right)\right) \ ,$$

with equality if and only if $H = \ln \rho$. From here, the rest is very simple, including the treatment of the normalized trace. ■

Petz [27] has shown how to extend Lemma 2.3 to a much more general context, and his result can be used to extend the validity Theorem 2.2 beyond the tracial case. However, since the examples in which we prove the generalized subadditivity inequality here are tracial, we shall not go into this.

Proof of Theorem 2.2: Suppose, first, that the non-commutative B-L inequality (1.5) holds. Then, for any probability density ρ in $\mathfrak{A}$, and any self-adjoint $H_j \in \mathfrak{A}_j$, $j = 1, \ldots, N$, apply (2.3) with $A = \rho$ and $H = \sum_{j=1}^N \phi_j(H_j)$ to obtain

$$
\begin{aligned}
-S(\rho) \quad &\geq \quad \lambda\left(\rho\left[\sum_{j=1}^N \phi_j(H_j)\right]\right) - \ln\left[\lambda\left(\exp\left[\sum_{j=1}^N \phi_j(H_j)\right]\right)\right] \\
&= \quad \sum_{j=1}^N \lambda_j(\rho_j H_j) - \ln\left[\lambda\left(\exp\left[\sum_{j=1}^N \phi_j(H_j)\right]\right)\right] \\
&\geq \quad \sum_{j=1}^N \lambda_j(\rho_j H_j) - \ln\left[C\prod_{j=1}^N \lambda_j\left(e^{p_j H_j}\right)^{1/p_j}\right] \\
&= \quad \sum_{j=1}^N \frac{1}{p_j}\left[\lambda_j(\rho_j[p_j H_j]) - \ln\left(\lambda_j\left(e^{[p_j H_j]}\right)\right)\right] - \ln C\ .
\end{aligned}
\tag{2.5}
$$

The first inequality here is (2.3), and the second is the non-commutative B-L inequality (1.5).

Now choosing $p_j H_j$ to maximize $\lambda_j(\rho_j[p_j H_j]) - \ln\left(\lambda_j\left(e^{[p_j H_j]}\right)\right)$, we get from (2.3) once more that

$$
\lambda_j(\rho_j[p_j H_j]) - \ln\left(\lambda_j\left(e^{[p_j H_j]}\right)\right) = -S(\rho_j) = \lambda_j(\rho_j \ln \rho_j)\ .
$$

Thus, we have proved (2.1) with the same $p_1, \ldots, p_N$ and C that we had in (1.5).

Next, suppose that (2.1) is true. We shall show that in this case the non-commutative B-L inequality (1.5) holds with the same $p_1, \ldots, p_N$ and C. To do this, let the self-sadjoint operators $H_1, \ldots, H_N$ be given, and define

$$
\rho = \left[\lambda\left(\exp\left[\sum_{j=1}^N \phi_j(H_j)\right]\right)\right]^{-1} \exp\left[\sum_{j=1}^N \phi_j(H_j)\right]\ .
$$

Then by Lemma 2.3,

$$\begin{aligned}
\ln\left[\lambda\left(\exp\left[\sum_{j=1}^N \phi_j(H_j)\right]\right)\right] &= \lambda\left(\rho\left[\sum_{j=1}^N \phi_j(H_j)\right]\right) + S(\rho) \\
&= \sum_{j=1}^N \lambda_j\left[\rho_j H_j\right] + S(\rho) \\
&\le \sum_{j=1}^N \frac{1}{p_j}\left[\lambda_j\left[\rho_j(p_j H_j)\right] + S(\rho_j)\right] + \ln C \\
&\le \sum_{j=1}^N \frac{1}{p_j}\ln\left[\lambda_j\left(\exp(p_j H_j)\right)\right] + \ln C
\end{aligned} \tag{2.6}$$

The first inequality is the generalized subadditivity of entropy inequality (2.1), and the second is (2.4).
Exponentiating both sides of (2.6), we obtain (1.5) with the same $p_1, \ldots, p_N$ and C that we had in (2.1). ■

3 Proof of the generalized subadditivity of entropy inequality for tensor products of Hilbert spaces

The crucial tool that we use in this section is the *strong subadditivity of entropy* [24], which we now recall in a formulation that is suited to our purposes.
Suppose, as in Example 1.3, that we are given n separable Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$. As before, let $\mathcal{K}$ denote their tensor product, and for any non-empty subset J of $\{1, \ldots, n\}$, let $\mathcal{K}_J$ denote $\bigotimes_{j\in J}\mathcal{H}_j$.
For a density matrix ρ on $\mathcal{K}$, and any non-empty subset J of $\{1, \ldots, n\}$, define $\rho_J = \mathrm{Tr}_{J^c}\rho$ to be the density matrix on $\mathcal{K}_J$ induced by the natural injection of $\mathfrak{B}(\mathcal{K}_J)$ into $\mathfrak{B}(\mathcal{K})$. As noted above, ρ_J is nothing other than the partial trace of ρ over the complementary product of Hilbert spaces, $\bigotimes_{j\notin J}\mathcal{H}_j$.
The strong subadditivity of entropy is expressed by the inequality stating that for all nonempty $J, K \subset \{1, \ldots, n\}$,

$$S(\rho_J) + S(\rho_K) \ge S(\rho_{J\cup K}) + S(\rho_{J\cap K})\ . \tag{3.1}$$

In case $\mathcal{J} \cap K = \emptyset$, it reduce to the ordinary subadditivity of entropy, which is the elementary inequality

$$S(\rho_J) + S(\rho_K) \ge S(\rho_{J\cup K}) \qquad \text{for} \quad J\cap K = \emptyset\ . \tag{3.2}$$

Combining these, we have

$$\begin{aligned}
S(\rho_{\{1,2\}}) + S(\rho_{\{2,3\}}) + S(\rho_{\{3,1\}}) &\ge S(\rho_{\{1,2,3\}}) + S(\rho_{\{2\}}) + S(\rho_{\{1,3\}}) \\
&\ge 2S(\rho_{\{1,2,3\}})\ ,
\end{aligned} \tag{3.3}$$

where the first inequality is the strong subadditivity (3.1) and the second is the ordinary subadditivity (3.2). Thus, for $n = 3$ and $J_1 = \{1,2\}$, $J_2 = \{2,3\}$ and $J_3 = \{3,1\}$, we obtain

$$\frac{1}{2}\sum_{j=1}^{N} S(\rho_{J_j}) \geq S(\rho) \ .$$

In this example, each index $i \in \{1,1,3\}$ belonged to exactly two of the set J_1, J_2 and J_3, and this is the source of the factor of $1/2$ in the inequality. The same procedure leads to the following result:

3.1 THEOREM. *Let $J_1,\ldots,J_N$ be N non-empty subsets of $\{1,\ldots,n\}$ For each $i \in \{1,\ldots,n\}$, let $p(i)$ denote the number of the sets $J_1,\ldots,J_N$ that contain i, and let p denote the minimum of the $p(i)$. Then*

$$\frac{1}{p}\sum_{j=1}^{N} S(\rho_{J_j}) \geq S(\rho) \tag{3.4}$$

for all density matrices ρ on $\mathcal{K} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$.

Proof: Simply use strong subadditivty to combine overlapping sets to produce as many "complete" sets as possible, as in the example above. Clearly, there can be no more than p of these. If $p(i) > p$ for some indices i, there will be "left over" partial sets. The entropy is always non-negative, and therefore, discarding the corresponding entropies gives us $\sum_{j=1}^{N} S(\rho_{J_j}) \geq pS(\rho)$, and hence the inequality. ■

It is now a very simple matter to prove Theorem 1.4:

Proof of Theorem 1.4: By the remarks made after the statement of the theorem, all that remains to be proved is the inequality (1.8) for $q = p$. However, this follows directly from Theorem 2.2 and Theorem 3.4. ■

4 On the generalized Young's inequality with a Gaussian reference measure

Before turning to the proof of our non-commutative B-L inequality in Clifford algebras, we discuss the commutative case in which the reference measures is Gaussian. We do this here for two reasons: First, as noted, a Clifford algebra $\mathfrak{C}$ with its normalized trace τ is a non-commutative analog of a Gaussian measure space. This analogy is strong enough that we shall be able to pattern our analysis in the Clifford algebra case on an analysis of the Gaussian case.

Second, the Gaussian inequality is of interest in itself, and seems not to have been fully studied before. Suppose that $V_1,\ldots,V_N$ are N non-zero subspaces of $\mathbb{R}^n$, and for each j, define $\phi_j = P_j$ to be the orthogonal projection of $\mathbb{R}^n$ onto V_j. Equip $\mathbb{R}^n$ and equip each V_j with Lebesgue measure. Then the problem of determining for which sets of indices $\{p_1,\ldots,p_N\}$ there exists a finite constant C so that (1.3) holds for all non-negative measurable functions f_j on V_j, $j = 1, \ldots, N$ is highly non trivial, and has only recently been fully solved [7, 8]. Moreover, determining the value of the best constant C for those choices of $\{p_1, \ldots, p_N\}$ is still a challenging finite dimensional variational problem for which there is no general explicit solution.

In contrast, suppose we are given a non-degenerate Gaussian measure on $\mathbb{R}^n$. It will be convenient to take the covariance matrix of the Gaussian to define the inner product, so that the Gaussian becomes a unit Gaussian. For each positive integer m, define $\gamma_m(x) = (2\pi)^{-m/2} e^{-|x|^2/2}$ on $\mathbb{R}^m$. Then equipping $\mathbb{R}^n$ with the measure $\gamma_n(x)\mathrm{d}x$ and equipping each V_j with the $\gamma_{d_j}(x)\mathrm{d}x$, d_j being the dimension of V_j, it turns out that there is a *very simple necessary and sufficient condition* on the indices $\{p_1, \ldots, p_N\}$ for the constant C to be finite, and better yet, *the best constant C is always 1 whenever it is finite*:

4.1 THEOREM. *Let $V_1, \ldots, V_N$ be N non-zero subspaces of $\mathbb{R}^n$, and for each j, and let d_j denote the dimension of V_j. Define P_j to be the orthogonal projection of $\mathbb{R}^n$ onto V_j. Given the numbers p_j, $1 \le p_j < \infty$ for $j = 1, \ldots, N$, there exists a finite constant C such that*

$$\int_{\mathbb{R}^n} \prod_{j=1}^N f_j \circ P_j(x) \gamma_n(x) \mathrm{d}x \le C \prod_{j=1}^N \left(\int_{V_j} f_j^{p_j}(y) \gamma_{d_j}(y) \mathrm{d}y \right)^{1/p_j} \tag{4.1}$$

holds for all non-negative f_j on V_j, $j = 1, \ldots, N$, if and only if

$$\sum_{j=1}^N \frac{1}{p_j} P_j \le \mathrm{Id}_{\mathbb{R}^n} \tag{4.2}$$

and in this case, $C = 1$.

We hasten to point out that this theorem is partially known. In the special case that each of the subspaces V_j is one dimensional, Barthe and Cordero-Erausquin [2], have proved that it is *sufficient* to have equality in (4.2); in this one dimensional context, their sufficient condition then is

$$\sum_{j=1}^N \frac{1}{p_j} u_j \otimes u_j = \mathrm{Id}_{\mathbb{R}^n} \tag{4.3}$$

with each u_j being a unit vector spanning V_j. They did this as an intermediate step in a short proof of the *Lebesgue measure* version of the B-L inequality under the condition (4.3) – the so-called *geometric case*. Perhaps because their main focus was the Lebesgue measure case, in which (4.3) is not a necessary condition for finiteness of the constant C, they did not address the necessity of this condition in the Gaussian case.

Indeed, under the condition that equality holds in (4.2), the inequality (4.1) is equivalent to its Lebesgue measure analog, which is known to hold with the constant $C = 1$ under this same condition [7, 8]. To see this, define $g_1, \ldots, g_N$ by

$$g_j(y) = f_j(y)(\gamma_j(y))^{1/d_j} \qquad j = 1, \ldots, N \ .$$

As noted in [2] for the one dimensional case, this change of variable allows one to pass back and forth between the Gaussian and Lebesgue measure version of the B-L inequality – under the condition (1.15).

It is also perhaps worth remarking that once one has proved the sufficiency of equality in (4.2), the inequality case follows easily: By the spectral theorem, one can write

$$I - \sum_{j=1}^{N} \frac{1}{p_j} P_j = \sum_{k=1}^{n} \nu_k Q_k$$

where the Q_k are rank one projections, and each ν_k satisfies $0 \leq \nu_k \leq 1$. For each k such that $\nu_k > 0$, define $q_k = 1/\nu_k$, and augment the sets of p_j's and P_j's by adding in all such q_k's and Q_k's. The augmented sets now satisfy (4.2) with equality, but the augmentation is "invisible" in the augmented version of (4.1) if we specialize to $f_k = 1$ identically for each index k in the augmentation. Hence, the equality case discussed in [2] really is the heart of the matter.

Nonetheless, it is worthwhile to give a proof of Theorem 4.1 here for two reasons: First, it may be surprising that the condition (1.15) is necessary for the inequality to hold with any finite constant at all. Second, the proof we will give of sufficiency of the condition (1.15) serves as a model for the proof of the corresponding theorem in the Clifford algebra case that we consider in the next section.

In proving Theorem 4.1, our first step is to pass to the problem of proving a generalized subadditivity inequality. Because the commutative version of Theorem 2.2 has been proved in [10], Theorem 4.2 below on subbadditivity of entropy with respect to a Gaussian reference measure is equivalent to Theorem 4.1. Hence, it suffices to prove one or the other.

Before stating and proving the subadditivty theorem, we first recall that for any probability density ρ on $(\mathbb{R}^m, \mathrm{d}\gamma_m)$, the entropy of ρ, is defined by

$$S(\rho) = -\int_{\mathbb{R}^m} \rho(y) \ln \rho(y) \gamma_m(y) \mathrm{d}y \ .$$

Note that the relative entropy of $\rho(y)\gamma_m(y)\mathrm{d}y$ to $\gamma_m(y)\mathrm{d}y$ is $-S(\rho)$; in the convention employed here, the entropy S is concave, and the relative entropy is convex.

4.2 THEOREM. *Let $V_1, \ldots, V_N$ be N non-zero subspaces of $\mathbb{R}^n$, and for each j, let d_j denote the dimension of V_j. Define P_j to be the orthogonal projection of $\mathbb{R}^n$ onto V_j. For any probability density ρ on $(\mathbb{R}^n, \mathrm{d}\gamma_n)$, let ρ_{V_j} denote the marginal density on $(V_j, \mathrm{d}\gamma_{d_j})$. Then, given the numbers p_j, $1 \leq p_j < \infty$ for $j = 1, \ldots, N$, there exists a finite constant C such that*

$$\sum_{j=1}^{N} \frac{1}{p_j} S(\rho_{V_j}) \geq S(\rho) - \ln(C) \tag{4.4}$$

holds for all probability densities ρ *on* $(\mathbb{R}^n, \mathrm{d}\gamma_n)$*, if and only if*

$$\sum_{j=1}^{N} \frac{1}{p_j} P_j \leq I \tag{4.5}$$

and in this case, $\ln(C) = 0$.

We first prove necessity of the condition (4.5):

4.3 LEMMA. *The condition (4.5) in Theorem 4.2 is necessary.*

Proof: It suffices to consider densities of the form

$$\rho(x) = \exp(b \cdot x - |b|^2/2) \ ,$$

for $b \in \mathbb{R}^n$. Then

$$\rho_{V_j}(x) = \exp(P_j b \cdot y - |P_j b|^2/2) \ ,$$

and we compute:

$$S(\rho) = -\frac{|b|^2}{2} \qquad \text{and} \qquad S(\rho_{V_j}) = -\frac{|P_j b|^2}{2} \ .$$

Thus

$$\sum_{j=1}^{N} \frac{1}{p_j} S(\rho_{V_j}) - S(\rho) = b \cdot \left[Id_{\mathbb{R}^n} - \sum_{j=1}^{N} \frac{1}{p_j} P_j \right] b \ ,$$

and evidently this is bounded below if and only if (4.5) is satisfied. ∎

4.1 Proof of sufficiency

The sufficiency of the condition (4.5) will be proved using an interpolation between an arbitrary density ρ and the uniform density that is provided by the Mehler semigroup. (Indeed, Barthe and Coredero-Erausquin used the Mehler semigroup in their work [2] mentioned above, but used it in a direct proof of the Gaussian B-L inequality inspired by the heat-flow method introduced in [13]. The heat flow approach to prove subadditivity inequalities was developed in [3] and [10].)

The Mehler semigroup is the strongly continuous semigroup of positivity preserving contractions on $L^2(\mathbb{R}^n, \gamma_n(x)\mathrm{d}x)$ whose generator $-\mathcal{N}$ is given by the Dirichlet form

$$\mathcal{E}(f,g) = \int_{\mathbb{R}^n} \nabla f^*(x) \cdot \nabla g(x) \gamma_n(x) \mathrm{d}x \tag{4.6}$$

through $\langle f, \mathcal{N} g\rangle_{L^2(\gamma_n)} = \mathcal{E}(f,g)$, where f^* is the complex conjugate of f. Integrating by parts, one finds

$$\mathcal{N} = -(\Delta - x \cdot \nabla) \ ,$$

The eigenvalues of $\mathcal{N}$ are the non-negative integers, and the eigenfunctions are the Hermite polynomials. (In certain physical contexts, the eigenvalues count occupancy of a quantum state and $\mathcal{N}$ is called the *Boson number operator*.) There is a simple explicit formula for the $e^{-t\mathcal{N}}$:

$$e^{-t\mathcal{N}}f(x) = \int_{\mathbb{R}^n} f\left(e^{-t}x + \sqrt{1-e^{2t}}y\right)\gamma_n(y)\mathrm{d}y \ , \tag{4.7}$$

which is easily checked.
Since evidently $\mathcal{N}1 = 0$, and $e^{-t\mathcal{N}}$ is self-adjoint, it also preserves integrals against $\gamma_n(x)\mathrm{d}x$, and hence, if ρ is any probability density, so is each $\rho_t := e^{-t\mathcal{N}}$. As one sees from (4.7),

$$\lim_{t\to\infty} e^{-t\mathcal{N}}\rho(x) = 1 \ , \tag{4.8}$$

the uniform probability density on $(\mathbb{R}^n, \gamma_n(x)\mathrm{d}x)$, and thus the Mehler semigroup provides us with an interpolation between any probability density ρ and the uniform density 1.
This interpolation is well-behaved with respect to the operation of taking marginals: Consider any probability density ρ on $(\mathbb{R}^n, \gamma_n(x)\mathrm{d}x)$, and any m dimensional subspace V of $\mathbb{R}^n$. Let ρ_V be the marginal density of ρ as in Theorem 4.2. Then of course, we may regard ρ_V as a probability density on $(\mathbb{R}^n, \gamma_n(x)\mathrm{d}x)$ that is constant along directions in $V^\perp$. (Simply compose ρ_V with P_V.) Interpreted this way, so that both ρ and ρ_V are functions on $\mathbb{R}^N$,

$$\left(e^{-t\mathcal{N}}\rho\right)_V = e^{-t\mathcal{N}}\left(\rho_V\right) \ . \tag{4.9}$$

That is, the process of taking marginals commutes with the action of the Mehler semigroup.
The next point to note is that the entropy is monotone increasing along this interpolation: Differentiating, with $\rho_t = e^{-t\mathcal{N}}\rho$,

$$\frac{\mathrm{d}}{\mathrm{d}t}S(\rho_t) = -\int_{\mathbb{R}^n} \ln(\rho_t)(\Delta - x\cdot\nabla)\rho_t\gamma_n\mathrm{d}x = \int_{\mathbb{R}^n} \nabla\ln\rho_t\cdot\nabla\rho_t\gamma_n\mathrm{d}x = \mathcal{E}(\ln\rho_t, \rho_t) \ .$$

For any smooth density ρ, $\mathcal{E}(\ln\rho,\rho) = \int_{\mathbb{R}^n}\nabla\ln\rho\cdot\nabla\rho\gamma_n\mathrm{d}x = \int_{\mathbb{R}^n}|\nabla\ln\rho|^2\rho\gamma_n\mathrm{d}x$, and hence $S(\rho_t)$ is strictly increasing for all t. Moreover, since $(x,t)\mapsto |x|^2/t$ is jointly convex on $\mathbb{R}^n\times\mathbb{R}_+$, $\rho\mapsto\mathcal{E}(\ln\rho,\rho)$ has a unique extension as a convex functional on the set of all probability densities on $(\mathbb{R}^n, \gamma_n(x)\mathrm{d}x)$.

4.4 DEFINITION (Entropy Production). The *entropy production* functional is the convex functional $D(\rho)$ on probability densities on $(\mathbb{R}^n, \gamma_n(x)\mathrm{d}x)$ given by

$$D(\rho) = \int_{\mathbb{R}^n} \ln\rho(x)\mathcal{N}\rho(x)\gamma_n(x)\mathrm{d}x = \mathcal{E}(\ln\rho,\rho) \ . \tag{4.10}$$

With this definition,

$$\frac{\mathrm{d}}{\mathrm{d}t}S(e^{-t\mathcal{N}}\rho) = D(e^{-t\mathcal{N}}\rho) \ .$$

Now because of (4.9), for any subspace V of $\mathbb{R}^n$,

$$\frac{\mathrm{d}}{\mathrm{d}t}S([e^{-t\mathcal{N}}\rho]_V) = D([e^{-t\mathcal{N}}\rho]_V) \ .$$

Since $[e^{-t\mathcal{N}}\rho]_V$ is constant along directions orthogonal to V, the derivatives in those directions that figure in $D([e^{-t\mathcal{N}}\rho]_V)$ are irrelevant; we need only take derivatives along directions in V. This consideration leads to the definitions of the *restricted number operator*, and the *restricted entropy production*:
Given an m dimensional subspace V of $\mathbb{R}^n$, let P_V be the orthogonal projection onto V. The restricted number operator $\mathcal{N}_V$ is the self-adjoint operator on $L^2(\mathbb{R}^n, \gamma_n(x)\mathrm{d}x)$ defined through

$$\langle f, \mathcal{N}_V g\rangle_{L^2(\gamma_n)} = \int_{\mathbb{R}^n} \nabla f^*(x) \cdot P_V \nabla g(x)\gamma_n(x)\mathrm{d}x \ , \tag{4.11}$$

and the *restricted entropy production functional* $D_V(\rho)$ is the convex functional given by

$$D_V(\rho) = \int_{\mathbb{R}^n} (\mathcal{N}_V \ln \rho(x))\, \rho(x)\gamma_n(x)\mathrm{d}x \ . \tag{4.12}$$

With this definition, $D(\rho_V) = D_V(\rho_V)$. However, there is a crucial difference between $D_V(\rho)$ and $D(\rho_V)$:

4.5 LEMMA. *For any smooth probability density ρ on $(\mathbb{R}^n, \gamma_n(x)\mathrm{d}x)$, and any m dimensional subspace V of $\mathbb{R}^n$, let ρ_V be the corresponding marginal density regarded as a probability density on $(\mathbb{R}^n, \gamma_n(x)\mathrm{d}x)$. Then*

$$D(\rho_V) \le D_V(\rho) \ . \tag{4.13}$$

Proof: Regard ρ_V as a function on $\mathbb{R}^n$ (by composing it with P_V). Assume that ρ is smooth and bounded above and below by strictly positive numbers. Notice that since ρ_V is constant along directions in $V^\perp$,

$$\mathcal{N} \ln \rho_V = \mathcal{N}_V \ln \rho_V \ .$$

Then, integrating by parts, and using the definition of ρ_V and the Schwarz inequality, we obtain:

$$D(\rho_V) = \int_{\mathbb{R}^n} [\mathcal{N}_V \ln \rho_V(x)]\, \rho_V(x)\gamma_n(x)\mathrm{d}x = \int_{\mathbb{R}^n} [\mathcal{N}_V \ln \rho_V(x)]\, \rho(x)\gamma_n(x)\mathrm{d}x \ ,$$

where we have used the definition of ρ_V to replace the second ρ_V by ρ itself. Then, by the definition of $\mathcal{N}_V$, and the Schwarz inequality,

$$
\begin{aligned}
D(\rho_V) &= \int_{\mathbb{R}^n} (\nabla \ln \rho_V(x)) \cdot P_V \nabla \rho(x) \gamma_n \mathrm{d}x \\
&= \int_{\mathbb{R}^n} (\nabla \ln \rho_V(x)) \cdot P_V \left(\nabla \ln \rho(x)\right) \rho(x) \gamma_n(x) \mathrm{d}x \\
&\leq \left(\int_{\mathbb{R}^n} |\nabla \ln \rho_V(x)|^2 \rho(x) \gamma_n(x) \mathrm{d}x \right)^{1/2} \left(\int_{\mathbb{R}^n} |P_V \nabla \ln \rho(x)|^2 \rho \gamma_n \mathrm{d}x \right)^{1/2}
\end{aligned} \tag{4.14}
$$

In the first factor in the last line, we may replace ρ by ρ_V since $|\nabla \ln \rho_V(x)|^2$ depends on x only through $P_V x$. Hence this factor is simply $\sqrt{D(\rho_V)}$, and the second factor is $\sqrt{D_V(\rho)}$. ■

The proof we have just given is patterned on the proof of an analogous result in the Lebesgue measure case in [10], which in turn is based on similar arguments in [9] and [3]. It is somewhat more complicated to adapt the argument to the Clifford algebra setting, but this is what we shall do in the next section. We are now ready to prove the sufficiency of condition (4.5):

4.6 LEMMA. *The condition (4.5) in Theorem 4.2 is sufficient.*

Proof: For a probability density ρ on $(\mathbb{R}^n, \mathrm{d}\gamma_n)$ with $S(\rho) > -\infty$, it is easy to see that

$$
\lim_{t\to\infty} S(e^{-t\mathcal{N}}\rho) = S(1) = 0
$$

and hence, $\lim_{t\to\infty} S(e^{-t\mathcal{N}}(\rho_{V_j})) = 0$ for each $j = 1, \ldots, N$. Therefore, it suffices to show that

$$
a(t) := \left[\sum_{j=1}^{N} \frac{1}{p_j} S(e^{-t\mathcal{N}}\rho_{V_j}) - S(e^{-t\mathcal{N}}\rho) \right]
$$

is monotone decreasing in t.

Differentiating, and using (4.9), and then Lemma 4.5,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} a(t) &= \left[\sum_{j=1}^{N} \frac{1}{p_j} D((e^{-t\mathcal{N}}\rho)_{V_j}) - D(e^{-t\mathcal{N}}\rho) \right] \\
&\leq \left[\sum_{j=1}^{N} \frac{1}{p_j} D_{V_j}(e^{-t\mathcal{N}}\rho) - D(e^{-t\mathcal{N}}\rho) \right]
\end{aligned} \tag{4.15}
$$

Now note that by (4.12), whenever (4.5) is satisfied,

$$
\sum_{j=1}^{N} \frac{1}{p_j} D_{V_j}(\sigma) \leq D(\sigma)
$$

for any smooth density σ. Hence the derivative of $\alpha(t)$ is negative for all $t > 0$. ∎

5 Generalized subadditivity of entropy in Clifford algebras

In this section we shall prove

5.1 THEOREM. *Let $V_1, \dots, V_N$ be N subspaces of $\mathbb{R}^n$, and let $\mathfrak{A}_j$ be the Clifford algebra over V_j with the inner product that V_j inherits from $\mathbb{R}^n$, and let $\mathfrak{A}_j$ be equipped with its unique tracial state τ_j. For any probability density $\rho \in \mathfrak{A}$, let ρ_{V_j} be the induced probability density in $\mathfrak{A}_j$. Let $S(\rho) = -\tau(\rho \ln \rho)$ and $S(\rho_{V_j}) = -\tau_j(\rho_{V_j} \ln \rho_{V_j})$*
Then

$$\sum_{j=1}^{N} \frac{1}{p_j} S(\rho_{V_j}) \geq S(\rho) \tag{5.1}$$

for all probability densities $\rho \in \mathfrak{A}$ if and only if

$$\sum_{j=1}^{N} \frac{1}{p_j} P_j \leq I_{\mathbb{R}^n} \ . \tag{5.2}$$

where P_j is the orthogonal projection onto V_j in $\mathbb{R}^n$.

Granted this result, we have:
Proof of Theorem 1.6: Theorem 2.2 and Theorem 5.1 together prove Theorem 1.6. ∎
We shall now prove Theorem 5.1. As before, we begin by proving the necessity of (5.2).

5.2 LEMMA. *The condition (5.2) in Theorem 5.1 is necessary.*

Proof: For any vector $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, define

$$\rho_a = I + \sum_{j=1}^{n} a_j Q_j = I + a \cdot Q \ .$$

Then ρ_a is a probability density if and only if $|a| \leq 1$. Indeed, ρ_a has only two eigenvalues, $1 \pm |a|$, with equal multiplicity.
Then $(\rho_a)_{V_j} = I + (P_j a) \cdot Q$, and so $(\rho_a)_{V_j}$ has only two eigenvalues, $1 \pm |P_j a|$, with equal multiplicity. Therefore,

$$S(\rho_a) = -\psi(|a|) \qquad \text{and} \qquad S((\rho_a)_{V_j}) = -\psi(|P_j a|) \ . \tag{5.3}$$

where $\psi(x)$ is the convex function defined by

$$\psi(x) = \begin{cases} \frac{1}{2}\left[(1+x)\ln(1+x) + (1-x)\ln(1-x)\right] & \text{if } |x| \leq 1. \\ \infty & \text{otherwise,} \end{cases} \tag{5.4}$$

Thus, for (5.1) to hold for each ρ_a, $|a| \leq 1$, it must be the case that

$$\sum_{j=1}^{N} \frac{1}{p_j}\psi(|P_j a|) \leq \psi(|a|) \qquad \text{for all } a \text{ with } |a| \leq 1\ . \tag{5.5}$$

Then, since $\psi(x) = x^2 + \mathcal{O}(x^4)$, replacing a by ta, $0 < t < 1$, we see that (5.2) must hold. ■

Because of (5.3), once we have proved Theorem 5.1, we will have a proof of (5.5). However, it is of interest to prove this inequality directly, and we do that next.

5.3 PROPOSITION. *The inequality (5.5) holds whenever (5.2) is satisfied.*

Proof: An easy calculation of derivatives shows that

$$\psi'(x) = \operatorname{arctanh}(x) \qquad \text{and} \qquad \psi''(x) = \frac{1}{1-x^2}$$

for $|x| < 1$.

Now fix any a with $|a| < 1$. Then, for $t > 0$, define

$$\eta(t) = \psi(e^{-t}|a|) - \sum_{j=1}^{N} \frac{1}{p_j}\psi(e^{-t}|P_j a|)\ .$$

We have to show that $\eta(t) > 0$ for all $t > 0$. Since evidently $\lim_{t\to\infty}\eta(t) = 0$, it suffices to show that $\eta'(t) < 0$ for all $t > 0$.

Differentiating, we find

$$\eta'(t) = -e^{-t}\left[|a|\ \operatorname{arctanh}(e^{-t}|a|) - \sum_{j=1}^{N}\frac{1}{p_j}|P_j a|\ \operatorname{arctanh}(e^{-t}|P_j a|)\right] := e^{-t}\theta(t).$$

Hence, it suffices to show that $\theta(t) \geq 0$ for all $t > 0$. Once again, since $\lim_{t\to\infty}\theta(t) = 0$, it suffices to show that $\theta'(t) < 0$ for all $t > 0$. Differentiating, we find

$$\theta'(t) = -e^{-t}\left[\frac{|a|^2}{1-e^{-2t}|a|^2} - \sum_{j=1}^{N}\frac{1}{p_j}\frac{|P_j a|^2}{1-e^{-2t}|P_j a|^2}\right]\ .$$

Multiplying through by e^{-t}, and absorbing a factor of e^{-t} into a, it suffices to show that

$$\frac{|a|^2}{1-|a|^2} \geq \sum_{j=1}^{N}\frac{1}{p_j}\frac{|P_j a|^2}{1-|P_j a|^2} \tag{5.6}$$

for all $|a| \leq 1$. However, since $|a| \geq |P_j a|$,

$$\frac{|P_j a|^2}{1-|a|^2} \geq \frac{|P_j a|^2}{1-|P_j a|^2}\ ,$$

and thus (5.6) follows from (5.2). ■

We are now in a position to give an elementary proof of Theorem 1.6 in the special case that each V_j is one dimensional. As explained in Example 1.5, it suffices in this case to prove the following:

5.4 PROPOSITION. *Suppose* $\{u_1, \dots, u_N\}$ *is any set of* N *unit vectors in* $\mathbb{R}^n$*, and* $\{p_1, \dots, p_N\}$ *is any set of* N *positive numbers such that*

$$\sum_{j=1}^{N} c_j u_j \otimes u_j = I_{\mathbb{R}^n} \ . \tag{5.7}$$

Then for any $b = (b_1, \dots, b_N)$ *in*$\mathbb{R}^N$*,*

$$\ln\cosh\left(\left|\sum_{j=1}^N b_j u_j\right|\right) \le \sum_{j=1}^{N} \frac{1}{p_j} \ln\cosh(p_j b_j) \ . \tag{5.8}$$

Proof: Let ψ^* denote the function $\psi^*(x) = \ln\cosh(x)$, $x \in \mathbb{R}$. The notation is meant to indicate the well known fact, easily checked, that ψ^* is the Legendre transform of the function ψ defined in (5.4).

Now, given a set of N orthogonal projections $\{P_1, \dots, P_N\}$ satisfying (5.2), we may make any choice of a unit vector u_j from the range of P_j, and then the N unit vectors $\{u_1, \dots, u_N\}$ will satisfy (5.7). Conversely, given any set of N unit vectors $\{u_1, \dots, u_N\}$ that satisfy (5.7), we may take $P_j = u_j \otimes u_j$, and then (5.2) is satisfied. Hence, we suppose we are given a set of N orthogonal projections $\{P_1, \dots, P_N\}$ satisfying (5.2), and for each j, u_j is a unit vector in the range of P_j.

Then for any $b \in \mathbb{R}^n$,

$$\begin{aligned}
\psi^*\left(\left|\sum_{j=1}^N b_j u_j\right|\right) &= \sup_{a\in\mathbb{R}^n}\left\{a\cdot\sum_{j=1}^N b_j u_j - \psi(|a|)\right\}\\
&= \sup_{|a|\le 1}\left\{\sum_{j=1}^N P_j a\cdot b_j u_j - \psi(|a|)\right\}\\
&\le \sup_{|a|\le 1}\left\{\sum_{j=1}^N P_j a\cdot b_j u_j - \sum_{j=1}^N \frac{1}{p_j}\psi(|P_j a|)\right\}\\
&\le \sup_{|a|\le 1}\left\{\sum_{j=1}^N |P_j a||b_j| - \sum_{j=1}^N \frac{1}{p_j}\psi(|P_j a|)\right\}\\
&= \sup_{|a|\le 1}\left\{\sum_{j=1}^N \frac{1}{p_j}\left[|P_j a| p_j |b_j| - \psi(|P_j a|)\right]\right\} \ ,
\end{aligned} \tag{5.9}$$

where the first inequality is from (5.5), and the second is from Schwarz. Then, by the definition of the Legendre transform, for any a,

$$\psi^*(p_j b_j) \geq |P_j a|(p_j|b_j|) - \psi(|P_j a|) \ ,$$

we obtain

$$\psi^*\left(\left|\sum_{j=1}^N b_j u_j\right|\right) \leq \sum_{j=1}^N \frac{1}{p_j}\psi^*(p_j b_j) \ ,$$

which is (5.8). ■

We now prove Theorem 5.1 in full generality. This gives another proof of the last two propositions, but by less elementary means. The proof will follow the basic pattern of the proof of Theorem 4.2 and use the Clifford algebra analog of the Mehler semigroup. This is the so-called Clifford–Mehler semigroup, about which we now recall a few relevant facts.

5.1 About the Clifford–Mehler semigroup

There is also a differential calculus in the Clifford algebra. Let $Q_1, \ldots, Q_n$ be any set of n generators for the Clifford algebra $\mathfrak{C}$ over $\mathbb{R}^n$. For $A \in \mathfrak{C}$, define

$$\nabla_i(A) = \frac{1}{2}\left[Q_i A - \Gamma(A) Q_i\right] \ ,$$

where Γ is the *grading operator* on $\mathfrak{C}$: That is, using the notation in (1.10),

$$\Gamma(Q^\alpha) = (-1)^{|\alpha|} Q^\alpha \ .$$

One computes that $\nabla_i(Q^\alpha) = 0$ if $\alpha(i) = 0$, and otherwise, $\nabla_i(Q^\alpha)$ is what one gets by anti-commuting the factor of Q_i through to the left, and then removing it. In this sense it is like a differentiation operator, and what is more, it is a skew derivation on $\mathfrak{C}$, which means that for all and A and B in $\mathfrak{C}$, $\nabla_j(AB) = \nabla_j(A)B + \Gamma(A)\nabla_j(B)$.

The Clifford algebra analog of the Gaussian energy integral (4.6) is given by

$$\mathcal{E}(A,B) = \tau\left(\sum_{j=1}^n \nabla_j A^* \nabla_j B\right) \ , \tag{5.10}$$

for all $A, B \in \mathfrak{C}$. This is the *Clifford Dirichlet form* studied by Gross [18]. Then, the fermionic number operator, also denoted $\mathcal{N}$, is defined by

$$\mathcal{E}(A,B) = \tau(A^* \mathcal{N}(B)) \ .$$

It is easy to see that the spectrum of $\mathcal{N}$ consists of the non negative integers $\{0, 1, \ldots, n\}$ and that

$$\mathcal{N} Q^\alpha = |\alpha| Q^\alpha \ . \tag{5.11}$$

The Clifford Mehler semigroup is then given by $e^{-t\mathcal{N}}$. It is clear from this definition, (1.11) and (5.11) that for any $A \in \mathfrak{C}$, $\lim_{t\to\infty} e^{-t\mathcal{N}}(A) = \tau(A)I$. Thus for any probability density ρ in $\mathfrak{C}$,

$$t \mapsto \rho_t = e^{-t\mathcal{N}}(\rho)$$

provides an interpolation between ρ and I, and each ρ_t is a probability density. This corresponds exactly to the Mehler semigroup interpolation that was used to prove Theorem 4.2, and we shall use it here in the same way, though some additional complications shall arise.

The operator $\mathcal{N}$ does not depend on the choice of the set of generators $Q_1, \dots, Q_n$. Indeed, if $\{u_1, \dots, u_n\}$ is any orthonormal basis of $\mathbb{R}^n$, and we define $\widetilde{Q}_j = u_j \cdot Q$ for $j = 1, \dots, n$, then the Clifford Dirichlet form that one obtains using this basis to define the derivatives is the same as the original.

In particular, given an m dimensional subspace V of $\mathbb{R}^n$, we may choose $\{u_1, \dots, u_n\}$ so that $\{u_1, \dots, u_m\}$ is an orthonormal basis for V, and then the first m generators will be a set of generators for $\mathfrak{C}_V$. We then define the *reduced Clifford Dirichlet form* $\mathcal{E}_V$ by

$$\mathcal{E}_V(A, B) = \tau\left(\sum_{i,j=1}^{m} \nabla_i A^*[P_V]_{i,j}\nabla_j B\right) , \tag{5.12}$$

where $[P_V]_{i,j}$ is the i,jth entry of the $n \times n$ matrix for P_V. The restricted number operator $\mathcal{N}_V$ is then the self-adjoint operator on $L^2(\mathfrak{C})$ given by $\tau(A^*\mathcal{N}_V(B)) = \mathcal{E}_V(A, B)$.

Now, for any probability density ρ in $\mathfrak{C}$ let ρ_V be the corresponding marginal density regarded as an operator in $\mathfrak{C}$ by identifying it with $\phi_V(\rho_V)$, where ϕ_V is the canonical embedding of $\mathfrak{C}(V)$ into $\mathfrak{C}(\mathbb{R}^n)$. Then it is an easy consequence of the definitions that

$$\left(e^{-t\mathcal{N}}\rho\right)_V = e^{-t\mathcal{N}}\left(\rho_V\right) = e^{-t\mathcal{N}_V}\left(\rho_V\right) . \tag{5.13}$$

Also, under the condition (5.2), it is easy to see that

$$\sum_{j=1}^{N} \frac{1}{p_j}\mathcal{N}_{V_j} \le \mathcal{N} . \tag{5.14}$$

Finally, we introduce *entropy production* $D(\rho)$: With $\rho_t := e^{-t\mathcal{N}}\rho$, we differentiate and find

$$\frac{\mathrm{d}}{\mathrm{d}t} S\left(\rho_t\right) = \tau\left(\ln(\rho_t)\mathcal{N}(\rho_t)\right) = \mathcal{E}(\ln(\rho_t), \rho_t) .$$

5.5 DEFINITION (Entropy Production). The *entropy production* functional at a probability density ρ is the functional defined by

$$D(\rho) = \tau\left(\ln(\rho)\mathcal{N}(\rho)\right) = \mathcal{E}(\ln(\rho), \rho) . \tag{5.15}$$

Given an m dimensional subspace V of $\mathbb{R}^n$, the *restricted entropy production* functional at a probability density ρ is the functional defined by

$$D_V(\rho) = \tau\left(\ln(\rho)\mathcal{N}_V(\rho)\right) = \mathcal{E}_V(\ln(\rho), \rho) \ . \tag{5.16}$$

The following lemma is the basis of our proof of the sufficiency of (5.2). In the course of proving it, we shall see that both $D(\rho)$ and $D_V(\rho)$ are convex functionals, which is somewhat less obvious than in the Gaussian case.

5.6 LEMMA. *For any any probability density ρ in $\mathfrak{C}(\mathbb{R}^n)$, and any m dimensional subspace V of $\mathbb{R}^n$, let ρ_V be the corresponding marginal probability density regarded as a probability density in $\mathfrak{C}(\mathbb{R}^n)$. Then*

$$D(\rho_V) \le D_V(\rho) \ .$$

PROOF: We choose an orthonormal basis $\{u_1, \dots, u_n\}$ for $\mathbb{R}^n$ such that $\{u_1, \dots, u_m\}$ is an orthonormal basis for V. Without loss of generality, we may suppose that $\{u_1, \dots, u_n\}$ is the standard basis so that $\{Q_1, \dots, Q_m\}$ is a set of generators for $\mathfrak{C}(V)$. Then,

$$\mathcal{E}(A, B) = \tau\left(\sum_{j=1}^{n} \nabla_j A^* \nabla_j B\right) \qquad \text{and} \qquad \mathcal{E}_V(A, B) = \tau\left(\sum_{j=1}^{m} \nabla_j A^* \nabla_j B\right) \ . \tag{5.17}$$

It will be convenient to define $\mathcal{N}_j = \nabla_j^* \nabla_j \qquad j = 1, \dots, n$. Then we have

$$\mathcal{N} = \sum_{j=1}^{n} \mathcal{N}_j \qquad \text{and} \qquad \mathcal{N}_V = \sum_{j=1}^{m} \mathcal{N}_j \ , \tag{5.18}$$

and so

$$D_V(\rho) = \sum_{j=1}^{m} \tau\left(\ln\rho, \mathcal{N}_j \rho\right) \ . \tag{5.19}$$

Since $\mathcal{N}_j Q^\alpha = \begin{cases} Q^\alpha & \text{if } \alpha(j) = 1, \\ 0 & \alpha(j) = 0 \end{cases}$, each $\mathcal{N}_j$ is an orthogonal projection, and so (5.19) can be rewritten as

$$D_V(\rho) = \sum_{j=1}^{m} \tau\left(\mathcal{N}_j(\ln\rho), \mathcal{N}_j \rho\right) \ . \tag{5.20}$$

To proceed, we use a formula of Gross [18] for $\mathcal{N}_j f(A)$ where $A \in \mathfrak{C}(\mathbb{R}^n)$, and f is a continuous function. To write down Gross's formula, first write $A = B + Q_j C$ where both B and C are linear combinations of the Q^α with $\alpha(j) = 0$. Then define $\widehat{A} = B - Q_j C$. Notice that if ρ is a probability density, then $\widehat{\rho}$ is again a probability density. Gross's formula is

$$\mathcal{N}_j f(A) = \frac{1}{2}\left[f(A) - f(\widehat{A}\,)\right] \ .$$

To prove this formula, notice that there is a unitary operator U such that $\widehat{A} = UAU^*$. (If the dimension n is odd, one can take U to be the product, in some order, of all of the Q_k for $k \neq j$; if the dimension is even, one can add another generator.) Therefore,

$$\widehat{f(A)} = Uf(A)U^* = f(UAU^*) = f(\widehat{A}\ ) \ .$$

Using this together with the fact that for any $A \in \mathfrak{A}$, $\mathcal{N}_j A = (1/2)[A - \widehat{A}]$, we obtain Gross's formula, which we now apply as follows:

$$\begin{aligned} \tau\left(\mathcal{N}_j(\ln\rho)\mathcal{N}_j\rho\right) &= \frac{1}{4}\tau\left([\ln(\rho) - \ln(\widehat{\rho})]\,[\rho - \widehat{\rho}]\right) \\ &= \frac{1}{4}\tau\left(\ln(\rho)\,[\rho - \widehat{\rho}]\right) + \frac{1}{4}\tau\left(\ln(\widehat{\rho})\,[\widehat{\rho} - \rho]\right) \\ &= \frac{1}{4}H[\rho|\widehat{\rho}\ ] + \frac{1}{4}H[\widehat{\rho}\ |\rho] \end{aligned} \tag{5.21}$$

where $H[\rho|\sigma] = \tau\rho(\ ln\rho - \ln\sigma)$ is the relative entropy of ρ with respect to σ. As is well known, $(\rho,\sigma) \mapsto H(\rho|\sigma)$ is jointly convex, and hence

$$\rho \mapsto \tau\left((\ln\rho)\mathcal{N}_j\rho\right)$$

is convex. Furthermore, by the fundamental monotonicity property of the relative entropy under conditional expectations [37],

$$H(\rho_V|\sigma_V) \leq H(\rho|\sigma)$$

for any two probability densities ρ and σ. It follows that $\tau\left((\ln\rho_V)\mathcal{N}_j\rho_V\right) \leq \tau\left((\ln\rho)\mathcal{N}_j\rho\right)$. Summing on j from 1 to m, we find

$$D(\rho_V) = D_V(\rho_V) = \sum_{j=1}^{m}\tau\left((\ln\rho_V)\mathcal{N}_j\rho_V\right) \leq \sum_{j=1}^{m}\tau\left((\ln\rho)\mathcal{N}_j\rho\right) = D_V(\rho) \ .$$

■

5.2 Proof of the sufficiency

5.7 LEMMA. *The condition (4.5) in Theorem 4.2 is sufficient.*

Proof: For a probability density ρ in $\mathfrak{C}(\mathbb{R}^n)$ it is easy to see that

$$\lim_{t\to\infty} S(e^{-t\mathcal{N}}\rho) = S(1) = 0$$

and hence, $\lim_{t\to\infty} S(e^{-t\mathcal{N}}(\rho_{V_j})) = 0$ for each $j = 1,\dots,N$. Therefore, it suffices to show that

$$a(t) := \left[\sum_{j=1}^{N}\frac{1}{p_j}S(e^{-t\mathcal{N}}\rho_{V_j}) - S(e^{-t\mathcal{N}}\rho)\right]$$

is monotone decreasing in t.
Differentiating, and using (5.13), and then Lemma 5.6,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}a(t) &= \left[\sum_{j=1}^{N}\frac{1}{p_j}D((e^{-t\mathcal{N}}\rho)_{V_j}) - D(e^{-t\mathcal{N}}\rho)\right]\\ &\leq \left[\sum_{j=1}^{N}\frac{1}{p_j}D_{V_j}(e^{-t\mathcal{N}}\rho) - D(e^{-t\mathcal{N}}\rho)\right]\end{aligned} \tag{5.22}$$

Now note that by (4.12), whenever (4.5) is satisfied, $\sum_{j=1}^{N}\frac{1}{p_j}D_{V_j}(\sigma) \leq D(\sigma)$ for any smooth density σ. Hence the derivative of $\alpha(t)$ is negative for all $t > 0$. ■
Notice that the proof is almost identical, symbol for symbol, with that of the corresponding proof in the Gaussian case. The main difference of course is that the proof of the main lemma, Lemma 5.6, is considerably more intricate than that of its Gaussian counterpart.
Proof of Theorem 5.1: This now follows immediately from Lemma 5.2 and 5.7. ■

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Profinite Homotopy Theory

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Abstract. We construct a model structure on simplicial profinite sets such that the homotopy groups carry a natural profinite structure. This yields a rigid profinite completion functor for spaces and pro-spaces. One motivation is the étale homotopy theory of schemes in which higher profinite étale homotopy groups fit well with the étale fundamental group which is always profinite. We show that the profinite étale topological realization functor is a good object in several respects.

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1. Introduction

Let $\mathcal{S}$ be the category of simplicial sets and let $\mathcal{H}$ be its homotopy category. The étale homotopy type of a scheme had been defined by Artin and Mazur [1] as a pro-object in the homotopy category $\mathcal{H}$. Friedlander [11] rigidified the definition such that the étale topological type is a pro-object in the category of simplicial sets itself. This theory has found very remarkable applications. Quillen and Friedlander used it to prove the Adams conjecture, a purely topological problem. Friedlander defined and studied étale K-theory, an étale topological analogue of algebraic K-theory. More recently, Dugger and Isaksen [8] proved a sums-of-squares formula over fields of positive characteristic using étale topological arguments. Schmidt [29] answered open questions in Galois theory using étale homotopy groups. In [25], a stable étale realization functor has been constructed and an étale cobordism theory for schemes has been defined. For smooth schemes over an algebraically closed field and with finite coefficients, étale cobordism agrees with algebraic cobordism after inverting a Bott element.

In almost every application, one considers a profinitely completed object, either with respect to finite or even with respect to p-groups for some prime p. The profinite (resp. pro-p-) completion is an object that is universal with respect to maps to spaces whose homotopy groups are finite (resp. p-) groups. Artin and Mazur realized the profinite completion of a space or pro-space only in pro-$\mathcal{H}$. There is no profinite completion for Friedlander's rigid objects in pro-$\mathcal{S}$. It is well known that in many respects it is preferable to work in a model category itself and not only in the corresponding homotopy category. Hence it is a fundamental question if there is a rigid model for the profinite completion, i.e. an object in $\mathcal{S}$ which is homotopy equivalent to the Artin-Mazur completion in pro-$\mathcal{H}$. Since the étale fundamental group of a scheme is always a profinite group, this is equivalent to the fundamental question if there is a space, not only a pro-object in some homotopy category, that yields profinite higher étale homotopy groups.

Bousfield-Kan [3] proved the existence of a $\mathbb{Z}/p$-completion for a simplicial set for every prime p in $\mathcal{S}$. Moreover, Morel [23] showed that there is even a $\mathbb{Z}/p$-model structure on the category $\hat{\mathcal{S}}$ of simplicial profinite sets such that the Bousfield-Kan completion yields a fibrant replacement functor in $\hat{\mathcal{S}}$. In [25], this rigid model for the pro-p-homotopy theory has been used. In this paper we prove that there is a model for arbitrary profinite completion. We do this by constructing a suitable model structure on the category of simplicial profinite sets such that the homotopy groups carry a natural profinite structure.

The plan of this paper is the following. First, we construct profinite fundamental groups and then we use the profinite topology to define continuous cohomology with local topological coefficients for profinite spaces. The main technical result is this: There is a model structure on simplicial profinite sets such that a weak equivalence is a map that induces isomorphisms on fundamental groups and in cohomology with local finite abelian coefficients. This model structure is fibrantly generated, simplicial and left proper. This result enables us to define higher profinite homotopy groups. It is an important property of the category of profinite spaces that the limit functor is homotopy invariant for cofiltering diagrams.

The fibrant replacement of the completion of a simplicial set in $\hat{\mathcal{S}}$ is a rigid model for the Artin-Mazur profinite completion of [1], and is equivalent in an appropriate sense to the completions of Bousfield-Kan [3] and Morel [23]. An important advantage of this approach is that, for profinite local coefficients, the continuous cohomology of this completion coincides with the continuous cohomology of Dwyer-Friedlander [9].

Although we had been motivated by étale homotopy theory, we have postponed this application to the last section of the paper. As in [25], we consider a profinite étale topological type of a locally noetherian scheme based on the work of Artin-Mazur and Friedlander. The resulting profinite space has three main advantages. First, its cohomology agrees with the continuous étale cohomology for a locally constant profinite sheaf defined by Dwyer-Friedlander in [9] and by Jannsen in [19]. Second, it provides a rigid model for the profinite higher étale homotopy groups of a scheme first defined in [1]. Third, it can be used for an étale realization functor of the motivic stable homotopy category. We discuss this last point briefly at the end of the paper, indicating that the étale realization also yields a derived functor from the flasque model structure of [18].

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2. Homotopy Theory of Profinite Spaces

For a category $\mathcal{C}$ with small limits, the pro-category of $\mathcal{C}$, denoted pro-$\mathcal{C}$, has as objects all cofiltering diagrams $X : I \to \mathcal{C}$. Its sets of morphisms are defined as

$$\mathrm{Hom}_{\mathrm{pro-}\mathcal{C}}(X,Y) := \lim_{j\in J} \operatorname*{colim}_{i\in I} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j).$$

A constant pro-object is one indexed by the category with one object and one identity map. The functor sending an object X of $\mathcal{C}$ to the constant pro-object with value X makes $\mathcal{C}$ a full subcategory of pro-$\mathcal{C}$. The right adjoint of this embedding is the limit functor lim: pro-$\mathcal{C} \to \mathcal{C}$, which sends a pro-object X to the limit in $\mathcal{C}$ of the diagram corresponding to X.

Let $\mathcal{E}$ denote the category of sets and let $\mathcal{F}$ be the full subcategory of finite sets. Let $\hat{\mathcal{E}}$ be the category of compact and totally disconnected topological spaces. We may identify $\mathcal{F}$ with a full subcategory of $\hat{\mathcal{E}}$ in the obvious way. The limit functor lim: pro-$\mathcal{F} \to \hat{\mathcal{E}}$ is an equivalence of categories.

We denote by $\hat{\mathcal{S}}$ (resp. $\mathcal{S}$) the category of simplicial objects in $\hat{\mathcal{E}}$ (resp. simplicial sets). The objects of $\hat{\mathcal{S}}$ (resp. $\mathcal{S}$) will be called *profinite spaces* (resp. *spaces*). The forgetful functor $\hat{\mathcal{E}} \to \mathcal{E}$ admits a left adjoint $\hat{(\cdot)} : \mathcal{E} \to \hat{\mathcal{E}}$. It induces a functor $\hat{(\cdot)} : \mathcal{S} \to \hat{\mathcal{S}}$, which is called *profinite completion*. It is left adjoint to the forgetful functor $|\cdot| : \hat{\mathcal{S}} \to \mathcal{S}$ which sends a profinite space to its underlying simplicial set.

For a profinite space X we define the set $\mathcal{R}(X)$ of simplicial open equivalence relations on X. An element R of $\mathcal{R}(X)$ is a simplicial profinite subset of the product $X \times X$ such that, in each degree n, R_n is an equivalence relation on X_n and an open subset of $X_n \times X_n$. It is ordered by inclusion. For every element R of $\mathcal{R}(X)$, the quotient X/R is a simplicial finite set and the map $X \to X/R$ is a map of profinite spaces. The canonical map $X \to \lim_{R\in\mathcal{R}(X)} X/R$ is an isomorphism in $\hat{\mathcal{S}}$, cf. [23], Lemme 1. Nevertheless, Isaksen pointed out that $\hat{\mathcal{S}}$ is not equivalent to the category of pro-objects of finite simplicial sets.

Let X be a profinite space. The continuous cohomology $H^*(X;\pi)$ of X with coefficients in the topological abelian group π is defined as the cohomology of the complex $C^*(X;\pi)$ of continuous cochains of X with values in π, i.e. $C^n(X;\pi)$ denotes the set $\mathrm{Hom}_{\hat{\mathcal{E}}}(X_n,\pi)$ of continuous maps $\alpha : X_n \to \pi$ and the differentials $\delta^n : C^n(X;\pi) \to C^{n+1}(X;\pi)$ are the morphisms associating to α the map $\sum_{i=0}^{n+1} \alpha \circ d_i$, where d_i denotes the ith face map of X. If π is a finite abelian group and Z a simplicial set, then the cohomologies $H^*(Z;\pi)$ and $H^*(\hat{Z};\pi)$ are canonically isomorphic.

If G is an arbitrary profinite group, we may still define the first cohomology of X with coefficients in G as done by Morel in [23] p. 355. The functor $X \mapsto \mathrm{Hom}_{\hat{\mathcal{E}}}(X_0, G)$ is represented in $\hat{\mathcal{S}}$ by a profinite space EG. We define the 1-cocycles $Z^1(X;G)$ to be the set of continuous maps $f : X_1 \to G$ such that $f(d_0x)f(d_2x) = f(d_1x)$ for every $x \in X_1$. The functor $X \mapsto Z^1(X;G)$ is represented by a profinite space BG. Explicit constructions of EG and BG may be given in the standard way, see [20]. Furthermore, there is a map $\delta : \mathrm{Hom}_{\hat{\mathcal{S}}}(X, EG) \to Z^1(X;G) \cong \mathrm{Hom}_{\hat{\mathcal{S}}}(X, BG)$ which sends $f : X_0 \to G$ to the 1-cocycle $x \mapsto \delta f(x) = f(d_0x)f(d_1x)^{-1}$. We define $B^1(X;G)$ to be the image of δ in $Z^1(X;G)$ and we define the pointed set $H^1(X;G)$ to be the quotient $Z^1(X;G)/B^1(X;G)$. Finally, if X is a profinite space, we define π_0X to be the coequalizer in $\hat{\mathcal{E}}$ of the diagram $d_0, d_1 : X_1 \rightrightarrows X_0$.

2.1. Profinite fundamental groups. For the definition of a profinite fundamental group of a profinite space X, we follow the ideas of Grothendieck in [14]. As in [12], we denote by R/X the category of coverings of X, i.e. the full subcategory of $\hat{\mathcal{S}}/X$ whose objects are the morphisms $p : E \to X$ such that for each commutative diagram

(1)
$$\begin{tikzcd} \Delta[0] \arrow[r, "u"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r, "v"'] & X \end{tikzcd}$$

there is a unique morphism $s : \Delta[n] \to E$ satisfying $p \circ s = v$ and $s \circ i = u$.
Let $g : X' \to X$ be a morphism of $\hat{\mathcal{S}}/X$ and $p : E \to X$ be a covering of X. The cartesian square

(2)
$$\begin{tikzcd} X' \times_X E \arrow[r] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ X' \arrow[r, "g"'] & X \end{tikzcd}$$

defines a covering p' of X'. The correspondence $R/g : R/X \to R/X'$ so defined is a covariant functor.
Let $x \in X_0$ be a vertex of X and let $\tilde{x} : \Delta[0] \to X$ be the corresponding continuous map. Since there is only one morphism $\Delta[n] \to \Delta[0]$ for each n, it follows from the definitions that each simplex of a covering E of $\Delta[0]$ is determined by one of its vertices, and hence $E = \mathrm{sk}_0 E$. Thus we can identify the category $R/\Delta[0]$ with the category of profinite sets. The functor $R/\tilde{x}$ is the fiber functor of R/X over x and we consider its range in $\hat{\mathcal{E}}$. As in [12] App. I 2.2, it is easy to show that a morphism $p : E \to X$ in $\hat{\mathcal{S}}$ is a covering if and only if it is a locally trivial morphism whose fibers are constant simplicial profinite sets.
In order to define the fundamental group of X at $x \in X_0$ we consider the full subcategory R_f/X of R/X of coverings with finite fibers together with the fiber functor $R_f/\tilde{x}$. We call an object of R_f/X a finite covering of X. The

pair $(R_f/X, R/\tilde{x})$ obviously satisfies the axioms $(G1)$ to $(G6)$ of [14] Expose V §4. Hence $R/\tilde{x}$ is pro-representable by a pro-object $(X_i)_{i\in I}$ in R_f/X, where the X_i are Galois, i.e. the action of $\mathrm{Aut}(X_i)$ on the fiber of X_i at x is simply transitive. The corresponding limit of this pro-object defines an element $\tilde{X}$ in R/X of profinite coverings of X and may be considered as the universal covering of X at x. We denote the automorphism group of $\tilde{X}$ by $\pi_1(X,x)$ and call it *the profinite fundamental group of X at x*. It has a canonical profinite structure as the limit of the finite automorphism groups $\mathrm{Aut}(X_i)$.
For varying x and y in X_0, we have two corresponding fiber functors R/x and R/y and representing objects $(\tilde{X},x)$ and $(\tilde{X},y)$. The morphisms between them are in fact isomorphisms. Hence we may consider the fundamental groupoid ΠX of X whose objects are the vertices of X and whose morphisms are the sets $\mathrm{Hom}((\tilde{X},x),(\tilde{X},y))$. Now let X be a pointed simplicial set and let $\hat{X}$ be its profinite completion. The finite coverings of X and $\hat{X}$ agree as coverings in $\mathcal{S}$. Since the automorphism group of a finite covering of X corresponds to a finite quotient of $\pi_1(X)$ and since $\pi_1(\hat{X})$ is the limit over these finite groups, we deduce the following result.

Proposition 2.1. *For a pointed simplicial set X, the canonical map from the profinite group completion of $\pi_1(X)$ to $\pi_1(\hat{X})$ is an isomorphism, i.e. $\widehat{\pi_1(X)} \cong \pi_1(\hat{X})$ as profinite groups.*

Moreover, we get a full description of coverings in $\hat{\mathcal{S}}$.

Proposition 2.2. *Let X be a connected pointed profinite space with fundamental group $\pi := \pi_1(X)$. Then the functor sending a profinite covering to its fiber is an equivalence between the category of profinite coverings of X and the category of profinite sets with a continuous π-action. Its inverse is given by $S \mapsto P \times_\pi S$.*

Proof. Since X is connected, a covering of X is determined up to isomorphism by its fiber as a π-set. Hence the category of pro-objects of R_f/X is equivalent to the category R/X of profinite coverings, i.e. the limit functor pro$-R_f/X \xrightarrow{\sim} R/X$ is an equivalence. Now the assertion follows from [14], Exposé V, Théorème 4.1 and Corollaire 5.9. □

Corollary 2.3. *Let X be a connected pointed profinite space. There is a bijective correspondence between the sets of profinite coverings of X and closed subgroups of $\pi_1(X)$.*

2.2. Local coefficient systems. Let Γ be a profinite groupoid, i.e. a small category whose morphisms are all isomorphisms and whose set of objects and morphisms are profinite sets. A natural example is ΠX for a profinite space X. A (topological) local coefficient system $\mathcal{M}$ on Γ is a contravariant functor from Γ to topological abelian groups. A morphism $(\mathcal{M},\Gamma) \to (\mathcal{M}',\Gamma')$ of local coefficient systems is a functor $f:\Gamma\to\Gamma'$ and natural transformation $\mathcal{M}'\to\mathcal{M}\circ f$. We call $\mathcal{M}$ a profinite local coefficient system if it takes values

in the category of profinite abelian groups. A local coefficient system $\mathcal{M}$ on a profinite space X is a coefficient system on the fundamental groupoid $\Gamma = \Pi X$ of X.
In order to define the continuous cohomology of a profinite space with local coefficients we use one alternative characterization of cohomology with local coefficients of a simplicial set following Goerss and Jardine [13]. We recall briefly the constructions of [13] VI §4. Let Γ be a profinite groupoid and let $B\Gamma$ be its profinite classifying space. If γ is an object of Γ, we form the category $\Gamma \downarrow \gamma$ whose objects are morphisms $\gamma' \to \gamma$ and whose morphisms are commutative diagrams in Γ. The forgetful functor $\Gamma \downarrow \gamma \to \Gamma$ induces a map $\pi_y : B(\Gamma \downarrow \gamma) \to B\Gamma$. A map $\gamma_1 \to \gamma_2$ induces a functor $B(\Gamma \downarrow \gamma_1) \to B(\Gamma \downarrow \gamma_2)$ which commutes with the forgetful functor. Hence we get a functor $\Gamma \to \hat{\mathcal{S}}$ defined by $\gamma \mapsto B(\Gamma \downarrow \gamma)$ which is fibered over $B\Gamma$.
Now let $\Phi : X \to B\Gamma$ be a map in $\hat{\mathcal{S}}$. One defines a collection of spaces $\tilde{X}_\gamma$ by forming pullbacks

$$\begin{tikzcd} \tilde{X}_\gamma \arrow[r] \arrow[d] & B(\Gamma \downarrow \gamma) \arrow[d] \\ X \arrow[r] & B\Gamma \end{tikzcd}$$

and gets a functor $\Gamma \to \hat{\mathcal{S}}$, called covering system for Φ. If Γ is ΠX and Φ is the canonical map, then $\tilde{X}_\gamma$ is just the universal covering $(\tilde{X}, x)$ for $x = \gamma$.
Moreover, if $Y : \Gamma \to \hat{\mathcal{S}}$ is a functor and $\mathcal{M}$ is a topological local coefficient system, there is a corresponding cochain complex $\hom_\Gamma(Y, \mathcal{M})$, having n-cochains given by the group of $\hom_\Gamma(Y_n, \mathcal{M})$ of all *continuous* natural transformations, i.e. for every $\gamma \in \Gamma$ we consider only the continuous maps $Y_n(\gamma) \to \mathcal{M}(\gamma)$ functorial in γ. The differentials are given by the alternating sum of the face maps as above. For $Y = \tilde{X}$, we denote this cochain complex by $C^*_\Gamma(X, \mathcal{M}) := \hom_\Gamma(\tilde{X}, \mathcal{M})$.

Definition 2.4. *For a topological local coefficient system $\mathcal{M}$ on Γ and a map $X \to B\Gamma$ in $\hat{\mathcal{S}}$, we define the* continuous cohomology of X with coefficients in $\mathcal{M}$*, denoted by $H^*_\Gamma(X, \mathcal{M})$, to be the cohomology of the cochain complex $C^*_\Gamma(X, \mathcal{M})$.*

If $\mathcal{M}$ is a local system on $\Gamma = \Pi X$, we write $H^*(X, \mathcal{M})$ for $H^*_{\Pi X}(X, \mathcal{M})$. If G is a profinite group and $X = BG$ is its classifying space, then a topological local coefficient system $\mathcal{M}$ on BG corresponds to a topological G-module M. The continuous cohomology of BG with coefficients in $\mathcal{M}$ equals then the continuous cohomology of G with coefficients in M as defined in [31], i.e.

$$H^*(BG, \mathcal{M}) = H^*(G, M).$$

The following proposition is a standard result. We restate it in our setting of profinite groups and continuous cohomology. The proof follows the arguments for [21] Theorem $8^{\mathrm{bis}}.9$.

Proposition 2.5. *Let Γ be a connected profinite groupoid, let $X \to B\Gamma$ be a profinite space over $B\Gamma$ and let $\mathcal{M}$ be a local coefficient system on Γ of topological abelian groups. For an object γ of Γ we denote by π the profinite group $\mathrm{Hom}_\Gamma(\gamma,\gamma)$. Let $p : \tilde{X} \to X$ be the corresponding covering space. Then π acts continuously on the discrete abelian group $H^q(\tilde{X};p^*\mathcal{M})$ and there is a strongly converging Cartan-Leray spectral sequence*

$$E_2^{p,q} = H^p(\pi; H^q(\tilde{X};p^*\mathcal{M})) \Rightarrow H_\Gamma^{p+q}(X;\mathcal{M}).$$

Proof. Since Γ is connected, $\mathcal{M}$ corresponds to a profinite π-module M. Moreover, the set $C^q(\tilde{X};p^*\mathcal{M})$ equals the set of continuous cochains $C^q(\tilde{X};M)$ denoted by C^q for short. We denote by $F^* \to \hat{\mathbb{Z}} \to 0$ a free profinite resolution of $\hat{\mathbb{Z}}$ by right π-modules, where $\hat{\mathbb{Z}}$ is considered as a trivial π-module. We define the double complex $\mathcal{C}^{*,*}$ by $\mathcal{C}^{p,q} := \mathrm{Hom}(F^p, C^q)$, $\delta_F \otimes 1 + (-1)^q 1 \otimes \delta_C$, where the Hom-set is taken in the category of continuous π-modules.
First, we filter $\mathcal{C}^{*,*}$ row-wise and get for $E_0^{*,q} = \mathrm{Hom}_{\pi-\mathrm{modules}}(F^*, C^q)$, the complex computing $H^*(\pi;C^q)$. Since C^q is a free π-module, the E_1-terms are concentrated in the 0-column, where we get $H^0(\pi, C^q) = C^q(\tilde{X};M)^\pi$, the π-fixed points of C^q. By [21] Theorem 2.15, the E_2-terms for this filtration are the continuous π-equivariant cohomology groups $H_\pi^*(\tilde{X};M)$ and the spectral sequence degenerates at E_2. Since Γ is connected, these cohomology groups are canonically isomorphic to $H_\Gamma^{p+q}(X;\mathcal{M})$.
Now we filter $\mathcal{C}^{*,*}$ column-wise and we get $E_0^{p,*} = \mathrm{Hom}_\pi(F^p, C^*)$. Since F^p is a free π-module, we may identify the corresponding $E_1^{p,*}$ with the complex computing $H_{\mathrm{cont}}^*(\pi; H^*(\tilde{X};M))$. Both spectral sequences strongly converge to the same target and the assertion is proved. □

2.3. The model structure on profinite spaces.

Definition 2.6. *A morphism $f : X \to Y$ in $\hat{\mathcal{S}}$ is called a* weak equivalence *if the induced map $f_* : \pi_0(X) \to \pi_0(Y)$ is an isomorphism of profinite sets, for every vertex $x \in X_0$ the map $f_* : \pi_1(X,x) \to \pi_1(Y, f(x))$ is an isomorphism of profinite groups and $f^* : H^q(Y;\mathcal{M}) \to H^q(X; f^*\mathcal{M})$ is an isomorphism for every local coefficient system $\mathcal{M}$ of finite abelian groups on Y for every $q \geq 0$.*

We want to show that this class of weak equivalences fits into a simplicial fibrantly generated left proper model structure on $\hat{\mathcal{S}}$. For every natural number $n \geq 0$ we choose a finite set with n elements, e.g. the set $\{0,1,\ldots,n-1\}$, as a representative of the isomorphism class of sets with n elements. We denote the set of these representatives by $\mathcal{T}$. Moreover, for every isomorphism class of finite groups, we choose a representative with underlying set $\{0,1,\ldots,n-1\}$. Hence for each n we have chosen as many groups as there are relations on the set $\{0,1,\ldots,n-1\}$. This ensures that the collection of these representatives forms a set which we denote by $\mathcal{G}$.

Let P and Q be the following two sets of morphisms:

P consisting of $EG \to BG, BG \to *,\ L(M,n) \to K(M,n+1)$,
$K(M,n) \to *,\ K(S,0) \to *$
for every finite set $S \in \mathcal{T},$ every finite group $G \in \mathcal{G}$,
every finite abelian group $M \in \mathcal{G}$ and every $n \geq 0$;

Q consisting of $EG \to *,\ L(M,n) \to *$ for every finite group $G \in \mathcal{G}$,
every finite abelian group $M \in \mathcal{G}$ and every $n \geq 0$.

Lemma 2.7. *The underlying profinite set of a profinite group G is an injective object in $\hat{\mathcal{E}}$.*

Proof. This can be deduced from Proposition 1 of [30]. Let $X \hookrightarrow Y$ be a monomorphism in $\hat{\mathcal{S}}$ and let $f : X \to G$ be a map in $\hat{\mathcal{E}}$. Since finite sets are injective objects in $\hat{\mathcal{E}}$ by [23] Lemme 2, there is a lift $Y \to G/U$ for every open (and hence closed) normal subgroup U of G. Let N be the set of pairs (S,s) of closed subgroups S of G such that there is a lift $s : Y \to G/S$ making the diagram

$$\begin{tikzcd} X \arrow[r, "f/S"] \arrow[d, "i"'] & G/S \\ Y \arrow[ur, "s"'] & \end{tikzcd}$$

commute. The set N contains the open normal subgroups and has a natural ordering given by $(S,s) \geq (S',s')$ if $S \subseteq S'$ and $s : Y \to G/S$ is the composite of s' and $G/S' \to G/S$. As shown in [30], N is an inductively ordered set and has a maximal element by Zorn's Lemma. We have to show that a maximal element (S,s) of N satisfies $S = \{1\}$.
Suppose $S \neq \{1\}$. Then there is an open subgroup U of G such that $S \cap U \neq S$ and $S/S\cap U$ is a finite group. By [30], Proposition 1, $G/S\cap U$ is isomorphic as a profinite set to the product $G/S \times S/S\cap U$. The map $f/S\cap U : X \to G/S\cap U \cong G/S \times S/S \cap U$ induces a compatible map $X \to S/S \cap U$. Since $S/S \cap U$ is a finite set, it is an injective object in $\hat{\mathcal{E}}$ and there is a lift $t : Y \to S/S \cap U$. Hence s and t define a lift $\tilde{s} : Y \to G/S \cap U$ in contradiction to the maximality of (S,s). Hence S is trivial and there is a lift of the initial map f.

□

Lemma 2.8. *1) The morphisms in Q have the right lifting property with respect to all monomorphisms. 2) The morphisms in P have the right lifting property with respect to all monomorphisms that are also weak equivalences.*

Proof. 1) Let $X \hookrightarrow Y$ be a monomorphism in $\hat{\mathcal{S}}$. We have to show that every $X \to EG$, resp. $X \to L(M,n)$, can be lifted to a map $Y \to EG$, resp. $Y \to L(M,n)$, in $\hat{\mathcal{S}}$. Hence we must prove that every map $X_n \to G$ may be lifted to a map $Y_n \to G$ in $\hat{\mathcal{E}}$ for every profinite group G. This follows from Lemma 2.7.
2) Let $X \hookrightarrow Y$ be a monomorphism in $\hat{\mathcal{S}}$ that is also a weak equivalence.

Let G be a finite group, which is supposed to be abelian if $n \geq 2$. We know by 1) that the morphism of complexes $C^*(Y;G) \to C^*(X;G)$ is surjective. By assumption, it also induces an isomorphism on the cohomology. Hence the maps $Z^n(Y;G) \to Z^n(X;G)$ and $C^n(Y;G) \to C^n(X;G) \times_{Z^{n+1}(X;G)} Z^{n+1}(Y;G)$ are surjective and the maps $L(G,n) \to K(G,n+1)$ and $K(G,n) \to *$ have the desired right lifting property.
For $K(S,0) \to *$ and a given map $X \to K(S,0)$, we recall that $K(S,0)_n$ is equal to S in each dimension and all face and degeneracy maps are identities. Hence a map $X \to K(S,0)$ in $\hat{\mathcal{S}}$ is completely determined by its values on $\pi_0 X$. Since f induces an isomorphism on π_0, there is a lift $Y \to K(S,0)$. Hence $K(S,0) \to *$ also has the desired right lifting property. □

We remind the reader of the following definitions of [23] p. 360. Let G be a simplicial profinite group and let E be a profinite G-space. We say that E is a principal profinite G-space if, for every n, the profinite G_n-set E_n is free.
A *principal G-fibration with base X* is a profinite G-space E and a morphism $f : E \to X$ that induces an isomorphism $E/G \cong X$. We denote by $\Phi^G(X)$ the set of isomorphism classes of principal G-fibrations with base X. The correspondence $X \mapsto \Phi^G(X)$ defines via pullback a contravariant functor $\hat{\mathcal{S}}^{\mathrm{op}} \to \mathcal{E}$.

Lemma 2.9. *Let X be a connected profinite space and let $x \in X_0$ be a vertex. Let G be a profinite group and let $\mathrm{Hom}(\pi_1(X,x),G)_G$ be the set of outer continuous homomorphisms. Then we have a natural isomorphism*

$$\phi : H^1(X;G) \xrightarrow{\cong} \mathrm{Hom}(\pi_1(X,x),G)_G.$$

Proof. Recall that $H^1(X;G)$ equals the quotient of $\mathrm{Hom}_{\hat{\mathcal{S}}}(X,BG)$ modulo maps that are induced by the canonical principal G-fibration $EG \to BG$. The map ϕ is defined as follows: Given a map $f : X \to BG$ we consider the induced map $X \times_G EG \to X$. This is a covering on which G acts freely. By Corollary 2.3, there is a quotient Q of $\pi = \pi_1(X,x)$ such that Q acts on the fibre G of $X \times_G EG \to X$. This action defines a homomorphism of profinite groups $\pi \to G$ up to inner automorphisms.
We define an inverse ψ for ϕ. Let $\alpha : \pi \to G$ be a homomorphism of profinite groups up to inner automorphisms. Again by Corollary 2.3, there is a covering $X(G) \to X$ on which G acts freely. Since its total space is cofibrant and $EG \to *$ has the left lifting property with respect to all cofibrations, there is a map $X(G) \to EG$. The corresponding quotient map is the map $\psi(\alpha) : X \to BG$. One can easily check that ϕ and ψ are mutually inverse to each other. □

The proof of the previous lemma also explains the following classification of principal G-fibrations.

Proposition 2.10. *Let G be a simplicial profinite group. For any profinite space X, the map*

$$\theta : H^1(X;G) \to \Phi^G(X),$$

sending the image of $f : X \to BG$ in $H^1(X;G)$ to the pullback of $EG \to BG$ along f, is a bijection.

Proposition 2.11. *Let $f : X \to Y$ be a map of profinite spaces. If $x \in X$ is a 0-simplex, let $y = f(x)$ and $p : (\tilde{X}, \tilde{x}) \to (X, x)$, resp. $q : (\tilde{Y}, \tilde{y}) \to (Y, y)$, be the universal coverings and $\tilde{f} : \tilde{X} \to \tilde{Y}$ be the unique covering of f with $\tilde{f}(\tilde{x}) = \tilde{y}$. The following assertions are equivalent:*
1) The map f is a weak equivalence in $\hat{\mathcal{S}}$ in the sense of Definition 2.6.
2) The induced maps $f^0 : H^0(Y;S) \to H^0(X;S)$ for every finite set S, $f^1 : H^1(Y;G) \to H^1(X;G)$ for every every finite group G and $f^* : H^q(Y;\mathcal{M}) \to H^q(X;f^*\mathcal{M})$ for every local coefficient system $\mathcal{M}$ of finite abelian groups on Y for every $q \geq 0$ are all isomorphisms.
3) The induced map $f_* : \pi_0(X) \to \pi_0(Y)$ is an isomorphism of profinite sets, $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is an isomorphism of profinite groups and the maps $\tilde{f}^* : H^q((\tilde{Y}, f(x)); M) \to H^q((\tilde{X}, x); M)$ are isomorphisms for every finite abelian group M for every $q \geq 0$ and every 0-simplex $x \in X_0$.

Proof. From $H^0(X;S) = \mathrm{Hom}_{\hat{\mathcal{E}}}(\pi_0(X), S)$ for every finite set S, we conclude that $\pi_0(f)$ is an isomorphism if and only if $H^0(f, S)$ is an isomorphism for every finite set S. From the previous lemma we get that $\pi_1(f)$ is an isomorphism if and only if $H^1(f, G)$ is an isomorphism for every finite group G. Hence (1) and (2) are equivalent.
In order to show (3) $\Rightarrow$ (1), we may assume that X and Y are connected profinite spaces and that $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is an isomorphism of profinite groups for every vertex $x \in X_0$. Let $x_0 \in X$ be a fixed 0-simplex and set $\pi := \pi_1(X, x_0) = \pi_1(Y, f(x_0))$. If $\mathcal{M}$ is a local coefficient system on Y, then there is a morphism of Cartan-Leray spectral sequences of Proposition 2.5

$$\begin{array}{rccc} E_2^{p,q} = & H^p(\pi, H^q(\tilde{Y}, q^*\mathcal{M})) & \Longrightarrow & H^{p+q}(Y, \mathcal{M}) \\ & \downarrow & & \downarrow \\ E_2^{p,q} = & H^p(\pi, H^q(\tilde{X}, p^*f^*\mathcal{M})) & \Longrightarrow & H^{p+q}(X, f^*\mathcal{M}). \end{array}$$

If we assume (3), then the map on the E_2-terms is an isomorphism and we get an isomorphism on the abutments. Hence (3) implies (1).
That (1) implies (3) follows from the definition of cohomology with local coefficients. $\square$

The following theorem had already been expected by Fabien Morel, see [23] §1.3.

Theorem 2.12. *There is a left proper fibrantly generated model structure on $\hat{\mathcal{S}}$ with the weak equivalences of Definition 2.6 for which P is the set of generating fibrations and Q is the set of generating trivial fibrations. The cofibrations are the levelwise monomorphisms. We denote the homotopy category by $\hat{\mathcal{H}}$.*

Proof. In order to prove that there is a fibrantly generated model structure, we check the four conditions of the dual of Kan's Theorem 11.3.1 in [15]. Since we use cosmall instead of small objects, it suffices that $\hat{\mathcal{S}}$ is closed under small limits and finite colimits. It is clear that the weak equivalences satisfy the 2-out-of-3 property and are closed under retracts. We denote by Q-cocell the subcategory of relative Q-cocell complexes consisting of limits of pullbacks of elements of Q. We write P-proj for the maps having the left lifting property with respect to all maps in P and P-fib for the maps having the right lifting property with respect to all maps in P-proj. Now we check the remaining hypotheses of Kan's Theorem 11.3.1 in [15].

(1) We have to show that the codomains of the maps in P and Q are cosmall relative to P-cocell and Q-cocell, respectively. This is clear for the terminal object $*$. It remains to check that the objects $K(M,n)$ and BG are cosmall relative to Q-cocell. By definition of cosmallness we have to show that the canonical map

$$f : \operatorname*{colim}_{\alpha} \mathrm{Hom}_{\hat{\mathcal{S}}}(Y_\alpha, K(M,n)) \to \mathrm{Hom}_{\hat{\mathcal{S}}}(\lim_{\alpha} Y_\alpha, K(M,n))$$

is an isomorphism for some cardinal κ, where Y_α is any projective system whose indexing category is of cardinality κ (and similarly for BG instead of $K(M,n)$). By the definition of the spaces $K(M,n)$ (resp. BG) this map is equal to the map $\mathrm{colim}_\alpha Z^n(Y_\alpha, M) \to Z^n(\lim_\alpha Y_\alpha, M)$ (resp. with $M = G$ and $n = 1$). But this map is already an isomorphism on the level of complexes C^n. For, Christensen and Isaksen have shown in [4], Lemma 3.4, that $\mathrm{colim}_\alpha \mathrm{Hom}_{\hat{\mathcal{E}}}(Y_\alpha, M) \cong \mathrm{Hom}_{\hat{\mathcal{E}}}(\lim_\alpha Y_\alpha, M)$ (resp. $M = G$, $n = 1$), since $\hat{\mathcal{E}}$ is equivalent to the pro-category of finite sets.

(2) We have to show that every Q-fibration is both a P-fibration and a weak equivalence. Let $i : A \to B$ be a map in P-proj. As in the proof of Lemma 2.8, this implies on the one hand that $i^* : Z^n(B,M) \to Z^n(A,M)$ (or G instead of M and $n = 1$) is surjective and on the other hand that $C^n(B,M) \to C^n(A,M) \times_{Z^{n+1}(A,M)} Z^{n+1}(B,M)$ is surjective for all $n \geq 0$ and every abelian finite group M (or $M = G$ and $n = 0$). It is easy to see that this implies that $i^* : C^n(B,M) \to C^n(A,M)$ (or $M = G$ and $n = 1$) is surjective as well. Hence i is an element in Q-proj. So P-proj $\subset$ Q-proj and hence Q-fib $\subset$ P-fib.

Furthermore, if i is a monomorphism in each dimension, then i is an element of Q-proj by Lemma 2.8. Hence Q-fib is contained in the class of maps that have the right lifting property with respect to all monomorphisms. By Lemme 3 of [23] this implies that the maps in Q-fib are simplicial homotopy equivalences and hence also weak equivalences.

(3) We have seen that every P-projective map is a Q-projective map. It remains to show P-proj $\subseteq W$. Let $f : A \to B$ be a map in P-proj. By the definition of P-proj, the maps $f^* : Z^1(B,G) \to Z^1(A,G)$ and $C^0(B,G) \to C^0(A,G) \times_{Z^1(A,G)} Z^1(B,G)$ are surjective. The latter implies that, if $f^*(\beta)$ is a boundary for an element $\beta \in Z^1(B,G)$, then β is already a boundary. Hence f induces an isomorphism $f^1 : H^1(B;G) \cong H^1(A;G)$. Moreover, if f has the left lifting property with respect to maps in P, then the same holds for its covering map $\tilde{f}$. The same argument for an abelian finite group M instead of

G shows that for every $n \geq 1$ the map $\tilde{f}^n : H^n(\tilde{B}; M) \to H^n(\tilde{A}; M)$ is an isomorphism. For $n = 0$, the left lifting property with respect to $K(S, 0) \to *$ for any finite set S, not only shows that $H^0(f; S) = Z^0(f; S)$ is surjective but that it is also an isomorphism since any two liftings $B \to K(S, 0)$ are simplicially homotopic and hence they agree. By Proposition 2.11, this implies that f is a weak equivalence.
(4) The remaining point is to show that $W \cap Q$-proj $\subseteq P$-proj. But this follows in almost the same way as we proved Lemma 2.8.
This proves that we have found a fibrantly generated model structure on $\hat{\mathcal{S}}$. It remains to show that it is left proper. In fact, the cofibrations are the maps in Q-proj and by Lemma 2.8 this class includes the monomorphisms. Hence every object in $\hat{\mathcal{S}}$ is cofibrant which implies that the model structure is left proper. That the cofibrations are exactly the monomorphisms will be proved in the following lemma. □

Lemma 2.13. *A map in $\hat{\mathcal{S}}$ is a cofibration if and only if it is a levelwise monomorphism. In particular, the maps $EG \to *$ and $L(M, n) \to *$ are trivial fibrations in $\hat{\mathcal{S}}$ for every profinite group G, every abelian profinite group M and every $n \geq 0$.*

Proof. We have proven in the theorem that the class of cofibrations equals the class of maps having the left lifting property with respect to all maps in Q. We have seen that if i is a monomorphism in each dimension, then i is a cofibration by Lemma 2.8.
So let $i : A \to B$ be a cofibration and suppose we have a map $f : A_n \to M$ in $\hat{\mathcal{E}}$ for an abelian profinite group. Then the induced map $\mathrm{Hom}_{\hat{\mathcal{E}}}(B_n, M/U) \to \mathrm{Hom}_{\hat{\mathcal{E}}}(A_n, M/U)$ is surjective for every $n \geq 0$ and every open (and closed) normal subgroup U of M, since $L(M/U, n)$ is an element of Q. Hence the set N of pairs (S, s) of closed subgroups S of M such that there is a lift $s : B_n \to M/S$ making the diagram

$$\begin{tikzcd} A_n \arrow[r, "f/S"] \arrow[d, "i"'] & M/S \\ B_n \arrow[ur, "s"'] & \end{tikzcd}$$

commute contains the open normal subgroups of M. Then the same argument as in Lemma 2.7 shows that there is in fact a lift $B_n \to M$. Thus the induced map $i^* : \mathrm{Hom}_{\hat{\mathcal{E}}}(B_n, M) \to \mathrm{Hom}_{\hat{\mathcal{E}}}(A_n, M)$ is surjective for every abelian profinite group M and every $n \geq 0$. The same argument works for an arbitrary profinite group G and $n = 0$.
By choosing M to be the free profinite group on the set A_n, see e.g. [27], we see that i must be injective in every dimension. Hence Q-proj equals the class of dimensionwise monomorphisms in $\hat{\mathcal{S}}$. □

An important property of this model structure is the following homotopy invariance of limits in $\hat{\mathcal{S}}$.

Proposition 2.14. *Let $f : X \to Y$ be a map of cofiltering diagrams $I \to \hat{\mathcal{S}}$ of profinite spaces such that each $f_i : X_i \to Y_i$ is a weak equivalence. Then $\lim f_i : \lim X_i \to \lim Y_i$ is a weak equivalence in $\hat{\mathcal{S}}$. In particular, the limit functor induces a functor on the homotopy category of $\hat{\mathcal{S}}$.*

Proof. It follows from the fact that π_0 and π_1 commute with cofiltering limits that the profinite groupoids $\Pi(\lim_i X_i)$ and $\Pi(\lim_i Y_i)$ are equivalent. This also follows from the fact that cohomology with finite coefficients transforms limits to colimits, i.e. $H^0(\lim_i X_i; S) \cong \operatorname{colim}_i H^0(X_i; S)$ for a finite set S and $H^1(\lim_i X_i; G) \cong \operatorname{colim}_i H^1(X_i; G)$ for a finite group G; hence the maps $H^0(f; S)$ and $H^1(f; G)$ are isomorphisms if each $H^0(f_i; S)$ and $H^1(f_i; G)$ is an isomorphism. The cohomology with finite abelian local coefficient systems commutes with limits in the same way. By Proposition 2.11, this implies that $\lim_i f_i$ is a weak equivalence. □

There is an obvious simplicial structure on $\hat{\mathcal{S}}$, cf. [23] and [5]. The function complex $\hom_{\hat{\mathcal{S}}}(X,Y)$ for $X,Y \in \hat{\mathcal{S}}$ is the simplicial set defined in degree n by $\operatorname{Hom}_{\hat{\mathcal{S}}}(X \times \Delta[n], Y)$. It is characterized by the isomorphism

$$\operatorname{Hom}_{\hat{\mathcal{S}}}(X \times W, Y) \cong \operatorname{Hom}_{\mathcal{S}}(W, \hom_{\hat{\mathcal{S}}}(X,Y))$$

which is natural in the simplicial finite set W and in $X, Y \in \hat{\mathcal{S}}$. Moreover, if $j : A \to B$ is a cofibration and $q : X \to Y$ a fibration in $\hat{\mathcal{S}}$, then the map

$$\hom_{\hat{\mathcal{S}}}(B,X) \longrightarrow \hom_{\hat{\mathcal{S}}}(A,X) \times_{\hom_{\hat{\mathcal{S}}}(A,Y)} \hom_{\hat{\mathcal{S}}}(B,Y)$$

is a fibration of simplicial sets which is also a weak equivalence if j or q is one. In fact, by adjunction this statement is equivalent to that for every cofibration $i : V \to W$ in $\hat{\mathcal{S}}$ the map

$$(A \times W) \cup_{A \times V} (B \times V) \longrightarrow B \times W$$

is a cofibration in $\hat{\mathcal{S}}$ which is a weak equivalence if j or i is one in $\hat{\mathcal{S}}$. The first point is clear and the second point follows from π_1 commuting with products, a van Kampen type theorem for profinite π_1, and the Mayer Vietoris long exact sequence for cohomology. Hence $\hat{\mathcal{S}}$ is a simplicial model category in the sense of [26]. In particular, for $X, Y \in \hat{\mathcal{S}}$, $\hom_{\hat{\mathcal{S}}}(X,Y)$ is fibrant in $\mathcal{S}$ if Y is fibrant. Of course, there is also a simplicial model structure on the category $\hat{\mathcal{S}}_*$ of pointed profinite spaces.

Furthermore, if W is a simplicial set and X is a profinite space, then by [23] the function complex $\hom(W, X)$ has the natural structure of a profinite space as the cofiltering limit of the simplicial finite sets $\hom(W_\alpha, X/Q)$ where W_α runs through the simplicial finite subsets of W.

As an example we consider the simplicial finite set S^1, defined as $\Delta[1]$ modulo its boundary. For a pointed profinite space X, we denote its smash product $S^1 \wedge X$ with S^1 by ΣX and by ΩX the profinite space $\hom_{\hat{\mathcal{S}}_*}(S^1, X)$. For $X, Y \in \hat{\mathcal{S}}_*$, there is a natural bijection $\hom_{\hat{\mathcal{S}}_*}(\Sigma X, Y) = \hom_{\hat{\mathcal{S}}_*}(X, \Omega Y)$. Proposition 2.11 motivates the following definition for profinite higher homotopy groups.

Definition 2.15. *Let X be a pointed profinite space and let RX be a functorial fibrant replacement of X in the above model structure on $\hat{\mathcal{S}}_*$. Then we define the nth* profinite homotopy group *of X for $n \geq 2$ to be the profinite group*

$$\pi_n(X) := \pi_0(\Omega^n(RX)).$$

It remains to show that the initial definition of the profinite fundamental group fits well with the definition of the higher homotopy groups, i.e. $\pi_1(X) \cong \pi_0(\Omega(RX))$.

Lemma 2.16. *For every $X \in \hat{\mathcal{S}}$, the canonical map $\mathrm{colim}_Q\, \mathcal{C}_f(X/Q) \to \mathcal{C}_f(X)$ is bijective, where the colimit is taken over all $Q \in \mathcal{R}(X)$.*

Proof. We have to show that any finite covering $E \to X$ of X is induced by a finite covering of X/Q via the quotient map $X \to X/Q$ for some $Q \in \mathcal{R}(X)$. We may assume that $E \to X$ is a Galois covering with finite Galois group G. Now $EG \to BG$ is the universal covering of BG. Hence by Proposition 2.10, it suffices to note that $X \to BG$ is isomorphic to some quotient map $X \to X/Q$. □

Proposition 2.17. *For a pointed profinite space X, the previously defined fundamental group $\pi_1 X$ and the group $\pi_0 \Omega X$ agree as profinite groups.*

Proof. The functors π_1, π_0 and Ω commute with cofiltering limits of fibrant objects by construction. The composed map

$$X \xrightarrow{\cong} \lim_{Q \in \mathcal{R}(X)} X/Q \xrightarrow{\simeq} \lim_{Q \in \mathcal{R}(X)} R(X/Q)$$

is a weak equivalence by the homotopy invariance of cofiltering limits in $\hat{\mathcal{S}}$ of Proposition 2.14. Hence we may assume that X is a fibrant simplicial finite set. In this case, $\pi_0 \Omega X$ agrees with the usual $\pi_1 |X|$ of the underlying simplicial finite set of X and hence we know that $\pi_0 \Omega X$ is equal to the group of automorphisms of the universal covering of X. □

Since $\hat{\mathcal{S}}$ is a simplicial model category, for any profinite abelian group M, every n and $X \in \hat{\mathcal{S}}_*$ there is an isomorphism

$$H^{n-q}(X; M) = \pi_q \hom_{\hat{\mathcal{S}}_*}(X, K(M, n)),$$

where π_q denotes the usual homotopy group of the simplicial set $\hom_{\hat{\mathcal{S}}_*}(X, K(M, n))$. For an arbitrary profinite group G there is a bijection of pointed sets

$$H^1(X; G) = \pi_0 \hom_{\hat{\mathcal{S}}_*}(X, BG).$$

Let $X : I \to \mathcal{S}$ be a functor from a small cofiltering category I to simplicial sets. By [3] XI §7.1, if each X_i fibrant there is a spectral sequence involving derived limits

$$E_2^{s,t} = \begin{cases} \lim_I^s \pi_t X_i & \text{for } 0 \leq s \leq t \\ 0 & \text{else} \end{cases} \tag{3}$$

converging to $\pi_{s+t}\operatorname{holim} X$. Using [3] XI §§4-7, one can construct a homotopy limit $\operatorname{holim} X \in \hat{\mathcal{S}}$ for a small cofiltering category I and a functor $X : I \to \hat{\mathcal{S}}$.

Lemma 2.18. *Let $X : I \to \hat{\mathcal{S}}$ be a small cofiltering diagram such that each X_i is fibrant in $\hat{\mathcal{S}}$. Then there is a natural isomorphism*

$$\pi_q(\operatorname{holim} X) \cong \lim_i \pi_q(X_i).$$

Proof. Since the underlying space of a fibrant profinite space is still fibrant in $\mathcal{S}$, the above spectral sequence exists also for a diagram of profinite spaces. But in this case all homotopy groups are profinite groups. Since the inverse limit functor is exact in the category of profinite groups, cf. [27] Proposition 2.2.4, the spectral sequence (3) degenerates to a single row and implies the desired isomorphism. □

Corollary 2.19. *Let $X : I \to \hat{\mathcal{S}}$ be a cofiltering diagram of profinite spaces such that each X_i is fibrant. Then the natural map* $\lim X \to \operatorname{holim} X$ *is a weak equivalence in $\hat{\mathcal{S}}$.*

Proof. This follows directly from Lemma 2.18 and the fact that the profinite homotopy groups commute with cofiltering limits of fibrant profinite spaces. □

Corollary 2.20. *Let I be a small cofiltering category and $\underline{M}$ (resp. $\underline{G}$) be a functor from I to the category of profinite abelian groups (resp. profinite groups). Then the canonical map $K(\lim \underline{M}, n) \to \operatorname{holim} K(\underline{M}, n)$ is a weak equivalence in $\hat{\mathcal{S}}$ for every $n \geq 0$ (resp. for $n = 0, 1$).*

Proof. The lemma shows that $\pi_q \operatorname{holim} K(\underline{M}, n)$ is equal to M for $q = n$ and vanishes otherwise. Hence the canonical map $\lim K(\underline{M}, n) \to \operatorname{holim} K(\underline{M}, n)$ is a weak equivalence. The same argument holds for G and $n = 1$ using the construction of $\lim^1$ of [3] XI §6. □

Since $\hom_{\hat{\mathcal{S}}_*}(X, -)$ commutes with homotopy limits of fibrant objects, this result and the Bousfield-Kan spectral sequence (3) imply the following result on the cohomology with profinite coefficients.

Proposition 2.21. *Let $f : X \to Y$ be a map in $\hat{\mathcal{S}}$, let G and M be profinite groups and let M be abelian. If $H^1(f; G/U)$, resp. $H^*(f; M/V)$, are isomorphisms for every normal subgroup of finite index $U \leq G$, resp. $V \leq M$, then $H^1(f; G)$, resp. $H^*(f; M)$, is an isomorphism.*

Proposition 2.22. *Let M be a profinite group and suppose that the topology of M has a basis consisting of a countable chain of open subgroups $M = U_0 \geq U_1 \geq \ldots$. Then there is a natural short exact sequence for every $Y \in \hat{\mathcal{S}}$ and every $i \geq 1$ ($i = 1$ if M is not abelian)*

$$0 \to \lim_n{}^1 H^{i-1}(Y; M/U_n) \to H^i(Y; M) \to \lim_n H^i(Y; M/U_n) \to 0.$$

Proof. This is the short exact sequence of [3] XI §7.4 for $X = K(M, i)$. Proposition 2.20 identifies $\pi_i \operatorname{holim} X$ with $H^i(X; M)$. □

The hypothesis of the previous proposition is satisfied for example if M is a finitely generated profinite abelian group. Finally, for a profinite group G and a profinite abelian G-module $M = \lim_U M/U$, the isomorphism

$$H^{n-q}(G;M) = \pi_q \hom_{\hat{\mathcal{S}}/BG}(BG, K(M,n))$$

for the continuous cohomology of G yields via (3) and Proposition 2.20 a natural spectral sequence for continuous group cohomology, U running through the open normal subgroups of M:

$$E_2^{p,q} = \lim_U{}^p H^q(G;M/U) \Rightarrow H^{p+q}(G;M).$$

Finally, we can generalize Lemma 2.8.

PROPOSITION 2.23. *For every profinite group G, every profinite abelian group M and every $n \geq 0$, the canonical maps $EG \to BG$ and $L(M,n) \to K(M,n+1)$ are fibrations in $\hat{\mathcal{S}}$.*

Proof. Let $X \hookrightarrow Y$ be a trivial cofibration in $\hat{\mathcal{S}}$. Let G be a profinite group, which is supposed to be abelian if $n \geq 2$. We know by Lemma 2.13 that the morphism of complexes $C^*(Y;G) \to C^*(X;G)$ is surjective. By assumption, it induces an isomorphism on the cohomology for every finite quotient G/U. Hence by Proposition 2.21 it also induces an isomorphism on cohomology with coefficients equal to G. Hence the maps $Z^n(Y;G) \to Z^n(X;G)$ and $C^n(Y;G) \to C^n(X;G) \times_{Z^{n+1}(X;G)} Z^{n+1}(Y;G)$ are surjective and the maps $L(G,n) \to K(G,n+1)$ and $K(G,n) \to *$ have the desired right lifting property as in Lemma 2.8. □

In terms of the homotopy category we can reformulate Proposition 2.10 as follows.

PROPOSITION 2.24. *Let G be a simplicial profinite group. For any profinite space X, the map*

$$\theta : \mathrm{Hom}_{\hat{\mathcal{H}}}(X, BG) \to \Phi^G(X),$$

sending the image of f in $\mathrm{Hom}_{\hat{\mathcal{H}}}(X, BG)$ to the pullback of $EG \to BG$ along f, is a bijection.

COROLLARY 2.25. *Let G be a simplicial profinite group and let $f : E \to X$ be a principal G-fibration. Then f is also a fibration in $\hat{\mathcal{S}}$.*

Proof. Since X is cofibrant and BG fibrant, the map $X \to BG$ in $\hat{\mathcal{H}}$ corresponding to the principal fibration $E \to X$ is represented by a map in $\hat{\mathcal{S}}$. Since $E \to X$ is the pullback of $EG \to BG$ under this map and since fibrations are stable under pullbacks, the assertion follows. □

The construction of the Serre spectral sequence of Dress in [7] can be easily translated to our profinite setting, see also [5] §1.5.

Proposition 2.26. *Let B be a simply connected profinite space, let $f : E \to B$ be a fibration in $\hat{\mathcal{S}}$ with fibre F and let M be an abelian profinite group. Then there is a strongly convergent Serre spectral sequence*

$$E_2^{p,q} = H^p(B; H^q(F; M)) \Rightarrow H^{p+q}(E; M).$$

Proposition 2.27. *Let G be a profinite group and let $p : E \to X$ be a principal G-fibration in $\hat{\mathcal{S}}$. Then the canonical map $f : E \times_G EG \to X$ is a weak equivalence.*

Proof. Since $E \to X$ is locally trivial, see [13] V Lemma 2.5, it is also a covering of X with fibre G. Hence we may assume that p is a Galois covering of X with $G = \mathrm{Aut}_X(E)$. It follows from the classification of coverings that there is a short exact sequence of groups

$$1 \to \pi_1(E) \to \pi_1(X) \to G \to 1.$$

On the other hand, we have a canonical G-equivariant map $EG \times_G (E \times_G EG) = EG \times E \to E$ which induces an isomorphism on fundamental groups since $\pi_1(EG)$ is trivial. Now $EG \times_G (E \times_G EG)$ is a principal G-fibration on $E \times_G EG$ and we have a corresponding short exact sequence

$$1 \to \pi_1(EG \times E) \to \pi_1(E \times_G EG) \to G \to 1.$$

Since the sequences are functorial, we conclude that $E \times_G EG \to X$ induces an isomorphism on fundamental groups. The remaining point to check follows from the Serre spectral sequence of Proposition 2.26 associated to the map of universal coverings of the fibration $E \times_G EG \to X$. □

2.4. Profinite completion of simplicial sets. We consider the category $\mathcal{S}$ of simplicial sets with the usual model structure of [26]. We denote its homotopy category by $\mathcal{H}$.

Proposition 2.28. *1. The completion functor $\hat{(\cdot)} : \mathcal{S} \to \hat{\mathcal{S}}$ preserves weak equivalences and cofibrations.*
2. The forgetful functor $|\cdot| : \hat{\mathcal{S}} \to \mathcal{S}$ preserves fibrations and weak equivalences between fibrant objects.
3. The induced completion functor $\hat{(\cdot)} : \mathcal{H} \to \hat{\mathcal{H}}$ and the right derived functor $R|\cdot| : \hat{\mathcal{H}} \to \mathcal{H}$ form a pair of adjoint functors.

Proof. Let $f : X \to Y$ be a map of simplicial sets. If f is a monomorphism and x and x' are two distinct n-simplices of X/Q for some $Q \in \mathcal{R}(X)$, then there is a finite quotient Y/R, $R \in \mathcal{R}(Y)$ such that $f(x)$ and $f(x')$ are not equal. Hence $\hat{f} : \hat{X} \to \hat{Y}$ is a monomorphism.
If f is a weak equivalence in $\mathcal{S}$, then $\pi_0(\hat{f}) = \widehat{\pi_0(f)}$ and $\pi_1(\hat{f}) = \widehat{\pi_1(f)}$ are isomorphisms for every basepoint by Proposition 2.1. Moreover, $H^n(f; \mathcal{M})$ are isomorphisms for every finite abelian coefficient system $\mathcal{M}$ on Y and every $n \geq 0$ by [26]. Since the profinite completion of the universal covering of a space X equals the universal profinite covering of the completion $\hat{X}$, we see that for a finite local system $\mathcal{M}$ the cohomologies $H^n(X; \mathcal{M})$ and $H^n(\hat{X}; \mathcal{M})$

agree. Hence $\hat{f}$ is a weak equivalence in $\hat{\mathcal{S}}$. The second and third assertion now follow from the first one since $\hat{(\cdot)}$ and $|\cdot|$ form a pair of adjoint functors. $\square$

For a simplicial set X, we have seen that the continuous cohomology of $\hat{X}$ agrees with the cohomology of X when the coefficients are finite. But for homotopical aspects, one should consider a fibrant replacement of $\hat{X}$ in $\hat{\mathcal{S}}$ and could call this the profinite completion of X.

2.5. Homology and the Hurewicz map. We define the homology $H_*(X) := H_*(X;\hat{\mathbb{Z}})$ of a profinite space X to be the homology of the complex $C_*(X)$ consisting in degree n of the profinite groups $C_n(X) := \hat{F}_{\mathrm{ab}}(X_n)$, the free abelian profinite group on the profinite set X_n. The differentials d are the alternating sums $\sum_{i=0}^{n} d_i$ of the face maps d_i of X. If M is a profinite abelian group, then $H_*(X;M)$ is defined to be the homology of the complex $C_*(X;M) := C_*(X)\hat{\otimes}M$, where $\hat{\otimes}$ denotes the completed tensor product, see e.g. [27] §5.5. As for simplicial sets, there is a natural isomorphism of profinite groups, $H_0(X;\hat{\mathbb{Z}}) = \hat{F}_{\mathrm{ab}}(\pi_0(X))$. For a pointed space $(X,*)$, we denote by $\tilde{H}_n(X;M)$ the reduced homology given by the complex $C_*(X;M)/C_*(*;M)$. For a fibrant pointed profinite space X, by Proposition 2.17 and by definition, the homotopy groups $\pi_n(X)$ are equal to the set $\mathrm{Hom}_{\hat{\mathcal{S}}_*}(S^n,X)/\sim$ of maps modulo simplicial homotopy, where S^n denotes the simplicial finite quotient $\Delta^n/\partial\Delta^n$. Hence an element $\alpha \in \pi_n(X)$ can be represented by an element $x \in X_n$. But we can view x also as a cycle of $\tilde{C}_n(X)$ with homology class $[x] \in \tilde{H}_n(X;\hat{\mathbb{Z}})$. One can show as in the case of simplicial sets that this correspondence $\alpha \mapsto [x]$ is well defined and is even a homomorphism of groups, cf. [20] §13. We call this map $h_n : \pi_n(X) \to \tilde{H}_n(X;\hat{\mathbb{Z}})$ the Hurewicz map.

Proposition 2.29. *Let X be a connected pointed fibrant profinite space. The induced map $\overline{h}_1 : \pi_1(X)/[\pi_1(X),\pi_1(X)] \to \tilde{H}_1(X;\hat{\mathbb{Z}})$ is an isomorphism.*

Proof. This follows as in [20] §13 or [13] III §3. Since X has a strong deformation retract Z that is reduced, i.e. Z_0 consists of a single element, we may assume that X is reduced. Then $\tilde{C}_1(X) = \tilde{Z}_1(X) = \hat{F}_{\mathrm{ab}}(X_1 - \{*\})$. The quotient $X_1 \to X_1/\sim$ induces a natural epimorphism $j : \tilde{Z}_1(X) \to \pi_1/[\pi_1(X),\pi_1(X)]$. If $x \in \tilde{C}_2(X)$, then by the definition of d and the definition of the group structure on $\pi_1(X) = \mathrm{Hom}_{\hat{\mathcal{S}}_*}(S^1,X)/\sim$, we get $j \circ d(x) = 0$. Thus j induces a map $\overline{j} : \tilde{H}_1(X) \to \pi_1/[\pi_1(X),\pi_1(X)]$ and one checks easily that $\overline{j}$ and $\overline{h}$ are mutually inverse to each other. $\square$

In exactly the same way as for simplicial sets, one proves the following Hurewicz theorem, see e.g. [20] §13.

Theorem 2.30. *Let $n \geq 1$ be an integer and let X be a fibrant pointed profinite space with $\pi_q(X) = 0$ for all $q < n$. Then the Hurewicz map $h : \pi_n(X) \to \tilde{H}_n(X;\hat{\mathbb{Z}})$ is an isomorphism of profinite groups.*

Let X be a pointed space. By the universal property of profinite completion, there is canonical map $\widehat{\pi_n(X)} \to \pi_n(\hat{X})$ of profinite groups. We have seen in Proposition 2.1 that this map is always an isomorphism for $n = 1$.

Proposition 2.31. *Let X be a pointed simplicial set. Suppose that $\pi_q(X) = 0$ for $q < n$. Then $\pi_n(\hat{X})$ is the profinite completion of $\pi_n(X)$, i.e.*

$$\widehat{\pi_n(X)} \cong \pi_n(\hat{X}).$$

Proof. This follows immediately from the Hurewicz theorem for profinite spaces. □

2.6. Pro-p-model structures. Morel has shown in [23] that there is a $\mathbb{Z}/p$-model structure on $\hat{\mathcal{S}}$ for every prime number p. The weak equivalences are maps that induce isomorphisms in the continuous cohomology with $\mathbb{Z}/p$-coefficients. The cofibrations are the levelwise monomorphisms. It is also a fibrantly generated model structure. The minimal sets of generating fibrations and generating trivial fibrations are given by the canonical maps $L(\mathbb{Z}/p, n) \to K(\mathbb{Z}/p, n+1)$, $K(\mathbb{Z}/p, n) \to *$ and by the maps $L(\mathbb{Z}/p, n) \to *$ for every $n \geq 0$, respectively. Moreover, Morel proved that a principal G-fibration $E \to X$ is a fibration in this model structure for any pro-p-group G. In particular, the maps $EG \to BG$, $BG \to *$ are fibrations and the maps $EG \to *$ are trivial fibrations for any (nonabelian) pro-p-group G. Hence we would not get a different structure if we added for example the maps $EG \to BG$ for a nonabelian p-group to the generating sets of fibrations.
However, for the model structure of Theorem 2.12 we cannot skip any map in the generating sets P and Q. The arguments of [23] rely on the fact that $\mathbb{Z}/p$ is a field and that every pro-p-group has a p-central descending filtration by normal subgroups such that all subquotients are cyclic of order p. For a general profinite group there is not such a nice description.
One can describe the structure of [23] as the left Bousfield localization of the model structure of Theorem 2.12 with respect to the set of fibrant objects $K(\mathbb{Z}/p, n)$ for every $n \geq 0$. The homotopy groups $\pi_n(X) := \pi_0(\Omega^n X)$ of a fibrant profinite space X for this model structure are pro-p-groups. For a simplicial set X, $\pi_1(\hat{X})$ is the pro-p-completion of $\pi_1(X)$. If $\pi_1(X)$ is finitely generated abelian, then $\pi_1(\hat{X})$ is isomorphic to $\mathbb{Z}_p \otimes_{\mathbb{Z}} \pi_1(X)$.

Remark 2.32. *The procedure for the proof of Theorem 2.12 may be applied to every complete class $\mathcal{C}$ of finite groups. It suffices to replace the word* finite *by the appropriate additional property $\mathcal{C}$. For the class of p-groups the situation simplifies in the way indicated above.*

2.7. Profinite completion of pro-spaces. We define a completion functor $\hat{(\cdot)} : \text{pro} - \mathcal{S} \to \hat{\mathcal{S}}$ as the composite of two functors. First we apply $\hat{(\cdot)} : \mathcal{S} \to \hat{\mathcal{S}}$ levelwise, then we take the limit in $\hat{\mathcal{S}}$ of the underlying diagram. This completion functor unfortunately has no right adjoint. But it obviously preserves monomorphisms and weak equivalences by Propositions 2.14 and 2.28 when we

equip pro-$\mathcal{S}$ with the model structure of [16].
We compare the above constructions with Artin-Mazur's point of view. We denote by $\mathcal{H}_{\mathrm{fin}}$ the full subcategory of $\mathcal{H}$ of spaces whose homotopy groups are all finite. If X is either a space or a pro-space, then the functor $\mathcal{H}_{\mathrm{fin}} \to \mathcal{E}$, $Y \mapsto \mathrm{Hom}_{\mathcal{H}}(Y, X)$ is pro-representable by [1] Theorem 3.4. The corresponding pro-object $\hat{X}^{\mathrm{AM}}$ of $\mathcal{H}_{\mathrm{fin}}$ is the Artin-Mazur profinite completion of X.
Now if X is a profinite space we can naturally associate to it a pro-object X^{AM} in $\mathcal{H}_{\mathrm{fin}}$. It is defined to be the functor $\mathcal{R}(RX) \to \mathcal{H}_{\mathrm{fin}}$, $Q \mapsto RX/Q$, where RX is a fibrant replacement of X in $\hat{\mathcal{S}}$. Since RX/Q is finite in each degree, the same is true for $\Omega^k(RX/Q)$. Hence $\pi_0(\Omega^k(RX/Q))$ is finite for every $k \geq 0$. This functor $X \mapsto X^{\mathrm{AM}}$ sends weak equivalences to isomorphisms and hence factors through $\hat{\mathcal{H}}$.
The composed functor $\mathcal{H} \to \hat{\mathcal{H}} \to \mathrm{pro}-\mathcal{H}_{\mathrm{fin}}$ is isomorphic to the Artin-Mazur completion of a space. In order to show this, it suffices to check that for a space X the fundamental groups $\pi_1(\hat{X}^{\mathrm{AM}})$ and $\pi_1(\hat{X})$ are isomorphic as profinite groups and that cohomology with finite local coefficients agree. The first point follows from Proposition 2.1 and [1], Corollary 3.7; and the second from the fact that cohomology with finite local coefficients transforms limits to colimits. In particular, this implies the following result.

Proposition 2.33. *Let X be connected pointed pro-space. Then the homotopy pro-groups of the Artin-Mazur profinite completion X^{AM} and the profinite homotopy groups of $\hat{X} \in \hat{\mathcal{S}}$ agree as profinite groups, i.e. for every $n \geq 1$*

$$\pi_n(\hat{X}) \cong \pi_n(X^{\mathrm{AM}}).$$

Nevertheless, the categories $\hat{\mathcal{H}}$ and $\mathrm{pro}-\mathcal{H}_{\mathrm{fin}}$ are not equivalent as the example of Morel in [23], p. 368, shows.
Finally, we show that the continuous cohomology with profinite coefficients of the completion of a pro-space is equal to the continuous cohomology of Dwyer and Friedlander [9] Definition 2.8, which we denote by $H^n_{\mathrm{DF}}(X; \mathcal{M})$.

Proposition 2.34. *Let Γ be a profinite groupoid and let $X \to B\Gamma$ be a pro-space over $B\Gamma$. Let $\mathcal{M}$ be a profinite coefficient system on Γ. For every $n \geq 0$ there is a natural isomorphism induced by completion*

$$H^n_{\Gamma}(\hat{X}; \mathcal{M}) \cong H^n_{\mathrm{DF}}(X; \mathcal{M}).$$

Proof. We may assume that Γ is connected since we can decompose the cohomology into the product of the cohomology of the connected components. We denote by π the profinite group $\mathrm{Hom}_{\Gamma}(\gamma, \gamma)$ of an object $\gamma \in \Gamma$. The continuous cohomology of [9] is then given by

$$H^n_{\mathrm{DF}}(X; \mathcal{M}) = \pi_0 \operatorname*{holim}_j \operatorname*{colim}_s \hom_{\mathcal{S}/B\pi}(X_s, K(M_j, n))$$

where $\mathcal{M} = \{M_j\}$ is given as an inverse system of finite π-modules and $\hom_{\mathcal{S}/B\pi}$ denotes the mapping space of spaces over $B\pi$. Because of the universal property of profinite completion and since each $K(M_j, n)$ is a simplicial finite set, we get a canonical identification of mapping spaces

$$\operatorname*{colim}_s \hom_{\mathcal{S}/B\pi}(X_s, K(M_j,n)) = \hom_{\hat{\mathcal{S}}/B\pi}(\hat{X}, K(M_j,n)).$$

Furthermore, from Proposition 2.20 we deduce

$$\pi_0 \operatorname*{holim}_j \hom_{\hat{\mathcal{S}}/B\pi}(\hat{X}, K(M_j,n)) \cong \pi_0 \hom_{\hat{\mathcal{S}}/B\pi}(\hat{X}, K(M,n))$$

where $M = \lim_j M_j$ is the profinite π-module corresponding to $\mathcal{M}$. Finally, the twisted cohomology of $\hat{X}$ is represented by the fibration $K(M,n) \to B\pi$ in $\hat{\mathcal{S}}$, i.e. there is a canonical and natural identification

$$H^n_\pi(\hat{X}; M) = \pi_0 \hom_{\hat{\mathcal{S}}/B\pi}(\hat{X}, K(M,n)).$$

This series of isomorphisms now yields the proof of the assertion. □

2.8. Stable profinite homotopy theory.

Definition 2.35. *A* profinite spectrum X *consists of a sequence* $X_n \in \hat{\mathcal{S}}_*$ *of pointed profinite spaces for* $n \geq 0$ *and maps* $\sigma_n : S^1 \wedge X_n \to X_{n+1}$ *in* $\hat{\mathcal{S}}_*$. *A* morphism $f : X \to Y$ *of spectra consists of maps* $f_n : X_n \to Y_n$ *in* $\hat{\mathcal{S}}_*$ *for* $n \geq 0$ *such that* $\sigma_n(1 \wedge f_n) = f_{n+1}\sigma_n$. *We denote by* $\mathrm{Sp}(\hat{\mathcal{S}}_*)$ *the corresponding category and call it the category of* profinite spectra. *A spectrum* $E \in \mathrm{Sp}(\hat{\mathcal{S}}_*)$ *is called an* Ω-spectrum *if each* E_n *is fibrant and the adjoint structure maps* $E_n \to \Omega E_{n+1}$ *are weak equivalences for all* $n \geq 0$.

The suspension $\Sigma^\infty : \hat{\mathcal{S}}_* \to \mathrm{Sp}(\hat{\mathcal{S}}_*)$ sends a profinite space X to the spectrum given in degree n by $S^n \wedge X$. Starting with the model structure on $\hat{\mathcal{S}}$ of Theorem 2.12, the localization theorem of [25] yields the following result.

Theorem 2.36. *There is a* stable model structure on $\mathrm{Sp}(\hat{\mathcal{S}}_*)$ *for which the prolongation of the suspension functor is a Quillen equivalence. The corresponding stable homotopy category is denoted by* $\hat{\mathcal{SH}}$. *In particular, the* stable equivalences *are the maps that induce an isomorphism on all generalized cohomology theories, represented by profinite* Ω*-spectra; the* stable cofibrations *are the maps* $i : X \to Y$ *such that* i_0 *and the induced maps* $j_n : X_n \amalg_{S^1 \wedge X_{n-1}} S^1 \wedge Y_{n-1} \to Y_n$ *are monomorphisms for all* n*; the* stable fibrations *are the maps with the right lifting property with respect to all maps that are both stable equivalences and stable cofibrations.*

Since the suspension is compatible with profinite completion, there is an analogous statement as in Proposition 2.28 on the pair of adjoint functors consisting of the levelwise completion functor $\mathrm{Sp}(\mathcal{S}_*) \to \mathrm{Sp}(\hat{\mathcal{S}}_*)$ and the forgetful functor where $\mathrm{Sp}(\mathcal{S}_*)$ denotes the category of simplicial spectra with the Bousfield-Friedlander model structure [2]. For a profinite spectrum X, we define the stable homotopy groups $\pi^s_n(X)$ to be the set $\mathrm{Hom}_{\hat{\mathcal{SH}}}(S^n, X)$ of maps in the stable homotopy category. Since Ω preserves fibrations in $\hat{\mathcal{S}}_*$, Proposition 2.31 implies the following characterization of the stable homotopy groups of the completion of a spectrum.

Proposition 2.37. *Let E be a connected simplicial spectrum, then the canonical map $\widehat{\pi_n^s(E)} \to \pi_n^s(\hat{E})$ is an isomorphism. In particular, if each $\pi_n^s(E)$ is finitely generated, then $\pi_n^s(\hat{E}) \cong \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \pi_n^s(E)$.*

Examples for the last assertion are Eilenberg-MacLane spectra HM for a finitely generated abelian group M or the simplicial spectrum of complex cobordism MU. Finally, the composition of completion and homotopy limit defines also a functor from pro-spectra to profinite spectra that induces a functor from the homotopy category of pro-spectra of Christensen and Isaksen in [4] to $\hat{\mathcal{SH}}$.

3. Applications in étale homotopy theory

3.1. Profinite étale homotopy groups. The construction of the étale topological type functor Et from locally noetherian schemes to pro-spaces is due to Artin-Mazur and Friedlander. We refer the reader to [11] and [17] for a detailed discussion of the category of rigid hypercoverings and rigid pullbacks. Let X be a locally noetherian scheme. The étale topological type of X is defined to be the pro-simplicial set $\mathrm{Et}\, X := \mathrm{Re} \circ \pi : HRR(X) \to \mathcal{S}$ sending a rigid hypercovering U of X to the simplicial set of connected components of U. If $f : X \to Y$ is a map of locally noetherian schemes, then the strict map $\mathrm{Et}\, f : \mathrm{Et}\, X \to \mathrm{Et}\, Y$ is given by the functor $f^* : HRR(Y) \to HRR(X)$ and the natural transformation $\mathrm{Et}\, X \circ \mathrm{Et}\, f \to \mathrm{Et}\, Y$. In order to get a profinite space, we compose Et with the completion from pro-spaces to $\hat{\mathcal{S}}$ and denote this functor by $\hat{\mathrm{Et}}$.

Let X be a pointed connected locally noetherian scheme. For $k > 0$, let $\pi_k(\mathrm{Et}\, X)$ be the pro-group, defined by the functor $\pi_k \circ \mathrm{Et}$ from $HRR(X)$ to the category of groups as in [11]. For $k = 1$ and G a group, Friedlander has shown in [11], 5.6, that the set of isomorphism classes of principal G-fibrations over X is isomorphic to the set $\mathrm{Hom}(\pi_1(\mathrm{Et}\, X), G)$ of homomorphisms of pro-groups. Furthermore, the locally constant étale sheaves on X, whose stalks are isomorphic to a set S, are in 1-1 correspondence to local coefficient systems on $\mathrm{Et}\, X$ with fibers isomorphic to S. This shows that $\pi_1(\mathrm{Et}\, X)$ is equal to the enlarged fundamental pro-group of [6], Exposé X, §6. It agrees with the profinite étale fundamental group of [14] if X is geometrically unibranched (e.g. if X is normal). More generally, Friedlander shows in [11], Theorem 7.3, that $\pi_k(\mathrm{Et}\, X)$ is a profinite group for $k > 0$ when X is connected and geometrically unibranched.

These arguments are easily transfered to the profinite setting. For a topological group G there is a bijection between the set of isomorphism classes of principal G-fibrations over X and the set of isomorphism classes of principal G-fibrations over $\hat{\mathrm{Et}}\, X$; and hence there is a bijection between the set of isomorphism classes of locally constant étale sheaves of profinite sets on X and the set of isomorphism classes of profinite local coefficient systems on $\hat{\mathrm{Et}}\, X$. Furthermore, we see that the finite coverings of $\hat{\mathrm{Et}}\, X$ are in 1-1-correspondence with the finite étale coverings of X. This implies that the finite quotients of $\pi_1(\hat{\mathrm{Et}}\, X)$ correspond to the finite étale coverings of X. Hence the profinite group $\pi_1(\hat{\mathrm{Et}}\, X)$

agrees with the usual profinite fundamental group $\pi_1^{\text{ét}}(X)$ of [14].
If F is a locally constant étale sheaf on X, we denote by F also the corresponding local system on $\operatorname{Et} X$ respectively $\hat{\operatorname{Et}} X$. Friedlander has shown in [11] Proposition 5.9, that the groups $H^*_{\text{ét}}(X;F)$ and $H^*(\operatorname{Et} X;F)$ are equal. Moreover, by the definition of $\hat{\operatorname{Et}} X$, the cohomology groups $H^*(\hat{\operatorname{Et}} X;F)$ and $H^*(\operatorname{Et} X;F)$ coincide if F is finite. But if F is not finite, $H^*(\hat{\operatorname{Et}} X;F)$ does not in general agree with the usual étale cohomology groups $H^*_{\text{ét}}(X;F)$ any more. But since, for example, ℓ-adic cohomology $H^i(X;\mathbb{Z}_\ell(j)) := \lim_n H^i_{\text{ét}}(X;\mathbb{Z}/\ell^n(j))$ of a scheme X has good properties only if the étale cohomology groups $H^i_{\text{ét}}(X;\mathbb{Z}/\ell^n(j))$ are finite, this may not be considered as a problem. In fact, it turns out that, if F is profinite, $H^*(\hat{\operatorname{Et}} X;F)$ is the continuous étale cohomology of X, which is a more sophisticated version of étale cohomology for inverse systems of coefficient sheaves defined by Jannsen in [19].

Proposition 3.1. *Let X be a locally noetherian scheme and let F be a locally constant étale sheaf on X whose stalks are profinite abelian groups. The cohomology $H^*(\hat{\operatorname{Et}} X;F)$ of $\hat{\operatorname{Et}} X$ with coefficients in the local system corresponding to F coincides with the continuous étale cohomology $H^*_{\text{cont}}(X;F)$ of Jannsen and of Dwyer and Friedlander. In particular, $H^*(\hat{\operatorname{Et}} X;\mathbb{Z}_\ell(j))$ equals $H^*_{\text{cont}}(X;\mathbb{Z}_\ell(j))$ of* [19].

Proof. This follows immediately from Lemma 3.30 of [19] and Proposition 2.34, since $\operatorname{Et} X$ is a pro-space over the profinite groupoid $\Pi_1^{\text{ét}}(X) = \Pi(\hat{\operatorname{Et}} X)$. □

The relation to the usual ℓ-adic cohomology of a locally noetherian scheme X is given by the exact sequence, as in [19],

$$0 \to \lim_n{}^1 H^{i-1}_{\text{ét}}(X;\mathbb{Z}/\ell^n(j)) \to H^i(\hat{\operatorname{Et}} X;\mathbb{Z}_\ell(j)) \to \lim_n H^i_{\text{ét}}(X;\mathbb{Z}/\ell^n(j)) \to 0.$$

If $p : Y \to X$ is a Galois covering with profinite Galois group G and F is a locally constant profinite étale sheaf on X, then there is a spectral sequence of continuous cohomology groups

$$E_2^{p,q} = H^p(G;H^q(\hat{\operatorname{Et}} Y;p^*F)) \Rightarrow H^{p+q}(\hat{\operatorname{Et}} X;F).$$

The previous discussion shows that $\hat{\operatorname{Et}} X$ is a good rigid model for the profinite homotopy type of a scheme. Since the étale fundamental group $\pi_1^{\text{ét}}(X)$ is always a profinite group and is equal to $\pi_1(\hat{\operatorname{Et}} X)$, we make the following definition.

Definition 3.2. *For a locally noetherian simplicial scheme X, a geometric point x and $n \geq 2$, we define the* profinite étale homotopy groups $\pi_n^{\text{ét}}(X,x)$ *of X to be the profinite groups $\pi_n(\hat{\operatorname{Et}} X,x)$.*

By Proposition 2.33, these profinite homotopy groups agree with the profinitely completed étale homotopy groups of Artin and Mazur in [1].

3.2. Example: $\hat{\operatorname{Et}} k$ and the absolute Galois group. Let k be a fixed field. For a Galois extension L/k we denote by $G(L/k)$ its Galois group. For a separable closure $\overline{k}$ of k we write G_k for the absolute Galois group $G(\overline{k}/k)$ of k.

We want to determine the well known homotopy type of $\hat{\mathrm{Et}}\, k$ as BG_k. We denote by $RC(k)$ the category of rigid coverings and by $HRR(k)$ the category of rigid hypercoverings of $\mathrm{Spec}\, k$. There are two canonical functors between them. On the one hand there is the restriction functor sending a hypercovering U to U_0; on the other hand we can send a rigid covering $U \to \mathrm{Spec}\, k$ to the hypercovering $\mathrm{cosk}_0^k(U)$. When we consider the rigid cover $\mathrm{Spec}\, L \to \mathrm{Spec}\, k$ associated to a finite Galois extension L/k, the simplicial set of connected components of the corresponding hypercover $\pi_0(\mathrm{cosk}_0^k(L))$ is equal to the classifying space $BG(L/k)$. Hence when we consider the functor $\pi_0 \circ \mathrm{cosk}_0^k(-) : RC(k) \to \mathcal{S}$ from rigid coverings of k, we see that it takes values in simplicial profinite sets $\hat{\mathcal{S}}$ and that the limit over all the rigid coverings equals BG_k.
Furthermore, the restriction functor defines a monomorphism $g : BG_k \hookrightarrow \hat{\mathrm{Et}}\, k$; and the coskeleton cosk_0^k defines a map in the other direction $f : \hat{\mathrm{Et}}\, k \to BG_k$. In particular, f and g are simplicial homotopy equivalences and g is a cofibration. For, it is clear that $f \circ g$ equals the identity of BG_k, whereas the composite $g \circ f$ is homotopic to the identity of $\hat{\mathrm{Et}}\, k$ by [11] Prop. 8.2.
For a k-scheme X we write $X_L := X \otimes_k L$. The group $G := G(L/k)$ acts via Galois-automorphisms $1 \otimes \sigma$ on X_L. The canonical map $X_L \to X$ is an étale Galois covering of X with Galois group G. Hence $\hat{\mathrm{Et}}\, X_L \to \hat{\mathrm{Et}}\, X$ is a principal G-fibration in $\hat{\mathcal{S}}$. This implies that the canonical map $\hat{\mathrm{Et}}\, X_L \times_G EG \to \hat{\mathrm{Et}}\, X$ is a weak equivalence by Proposition 2.27 and that the cohomology of $\hat{\mathrm{Et}}\, X$ is equal to the Borel-cohomology of $\hat{\mathrm{Et}}\, X_L$ under the action of $G(L/k)$.
An interesting and well known example is the étale realization of a finite field $\mathbb{F}_q$ with q elements. Its Galois group is $\hat{\mathbb{Z}}$ and hence $\hat{\mathrm{Et}}\, \mathbb{F}_q = K(\hat{\mathbb{Z}}, 1)$. In fact, there is a canonical homotopy equivalence $S^1 \to \hat{\mathrm{Et}}\, \mathbb{F}_q$ sending the generator of $\pi_1 S^1 = \hat{\mathbb{Z}}$ to the Frobenius map.
Finally, for a base field k, $\hat{\mathrm{Et}}\, \mathbb{P}^1_{\overline{k}}$ is an Eilenberg MacLane G_k-space $K(\hat{\mathbb{Z}}(1), 2)$ with the canonical G_k-action on the profinite group $\hat{\mathbb{Z}}(1) = \mu(\overline{k})$ of all roots of unity in $\overline{k}$. More examples may be found in [10].

3.3. Etale realization of the flasque motivic model structure. The construction of the $\mathbb{A}^1$-homotopy category of schemes gave rise to the question if this functor may be enlarged to the category of motivic spaces. This has been answered independently by Dugger-Isaksen and Schmidt. The latter one constructed a geometric functor to the category pro-$\mathcal{H}$ as Artin-Mazur. The idea of Dugger and Isaksen for the less intuitive extension of $\mathrm{Et}\, X$ is that it should be the usual $\mathrm{Et}\, X$ on a representable presheaf X and should preserve colimits and the simplicial structure. Isaksen then showed that Et induces a left Quillen functor on the projective model structure on simplicial presheaves.
Let S be a base scheme and let Sm/S be the category of smooth quasi-projective schemes of finite type over S. Isaksen has shown in [18] that the flasque model structure on presheaves on Sm/S is a good model for the $\mathbb{A}^1$-homotopy category, in particular for the construction of the stable motivic homotopy category, since $\mathbb{P}^1$ is a flasque cofibrant space. It was shown in [25] that $\hat{\mathrm{Et}}$ yields a stable

étale realization functor of the stable motivic homotopy category. In order to simplify the constructions, we will deduce from [17] that $\hat{\mathrm{Et}}$ induces a derived functor on the flasque motivic structure as well.

Lemma 3.3. *Let $i : Y \hookrightarrow X$ be an open or closed immersion of locally noetherian schemes. Then* $\mathrm{Et}\, i : \mathrm{Et}\, Y \hookrightarrow \mathrm{Et}\, X$ *is a monomorphism of pro-simplicial sets.*

Proof. The map $\mathrm{Et}\, i$ is by definition the map of pro-simplicial sets which is induced by the natural transformation of indexing categories given by the rigid pullback functor $i^* : HRR(X) \to HRR(Y)$. If $\coprod_x U_x, x \to X$ is a rigid cover of X indexed by the geometric points of X, the connected component $(i^*U)_y$ of the pullback cover of Y at the geometric point y of Y is sent to the connected component $U_{i(y)}$ over X. If i is an open, resp. closed, immersion, this map on the sets of connected components is obviously injective. Since open, resp. closed, immersions are stable under base change and since for a rigid hypercover V of X, $(\mathrm{cosk}_{t-1}^{X_s} V)_t$ is a finite fiber product involving the $V_{s,r}$'s for $r \leq t$ and X_s, we deduce that the induced map $\mathrm{Et}\, i$ is a monomorphism as well. □

Theorem 3.4. *The étale topological type functor* Et *is a left Quillen functor on the Nisnevich (resp. étale) local flasque model structure on simplicial presheaves on* Sm/S *to the model structure on pro-simplicial sets of* [16].

Proof. Let $i : \cup_{k=1}^n U_n \to X$ be an acceptable monomorphism as defined in [18]. We have to show that $\mathrm{Et}\, i$ is a monomorphism in pro-$\mathcal{S}$. By compatibility with colimits, $\mathrm{Et}\, \cup_{k=1}^n U_n$ is the coequalizer of the diagram

$$\mathrm{Et}\,(\coprod_{k,k'} U_k \times_X U_{k'}) \to \mathrm{Et}\,(\coprod_k U_k).$$

Hence in order to show that $\mathrm{Et}\, i$ is a cofibration, it is enough to show that $\mathrm{Et}\,(U_k \times_X U_{k'}) \to \mathrm{Et}\, U_k$ is a monomorphism for every k and k'. But since open immersions of schemes are stable under base change, this follows from Lemma 3.3. Moreover, Et is compatible with pushout products and hence Et sends the generating (trivial) cofibrations in the flasque model structure on presheaves to (trivial) cofibrations of pro-simplicial sets.
The Nisnevich (resp. étale) local flasque model structure is constructed via the left Bousfield localizations with respect to all Nisnevich (resp. étale) hypercovers. In fact, Isaksen shows that Et sends Nisnevich and étale hypercovers to weak equivalences in pro-$\mathcal{S}$, [17], Theorem 12. □

Corollary 3.5. *If* $\mathrm{char}\ S = \{0\}$*, the functor* $\hat{\mathrm{Et}}$ *also induces a total left derived functor from the motivic flasque model structure on* $\mathrm{sPre}(\mathrm{Sm}/S)$ *to* $\hat{\mathcal{S}}$ *with the general model structure of Theorem 2.12.*
If $\mathrm{char}\ S$ *contains a prime $p > 0$, it induces a total derived functor to $\hat{\mathcal{S}}$ with the $\mathbb{Z}/\ell$-model structure of* [23] *for any prime number ℓ which is prime to* $\mathrm{char}\ S$.

Proof. Since Et induces a total left derived functor on the local structures and since the above completion induces a functor on the homotopy categories, it is

clear that $\hat{\mathrm{E}}\mathrm{t}$ induces the desired functor for the local structures.
It remains to check that $\hat{\mathrm{E}}\mathrm{t}$ sends the projection $p_X : \mathbb{A}^1 \times_k X \to X$ to a weak equivalence in $\hat{\mathcal{S}}$ for all $X \in \mathrm{Sm}/k$. If char $k = 0$, p_X induces an isomorphism on étale cohomology $H^*_{\text{ét}}(-; F)$ for all torsion sheaves F, see e.g. [22] VI, Corollary 4.20, and an isomorphism on étale fundamental groups. Hence in this case $\hat{\mathrm{E}}\mathrm{t}\, p_X$ is a weak equivalence in $\hat{\mathcal{S}}$.
If char $k = p > 0$, π_X induces an isomorphism on étale cohomology $H^*_{\text{ét}}(-; F)$ for all torsion sheaves F whose torsion is prime to p. But it does not induce an isomorphism on étale fundamental groups. Hence in this case $\hat{\mathrm{E}}\mathrm{t}\, p_X$ is only a weak equivalence in the mod ℓ-model structure on $\hat{\mathcal{S}}$. □

The space $\mathbb{P}^1_k$ pointed at ∞ is flasque cofibrant and may be used as in [18] to construct the stable motivic homotopy category of $\mathbb{P}^1$-spectra starting from the flasque model structure on motivic spaces. As in [25] this yields a stable motivic realization functor without taking a cofibrant replacement of $\mathbb{P}^1$.

Corollary 3.6. *Let k be a field with* char $k = 0$. *The functor* $\hat{\mathrm{E}}\mathrm{t}$ *induces a stable étale realization functor of the stable motivic homotopy category to the homotopy category $\hat{\mathcal{SH}}$ of Theorem 2.36.*

Remark 3.7. *1. For a pointed presheaf $\mathcal{X}$ on* Sm/S *there are profinite étale homotopy groups for every $n \geq 1$: $\pi_n^{\text{ét}}(\mathcal{X}) := \pi_n(\hat{\mathrm{E}}\mathrm{t}\, \mathcal{X})$.*
2. All statements of this subsection also hold for the closed (motivic) model structure of [24].

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Around the Gysin Triangle II.

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Abstract. The notions of orientation and duality are well understood in algebraic topology in the framework of the stable homotopy category. In this work, we follow these lines in algebraic geometry, in the framework of motivic stable homotopy, introduced by F. Morel and V. Voevodsky. We use an axiomatic treatment which allows us to consider both mixed motives and oriented spectra over an arbitrary base scheme. In this context, we introduce the Gysin triangle and prove several formulas extending the traditional panoply of results on algebraic cycles modulo rational equivalence. We also obtain the Gysin morphism of a projective morphism and prove a duality theorem in the (relative) pure case. These constructions involve certain characteristic classes (Chern classes, fundamental classes, cobordism classes) together with their usual properties. They imply statements in motivic cohomology, algebraic K-theory (assuming the base is regular) and "abstract" algebraic cobordism as well as the dual statements in the corresponding homology theories. They apply also to ordinary cohomology theories in algebraic geometry through the notion of a mixed Weil cohomology theory, introduced by D.-C. Cisinski and the author in [CD06], notably rigid cohomology.

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Notations

We fix a noetherian base scheme S. The schemes considered in this paper are always assumed to be finite type S-schemes. Similarly, a smooth scheme (resp. morphism of schemes) means a smooth S-scheme (resp. S-morphism of S-schemes). We eliminate the reference to the base S in all notation (e.g. $\times$, $\mathbb{P}^n$, ...)
An immersion i of schemes will be a locally closed immersion and we say i is an open (resp. closed) immersion when i is open (resp. closed). We say a morphism $f : Y \to X$ is projective[2] if Y admits a closed X-immersion into a *trivial* projective bundle over X.
Given a smooth closed subscheme Z of a scheme X, we denote by $N_Z X$ the normal vector bundle of Z in X. Recall a morphism $f : Y \to X$ of schemes is said to be transversal to Z if $T = Y \times_X Z$ is smooth and the canonical morphism $N_T Y \to T \times_Z N_Z X$ is an isomorphism.
For any scheme X, we denote by $\mathrm{Pic}(X)$ the Picard group of X.
Suppose X is a smooth scheme. Given a vector bundle E over X, we let $P = \mathbb{P}(E)$ be the projective bundle of lines in E. Let $p : P \to X$ be the canonical projection. There is a canonical line bundle λ_P on P such that $\lambda_P \subset p^{-1}(E)$. We call it the *canonical line bundle* on P. We set $\xi_P = p^{-1}(E)/\lambda_P$, called the *universal quotient bundle*. For any integer $n \geq 0$, we also use the abbreviation $\lambda_n = \lambda_{\mathbb{P}^n_S}$. We call the projective bundle $\mathbb{P}(E \oplus 1)$, with its canonical open immersion $E \to \mathbb{P}(E \oplus 1)$, the *projective completion* of E.

1. Introduction

In algebraic topology, it is well known that oriented multiplicative cohomology theories correspond to algebras over the complex cobordism spectrum **MU**. Using the stable homotopy category allows a systematic treatment of this kind of generalized cohomology theory, which are considered as oriented ring spectra.
In algebraic geometry, the motive associated to a smooth scheme plays the role of a universal cohomology theory. In this article, we unify the two approaches : on the one hand, we replace ring spectra by spectra with a structure of modules over a suitable oriented ring spectra - e.g. the spectrum **MGL** of algebraic cobordism. On the other hand, we introduce and consider formal group laws in the motivic theory, generalizing the classical point of view.
More precisely, we use an axiomatic treatment based on homotopy invariance and excision property which allows to formulate results in a triangulated category which models both stable homotopy category and mixed motives. A suitable notion of orientation is introduced which implies the existence of Chern classes together with a formal group law. This allows to prove a purity theorem which implies the existence of Gysin morphisms for closed immersions and their companion residue morphisms. We extend the definition of the Gysin morphism to the case of a projective morphism, which involves a delicate study of cobordism classes in the case of an arbitrary formal group law. This theory then implies very neatly the duality statement in the projective smooth case. Moreover, these

[2] If X admits an amble line bundle, this definition coincide with that of [EGA2].

constructions are obtained over an arbitrary base scheme, eventually singular and with unequal characteristics.
Examples are given which include triangulated mixed motives, generalizing the constructions and results of V. Voevodsky, and **MGL**-modules. Thus, this work can be applied in motivic cohomology (and motivic homology), as well as in algebraic cobordism. It also applies in homotopy algebraic K-theory[3] and some of the formulas obtained here are new in this context. It can be applied finally to classical cohomology theories through the notion of a *mixed Weil theory* introduced in [CD06]. In the case of rigid cohomology, the formulas and constructions given here generalize some of the results obtained by P. Berthelot and D. Pétrequin. Moreover, the theorems proved here are used in an essential way in [CD06].

1.1. **The axiomatic framework.** We fix a triangulated symmetric monoidal category $\mathscr{T}$, with unit $\mathbb{1}$, whose objects are simply called *motives*[4]. To any pair of smooth schemes (X, U) such that $U \subset X$ is associated a motive $M(X/U)$ functorial with respect to $U \subset X$, and a canonical distinguished triangle :

$$M(U) \to M(X) \to M(X/U) \xrightarrow{\partial} M(U)[1],$$

where we put $M(U) := M(U/\emptyset)$ and so on. The first two maps are obtained by functoriality. As usual, the Tate motive is defined to be $\mathbb{1}(1) := M(\mathbb{P}^1_S/S_\infty)[-2]$ where S_∞ is the point at infinity.
The axioms we require are, for the most common, additivity (Add), homotopy invariance (Htp), Nisnevich excision (Exc), Künneth formula for pairs of schemes (Kun) and stability (Stab) – *i.e.* invertibility of $\mathbb{1}(1)$ (see paragraph 2.1 for the precise statement). All these axioms are satisfied by the stable homotopy category of schemes of F. Morel and V. Voevodsky. However, we require a further axiom which is in fact our principal object of study, the orientation axiom (Orient) : to any line bundle L over a smooth scheme X is associated a morphism $c_1(L)$: $M(X) \to \mathbb{1}(1)[2]$ – the first Chern class of L – compatible with base change and constant on the isomorphism class of L/X.
The best known example of a category satisfying this set of axioms is the triangulated category of (geometric) mixed motives over S, denoted by $DM_{gm}(S)$. It is defined according to V. Voevodsky along the lines of the case of a perfect base field but replacing Zariski topology by the Nisnevich one (cf section 2.3.1). Another example can be obtained by considering the category of oriented spectra in the sense of F. Morel (see [Vez01]). However, in order to define a monoidal structure on that category, we have to consider modules over the algebraic cobordism spectrum **MGL**, in the E_∞-sense. One can see that oriented spectra are equivalent to **MGL**-modules, but the tensor product is given with respect to the **MGL**-module structure.

[3]Recall homotopy algebraic K-theory was introduced by Weibel in [Wei89]. This cohomology theory coincide with algebraic K-theory when S is regular.
[4] A correct terminology would be to call these objects *generalized triangulated motives* or *triangulated motives with coefficients* as the triangulated mixed motives defined by Voevodsky are particular examples.

Any object $\mathbb{E}$ of the triangulated category $\mathscr{T}$ defines a bigraded cohomology (resp. homology) theory on smooth schemes by the formulas

$$\mathbb{E}^{n,p}(X) = \operatorname{Hom}_{\mathscr{T}}\big(M(X), \mathbb{1}(p)[n]\big) \textit{ resp. } \mathbb{E}_{n,p}(X) = \operatorname{Hom}_{\mathscr{T}}\big(\mathbb{1}(p)[n], \mathbb{E} \otimes M(X)\big).$$

As in algebraic topology, there is a rich algebraic structure on these graded groups (see section 2.2). The Künneth axiom (Kun) implies that, in the case where $\mathbb{E}$ is the unit object $\mathbb{1}$, we obtain a multiplicative cohomology theory simply denoted by H^{**}. It also implies that for any smooth scheme X, $\mathbb{E}^{**}(X)$ has a module structure over $H^{**}(X)$. More generally, if we put $A = H^{**}(S)$, called the ring of (universal) coefficients, cohomology and homology groups of the previous kind are graded A-modules.

1.2. **Central constructions.** These axioms are sufficient to establish an essential basic fact, the projective bundle theorem :

(**th. 3.2**)[5] Let X be a smooth scheme, $P \xrightarrow{p} X$ be a projective bundle of dimension n, and c be the first Chern class of the canonical line bundle. Then the map: $\sum_{0 \leq i \leq n} p_* \boxtimes c^i : M(P) \to \bigoplus_{0 \leq i \leq n} M(X)(i)[2i]$ is an isomorphism.

Remark that considering any motive $\mathbb{E}$, even without ring structure, we obtain $\mathbb{E}^{**}(P) = \mathbb{E}^{**}(X) \otimes_{H^{**}(X)} H^{**}(P)$ where tensor product is taken with respect to the $H^{**}(X)$-module structure. In the case $\mathbb{E} = \mathbb{1}$, we thus obtain the projective bundle formula for H^{**} which allows the definition of *(higher) Chern classes* following the classical method of Grothendieck :

(**def. 3.10**) For any smooth scheme X, any vector bundle E over X and any integer $i \geq 0$, $c_i(E) : M(X) \to \mathbb{1}(i)[2i]$.

Moreover, the projective bundle formula leads to the following constructions :

(i) (**3.7 & 3.8**) A *formal group law* $F(x, y)$ over A such that for any smooth scheme X which admits an ample line bundle, for any line bundles L, L' over X, the formula
$$c_1(L \otimes L') = F(c_1(L), c_1(L'))$$
is well defined and holds in the A-algebra $H^{**}(X)$.

(ii) (**def. 5.12**) For any smooth schemes X, Y and any projective morphism $f : Y \to X$ of relative dimension n, the associated *Gysin morphism* $f^* : M(X) \to M(Y)(-n)[-2n]$.

(iii) (**def. 4.6**) For any closed immersion $i : Z \to X$ of codimension n between smooth schemes, with complementary open immersion j, the Gysin triangle :
$$M(X - Z) \xrightarrow{j_*} M(X) \xrightarrow{i^*} M(Z)(n)[2n] \xrightarrow{\partial_{X,Z}} M(X - Z)[1].$$
The last morphism in this triangle is called the *residue morphism.*

The Gysin morphism permits the construction of a *duality pairing* in the pure case :

(**th. 5.23**) For any smooth projective scheme $p : X \to S$ of relative dimension n,

[5] The proof is essentially based on a very elegant lemma due to F. Morel.

with diagonal embedding $\delta : X \to X \times X$, there is a strong duality[6] (in the sense of Dold-Puppe) :

$$\text{(iv)} \qquad \begin{aligned} &\mu_X : \mathbb{1} \xrightarrow{p^*} M(X)(-n)[-2n] \xrightarrow{\delta_*} M(X)(-n)[-2n] \otimes M(X) \\ &\epsilon_X : M(X) \otimes M(X)(-n)[-2n] \xrightarrow{\delta^*} M(X) \xrightarrow{p_*} \mathbb{1}. \end{aligned}$$

In particular, the Hom object $\underline{\mathrm{Hom}}(M(X), \mathbb{1})$ is defined in the monoidal category $\mathscr{T}$ and μ_X induces a canonical duality isomorphism :

$$\underline{\mathrm{Hom}}(M(X), \mathbb{1}) \to M(X)(-n)[-2n].$$

This explicit duality allows us to recover the usual form of duality between cohomology and homology as in algebraic topology, in terms of the fundamental class of X and cap-product on one hand and in terms of the fundamental class of δ and slant product on the other hand. Moreover, considering a motive $\mathbb{E}$ with a monoid structure in $\mathscr{T}$ and such that the cohomology $\mathbb{E}^{**}$ satisfies the Künneth formula, we obtain the usual Poincaré duality theorem in terms of the trace morphism (induced by the Gysin morphism $p^* : \mathbb{1} \to M(X)(n)[2n]$) and cup-product (see **paragraph 5.24**).

Note also we deduce easily from our construction that the Gysin morphism associated to a morphism f between smooth projective schemes is the dual of f_* (**prop. 5.26**).

Remark finally that, considering any closed subscheme Z_0 of S, and taking tensor product with the motive $M(S/S - Z_0)$ in the constructions (ii), (iii) and (iv), we obtain a Gysin morphism and a Gysin triangle with *support*. For example, given a projective morphism $f : Y \to X$ as in (*ii*), $Z = X \times_S Z_0$ and $T = Y \times_S Z_0$, we obtain the morphism $M_Z(X) \to M_T(Y)(-n)[-2n]$. Similarly, if X is projective smooth of relative dimension n, $M_Z(X)$ admits a strong dual, $M_Z(X)(-n)[-2n]$. Of course, all the other formulas given below are valid for these motives with support.

1.3. **Set of formulas.** The advantage of the motivic point of view is to obtain universal formulae which imply both cohomological and homological statements, with a minimal amount of algebraic structure involved.

1.3.1. *Gysin morphism.* We prove the basic properties of the Gysin morphisms such as functoriality ($g^*f^* = (fg)^*$), compatibility with the monoidal structure $(f \times g)^* = f^* \otimes g^*$), the projection formula ($(1_{Y*} \boxtimes f_*)f^* = f^* \boxtimes 1_{X*}$) and the base change formula in the transversal case ($f^*p_* = q_*g^*$).

For the needs of the following formulae, we introduce a useful notation which appear in the article. For any smooth scheme X, any cohomology class $\alpha \in H^{n,p}(X)$ and any morphism $\phi : M(X) \to M$ in $\mathscr{T}$, we put

$$\phi \boxtimes \alpha := (\phi \otimes \alpha) \circ \delta_* : M(X) \to M(p)[n]$$

where α is considered as a morphism $M(X) \to \mathbb{1}(p)[n]$, and $\delta_* : M(X) \to M(X) \otimes M(X)$ is the morphism induced by the diagonal of X/S and by the Künneth axiom (Kun).

[6]Note we use essentially the axiom (Kun) here.

More striking are the following formulae which express the *defect* in base change formulas. Fix a commutative square of smooth schemes

$$
\begin{tikzcd}
T \arrow[r, "q"] \arrow[d, "g"'] \arrow[dr, phantom, "\Delta"] & Y \arrow[d, "f"] \\
Z \arrow[r, "p"'] & X
\end{tikzcd}
\tag{1.1}
$$

which is cartesian on the underlying topological spaces, and such that p (resp. q) is projective of relative dimension n (resp. m).

Excess of intersection (**prop. 4.16**).– Suppose the square Δ is cartesian. We then define the *excess intersection bundle* ξ associated to Δ as follows. Choose a projective bundle P/X and a closed immersion $Z \xrightarrow{i} P$ over X with normal bundle $N_Z P$. Consider the pullback Q of P over Y and the normal bundle $N_Y Q$ of Y in Q. Then $\xi = N_Y Q/g^{-1}N_Z P$ is independent up to isomorphism of the choice of P and i. The rank of ξ is the integer $e = n - m$.

Then, $p^* f_* = \big(g_* \boxtimes c_e(\xi)\big)q^*$.

Ramification formula (**th. 4.26**).– Consider the square Δ and assume $n = m$. Suppose that T admits an ample line bundle and (for simplicity) that S is integral[7].

Let $T = \cup_{i \in I} T_i$ be the decomposition of T into connected components. Consider an index $i \in I$. We let p_i and g_i be the restrictions of p and g to T_i. The canonical map $T \to Z \times_X Y$ is a thickening. Thus, the connected component T_i corresponds to a unique connected component T'_i of $Z \times_X Y$. According to the classical definition, the *ramification index* of f along T_i is the geometric multiplicity $r_i \in \mathbb{N}^*$ of T'_i. We define (cf **def. 4.24**) a generalized intersection multiplicity for T_i which takes into account the formal group law F, called for this reason the *F-intersection multiplicity.* It is an element $r(T_i; f, g) \in H^{0,0}(T_i)$. We then prove the formula :

$$p^* f_* = \sum_{i \in I} \big(r(T_i; f, g) \boxtimes_{T_i} g_{i*}\big) q_i^*.$$

In general, $r(T_i; f, g) = r_i + \epsilon$ where the correction term ϵ is a function of the coefficients of F – it is zero when F is additive.

1.3.2. *Residue morphism.* A specificity of the present work is the study of the Gysin triangle, notably its boundary morphism, called the residue morphism. Consider a square Δ as in (1.1). Put $U = X - Z$, $V = Y - T$ and let $h : V \to U$ be the morphism induced by f.

We obtain the following formulas :

(1) $(j_* \boxtimes 1_{U*})\partial_{X,Z} = \partial_{X,Z} \boxtimes i_*$.
(2) For any smooth scheme Y, $\partial_{X \times Y, Z \times Y} = \partial_{X,Z} \otimes 1_{Y*}$.
(3) If f is a closed immersion, $\partial_{X-Z,Y-T}\partial_{Y,T} + \partial_{X-Y,Z-T}\partial_{Z,T} = 0$.
(4) If f is projective, $\partial_{Y,T} g^* = h^* \partial_{X,Z}$.
(5) When f is transversal to i, $h_* \partial_{Y,T} = \partial_{X,Z} g_*$.
(6) When Δ satisfies the hypothesis of *Excess of intersection*,
$h_* \partial_{Y,T} = \partial_{X,Z}\big(g_* \boxtimes c_e(\xi)\big)$.

[7] We prove in the text a stronger statement assuming only that S is reduced.

(7) When Δ satisfies the hypothesis of *Ramification formula*,
$\sum_{i\in I} h_*\partial_{Y,T_i} = \sum_{i\in I} \partial_{X,Z}\big(r(T_i; f,g) \boxtimes g_{i*}\big)$.

The differential taste of the residue morphism appears clearly in the last formula (especially in the cohomological formulation) where the multiplicity $r(T_i; f,g)$ takes into account the ramification index r_i. Even in algebraic K-theory, this formula seems to be new.

1.3.3. *Blow-up formulas.* Let X be a smooth scheme and $Z \subset X$ be a smooth closed subscheme of codimension n. Let B be the blow-up of X with center Z and consider the cartesian square $\begin{array}{ccc} P & \xrightarrow{k} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & X \end{array}$. Let e be the top Chern class of the canonical quotient bundle on the projective space P/Z.

(1) (**prop. 5.38**) Let $M(P)/M(Z)$ be the kernel of the split monomorphism p_*. The morphism (k_*, f^*) induces an isomorphism :
$$M(P)/M(Z) \oplus M(X) \to M(B).$$
(2) (**prop. 5.39**) The short sequence
$$0 \to M(B) \xrightarrow{\binom{k^*}{f_*}} M(P)(1)[2] \oplus M(X) \xrightarrow{(p_* \boxtimes e, -i^*)} M(Z)(n)[2n] \to 0$$
is split exact. Moreover, $(p_* \boxtimes e, -i^*) \circ \binom{p^*}{0}$ is an isomorphism[8].

The first formula was obtained by V. Voevodsky using resolution of singularities in the case where S is the spectrum of a perfect field. The second formula is the analog of a result of W. Fulton on Chow groups (cf [Ful98, 6.7]).

1.4. **Characteristic classes.** Besides Chern classes, we can introduce the following characteristic classes in our context.
Let $i : Z \to X$ be a closed immersion of codimension n between smooth schemes, $\pi : Z \to S$ the canonical projection. We define the *fundamental class* of Z in X (**paragraph 4.14**) as the cohomology class represented by the morphism
$$\eta_X(Z) : M(X) \xrightarrow{i^*} M(Z)(n)[2n] \xrightarrow{\pi_*} \mathbb{1}(n)[2n].$$
It is a cohomology class in $H^{2n,n}(X)$ satisfying the more classical expression $\eta_X(Z) = i_*(1)$.
Considering the hypothesis of the ramification formula above, when $n = m = 1$, we obtain the enlightening formula (cf **cor. 4.28**) :
$$f^*(\eta_X(Z)) = \sum_{i\in I} [r_i]_F \cdot \eta_Y(T_i)$$
where r_i is the ramification index of f along T_i and $[r_i]_F$ is the r_i-th formal sum with respect to F applied to the cohomological class $\eta_Y(T_i)$. Indeed, the fact T admits an ample line bundle implies this class is nilpotent.
The most useful fundamental class in the article is the Thom class of a vector bundle E/X of rank n. Let $P = \mathbb{P}(E \oplus 1)$ be its projective completion and

[8]This isomorphism is the identity at least in the case when $F(x,y) = x + y$

consider the canonical section $X \xrightarrow{s} P$. The *Thom class* of E/X is $t(E) := \eta_P(X)$. By the projection formula, $s^* = p_* \boxtimes t(E)$, where $p : P \to X$ is the canonical projection. Let λ (resp. ξ) be the canonical line bundle (resp. universal quotient bundle) on P/X. We also obtain the following equalities[9] :

$$t(E) = c_n(\lambda^\vee \otimes p^{-1}E) \overset{(*)}{=} c_n(\xi) = \sum_{i=0}^{n} c_i(p^{-1}E) \cup \big(- c_1(\lambda)\big)^i.$$

This is straightforward in the case where $F(x, y) = x + y$ but more difficult in general.

We also obtain a computation which the author has not seen in the literature (even in complex cobordism). Write $F(x, y) = \sum_{i,j} a_{ij}.x^i y^j$ with $a_{ij} \in A$. Consider the diagonal embedding $\delta : \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$. Let λ_1 and λ_2 be the respective canonical line bundle on the first and second factor of $\mathbb{P}^n \times \mathbb{P}^n$. Then (**prop. 5.30**) the fundamental class of δ satisfies

$$\eta_{\mathbb{P}^n \times \mathbb{P}^n}(\mathbb{P}^n) = \sum_{0 \leq i,j \leq n} a_{1,i+j-n}.c_1(\lambda_1^\vee)^i c_1(\lambda_2^\vee)^j.$$

Another kind of characteristic classes are *cobordism classes*. Let $p : X \to S$ be a smooth projective scheme of relative dimension n. The cobordism class of X/S is the cohomology class represented by the morphism

$$[X] : \mathbb{1} \xrightarrow{p^*} M(X)(-n)[-2n] \xrightarrow{p_*} \mathbb{1}(-n)[-2n].$$

It is a class in $A^{-2n,-n}$. As an application of the previous equality, we obtain the following computation (**cor. 5.31**) :

$$[\mathbb{P}^n] = (-1)^n. \det \begin{pmatrix} 0 & \cdots & 0 & 1 & a_{1,1} \\ \vdots & & & & a_{1,2} \\ 0 & & & & \vdots \\ 1 & & & & \\ a_{1,1} & a_{1,2} & \cdots & & a_{1,n} \end{pmatrix}$$

which of course coincides with the expression given by the classical theorem of Myschenko in complex cobordism. In fact, our method gives a new proof of the latter theorem.

1.5. **Outline of the work.** In section 2, we give the list of axioms (cf 2.1) satisfied by the category $\mathscr{T}$ and discuss the first consequences of these. Remark an originality of our axiomatic is that we not only consider pairs of schemes but also quadruples (used in the proof of 4.32). The last subsection 2.3 gives the principle examples which satisfy the axiomatic 2.1. Section 3 contains the projective bundle theorem and its consequences, the formal group law and Chern classes.

Section 4 contains the study of the Gysin triangle. The fundamental result in this section is the purity theorem 4.3. Usually, one constructs the Thom isomorphism using the Thom class (4.4). Here however, we directly construct the former isomorphism from the projective bundle theorem and the deformation to the normal

[9] This corrects an affirmation of I. Panin in the introduction of [Pan03a, p. 268] where equality $(*)$ is said not to hold.

cone. This makes the construction more canonical – though there is a delicate choice of signs hidden (cf beginning of section 4.1) – and it thus gives a canonical Thom class. We then study the two principle subjects around the Gysin triangle : the base change formula and its defect (section 4.2 which contains notably 4.26 and 4.16 cited above) and the interaction (containing notably the functoriality of the Gysin morphism) of two Gysin triangles attached with smooth subschemes of a given smooth scheme (th. 4.32).
In section 5, we first recall the notion of *strong duality* introduced by A. Dold and D. Puppe and give some complements. Then we give the construction of the Gysin morphism in the projective case and the duality statement. The general situation is particularly complicated when the formal group law F is not the additive one, as the Gysin morphism associated to the projection p of $\mathbb{P}^n$ is not easy to handle. Our method is to exploit the strong duality on $\mathbb{P}^n$ implied by the projective bundle theorem. We show that the fundamental class of the diagonal δ of $\mathbb{P}^n/S$ determines canonically the Gysin morphism of the projection (see def. 5.6). This is due to the explicit form of the duality pairing for $\mathbb{P}^n$ cited above : the motive $M(\mathbb{P}^n)$ being strongly dualizable, one morphism of the duality pairing (μ_X, ϵ_X) determines the other; the first one is induced by δ^* and the other one by p^*. Once this fact is determined, we easily obtain all the properties required to define the Gysin morphism and then the general duality pairing. The article ends with the explicit determination of the cobordism class of $\mathbb{P}^n$ and the blow-up formulas as illustrations of the theory developed here.

1.6. **Final commentary.** In another work [Dég08], we study the Gysin triangle directly in the category of geometric mixed motives over a perfect field. In the latter, we used the isomorphism of the relevant part of motivic cohomology groups and Chow groups and prove our Gysin morphism induces the usual pushout on Chow groups via this isomorphism (cf [Dég08, 1.21]). This gives a shortcut for the definitions and propositions proved here in the particular case of motives over a perfect field. In *loc. cit.* moreover, we also use the isomorphism between the diagonal part of the motivic cohomology groups of a field L and the Milnor K-theory of L and prove our Gysin morphism induces the usual norm morphism on Milnor K-theory (cf [Dég08, 3.10]) – after a limit process, considering L as a function field.
The present work is obviously linked with the fundamental book on algebraic cobordism by Levine and Morel [LM07] (see also [Lev08b]), but here, we study oriented cohomology theories from the point of view of stable homotopy. This point of view is precisely that of [Lev08a]. It is more directly linked with the prepublication [Pan03b] of I. Panin which was mainly concerned with the construction of pushforwards in cohomology, corresponding to our Gysin morphism (see also [Smi06] and [Pim05] for extensions of this work). Our study gives a unified self-contained treatment of all these works, except that we have not considered here the theory of transfers and Chern classes with support (see [Smi06], [Lev08a, part 5]).

The final work we would like to mention is the thesis of J. Ayoub on cross functors ([Ayo07]). In fact, it is now folklore that the six functor formalism yields a construction of the Gysin morphism. In the work of Ayoub however, the questions of orientability are not treated. In particular, the Gysin morphism we obtain takes value in a certain Thom space. To obtain the Gysin morphism in the usual form, we have to consider the Thom isomorphism introduced here. Moreover, we do not need the localization property in our study whereas it is essentially used in the formalism of cross functors. This is a strong property which is not known in general for triangulated mixed motives. Finally, the interest of this article relies in the study of the defect of the base change formula which is not covered by the six functor formalism.

Remerciements. J'aimerais remercier tout spécialement Fabien Morel car ce travail, commencé à la fin de ma thèse, a bénéficié de ses nombreuses idées et de son support. Aussi, l'excellent rapport d'une version préliminaire de l'article [Dég08] m'a engagé à le généraliser sous la forme présente; j'en remercie le rapporteur, ainsi que Jörg Wildeshaus pour son soutien. Je tiens à remercier Geoffrey Powell pour m'avoir grandement aidé à clarifier l'introduction de cet article et Joël Riou pour m'avoir indiqué une incohérence dans une première version de la formule de ramification. Mes remerciements vont aussi au rapporteur de cet article pour sa lecture attentive qui m'a notamment aidée à clarifier les axiomes. Je souhaite enfin adresser un mot à Denis-Charles Cisinski pour notre amitié mathématique qui a été la meilleure des muses.

2. The general setting : homotopy oriented triangulated systems

2.1. **Axioms and notations.** Let $\mathscr{D}$ be the category whose objects are the cartesian squares

$$(*) \qquad \begin{array}{ccc} W & \to & V \\ \downarrow & \Delta & \downarrow \\ U & \to & X \end{array}$$

made of immersions between smooth schemes. The morphisms in $\mathscr{D}$ are the evident commutative cubes. We will define the *transpose* of the square Δ, denoted by Δ', as the square

$$\begin{array}{ccc} W & \to & U \\ \downarrow & \Delta' & \downarrow \\ V & \to & X \end{array}$$

made of the same immersions. This defines an endofunctor of $\mathscr{D}$.
In all this work, we consider a triangulated symmetric monoidal category $(\mathscr{T}, \otimes, \mathbb{1})$ together with a covariant functor $M : \mathscr{D} \to \mathscr{T}$. Objects of $\mathscr{T}$ are called *premotives*. Considering a square as in $(*)$, we adopt the suggesting notation

$$M\left(\frac{X/U}{V/W}\right) = M(\Delta).$$

We simplify this notation in the following two cases :

(1) If $V = W = \emptyset$, we put $M(X/U) = M(\Delta)$.
(2) If $U = V = W = \emptyset$, we put $M(X) = M(\Delta)$.

We call *closed pair* any pair (X, Z) of schemes such that X is smooth and Z is a closed (not necessarily smooth) subscheme of X. As usual, we define the premotive of X with support in Z as $M_Z(X) = M(X/X - Z)$.
A *pointed scheme* is a scheme X together with an S-point $x : S \to X$. When X is smooth, the reduced premotive associated with (X, x) will be $\tilde{M}(X, x) = M(S \xrightarrow{x} X)$. Let $n > 0$ be an integer. We will always assume the smooth scheme $\mathbb{P}^n_S$ is pointed by the infinity. We define the *Tate twist* as the premotive $\mathbb{1}(1) = \tilde{M}(\mathbb{P}^1_S)[-2]$ of $\mathscr{T}$.

2.1. We suppose the functor M satisfies the following axioms :

(Add) For any finite family of smooth schemes $(X_i)_{i\in I}$,
$M(\sqcup_{i\in I} X_i) = \oplus_{i\in I} M(X_i)$.

(Htp) For any smooth scheme X, the canonical projection of the affine line induces an isomorphism $M(\mathbb{A}^1_X) \to M(X)$.

(Exc) Let (X, Z) be a closed pair and $f : V \to X$ be an étale morphism. Put $T = f^{-1}(Z)$ and suppose the map $T_{red} \to Z_{red}$ obtained by restriction of f is an isomorphism. Then the induced morphism $\phi : M_T(V) \to M_Z(X)$ is an isomorphism.

(Stab) The Tate premotive $\mathbb{1}(1)$ admits an inverse for the tensor product denoted by $\mathbb{1}(-1)$.

(Loc) For any square Δ as in $(*)$, a morphism $\partial_\Delta : M\left(\frac{X/U}{V/W}\right) \to M(V/W)[1]$ is given natural in Δ and such that the sequence of morphisms

$$M(V/W) \to M(X/U) \to M\left(\frac{X/U}{V/W}\right) \xrightarrow{\partial_\Delta} M(V/W)[1]$$

made of the evident arrows is a distinguished triangle in $\mathscr{T}$.

(Sym) Let Δ be a square as in $(*)$ and consider its transpose Δ'. There is given a morphism $\epsilon_\Delta : M\left(\frac{X/U}{V/W}\right) \to M\left(\frac{X/V}{U/W}\right)$ natural in Δ.
If in the square Δ, $V = W = \emptyset$, we put

$$\partial_{X/U} = \partial_{\Delta'} \circ \epsilon_\Delta : M(X/U) \to M(U)[1].$$

We ask the following coherence properties :
(a) $\epsilon_{\Delta'} \circ \epsilon_\Delta = 1$.
(b) If $\Delta = \Delta'$ then $\epsilon_\Delta = 1$.
(c) The following diagram is anti-commutative :

$$\begin{tikzcd} M\left(\frac{X/U}{V/W}\right) \arrow[r, "\epsilon_\Delta"] \arrow[d, "\partial_\Delta"'] & M\left(\frac{X/V}{U/W}\right) \arrow[r, "\partial_{\Delta'}"] & M(U/W)[1] \arrow[d, "\partial_{U/W}[1]"] \\ M(V/W)[1] \arrow[rr, "\partial_{V/W}[1]"] & & M(W)[2]. \end{tikzcd}$$

(Kun) (a) For any open immersions $U \to X$ and $V \to Y$ of smooth schemes, there are canonical isomorphisms:
$M(X/U) \otimes M(Y/V) = M(X \times Y/X \times V \cup U \times Y), \quad M(S) = \mathbb{1}$
satisfying the coherence conditions of a monoidal functor.

(b) Let X and Y be smooth schemes and $U \to X$ be an open immersion. Then, $\partial_{X\times Y/U\times Y} = \partial_{X/U} \otimes 1_{Y*}$ through the preceding canonical isomorphism.

(Orient) For any smooth scheme X, there is an application, called the *orientation*,

$$c_1 : \mathrm{Pic}(X) \to \mathrm{Hom}_{\mathscr{T}}(M(X), \mathbb{1}(1)[2])$$

which is functorial in X and such that the class $c_1(\lambda_1) : M(\mathbb{P}^1_S) \to \mathbb{1}(1)[2]$ is the canonical projection.

For any integer $n \in \mathbb{N}$, we let $\mathbb{1}(n)$ (resp. $\mathbb{1}(-n)$) be the n-th tensor power of $\mathbb{1}(1)$ (resp. $\mathbb{1}(-1)$). Moreover, for an integer $n \in \mathbb{Z}$ and a premotive $\mathbb{E}$, we put $\mathbb{E}(n) = \mathbb{E} \otimes \mathbb{1}(n)$.

2.2. Using the excision axiom (Exc) and an easy noetherian induction, we obtain from the homotopy axiom (Htp) the following stronger result :

(Htp') For any fiber bundle E over a smooth scheme X, the morphism induced by the canonical projection $M(E) \to M(X)$ is an isomorphism.

We further obtain the following interesting property :

(Add') Let X be a smooth scheme and Z, T be disjoint closed subschemes of X. Then the canonical map $M_{Z\sqcup T}(X) \to M_Z(X) \oplus M_T(X)$ induced by naturality is an isomorphism.

Indeed, using (Loc) with $V = X - T$, $W = X - (Z \sqcup T)$ and $U = W$, we get a distinguished triangle

$$M_Z(V) \to M_{Z\sqcup T}(X) \xrightarrow{\pi} M\left(\frac{X/W}{V/W}\right) \to M_Z(V)[1].$$

Using (Exc), we obtain $M_Z(V) = M_Z(X)$. The natural map $M_{Z\sqcup T}(X) \to M_Z(X)$ induces a retraction of the first arrow. Moreover, we get $M\left(\frac{X/W}{V/W}\right) = M_T(X)$ from the symmetry axiom (Sym). Note that we need (Sym)(b) and the naturality of ϵ_Δ to identify π with the natural map $M_{Z\sqcup T}(X) \to M_T(X)$.

Remark 2.3. *About the axioms.—*

(1) There is a stronger form of the excision axiom (Exc) usually called the Brown-Gersten property (or distinguished triangle). In the situation of axiom (Exc), with $U = X - Z$ and $W = V - T$, we consider the cone in the sense of [Nee01] of the morphism of distinguished triangles

$$\begin{tikzcd} M(W) \arrow[r] \arrow[d] & M(V) \arrow[r] \arrow[d] & M(V/W) \arrow[r] \arrow[d] & M(V)[1] \arrow[d] \\ M(U) \arrow[r] & M(X) \arrow[r] & M(X/U) \arrow[r] & M(U)[1] \end{tikzcd}$$

This is a *candidate triangle* in the sense of *op. cit.* of the form

$$M(W) \to M(U) \oplus M(V) \to M(X) \to M(W).$$

Thus, in our abstract setting, it is not necessarily a distinguished triangle. We call (BG) the hypothesis that in every such situation, the candidate triangle obtained above is a distinguished triangle. We will not need the hypothesis (BG) ; however, in the applications, it is always true and the reader may use this stronger form for simplification.

(2) We can replace axiom (Kun)(a) by a weaker one

(wKun) The restriction of M to the category of pairs of schemes (X, U) is a *lax* monoidal symmetric functor.

(Kun)(b) is then replaced by an obvious coherence property of the boundary operator in (Loc). This hypothesis is sufficient for the needs of the article with a notable exception of the duality pairing 5.23. For example, if one wants to work with cohomology theories directly, one has to use rather this axiom, replace $\mathscr{T}$ by an abelian category and "distinguished triangle" by "long exact sequence" everywhere. The arguments given here covers equally this situation, except for the general duality pairing.

(3) The symmetry axiom (Sym) encodes a part of a richer structure which possess the usual examples (all the ones considered in section 2.3). This is the structure of a derivator as the object $M(\Delta)$ may be seen as a homotopy colimit. The coherence axioms which appear in (Sym) are very natural from this point of view.

Definition 2.4. Let $\mathbb{E}$ be a premotive. For any smooth scheme X and any couple $(n, p) \in \mathbb{Z} \times \mathbb{Z}$, we define respectively the cohomology and the homology groups of X with coefficient in $\mathbb{E}$ as

$$\mathbb{E}^{n,p}(X) = \mathrm{Hom}_{\mathscr{T}}\big(M(X), \mathbb{E}(p)[n]\big),$$
$$\text{resp. } \mathbb{E}_{n,p}(X) = \mathrm{Hom}_{\mathscr{T}}\big(\mathbb{1}(p)[n], \mathbb{E} \otimes M(X)\big).$$

We refer to the corresponding bigraded cohomology group (resp. homology group) by $\mathbb{E}^{**}(X)$ (resp. $\mathbb{E}_{**}(X)$). The first index is usually refered to as the cohomological (resp. homological) *degree* and the second one as the cohomological (resp. homological) *twist*. We also define the module of coefficients attached to $\mathbb{E}$ as $\mathbb{E}^{**} = \mathbb{E}^{**}(S)$.
When $\mathbb{E} = \mathbb{1}$, we use the notations $H^{**}(X)$ (resp. $H_{**}(X)$) for the cohomology (resp. homology) with coefficients in $\mathbb{1}$. Finally, we simply put $A = H^{**}(S)$.

Remark that, from axiom (Kun)(a), A is a bigraded ring. Moreover, using the axiom (Stab), $A = H_{**}(S)$. Thus, there are two bigraduations on A, one cohomological and the other homological, and the two are exchanged as usual by a change of sign. The tensor product of morphisms in $\mathscr{T}$ induces a structure of left bigraded A-module on $\mathbb{E}^{**}(X)$ (resp. $\mathbb{E}_{**}(X)$). There is a lot more algebraic structures on these bigraded groups that we have gathered in section 2.2.
The axiom (Orient) gives a natural transformation

$$c_1 : \mathrm{Pic} \to H^{2,1}$$

of presheaves of *sets* on $\mathscr{S}m_S$, or in other words, an *orientation* on the fundamental cohomology $H^{*,*}$ assocciated with the functor M. In our setting, cohomology classes are morphisms in $\mathscr{T}$: for any element $L \in \mathrm{Pic}(X)$, we view $c_1(L)$ both as a cohomology class, the *first Chern class*, and as a morphism in $\mathscr{T}$.

Remark 2.5. In the previous definition, we can replace the premotive $M(X)$ by any premotive $\mathcal{M}$. This allows to define as usual the cohomology/homology of an (arbitrary) pair (X, U) made by a smooth scheme X and a smooth subscheme U of X. Particular cases of this general definition is the cohomology/homology of a smooth scheme X with support in a closed subscheme Z and the reduced cohomology/homology associated with a pointed smooth scheme.

2.2. **Products.** Let X be a smooth scheme and $\delta : X \to X \times X$ its associated diagonal embedding. Using axiom (Kun)(a) and functoriality, we get a morphism $\delta'_* : M(X) \to M(X) \otimes M(X)$. Given two morphisms $x : M(X) \to \mathbb{E}$ and $y : M(X) \to \mathbb{F}$ in $\mathscr{T}$, we can define a product

$$x \boxtimes y = (x \otimes y) \circ \delta'_* : M(X) \to \mathbb{E} \otimes \mathbb{F}.$$

2.6. By analogy with topology, we will call *ringed premotive* any premotive $\mathbb{E}$ equipped with a commutative monoid structure in the symmetric monoidal category $\mathscr{T}$. This means we have a product map $\mu : \mathbb{E} \otimes \mathbb{E} \to \mathbb{E}$ and a unit map $\eta : \mathbb{1} \to \mathbb{E}$ satisfying the formal properties of a commutative monoid.
For any smooth scheme X and any couple of integer $(n, p) \in \mathbb{Z}^2$, the unit map induces morphisms

$$\begin{aligned} \varphi_X :& H^{n,p}(X) \to \mathbb{E}^{n,p}(X) \\ \psi_X :& H_{n,p}(X) \to \mathbb{E}_{n,p}(X) \end{aligned}$$

which we call the *regulator maps*.
Giving such a ringed premotive $\mathbb{E}$, we define[10] the following products :

- *Exterior products* :
$$\begin{aligned} \mathbb{E}^{n,p}(X) \otimes \mathbb{E}^{m,q}(Y) &\to \mathbb{E}^{n+m,p+q}(X \times Y), \\ (x, y) &\mapsto x \times y := \mu \circ x \otimes y \\ \mathbb{E}_{n,p}(X) \otimes \mathbb{E}_{m,q}(Y) &\to \mathbb{E}_{n+m,p+q}(X \times Y), \\ (x, y) &\mapsto x \times y := (\mu \otimes 1_{X \times Y*}) \circ (x \otimes y) \end{aligned}$$
- *Cup-product* :
$$\mathbb{E}^{n,p}(X) \otimes \mathbb{E}^{m,q}(X) \to \mathbb{E}^{n+m,p+q}(X), (x, x') \mapsto x \cup x' := \mu \circ (x \boxtimes x').$$
Then $\mathbb{E}^{**}$ is a bigraded ring and $\mathbb{E}^{**}(X)$ is a bigraded $\mathbb{E}^{**}$-algebra. Moreover, $\mathbb{E}^{**}$ is a bigraded A-algebra and the regulator map is a morphism of bigraded A-algebra.
- *Slant products*[11] :
$$\begin{aligned} \mathbb{E}^{n,p}(X \times Y) \otimes \mathbb{E}_{m,q}(X) &\to \mathbb{E}^{n-m,p-q}(Y), \\ (w, x) &\mapsto w/x := \mu \circ (1_{\mathbb{E}} \otimes w) \circ (x \otimes 1_{Y*}) \\ \mathbb{E}^{n,p}(X) \otimes \mathbb{E}_{m,q}(X \times Y) &\to \mathbb{E}_{m-n,q-p}(Y), \\ (x, w) &\mapsto x \backslash w := (\mu \otimes 1_{Y*}) \circ (x \otimes 1_{\mathbb{E}} \otimes 1_{Y*}) \circ w. \end{aligned}$$

[10] We do not indicate the commutativity isomorphisms for the tensor product and the twists in the formulas to make them shorter.

[11] For the first slant product defined here, we took a slightly different covention than [Swi02, 13.50(ii)] in order to obtain formula (5.3). Of course, the two conventions coincide up to the isomorphism $X \times Y \simeq Y \times X$.

- *Cap-product* :

$$\mathbb{E}^{n,p}(X) \otimes \mathbb{E}_{m,q}(X) \to \mathbb{E}_{m-n,q-p}(X), (x, x') \mapsto x \cap x' := x \backslash \big((1_{\mathbb{E}} \otimes \delta_*) \circ x'\big).$$

- *Kronecker product* :

$$\mathbb{E}^{n,p}(X) \otimes \mathbb{E}_{m,q}(X) \to A^{n-m,p-q}, (x, x') \mapsto \langle x, x' \rangle := x/x'$$

where y is identified to a homology class in $\mathbb{E}_{m,q}(S \times X)$.

The regulator maps (cohomological and homological) are compatible with these products in the obvious way.

Remark 2.7. These products satisfy a lot of formal properties. We will not use them in this text but we refer the interested reader to [Swi02, chap. 13] for more details (see more precisely 13.57, 13.61, 13.62).

2.8. We can extend the definition of these products to the cohomology of an open pair (X, U). We refer the reader to *loc. cit.* for this extension but we give details for the cup-product in the case of cohomology with supports as this will be used in the sequel.

Let X be a smooth scheme and Z, T be two closed subschemes of X. Then the diagonal embedding of X/S induces using once again axiom (Kun)(a) a morphism $\delta''_* : M_{Z\cap T}(X) \to M_Z(X) \otimes M_T(X)$. This allows to define a product of motives with support. Given two morphisms $x : M_Z(X) \to \mathbb{E}$ and $y : M_T(X) \to \mathbb{F}$ in $\mathscr{T}$, we define

$$x \boxtimes y = (x \otimes y) \circ \delta''_* : M_{Z\cap T}(X) \to \mathbb{E} \otimes \mathbb{F}.$$

In cohomology, we also define the cup-product with support :

$$\mathbb{E}^{n,p}_Z(X) \otimes E^{m,q}_T(X) \to \mathbb{E}^{n+m,p+q}_{Z\cap T}(X), (x, y) \mapsto x \cup_{Z,T} y = \mu \circ (x \otimes y) \circ \delta''_*.$$

Note that considering the canonical morphism $\nu_{X,W} : \mathbb{E}^{n,p}_W(X) \to \mathbb{E}^{n,p}(X)$, for any closed subscheme W of X, we obtain easily :

$$\nu_{X,Z}(x) \cup \nu_{X,T}(y) = \nu_{X,Z\cap T}(x \cup_{Z,T} y). \tag{2.1}$$

2.9. Suppose now that $\mathbb{E}$ has no ring structure. It nethertheless always has a module structure over the ringed premotive $\mathbb{1}$ – given by the structural map (isomorphism) $\eta : \mathbb{1} \otimes \mathbb{E} \to \mathbb{E}$.

This induces in particular a structure of left $H^{**}(X)$-module on $\mathbb{E}^{**}(X)$ for any smooth scheme X. Moreover, it allows to extend the definition of slant products and cap products. Explicitely, this gives in simplified terms :

- *Slant products* :

$$\begin{aligned} H^{n,p}(X \times Y) \otimes \mathbb{E}_{m,q}(X) &\to \mathbb{E}^{n-m,p-q}(Y), \\ (w, y) &\mapsto w/y := \eta \circ (1_{\mathbb{E}} \otimes w) \circ (x \otimes 1_{Y*}) \end{aligned}$$

- *Cap-products* :

$$\mathbb{E}^{n,p}(X) \otimes H_{m,q}(X) \to \mathbb{E}_{m-n,q-p}(X), (x, x') \mapsto x \cap x' := (x \otimes 1_{X*}) \circ \delta_* \circ x'.$$

These generalized products will be used at the end of the article to formulate duality with coefficients in $\mathbb{E}$ (cf paragraph 5.24).

Note finally that, analog to the cap-product, we have a $H^{**}(X)$-module structure on $E_{**}(X)$ that can be used to describe the projective bundle formula in $\mathbb{E}$-homology (cf formula (2) of 3.4).

2.3. **Examples.**

2.3.1. *Motives.* Suppose S is a regular scheme. Below, we give the full construction of the category of geometric motives of Voevodsky over S, and indicate how to check the axioms of 2.1. Note however we will give a full construction of this category, together with the category of motivic complexes and spectra, over any noetherian base S in [CD07]. Here, the reader can find all the details for the proof of the axioms 2.1 (especially axiom (Orient)).
For any smooth schemes X and Y, we let $c_S(X,Y)$ be the abelian group of cycles in $X \times_S Y$ whose support is finite equidimensional over X. As shown in [Dég07, sec. 4.1.2], this defines the morphisms of a category denoted by $\mathscr{S}m^{\mathrm{cor}}_S$. The category $\mathscr{S}m^{\mathrm{cor}}_S$ is obviously additive. It has a symmetric monoidal structure defined by the cartesian product on schemes and by the exterior product of cycles on morphisms.
Following Voevodsky, we define the category of effective geometric motives $DM^{eff}_{gm}(S)$ as the pseudo-abelian envelope[12] of the Verdier triangulated quotient

$$\mathrm{K}^b(\mathscr{S}m^{\mathrm{cor}}_S)/\mathscr{T}$$

where $\mathrm{K}^b(\mathscr{S}m^{\mathrm{cor}}_S)$ is the category of bounded complexes up to chain homotopy equivalence and $\mathscr{T}$ is the thick subcategory generated by the following complexes :

(1) For any smooth scheme X,

$$\dots 0 \to \mathbb{A}^1_X \xrightarrow{p} X \to 0 \dots$$

with p the canonical projection.

(2) For any cartesian square of smooth schemes

$$\begin{tikzcd} W \arrow[r, "k"] \arrow[d, "g"'] & V \arrow[d, "f"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

such that j is an open immersion, f is étale and the induced morphism $f^{-1}(X-U)_{red} \to (X-U)_{red}$ is an isomorphism,

$$\dots 0 \to W \xrightarrow{\binom{g}{-k}} U \oplus V \xrightarrow{(j,f)} X \to 0 \dots \tag{2.2}$$

Consider a cartesian square of immersions

$$\begin{tikzcd} W \arrow[r, "k"] \arrow[d, "g"'] \arrow[dr, phantom, "\Delta"] & V \arrow[d, "f"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

[12] Recall that according to the result of [BS01], the pseudo-abelian envelope of a triangulated category is still triangulated

This defines a morphism of complexes in $\mathscr{S}m_S^{\mathrm{cor}}$:

$$\psi : \left\{ \begin{tikzcd} \dots 0 \arrow[r] & W \arrow[r, "k"] \arrow[d, "g"'] & V \arrow[r] \arrow[d, "f"] & 0 \dots \\ \dots 0 \arrow[r] & U \arrow[r, "j"] & X \arrow[r] & 0 \dots \end{tikzcd} \right.$$

We let $M(\Delta)$ be the cone of ψ and see it as an object of $DM_{gm}^{eff}(S)$. To fix the convention, we define this cone as the triangle (2.2) above. With this convention, we define ϵ_Δ as the following morphism :

$$\begin{tikzcd} \dots 0 \arrow[r] & W \arrow[r] \arrow[d, "-1"'] & U \oplus V \arrow[r] \arrow[d, "\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}"] & X \arrow[r] \arrow[d, "+1"] & 0 \dots \\ \dots 0 \arrow[r] & W \arrow[r] & V \oplus U \arrow[r] & X \arrow[r] & 0 \dots \end{tikzcd}$$

The reader can now check easily that the resulting functor $M : \mathscr{D} \to DM_{gm}^{eff}(S)$, satisfies all the axioms of 2.1 except (Stab) and (Orient). We let $\mathbb{Z} = M(S)$ be the unit object for the monoidal structure of $DM_{gm}^{eff}(S)$.
To force axiom (Stab), we formally invert the motive $\mathbb{Z}(1)$ in the monoidal category $DM_{gm}^{eff}(S)$. This defines the *triangulated category of (geometric) motives* denoted by $DM_{gm}(S)$. Remark that according to the proof of [Voe02, lem. 4.8], the cyclic permutation of the factors of $\mathbb{Z}(3)$ is the identity. This implies the monoidal structure on $DM_{gm}^{eff}(S)$ induces a unique monoidal structure on $DM_{gm}(S)$ such that the obvious triangulated functor $DM_{gm}^{eff}(S) \to DM_{gm}(S)$ is monoidal. Now, the functor $M : \mathscr{D} \to DM_{gm}(S)$ still satisfies all axioms of 2.1 mentioned above but also axiom (Stab).
To check the axiom (Orient), it is sufficient to construct a natural application

$$\mathrm{Pic}(X) \to \mathrm{Hom}_{DM_{gm}^{eff}(S)}(M(X), \mathbb{Z}(1)[2]).$$

We indicate how to obtain this map. Note moreover that, from the following construction, it is a morphism of abelian group.
Still following Voevodsky, we have defined in [Dég07] the abelian category of sheaves with transfers over S, denoted by $Sh(\mathscr{S}m_S^{\mathrm{cor}})$. We define the cateogy $DM^{eff}(S)$ of motivic complexes as the $\mathbb{A}^1$-localization of the derived category of $Sh(\mathscr{S}m_S^{\mathrm{cor}})$. The Yoneda embedding $\mathscr{S}m_S^{\mathrm{cor}} \to Sh(\mathscr{S}m_S^{\mathrm{cor}})$ sends smooth schemes to *free* abelian groups. For this reason, the canonical functor

$$DM_{gm}^{eff}(S) \to DM^{eff}(S)$$

is fully faithful. Let $\mathbb{G}_m$ be the sheaf with transfers which associates to a smooth scheme its group of invertible (global) functions. Following Suslin and Voevodsky (cf also [Dég05, 2.2.4]), we construct a morphism in $DM^{eff}(S)$:

$$\mathbb{G}_m \to M(\mathbb{G}_m) = \mathbb{Z} \oplus \mathbb{Z}(1)[1]$$

This allows to define the required morphism :

$$\begin{aligned} \mathrm{Pic}(X) = H^1_{\mathrm{Nis}}(X; \mathbb{G}_m) &\simeq \mathrm{Hom}_{DM^{eff}(S)}(M(X), \mathbb{G}_m[1]) \\ &\to \mathrm{Hom}_{DM^{eff}(S)}(M(X), \mathbb{Z}(1)[2]) \simeq \mathrm{Hom}_{DM_{gm}^{eff}(S)}(M(X), \mathbb{Z}(1)[2]). \end{aligned}$$

The first isomorphism uses that the sheaf $\mathbb{G}_m$ is $\mathbb{A}^1$-local and that the functor forgetting transfers is exact (cf [Dég07, prop. 2.9]).

2.3.2. *Stable homotopy exact functors.* In this example, S is any noetherian scheme. For any smooth scheme X, we let X_+ be the pointed sheaf of sets on $\mathscr{S}m_S$ represented by X with a (disjoint) base point added.
Consider an immersion $U \to X$ of smooth schemes. We let X/U be the pointed sheaf of sets which is the cokernel of the pointed map $U_+ \to X_+$.
Suppose moreover given a square Δ as in $(*)$. Then we obtain an induced morphism of pointed sheaves of sets $V/W \to X/U$ which is injective. We let $\frac{X/U}{V/W}$ be the cokernel of this monomorphism. Thus, we obtain a cofiber sequence in $\mathscr{H}_\bullet(S)$

$$V/W \to X/U \to \frac{X/U}{V/W} \xrightarrow{\partial_\Delta} S^1_s \wedge V/W.$$

Moreover, the functor

$$\mathscr{D} \to \mathscr{H}_\bullet(S), \Delta \mapsto \frac{X/U}{V/W}$$

satisfies axioms (Add), (Htp), (Exc) and (Kun) from [MV99].
Consider now the stable homotopy category of schemes $S\mathscr{H}(S)$ (cf [Jar00]) together with the infinite suspension functor

$$\Sigma^\infty : \mathscr{H}_\bullet(S) \to S\mathscr{H}(S).$$

The category $S\mathscr{H}(S)$ is a triangulated symmetric monoidal category. The canonical functor $\mathscr{D} \to S\mathscr{H}(S)$ satisfies all the axioms of 2.1 except axiom (Orient). In fact, (Loc) and (Sym) follows easily from the definitions and (Stab) was forced in the construction of $S\mathscr{H}(S)$.
Suppose we are given a triangulated symmetric monoidal category $\mathscr{T}$ together with a triangulated symmetric monoidal functor

$$R : S\mathscr{H}(S) \to \mathscr{T}.$$

This induces a canonical functor

$$M : \mathscr{D} \to \mathscr{T}, \frac{X/U}{V/W} \mapsto M\left(\frac{X/U}{V/W}\right) := R\left(\Sigma^\infty\left(\frac{X/U}{V/W}\right)\right)$$

and $(M, \mathscr{T})$ satisfies formally all the axioms 2.1 except (Orient).
Let $B\mathbb{G}_m$ be the classifying space of $\mathbb{G}_m$ defined in [MV99, section 4]. It is an object of the simplicial homotopy category $\mathscr{H}^s_\bullet(S)$ and from *loc. cit.*, proposition 1.16,

$$Pic(X) = \mathrm{Hom}_{\mathscr{H}^s_\bullet(S)}(X_+, B\mathbb{G}_m).$$

Let $\pi : \mathscr{H}^s_\bullet(S) \to \mathscr{H}_\bullet(S)$ be the canonical $\mathbb{A}^1$-localisation functor. Applying proposition 3.7 of *loc. cit.*, $\pi(B\mathbb{G}_m) = \mathbb{P}^\infty$ where $\mathbb{P}^\infty$ is the tower of pointed schemes

$$\mathbb{P}^1 \to .. \to \mathbb{P}^n \xrightarrow{\iota_n} \mathbb{P}^{n+1} \to ...$$

made of the inclusions onto the corresponding hyperplane at infinity. We let $M(\mathbb{P}^\infty)$ (resp. $\tilde{M}(\mathbb{P}^\infty)$) be the ind-object of $\mathscr{T}$ obtained by applying M (resp. $\tilde{M}$) on each degree of the tower above.

Using this, we can define an application

$$\begin{aligned}\rho_X :& Pic(X) = \mathrm{Hom}_{\mathscr{H}^s_\bullet(S)}(X_+, B\mathbb{G}_m) \\ &\to \mathrm{Hom}_{\mathscr{H}_\bullet(S)}(X_+, \pi(B\mathbb{G}_m)) = \mathrm{Hom}_{\mathscr{H}_\bullet(S)}(X_+, \mathbb{P}^\infty) \\ &\to \mathrm{Hom}_{\mathscr{T}}\Big(M(X), \tilde{M}(\mathbb{P}^\infty)\Big)\end{aligned}$$

where the last group of morphisms denotes by abuse of notations the group of morphisms in the category of ind-objects of $\mathscr{T}$ – and similarly in what follows.

Remark 2.10. Note that the sequence $(\lambda_n)_{n\in\mathbb{N}}$ of line bundles is sent by $\rho_{\mathbb{P}^\infty}$ to the canonical projection $M(\mathbb{P}^\infty) \to \tilde{M}(\mathbb{P}^\infty)$ – this follows from the construction of the isomorphism of *loc. cit.*, prop. 1.16.

Recall that $\mathbb{1}(1)[2] = \tilde{M}(\mathbb{P}^1)$ in $\mathscr{T}$. Let $\pi : M(\mathbb{P}^1) \to \tilde{M}(\mathbb{P}^1)$ be the canonical projection and $\iota : \mathbb{P}^1 \to \mathbb{P}^\infty$ be the canonical morphism of pointed ind-schemes. We introduce the following two sets :

- (S_1) The transformations $c_1 : \mathrm{Pic}(X) \to \mathrm{Hom}_{\mathscr{T}}(M(X), \mathbb{1}(1)[2])$ natural in the smooth scheme X such that $c_1(\lambda_1) = \pi$.
- (S_2) The morphisms $c_1' : \tilde{M}(\mathbb{P}^\infty) \to \tilde{M}(\mathbb{P}^1)$ such that $c_1' \circ \iota_* = 1$.

We define the following applications :

(1) $\varphi : (S_1) \to (S_2)$.
Consider an element c_1 of (S_1). The collection $\big(c_1(\lambda_n)\big)_{n\in\mathbb{N}}$ defines a morphism $M(\mathbb{P}^\infty) \to \tilde{M}(\mathbb{P}^1)$. Moreover, the restriction of this latter morphism $\tilde{M}(\mathbb{P}^\infty) \to \tilde{M}(\mathbb{P}^1)$ is obviously an element of (S_2), denoted by $\varphi(c_1)$.

(2) $\psi : (S_2) \to (S_1)$.
Let c_1' be an element of (S_2). For any smooth scheme X, we define

$$\psi(c_1') : \mathrm{Pic}(X) \xrightarrow{\rho_X} \mathrm{Hom}_{\mathscr{T}}\Big(M(X), \tilde{M}(\mathbb{P}^\infty)\Big) \xrightarrow{c'_{1*}} \mathrm{Hom}_{\mathscr{T}}\Big(M(X), \tilde{M}(\mathbb{P}^1)\Big).$$

Using remark 2.10, we check easily that $\psi(c_1')$ belongs to (S_1).

The following lemma is obvious from these definitions :

Lemma 2.11. *Given, the hypothesis and definitions above,* $\varphi \circ \psi = 1$.

Thus, an element of (S_2) determines canonically an element of (S_1). This gives a way to check the axiom (Orient) for a functor R as above. Moreover, we will see below (cf paragraph 3.7) that given an element of (S_1), we obtain a canonical isomorphism $H^{**}(\mathbb{P}^\infty) = A[[t]]$ of bigraded algebra, t having bidegree $(2,1)$. Then elements of (S_2) are in bijection with the set of generators of the the bigraded algebra $H^{**}(\mathbb{P}^\infty)$. Thus in this case, elements of (S_2) are equivalent to *orientations* of the cohomology H^{**} in the classical sense of algebraic topology.

Example 2.12. (1) Let $S = \mathrm{Spec}(k)$ be the spectrum of a field, or more generally any regular scheme. In [CD06, 2.1.4], D.C. Cisinski and the author introduce the notion of mixed Weil theory (and more generally of stable theory) as axioms for cohomology theories on smooth S-schemes which extends the classical axioms of Weil. Examples of such cohomology theories are algebraic De Rham cohomology if k has characteristic 0, rigid cohomology if k has caracteristic p and étale l-adic cohomology in any case,

l being invertible in k (cf part 3 of *loc. cit.*). To a mixed Weil theory (or more generally a stable theory) is associated a commutative ring spectrum (cf *loc. cit.* 2.1.5) and a triangulated closed symmetric monoidal category $D_{\mathbb{A}^1}(S, \mathcal{E})$ – which is obtained by localization of a derived category. By construction (see *loc. cit.* (1.5.3.1)), we have a triangulated monoidal symmetric functor

$$S\mathscr{H}(S) \to D_{\mathbb{A}^1}(S, \mathcal{E}).$$

In *loc. cit.* 2.2.9, we associate a canonical element of the set (S_2) for this functor. Thus the resulting functor $\mathscr{D} \to D_{\mathbb{A}^1}(S, \mathcal{E})$ satisfies all the axioms of 2.1.

(2) Consider a noetherian scheme S and the model category of symmetric T-spectra Sp_S over S defined by R. Jardine in [Jar00]. It is a cofibrantly generated, symmetric monoidal model category which satisfies the monoid axiom of [SS00, 3.1] (cf [Jar00, 4.19] for this latter fact).

A commutative monoid $\mathbb{E}$ in the category Sp_S will be called a (homotopy) *coherent* ring spectrum. Given such a ring spectrum, according to [SS00, 4.1(2)], the category of $\mathbb{E}$-modules in the symmetric monoidal category Sp_S carries a structure of a cofibrantly generated, symmetric monoidal model category such that the pair of adjoint functors $(F, \mathcal{O})$ given by the free $\mathbb{E}$-module functor and the obvious forgetful functor is a Quillen adjunction. We denote by $S\mathscr{H}(S; \mathbb{E})$ the associated homotopy category and consider the left derived free $\mathbb{E}$-module functor

$$S\mathscr{H}(S) \to S\mathscr{H}(S; \mathbb{E}).$$

It is a triangulated symmetric monoidal functor. Then, as indicated in the previous remark, an element of (S_2) relative to this functor is equivalent to an orientation on the ring spectrum $\mathbb{E}$ in the classical sense (see [Vez01, 3.1]).

The basic example of such a ring spectrum is the cobordism ring spectrum **MGL**. Indeed, **MGL** has a structure of a coherent ring spectrum in our sense and is evidently oriented (see [PPR07, 1.2.3 and 2.1] for details). Thus the homotopy category $S\mathscr{H}(S; \mathbf{MGL})$ of **MGL**-modules satisfies the axioms 2.1.

Another example is given by the spectrum **BGL** introduced by Voevodsky in [Voe98, par. 6.2]. According to *loc. cit.*, th. 6.9, it represents the homotopy invariant algebraic K-theory defined by Weibel (cf [Wei89]). However, it is not at all clear to get a coherent structure on the ring spectrum **BGL** with the definition given in *loc. cit.* To obtain such a coherent ring structure on **BGL** we invoke a recent result of Gepner and Snaith which construct a coherent ring spectrum homotopy equivalent to **BGL** in [DV07, 5.9].

3. Chern classes

3.1. **The projective bundle theorem.** Let X be a smooth scheme and P be a projective bundle over X of rank n. We denote by $p : P \to X$ the canonical projection and by λ the canonical line bundle on P. Put $c = c_1(\lambda) : M(P) \to \mathbb{1}(1)[2]$. We can define a canonical map :

$$\epsilon_P := \sum_{0 \leq i \leq n} p_* \boxtimes c^i : M(P) \to \bigoplus_{0 \leq i \leq n} M(X)(i)[2i]$$

Consider moreover an open subscheme $U \subset X$, $P_U = P \times_X U$. We let $\pi : P/P_U \to X/U$ be the canonical projection and $\nu : P/P_U \to (P \times P)/(P \times P_U)$ the morphism induced by the diagonal embedding and the graph of the immersion $P_U \to P$. Using the product of motives with support (cf 2.8), we also define a canonical map :

$$\epsilon_{P/P_U} := \sum_{0 \leq i \leq n} \pi_* \boxtimes c^i : M(P/P_U) \to \bigoplus_{0 \leq i \leq n} M(X/U)(i)[2i]$$

Lemma 3.1. *Using the above notations, the following diagram is commutative :*

$$\begin{tikzcd}
M(P_U) \arrow[r] \arrow[d, "\epsilon_{P_U}"'] \arrow[dr, phantom, "(1)"] & M(P) \arrow[r] \arrow[d, "\epsilon_P"'] \arrow[dr, phantom, "(2)"] & M(P/P_U) \arrow[r] \arrow[d, "\epsilon_{P/P_U}"'] \arrow[dr, phantom, "(3)"] & M(P_U)[1] \arrow[d, "\epsilon_{P_U}"] \\
\bigoplus_i M(U)(i)[2i] \arrow[r] & \bigoplus_i M(X)(i)[2i] \arrow[r] & \bigoplus_i M(X/U)(i)[2i] \arrow[r] & \bigoplus_i M(U)(i)[2i+1]
\end{tikzcd}$$

where the top (resp. bottom) line is the distinguished triangle (resp. sum of distinguished triangles) obtained using (Loc) *(resp. and tensoring with* $\mathbb{1}(i)[2i]$*).*

Proof. Coming back to the definition of product and product with supports, squares (1) and (2) are commutative by functoriality of M. For square (3), besides this functoriality, we have to use axiom (Kun)(b). □

Theorem 3.2. *With the above hypothesis and notations, the morphism* $\epsilon_P : M(P) \to \bigoplus_{0 \leq i \leq n} M(X)(i)[2i]$ *is an isomorphism in* $\mathscr{T}$*.*

Proof. Consider an open cover $X = U \cup V$, $W = U \cap V$. Assume that ϵ_{P_U}, ϵ_{P_V} and ϵ_{P_W} are isomorphisms. Then according to the previous lemma, ϵ_{P_V/P_W} is an isomorphism. Using the compatibility of the first Chern class with pullback, we obtain a commutative diagram

$$\begin{tikzcd}
M(P_V/P_W) \arrow[r] \arrow[d, "\epsilon_{P_V/P_W}"'] & M(P/P_U) \arrow[d, "\epsilon_{P/P_U}"] \\
\bigoplus_i M(V/W)(i)[2i] \arrow[r] & \bigoplus_i M(X/U)(i)[2i]
\end{tikzcd}$$

where the horizontal maps are obtained by functoriality. According to axiom (Exc), these maps are isomorphisms which implies ϵ_{P/P_U} is an isomorphism. Applying ance again the previous lemma, we deduce that ϵ_P is an isomorphism.

This reasoning shows that we can argue locally on X and assume P is trivializable as a projective bundle over X. Then, as the map depends only on the isomorphism class of the projective bundle P, we can assume $P = \mathbb{P}^n_X$. Finally, by property (Kun)(a), $\epsilon_{\mathbb{P}^n_X} = M(X) \otimes \epsilon_{\mathbb{P}^n}$ and we can assume $X = S$. Put simply $\epsilon_n = \epsilon_{\mathbb{P}^n}$.

For $n = 0$, the statement is trivial. Assume $n > 0$. Recall we consider the scheme $\mathbb{P}^n$ pointed by the infinite point. The morphism ϵ_n induces a map $\tilde{M}(\mathbb{P}^n) \to \oplus_{0<i\leq n} \tilde{M}(\mathbb{P}^1)^{\otimes, i}$ still denoted by ϵ_n and we have to prove this later is an isomorphism. Put $c_{1,n} = c_1(\lambda_n)$ for any integer $n \geq 0$.

The canonical inclusion $\mathbb{P}^{n-1} \to \mathbb{P}^n - \{0\}$ is the zero section of a vector bundle. For any integer $i \in [1,n]$, we put $U_i = \{(x_1, ..., x_n) \in \mathbb{A}^n \mid x_i \neq 0\}$ considered as an open subscheme of $\mathbb{A}^n$. We obtain the canonical isomorphism denoted by τ_n :

$$\begin{aligned} M(\mathbb{P}^n/\mathbb{P}^{n-1}) &\overset{(1)}{\simeq} M(\mathbb{P}^n/\mathbb{P}^n - \{0\}) \overset{(2)}{\simeq} M(\mathbb{A}^n/\mathbb{A}^n - \{0\}) \\ &= M(\mathbb{A}^n/\cup_i U_i) \overset{(3)}{=} M(\mathbb{A}^1/\mathbb{A}^1 - \{0\})^{\otimes,n} \overset{(4)}{\simeq} \tilde{M}(\mathbb{P}^1)^{\otimes,n} \end{aligned}$$

where (1) follows from (Htp) and (Loc), (2) from (Exc), (3) from (Kun)(a) and (4) from (Exc), (Htp) and (Loc).
Consider the following diagram

$$\text{(3.1)} \qquad \begin{tikzcd} \tilde{M}(\mathbb{P}^{n-1}) \arrow[r, "\iota_{n-1*}"] \arrow[d, "\epsilon_{n-1}"'] \arrow[dr, phantom, "(a)"] & \tilde{M}(\mathbb{P}^n) \arrow[r, "\pi_n"] \arrow[d, "\epsilon_n"'] \arrow[dr, phantom, "(b)"] & M(\mathbb{P}^n/\mathbb{P}^{n-1}) \arrow[d, "\tau_n"] \\ \oplus_{0<i<n}\tilde{M}(\mathbb{P}^1)^{\otimes,i} \arrow[r] & \oplus_{0<i\leq n}\tilde{M}(\mathbb{P}^1)^{\otimes,i} \arrow[r] & \tilde{M}(\mathbb{P}^1)^{\otimes,n} \end{tikzcd}$$

where ι_{n-1} is the canonical inclusion, π_n is the obvious morphism obtained by functoriality in $\mathscr{D}$, and the bottom line is made up of the evident split distinguished triangle. We prove by induction on $n > 0$ the following statement :

(3.2)
- (i) ι_{n-1*} is a split monomorphism.
- (ii) $c^n_{1,n-1} = 0$ which means square (a) is commutative.
- (iii) $c^n_{1,n} = \tau_n\pi_n$ which means square (b) is commutative.
- (iv) ϵ_n is an isomorphism.

For $n = 1$, this is obvious as (iii) is a part of axiom (Orient).
The induction relies on the following lemma due to Morel.

Lemma 3.3. *Let $\delta_n : \mathbb{P}^n \to (\mathbb{P}^n)^n$ be the iterated n-th diagonal of $\mathbb{P}^n/S$ and denote by $\delta_{n*} : \tilde{M}(\mathbb{P}^n) \to \tilde{M}(\mathbb{P}^n)^{\otimes,n}$ the morphism induced by δ_n and axiom* (Kun)(a). *Let $\iota_{1,n} : \mathbb{P}^1 \to \mathbb{P}^n$ be the canonical inclusion.*
Then the following square commutes :

$$\begin{tikzcd} \tilde{M}(\mathbb{P}^n) \arrow[r, "\delta_{n*}"] \arrow[d, "\pi_n"'] & \tilde{M}(\mathbb{P}^n)^{\otimes,n} \\ M(\mathbb{P}^n/\mathbb{P}^{n-1}) \arrow[r, "\tau_n"] & \tilde{M}(\mathbb{P}^1)^{\otimes,n}. \arrow[u, "(\iota_{1,n*})^{\otimes,n}"'] \end{tikzcd}$$

Consider an integer $i \in [1,n]$ and let $\bar{U}_i$ be the open subscheme of $\mathbb{P}^n$ made of points $(x_1 : ... : x_n : x_{n+1})$ such that $x_i \neq 0$ and put $\Omega_i = \mathbb{P}^{i-1} \times U_i \times \mathbb{P}^{n-i}$.
We consider the following commutative diagram :

$$\begin{tikzcd} \tilde{M}(\mathbb{P}^n) \arrow[r, "(1)"] \arrow[d, "\pi_n"] & M\big((\mathbb{P}^n)^n/\cup_i \Omega_i\big) & \\ M(\mathbb{P}^n/\mathbb{P}^{n-1}) \arrow[r, "\sim"] & M\big(\mathbb{P}^n/\cup_i \bar{U}_i\big) \arrow[u] & M(\mathbb{A}^n/\cup_i U_i) \arrow[l, "\sim"'] \arrow[ul, "(2)"'] \end{tikzcd}$$

where the map (1) is induced by δ_n, the maps on the lower horizontal line are isomorphisms given respectively by the inclusions $\mathbb{P}^{n-1} \subset \cup_i\bar{U}_i$ and $U_i \subset \bar{U}_i$.
Consequently, the map (2) is induced by the restriction of δ_n. However, this map is

$\mathbb{A}^1$-homotopic to the product $\iota^{(1)} \times ... \times \iota^{(n)}$ where $\iota^{(i)} : \mathbb{A}^1 \to \mathbb{P}^n$ is the embedding defined by $\iota^{(i)}(x) = (x_1 : ... : x_{n+1})$ with $x_j = 0$ if $j \notin \{i, n+1\}$, $x_i = x$, $x_{n+1} = 1$. It follows from property (Htp) and (Kun)(a) that the map (2) is equal to the morphism

$$M(\mathbb{A}^1/\mathbb{A}^1 - \{0\})^{\otimes, n} \xrightarrow{\iota_*^{(1)} \otimes ... \otimes \iota_*^{(n)}} M(\mathbb{P}^n/\bar{U}_1) \otimes ... \otimes M(\mathbb{P}^n/\bar{U}_n).$$

Note finally the scheme $\bar{U}_i \simeq \mathbb{A}^n$ is contractible and, from property (Htp), the corresponding map $\iota_*^{(i)} : M(\mathbb{A}^1/\mathbb{A}^1 - \{0\}) \to \tilde{M}(\mathbb{P}^n)$ does not depend on the integer i. Thus the preceding commutative diagram together with the identifications just described allows to conclude.

With that lemma in hand, we conclude as follows. Suppose the property (3.2) is true for $n-1$.

The composite map $\left(\sum_{0<i<n} p_* \boxtimes c_n^i\right) \circ \iota_{n-1}$ is equal to ϵ_{n-1} as $c_n \circ \iota_{n-1*} = c_{n-1}$. This shows (3.2)(i). Then, the preceding lemma implies properties (ii) and (iii). Now, using (Loc) and (Sym), the upper horizontal line of diagram (3.1) is a split distinguished triangle which concludes. □

Using axiom (Stab), we obtain the following corollary :

Corollary 3.4. *Consider the hypothesis and notations of the previous theorem. Then $H^{**}(P)$ is a free $H^{**}(X)$-module with base $1, ..., c^n$.*
Let $\mathbb{E}$ be a motive.

(1) *The map*

$$\mathbb{E}^{**}(X) \otimes_{H^{**}(X)} H^{**}(P) \to \mathbb{E}^{**}(P), x \otimes \lambda \to \lambda.p^*(x)$$

is an isomorphism. If moreover $\mathbb{E}$ has a ringed motive structure, it is an isomorphism of $\mathbb{E}^{**}(X)$-algebra.

(2) *Considering the $H^{**}(X)$-module structure on $\mathbb{E}_{**}(X)$ (cf the end of 2.9), the map*

$$\mathbb{E}_{**}(P) \to \bigoplus_{0\leq i\leq n} \mathbb{E}_{**}(X), \varphi \mapsto \sum_i c^i \cap p_*(\varphi)$$

is an isomorphism.

Remark 3.5. It can be seen actually that the first assertion of this corollary is equivalent to the fact $H^{n,m}(M(X)(r)) = H^{n,m-r}(M(X))$ which is a weak form of the stability axiom (Stab).

A corollary of the projective bundle theorem is the following result, classical in topology and first exploited in the homotopy category of schemes by Morel :

Corollary 3.6. *Consider the permutation isomorphism $\eta : \mathbb{1}(1) \otimes \mathbb{1}(1) \to \mathbb{1}(1) \otimes \mathbb{1}(1)$ in the symmetric monoidal category $\mathscr{T}$. Then $\eta = 1$.*
Let $\mathbb{E}$ be a ringed motive and X be a smooth scheme.
For any $x \in \mathbb{E}^{n,p}(X)$ and $y \in \mathbb{E}^{m,q}(X)$, $x \cup y = (-1)^{nm}.y \cup x$.

Proof. In general, for $x \in \mathbb{E}^{n,p}(X)$ and $y \in \mathbb{E}^{m,q}(X)$, we have $x \cup y = (-1)^{nm}\eta^{pq}.y \cup x$. In particular, when $X = \mathbb{P}^2$ and $c = c_1(\lambda_2)$, we get $c^2 = \eta.c^2$. This implies $\eta = 1$ from the previous corollary and the other assertion follows. □

3.2. **The associated formal group law.**

3.7. Put $H^{**}(\mathbb{P}^\infty) = \varprojlim_{n>0} H^{**}(\mathbb{P}^n)$. Then corollary 3.4 together with the relation (3.2)(ii) implies $H^{**}(\mathbb{P}^\infty) = A[[c]]$, free ring of power series over A with generator $c = (c_{1,n})_{n>0}$ of degree $(2,1)$. Moreover, $H^{**}(\mathbb{P}^\infty \times \mathbb{P}^\infty) = A[[x,y]]$.
Consider the Segre embeddings $\sigma_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{n+m+nm}$ for $(n,m) \in \mathbb{N}^2$ and the induced map on ind-schemes $\sigma : \mathbb{P}^\infty \times \mathbb{P}^\infty \to \mathbb{P}^\infty$. Then the map $\sigma^* : H^{**}(\mathbb{P}^\infty) \to H^{**}(\mathbb{P}^\infty \times \mathbb{P}^\infty)$ corresponds to a power series

$$F = \sum_{i,j} a_{ij}.x^i y^j \in A[[x,y]]$$

which according to the classical situation[13] in algebraic topology is a commutative formal group law :

$$F(x,0) = x, F(x,y) = F(y,x), F(x,F(y,z)) = F(F(x,y),z).$$

For any $(i,j) \in \mathbb{N}^2$, the element $a_{i,j} \in A$ is of homological degree $(2(i+j-1), i+j-1)$ and the first two relations above are equivalent to

$$a_{0,1} = 1, a_{0,i} = 0 \text{ if } i \neq 1, a_{i,j} = a_{j,i}.$$

Recall also there is a formal inverse associated to F, that is a formal power series $m \in A[[x]]$ such that $F(x, m(x)) = 0$. We can find the notation $x +_F y = F(x,y)$ in the litterature. For an integer $n \geq 0$, we put $[n]_F \cdot x = x +_F \ldots +_F x$, that is the power series in x equal to the formal n-th addition of x with itself. These notations will be fixed through the rest of the article.

Proposition 3.8. *Let X be a smooth scheme.*

(1) *For any line bundle L/X, the class $c_1(L)$ is nilpotent in $H^{**}(X)$.*
(2) *Suppose X admits an ample line bundle. For any line bundles L, L' over X,*

$$c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2)) \in H^{2,1}(X).$$

Proof. For the first point, we first remark the question is local in X. As X is noetherian, we are reduced by induction to consider an open covering $X = U \cup V$, such that $c_1(L_U)$ (resp. $c_1(L_V)$) is nilpotent in $H^{**}(U)$ (resp. $H^{**}(V)$) where L_U (resp. L_V) is the restrion of L to U (resp. V). Let n (resp. m) be the order of nilpotency of $c_1(L_U)$ (resp. $c_1(L_V)$). Let $Z = X - U$ (resp. $T = X - V$) and consider the canonical morphism $\nu_{X,W} : H^{**}_W(X) \to H^{**}(X)$ for $W = Z, T$. From axiom (Loc), there exists a class a (resp. b) in $H^{**}_Z(X)$ (resp. $H^{**}_T(X)$ such that $a = c_1(L)^n$ (resp. $b = c_1(L)^m$). As $Z \cap T = \emptyset$, axiom (Loc) implies $a \cup_{Z,T} b = 0$. Thus, relation (2.1) implies $c_1(L)^{n+m} = 0$ as wanted.
The first point follows, as λ is locally trivial and the Chern class of a trivial line bundle is 0 by definition.
For the second point, the assumption implies there is a torsor $\pi : X' \to X$ under a vector bundle over X such that X' is affine. From axioms (Htp') and (Exc), we

[13]Recall these properties follows from the fact that the coefficients a_{ij} for $i \leq n$, $j \leq m$ are determined by the map $\sigma_{n,m}$. The reader can find a more detailed proof in [LM07], proof of cor. 10.6.

obtain that $\pi_* : M(X') \to M(X)$ is an isomorphism. Thus we are reduced to the case where X is affine.
Then, the line bundle L is generated by its section (cf [EGA2, 5.1.2,e]), which means there is a closed immersion $L \xrightarrow{\iota} \mathbb{A}^{n+1}_X$ where $n+1$ is the cardinal of a generating family. In particular, we get a morphism

$$f : X \simeq \mathbb{P}(L) \xrightarrow{\iota} \mathbb{P}^n_X \to \mathbb{P}^n$$

with the property that $f^{-1}(\lambda_n) = L$. In the same way, we can find a morphism $g : X \to \mathbb{P}^m$ such that $g^{-1}(\lambda_m) = L'$. We consider the morphism

$$\varphi : X \to X \times X \xrightarrow{f \times g} \mathbb{P}^n \times \mathbb{P}^m \xrightarrow{\sigma_{n,m}} \mathbb{P}^{nm+n+m}.$$

By construction, $\varphi^{-1}(\lambda_{nm+n+m}) = L \otimes L'$ and this concludes, computing in two ways the Chern class of this line bundle. $\square$

Consider a ringed motive $\mathbb{E}$ with regulator map $\varphi : H^{**} \to \mathbb{E}^{**}$.
The map $\sigma^* : \mathbb{E}^{**}(\mathbb{P}^\infty) \to \mathbb{E}^{**}(\mathbb{P}^\infty \times \mathbb{P}^\infty)$ defines a formal group law $F_{\mathbb{E}}$ with coefficients in $\mathbb{E}^{**}$ and $F_{\mathbb{E}} = \sum_{i,j} \varphi_S(a_{i,j}).x^i y^j$. Thus the regulator map induces a morphism of formal group law $(A, F) \to (\mathbb{E}^{**}, F_{\mathbb{E}})$.

Remark 3.9. In case F is the additive formal group law, $F(x, y) = x + y$, for any ringed motive $\mathbb{E}$, $F_{\mathbb{E}}$ is the additive formal group law. This is the case for example if $\mathscr{T} = DM_{gm}(S)$ or $\mathscr{T}$ is the category of modules over a mixed Weil theory.
When F is the universal multiplicative formal group law $F = x + y + \beta.xy$, the obstruction for $F_{\mathbb{E}}$ to be additive is the element $\varphi(\beta)$.

3.3. **Higher Chern classes.** We now follow the classical approach of Grothendieck to define higher Chern classes. Consider a vector bundle E of rank $n > 0$ over a smooth scheme X. Let λ (resp. p) be the canonical invertible sheaf (resp. projection) of the projective bundle $\mathbb{P}(E)/X$. From corollary 3.4, there are unique classes $c_i(E) \in H^{2i,i}(X)$ for $i = 0, ..., n$, such that

$$\sum_{i=0}^{n} p^*(c_i(E)) \cup \big(-c_1(\lambda)\big)^{n-i} = 0 \tag{3.3}$$

and $c_0(E) = 1$.

Definition 3.10. With the above notations, we call $c_i(E)$ the i-th Chern class of E. We also put $c_i(E) = 0$ for any integer $i > n$.

Remark 3.11. In the case $n = 1$, due to our choice of conventions, $\lambda = p^{-1}(E)$. The previous relation is not a definition, but a tautology. This enlighten particularly our choice of sign in the previous relation. Besides, when $c_1(\lambda^\vee) = -c_1(\lambda)$ (in particular when the formal group law F is additive), relation (3.3) agrees precisely with that of [Gro58].

Remark 3.12. Considering any ringed motive $\mathbb{E}$, with regulator map $\varphi : H \to \mathbb{E}$, $\varphi \circ c_i$ defines Chern classes for cohomology with coefficients in $\mathbb{E}$. When no ringed structure is given on $\mathbb{E}$, we still get an action of the former Chern classes on the $\mathbb{E}$-cohomology using the action of the cohomology theory H (cf 2.9).

The Chern classes are obviously functorial with respect to pullback and invariant under isomorphism of vector bundles. They also satisfy the Whitney sum formula ; we recall the proof to the reader as it uses the axiom (Kun)(a) in an essential way.

Lemma 3.13. *Let X be a smooth scheme and consider an exact sequence of vector bundles over X :*

$$0 \to E' \to E \to E'' \to 0$$

Then for any $k \in \mathbb{N}$, $c_k(E) = \sum_{i+j=k} c_i(E') \cup c_j(F'')$.

Proof. By compatibility of Chern classes with pullback we can assume the sequence above is split. Let n (resp. m) be the rank of E'/X (resp. E''/X). Put $P = \mathbb{P}(E)$ and consider $c \in H^{2,1}(P)$ (resp. $p : P \to X$) the first Chern class of the canonical line bundle on (resp. canonical projection of) P/X.
Put $a = \sum_{i=0}^{n} p^*(c_i(E')).c^{n-i}$ and $b = \sum_{j=0}^{m} p^*(c_j(E'')).c^{m-j}$ as cohomology classes in $H^{**}(P)$. We have to prove $a \cup b = 0$.
Consider the canonical embeddings $i : \mathbb{P}(E') \to P$ and $j : P - \mathbb{P}(E'') \to P$. Then $i^*(a) = 0$ which implies by property (Htp') that $j^*(a) = 0$. Thus there exists $a' \in H^{*,*}_{\mathbb{P}(E'')}(P)$ such that $a = \nu_F(a')$ where $\nu_F : H^{*,*}_{\mathbb{P}(E'')}(P) \to H^{**}(P)$ is the canonical morphism. Similarly, there exists $b' \in H^{*,*}_{\mathbb{P}(E')}(P)$ such that $b = \nu_E(b')$ where $\nu_E : H^{*,*}_{\mathbb{P}(E')}(P) \to H^{**}(P)$ is the canonical morphism. Then, relation (2.1) allows[14] to conclude because $\mathbb{P}(E') \cap \mathbb{P}(E'') = \emptyset$ in P and $H^{*,*}_{\emptyset}(P) = 0$ from property (Loc). □

Remark 3.14. Suppose X admits an ample line bundle and consider a vector bundle E/X. As a corollary of the first point of proposition 3.8 and the usual splitting principle, we obtain that the class $c_n(E)$ is nilpotent in $H^{**}(X)$ for any integer $n \geq 0$.

4. The Gysin triangle

In this section, we consider closed pairs (X, Z) – recall X is assumed to be smooth and Z is a closed subscheme of X. We say (X, Z) is *smooth* (resp. of *codimension* n) if Z is smooth (resp. has everywhere codimension n in X). A *morphism* of closed pair $(f, g) : (Y, T) \to (X, Z)$ is a commutative square

$$\begin{tikzcd} T \arrow[r] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ Z \arrow[r] & X \end{tikzcd}$$

which is cartesian on the underlying topological space. This means the canonical embedding $T \to Z \times_X Y$ is a thickening. We say the morphism is *cartesian* if the square is cartesian.
The premotive $M_Z(X)$ is functorial with respect to morphisms of closed pairs.

[14]This is where axiom (Kun)(a) is used.

4.1. **Purity isomorphism.** Consider a projective bundle over a smooth scheme X of rank n. For any integer $0 \leq r \leq n$, we will consider the embedding[15]

$$\mathfrak{l}_r(P) : M(X)(r)[2r] \xrightarrow{(-1)^r} \bigoplus_{0 \leq i \leq n} M(X)(i)[2i] \xrightarrow{\epsilon_{P/X}^{-1}} M(P).$$

where the first map is the canonical embedding time $(-1)^r$ and the second one is induced by the isomorphism of theorem 3.2.

4.1. Consider a smooth closed pair (X, Z). Let $N_Z X$ (resp. $B_Z X$) be the normal bundle (resp. blow-up) of (X, Z) and $P_Z X$ be the projective completion of $N_Z X$. We denote by $B_Z(\mathbb{A}^1_X)$ the blow-up of $\mathbb{A}^1_X$ with center $\{0\} \times Z$. It contains as a closed subscheme the trivial blow-up $\mathbb{A}^1_Z = B_Z(\mathbb{A}^1_Z)$. We consider the closed pair $(B_Z(\mathbb{A}^1_X), \mathbb{A}^1_Z)$ over $\mathbb{A}^1$. Its fiber over 1 is the closed pair (X, Z) and its fiber over 0 is $(B_Z X \cup P_Z X, Z)$. Thus we can consider the following deformation diagram :

$$(X, Z) \xrightarrow{\bar{\sigma}_1} (B_Z(\mathbb{A}^1_X), \mathbb{A}^1_Z) \xleftarrow{\bar{\sigma}_0} (P_Z X, Z). \tag{4.1}$$

We will also consider the open subscheme $D_Z X = B_Z(\mathbb{A}^1_X) - B_Z X$, which still contains $\mathbb{A}^1_Z$ as a closed subscheme. The previous diagram then gives by restriction a second deformation diagram :

$$(X, Z) \xrightarrow{\sigma_1} (D_Z X, \mathbb{A}^1_Z) \xleftarrow{\sigma_0} (N_Z X, Z). \tag{4.2}$$

Note these two deformation diagrams are functorial in (X, Z) with respect to cartesian morphisms of closed pairs.

Remark 4.2. As we will see in the followings, one of the advantage to consider the deformation space $D_Z X$ is that, when X is a vector bundle over Z and the embedding $Z \subset X$ is the 0-section, we can define a canonical isomorphism $D_Z X \simeq \mathbb{A}^1 \times X$. In fact, when $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$, $D_Z X = \mathrm{Spec}(\oplus_{n \in \mathbb{Z}} I^n.t^{-n})$ with the convention that for $n < 0$, $I^n = A$ (t is an indeterminate). Thus, if $A = A_0[x_1, ..., x_n]$, $I = (x_1, ..., x_n)$, we get an isomorphism defined on the affine level by

$$A[t', x'_1, ..x'_n] \to \oplus_{n \in \mathbb{Z}} I^n.t^{-n}, t' \mapsto t, x'_i \mapsto t^{-1} x_i.$$

This isomorphism is independant on the regular sequence parametrizing I. Thus, in the case when X is an arbitrary vector bundle, we can glue the isomorphisms obtained by choosing local parametrizations.

Proposition 4.3. *Let n be a natural integer.*
There exists a unique family of isomorphisms of the form

$$\mathfrak{p}_{(X,Z)} : M_Z(X) \to M(Z)(n)[2n]$$

indexed by smooth closed pairs of codimension n such that :

[15] The change of sign which appears in this formula amounts to take $-c$ instead of c as a generator of the algebra $H^{**}(P)$.

(1) *for every cartesian morphism $(f,g) : (Y,T) \to (X,Z)$ of smooth closed pairs of codimension n, the following diagram is commutative :*

$$\begin{tikzcd}
M_T(Y) \arrow[r, "(f,g)_*"] \arrow[d, "\mathfrak{p}_{(Y,T)}"'] & M_Z(X) \arrow[d, "\mathfrak{p}_{(X,Z)}"] \\
M(T)(n)[2n] \arrow[r, "g_*(n)[2n]"] & M(Z)(n)[2n].
\end{tikzcd}$$

(2) *Let X be a smooth scheme, E be a vector bundle over X of rank n. Put $P = \mathbb{P}(E \oplus 1)$. Consider the closed pair (P, X) corresponding to the canonical section of P/X. Then $\mathfrak{p}_{(P,X)}$ is the inverse of the following composition*

$$M(X)(n)[2n] \xrightarrow{\mathfrak{l}_n(P)} M(P) \xrightarrow{\pi} M_X(P)$$

where the second arrow is obtained by functoriality in $\mathscr{D}$.

Proof. Uniqueness : Consider a smooth closed pair (X,Z) of codimension n. Applying property (1) above to the deformation diagram (4.1), we obtain the following commutative diagram :

$$\begin{tikzcd}
M_Z(X) \arrow[r, "\bar\sigma_{1*}"] \arrow[d, "\mathfrak{p}_{(X,Z)}"'] & M_{\mathbb{A}^1_Z}(B_Z(\mathbb{A}^1_X)) \arrow[d, dashed, "\mathfrak{p}_{(B_Z(\mathbb{A}^1_X),\mathbb{A}^1_Z)}"] & M_Z(P_Z X) \arrow[l, "\bar\sigma_{0*}"'] \arrow[d, "\mathfrak{p}_{(P_Z X,Z)}"] \\
M(Z)(n)[2n] \arrow[r, "s_{1*}(n)[2n]"] & M(\mathbb{A}^1_Z)(n)[2n] & M(Z)(n)[2n] \arrow[l, "s_{0*}(n)[2n]"']
\end{tikzcd}$$

The morphisms $s_0, s_1 : Z \to \mathbb{A}^1_Z$ are respectively the zero section and the unit section of $\mathbb{A}^1_Z/Z$. Using axiom (Htp), $s_{0*} = s_{1*}$. Thus in the above diagram, all morphisms are isomorphisms. Now, property (2) stated previously determines uniquely $\mathfrak{p}_{(P_Z X,Z)}$, thus $\mathfrak{p}_{(X,Z)}$ is also uniquely determined.

Existence : Consider property (2). Let $i : \mathbb{P}(E) \to P$ be the canonical embedding. Its corestriction $i' : \mathbb{P}(E) \to P - X$ is the zero section of a vector bundle, thus it induces an isomorphism on premotives from property (Htp'). By (Loc), we then obtain the distinguished triangle :

$$M(\mathbb{P}(E)) \xrightarrow{i_*} M(P) \xrightarrow{\pi} M_X(P) \xrightarrow{+1}$$

We easily obtain $\mathfrak{l}_r(\mathbb{P}(E)) \circ i_* = \mathfrak{l}_r(P)$ for any integer $r < n$. Thus the composite $\mathfrak{l}_n(P) \circ \pi$ is an isomorphism as required. We put: $\mathfrak{p}_{(P,X)} = \big(\mathfrak{l}_n(P) \circ \pi\big)^{-1}$.

Considering the proof of uniqueness, we have to show that $\bar\sigma_{0*}$ and $\bar\sigma_{1*}$ are isomorphisms. Considering the excision axiom (Exc), this is equivalent to prove the morphisms

$$M_Z(X) \xrightarrow{\sigma_{1*}} M_{\mathbb{A}^1_Z}(D_Z(X)) \xleftarrow{\sigma_{0*}} M_Z(N_Z X)$$

induced by diagram (4.2) are isomorphisms. In the case $X = \mathbb{A}^n_Z$ and the inclusion $Z \subset X$ is the 0-section, the result follows from remark 4.2 and axiom (Htp).

We can argue locally for the Zariski topology on X. In fact, consider an open cover $X = U \cup V$, $W = U \cap V$, such that the case of $(U, Z \cap U)$, $(V, Z \cap V)$ and $(W, Z \cap W)$ are known. Using axiom (Sym), (Exc) and (Loc), the canonical map

$$M\left(\frac{V/V - Z \cap V}{W/W - Z \cap W}\right) \to M\left(\frac{X/X - Z}{U/U - Z \cap U}\right)$$

is an isomorphism, and the same is true when we replace (X,Z) by $(D_Z X, \mathbb{A}^1_Z)$. This fact, together with the above three assumptions and axiom (Loc), allows to obtain the result for $(D_Z X, \mathbb{A}^1_Z)$.
Thus we can assume there exists a parametrisation of the closed pair (X,Z), that is to say a cartesian morphism $(f,g) : (X,Z) \to (\mathbb{A}^{d+n}_S, \mathbb{A}^d_S)$ such that f is étale. Consider the pullback square

$$\begin{tikzcd} X' \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d, "f"] \\ \mathbb{A}^n_Z \arrow[r, "1\times g"] & \mathbb{A}^{n+d}_S. \end{tikzcd}$$

There is an obvious closed immersion $Z \to X'$ and its image is contained in $q^{-1}(Z)$. As q is étale, Z is a direct factor of $q^{-1}(Z)$. Put $W = q^{-1}(Z) - Z$ and $\Omega = X' - W$. Thus Ω is an open subscheme of X', and the reader can check that p and q induce cartesian étale morphisms

$$(X,Z) \leftarrow (\Omega, Z) \rightarrow (\mathbb{A}^n_Z, Z).$$

The functorialty of (4.2) and axiom (Exc) allow to conclude in view of the previous case.
To sum up, the purity isomorphism $\mathfrak{p}_{(X,Z)}$ is defined as the composite

$$M_Z X \xrightarrow{\bar{\sigma}_{0*}} M_{\mathbb{A}^1_Z}\big(B_Z(\mathbb{A}^1_X)\big) \xrightarrow{\bar{\sigma}_{1*}^{-1}} M_Z(P_Z X) \xrightarrow{\mathfrak{p}_{(Z,P_Z X)}} M(Z)(n)[2n].$$

We finally have to check the coherence of this definition in the case of the closed pair (P,X), $P = \mathbb{P}(E \oplus 1)$, appearing in property (2). Explicitely, we have to check that in this case $\bar{\sigma}_{1*}^{-1} \circ \bar{\sigma}_{0*} = 1$. This is easily seen considering the commutative diagram :

$$\begin{tikzcd} M_X(P) \arrow[r, "\bar{\sigma}_{1*}"] & M_{\mathbb{A}^1_X}(B_Z(\mathbb{A}^1_X)) & M_X(P) \arrow[l, "\bar{\sigma}_{0*}"'] \\ M_X(E) \arrow[u] \arrow[r, "\sigma_{1*}"] & M_{\mathbb{A}^1_X}(D_X E) \arrow[u] & M_X(E). \arrow[u] \arrow[l, "\sigma_{0*}"'] \end{tikzcd}$$

We have identified the projective normal bundle of (P,X) (resp. the normal bundle of (E,X)) with P (resp. E). According to remark 4.2, there is a canonical isomorphism $D_X E \simeq \mathbb{A}^1 \times E$ through which σ_0 (resp. σ_1) corresponds to the zero (resp. unit) section. The homotopy axiom (Htp) allows to conclude. $\square$

4.4. Let X be a smooth scheme, E be a vector bundle over X of rank n and put $P = \mathbb{P}(E \oplus 1)$. Let λ be the canonical line bundle on P, and $p : P \to X$ be the canonical projection. We define the *Thom class* of E/X as the cohomology class

$$t(E) = \sum_{i=0}^{n} p^*(c_i(E)) \cup (-c_1(\lambda))^{n-i}$$

in $H^{2n}(P)$. This is in fact a morphism $M(P) \to \mathbb{1}(n)[2n]$ whose restriction to $M(\mathbb{P}(E))$ is zero. This implies the morphism

$$p_* \boxtimes t(E) : M(P) \to M(X)(n)[2n]$$

factors as a morphism $M_X(P) \to M(X)(n)[2n]$ and this latter is equal to $\mathfrak{p}_{(P,X)}$. Indeed, $p_* \boxtimes t(E)$ is a split epimorphism with splitting $\mathfrak{l}_n(P)$.

We introduce the *Thom premotive*[16] as $M\mathrm{Th}(E) := M_X(E)$ - remark it is functorial with respect to monomorphisms of vector bundles. Using property (Exc), the natural morphism $M\mathrm{Th}(E) \to M_X(P)$ is an isomorphism. As a consequence, the morphism $p_* \boxtimes t(E)$ induces an isomorphism $M\mathrm{Th}(E) : M\mathrm{Th}(E) \to M(X)(n)[2n]$ which is precisely the purity isomorphism $\mathfrak{p}_{(E,X)}$. In the litterature, this arrow is called the *Thom isomorphism*.

Remark 4.5. Recall the universal quotient bundle ξ on P is defined by the exact sequence
$$0 \to \lambda \to p^{-1}(E \oplus 1) \to \xi \to 0.$$
Thus the Whitney sum formula 3.13 gives: $t(E) = c_n(\xi)$.

Definition 4.6. Let (X, Z) be a smooth closed pair of codimension n. Put $U = X - Z$ and consider the obvious immersions $i : Z \to X$ and $j : U \to X$. Considering the notations of the previous proposition, we call $\mathfrak{p}_{(X,Z)}$ the *purity isomorphism* associated with (X, Z). Using this isomorphism together with property (Loc) we obtain a distinguished triangle
$$M(X - Z) \xrightarrow{j_*} M(X) \xrightarrow{i^*} M(Z)(n)[2n] \xrightarrow{\partial_{X,Z}} M(X - Z)[1]$$
called the *Gysin triangle*. The morphism i^* (resp. $\partial_{X,Z}$) is called the *Gysin morphism* (resp. *residue morphism*) associated with (X, Z).

Example 4.7. Let X be a smooth scheme and E/X be a vector bundle of rank n. Put $P = \mathbb{P}(E \oplus 1)$ and consider the canonical section $s : X \to P$ of P/X. Then property (2) of proposition 4.3 implies $s^* \circ \mathfrak{l}_n(P) = 1 :$ the Gysin triangle of (P, X) is split and $\partial_{P,X} = 0$. Moreover, remark 4.4 and the previous definition implies that
$$s^* = p_* \boxtimes t(E).$$

4.2. **Base change.**

Definition 4.8. Let (X, Z) (resp. (Y, T)) be a smooth closed pair of codimension n (resp. m). Let $(f, g) : (Y, T) \to (X, Z)$ be a morphism of closed pairs. We define the morphism $(f, g)_! : M(T)(m)[2m] \to M(Z)(n)[2n]$ by the equality $(f, g)_! := \mathfrak{p}_{(X,Z)} \circ (f, g)_* \circ \mathfrak{p}^{-1}_{(Y,T)}$.

Thus we obtain a commutative diagram

$$\begin{tikzcd}
M(Y-T) \arrow[r, "l_*"] \arrow[d, "h_*"'] & M(Y) \arrow[r, "k^*"] \arrow[d, "f_*"'] & M(T)(n)[2n] \arrow[r, "\partial_{Y,T}"] \arrow[d, "(f,g)_!"] & M(Y-T)[1] \arrow[d, "h_*[1]"] \\
M(X-Z) \arrow[r, "j_*"] & M(X) \arrow[r, "i^*"] & M(Z)(n)[2n] \arrow[r, "\partial_{X,Z}"] & M(X-Z)[1]
\end{tikzcd} \tag{4.3}$$

where i, j, k, l are the obvious immersions and h is the restriction of f.
In what follows, we will compute the morphism $(f, g)_!$ in various cases. The commutativity of the second square will give us *refined projection formulas*. The new thing in our study is that any such formula corresponds to another formula involving residue morphisms as we see by considering the third commutative square.

[16] Analog of the Thom space in algebraic topology.

Remark 4.9. The notation $(f,g)_!$ is to be compared with the notation of [Ful98] for the "refined Gysin morphism". In fact, the reader will notice that in the case of motivic cohomology, our formulas extend the formulas of Fulton to the case of arbitrary weights (and arbitrary base). Be careful however that our Gysin morphism $i^* : M(X) \to M(Z)(n)[2n]$ corresponds to the usual pushout on Chow groups (cf [Dég08][1.21]). The Gysin morphism considered by Fulton is induced by the usual functoriality of motives. This fact can be understand if we thought of Chow groups over a field studied by Fulton as motivic homology with compact support.

4.2.1. *The transversal case.*

Proposition 4.10. *Consider the hypothesis of definition 4.8.*
Suppose (f,g) is cartesian and $n = m$. Then $(f,g)_! = g_*(n)[2n]$.

Proof. Diagram (4.1) is functorial with respect to cartesian morphism. Let $p : P_TY \to P_ZX$ be the morphism induced by (f,g) on the projective completions of the normal bundles. Through the morphisms $\bar{\sigma}_{0*}$ and $\bar{\sigma}_{1*}$ for the closed pairs (X,Z) and (Y,T), the morphism $(f,g)_*$ is isomorphic to

$$(p,g)_* : M(P_TY,T) \to M(P_ZX,Z).$$

As $n = m$ and $Y = X \times_Z T$, one has $P_TY = P_ZX \times_Z T$. Using the compatibility of the projective bundle isomorphism with base change, we see that the following diagram commutes

$$\begin{tikzcd} M(T)(n)[2n] \arrow[r, "\mathfrak{l}_n(P_TY)"] \arrow[d, "g_*(n)[2n]"'] & M(P_TY) \arrow[d, "p_*"] \\ M(Z)(n)[2n] \arrow[r, "\mathfrak{l}_n(P_ZX)"] & M(P_ZX) \end{tikzcd}$$

which concludes in view of the property (2) in proposition 4.3. □

Corollary 4.11. *Consider a smooth closed pair (X,Z) of codimension n and $i : Z \to X$ the corresponding immersion. Put $U = X - Z$.*
Then $(1_{Z*} \boxtimes i_*) \circ i^* = i^* \boxtimes 1_{X*}$ as a morphism $M(X) \to M(Z \times X)(n)[2n]$,
and $(j_* \boxtimes 1_{U*}) \circ \partial_{X,Z} = \partial_{X,Z} \boxtimes i_*$ as a morphism $M(Z)(n)[2n] \to M(U \times X)[1]$.

Proof. We consider the cartesian square

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "\gamma_i"'] & X \arrow[d, "\delta_X"] \\ Z \times X \arrow[r, "i \times 1_X"] & X \times X \end{tikzcd}$$

where δ_X is the diagonal embedding of X/S. The two formulas then follow from the previous proposition applied to the morphism of closed pairs $(\delta_X, \gamma_i) : (X,Z) \to (X \times X, Z \times X)$ with the help of the following elementary lemma :

Lemma 4.12. *Let (X,Z) be a smooth closed pair of codimension n and Y be a smooth scheme.*
Then $(i \times 1_Y)^* = i^* \otimes 1_{Y*}$ and $\partial_{X \times Y, Z \times Y} = \partial_{X,Z} \otimes 1_{Y*}$.

Using axiom (Kun)(a) and (Kun)(b), the lemma is reduced to prove that $\mathfrak{p}_{(X\times Y, Z\times Y)} = \mathfrak{p}_{(X,Z)} \otimes Y$. From the construction of the purity isomorphism, we are reduced to show that for a projective bundle P/X, $\epsilon_{P\times Y} = \epsilon_P \otimes 1_{X*}$ using the notations of theorem 3.2. This last equality follows finally from axiom (Kun)(a) and the functoriality of the first Chern class in axiom (Orient). □

Remark 4.13.
1. In the formula of this lemma, there is hidden a permutation isomorphism for the tensor product. In this paper, we will not need to care about this isomorphism. However, in some cases, it may result in a change of sign (see [Dég05], rem. 2.6.2).
2. Considering a ringed premotive $\mathbb{E}$, the previous corollary gives the usual projection formula for i : for any $z \in \mathbb{E}^{**}(Z)$ and any $x \in \mathbb{E}^{**}(X)$, $i_*(z \cup i^*(x)) = i_*(z) \cup x$.

4.14. Let (X, Z) be a smooth closed pair of codimension n, $i : Z \to X$ the corresponding closed immersion. Following Grothendieck (see [Gro58]), we define the *fundamental class* of Z in X as the cohomology class $\eta_X(Z) = i_*(1)$ in $H^{2n,n}(X)$. As a morphism, it is equal to the composite

$$M(X) \xrightarrow{i^*} M(Z)(n)[2n] \xrightarrow{\pi_{Z*}} \mathbb{1}(n)[2n]$$

where $\pi_Z : Z \to S$ is the structural morphism of Z/S.
Suppose that i admits a retraction $p : X \to Z$. Then corollary 3.10 gives the following computation[17] of the Gysin morphism :

$$i^* = p_* \boxtimes \eta_X(Z). \tag{4.4}$$

Suppose given a vector bundle E/X and put $P = \mathbb{P}(E \oplus 1)$. Applying example 4.7, we get

$$\eta_P(X) = t(E)$$

where X is embedded in P through the canonical section. Indeed example 4.7 is a particular case of the formula (4.4).
More generally, we can define the *localised fundamental class* of Z in X as the cohomology class $\bar{\eta}_X(Z) \in H^{2n,n}_Z(X)$ equal to the composite

$$M_Z(X) \xrightarrow{\mathfrak{p}_{(X,Z)}} M(Z)(n)[2n] \xrightarrow{\pi_{Z*}} \mathbb{1}(n)[2n].$$

Considering the canonical morphism $\nu_{X,Z} : H^{2n,n}_Z(X) \to H^{2n,n}(X)$, we have tautologically $\nu_{X,Z}(\bar{\eta}_X(Z)) = \eta_X(Z)$.
For any vector bundle E/X of rank n, $P = \mathbb{P}(E \oplus 1)$, the *localised Thom class* $\bar{t}(E) = \bar{\eta}_P(X)$ is uniquely determined by the Thom class $t(E)$. Usually, $\bar{t}(E)$ is considered as an element of $H^{2n,n}_X(E)$ using axiom (Exc).

As a last application of the previous corollary, let us remark the following :

Corollary 4.15. *Let (X, Z) be a smooth closed pair of codimension m, and P be a projective bundle of rank n over X.*

[17] Considered in cohomology, this is a well known formula.

Then for any integer $r \in [0, n]$, the following diagram is commutative :

$$\begin{tikzcd}
M(P_V) \arrow[r, "\nu_*"] \arrow[d, "p_{V*} \boxtimes c_1(\lambda_V)^r"'] & M(P) \arrow[r, "\iota^*"] \arrow[d, "p_* \boxtimes c_1(\lambda)^r"'] & M(P_Z)(m)[2m] \arrow[r, "\partial_\iota"] \arrow[d, "p_{Z*} \boxtimes c_1(\lambda_Z)^r"'] & M(P_V)[1] \arrow[d, "p_{V*}[1] \boxtimes c_1(\lambda_V)^r"'] \\
M(V)(r)[2r] \arrow[r, "j_*"] & M(Y)(r)[2r] \arrow[r, "i^*"] & M(Z)(r+m)[2(r+m)] \arrow[r, "\partial_i"] & M(V)(r)[2r+1].
\end{tikzcd}$$

In particular, the Gysin triangle is compatible with the projective bundle isomorphisms and with the induced embeddings $\mathfrak{l}_r(P_?)$.

4.2.2. *The excess intersection case.* Remark that in the hypothesis of definition 4.8, we have a canonical closed immersion

$$N_T Y \xrightarrow{\nu} g^*(N_Z X).$$

In particular, we have necessarily the inequality $n \geq m$.

Proposition 4.16. *Consider the hypothesis of definition 4.8. Suppose (f, g) is cartesian.*
Put $e = n - m$ and consider $\xi = g^{-1}(N_Z X)/N_T Y$, quotient vector bundle over T.
Then $(f, g)_! = (g_* \boxtimes_T c_e(\xi))\,(m)[2m]$.

Remark 4.17. The integer e is usually called the *excess of intersection*, and ξ the *excess intersection bundle*.

Proof. The morphism (f, g) induces the following composite morphism on normal bundles :

$$N_T Y \xrightarrow{\nu} g^{-1}(N_Z X) \xrightarrow{g'} N_Z X.$$

Thus, considering now the functoriality of diagram (4.2) with respect to the cartesian morphism (f, g), we obtain $(f, g)_! = (\nu, 1_T)_!(g', g)_!$. From proposition 4.10, $(g', g)_! = g_*(n)[2n]$. We conclude using the following lemma :

Lemma 4.18. *Let E and F be vector bundles over a smooth scheme T of respective rank n and m. Consider a monomorphism $\nu : F \to E$ of vector bundles and put $e = n - m$.*
Then $(\nu, 1_T)_! = \big(1_{T*} \boxtimes c_e(E/F)\big)(m)[2m]$.

To prove the lemma, we use the description of $\mathfrak{p}_{(F,T)}$ and $\mathfrak{p}_{(E,T)}$ using the Thom class (cf 4.4). Let P, Q and $\bar{\nu} : Q \to P$ be the respective projective completions of E, F and ν. Let $p : P \to T$ and $q : Q \to T$ be the canonical projections. We are reduced to prove the relation $\bar{\nu}^*(t(E)) = (q^* c_e(E/F)) \cup t(F)$ in $H^{2n,n}(Q)$.
From remark 4.5, we get $t(E) = c_n(\xi_P)$ (resp. $t(F) = c_m(\xi_Q)$) where ξ_P (resp. ξ_Q) is the universal quotient bundle on P (resp. Q). Thus, the relation follows from the Whitney sum formula 3.13 and the following exact sequence of vector bundles over Q :

$$0 \to \xi_Q \to \bar{\nu}^{-1}\xi_P \to q^{-1}(E/F) \to 0.$$

□

Corollary 4.19. *Let (X, Z) be a smooth closed pair of codimension n. Then :*
(1) $i^* i_* = 1_Z \boxtimes c_n(N_Z X)$ as a morphism $M(Z) \to M(Z)(n)[2n]$.
(2) $\partial_{X,Z} \circ (1_Z \boxtimes c_n(N_Z X)) = 0$.

This follows from the previous proposition applied with $(f, g) = (i, 1_Z)$. We usually refer to the first formula as the *self-intersection formula*.

4.20. Consider a vector bundle E over a smooth scheme X of rank n. Let $E^\times$ be the complement of the zero section in E and $\pi : E^\times \to X$ be the obvious projection. Then using property (Htp') and the previous corollary, we obtain from the Gysin triangle for (E, X) the following distinguished triangle

$$M(E^\times) \xrightarrow{\pi_*} M(X) \xrightarrow{1_X \boxtimes c_n(E)} M(X)(n)[2n] \xrightarrow{\partial_{E,X}} M(E^\times)[1]$$

which we shall call the *Euler distinguished triangle*. Indeed, in cohomology with coefficients in a ringed premotive $\mathbb{E}$, it corresponds to a long exact sequence where one of the arrow is the cup product by $c_n(E)$.
As a corollary of the self-intersection formula 4.19, we obtain the following tool to compute fundamental classes which generalises in our setting a theorem of Grothendieck (cf [Gro58, th. 2]).

Corollary 4.21. *Consider a smooth closed pair (X, Z) of codimension n. Let i be the corresponding closed immersion and $\eta_X(Z) = i_*(1) \in H^{2n,n}(X)$ be the fundamental class of Z in X (cf 4.14).*
Suppose there exists a vector bundle E on X and a section s of E/X such that s is transversal to the zero section s_0 of E and $Z = s^{-1}(s_0(X))$.
Then, $\eta_X(Z) = c_n(E)$.

It simply follows from corollary 4.19 applied to s_0 together with proposition 4.10 applied to the following transversal square :

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ \downarrow & & \downarrow{\scriptstyle s_0} \\ X & \xrightarrow{s} & E. \end{array}$$

Example 4.22. Let E be a vector bundle of rank n over a smooth scheme X. Put $P = \mathbb{P}(E \oplus 1)$ and consider $p : P \to X$ (resp. $s : X \to P$, λ) the canonical projection (resp. section, line bundle) of P/X. Consider finally the vector bundle $F = \lambda^\vee \otimes p^{-1}(E)$ over P. The sequence of morphisms of vector bundles over P,

$$\lambda \to p^{-1}(E \oplus 1) \to p^{-1}(E)$$

gives a section σ of F/P. We check easily it is transversal to the zero section and we have $\sigma^{-1}(0) = X$, while the embedding $\sigma^{-1}(0) \to P$ is s. Thus we obtain from the previous corollary $\eta_X(P) = c_n(F)$. Considering paragraph 4.14 and remark 4.5 we thus obtain three expressions of the fundamental class of X in P :

$$t(E) = c_n\big(p^{-1}(E \oplus 1)/\lambda\big) = c_n\big(\lambda^\vee \otimes p^{-1}(E)\big).$$

Note the last equality, though obvious in the case where F is the additive formal group, is not evident to check directly in the general case. However, we left as an exercice to the reader to check it using the inverse series of the formal group law F in the case of a line bundle. This implies the general case by the splitting principle.

4.2.3. *The ramified case.* In this section, we study the case of a morphism $(f,g) : (Y,T) \to (X,Z)$ of smooth closed pairs of same codimension n. This corresponds to the *proper case* in the operation of pullback of Z along f. We put $T' = Z \times_X Y$ and consider the canonical thickening $T' \to T$ induced by (f,g).
We first need an assumtion. Let $T' = \bigcup_{i \in I} T'_i$ be the decomposition into connected components. For any $i \in I$, we also consider the decomposition $T'_i = \bigcup_{j \in J_i} T'_{ij}$ into irreducible components. Put $T_{ij} = T'_{ij} \times_{T'} T$. As $T \to T'$ is a thickening, the geometric multiplicity $m(T'_{ij})$ of T'_{ij} is an integral multiple of the geometric multiplicity $m(T_{ij})$ of T_{ij}. We introduce the following condition on the morphism (f,g):

(Special) For any $i \in I$, there exists an integer $r_i \geq 0$ such that for any $j \in J_i$, $m(T'_{ij}) = r_i.m(T_{ij})$.

The integer r_i will be called the *ramification index* of f along T_i.

Remark 4.23. When S is irreducible, this condition is always fulfilled. When S is integral, T'_i is irreducible and the integer r_i is nothing else than the geometric multiplicity of T'_i.

Under this assumption, we define intersection multiplicities which take into account the formal group law F introduced in paragraph 3.7.
Let B be the blow-up of $\mathbb{A}^1_X$ with center $\{0\} \times Z$, and P its exceptional divisor. Put $C = B \times_X Y$, and for any $i \in I$, $Q_i = P \times_T T_i$. Remark that Q_i/T_i admits a canonical section s_i. We denote by L_i the line bundle over T_i obtain by the pullback of the normal bundle $N_{Q_i}(C)$ along s_i. We consider the localised Thom class $\bar{t}(L_i) \in H^{2,1}_{T_i}(L_i)$ (cf 4.14); we recall it is sent to 1 by the purity isomorphism $\mathfrak{p}^*_{(L_i,T_i)} : H^{2,1}_{T_i}(L_i) \to H^{0,0}(T_i)$.
Note that, according to remark 3.14, the Thom class $t(L_i)$ is nilpotent. Thus, the same is true for $\bar{t}(L_i)$. In particular, we can apply the power series $[r_i]_F$ (see paragraph 3.7) to the element $\bar{t}(L_i)$ of the A-algebra $H^{**}_{T_i}(L_i)$. This defines an element $[r_i]_F \cdot \bar{t}(L_i) \in H^{**}_{T_i}(L_i)$ of bidegree $(2,1)$.

Definition 4.24. Consider a morphism $(f,g) : (Y,T) \to (X,Z)$ which satisfies the condition (Special). Assume T admits an ample line bundle.
We consider the notations introduced above. For any $i \in I$, we define the *F-intersection multiplicity* of T_i in $f^{-1}(Z)$ as the element

$$r(T_i; f, g) = \mathfrak{p}^*_{(L_i,T_i)}\big([r_i]_F \cdot \bar{t}(L)\big) \in H^{0,0}(T_i)$$

where r_i is the ramification index of f along T_i.

A straightforward check shows the F-intersection multiplicities are compatible with flat base change. When the formal group law F is additive, we easily get that $r(T_i; f, g) = r_i$.
In the codimension $n = 1$ case, we can also consider the localised fundamental class $\bar{\eta}_Y(T_i) \in H^{2,1}_{T_i}(Y)$ introduced in paragraph 4.14. It corresponds to the localised Thom class $\bar{t}(N_{T_i}(Y))$ under the isomorphisms given by the deformation diagram

(4.2). Thus applying remark 3.14 as above, we obtain that the class $\bar{\eta}_Y(T_i)$ is nilpotent. In particular, we can consider the class $[r_i]_F \cdot \bar{\eta}_Y(T_i) \in H^{2,1}_{T_i}(Y)$ obtained by applying the power series $[r_i]_F$ of 3.7. We then obtain a natural expression of the F-intersection multiplicity :

Lemma 4.25. *Consider the hypothesis and assumptions of the previous definition and assume $n = 1$. Let $\bar{\eta}_Y(T_i) \in H^{2,1}_{T_i}(Y)$ be the localised fundamental class of T_i in Y (cf paragraph 4.14) and $\mathfrak{p}^*_{(Y,T_i)} : H^{2,1}_{T_i}(Y) \to H^{0,0}(T_i)$ be the purity isomorphism in cohomology.*
Then, $r(T_i; f, g) = \mathfrak{p}^*_{(Y,T_i)}\big([r_i]_F \cdot \bar{\eta}_Y(T_i)\big)$.

Proof. We may assume T is connected. Thus $I = \{i\}$ and we put $L = L_i$, $r = r_i$ with the notations of the previous definition. As $n = 1$, the zero section of $\mathbb{A}^1_X/X$ induces the following transversal square

$$\begin{tikzcd} Z = \mathbb{P}(N_Z X) \arrow[r, "s"] \arrow[d] & \mathbb{P}(N_Z X \oplus 1) = P \arrow[d] \\ X = B_Z X \arrow[r] & B_Z(\mathbb{A}^1_X) = B \end{tikzcd}$$

which, after pullback above Y gives a cartesian square, still transversal, $\begin{tikzcd} T \arrow[r, "t"] \arrow[d] & Q \arrow[d] \\ Y \arrow[r] & C \end{tikzcd}$

with t the canonical section of Q/T. Thus we get :

$$\begin{aligned} \mathfrak{p}^*_{(L,T)}([r]_F \cdot \bar{t}(L)) &= t^*\mathfrak{p}^*_{(N_Q C,Q)}([r]_F \cdot \bar{t}(N_Q C)) = t^*\mathfrak{p}^*_{(C,Q)}([r]_F \cdot \bar{\eta}_C(Q)) \\ &= \mathfrak{p}^*_{(Y,T)}([r]_F \cdot \bar{\eta}_Y(T)) \end{aligned}$$

where the last equality follows from the transversal square above and proposition 4.10 whereas the other equalities follow from the definitions. □

Before stating the main result of this section, we need to recall an extension of the functoriality of the deformation diagram (4.2) to certain morphisms of closed pairs (see also [Dég03, proof of 3.3]).
Consider a morphism $(f, g) : (Y, T) \to (X, Z)$ of smooth closed pairs of codimension 1. Let $\mathcal{I}$ (resp. $\mathcal{J}$, $\mathcal{J}'$) be the ideal defining Z in X (resp. T in Y, T' in Y). The map f induces a morphism $\varphi : \mathcal{I} \to f_*\mathcal{J}'$ of sheaves over X.
We consider the second deformation space $D_Z X = B_Z(\mathbb{A}^1_X) - B_Z X$ as in 4.1. An easy computation shows

$$D_Z X = \mathrm{Spec}_X \left(\bigoplus_{n \in \mathbb{Z}} \mathcal{I}^n.u^{-n} \right)$$

where $\mathcal{I}^n = \mathcal{O}_X$ for $n < 0$, and u is an indeterminate.
Assume[18] $\underline{\mathcal{J}' = \mathcal{J}^r}$. Then we can define a morphism of sheaves of rings over X :

$$\bigoplus_{n \in \mathbb{Z}} \mathcal{I}^n.u^{-n} \to \bigoplus_{n \in \mathbb{Z}} f_*(\mathcal{J}'^n).u^{-n} \to \bigoplus_{m \in \mathbb{Z}} f_*(\mathcal{J}^m).v^{-m}$$

[18] As the immersion $T \to Y$ is regular, this can happen only in the codimension 1 case. Note it implies (f, g) satisfies the condition (Special) and the ramification indexes are all equal to r.

where the first arrow is induced by φ and the second is the obvious inclusion which maps u to v^r as $\mathcal{J}' = \mathcal{J}^r$.
Taking the spectrum of these morphisms over X, we get a morphism

$$\rho_r(f,g) : D_T Y \to D_Z X$$

of schemes over $\mathbb{A}^1$. The fibre of $\rho_r(f,g)$ over 1 is simply f and one can check that its fiber over 0 is a composite morphism

$$\sigma_r(f,g) : N_T Y \xrightarrow{\nu} g^* N_Z X \xrightarrow{\mu} N_Z X$$

such that μ is induced by g and ν is a homogenous morphism of degree r. Thus, considering the respective deformation diagrams (4.2) for (X,Z) and (Y,T) we obtain a commutative diagram of closed pairs

$$\begin{tikzcd}
(Y,T) \arrow[r, "\sigma'_1"] \arrow[d, "{(f,g)}"'] & (D_T Y, \mathbb{A}^1_T) \arrow[d, dashed, "{(\rho_r(f,g),1\times g)}"] & (N_T Y, T) \arrow[l, "\sigma'_0"'] \arrow[d, "{(\sigma_r(f,g),g)}"] \\
(X,Z) \arrow[r, "\sigma_1"] & (D_Z X, \mathbb{A}^1_Z) & (N_Z X, Z). \arrow[l, "\sigma_0"']
\end{tikzcd} \tag{4.5}$$

Theorem 4.26. *Let $(f,g) : (Y,T) \to (X,Z)$ be a morphism of smooth closed pairs of codimension n. We assume T admits an ample line bundle and (f,g) satisfies condition (Special).*
Then

$$(f,g)_! = \sum_{i\in I} r(T_i; f,g) \boxtimes_T g_{i*}$$

where $T = \bigcup_{i\in I} T_i$ is the decomposition into connected component, $g_i = g|_{T_i}$ and $r(T_i; f,g)$ is the F-intersection multiplicity of T_i in $f^{-1}(Z)$.

Proof. Using axiom (Add'), we can assume T is connected.
We first reduce to the codimension $n = 1$ case. Consider the blow-up $B = B_Z(\mathbb{A}^1_X)$ and its exceptional divisor $P = \mathbb{P}(N_Z X \oplus 1)$. Consider also the cartesian morphism $(p,q) : (B,P) \to (X,Z)$. If we put $B_Y = B \times_X Y$, $Q = P \times_Z T$, we obtain the following commutative diagram of morphisms closed pairs :

$$\begin{tikzcd}
(B_Y, Q) \arrow[r, "{(f',g')}"] \arrow[d, "{(\pi',q)}"'] & (B,P) \arrow[d, "{(\pi,p)}"] \\
(Y,T) \arrow[r, "{(f,g)}"] & (X,Z).
\end{tikzcd}$$

By definition, $(f,g)_!(\pi',q)_! = (\pi,p)_!(f',g')_!$.
Note that (π,p) and (π',q) are cartesians. We can apply proposition 4.16 to (π,p) : the excess intersection bundle is the universal quotient bundle ξ_0 on P and $(\pi,p)_! = p_* \boxtimes c_n(\xi_0)$. Thus, according to remark 4.5 and paragraph 4.14, $(\pi,p)_! = s^*$ where $s : X \to P$ is the canonical section.
Similarly, if we put $\xi = {g'}^{-1}(\xi_0)$, we get $(\pi',q)_! = q_* \boxtimes c_n(\xi) = t^*$ with $t : T \to Q$ the canonical section. Note this latter morphism is a split epimorphism with splitting $\mathfrak{l}_n(Q)$. Thus we get

$$(f,g)_! = s^* \circ (f',g')_! \circ \mathfrak{l}_n(Q).$$

Remark that $Q = P \times_B B_Y$. Thus the morphism (f', g') of smooth closed pairs of codimension 1 satifies the condition (Special) and the ramification indexes of f along T and f' along Q are equal. Assume $(f', g') = r(Q; f', g') \boxtimes g'_*$. According to the expression above, we get

$$(f,g)_! = \big(r(Q;f',g') \boxtimes s^*g'_*\big) \circ \mathfrak{l}_n(Q) \overset{(1)}{=} \big(r(Q;f',g') \boxtimes g_*t^*\big) \circ \mathfrak{l}_n(Q)$$
$$\overset{(2)}{=} \big((r(Q;f',g') \circ t_*) \boxtimes g_*\big)t^* \circ \mathfrak{l}_n(Q) = (r(Q;f',g') \circ t_*) \boxtimes g_*.$$

where equality (1) follows from the projection formula of proposition 4.10 and equality (2) from the other projection formula of corollary 4.11. From definition 4.24, the reader can now easily check the equality of the cohomological classes $t^*[r(Q; f', g')] = r(T; f, g)$.
Thus we are reduced to the case $n = 1$, T still being connected. Let r be the ramification index of f along T. Let $\mathcal{J}$ (resp. $\mathcal{J}'$) be the ideal sheaf of T (resp. T') in Y. As $Z \to X$ and $T \to Y$ are regular immersions of a divisor, we see that necessarily, $\mathcal{J}' = \mathcal{J}^r$. Considering now diagram (4.5), we obtain that $(f,g)_! = (\sigma_r(f,g), g)_!$. In view of the factorization of the morphism $\sigma_r(f,g)$, we then are reduced to the following lemma :

Lemma 4.27. *Let T be a smooth scheme which admits an ample line bundle. Consider a line bundle N over T and $N^{\otimes r}$ be its r-th tensor power over T.*
Let $\nu : N \to N^{\otimes r}$ be the obvious homogenous morphism of degree r, and $(\nu, 1_T) : (N,T) \to (N^{\otimes r}, T)$ be the corresponding morphism of closed pairs.
Then $(\nu, 1_T)_! = \rho \otimes 1_{T*}$ where ρ is the unique element of $H^{00}(T)$ such that $[r]_F \cdot t(N) = \rho.t(N)$.

Put $P = \mathbb{P}(N \oplus 1)$, $P' = \mathbb{P}(N^{\otimes r} \oplus 1)$ and consider the projective completion $\bar{\nu} : P \to P'$ of ν. Let λ (resp. λ') be the canonical line bundle and p (resp. p') be the canonical projection of P/T (resp. P'/T). An easy computation shows that $\bar{\nu}^*(\lambda') = \lambda^{\otimes r}$. Recall from 4.22 that the Thom class of N (resp. $L^{\otimes r}$) is equal to $t(N) = c_1(\lambda^\vee \otimes p^{-1}N)$ (resp. $t(N^{\otimes r}) = c_1(\lambda'^\vee \otimes p'^{-1}N^{\otimes r})$). Thus, from the second point of proposition 3.8, $\bar{\nu}^* t(N^{\otimes r}) = [r]_F \cdot t(N)$. This latter class is zero on $P - T$, thus we get the relation $[r]_F \cdot t(N) = \rho.t(N)$ in $H^{2,1}(P)$. The conclusion now follows according to the computation of the Thom isomorphism 4.4.
To finish the proof with that lemma, we remark that $\rho = r(T; \nu, 1_T) = r(T; f, g)$. $\square$

Corollary 4.28. *Let $(f,g) : (Y,T) \to (X,Z)$ be a morphism of smooth closed pairs of codimension 1. We assume T admits an ample line bundle and (f,g) satisfies condition (Special). Let $(T_i)_{i \in I}$ be the connected components of T, and $r_i \in \mathbb{N}$ be the ramification index of f along T_i.*
Then, for any $i \in I$, the fundamental class $\eta_Y(T_i)$ is nilpotent and

$$f^*(\eta_X(Z)) = \sum_{i \in I} [r_i]_F \cdot \eta_T(T_i)$$

where $[r_i]_F$ is the power series equal to the r_i-th formal sum with respect to the formal group law F.

Proof. Let $i : Z \to X$ and $j_i : T_i \to Y$ be the canonical immersions. We simply put $\rho_i = r(T_i; f, g) \in H^{0,0}(T_i)$. In cohomology, the preceding theorem applied to (f, g) gives the relation

$$f^* i_*(z) = \sum_{i \in I} j_{i*}(\rho_i \boxtimes g_i^*(z))$$

for $z \in H^{**}(Z)$. Applied with $z = 1$, this gives $f^*(\eta_X(Z)) = \sum_i j_{i*}(\rho_i)$. Recall from lemma 4.25 that $\rho_i = \mathfrak{p}^*_{(Y,T_i)}([r_i]_F \cdot \bar{\eta}_Y(T_i))$. Tautologically, the composition $j_{i*}\mathfrak{p}^*_{(Y,T_i)}$ is equal to the canonical morphism $H^{**}_{T_i}(Y) \to H^{**}(Y)$ simply obtained by functoriality. For conclusion, it is sufficient to recall this latter is a morphism of A-algebra (cf paragraph 2.8). $\square$

Remark 4.29. In the previous corollary, the integers r_i can be understood as follows: locally, Z is parametrized by a S-regular function $a : X \to \mathbb{A}^1$. Then, (f, g) is special if $a \circ f$ can be written locally $u. \prod_{i \in I} b_i^{r_i}$ where u is a unit and $b_i : Y \to \mathbb{A}^1$ is a S-regular function parametrizing T_i – this expression should remain the same when we change any of the parameters b_i or a.

4.3. **Crossing Gysin triangles.** The following lemma will be the key point of the main result of this section. Though it will appear finally as a particular case, we begin by proving it to enlighten the proof of theorem 4.32.

Lemma 4.30. *Let Z be a smooth scheme, E and E' be vector bundles over Z of respective ranks n and m. Put $Q = \mathbb{P}(E \oplus 1)$, $Q' = \mathbb{P}(E' \oplus 1)$ and $P = Q \times_Z Q'$. Consider the fundamental class (see paragraph 4.14) $\eta_P(Z)$ (resp. $\eta_P(Q)$, $\eta_P(Q')$) of the canonical embedding of Z (resp. Q, Q') in P, as an element of $H^{**}(P)$. Then $\eta_P(Z) = \eta_P(Q) \cup \eta_P(Q')$.*

Proof. Put $d = n + m$. Let $\bar{\eta}_P(Z)$ be the localised fundamental class of Z in P (cf paragraph 4.14). Consider the deformation diagram (4.1) for the closed pair (P, Z), with $B = B_Z(\mathbb{A}^1_P)$:

$$(P, Z) \xrightarrow{\bar{\sigma}_1} (B, \mathbb{A}^1_Z) \xleftarrow{\bar{\sigma}_0} (P, Z).$$

As $\bar{\sigma}_0^*$ and $\bar{\sigma}_1^*$ are isomorphisms, $\bar{\eta}_P(Z)$ is uniquely determined by the class $\bar{t} = \bar{\sigma}_1^*(\bar{\eta}_P(Z))$ and $\bar{t}$ is uniquely determined by the fact that $\bar{\sigma}_0^*(\bar{t})$ corresponds to the Thom class $t(E \oplus E')$ in $H^{2d,d}(P)$.
Consider the divisor $D = B_Z(\mathbb{A}^1 \times \mathbb{P}(E) \times \mathbb{P}(E' \oplus 1))$ (resp. $D' = B_Z(\mathbb{A}^1 \times \mathbb{P}(E \oplus 1) \times \mathbb{P}(E'))$ in B and the class $c = -c_1(D)$ (resp. $c' = -c_1(D')$) in $H^{2,1}(B)$. Let π be the canonical projection of P/X. We define a cohomology class in $H^{2,1}(B)$:

$$t = \left(\sum_{0 \le i \le n} \pi^*(c_i(E)) \cup c^{n-i} \right) \cup \left(\sum_{0 \le j \le m} \pi^*(c_j(E')) \cup c'^{m-j} \right).$$

Then t vanishes on $B - \mathbb{A}^1_Z$ and, by construction, its pullback by $\bar{\sigma}_0$ is equal to $t(E \oplus E')$. Thus t corresponds to the class $\bar{t}$ mentionned above, through the map $H^{2d,d}_{\mathbb{A}^1_Z}(B) \to H^{2d,d}(B)$. The computation of its pullback by $\bar{\sigma}_1$ gives the desired formula. $\square$

Remark 4.31. Another way to obtain this lemma is to apply corollary 4.21 with $X = P$ and $E = \xi \times_Z \xi'$ where ξ (resp. ξ') is the universal quotient bundle of Q (resp. Q') — compare with remark 4.5.

Theorem 4.32. *Consider a cartesian square of smooth schemes* $\begin{tikzcd} Z \arrow[r, "k"] \arrow[d, "l"'] & Y' \arrow[d, "j"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$ *such that i,j,k,l are closed immersions of respective pure codimension* n, m, s, t. *We put* $d = n+s = m+t$ *and consider the closed immersion* $h : (Y-Z) \longrightarrow (X-Y')$ *induced by* i.
Then, in the following diagram :

$$\begin{tikzcd}
M(X) \arrow[r, "j^*"] \arrow[d, "i^*"'] \arrow[dr, phantom, "(1)"] & M(Y')(m)[2m] \arrow[r, "\partial_{X,Y'}"] \arrow[d, "k^*"] \arrow[dr, phantom, "(2)"] & M(X-Y')[1] \arrow[d, "h^*"] \\
M(Y)(n)[2n] \arrow[r, "l^*"'] & M(Z)(d)[2d] \arrow[r, "\partial_{Y,Z}"] \arrow[d, "\partial_{Y',Z}"'] \arrow[dr, phantom, "(3)"] & M(Y-Z)(n)[2n+1] \arrow[d, "\partial_{X-Y',Y-Z}"] \\
& M(Y-Z)(m)[2m+1] \arrow[r, "\partial_{X-Y,Y-Z}"'] & M(X-Y\cup Y')[2]
\end{tikzcd}$$

squares (1) and (2) are commutative and square (3) is anti-commutative.

Proof. Put $Y'' = Y \cup Y'$. Using axiom (Loc) and (Sym)(c), we obtain the following diagram :

$$\begin{tikzcd}
(\mathcal{D}) : M(X-Y'') \arrow[r] \arrow[d] & M(X-Y) \arrow[r] \arrow[d] & M\left(\frac{X-Y}{X-Y''}\right) \arrow[r] \arrow[d] & M(X-Y'')[1] \arrow[d] \\
M(X-Y') \arrow[r] \arrow[d] & M(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "(1)"] & M\left(\frac{X}{X-Y'}\right) \arrow[r] \arrow[d] \arrow[dr, phantom, "(2)"] & M(X-Y')[1] \arrow[d] \\
M\left(\frac{X-Y'}{X-Y''}\right) \arrow[r] \arrow[d] & M\left(\frac{X}{X-Y}\right) \arrow[r] \arrow[d] & M\left(\frac{X/X-Y}{X-Y'/X-Y''}\right) \arrow[r] \arrow[d] \arrow[dr, phantom, "(3)"] & M\left(\frac{X-Y'}{X-Y''}\right)[1] \arrow[d] \\
M(X-Y'')[1] \arrow[r] & M(X-Y)[1] \arrow[r] & M\left(\frac{X-Y}{X-Y''}\right)[1] \arrow[r] & M(X-Y'')[2],
\end{tikzcd}$$

in which any line or any row is a distinguished triangle, every square is commutative except square (3) which is anticommutative.
We put $M(X;Y,Y') = M\left(\frac{X/X-Y}{X-Y'/X-Y''}\right)$ for short. The proof will consist in constructing a purity isomorphism $\mathfrak{p}_{(X;Y,Y')} : M(X;Y,Y') \to M(Z)(d)[2d]$ which satisfies the following properties :

(i) *Functoriality :* The morphism $\mathfrak{p}_{(X;Y,Y')}$ is functorial with respect to morphisms in X which are transversal to Y, Y' and Z respectively.

(ii) *Symmetry :* The following diagram is commutative :

$$\begin{tikzcd}
M(X;Y,Y') \arrow[rr, "\epsilon"] \arrow[dr, "\mathfrak{p}_{(X;Y,Y')}"'] & & M(X;Y',Y) \arrow[dl, "\mathfrak{p}_{(X;Y',Y)}"] \\
& M(Z)(d)[2d], &
\end{tikzcd}$$

where ϵ is the isomorphism given in axiom (Sym).

(iii) *Compatibility :* The following diagram is commutative :

$$\begin{tikzcd}
M(\frac{X-Y'}{X-\Omega}) \arrow[r] \arrow[d, "\mathfrak{p}_{(X-Y',Y-Z)}"] & M(\frac{X}{X-Y}) \arrow[r] \arrow[d, "\mathfrak{p}_{(X,Y)}"] & M(X;Y,Y') \arrow[r] \arrow[d, "\mathfrak{p}_{(X;Y,Y')}"] & M(\frac{X-Y'}{X-\Omega})[1] \arrow[d, "\mathfrak{p}_{(X-Y',Y-Z)}"] \\
M(Y-Z)(n)[2n] \arrow[r] & M(Y)(n)[2n] \arrow[r, "j^*"] & M(Z)(d)[2d] \arrow[r, "\partial_{Y,Z}"] & M(Y-Z)(n)[2n+1]
\end{tikzcd}$$

With this isomorphism, we can deduce the three relations of the theorem by considering squares (1), (2), (3) in the above diagram when we apply the evident purity isomorphisms where we can. We then are reduced to construct the isomorphism and to prove the above relations. The difficult one is the second relation because we have to show that two isomorphisms in a triangulated category are equal. This forces to be very precise in the construction of the isomorphism.

We use a construction analog to the construction of the purity isomorphism in proposition 4.3. The first deformation space (cf paragraph 4.1) for the pair (X,Y) is $B = B_Y(\mathbb{A}^1_X)$. We let $P = P_Y X$ be the projective completion of the normal bundle of (X,Y). Consider also the closed pair $(U,V) = (X-Y', Y-Z)$. The analog deformation space for (U,V) is $B_U = B \times_X U$ and the projective completion of its normal bundle is $P_V = P \times_Y V$.

The deformation diagrams (4.1) for (X,Y) and (U,V) induce the following morphisms

$$M(X;Y,Y') = M\left(\frac{X/X-Y}{U/U-V}\right) \xrightarrow{\bar{\sigma}_{1*}} M\left(\frac{B/B-\mathbb{A}^1_Y}{B_U/B_U-\mathbb{A}^1_V}\right) \xleftarrow{\bar{\sigma}_{0*}} M\left(\frac{P/P-Y}{P_V/P_V-V}\right)$$

and the axiom (Loc) together with the purity theorem 4.3 shows $\bar{\sigma}_{0*}$ and $\bar{\sigma}_{1*}$ are isomorphisms.

Using the compatibility of the Gysin triangle with the projective bundle isomorphism (cf corollary 4.15), we obtain a commutative diagram :

$$\begin{tikzcd}
M(P_V/P_V-V) \arrow[r] & M(P/P-U) \arrow[r] & M\left(\frac{P/P-Y}{P_V/P_V-V}\right) \arrow[r, "+1"] & {} \\
M(P_V) \arrow[r] \arrow[u] & M(P) \arrow[r] \arrow[u] & M(\frac{P}{P_V}) \arrow[r, "+1"] \arrow[u] & {} \\
M(P_V) \arrow[r] \arrow[u, equal] & M(P) \arrow[r] \arrow[u, equal] & M(P_Z)(s)[2s] \arrow[r, "+1"] \arrow[u, "\mathfrak{p}^{-1}_{(P,P_Z)}"'] & {} \\
M(Y-Z)(n)[2n] \arrow[r] \arrow[u, "\mathfrak{l}_n(P_V)"] & M(Y)(n)[2n] \arrow[r, "j^*"] \arrow[u, "\mathfrak{l}_n(P)"] & M(Z)(d)[2d] \arrow[r, "+1"] \arrow[u, "\mathfrak{l}_n(P_Z)(s)[2s]"'] & {}
\end{tikzcd}$$

The composite of the vertical maps thus gives a morphism of triangles. Using property (2) of proposition 4.3, the first two maps of this morphism are isomorphisms and so is the third. This last isomorphism together with the maps $\bar{\sigma}_{1*}$ and $\bar{\sigma}_{0*}$ gives the desired isomorphism $\mathfrak{p}_{(X;Y,Y')}$.

Note that property (iii) is obvious by construction. Property (i) is easily obtained as in proposition 4.10.

Thus we have only to prove property (ii). First of all, we remark that the previous construction implies immediately the commutativity of the diagram :

$$\begin{tikzcd} M(X;Y,Y') \arrow[rr, "\alpha_{(X;Y,Y')}"] \arrow[dr, "\mathfrak{p}_{(X;Y,Y')}"'] & & M(X;Y,Z) \arrow[dl, "\mathfrak{p}_{(X;Y,Z)}"] \\ & M(Z)(d)[2d], & \end{tikzcd}$$

where $\alpha_{(X;Y,Y')}$ is induced by the evident open immersions.
Consider the following map

$$\beta_{(X;Y,Y')} : M_Z(X) \xrightarrow{\pi_{(X,Y,Z)}} M(X;Y,Z) \xrightarrow{\alpha^{-1}_{(X;Y,Y')}} M(X;Y,Y')$$

where $\pi_{(X,Y,Z)}$ is obtained by functoriality as usual – it is an isomorphism from axioms (Loc) and (Sym). Using the coherence axiom (Sym)(b), one checks that the following diagram is commutative

$$\begin{tikzcd} & M_Z(X) \arrow[dl, "\beta_{(X;Y,Y')}"'] \arrow[dr, "\beta_{(X;Y',Y)}"] & \\ M(X;Y,Y') \arrow[rr, "\epsilon"] & & M(X;Y',Y). \end{tikzcd}$$

Thus, it will be sufficient to prove the commutativity of the following diagram :

$$\begin{tikzcd} M_Z(X) \arrow[rr, "\pi_{(X,Y,Z)}"] \arrow[dr, "\mathfrak{p}_{(X,Z)}"'] & (*) & M(X;Y,Z) \arrow[dl, "\mathfrak{p}_{(X;Y,Z)}"] \\ & M(Z)(d)[2d]. & \end{tikzcd}$$

In the remainings of the proof, we consider the triples of smooth schemes (X',Y',Z') such that $Z' \subset Y' \subset X'$ are closed subschemes. A morphism of triples $(f,g,h) : (X'',Y'',Z'') \to (X',Y',Z')$ is a morphism of schemes $f : X'' \to X'$ which is transversal to Y' and Z', and such that $Y'' = f^{-}(Y')$, $Z'' = f^{-1}(Z')$. Using the functoriality of $\mathfrak{p}_{(X;Y,Z)}$, we remark that diagram $(*)$ is natural with respect to morphisms of triples.
We use the notations of paragraph 4.1. We also put $B(X',Z') := B_{Z'}(X')$, for a closed pair (X',Z'), and so on for the other schemes depending on a closed pair, to clarify the following considerations. We consider the evident closed pair $(D_ZX, D_ZX|_Y)$ and we put $D(X,Y,Z) = D(D_ZX, D_ZX|_Y)$. This scheme is in fact fibered over $\mathbb{A}^2$. The fiber over $(1,1)$ is X and the fiber over $(0,0)$ is $B(B_ZX \cup P_ZX, B_ZX|_Y \cup P_ZX|_Y)$. In particular, the $(0,0)$-fiber contains the scheme $P(P_ZX, P_ZY)$.
We now put : $D = D(X,Y,Z)$, $D' = D(Y,Y,Z)$. Remark that $D(Z,Z,Z) = \mathbb{A}^2_Z$. Similarly, we put $P = P(P_ZX, P_ZY)$, $Q = P_ZY$. Remark finally that if we consider $Q' = P_YX|_Z$, then[19] $P = Q \times_Z Q'$.
From the above description of fibers, we obtain a deformation diagram of triples :

$$(X,Y,Z) \to (D,D',\mathbb{A}^2_Z) \leftarrow (E,G,Z).$$

[19]This is equivalent to the canonical isomorphism $N(N_ZX, N_ZY) = N_ZY \oplus N_YX|_Z$.

Note that these morphisms are on the smaller closed subscheme the $(0,0)$-section and $(1,1)$-section of $\mathbb{A}^2_Z$ over Z, denoted respectively by s_1 and s_0. Now we apply these morphisms to diagram $(*)$ obtaining the following commutative diagram :

$$\begin{tikzcd}
M_Z(X) \arrow[rr] \arrow[dr, "\pi_{X,Y,Z}"] \arrow[dd, "\mathfrak{p}_{(X,Z)}"'] & & M_{\mathbb{A}^2_Z}(D) \arrow[dr] \arrow[dd, "\mathfrak{p}_{(D,\mathbb{A}^2_Z)}"'] & & M_Z(P) \arrow[ll] \arrow[dr, "\pi_{P,Q,Z}"] \arrow[dd, "\mathfrak{p}_{(P,Z)}"'] & \\
& M(X;Y,Z) \arrow[rr] \arrow[dl, "\mathfrak{p}_{(X,Y,Z)}"] & & M(D;D',\mathbb{A}^2_Z) \arrow[dl, "\mathfrak{p}_{(D,D',Z)}"] & & M(P;Q,Z) \arrow[ll] \arrow[dl, "\mathfrak{p}_{(P,Q,Z)}"] \\
M(Z)(d)[2d] \arrow[rr, "s_{1*}"'] & & M(\mathbb{A}^2_Z)(d)[2d] & & M(Z)(d)[2d]. \arrow[ll, "s_{0*}"] &
\end{tikzcd}$$

The square parts of this prism are commutative. As morphisms s_{1*} and s_{0*} are isomorphisms, the commutativity of the left triangle is equivalent to the commutativity of the right one.
Thus, we are reduced to the case of the smooth triple (P,Q,Z). Now, using the canonical split epimorphism $M(P) \to M(P/P-Z)$, we are reduced to prove the commutativity of the diagram :

$$\begin{tikzcd}
M(P) \arrow[rd] \arrow[d, "i^*"'] & \\
M(Z)(d)[2d] & M\left(\frac{P/P-Q}{P-Z/P-Q}\right) \arrow[l, "\mathfrak{p}_{(P,Q,Z)}"]
\end{tikzcd}$$

where $i : Z \to P$ denotes the canonical closed immersion.
Using property (iii) of the isomorphism $\mathfrak{p}_{(P,Q,Z)}$, we are finally reduced to prove the commutativity of the triangle

$$\begin{tikzcd}
M(P) \arrow[rr, "j^*"] \arrow[dr, "i^*"'] & & M(Q)(n)[2n] \arrow[dl, "k^*"] \\
& M(Z)(d)[2d] &
\end{tikzcd}$$

where we considered $Z \xrightarrow{k} Q \xrightarrow{j} P$ the canonical closed embeddings. This now simply follows from paragraph 4.14 and lemma 4.30. $\square$

As a corollary (apply commutativity of square (1) in the case $Y' = Z$), we get the functoriality of the Gysin morphism of a closed immersion :

Corollary 4.33. *Let $Z \xrightarrow{l} Y \xrightarrow{i} X$ be closed immersions between smooth schemes of respective pure codimension n and m.*
Then, $l^* \circ i^* = (i \circ l)^*$ as a morphism $M(X) \to M(Z)(n+m)[2(n+m)]$.

A corollary of this result, using lemma 4.12, is the compatibility of the Gysin morphism with products :

Corollary 4.34. *Consider a closed immersion $i : Z \to X$ (resp. $k : T \to Y$) between smooth schemes of pure codimension n (resp. m).*
Then $(i \times k)^* = i^* \otimes k^*$ as a morphism[20] *:*

$$M(X) \otimes M(Y) \to M(Z) \otimes M(T)(n+m)[2(n+m)].$$

[20] When we identify $M(Z)(n)[2n] \otimes M(T)(m)[2m]$ with $M(Z) \otimes M(T)(n+m)[2(n+m)]$ through the canonical isomorphism.

Remark 4.35. In the hypothesis of the previous corollary, we obtain in terms of fundamental classes :

$$\eta_{X\times Y}(Z\times T)=\eta_X(Z)\otimes\eta_Y(T).$$

We also obtain a result of intersection of fundamental classes in the case of smooth cycles :

Corollary 4.36. *Let X be a smooth scheme, Z and T be smooth closed subschemes of X. We assume that :*

1. *The intersection of Z and T in X is proper.*
2. *There is a closed subscheme W in $Z\cap T$ which is smooth, homeomorphic to $Z\cap T$ and admits an ample line bundle.*
3. *The induced morphism of closed pairs $(T,W)\to(X,Z)$ satisfies condition (Special).*

Let $\nu_{X,W}:H^{**}_W(X)\to H^{**}(X)$ be the canonical morphism. According to (Add′), $H^{**}_W(X)=\bigoplus_{i\in I}H^{**}_{W_i}(X)$ where $(W_i)_{i\in I}$ are the connected components of W. For any $i\in I$, we can consider the localised fundamental class $\bar\eta_X(W_i)$ as an element of $H^{**}_W(X)$ (see paragraph 4.14). We let $\rho_i\in H^{0,0}(W_i)$ be the F-intersection multiplicity of W_i in $Z\cap T$ (see definition 4.24). Then,

$$\eta_X(Z)\cup\eta_X(T)=\nu_{X,W}\Big(\sum\nolimits_{i\in I}\rho_i.\bar\eta_X(W_i)\Big)$$

using the $H^{0,0}(W)$-module structure of $H^{**}_W(X)$ obtained through the purity isomorphism.

Proof. We apply theorem 4.26 to the obvious square :

$$\begin{array}{ccc} W & \xrightarrow{\nu'} & T \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{\nu} & X. \end{array}$$

For any $i\in I$, we let ν'_i (resp. μ_i) be the immersion of W_i in T (resp. X). We thus obtain the formula in $H^{**}(T)$: $f^*\nu_*(1)=\sum_{i\in I}\nu'_{i*}(\rho_i)$.
Applying f_* to this formula and using corollary 4.11 for the left hand side, corollary 4.33 for the right hand side, we obtain $\eta_X(Z)\boxtimes\eta_X(T)=\sum_{i\in I}\mu_{i*}(\rho_i)$. By the very definition now, $\mu_{i*}(\rho_i)=\nu_{X,W}(\rho_i.\bar\eta_X(W_i))$. □

5. Duality and Gysin morphism

5.1. **Preliminaries.** For the rest of the section, we fix a monoidal category $\mathscr{C}$ with unit $\mathbb{1}$.

Definition 5.1. Let M an object of $\mathscr{C}$.
We say M is *strongly dualizable* if the following conditions are fulfilled :

1. The functor $M\otimes .$ admits a right adjoint $\underline{\mathrm{Hom}}(M,.)$.
2. For any object N of $\mathscr{C}$, consider the map

   $$M\otimes\underline{\mathrm{Hom}}(M,\mathbb{1})\otimes N\xrightarrow{ad\otimes 1_N}N$$

   induced by the evident adjunction morphism. Then the adjoint map

   $$\underline{\mathrm{Hom}}(M,\mathbb{1})\otimes N\to\underline{\mathrm{Hom}}(M,N)$$

is an isomorphism.

This definition coincides with definition 1.2 of [DP80]. Obviously, strongly dualizable objects are stable by finite sums and tensor product. Remark also that any invertible object of $\mathscr{C}$ for the tensor product is *a fortiori* strongly dualizable.

Definition 5.2. Consider an object M of $\mathscr{C}$.
A *strong dual* of M is an object $M^\vee$ of $\mathscr{C}$ and two morphisms $\mu : M \otimes M^\vee \to \mathbb{1}$, $\epsilon : \mathbb{1} \to M^\vee \otimes M$ such that the following composites

$$(i) \quad M \xrightarrow{1 \otimes \epsilon} M \otimes M^\vee \otimes M \xrightarrow{\mu \otimes 1} M$$

$$(ii) \quad M^\vee \xrightarrow{\epsilon \otimes 1} M^\vee \otimes M \otimes M^\vee \xrightarrow{1 \otimes \mu} M^\vee$$

are the identity morphisms.

The conditions of the definition imply that $M^\vee \otimes .$ is right adjoint to $M \otimes .$ and the natural transformations $\epsilon \otimes .$ and $\mu \otimes .$ are the adjunction transformations. Moreover, M is strongly dualizable as condition (2) of the first definition simply follows from the structural isomorphism $(M^\vee \otimes \mathbb{1}) \otimes N \simeq M^\vee \otimes N$ (see also [DP80, 1.3]).
Remark we also obtain that $. \otimes M^\vee$ is left adjoint to $. \otimes M$ with natural transformation $. \otimes \mu$ and $. \otimes \epsilon$. This gives the following reciprocal isomorphisms which we describe for future needs :

$$(5.1) \quad \begin{aligned} &\mathrm{Hom}_{\mathscr{C}}(M^\vee, \mathbb{E}) \to \mathrm{Hom}_{\mathscr{C}}(\mathbb{1}, \mathbb{E} \otimes M), \varphi \mapsto (\varphi \otimes 1_M) \circ \epsilon \\ &\mathrm{Hom}_{\mathscr{C}}(\mathbb{1}, \mathbb{E} \otimes M) \to \mathrm{Hom}_{\mathscr{C}}(M^\vee, \mathbb{E}), \psi \mapsto (1_{\mathbb{E}} \otimes \mu) \circ (\psi \otimes 1_{M^\vee}), \end{aligned}$$

where $\mathbb{E}$ is any object of $\mathscr{C}$.
The following lemma gives some precisions on the relation between "strongly dualizable" and "strong dual" :

Lemma 5.3. *Consider a strongly dualizable object M of $\mathscr{C}$. Let $M^\vee$ be an object of $\mathscr{C}$.*
Consider the following sets :

1. *Couples of morphisms $\mu : M \otimes M^\vee \to \mathbb{1}$ and $\epsilon : M^\vee \otimes M \to \mathbb{1}$ such that $(M^\vee, \mu, \epsilon)$ is a strong dual of M.*
2. *Morphisms $\mu : M \otimes M^\vee \to \mathbb{1}$ such that the adjoint map $\phi : M^\vee \to \underline{\mathrm{Hom}}(M, \mathbb{1})$ is an isomorphism.*

We associate to any morphism μ in (2) the following composite

$$\epsilon_\mu : \mathbb{1} \xrightarrow{ad'} \underline{\mathrm{Hom}}(M, M) \to \underline{\mathrm{Hom}}(M, \mathbb{1}) \otimes M \xrightarrow{\phi^{-1} \otimes 1} M^\vee \otimes M$$

where the first map is the evident adjunction morphism and the second one is induced by the isomorphism obtained by the property of the strongly dualizable object M.
Then (μ, ϵ_μ) is an element of (1) and the application

$$(2) \to (1), \mu \mapsto (\mu, \epsilon_\mu)$$

is a bijection.

We left the easy check to the reader.

Definition 5.4. Let M (resp. N) be an object of $\mathscr{C}$ and $(M^\vee, \mu_M, \epsilon_M)$ (resp. $(N^\vee, \mu_N, \epsilon_N)$) be a strong dual of M (resp. N).
For any morphism $f : M \to N$, we define the transpose morphism of f (with respect to the chosen strong duals) as the composite

$$^{\mathrm{t}}f : N^\vee \xrightarrow{\mu_M \otimes 1} M^\vee \otimes M \otimes N^\vee \xrightarrow{1 \otimes f \otimes 1} M^\vee \otimes N \otimes N^\vee \xrightarrow{1 \otimes \epsilon_N} M^\vee.$$

Remark that the morphism $^{\mathrm{t}}f$ in the previous definition is characterized by either one of the next two properties :

(i) The following diagram is commutative :

$$\begin{tikzcd} M \otimes N^\vee \arrow[r, "f \otimes 1"] \arrow[d, "1 \otimes {}^{\mathrm{t}}f"'] & N \otimes N^\vee \arrow[d, "\mu_N"] \\ M \otimes M^\vee \arrow[r, "\mu_M"] & \mathbb{1}. \end{tikzcd}$$

(ii) The following diagram is commutative :

$$\begin{tikzcd} N^\vee \arrow[r, "{}^{\mathrm{t}}f"] \arrow[d] & M^\vee \arrow[d] \\ \underline{\mathrm{Hom}}(N, \mathbb{1}) \arrow[r, "\underline{\mathrm{Hom}}(f, \mathbb{1})"] & \underline{\mathrm{Hom}}(M, \mathbb{1}) \end{tikzcd}$$

where the vertical maps are induced by adjunction from μ_N and μ_M – cf lemma 5.3.

5.2. **The projective bundle case.** Fix an integer $n \geq 0$. Using the projective bundle theorem 3.2 and axiom (Stab), we obtain that the motive $M(\mathbb{P}^n)$ is strongly dualizable, as a finite sum of invertible motives.
Let λ_n be the canonical line bundle on $\mathbb{P}^n$, $c' = c_1(\lambda_n)$. From the projective bundle theorem 3.2, c' is a generator of the A-algebra $H^{**}(\mathbb{P}^n)$. Let $c = c_1(\lambda_n^\vee)$. According to paragraph 3.7, $c = m(c') = -c' \mod c'^2$ where m is the inverse series associated to the formal group law F. Thus, the class c is still a generator of $H^{**}(\mathbb{P}^n)$ and also satisfies the relation $c^{n+1} = 0$. In all this section on duality, we systematically use this generator.
We consider the following morphism

$$\mu_n : M(\mathbb{P}^n) \otimes M(\mathbb{P}^n)(-n)[-2n] \xrightarrow{\delta^*} M(\mathbb{P}^n) \xrightarrow{p_*} \mathbb{1}$$

where $p : \mathbb{P}^n \to S$ is the canonical projection and $\delta : \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$ the diagonal embedding of $\mathbb{P}^n/S$.
If we consider this morphism as a cohomological class in $H^{2n,n}(\mathbb{P}^n \times \mathbb{P}^n)$, it is the fundamental class $\eta_{\mathbb{P}^n \times \mathbb{P}^n}(\mathbb{P}^n) = \delta_*(1) \in H^{2n,n}(\mathbb{P}^n \times \mathbb{P}^n)$ of the diagonal. Using the projective bundle theorem 3.2, it can be written

$$\eta_{\mathbb{P}^n \times \mathbb{P}^n}(\mathbb{P}^n) = \sum_{0 \leq i,j \leq n} \eta_{i,j}^{(n)}.c^i \cup d^j$$

where $\eta_{i,j}^{(n)}$ is an element in $A^{2(n-i-j),n-i-j}$ and c (resp. d) is the first Chern class of the canonical dual line bundle on the first (resp. second) factor of $\mathbb{P}^n \times \mathbb{P}^n$.
We define the $(n+1)$-dimensional square matrix $M_n = \left(\eta_{i,j}^{(n)}\right)_{0 \leq i,j \leq n}$ over the bigraded ring A. Note that M_n is symmetric. Remark finally that the morphism

induced by adjunction from μ_n gives by another application of theorem 3.2 a morphism

$$\bigoplus_{i=0}^{n} \mathbb{1}(n-i)[2(n-i)] \to \bigoplus_{j=0}^{n} \mathbb{1}(j)[2j]$$

whose matrix is precisely M_n.

Lemma 5.5. *For any integer $i \geq 0$, put $\eta_i = \eta_{ii}^{(i)} \in A^{-2i,-i}$. The matrix M_n has the form*

$$\begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & \eta_1 \\ 0 & \iddots & \iddots & \vdots \\ 1 & \eta_1 & \cdots & \eta_n \end{pmatrix}$$

Proof. First remark the lemma is clear when $n = 0$.
Consider the canonical embedding $\sigma : \mathbb{P}^n \to \mathbb{P}^{n+1}$. We apply the excess intersection formula 4.16 in the case of the following square

$$\begin{tikzcd} \mathbb{P}^n \arrow[r, "\delta"] \arrow[d, "\sigma"'] & \mathbb{P}^n \times \mathbb{P}^n \arrow[d, "\sigma\times\sigma"] \\ \mathbb{P}^{n+1} \arrow[r, "\delta'"] & \mathbb{P}^{n+1} \times \mathbb{P}^{n+1}. \end{tikzcd}$$

In this case, the excess of codimension is 1 and the excess intersection bundle on $\mathbb{P}^n$ is the canonical dual line bundle $\lambda_n^\vee$. Proposition 4.16 then gives the formula $(\sigma \times \sigma)^*(\delta'_*(1)) = \delta_*(c_1(\lambda_n^\vee))$.
The projection on the first factor $p_1 : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ gives a retraction of δ, and consequently, $\delta_*(c_1(\lambda_n^\vee)) = c \cup \delta_*(1)$. Thus the previous relation reads :

$$\sum_{0 \leq i,j \leq n} \eta_{i,j}^{(n+1)}.c^i \cup d^j = \sum_{0 \leq i,j \leq n} \eta_{i,j}^{(n)}.c^{i+1} \cup d^j$$

with the notations which precede the lemma. This in turn gives the relations

$$\begin{cases} \eta_{0,j}^{(n+1)} = 0 & \text{if } 0 \leq j \leq n, \\ \eta_{i,j}^{(n+1)} = \eta_{i-1,j}^{(n)} & \text{if } 0 < i \leq n \text{ and } 0 \leq j \leq n, \end{cases}$$

which allow to conclude by induction on the integer n. □

As a corollary, we obtain from lemma 5.3 that $\mu_n : M(\mathbb{P}^n) \otimes M(\mathbb{P}^n)(-n)[-2n] \to \mathbb{1}$ turns $M(\mathbb{P}^n)(-n)[-2n]$ into a strong dual of $M(\mathbb{P}^n)$.

Definition 5.6. We define the Gysin morphism $p^* : \mathbb{1} \to M(\mathbb{P}^n)(-n)[-2n]$ associated to the projection $p : \mathbb{P}^n \to S$ as the transpose of the morphism $p_* : M(\mathbb{P}^n) \to \mathbb{1}$ with respect to the strong duality on $M(\mathbb{P}^n)$ induced by μ_n.
Moreover, for any smooth scheme X, considering the projection $p_X : \mathbb{P}^n_X \to X$, we define the Gysin morphism associated to p_X as the morphism

$$p_X^* := 1 \otimes p^* : M(X) \to M(\mathbb{P}^n_X)(-n)[-2n].$$

Using property (ii) after definition 5.4, we obtain the following way to compute p^*. Consider the inverse matrix

$$M_n^{-1} = \begin{pmatrix} \eta'_n & \cdots & \eta'_1 & 1 \\ \vdots & \iddots & \iddots & 0 \\ \eta'_1 & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

where $\eta'_i \in A^{-2i,-i}$ is given by the determinant of the matrix obtained by removing line 0 and column $n-i$ from M_n times $(-1)^i$. Then

$$p^* : \mathbb{1} \to \bigoplus_{i=0}^{n} \mathbb{1}(i-n)[2(i-n)]$$

is given by the vector $\begin{pmatrix} \eta'_n \\ \vdots \\ \eta'_1 \\ 1 \end{pmatrix}$.

Note we have the fundamental relation in $A^{-2n,-n}$:

$$\sum_{i=0}^{n} \eta_i \cup \eta'_{n-i} = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.2}$$

Remark 5.7. The coefficients η_i and η'_i will be determined in proposition 5.30 and corollary 5.31.

5.3. **The Gysin morphism associated to a projective morphism.**

5.3.1. *Preliminary lemmas.*

Lemma 5.8. *Fix a couple of integers $n, m \in \mathbb{N}$ and a smooth scheme X. Consider the projection morphisms*

$$\begin{tikzcd} \mathbb{P}^n_X \times_X \mathbb{P}^m_X \arrow[r, "q'"] \arrow[d, "p'"'] & \mathbb{P}^n_X \arrow[d, "p"] \\ \mathbb{P}^m_X \arrow[r, "q"] & X. \end{tikzcd}$$

Then ${q'}^* p^* = {p'}^* q^*$.

Obvious from definition 5.6.

Lemma 5.9. *Consider a closed immersion $i : Z \to X$ between smooth schemes and an integer $n \geq 0$. Consider the pullback square*

$$\begin{tikzcd} \mathbb{P}^n_Z \arrow[r, "l"] \arrow[d, "q"'] & \mathbb{P}^n_X \arrow[d, "p"] \\ Z \arrow[r, "i"] & X. \end{tikzcd}$$

Then $l^* p^* = q^* i^*$.

It follows easily from definition 5.6 and lemma 4.12.

Lemma 5.10. *Consider and integer $n \geq 0$ and a smooth scheme X. Consider the canonical projection $p : \mathbb{P}^n_X \to X$.*
Then for any section $s : X \to \mathbb{P}^n_X$ of p, we have : $s^* p^* = 1$.

Proof. Recall from paragraph 4.14 that $s^* = p_* \boxtimes_{\mathbb{P}^n_X} (\pi_{X*} s^*)$ where $\pi_X : X \to S$ is the structural morphism of X/S. We easily obtain the following relation :

$$(p_* \boxtimes_{\mathbb{P}^n_X} 1) \circ p^* = 1 \boxtimes_X p^*.$$

Thus : $s^* p^* = 1 \boxtimes_X (\pi_{X*} s^* p^*)$.
As s is a section of p, it can be written $s = \nu \times 1_X$ for a closed immersion $\nu : X \to \mathbb{P}^n_S$. Consider the following cartesian squares :

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "\nu"'] & \mathbb{P}^n_S \times X \arrow[r, "p"] \arrow[d, "1_{\mathbb{P}^n_S} \times \nu"] & X \arrow[d, "\nu"] \\ \mathbb{P}^n_S \arrow[r, "\delta"] & \mathbb{P}^n_S \times \mathbb{P}^n_S \arrow[r, "\pi"] & \mathbb{P}^n_S \end{tikzcd}$$

where δ is the diagonal embedding and π the canonical projection on the second factor. Using the projection formula for each square – for the first square, this is 4.10, for the second square, it follows easily from definition 5.6 – we obtain : $\nu_* s^* p^* = \delta^* \pi^* \nu_*$.
As $\pi_{X*} = \pi_{\mathbb{P}^n_S *} \nu_*$, we thus are reduced to prove $\delta^* \pi^* = 1$. To conclude, the reader has the choice :

(1) A direct computation shows that the matrix of π^* (resp. δ^*), through the projective bundle isomorphism 3.2, is

$$\left(\delta_i^k . \eta'_{n-j}\right)_{(j,k) \in [0,n]^2,\ i \in [0,n]}$$
$$\text{resp.}\ \ \left(\eta_{j+k-l-n}\right)_{l \in [0,n],\ (j,k) \in [0,n]^2}.$$

The fundamental relation (5.2) allows to conclude.
(2) Use definition 5.4 to compute $\pi^* = 1 \otimes p^*$ in terms of the duality pairing $(\mu_n, \epsilon_{\mu_n})$ (cf lemma 5.3). Apply the projection formula 4.10 to compute directly $\delta^* \pi^*$; the second relation of definition 5.2 concludes.
(3) Prove $\delta^* = {}^t(\delta_*)$ using characterization (i) after definition 5.4 (and the usual projection formula 4.10).

□

5.3.2. *Definition.*

Lemma 5.11. *Consider a commutative diagram :*

$$\begin{tikzcd} & \mathbb{P}^n_X \arrow[dr, "p"] & \\ Y \arrow[ur, "k"] \arrow[dr, "i"'] & & X \\ & \mathbb{P}^m_X \arrow[ur, "q"'] & \end{tikzcd}$$

where i (resp. k) is a closed immersion of codimension r (resp. s) and p (resp. q) is the canonical projection. Then, $k^* p^* = i^* q^*$.

Proof. Let us introduce the following morphisms :

$$\begin{tikzcd}
 & & \mathbb{P}^n_X \arrow[dr, "p"] & \\
Y \arrow[urr, bend left, "k"] \arrow[r, "\nu"] \arrow[drr, bend right, "i"'] & \mathbb{P}^n_X \times_X \mathbb{P}^m_X \arrow[ur, "q'"] \arrow[dr, "p'"'] & & X. \\
 & & \mathbb{P}^m_X \arrow[ur, "q"'] &
\end{tikzcd}$$

Applying lemma 5.8, we are reduced to prove $k^* = \nu^* {q'}^*$ and $i^* = \nu^* {p'}^*$. In other words, we are reduced to the case $m = 0$ and $q = 1_X$.
In this case, we introduce the following morphisms :

$$\begin{tikzcd}
Y \arrow[dr, "s"] \arrow[drr, bend left, "k"] \arrow[ddr, equal] & & \\
 & \mathbb{P}^n_Y \arrow[r, "l"'] \arrow[d, "q"] & \mathbb{P}^n_X \arrow[d, "p"] \\
 & Y \arrow[r, "i"] & X
\end{tikzcd}$$

Then the lemma follows from lemma 5.9, lemma 5.10 and corollary 4.33. ☐

Consider smooth schemes X and Y and a projective morphism $f : Y \to X$ of codimension d. Consider an arbitrary factorization $Y \xrightarrow{i} \mathbb{P}^n_X \xrightarrow{p} X$ of f into a closed immersion of codimension $d+n$ and the canonical projection. The preceding lemma shows that the composite morphism

$$M(X) \xrightarrow{p^*} M(\mathbb{P}^n_X)(-n)[-2n] \xrightarrow{i^*} M(Y)(d)[2d]$$

is independent of the chosen factorization.

Definition 5.12. Considering the above notations, we define the Gysin morphism associated to f as the morphism

$$f^* := i^* p^* : M(X) \to M(Y)(d)[2d].$$

5.3.3. *Properties.*

5.13. Let us first remark that, as a corollary of 4.34, we obtain : $(f \times g)^* = f^* \otimes g^*$ for any projective morphisms f and g.

Proposition 5.14. *Consider projective morphisms $Z \xrightarrow{g} Y \xrightarrow{f} X$ between smooth schemes.*
Then $g^* f^* = (fg)^*$.

Proof. We choose a factorization $Y \xrightarrow{i} \mathbb{P}^n_X \xrightarrow{p} X$ (resp. $Z \xrightarrow{j} \mathbb{P}^m_X \xrightarrow{q} X$) of f (resp. fg) and we introduce the diagram

$$\begin{tikzcd}
 & & \mathbb{P}^m_X \arrow[ddrr, bend left, "q"] & & \\
 & & \mathbb{P}^n_X \times_X \mathbb{P}^m_X \arrow[u, "p'"] \arrow[dr, "q'"] & & \\
 & \mathbb{P}^m_Y \arrow[ur, "i'"] \arrow[dr, "q''"] & & \mathbb{P}^n_X \arrow[dr, "p"] & \\
Z \arrow[uuurr, bend left, "j"] \arrow[ur, "k"] \arrow[rr, "g"] & & Y \arrow[ur, "i"] \arrow[rr, "f"] & & X.
\end{tikzcd}$$

in which p' is deduced from p by base change and so on for q' and q''.
Then, by using the factorizations given in the preceding diagram, the proposition follows directly using 5.9, 5.8, 4.33 and finally 5.11. □

Proposition 5.15. *Consider a commutative square of smooth schemes*

$$\begin{array}{ccc} T & \xrightarrow{k} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & X \end{array}$$

such that i is a closed immersion and f is a projective morphism. Let h be the pullback of f on $X - Z$. Let n, m, s, t be the respective codimension of i, k, f, g. Note that $n + s = m + t$ and put $d = n + s$.
Then the following square is commutative :

$$\begin{array}{ccc} M(T)(d)[2d] & \xrightarrow{\partial_{Y,T}} & M(Y-T)(s)[2s+1] \\ {\scriptstyle g^*}\uparrow & & \uparrow{\scriptstyle h^*} \\ M(Z)(n)[2n] & \xrightarrow{\partial_{X,Z}} & M(X-Z)[1] \end{array}$$

Proof. By construction of the Gysin morphism, we have only to consider the case where f is the projection of a projective bundle or a closed immersion. It follows from lemma 4.12 in the first case and from theorem 4.32 in the second. □

Remark 5.16. Applying the two preceding propositions and case (i) of the following proposition, we obtain that the Gysin triangle is functorial with respect to the Gysin morphism of a projective morphism in the case of a cartesian square as in the preceding statement.

Proposition 5.17. *Consider a cartesian square of smooth schemes*

$$\begin{array}{ccc} Y' & \xrightarrow{g} & X' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{f} & X \end{array}$$

such that f (resp. g) is a projective morphism of codimension n (resp. m). Note that necessarily, $n \geq m$.

(i) *Suppose $n = m$ and $Y \times_X X'$ is smooth (*i.e. *$Y' = Y \times_X X'$).*
Then $f^* p_* = q_* g^*$.

(ii) *Suppose $Y \times_X X'$ is smooth and $n > m$. Put $e = n - m$. We attach to the above square a vector bundle ξ of rank e called the excess intersection bundle : choose a projective bundle P/X and a factorization $Y \xrightarrow{i} P \xrightarrow{p} X$ of f into a closed immersion followed by the canonical projection. We obtain a canonical embedding $N_{Y'}(P \times_X X') \to q^* N_Y(P)$ and denote by ξ the quotient bundle over Y'. This definition is independent of the choice of the factorization as shown in* [Ful98]*, proof of prop. 6.6.*
Then, $f^* p_* = \big(q_* \boxtimes c_e(\xi)\big) \circ g^*$.

Proof. In each case, we reduce to the corresponding assertion for a closed immersion (4.10, 4.16 and 4.26) by choosing a factorization of f into a closed immersion followed by a projection and by considering its pullback on X'. Indeed, the assertion (i) when f is the projection of a projective bundle is trivial. $\square$

We obtain finally the analog of corollary 4.11 :

Corollary 5.18. *Let $f : Y \to X$ be a projective morphism between smooth scheme of pure codimension d.*
Then $(1_{Y*} \boxtimes f_*) \circ f^* = f^* \boxtimes 1_{X*}$ as a morphism $M(X) \to M(Y \times X)(d)[2d]$.

The proof is the same as for 4.11 using assertion (i) of the proposition above and the formula of 5.13.

5.19. We now consider the analog of the ramification formula 4.26. Consider a commutative square of smooth schemes

$$\begin{tikzcd} T \arrow[r, "q"] \arrow[d, "g"'] \arrow[dr, phantom, "\Delta"] & Y \arrow[d, "f"] \\ Z \arrow[r, "p"'] & X \end{tikzcd}$$

which is cartesian on the underlying topological spaces and such that p and q are projective morphisms of codimension n. We assume T admits an ample line bundle.
Put $T' = T \times_X Y$ and note the morphism $T \to T'$ induce by Δ is a thickening. Let $T' = \bigcup_{i \in I} T'_i$ (resp. $T'_i = \bigcup_{j \in J_j} T'_{ij}$) be the decomposition into connected (resp. irreducible) components. Put $T_i = T'_i \times_{T'} T$ and $T_{ij} = T'_{ij} \times_{T'} T$. We introduce the following condition on Δ :

(Special) For any $i \in I$, there exists an integer $r_i \geq 0$ such that for any $j \in J_i$, $m(T'_{ij}) = r_i.m(T_{ij})$.

In this case, the integer r_i will be called the *ramification index* of f along T_i.
Consider a factorization $Z \xrightarrow{i} P \xrightarrow{\pi} X$ of p into a closed immersion and the projection of a projective bundle. We put $Q = P \times_X Y$ and consider the obvious morphism of closed pairs $(h, g) : (Q, T) \to (P, Z)$. Of course, Δ satisfies (Special) if and only if (h, g) satisfies (Special). Moreover, for any $i \in I$, the element $r(T_i; h, g)$ is independent of the chosen factorization. Indeed, taking into account the compatibility of F-intersection multiplicity with flat base change, this boils down to the following lemma :

Lemma 5.20. *Consider a commutative diagram of smooth schemes*

$$\begin{tikzcd} & T' \arrow[dl, "t"'] \arrow[dr] \arrow[dd, "g'"] & \\ T \arrow[rr] \arrow[dd, "g"'] & & Y \arrow[dd, "f"] \\ & Z' \arrow[dl, "s"'] \arrow[dr] & \\ Z \arrow[rr] & & X \end{tikzcd}$$

such that T and T' are connected and admits an ample line bundle, $t = s \times_Z T$ and (f, g) (resp. (f, g')) is a morphism of smooth closed pairs satisfying condition (Special) with ramification index r.
Then, $r(T'; f, g') = t^* r(T; f, g) \in H^{0,0}(T')$.

Proof. Consider the blow-up $B = B_Z(\mathbb{A}^1_X)$ (resp. $B' = B_{Z'}(\mathbb{A}^1_X)$) and its exceptional divisor P (resp. P'). As $Z' \subset Z$, we get a cartesian transversal square, together with its pullback over Y

$$\begin{tikzcd} P' \arrow[r] \arrow[d] & B' \arrow[d] \\ P \arrow[r] & B. \end{tikzcd}, \quad \text{pullback above } Y : \quad \begin{tikzcd} Q' \arrow[r] \arrow[d] & C' \arrow[d] \\ Q \arrow[r] & C \end{tikzcd}.$$

The second square is still transversal. Put $L = N_Q C|_Z$ and $L' = N_{Q'}(C')|_{Z'}$. Thus, $L' = L|_{Z'}$. According to this equality, the lemma follows from the definition of F-intersection multiplicities. □

Definition 5.21. Consider the notations and hypothesis of 5.19, assuming the square Δ satisfies condition (Special). For any $i \in I$, we define the F-intersection multiplicity $r(T_i; \Delta)$ of T_i in the pullback square Δ as the well defined element $r(T_i; h, g)$ according to the notations above.

The following proposition is now a corollary of 4.26 :

Proposition 5.22. *Consider the hypothesis and notations of the preceding definition. Put $g_i = g|_{T_i}$ and $q_i = q|_{T_i}$.*
Then, $p^* f_* = \sum_{i \in I} \big(r(T_i; \Delta) \boxtimes g_{i*}\big) q_i^*$.

5.4. **The duality pairing.** Let X/S be a smooth projective scheme of pure dimension n. Let $p : X \to S$ (resp. $\delta : X \to X \times X$) be its structural morphism (resp. its diagonal embedding).
Then we define morphisms

$$\mu_X : \mathbb{1} \xrightarrow{p^*} M(X)(-n)[-2n] \xrightarrow{\delta_*} M(X \times X)(-n)[-2n] = M(X)(-n)[-2n] \otimes M(X)$$

$$\epsilon_X : M(X) \otimes M(X)(-n)[-2n] \xrightarrow{\delta^*} M(X) \xrightarrow{p_*} \mathbb{1}.$$

The following result is now a formality :

Theorem 5.23. *Consider the notations above.*
Then $\big(M(X)(-n)[-2n], \mu_X, \epsilon_X\big)$ is a strong dual of $M(X)$.

Proof. Each identity of definition 5.2 is an easy application of 5.13, 5.17(i) (the usual projection formula) and proposition 5.14. □

5.24. *Applications* : Consider the notations of the previous proposition and let $\mathbb{E}$ be a motive.

(1) We define the fundamental class $\tau_X \in H_{2n,n}(X)$ of X as the element

$$p^* : \mathbb{1} \to M(X)(-n)[-2n].$$

We also consider $\eta \in H^{2n,n}(X \times X)$ the fundamental class of the diagonal δ.

Then the isomorphisms of (5.1) with $M = M(X)$ gives exactly, considering the definitions of cap-product and slant product (cf 2.9), the following reciprocal isomorphisms :

$$\begin{aligned} \mathbb{E}^{r,p}(X) &\leftrightarrows \mathbb{E}_{2n-r,p-n}(X) \\ x &\mapsto x \cap \tau_X \\ \eta/y &\leftarrow\!\shortmid y. \end{aligned} \tag{5.3}$$

This is the *Poincaré duality isomorphism*, as it appears in algebraic topology (cf [Ada74], [Swi02, 14.41, 14.42]). To the knowledge of the author, the first appearance of this precise form of duality in algebraic geometry is in [PY02].

(2) Suppose $\mathbb{E}$ is a ringed motive. In this case, the regulator maps

$$\begin{aligned} \varphi_X &: H^{2n,n}(X) \rightarrow \mathbb{E}^{2n,n}(X) \\ \psi_X &: H_{2n,n}(X) \rightarrow \mathbb{E}_{2n,n}(X) \end{aligned}$$

allow to define the fundamental class of X (resp. the fundamental class of the diagonal) with coefficients in $\mathbb{E}$ as the image $\psi_X(\tau_X)$ (resp. $\varphi_X(\eta)$) of the corresponding class with coefficients in H. Moreover, we can obviously express the isomorphisms above with this classes (cf number (1) above), obtaining a Poincaré duality purely in terms of the cohomology theory $\mathbb{E}^{**}$.

(3) Suppose $\mathbb{E}$ is a ringed motive.

The morphism

$$p_* : \mathbb{E}^{**}(X) \rightarrow A$$

induced by the Gysin morphism of p is usually called the *trace morphism* (relative to S).

We suppose the cohomology $\mathbb{E}^{**}$ satisfies the following Künneth property : for any motives $M, N, P \in \{\mathbb{1}, M(X), M(X)(-n)[-2n]\}$, the pairing

$$\mathbb{E}^{**}(M) \otimes_A \mathbb{E}^{**}(N) \otimes_A \mathbb{E}^{**}(P) \rightarrow \mathbb{E}^{**}(M \otimes N \otimes P)$$

is an isomorphism.

Then it follows formally that

$$\big(\mathbb{E}^{**}(M(X)(-n)[-2n]), \mathbb{E}^{**}(\mu), \mathbb{E}^{**}(\epsilon)\big)$$

is a strong dual of $\mathbb{E}^{**}(M(X))$ in the category of graded A-modules.

More concretely, the pairing (induced by $\mathbb{E}^{**}(\mu)$)

$$\mathbb{E}^{**}(X) \otimes_A \mathbb{E}^{**}(X) \rightarrow A, x \otimes y \mapsto p_*(x \cup y)$$

is a perfect pairing of graded A-modules. This is usually called the *Poincaré duality pairing*[21] for the cohomology theory $\mathbb{E}^{**}$.

Note it implies that $E^{**}(X)$ is a projective finitely generated graded A-module (see [DP80, 1.4]).

Example 5.25. The conditions of point (3) are fulfilled when X is a Grassmanian scheme over S, or more generally a cellular variety over S, without any assumption on $\mathbb{E}$. In [CD06], we study cohomology theories $\mathbb{E}^{**}$ which satisfies the Künneth formula.

[21] This way of deducing the Poincaré duality pairing from the abstract duality of theorem 5.23 and the Künneth formula was explained to me by D.C.Cisinski.

The Gysin morphism determine the duality pairing defined above. Reciprocally, this duality determines the Gysin morphism as shown in the next proposition.

Proposition 5.26. *Let $f : Y \to X$ be a morphism between smooth projective S-schemes. Suppose X (resp. Y) is of constant relative dimension n (resp. m) over S.*
Then
$$f^* = {}^{\mathrm{t}}(f_*)(-n)[-2n]$$
where the transpose morphism on the right hand side is taken with respect to the strong duals of $M(X)$ and $M(Y)$ obtained in the previous theorem.

Proof. Consider the structural projections $p : X \to S$, $q : Y \to S$ and the diagonal embeddings $\delta_X : X \to X \times X$, $\delta_Y : Y \to Y \times Y$. Let n be the dimension of X Put $M(X)^\vee = M(X)(-n)[-2n]$ and $M(Y)^\vee = M(Y)(-m)[-2m]$.
According to the first point which follows definition 5.4, we have to prove the following square is commutative :
$$\begin{tikzcd} M(Y) \otimes M(X)^\vee \arrow[r, "f_* \otimes 1"] \arrow[d, "1 \otimes f^*"'] & M(X) \otimes M(X)^\vee \arrow[d, "p_* \delta_X^*"] \\ M(Y) \otimes M(Y)^\vee \arrow[r, "q_* \delta_Y^*"] & \mathbb{1}. \end{tikzcd}$$
We introduce the following cartesian square :
$$\begin{tikzcd} Y \arrow[r, "\gamma"] \arrow[d, "f"'] & Y \times X \arrow[d, "f \times 1"] \\ X \arrow[r, "\delta_X"] & X \times X \end{tikzcd}$$
Note that $f_* \otimes 1 = (f \times 1)_*$ and $1 \otimes f^* = (1 \times f)^*$ (cf 5.13). The result follows from the computation :
$$p_* \delta_X^* (f \times 1)_* = p_* f_* \gamma^* = q_* \delta_Y^* (1 \times f)^*$$
which uses 5.17(i) and 5.14. □

5.5. **Two illustrations.**

5.27. *Cobordism classes.—*

Definition 5.28. Let X be a smooth projective scheme of pure dimension n. Let $p : X \to S$ be its structural projection.
We define the cobordism class of X/S as the element of A, of (cohomological) degree $(-2n, -n)$,
$$[X] = \mathbb{1} \xrightarrow{p^*} M(X)(-n)[-2n] \xrightarrow{p_*} \mathbb{1}(-n)[-2n].$$
In other words, $[X] = p_*(1)$ as a cohomological class. Note that according to definition 5.6 and what follows it, we obtain that $[\mathbb{P}^n] = \eta'_n$. Note also that $[X \sqcup Y] = [X] + [Y]$ (from axiom (Add)) and $[X \times_S Y] = [X] \cup [Y]$ (from 5.13).

Example 5.29. Consider a factorization $X \xrightarrow{i} \mathbb{P}^N \xrightarrow{\pi} S$ of the morphism p into a closed immersion followed by the canonical projection. Let $c = N - n$ be the codimension of i. Let $\eta_{\mathbb{P}^N}(X) \in H^{2c,c}(\mathbb{P}^N)$ be the fundamental class associated to the embedding i. Then from corollary 4.33, $[X] = p_*(\eta_{\mathbb{P}^N}(X))$ (as a cohomological class).

Thus, to compute this cobordism class, we can use the projective bundle theorem, which implies we can write $\eta_{\mathbb{P}^N}(X) = \sum_{i=0}^N x_i.c^i$ where c is the Chern class of the dual canonical line bundle, and x_i is an element of A. Then,

$$[X] = \sum_{i=0}^{N} x_i \cup [\mathbb{P}^{N-i}]$$

as $p_*(c^i) = [\mathbb{P}^{N-i}]$ according to definition 5.6.

We want to compute now the cobordism class $[\mathbb{P}^n]$. Let $\delta : \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$ be the diagonal embedding. According to definition 5.6, we have to compute

$$\delta_*(1) = \sum_{0 \leq i,j \leq n} \eta_{i+j-n}.c^i \cup d^j \tag{5.4}$$

with the notations preceding lemma 5.5.

Let $p_1, p_2 : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ be the projections respectively on the first and second factor. Let λ (resp. ξ) be the canonical line bundle (resp. quotient bundle) on $\mathbb{P}^n$. Consider the vector bundles $\lambda^{(i)} = p_i^{-1}(\lambda)$ and $\xi^{(i)} = p_i^{-1}(\xi)$ for $i = 1, 2$. In the preceding expression, $c = c_1(\lambda_1^\vee)$ and $d = c_1(\lambda_2^\vee)$. Put $E = \underline{\mathrm{Hom}}(\lambda_1, \xi_2) = \lambda_1^\vee \otimes \xi_2$. We get a section s of this vector bundle considering the canonical morphism

$$\lambda_1 \to \mathbb{A}^{n+1} \times \mathbb{P}^n = \mathbb{P}^n \times \mathbb{A}^{n+1} \to \xi_2.$$

It is well known (see [PSP]) that $\delta(\mathbb{P}^n)$ is the subscheme defined by $s = 0$. Thus according to corollary 4.21, $\delta_*(1) = c_n(E) = c_n(\lambda_1^\vee \otimes \xi_2)$. From this expression, we obtain easily :

(1) *Additive case* : When the formal group law is additive[22] (*i.e.* $F(x,y) = x+y$), according to a well known formula (cf [Ful98, ex. 3.2.2]),

$$c_n(\lambda_1 \otimes \xi_2) = \sum_{i=0}^{n} c_1(\lambda_1)^i \cup c_{n-i}(\xi_2) = \sum_{i=0}^{n} c^i \cup d^{n-i}.$$

Thus, $[\mathbb{P}^n] = 0$ if $n > 0$.

(2) *Case* $n = 1$: As $c^2 = d^2 = 0$, we simply obtain :
$c_1(\lambda_1 \otimes \xi_2) = F(c_1(\lambda_1), c_1(\xi_2)) = c + d + a_{1,1}.c \cup d.$
Thus $\eta_1 = a_{1,1}$ which implies $[\mathbb{P}^1] = -a_{1,1}$.

In the general case, we obtain the following computation :

Proposition 5.30. *With the notations introduced above,*

$$\delta_*(1) = \sum_{0 \leq i,j \leq n} a_{1,i+j-n}.c^i \cup d^j.$$

Proof. Consider the ind-scheme $\mathbb{P}^\infty \times \mathbb{P}^n$ and the embedding $\tau : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^\infty \times \mathbb{P}^n$. Let $\tilde{p}_1$ (resp. p_2) be the projection on the first (resp. second) factor of $\mathbb{P}^\infty \times \mathbb{P}^n$. Put $\tilde{\lambda}_1 = \tilde{p}_1^{-1}(\lambda)$, $\lambda_2 = p_2^{-1}(\lambda)$ and $\xi_2 = p_2^{-1}(\xi)$. Thus, with a little abuse of notation, $c_n(\lambda_1 \otimes \xi_2) = \tau^* c_n\left(\tilde{\lambda}_1 \otimes \xi_2\right)$.

[22] This is the case for the category $DM(S)$.

According to the definition of ξ, we consider the short exact sequence :

$$0 \to \tilde{\lambda}_1^\vee \otimes \lambda_2 \to \tilde{\lambda}_1^\vee \otimes \mathbb{A}^{n+1} \to \tilde{\lambda}_1^\vee \otimes \xi_2 \to 0.$$

From the Whitney sum formula 3.13, we thus obtain the relation :

$$c_{n+1}(\tilde{\lambda}_1^\vee \otimes \mathbb{A}^{n+1}) = c_1(\tilde{\lambda}_1^\vee \otimes \lambda_2) \cup c_n(\tilde{\lambda}_1^\vee \otimes \xi_2).$$

Put $\tilde{c} = c_1(\tilde{\lambda}_1)$, $d = c_1(\lambda_2)$ as cohomology classes in $B = H^{**}(\mathbb{P}^\infty \times \mathbb{P}^n)$. Note moreover the A-algebra B is equal to $(A[d]/d^{n+1})[[\tilde{c}]]$. In terms of the fundamental group law F and its inverse power series m, the preceding relation reads[23] $\tilde{c}^{n+1} = F(\tilde{c}, m(d)) \cup c_n(\tilde{\lambda}_1^\vee \otimes \xi_2)$.
We have to prove :

$$c_n(\tilde{\lambda}_1^\vee \otimes \xi_2) = \sum_{0 \leq i,j \leq n} a_{1,i+j-n}.\tilde{c}^i \cup d^j \quad \mathrm{mod}\ \tilde{c}^{n+1}.$$

Let $m(x) = \sum_{k>0} m_k.x^k$ (thus $m_1 = -1, m_2 = a_{1,1}$, etc). For any integers l, s, we put

$$M_{l,s} = \sum_{\substack{k_1+\ldots+k_l=s \\ k_1,\ldots,k_l>0}} m_{k_1}...m_{k_l}$$

when $(l,s) \neq (0,0)$, and $M_{0,0} = 1$. Thus, $F(\tilde{c}, m(d)) = \sum_{k,l,s} a_{k,l} M_{l,s}.\tilde{c}^k \cup d^s$. In particular, $F(\tilde{c}, m(d)) = u.\tilde{c} + v$ where u is invertible in B and v is nilpotent. This implies $F(\tilde{c}, m(d))$ is a non zero divisor in B and we are reduced to prove :

$$F(\tilde{c}, m(d)) \cup \sum_{0 \leq i,j \leq n} a_{1,i+j-n}.\tilde{c}^i \cup d^j = 0 \quad \mathrm{mod}\ \tilde{c}^{n+1}.$$

The left hand side can be expanded (modulo $\tilde{c}^{n+1}$) as the sum :

$$\sum_{0 \leq u,v \leq n} \left(\sum_{k,l,s} a_{k,l} M_{l,s} a_{1,u+v-n-k-s} \right) .\tilde{c}^u \cup d^v.$$

Finally, for any integers $u, v \in [0, n]$, the coefficient of $\tilde{c}^u \cup d^v$ in the preceding sum can be written

$$\sum_w \left(\sum_{k,l} a_{k,l} M_{l,w-k} \right) a_{1,u+v-n-w}.$$

This is zero according to the relation $F(x, m(x)) = 0$. $\square$

From definition 5.6, the previous proposition reads $\eta_i = a_{1,i}$. As a corollary (cf relation (5.2)), we recover the classical Myschenko theorem together with a nice expression of $[\mathbb{P}^n]$ as a determinant :

[23] This expression for computing $\delta_*(1)$ was also obtained in [Pan03b].

Corollary 5.31. (1) *For any integer $n \geq 0$,*

$$[\mathbb{P}^n] = (-1)^n . \det \begin{pmatrix} 0 & \cdots & 0 & 1 & a_{1,1} \\ \vdots & \iddots & \iddots & \iddots & a_{1,2} \\ 0 & \iddots & \iddots & \iddots & \vdots \\ 1 & \iddots & \iddots & & \vdots \\ a_{1,1} & a_{1,2} & \cdots & \cdots & a_{1,n} \end{pmatrix}.$$

(2) *For any integer $n > 0$, $\sum_{0 \leq i \leq n} a_{1,i}.[\mathbb{P}^{n-i}] = 0$.*

The usual formulation of the relations given in (2) uses the series $p(x) = \sum_i [\mathbb{P}^i].x^i$ and $\omega(x) = \frac{\partial F}{\partial y}(x, 0)$. It reads $p(x) = \omega(x)^{-1}$.

Remark 5.32. An interesting problem is to extend this computation to the case of an arbitrary projective bundle $\mathbb{P}(E)$. We hope the fundamental class $\eta_{\mathbb{P}(E) \times \mathbb{P}(E)}(\mathbb{P}(E))$ as an explicit description in terms of the coefficients $a_{1,i}$ and the Chern classes of E which would give an expression of $[\mathbb{P}(E)]$ as a determinant analog of the above. This will give a counter-part to a classical formula of Quillen.

5.33. *Blow-up formulas.—*

Proposition 5.34. *Let (X, Z) be a smooth closed pair and B be the blow-up of X with center Z. Let $f : B \to X$ be the canonical projection.*
Then, $f_* f^* = 1$.

Proof. Let s_1 (resp. s_0, π) be the unit section (resp. zero section, canonical projection) of $\mathbb{A}^1_X/X$. Let B' be the blow-up of $\mathbb{A}^1_X$ with center $0 \times Z$. We consider the following cartesian square :

$$\begin{tikzcd} X \arrow[r, "\bar{\sigma}_1"] \arrow[d, equal] & B' \arrow[d, "f'"] \\ X \arrow[r, "s_1"] & \mathbb{A}^1_X . \end{tikzcd}$$

From the projection formula 5.17(i), we obtain $f'^* s_{1*} = \bar{\sigma}_{1*}$ which implies $f'^* = \bar{\sigma}_{1*} \pi_*$ by the axiom (Htp).
Thus we deduce easily : $f'_* f'^* = f'_* \bar{\sigma}_{1*} \pi_* = s_{1*} \pi_* = 1$.
Finally we consider the cartesian diagram :

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "\nu"'] & X \arrow[d, "s_0"] \\ B' \arrow[r, "f'"] & \mathbb{A}^1_X \end{tikzcd}$$

Using once again the projection formula *loc. cit.* we get : $f'_* f'^* s_{0*} = s_{0*} f_* f^*$. This concludes using axiom (Htp). $\square$

Lemma 5.35. *Let P/X be a projective bundle over a smooth scheme X of pure dimension d. Let ξ be the canonical quotient bundle of P/X and put $e = c_d(\xi)$ seen as a morphism $M(P) \to \mathbb{1}(d)[2d]$.*
Then, $(p_* \boxtimes e) \circ p^*$ is an isomorphism.

Proof. Using the projection formula 5.18, we have to prove that the cohomological class $p_*(e) \in H^{00}(X)$ is a unit. By compatibility of this class with base change and invariance under isomorphisms of projective bundles, we reduce to the case of $P = \mathbb{P}^d_S$. Let $s : S \to \mathbb{P}^d_S$ be the canonical section. Then, $e = s_*(1)$ (cf remark 4.5 combined with example 4.7). Thus, following lemma 5.10, $p_*(e) = 1$. $\square$

Remark 5.36. In the case of an additive formal group law, we can easily see that $p_*(e) = 1$ for any projective bundle P/X which implies the composite isomorphism of the lemma is just the identity.

5.37. Let X be a smooth scheme, Z be a smooth closed subscheme of X of pure codimension n. Let B be the blow-up of X with center Z and P be the exceptional divisor. Consider the cartesian squares :

$$\begin{tikzcd} P \arrow[r, "k"] \arrow[d, "p"'] & B \arrow[d, "f"] & B - P \arrow[l, "l"'] \arrow[d, "h"] \\ Z \arrow[r, "i"] & X & X - Z. \arrow[l, "j"'] \end{tikzcd}$$

We let λ (resp. $\xi = p^{-1}(N_Z X)/\lambda$) be the canonical line bundle (resp. quotient bundle) on $P = \mathbb{P}(N_Z X)$ and we put : $e = c_{n-1}(\xi)$.

Proposition 5.38. *Using the notations above, the short sequence*

$$0 \to M(P) \xrightarrow{\binom{p_*}{k_*}} M(Z) \oplus M(B) \xrightarrow{(-i_*, f_*)} M(X) \to 0$$

is split exact with splitting $\binom{0}{f^*}$.
By abuse of notation, we denote by $M(P/Z)$ the kernel[24] *of the split monomorphism p_* and let $\tilde{k}_* : M(P/Z) \to M(B)$ be the morphism induced by k_*. Then, we obtain an isomorphism*

$$M(P/Z) \oplus M(X) \xrightarrow{(\tilde{k}_*, f^*)} M(B).$$

Proof. The previous short sequence is obviously a complex. The fact $\binom{0}{f^*}$ is a splitting is proposition 5.34.
We directly prove the last assertion of the proposition which then concludes. Consider the following diagram :

$$\begin{tikzcd}[column sep=large] M(X-Z) \arrow[r, "\binom{0}{j_*}"] \arrow[d, "h^*"'] \arrow[dr, phantom, "(1)"] & \begin{matrix} M(P/Z) \\ \oplus \\ M(X) \end{matrix} \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & i^* \end{pmatrix}"] \arrow[d, "(\tilde{k}_*, f^*)"] \arrow[dr, phantom, "(2)"] & \begin{matrix} M(P/Z) \\ \oplus \\ M(Z)(n)[2n] \end{matrix} \arrow[r, "(0, \partial_{X,Z})"] \arrow[d, "(k^*\tilde{k}_*, p^*)"] \arrow[dr, phantom, "(3)"] & M(X-Z)[1] \arrow[d, "h^*"] \\ M(B-P) \arrow[r, "l_*"'] & M(B) \arrow[r, "k^*"'] & M(P)(1)[2] \arrow[r, "\partial_{B,P}"'] & M(B-P)[1] \end{tikzcd}$$

The two horizontal lines are distinguished triangles. It is commutative : for square (1), use the projection formula 5.17(i), for square (2), the functoriality of the Gysin

[24] If we had a splitting $s : Z \to P$ of p, this will be the motive associated to the immersion s.

morphism 5.14, for square (3), the compatibility of residues and Gysin morphism 5.15 and the defining property of the residue $\partial_{B,P}$.
As h is an isomorphism, we are reduced to prove $(k^*\tilde{k}_*, p^*)$ is an isomorphism.
The normal bundle of $k : P \to B$ is the canonical line bundle λ. Thus, from the self-intersection formula 4.19, $k^*k_* = 1_{P*} \boxtimes c$ with $c = c_1(\lambda)$. The remaining assertion is local in X so that we can assume that $N_Z X$ is trivializable. Finally, we compute easily the matrix of

$$(k^*k_*, p^*) : \oplus_{i=0}^{n-1} M(Z)(i)[2i] \oplus M(Z)(n)[2n] \to \oplus_{i=0}^{n-1} M(Z)(i+1)[2i+1]$$

obtained through the projective bundle isomorphism 3.2 :

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & [\mathbb{P}^n] \\ \vdots & & \ddots & \ddots & \vdots & [\mathbb{P}^{n-1}] \\ & & & \ddots & 0 & \vdots \\ & & & & 1 & \\ 0 & \cdots & & & 0 & 1 \end{pmatrix}.$$

As the matrix of $(k^*\tilde{k}_*, p^*)$ is obtained from the above one removing the first column, it is obviously invertible. $\square$

Proposition 5.39. *Consider the notations 5.37. The short sequence*

$$0 \to M(B) \xrightarrow{\binom{k^*}{f_*}} M(P)(1)[2] \oplus M(X) \xrightarrow{(p_* \boxtimes e, -i^*)} M(Z)(n)[2n] \to 0$$

is split exact with pseudo-splitting $\binom{p^*}{0}$.
Let C be the cokernel of the split mono $p^* : M(Z)(n-1)[2n-2] \to M(P)$ and $\tilde{k}^* : M(B) \to C(1)[2]$ the morphism induced by k^*. Then the following morphism is an isomorphism :

$$M(B) \xrightarrow{\binom{\tilde{k}^*}{f_*}} C(1)[2] \oplus M(X).$$

Remark 5.40. This second blow-up formula is a generalization of [Ful98, 6.7(a)]. In case X and Z are projective smooth, it is simply the dual statement of the previous proposition using 5.26. More precisely, from 5.38 (resp. 5.39) the morphism

$$\begin{pmatrix} k_* & p_* \\ f^* & 0 \end{pmatrix} \text{ (resp. } \begin{pmatrix} k^* & f_* \\ p^* & 0 \end{pmatrix})$$

is an isomorphism. These two matrices are dual.

Proof. The above sequence is a complex from the excess intersection formula 4.16 applied to the morphism (f, p). The pseudo-splitting of this sequence is exactly lemma 5.35. We thus are reduced to the last assertion.
Let $\pi : M(P)(1)[2] \to C(1)[2]$ be the canonical projection. Consider the following

diagram :

$$\begin{tikzcd}
M(B-P) \arrow[r, "l_*"] \arrow[d, "h^*"'] \arrow[dr, phantom, "(1)"] & M(B) \arrow[r, "k^*"] \arrow[d, "\binom{\tilde{k}^*}{f_*}"'] \arrow[dr, phantom, "(2)"] & M(P)(1)[2] \arrow[r, "\partial_{B,P}"] \arrow[d, "\binom{\pi}{p_* \boxtimes e}"] \arrow[dr, phantom, "(3)"] & M(B-P)[1] \arrow[d, "h^*"] \\
M(X-Z) \arrow[r, "\binom{0}{j_*}"'] & \begin{array}{c} C(1)[2] \\ \oplus \\ M(X) \end{array} \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & i^* \end{pmatrix}"'] & \begin{array}{c} C(1)[2] \\ \oplus \\ M(Z)(n)[2n] \end{array} \arrow[r, "(0,\partial_{X,Z})"'] & M(X-Z)[1]
\end{tikzcd}$$

The horizontal lines are distinguished triangles. The diagram is commutative : (1) follows from definitions, (2) is a consequence of the excess intersection formula 4.16 for (f,p) and (3) is a consequence of the same formula, considered for residues. Finally, we are reduced to prove that $\binom{\pi}{p_* \boxtimes e}$ is an isomorphism. But $\mathrm{coKer}(p^*) \simeq \mathrm{Ker}(\mathrm{p}_* \boxtimes \mathrm{e})$ by a canonical isomorphism so that the latter morphism is simply the decomposition isomorphism associated to the split epimorphism $p_* \boxtimes e$. $\square$

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Comparison of Spectral Sequences Involving Bifunctors

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Abstract. Suppose given functors $\mathcal{A} \times \mathcal{A}' \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ between abelian categories, an object X in $\mathcal{A}$ and an object X' in $\mathcal{A}'$ such that $F(X,-)$, $F(-,X')$ and G are left exact, and such that further conditions hold. We show that, E_1-terms exempt, the Grothendieck spectral sequence of the composition of $F(X,-)$ and G evaluated at X' is isomorphic to the Grothendieck spectral sequence of the composition of $F(-,X')$ and G evaluated at X. The respective E_2-terms are a priori seen to be isomorphic. But instead of trying to compare the differentials and to proceed by induction on the pages, we rather compare the double complexes that give rise to these spectral sequences.

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Contents

0 Introduction

To calculate $\mathrm{Ext}^*(X, Y)$, one can either resolve X projectively or Y injectively; the result is, up to isomorphism, the same. To show this, one uses the double complex arising when one resolves both X and Y; cf. [5, Chap. V, Th. 8.1].
Two problems in this spirit occur in the context of Grothendieck spectral sequences; cf. §§ 0.2, 0.3.

0.1 Language

In §3, we give a brief introduction to the Deligne-Verdier spectral sequence language; cf. [17, II.§4], [6, App.]; or, on a more basic level, cf. [11, Kap. 4]. This language amounts to considering a diagram $\mathrm{E}(X)$ containing all the images between the homology groups of the subquotients of a given filtered complex X, instead of, as is classical, only selected ones. This helps to gain some elbow room in practice: to govern the objects of the diagram $\mathrm{E}(X)$ we can make use of a certain short exact sequence; cf. §3.4.
Dropping the E_1-terms and similar ones, we obtain the *proper* spectral sequence $\dot{\mathrm{E}}(X)$ of our filtered complex X. Amongst others, it contains all E_k-terms for $k \geq 2$ in the classical language; cf. §§ 3.6, 3.5.

0.2 First comparison

Suppose given abelian categories $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{B}$ with enough injectives and an abelian category $\mathcal{C}$. Suppose given objects $X \in \mathrm{Ob}\,\mathcal{A}$ and $X' \in \mathrm{Ob}\,\mathcal{A}'$. Let $\mathcal{A} \times \mathcal{A}' \xrightarrow{F} \mathcal{B}$ be a biadditive functor such that $F(X, -)$ and $F(-, X')$ are left exact. Let $\mathcal{B} \xrightarrow{G} \mathcal{C}$ be a left exact functor. Suppose further conditions to hold; see §5.1.

We have a Grothendieck spectral sequence for the composition $G \circ F(X,-)$ and a Grothendieck spectral sequence for the composition $G \circ F(-,X')$. We evaluate the former at X' and the latter at X.
In both cases, the E_2-terms are $(\mathrm{R}^i G)(\mathrm{R}^j F)(X,X')$. Moreover, they both converge to $\big(\mathrm{R}^{i+j}(G \circ F)\big)(X,X')$. So the following assertion is well-motivated.

Theorem 31. *The proper Grothendieck spectral sequences just described are isomorphic; i.e.* $\dot{\mathrm{E}}^{\mathrm{Gr}}_{F(X,-),G}(X') \simeq \dot{\mathrm{E}}^{\mathrm{Gr}}_{F(-,X'),G}(X)$.

So instead of "resolving X' twice", we may just as well "resolve X twice".
In fact, the underlying double complexes are connected by a chain of double homotopisms, i.e. isomorphisms in the homotopy category as defined in [5, IV.§4], and rowwise homotopisms (the proof uses a chain $\bullet \xleftarrow{\text{double}} \bullet \xleftarrow{\text{roww.}} \bullet \xrightarrow{\text{roww.}} \bullet \xrightarrow{\text{double}} \bullet$). These morphisms then induce isomorphisms on the associated proper first spectral sequences.

0.3 Second comparison

Suppose given abelian categories $\mathcal{A}$ and $\mathcal{B}'$ with enough injectives and abelian categories $\mathcal{B}$ and $\mathcal{C}$. Suppose given objects $X \in \operatorname{Ob}\mathcal{A}$ and $Y \in \operatorname{Ob}\mathcal{B}$. Let $\mathcal{A} \xrightarrow{F} \mathcal{B}'$ be a left exact functor. Let $\mathcal{B} \times \mathcal{B}' \xrightarrow{G} \mathcal{C}$ be a biadditive functor such that $G(Y,-)$ is left exact.
Let $B \in \operatorname{Ob}\mathrm{C}^{[0}(\mathcal{B})$ be a resolution of Y, i.e. a complex B admitting a quasiisomorphism $\operatorname{Conc} Y \longrightarrow B$. Suppose that $G(B^k,-)$ is exact for all $k \geq 0$. Let $A \in \operatorname{Ob}\mathrm{C}^{[0}(\mathcal{A})$ be, say, an injective resolution of X. Suppose further conditions to hold; see §6.1.
We have a Grothendieck spectral sequence for the composition $G(Y,-) \circ F$, which we evaluate at X. On the other hand, we can consider the double complex $G(B,FA)$, where the indices of B count rows and the indices of A count columns. To the first filtration of its total complex, we can associate the proper spectral sequence $\dot{\mathrm{E}}_{\mathrm{I}}\big(G(B,FA)\big)$.
If $\mathcal{B}$ has enough injectives and B is an injective resolution of Y, then in both cases the E_2-terms are a priori seen to be $(\mathrm{R}^i G)\big(Y,(\mathrm{R}^j F)(X)\big)$. So also the following assertion is well-motivated.

Theorem 34. *We have* $\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G(Y,-)}(X) \simeq \dot{\mathrm{E}}_{\mathrm{I}}\big(G(B,FA)\big)$.

So instead of "resolving X twice", we may just as well "resolve X once and Y once".
The left hand side spectral sequence converges to $\big(\mathrm{R}^{i+j}(G(Y,-) \circ F)\big)(X)$. By this theorem, so does the right hand side one.
The underlying double complexes are connected by two morphisms of double complexes (in the directions $\bullet \longrightarrow \bullet \longleftarrow \bullet$) that induce isomorphisms on the associated proper spectral sequences.
Of course, Theorems 31 and 34 have dual counterparts.

0.4 Results of Beyl and Barnes

Let R be a commutative ring. Let G be a group. Let $N \trianglelefteq G$ be a normal subgroup. Let M be an RG-module.
Beyl generalises Grothendieck's setup, allowing for a variant of a Cartan-Eilenberg resolution that consists of acyclic, but no longer necessarily injective objects [4, Th. 3.4]. We have documented Beyl's Theorem as Theorem 40 in our framework, without claiming originality.
Beyl uses his Theorem to prove that, from the E_2-term on, the Grothendieck spectral sequence for $RG\text{-Mod} \xrightarrow{(-)^N} RN\text{-Mod} \xrightarrow{(-)^{G/N}} R\text{-Mod}$ at M is isomorphic to the Lyndon-Hochschild-Serre spectral sequence, i.e. the spectral sequence associated to the double complex ${}_{RG}(\mathrm{Bar}_{G/N;R} \otimes_R \mathrm{Bar}_{G;R}\, , M)$; cf. [4, Th. 3.5], [3, §3.5]. This is now also a consequence of Theorems 31 and 34, as explained in §§ 8.2, 8.3.
Barnes works in a slightly different setup. He supposes given a commutative ring R, abelian categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ of R-modules, and left exact functors $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{C}$, where F is supposed to have an exact left adjoint $J : \mathcal{B} \longrightarrow \mathcal{A}$ that satisfies $F \circ J = 1_{\mathcal{B}}$. Moreover, he assumes $\mathcal{A}$ to have ample injectives and $\mathcal{C}$ to have enough injectives. In this setup, he obtains a general comparison theorem. See [2, Sec. X.5, Def. X.2.5, Th. X.5.4].
Beyl [4] and Barnes [2] also consider cup products; in this article, we do not.

0.5 Acknowledgements

Results of Beyl and Haas are included for sake of documentation that they work within our framework; cf. Theorem 40 and §4. No originality from my part is claimed.
I thank B. Keller for directing me to [12, XII.§11]. I thank the referee for helping to considerably improve the presentation, and for suggesting Lemma 47 and §8.2. I thank G. Carnovale and G. Hiss for help with Hopf algebras.

Conventions

Throughout these conventions, let $\mathcal{C}$ and $\mathcal{D}$ be categories, let $\mathcal{A}$ be an additive category, let $\mathcal{B}$ and $\mathcal{B}'$ be abelian categories, and let $\mathcal{E}$ be an exact category in which all idempotents split.

- For $a,\, b \in \mathbf{Z}$, we write $[a,b] := \{c \in \mathbf{Z} \;:\; a \leq c \leq b\}$, $[a,b[\;:= \{c \in \mathbf{Z} \;:\; a \leq c < b\}$, etc.
- Given $I \subseteq \mathbf{Z}$ and $i \in \mathbf{Z}$, we write $I_{\geq i} := \{j \in I \;:\; j \geq i\}$ and $I_{<i} := \{j \in I \;:\; j < i\}$.
- The disjoint union of sets A and B is denoted by $A \sqcup B$.
- Composition of morphisms is written on the right, i.e. $\xrightarrow{a}\xrightarrow{b} = \xrightarrow{ab}$.
- Functors act on the left. Composition of functors is written on the left, i.e. $\xrightarrow{F}\xrightarrow{G} = \xrightarrow{G\circ F}$
- Given objects X, Y in $\mathcal{C}$, we denote the set of morphisms from X to Y by ${}_{\mathcal{C}}(X,Y)$.
- The category of functors from $\mathcal{C}$ to $\mathcal{D}$ and transformations between them is denoted by $[\![\mathcal{C},\mathcal{D}]\!]$.

- Denote by $\mathrm{C}(\mathcal{A})$ the category of complexes
$$X \;=\; (\cdots \xrightarrow{d} X^{i-1} \xrightarrow{d} X^i \xrightarrow{d} X^{i+1} \xrightarrow{d} \cdots)$$
with values in $\mathcal{A}$. Denote by $\mathrm{C}^{[0}(\mathcal{A})$ the full subcategory of $\mathrm{C}(\mathcal{A})$ consisting of complexes X with $X^i = 0$ for $i < 0$. We have a full embedding $\mathcal{A} \xrightarrow{\mathrm{Conc}} \mathrm{C}^{[0}(\mathcal{A})$, where, given $X \in \operatorname{Ob}\mathcal{A}$, the complex $\operatorname{Conc} X$ has entry X at position 0 and zero elsewhere.
- Given a complex $X \in \operatorname{Ob}\mathrm{C}(\mathcal{A})$ and $k \in \mathbf{Z}$, we denote by $X^{\bullet+k}$ the complex that has differential $X^{i+k} \xrightarrow{(-1)^k d} X^{i+1+k}$ between positions i and $i+1$. We also write $X^{\bullet-1} := X^{\bullet+(-1)}$ etc.
- Suppose given a full additive subcategory $\mathcal{M} \subseteq \mathcal{A}$. Then $\mathcal{A}/\mathcal{M}$ denotes the quotient of $\mathcal{A}$ by $\mathcal{M}$, which has the same objects as $\mathcal{A}$, and which has as morphisms residue classes of morphisms of $\mathcal{A}$, where two morphisms are in the same residue class if their difference factors over an object of $\mathcal{M}$.
- A morphism in $\mathcal{A}$ is *split* if it isomorphic, as a diagram on $\bullet \longrightarrow \bullet$, to a morphism of the form $X \oplus Y \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)} X \oplus Z$. A complex $X \in \operatorname{Ob}\mathrm{C}(\mathcal{A})$ is *split* if all of its differentials are split.
- An *elementary split acyclic* complex in $\mathrm{C}(\mathcal{A})$ is a complex of the form
$$\cdots \longrightarrow 0 \longrightarrow T \xrightarrow{1} T \longrightarrow 0 \longrightarrow \cdots\,,$$
where the entry T is at positions k and $k+1$ for some $k \in \mathbf{Z}$. A *split acyclic* complex is a complex isomorphic to a direct sum of elementary split acyclic complexes, i.e. a complex isomorphic to a complex of the form
$$\cdots \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} T^i \oplus T^{i+1} \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} T^{i+1} \oplus T^{i+2} \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} T^{i+2} \oplus T^{i+3} \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)} \cdots$$
Let $\mathrm{C}_{\mathrm{sp\,ac}}(\mathcal{A}) \subseteq \mathrm{C}(\mathcal{A})$ denote the full additive subcategory of split acyclic complexes. Let $\mathrm{K}(\mathcal{A}) := \mathrm{C}(\mathcal{A})/\mathrm{C}_{\mathrm{sp\,ac}}(\mathcal{A})$ denote the homotopy category of complexes with values in $\mathcal{A}$. Let $\mathrm{K}^{[0}(\mathcal{A})$ denote the image of $\mathrm{C}^{[0}(\mathcal{A})$ in $\mathrm{K}(\mathcal{A})$. A morphism in $\mathrm{C}(\mathcal{A})$ is a *homotopism* if its image in $\mathrm{K}(\mathcal{A})$ is an isomorphism.
- We denote by $\operatorname{Inj}\mathcal{B} \subseteq \mathcal{B}$ the full subcategory of injective objects.
- Concerning exact categories, introduced by Quillen [14, p. 15], we use the conventions of [10, Sec. A.2]. In particular, a commutative quadrangle in $\mathcal{E}$ being a pullback is indicated by
$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d] \\ C \arrow[r] & D\,, \end{tikzcd}$$
a commutative quadrangle being a pushout by
$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & B \arrow[d] \\ C \arrow[r] & D\,. \end{tikzcd}$$
- Given $X \in \operatorname{Ob}\mathrm{C}(\mathcal{E})$ with pure differentials, and given $k \in \mathbf{Z}$, we denote by $\mathrm{Z}^k X$ the kernel of the differential $X^k \longrightarrow X^{k+1}$, by $\mathrm{Z}'^k X$ the cokernel of the differential $X^{k-1} \longrightarrow X^k$, and by $\mathrm{B}^k X$ the image of the differential $X^{k-1} \longrightarrow X^k$. Furthermore, we have pure short exact sequences $\mathrm{B}^k X \rightarrowtail \mathrm{Z}^k X \twoheadrightarrow \mathrm{H}^k X$ and $\mathrm{H}^k X \rightarrowtail \mathrm{Z}'^k X \twoheadrightarrow \mathrm{B}^{k+1} X$.

- A morphism $X \longrightarrow Y$ in $\mathrm{C}(\mathcal{E})$ between complexes X and Y with pure differentials is a *quasiisomorphism* if H^k applied to it yields an isomorphism for all $k \in \mathbf{Z}$. A complex X with pure differentials is *acyclic* if $\mathrm{H}^k X \simeq 0$ for all $k \geq 0$. Such a complex is also called a *purely acyclic* complex.

- Suppose that $\mathcal{B}$ has enough injectives. Given a left exact functor $\mathcal{B} \xrightarrow{F} \mathcal{B}'$, an object $X \in \mathrm{Ob}\,\mathcal{B}$ is *F-acyclic* if $\mathrm{R}^i FX \simeq 0$ for all $i \geq 1$. In other words, X is F-acyclic if for an injective resolution $I \in \mathrm{C}^{[0}(\mathrm{Inj}\,\mathcal{B})$ of X (and then for all such injective resolutions), we have $\mathrm{H}^i FI \simeq 0$ for all $i \geq 1$.

- By a module, we understand a left module, unless stated otherwise. If A is a ring, we abbreviate ${}_A(-,=) := {}_{A\text{-Mod}}(-,=) = \mathrm{Hom}_A(-,=)$.

1 Double and triple complexes

We fix some notations and sign conventions.

Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories. Let $\mathrm{C}(\mathcal{A}) \xrightarrow{H} \mathcal{B}$ be an additive functor.

1.1 Double complexes

1.1.1 Definition

A *double complex* with entries in $\mathcal{A}$ is a diagram

$$
X \;=\; \begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r, "d"] & X^{i+2,j} \arrow[r, "d"] \arrow[u, "\partial"] & X^{i+2,j+1} \arrow[r, "d"] \arrow[u, "\partial"] & X^{i+2,j+2} \arrow[r, "d"] \arrow[u, "\partial"] & \cdots \\
\cdots \arrow[r, "d"] & X^{i+1,j} \arrow[r, "d"] \arrow[u, "\partial"] & X^{i+1,j+1} \arrow[r, "d"] \arrow[u, "\partial"] & X^{i+1,j+2} \arrow[r, "d"] \arrow[u, "\partial"] & \cdots \\
\cdots \arrow[r, "d"] & X^{i,j} \arrow[r, "d"] \arrow[u, "\partial"] & X^{i,j+1} \arrow[r, "d"] \arrow[u, "\partial"] & X^{i,j+2} \arrow[r, "d"] \arrow[u, "\partial"] & \cdots \\
 & \vdots \arrow[u, "\partial"] & \vdots \arrow[u, "\partial"] & \vdots \arrow[u, "\partial"] &
\end{tikzcd}
$$

in $\mathcal{A}$ such that $dd = 0$, $\partial\partial = 0$ and $d\partial = \partial d$ everywhere. As morphisms between double complexes, we take all diagram morphisms. Let $\mathrm{CC}(\mathcal{A})$ denote the category of double complexes. We may identify $\mathrm{CC}(\mathcal{A}) = \mathrm{C}(\mathrm{C}(\mathcal{A}))$.
The double complexes considered in this §1.1 are stipulated to have entries in $\mathcal{A}$.
Let $\mathrm{CC}^{\llcorner}(\mathcal{A}) := \mathrm{C}^{[0}(\mathrm{C}^{[0}(\mathcal{A}))$ be the category of *first quadrant double complexes*, consisting of double complexes X such that $X^{i,j} = 0$ whenever $i < 0$ or $j < 0$.

Given a double complex X and $i \in \mathbf{Z}$, we let $X^{i,*} \in \operatorname{Ob} \mathrm{C}(\mathcal{A})$ denote the complex that has entry $X^{i,j}$ at position $j \in \mathbf{Z}$, the differentials taken accordingly; $X^{i,*}$ is called the ith *row* of X.
Similarly, given $j \in \mathbf{Z}$, $X^{*,j} \in \operatorname{Ob} \mathrm{C}(\mathcal{A})$ denotes the jth *column* of X.

1.1.2 Applying H in different directions

Given $X \in \operatorname{Ob} \mathrm{CC}(\mathcal{A})$, we let $H(X^{*,-}) \in \operatorname{Ob} \mathrm{C}(\mathcal{A})$ denote the complex that has $H(X^{*,j})$ at position $j \in \mathbf{Z}$, and as differential $H(X^{*,j}) \longrightarrow H(X^{*,j+1})$ the image of the morphism $X^{*,j} \longrightarrow X^{*,j+1}$ of complexes under H. Similarly, $H(X^{-,*}) \in \operatorname{Ob} \mathrm{C}(\mathcal{A})$ has $H(X^{j,*})$ at position $j \in \mathbf{Z}$.

In other words, a "$*$" denotes the index direction to which H is applied, a "$-$" denotes the surviving index direction. For short, "$*$" before "$-$".

1.1.3 Concentrated double complexes

Given a complex $U \in \operatorname{Ob} \mathrm{C}^{[0}(\mathcal{A})$, we denote by $\operatorname{Conc}_2 U \in \operatorname{Ob} \mathrm{CC}^{\llcorner}(\mathcal{A})$ the double complex whose 0th row is given by U, and whose other rows are zero; i.e. given $j \in \mathbf{Z}$, then $(\operatorname{Conc}_2 U)^{i,j}$ equals U^j if $i = 0$, and 0 otherwise, the differentials taken accordingly. Similarly, $\operatorname{Conc}_1 U \in \operatorname{Ob} \mathrm{CC}^{\llcorner}(\mathcal{B})$ denotes the double complex whose 0th column is given by U, and whose other columns are zero.

1.1.4 Row- and columnwise notions

A morphism $X \xrightarrow{f} Y$ of double complexes is called a *rowwise homotopism* if $X^{i,*} \xrightarrow{f^{i,*}} Y^{i,*}$ is a homotopism for all $i \in \mathbf{Z}$. Provided $\mathcal{A}$ is abelian, it is called a *rowwise quasiisomorphism* if $X^{i,*} \xrightarrow{f^{i,*}} Y^{i,*}$ is a quasiisomorphism for all $i \in \mathbf{Z}$.
A morphism $X \xrightarrow{f} Y$ of double complexes is called a *columnwise homotopism* if $X^{*,j} \xrightarrow{f^{*,j}} Y^{*,j}$ is a homotopism for all $j \in \mathbf{Z}$. Provided $\mathcal{A}$ is abelian, it is called a *columnwise quasiisomorphism* if $X^{*,j} \xrightarrow{f^{*,j}} Y^{*,j}$ is a quasiisomorphism for all $j \in \mathbf{Z}$.
Provided $\mathcal{A}$ is abelian, a double complex X is called *rowwise split* if $X^{i,*}$ is split for all $i \in \mathbf{Z}$; a short exact sequence $X' \longrightarrow X \longrightarrow X''$ of double complexes is called *rowwise split short exact* if $X'^{i,*} \longrightarrow X^{i,*} \longrightarrow X''^{i,*}$ is split short exact for all $i \in \mathbf{Z}$.
A double complex X is called *rowwise split acyclic* if $X^{i,*}$ is a split acyclic complex for all $i \in \mathbf{Z}$. It is called *columnwise split acyclic* if $X^{*,j}$ is a split acyclic complex for all $j \in \mathbf{Z}$.

1.1.5 Horizontally and vertically split acyclic double complexes

An *elementary horizontally split acyclic* double complex is a double complex of the form

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[u] & T^{i+1} \arrow[r, equal] \arrow[u] & T^{i+1} \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[u] & T^{i} \arrow[r, equal] \arrow[u, "\partial"] & T^{i} \arrow[r] \arrow[u, "\partial"] & 0 \arrow[r] \arrow[u] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & .
\end{tikzcd}$$

A *horizontally split acyclic* double complex is a double complex isomorphic to a direct sum of elementary horizontally split acyclic double complexes, i.e. to one of the form

$$\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r] & T^{i+1,j} \oplus T^{i+1,j+1} \arrow[r, "\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)"] \arrow[u] & T^{i+1,j+1} \oplus T^{i+1,j+2} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & T^{i,j} \oplus T^{i,j+1} \arrow[r, "\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)"] \arrow[u, "\left(\begin{smallmatrix} \partial & 0 \\ 0 & \partial \end{smallmatrix}\right)"] & T^{i,j+1} \oplus T^{i,j+2} \arrow[r] \arrow[u, "\left(\begin{smallmatrix} \partial & 0 \\ 0 & \partial \end{smallmatrix}\right)"] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] & .
\end{tikzcd}$$

An *elementary vertically split acyclic* double complex is a double complex of the form

$$
\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & T^i \arrow[r, "d"] \arrow[u] & T^{i+1} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & T^i \arrow[r, "d"] \arrow[u, equal] & T^{i+1} \arrow[r] \arrow[u, equal] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] & .
\end{tikzcd}
$$

A *vertically split acyclic* double complex is a double complex isomorphic to a direct sum of elementary vertically split acyclic double complexes, i.e. to one of the form

$$
\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r] & T^{i+1,j} \oplus T^{i+2,j} \arrow[r, "\left(\begin{smallmatrix} d & 0 \\ 0 & d \end{smallmatrix}\right)"] \arrow[u] & T^{i+1,j+1} \oplus T^{i+2,j+1} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & T^{i,j} \oplus T^{i+1,j} \arrow[r, "\left(\begin{smallmatrix} d & 0 \\ 0 & d \end{smallmatrix}\right)"] \arrow[u, "\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)"] & T^{i,j+1} \oplus T^{i+1,j+1} \arrow[r] \arrow[u, "\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)"] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] & .
\end{tikzcd}
$$

A horizontally split acyclic double complex is in particular rowwise split acyclic. A vertically split acyclic double complex is in particular columnwise split

acyclic.
A double complex is called *split acyclic* if it is isomorphic to the direct sum of a horizontally and a vertically split acyclic double complex. Let $\mathrm{CC}_{\mathrm{sp\,ac}}(\mathcal{A})$ denote the full additive subcategory of split acyclic double complexes. Let

$$\mathrm{KK}(\mathcal{A}) \;:=\; \mathrm{CC}(\mathcal{A})/\mathrm{CC}_{\mathrm{sp\,ac}}(\mathcal{A})\;;$$

cf. [5, IV.§4]. A morphism in $\mathrm{CC}(\mathcal{A})$ that is mapped to an isomorphism in $\mathrm{KK}(\mathcal{A})$ is called a *double homotopism*.

A speculative aside. The category $\mathrm{K}(\mathcal{A})$ is Heller triangulated; cf. [10, Def. 1.5.(i), Th. 4.6]. Such a Heller triangulation hinges on two induced shift functors, one of them induced by the shift functor on $\mathrm{K}(\mathcal{A})$. Now $\mathrm{KK}(\mathcal{A})$ carries two shift functors, and so there might be more isomorphisms between induced shift functors one can fix. How can the formal structure of $\mathrm{KK}(\mathcal{A})$ be described?

1.1.6 Total complex

Let $\mathrm{KK}^{\llcorner}(\mathcal{A})$ be the full image of $\mathrm{CC}^{\llcorner}(\mathcal{A})$ in $\mathrm{KK}(\mathcal{A})$.
The *total complex* $\mathrm{t}X$ of a double complex $X \in \mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathcal{A})$ is given by the complex

$$\mathrm{t}X \;=\; \Big(X^{0,0} \xrightarrow{(d\;\partial)} X^{0,1}\oplus X^{1,0} \xrightarrow{\left(\begin{smallmatrix} d & \partial & 0\\ 0 & -d & -\partial\end{smallmatrix}\right)} X^{0,2}\oplus X^{1,1}\oplus X^{2,0} \xrightarrow{\left(\begin{smallmatrix} d & \partial & 0 & 0\\ 0 & -d & -\partial & 0\\ 0 & 0 & d & \partial\end{smallmatrix}\right)} X^{0,3}\oplus X^{1,2}\oplus X^{2,1}\oplus X^{3,0} \longrightarrow \cdots \Big)$$

in $\mathrm{Ob}\,\mathrm{C}^{[0}(\mathcal{A})$. Using the induced morphisms, we obtain a total complex functor $\mathrm{CC}^{\llcorner}(\mathcal{A}) \xrightarrow{\mathrm{t}} \mathrm{C}^{[0}(\mathcal{A})$. Since t maps elementary horizontally or vertically split acyclic double complexes to split acyclic complexes, it induces a functor $\mathrm{KK}^{\llcorner}(\mathcal{A}) \xrightarrow{\mathrm{t}} \mathrm{K}^{[0}(\mathcal{A})$. If, in addition, $\mathcal{A}$ is abelian, the total complex functor maps rowwise quasiisomorphisms and columnwise quasiisomorphisms to quasiisomorphisms, as one sees using the long exact homology sequence and induction on a suitable filtration.
Note that we have an isomorphism $U \xrightarrow{\sim} \mathrm{t}\,\mathrm{Conc}_1 U$, natural in $U \in \mathrm{Ob}\,\mathrm{C}^{[0}(\mathcal{A})$, having entries 1_{U_0}, 1_{U_1}, -1_{U_2}, -1_{U_3}, 1_{U_4}, etc. Moreover, $U = \mathrm{t}\,\mathrm{Conc}_2 U$, natural in $U \in \mathrm{Ob}\,\mathrm{C}^{[0}(\mathcal{A})$.

1.1.7 The homotopy category of first quadrant double complexes as a quotient

Lemma 1 *The residue class functor* $\mathrm{CC}(\mathcal{A}) \longrightarrow \mathrm{KK}(\mathcal{A})$, *restricted to* $\mathrm{CC}^{\llcorner}(\mathcal{A}) \longrightarrow \mathrm{KK}^{\llcorner}(\mathcal{A})$, *induces an equivalence*

$$\mathrm{CC}^{\llcorner}(\mathcal{A})/\big(\mathrm{CC}_{\mathrm{sp\,ac}}(\mathcal{A})\cap \mathrm{CC}^{\llcorner}(\mathcal{A})\big) \;\xrightarrow{\sim}\; \mathrm{KK}^{\llcorner}(\mathcal{A})\,.$$

Proof. We have to show faithfulness; i.e. that if a morphism $X \longrightarrow Y$ in $\mathrm{CC}^{\llcorner}(\mathcal{A})$ factors over a split acyclic double complex, then it factors over a split acyclic double complex that lies in $\mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathcal{A})$. By symmetry and additivity, it suffices to show that if a morphism $X \longrightarrow Y$ in $\mathrm{CC}^{\llcorner}(\mathcal{A})$ factors over a horizontally split acyclic double complex, then it factors over a horizontally split acyclic double complex that lies in $\mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathcal{A})$. Furthermore, we may assume $X \longrightarrow Y$ to factor over an elementary horizontally split acyclic double complex S concentrated in the columns k and $k+1$ for some $k \in \mathbf{Z}$. We may assume that $S^{i,j} = 0$ for $i < 0$ and $j \in \mathbf{Z}$. If $k < 0$, and in particular, if $k = -1$, then $X \longrightarrow Y$ is zero because $S \longrightarrow Y$ is zero, so that in this case we may assume $S = 0$. On the other hand, if $k \geq 0$, then $S \in \mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathcal{A})$. $\square$

Cf. also the similar Remark 2.

1.2 Triple complexes

1.2.1 Definition

Let $\mathrm{CCC}(\mathcal{A}) := \mathrm{C}(\mathrm{C}(\mathrm{C}(\mathcal{A})))$ be the category of *triple complexes.* A triple complex Y has entries $Y^{k,\ell,m}$ for $k,\ \ell,\ m \in \mathbf{Z}$.

We denote the differentials in the three directions by $Y^{k,\ell,m} \xrightarrow{d_1} Y^{k+1,\ell,m}$, $Y^{k,\ell,m} \xrightarrow{d_2} Y^{k,\ell+1,m}$ and $Y^{k,\ell,m} \xrightarrow{d_3} Y^{k,\ell,m+1}$, respectively.

Let $k,\ \ell,\ m \in \mathbf{Z}$. We shall use the notation $Y^{-,\ell,=}$ for the double complex having at position (k,m) the entry $Y^{k,\ell,m}$, differentials taken accordingly. Similarly the complex $Y^{k,\ell,*}$ etc.

Given a triple complex $Y \in \mathrm{Ob}\,\mathrm{CCC}(\mathcal{A})$, we write $HY^{-,=,*} \in \mathrm{Ob}\,\mathrm{CC}(\mathcal{A})$ for the double complex having at position (k,ℓ) the entry $H(Y^{k,\ell,*})$, differentials taken accordingly.

Denote by $\mathrm{CCC}^{\llcorner}(\mathcal{A}) \subseteq \mathrm{CCC}(\mathcal{A})$ the full subcategory of *first octant triple complexes;* i.e. triple complexes Y having $Y^{k,\ell,m} = 0$ whenever $k < 0$ or $\ell < 0$ or $m < 0$.

1.2.2 Planewise total complex

For $Y \in \mathrm{Ob}\,\mathrm{CCC}^{\llcorner}(\mathcal{A})$ we denote by $\mathrm{t}_{1,2}Y \in \mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathcal{A})$ the *planewise total complex* of Y, defined for $m \in \mathbf{Z}$ as

$$(\mathrm{t}_{1,2}Y)^{*,m} := \mathrm{t}(Y^{-,=,m}) ,$$

with the differentials of $\mathrm{t}_{1,2}Y$ in the horizontal direction being induced by the differentials in the third index direction of Y, and with the differentials of $\mathrm{t}_{1,2}Y$ in the vertical direction being given by the total complex differentials. Explicitly, given $k,\ \ell \geq 0$, we have

$$(\mathrm{t}_{1,2}Y)^{k,\ell} = \bigoplus_{i,\,j\,\geq\,0,\ i+j\,=\,k} Y^{i,j,\ell} .$$

By means of induced morphisms, this furnishes a functor

$$\begin{array}{rcl} \mathrm{CCC}^{\llcorner\!\swarrow}(\mathcal{A}) & \xrightarrow{\mathrm{t}_{1,2}} & \mathrm{CC}^{\llcorner}(\mathcal{A}) \\ Y & \longmapsto & \mathrm{t}_{1,2}Y\ . \end{array}$$

2 Cartan-Eilenberg resolutions

We shall use Quillen's language of exact categories [14, p. 15] to deal with Cartan-Eilenberg resolutions [5, XVII.§1], as it has been done by Mac Lane already before this language was available; cf. [12, XII.§11]. The assertions in this section are for the most part wellknown.

2.1 A remark

Remark 2 *Let $\mathcal{A}$ be an additive category. Then*

$$\mathrm{C}^{[0}(\mathcal{A})/\big(\mathrm{C}^{[0}(\mathcal{A}) \cap \mathrm{C}_{\mathrm{sp\,ac}}(\mathcal{A})\big) \longrightarrow \mathrm{K}^{[0}(\mathcal{A})$$

is an equivalence.

Proof. Faithfulness is to be shown. A morphism $X \longrightarrow Y$ in $\mathrm{C}^{[0}(\mathcal{A})$ that factors over an elementary split acyclic complex of the form $(\cdots \longrightarrow 0 \longrightarrow T =\!= T \longrightarrow 0 \longrightarrow \cdots)$ with T in positions k and $k+1$ is zero, provided $k < 0$. ▫

2.2 Exact categories

Concerning the terminology of exact categories, introduced by Quillen [14, p. 15], we refer to [10, Sec. A.2].

Let $\mathcal{E}$ be an exact category in which all idempotents split. An object $I \in \mathrm{Ob}\,\mathcal{E}$ is called *relatively injective,* or a *relative injective* (relative to the set of pure short exact sequences, that is), if ${}_{\mathcal{E}}(-, I)$ maps pure short exact sequences of $\mathcal{E}$ to short exact sequences. We say that $\mathcal{E}$ has *enough relative injectives,* if for all $X \in \mathrm{Ob}\,\mathcal{E}$, there exists a relative injective I and a pure monomorphism $X \rightarrowtail I$.

In case $\mathcal{E}$ is an abelian category, with all short exact sequences stipulated to be pure, then we omit "relative" and speak of "injectives" etc.

Definition 3 Suppose given a complex $X \in \mathrm{Ob}\,\mathrm{C}^{[0}(\mathcal{E})$ with pure differentials. A *relatively injective complex resolution* of X is a complex $I \in \mathrm{Ob}\,\mathrm{C}^{[0}(\mathcal{E})$, together with a quasiisomorphism $X \longrightarrow I$, such that the following properties are satisfied.

(1) The object entries of I are relatively injective.

(2) The differentials of I are pure.

(3) The quasiisomorphism $X \longrightarrow I$ consists of pure monomorphisms.

We often refer to such a relatively injective complex resolution just by I.
A *relatively injective object resolution*, or just a *relatively injective resolution,* of an object $Y \in \operatorname{Ob}\mathcal{E}$ is a relatively injective complex resolution of $\operatorname{Conc} Y$.
A *relatively injective resolution* is the complex of a relatively injective object resolution of some object in $\mathcal{E}$.

Remark 4 *Suppose that $\mathcal{E}$ has enough relative injectives. Every complex $X \in \operatorname{Ob} \mathrm{C}^{[0}(\mathcal{E})$ with pure differentials has a relatively injective complex resolution $I \in \operatorname{Ob} \mathrm{C}^{[0}(\mathcal{E})$. In particular, every object $Y \in \operatorname{Ob}\mathcal{E}$ has a relatively injective resolution $J \in \operatorname{Ob} \mathrm{C}^{[0}(\mathcal{E})$.*

Proof. Let $X^0 \rightarrowtail I^0$ be a pure monomorphism into a relatively injective object I^0. Forming a pushout along $X^0 \rightarrowtail I^0$, we obtain a pointwise purely monomorphic morphism of complexes $X \longrightarrow X'$ with $X'^0 = I^0$ and $X'^k = X^k$ for $k \geq 2$. By considering its cokernel, we see that it is a quasiisomorphism. So we may assume X^0 to be relatively injective.
Let $X^1 \rightarrowtail I^1$ be a pure monomorphism into a relatively injective object I^1. Form a pushout along $X^1 \rightarrowtail I^1$ etc. $\square$

Remark 5 *Suppose given $X \in \operatorname{Ob} \mathrm{C}^{[0}(\mathcal{E})$ with pure differentials such that $\mathrm{H}^k X \simeq 0$ for $k \geq 1$. Suppose given $I \in \operatorname{Ob} \mathrm{C}^{[0}(\mathcal{E})$ such that I^k is purely injective for $k \geq 0$, and such that the differential $I^0 \xrightarrow{d} I^1$ has a kernel in $\mathcal{E}$. Then the map*

$$ {}_{\mathrm{K}^{[0}(\mathcal{E})}(X, I) \longrightarrow {}_{\mathcal{E}}\big(\operatorname{Kern}(X^0 \xrightarrow{d} X^1),\, \operatorname{Kern}(I^0 \xrightarrow{d} I^1)\big) $$

that sends a representing morphism of complexes to the morphism induced on the mentioned kernels, is bijective.

Suppose $\mathcal{E}$ to have enough relative injectives. Let $\mathcal{I} \subseteq \mathcal{E}$ denote the full subcategory of relative injectives. Let $\mathrm{C}^{[0,\,\mathrm{res}}(\mathcal{I})$ denote the full subcategory of $\mathrm{C}^{[0}(\mathcal{I})$ consisting of complexes X with pure differentials such that $\mathrm{H}^k X \simeq 0$ for $k \geq 1$. Let $\mathrm{K}^{[0,\,\mathrm{res}}(\mathcal{I})$ denote the image of $\mathrm{C}^{[0,\,\mathrm{res}}(\mathcal{I})$ in $\mathrm{K}(\mathcal{E})$.

Remark 6 *The functor $\mathrm{C}^{[0,\,\mathrm{res}}(\mathcal{I}) \longrightarrow \mathcal{E}$, $X \longmapsto \mathrm{H}^0(X)$, induces an equivalence*

$$ \mathrm{K}^{[0,\,\mathrm{res}}(\mathcal{I}) \xrightarrow{\sim} \mathcal{E}\,. $$

Proof. This functor is dense by Remark 4, and full and faithful by Remark 5. $\square$

Remark 7 (exact Horseshoe Lemma)
Given a pure short exact sequence $X' \longrightarrow X \longrightarrow X''$ and relatively injective resolutions I' of X' and I'' of X'', there exists a relatively injective resolution I of X and a pointwise split short exact sequence $I' \longrightarrow I \longrightarrow I''$ that maps under H^0 to $X' \longrightarrow X \longrightarrow X''$.

Proof. Choose pure monomorphisms $X' \rightarrowtail I'^0$ and $X'' \rightarrowtail I''^0$ into relative injectives I'^0 and I''^0. Embed them into a morphism from the pure short exact sequence

$$X' \rightarrowtail X \twoheadrightarrow X''$$

to the split short exact sequence

$$I' \xrightarrow{(1\,0)} I' \oplus I'' \xrightarrow{\binom{0}{1}} I'' .$$

Insert the pushout T of $X' \rightarrowtail X$ along $X' \rightarrowtail I'^0$ and the pullback of $I'^0 \oplus I''^0 \twoheadrightarrow I''^0$ along $X'' \rightarrowtail I''^0$ to see that $X \longrightarrow I'^0 \oplus I''^0$ is purely monomorphic. So we can take the cokernel $\mathrm{B}^1 I' \longrightarrow \mathrm{B}^1 I \longrightarrow \mathrm{B}^1 I''$ of this morphism of pure short exact sequences. Considering the cokernels on the commutative triangle $(X, T, I'^0 \oplus I''^0)$ of pure monomorphisms, we obtain a bicartesian square $(T,\, I'^0 \oplus I''^0,\, \mathrm{B}^1 I',\, \mathrm{B}^1 I)$ and conclude that the sequence of cokernels is itself purely short exact. So we can iterate. ▫

2.3 An exact category structure on $\mathrm{C}(\mathcal{A})$

Let $\mathcal{A}$ be an abelian category with enough injectives.

Remark 8 *The following conditions on a short exact sequence $X' \longrightarrow X \longrightarrow X''$ in $\mathrm{C}(\mathcal{A})$ are equivalent.*

(1) *All connectors in its long exact homology sequence are equal to zero.*

(2) *The sequence $\mathrm{B}^k X' \longrightarrow \mathrm{B}^k X \longrightarrow \mathrm{B}^k X''$ is short exact for all $k \in \mathbf{Z}$.*

(3) *The morphism $\mathrm{Z}^k X \longrightarrow \mathrm{Z}^k X''$ is epimorphic for all $k \in \mathbf{Z}$.*

(3′) *The morphism $\mathrm{Z}'^k X' \longrightarrow \mathrm{Z}'^k X$ is monomorphic for all $k \in \mathbf{Z}$.*

(4) *The diagram*

$$\begin{tikzcd}
\mathrm{B}^k X' \arrow[r] \arrow[d] & \mathrm{Z}^k X' \arrow[r] \arrow[d] & \mathrm{H}^k X' \arrow[d] \\
\mathrm{B}^k X \arrow[r] \arrow[d] & \mathrm{Z}^k X \arrow[r] \arrow[d] & \mathrm{H}^k X \arrow[d] \\
\mathrm{B}^k X'' \arrow[r] & \mathrm{Z}^k X'' \arrow[r] & \mathrm{H}^k X''
\end{tikzcd}$$

has short exact rows and short exact columns for all $k \in \mathbf{Z}$.

Proof. We consider the diagram in (4) as a (horizontal) short exact sequence of (vertical) complexes and regard its long exact homology sequence. Taking into account that all assertions are supposed to hold for all $k \in \mathbf{Z}$, we can employ the long exact homology sequence on $X' \longrightarrow X \longrightarrow X''$ to prove the equivalence of (1), (2), (3) and (4).
Now the assertion (1) $\Longleftrightarrow$ (3) is dual to the assertion (1) $\Longleftrightarrow$ (3′). ▫

REMARK 9 *The category* $\mathrm{C}(\mathcal{A})$*, equipped with the set of short exact sequences that have zero connectors on homology as pure short exact sequences, is an exact category with enough relatively injective objects in which all idempotents split. With respect to this exact category structure on* $\mathrm{C}(\mathcal{A})$*, a complex is relatively injective if and only if it is split and has injective object entries.*

Cf. [12, XII.§11], where pure short exact sequences are called *proper*. A relatively injective object in $\mathrm{C}(\mathcal{A})$ is also referred to as an *injectively split complex.* To a relatively injective resolution of a complex $X \in \operatorname{Ob}\mathrm{C}(\mathcal{A})$, we also refer as a *Cartan-Eilenberg-resolution,* or, for short, as a *CE-resolution* of X; cf. [5, XVII.§1]. A *CE-resolution* is a CE-resolution of some complex. Considered as a double complex, it is in particular rowwise split and has injective object entries.

Given a morphism $X \xrightarrow{f} X'$ in $\mathrm{C}(\mathcal{A})$, CE-resolutions J of X and J' of X', a morphism $J \xrightarrow{\hat{f}} J'$ in $\mathrm{CC}(\mathcal{A})$ such that $(J^{i,j} \xrightarrow{\hat{f}^{i,j}} J'^{i,j}) = (0 \longrightarrow 0)$ for $i < 0$ and such that

$$\mathrm{H}^0(J^{*,-} \xrightarrow{\hat{f}^{*,-}} J'^{*,-}) \;=\; (X \xrightarrow{f} X')$$

is called a *CE-resolution* of $X \xrightarrow{f} X'$. By Remarks 9 and 6, each morphism in $\mathrm{C}(\mathcal{A})$ has a CE-resolution.

Proof of Remark 9. We *claim* that $\mathrm{C}(\mathcal{A})$, equipped with the said set of short exact sequences, is an exact category. We verify the conditions (Ex $1, 2, 3$) listed in [10, Sec. A.2]. The conditions (Ex $1^\circ, 2^\circ, 3^\circ$) then follow by duality.

Note that by Remark 8.($3'$), a monomorphism $X \longrightarrow Y$ in $\mathrm{C}(\mathcal{A})$ is pure if and only if $\mathrm{Z}'^k(X \longrightarrow Y)$ is monomorphic in $\mathcal{A}$ for all $k \in \mathbf{Z}$.

Ad (Ex 1). To see that a split monomorphism is pure, we may use additivity of the functor Z'^k for $k \in \mathbf{Z}$.

Ad (Ex 2). To see that the composition of two pure monomorphisms is pure, we may use Z'^k being a functor for $k \in \mathbf{Z}$.

Ad (Ex 3). Suppose given a commutative triangle

$$\begin{tikzcd} & Y \arrow[dr, "\shortmid" marking] & \\ X \arrow[ur] \arrow[rr, "\bullet" marking] & & Z \end{tikzcd} ,$$

in $\mathrm{C}(\mathcal{A})$. Applying the functor Z'^k to it, for $k \in \mathbf{Z}$, we conclude that $\mathrm{Z}'^k(X \longrightarrow Y)$ is monomorphic, whence $X \longrightarrow Y$ is purely monomorphic. So we may complete to

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "\bullet" marking] & & B \\ & Y \arrow[ur, "|" marking] \arrow[dr, "|" marking] & \\ X \arrow[ur, "\bullet" marking] \arrow[rr, "\bullet" marking] & & Z \end{tikzcd}$$

in $\mathrm{C}(\mathcal{A})$ with (X, Y, B) and (A, Y, Z) pure short exact sequences. Applying Z'^k to this diagram, we conclude that $\mathrm{Z}'^k(A \longrightarrow B)$ is a monomorphism for $k \in \mathbf{Z}$, whence $A \longrightarrow B$ is a pure monomorphism.
This proves the *claim.*
Note that idempotents in $\mathrm{C}(\mathcal{A})$ are split since $\mathrm{C}(\mathcal{A})$ is also an abelian category. We *claim* relative injectivity of complexes with split differentials and injective object entries. By a direct sum decomposition, and using the fact that any monomorphism from an elementary split acyclic complex with injective entries to an arbitrary complex is split, we are reduced to showing that a pure monomorphism from a complex with a single nonzero injective entry, at position 0, say, to an arbitrary complex is split. So suppose given $I \in \operatorname{Ob}\operatorname{Inj}\mathcal{A}$, $X \in \operatorname{Ob}\mathrm{C}(\mathcal{A})$ and a pure monomorphism $\operatorname{Conc} I \overset{\bullet}{\longrightarrow} X$. Using Remark 8.(3′), we may choose a retraction to the composite $(I \longrightarrow X^0 \longrightarrow \mathrm{Z}'^0 X)$. This yields a retraction to $I \longrightarrow X^0$ that composes to 0 with $X^{-1} \longrightarrow X^0$, which can be employed for the sought retraction $X \longrightarrow \operatorname{Conc} I$. This proves the *claim.*
Let $X \in \operatorname{Ob}\mathrm{C}(\mathcal{A})$. We *claim* that there exists a pure monomorphism from X to a relatively injective complex. Since $\mathcal{A}$ has enough injectives, by a direct sum decomposition we are reduced to finding a pure monomorphism from X to a split complex. Consider the following morphism ϕ_k of complexes for $k \in \mathbf{Z}$,

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & X^k \arrow[r, "(1\,0)"] & X^k \oplus \mathrm{Z}'^k X \arrow[r] & 0 \arrow[r] & \cdots \\ \cdots \arrow[r] & X^{k-2} \arrow[r, "d"] \arrow[u] & X^{k-1} \arrow[r, "d"] \arrow[u, "d"] & X^k \arrow[r, "d"] \arrow[u, "(1\,p)"'] & X^{k+1} \arrow[r] \arrow[u] & \cdots \end{tikzcd} ,$$

where $X^k \xrightarrow{p} \mathrm{Z}'^k X$ is taken from X. The functor Z'^k maps it to the identity. We take the direct sum of the upper complexes over $k \in \mathbf{Z}$ and let the morphisms ϕ_k be the components of a morphism ϕ from X to this direct sum. At position k, this morphism ϕ is monomorphic because ϕ_k is. Moreover, $\mathrm{Z}'^k(\phi)$ is a monomorphism because $\mathrm{Z}'^k(\phi_k)$ is. Hence ϕ is purely monomorphic by condition (3′) of Remark 8. This proves the *claim.* ▫

Remark 10 *Write* $\mathcal{E} := \mathrm{C}(\mathcal{A})$. *Given* $\ell \geq 0$, *we have a homology functor* $\mathcal{E} \xrightarrow{\mathrm{H}^\ell} \mathcal{A}$, *which induces a functor* $\mathrm{C}(\mathcal{E}) \xrightarrow{\mathrm{C}(\mathrm{H}^\ell)} \mathrm{C}(\mathcal{A})$. *Suppose given a purely acyclic complex* $X \in \operatorname{Ob}\mathrm{C}(\mathcal{E})$. *Then* $\mathrm{C}(\mathrm{H}^\ell)X \in \operatorname{Ob}\mathrm{C}(\mathcal{A})$ *is acyclic.*

Proof. This follows using the definition of pure short exact sequences, i.e. Remark 8.(1). ▫

2.4 An exact category structure on $\mathrm{C}^{[0}(\mathcal{A})$

Write $\mathrm{CC}^{\llcorner,\,\mathrm{CE}}(\mathrm{Inj}\,\mathcal{A})$ for the full subcategory of $\mathrm{CC}^{\llcorner}(\mathcal{A})$ whose objects are CE-resolutions. Write $\mathrm{KK}^{\llcorner,\,\mathrm{CE}}(\mathrm{Inj}\,\mathcal{A})$ for the full subcategory of $\mathrm{KK}^{\llcorner}(\mathcal{A})$ whose objects are CE-resolutions.

Remark 11 *The category $\mathrm{C}^{[0}(\mathcal{A})$, equipped with the short exact sequences that lie in $\mathrm{C}^{[0}(\mathcal{A})$ and that are pure in $\mathrm{C}(\mathcal{A})$ in the sense of Remark 9 as pure short exact sequences, is an exact category wherein idempotents are split. It has enough relative injectives, viz. injectively split complexes that lie in $\mathrm{C}^{[0}(\mathcal{A})$.*

Proof. To show that it has enough relative injectives, we replace ϕ_0 in the proof of Remark 9 by $X \xrightarrow{\phi_0'} \mathrm{Conc}\, X^0$, defined by $X_0 \xrightarrow{1_{X_0}} X_0$ at position 0. ▫

2.5 The Cartan-Eilenberg resolution of a quasiisomorphism

Abbreviate $\mathcal{E} := \mathrm{C}(\mathcal{A})$, which is an exact category as in Remark 9. Consider $\mathrm{CC}^{\llcorner}(\mathcal{A}) \subseteq \mathrm{C}^{[0}(\mathcal{E})$, where the second index of $X \in \mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathcal{A})$ counts the positions in $\mathcal{E} = \mathrm{C}(\mathcal{A})$; i.e. when X is viewed as a complex with values in $\mathcal{E}$, its entry at position k is given by $X^{k,*} \in \mathcal{E} = \mathrm{C}(\mathcal{A})$.

Remark 12 *Suppose given a split acyclic complex $X \in \mathrm{Ob}\,\mathrm{C}^{[0}(\mathcal{A})$. There exists a horizontally split acyclic CE-resolution $J \in \mathrm{Ob}\,\mathrm{CC}^{\llcorner,\,\mathrm{CE}}(\mathrm{Inj}\,\mathcal{A})$ of X.*

Proof. This holds for an elementary split acyclic complex, and thus also in the general case by taking a direct sum. ▫

Lemma 13 *Suppose given $X \in \mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathcal{A})$ with pure differentials when considered as an object of $\mathrm{C}^{[0}(\mathcal{E})$, and with $\mathrm{H}^k\big(X^{*,-}\big) \simeq 0$ in $\mathrm{C}^{[0}(\mathcal{A})$ for $k \geq 1$. Suppose given $J \in \mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathrm{Inj}\,\mathcal{A})$ with split rows $J^{k,*}$ for $k \geq 1$. In other words, J is supposed to consist of relative injective object entries when considered as an object of $\mathrm{C}^{[0}(\mathcal{E})$.*
Then the map

$$(*) \qquad {}_{\mathrm{KK}^{\llcorner}(\mathcal{A})}(X, J) \xrightarrow{\mathrm{H}^0((-)^{*,-})} {}_{\mathrm{K}^{[0}(\mathcal{A})}\big(\mathrm{H}^0\big(X^{*,-}\big), \mathrm{H}^0\big(J^{*,-}\big)\big)$$

is bijective.

Proof. First, we observe that by Remark 5, we have

$$(**) \qquad {}_{\mathrm{K}^{[0}(\mathcal{E})}(X, J) \underset{\sim}{\xrightarrow{\mathrm{H}^0((-)^{*,-})}} {}_{\mathcal{E}}\big(\mathrm{H}^0\big(X^{*,-}\big), \mathrm{H}^0\big(J^{*,-}\big)\big) .$$

So it remains to show that $(*)$ is injective. Let $X \xrightarrow{f} J$ be a morphism that vanishes under $(*)$. Then $\mathrm{H}^0\big(X^{*,-}\big) \longrightarrow \mathrm{H}^0\big(J^{*,-}\big)$ factors over a split acyclic complex $S \in \mathrm{Ob}\,\mathrm{C}^{[0}(\mathcal{A})$; cf. Remark 2. Let K be a horizontally split acyclic CE-resolution of S; cf. Remark 12. By Remark 5, we obtain a morphism $X \longrightarrow K$ that lifts $\mathrm{H}^0\big(X^{*,-}\big) \longrightarrow S$ and a morphism $K \longrightarrow J$ that lifts $S \longrightarrow \mathrm{H}^0\big(J^{*,-}\big)$. The composite $X \longrightarrow K \longrightarrow J$ vanishes in $\mathrm{KK}^{\llcorner}(\mathcal{A})$. The difference

$$(X \xrightarrow{f} J) - (X \longrightarrow K \longrightarrow J)$$

lifts $\mathrm{H}^0\big(X^{*,-}\big) \xrightarrow{0} \mathrm{H}^0\big(J^{*,-}\big)$. Hence by $(**)$, it vanishes in $\mathrm{K}^{[0}(\mathcal{E})$ and so a fortiori in $\mathrm{KK}^{\llcorner}(\mathcal{A})$. Altogether, $X \xrightarrow{f} J$ vanishes in $\mathrm{KK}^{\llcorner}(\mathcal{A})$. ▫

Proposition 14 *The functor* $\mathrm{CC}^{\llcorner,\,\mathrm{CE}}(\mathrm{Inj}\,\mathcal{A}) \xrightarrow{\mathrm{H}^0((-)^{*,-})} \mathrm{C}^{[0}(\mathcal{A})$ *induces an equivalence*

$$\mathrm{KK}^{\llcorner,\,\mathrm{CE}}(\mathrm{Inj}\,\mathcal{A}) \xrightarrow[\sim]{\mathrm{H}^0((-)^{*,-})} \mathrm{K}^{[0}(\mathcal{A})\ .$$

Proof. By Lemma 13, this functor is full and faithful. By Remark 4, it is dense. ▫

Corollary 15 *Suppose given* $X,\, X' \in \mathrm{Ob}\,\mathcal{C}^{[0}(\mathcal{A})$. *Let J be a CE-resolution of X. Let J' be a CE-resolution of X'. If X and X' are isomorphic in* $\mathrm{K}^{[0}(\mathcal{A})$, *then J and J' are isomorphic in* $\mathrm{KK}^{\llcorner}(\mathcal{A})$.

The following lemma is to be compared to Remark 12.

Lemma 16 *Suppose given an acyclic complex* $X \in \mathrm{Ob}\,\mathrm{C}^{[0}(\mathcal{A})$. *There exists a rowwise split acyclic CE-resolution J of X. Each CE-resolution of X is isomorphic to J in* $\mathrm{KK}^{\llcorner}(\mathcal{A})$.

Proof. By Corollary 15, it suffices to show that there exists a rowwise split acyclic CE-resolution of X. Recall that a CE-resolution of an arbitrary complex $Y \in \mathrm{Ob}\,\mathrm{C}^{[0}(\mathcal{A})$ can be constructed by a choice of injective resolutions of H^kY and B^kY for $k \in \mathbf{Z}$, followed by an application of the abelian Horseshoe Lemma to the short exact sequences $\mathrm{B}^kY \longrightarrow \mathrm{Z}^kY \longrightarrow \mathrm{H}^kY$ for $k \in \mathbf{Z}$ and then to $\mathrm{Z}^kY \longrightarrow Y^k \longrightarrow \mathrm{B}^{k+1}Y$ for $k \in \mathbf{Z}$; cf. [5, Chap. XVII, Prop. 1.2]. Since $\mathrm{H}^kX = 0$ for $k \in \mathbf{Z}$, we may choose the zero resolution for it. Applying this construction, we obtain a rowwise split acyclic CE-resolution. ▫

Given $X \xrightarrow{f} X'$ in $\mathrm{C}^{[0}(\mathcal{A})$, a morphism $J \xrightarrow{\hat{f}} J'$ in $\mathrm{CC}^{\llcorner}(\mathcal{A})$ is called a *CE-resolution* of $X \xrightarrow{f} X'$ if $\mathrm{H}^0(\hat{f}^{*,-}) \simeq f$, as diagrams of the form $\bullet \longrightarrow \bullet$. By Remark 5, given CE-resolutions J of X and J' of X', there exists a CE-resolution $J \xrightarrow{\hat{f}} J'$ of $X \xrightarrow{f} X'$.

Proposition 17 *Let $X \xrightarrow{f} X'$ be a quasiisomorphism in $\mathrm{C}^{[0}(\mathcal{A})$. Let $J \xrightarrow{\hat{f}} J'$ be a CE-resolution of $X \xrightarrow{f} X'$. Then $\hat{f}$ can be written as a composite in $\mathrm{CC}^{\llcorner,\,\mathrm{CE}}(\mathrm{Inj}\,\mathcal{A})$ of a rowwise homotopism, followed by a double homotopism.*

Proof. Choose a pointwise split monomorphism $X \xrightarrow{a} A$ into a split acyclic complex X. We can factor

$$(X \xrightarrow{f} X') \;=\; \left(X \xrightarrow{(f\;a)} X' \oplus A \xrightarrow{\binom{1}{0}} X'\right) ,$$

so that $(f\;a)$ is a pointwise split monomorphism. Let B be a CE-resolution of A. Choosing a CE-resolution b of a, we obtain the factorisation

$$(J \xrightarrow{\hat{f}} J') \;=\; \left(J \xrightarrow{(\hat{f}\;b)} J' \oplus B \xrightarrow{\binom{1}{0}} J'\right) .$$

Since $X' \oplus A \xrightarrow{\binom{1}{0}} X'$ is a homotopism, $J' \oplus B \xrightarrow{\binom{1}{0}} J$ is a double homotopism; cf. Corollary 15. Hence $\hat{f}$ is a composite of a rowwise homotopism and a double homotopism if and only if this holds for $(\hat{f}\;b)$. So we may assume that f is pointwise split monomorphic, so in particular, monomorphic.

By Proposition 14, we may replace the given CE-resolution $\hat{f}$ by an arbitrary CE-resolution of f between J and an arbitrarily chosen CE-resolution of X' without changing the property of being a composite of a rowwise homotopism and a double homotopism for this newly chosen CE-resolution of f.

Let $X \xrightarrow{f} X' \longrightarrow \bar{X}$ be a short exact sequence in $\mathrm{C}^{[0}(\mathcal{A})$. Since f is a quasi-isomorphism, $\bar{X} \in \mathrm{Ob}\,\mathrm{C}^{[0}(\mathcal{A})$ is acyclic. Let $\bar{J}$ be a rowwise split acyclic CE-resolution of $\bar{X}$; cf. Lemma 16. The short exact sequence $X \xrightarrow{f} X' \longrightarrow \bar{X}$ is pure by acyclicity of $\bar{X}$; cf. Remark 8.(1). Hence by the exact Horseshoe Lemma, there exists a rowwise split short exact sequence $J \longrightarrow \tilde{J}' \longrightarrow \bar{J}$ of CE-resolutions that maps to $X \xrightarrow{f} X' \longrightarrow \bar{X}$ under $\mathrm{H}^0\big((-)^{*,-}\big)$; cf. Remark 7. Since $\bar{J}$ is rowwise split acyclic and since the sequence $J \longrightarrow \tilde{J}' \longrightarrow \bar{J}$ is rowwise split short exact, $J \longrightarrow \tilde{J}'$ is a rowwise homotopism. Since $J \longrightarrow \tilde{J}'$ is a CE-resolution of $X \xrightarrow{f} X'$, this proves the proposition. □

3 Formalism of spectral sequences

We follow essentially Verdier [17, II.4]; cf. [6, App.]; on a more basic level, cf. [11, Kap. 4].

Let $\mathcal{A}$ be an abelian category.

3.1 Pointwise split and pointwise finitely filtered complexes

Let $\mathbf{Z}_\infty := \{-\infty\} \sqcup \mathbf{Z} \sqcup \{\infty\}$, considered as a linearly ordered set, and thus as a category. Write $]\alpha, \beta] := \{\sigma \in \mathbf{Z}_\infty \,:\, \alpha < \sigma \leq \beta\}$ for α, $\beta \in \mathbf{Z}_\infty$ such that $\alpha \leq \beta$; etc.

Given $X \in \operatorname{Ob} [\![\mathbf{Z}_\infty, \mathrm{C}(\mathcal{A})]\!]$, the morphism of X on $\alpha \leq \beta$ in $\mathbf{Z}_\infty$ shall be denoted by $X(\alpha) \xrightarrow{x} X(\beta)$.

An object $X \in \operatorname{Ob} [\![\mathbf{Z}_\infty, \mathrm{C}(\mathcal{A})]\!]$ is called a *pointwise split and pointwise finitely filtered complex (with values in $\mathcal{A}$),* provided (SFF $1, 2, 3$) hold.

(SFF 1) We have $X(-\infty) = 0$.

(SFF 2) The morphism $X(\alpha)^i \xrightarrow{x^i} X(\beta)^i$ is split monomorphic for all $i \in \mathbf{Z}$ and all $\alpha \leq \beta$ in $\mathbf{Z}_\infty$.

(SFF 3) For all $i \in \mathbf{Z}$, there exist β_0, $\alpha_0 \in \mathbf{Z}$ such that $X(\alpha)^i \xrightarrow{x^i} X(\beta)^i$ is an identity whenever $\alpha \leq \beta \leq \beta_0$ or $\alpha_0 \leq \alpha \leq \beta$ in $\mathbf{Z}_\infty$.

The pointwise split and pointwise finitely filtered complexes with values in $\mathcal{A}$ form a full subcategory $\mathrm{SFFC}(\mathcal{A}) \subseteq [\![\mathbf{Z}_\infty, \mathrm{C}(\mathcal{A})]\!]$.

Suppose given a pointwise split and pointwise finitely filtered complex X with values in $\mathcal{A}$ for the rest of the present §3.

Let $\alpha \in \mathbf{Z}_\infty$. Write $\bar{X}(\alpha) := \operatorname{Cokern}\big(X(\alpha - 1) \longrightarrow X(\alpha)\big)$ for $\alpha \in \mathbf{Z}$. Given $i \in \mathbf{Z}$, we obtain $X(\alpha)^i \simeq \bigoplus_{\sigma \in]-\infty, \alpha]} \bar{X}(\sigma)^i$, which is a finite direct sum. We identify along this isomorphism. In particular, we get as a matrix representation for the differential

$$
\begin{aligned}
&\big(X(\alpha)^i \xrightarrow{d} X(\alpha)^{i+1}\big) \\
&= \left(\bigoplus_{\sigma \in]-\infty, \alpha]} \bar{X}(\sigma)^i \xrightarrow{\quad (d^i_{\sigma,\tau})_{\sigma,\tau} \quad} \bigoplus_{\tau \in]-\infty, \alpha]} \bar{X}(\tau)^{i+1} \right) ,
\end{aligned}
$$

where $d^i_{\sigma,\tau} = 0$ whenever $\sigma < \tau$; a kind of lower triangular matrix.

3.2 Spectral objects

Let $\bar{\mathbf{Z}}_\infty := \mathbf{Z}_\infty \times \mathbf{Z}$. Write $\alpha^{+k} := (\alpha, k)$, where $\alpha \in \mathbf{Z}_\infty$ and $k \in \mathbf{Z}$. Let $\alpha^{+k} \leq \beta^{+\ell}$ in $\bar{\mathbf{Z}}_\infty$ if $k < \ell$ or ($k = \ell$ and $\alpha \leq \beta$), i.e. let $\bar{\mathbf{Z}}_\infty$ be linearly ordered via a lexicographical ordering. We have an automorphism $\alpha^{+k} \longmapsto \alpha^{+k+1}$ of the poset $\bar{\mathbf{Z}}_\infty$, to which we refer as *shift.* Note that $-\infty^{+k} = (-\infty)^{+k}$.

We have an order preserving injection $\mathbf{Z}_\infty \longrightarrow \bar{\mathbf{Z}}_\infty$, $\alpha \longmapsto \alpha^{+0}$. We use this injection as an identification of $\mathbf{Z}_\infty$ with its image in $\bar{\mathbf{Z}}_\infty$, i.e. we sometimes write $\alpha := \alpha^{+0}$ by abuse of notation.

Let $\bar{\mathbf{Z}}^\#_\infty := \{(\alpha, \beta) \in \bar{\mathbf{Z}}_\infty \times \bar{\mathbf{Z}}_\infty \;:\; \beta^{-1} \leq \alpha \leq \beta \leq \alpha^{+1}\}$. We usually write $\beta/\alpha := (\alpha, \beta) \in \bar{\mathbf{Z}}^\#_\infty$; reminiscent of a quotient. The set $\bar{\mathbf{Z}}^\#_\infty$ is partially ordered by $\beta/\alpha \leq \beta'/\alpha' :\Longleftrightarrow (\beta \leq \beta'$ and $\alpha \leq \alpha')$. We have an automorphism $\beta/\alpha \longmapsto (\beta/\alpha)^{+1} := \alpha^{+1}/\beta$ of the poset $\bar{\mathbf{Z}}^\#_\infty$, to which, again, we refer as *shift.*

We write $\mathbf{Z}^\#_\infty := \{\beta/\alpha \in \bar{\mathbf{Z}}^\#_\infty \;:\; -\infty \le \alpha \le \beta \le \infty\}$. Note that any element of $\bar{\mathbf{Z}}^\#_\infty$ can uniquely be written as $(\beta/\alpha)^{+k}$ for some $\beta/\alpha \in \mathbf{Z}^\#_\infty$ and some $k \in \mathbf{Z}$. We shall construct the *spectral object* $\operatorname{Sp}(X) \in \operatorname{Ob} [\![\bar{\mathbf{Z}}^\#_\infty, \mathrm{K}(\mathcal{A})]\!]$. The morphism of $\operatorname{Sp}(X)$ on $\beta/\alpha \le \beta'/\alpha'$ in $\bar{\mathbf{Z}}^\#_\infty$ shall be denoted by $X(\beta/\alpha) \xrightarrow{x} X(\beta'/\alpha')$. We require that

$$\Big(X\big((\beta/\alpha)^{+k}\big) \xrightarrow{x} X\big((\beta'/\alpha')^{+k}\big)\Big) \;=\; \Big(X(\beta/\alpha) \xrightarrow{x} X(\beta'/\alpha')\Big)^{\bullet+k}$$

for $\beta/\alpha \le \beta'/\alpha'$ in $\bar{\mathbf{Z}}^\#_\infty$; i.e., roughly put, that $\operatorname{Sp}(X)$ be periodic up to shift of complexes.
Define

$$X\big(\beta/\alpha\big) \;:=\; \operatorname{Cokern}\big(X(\alpha) \xrightarrow{x} X(\beta)\big)$$

for $\beta/\alpha \in \mathbf{Z}^\#_\infty$. By periodicity, we conclude that $X\big(\alpha/\alpha\big) = 0$ and $X\big(\alpha^{+1}/\alpha\big) = 0$ for all $\alpha \in \bar{\mathbf{Z}}_\infty$.
Write

$$D^i_{\beta/\alpha,\,\beta'/\alpha'} \;:=\; (d^i_{\sigma,\tau})_{\sigma\in]\alpha,\beta],\;\tau\in]\alpha',\beta']} \;:\; X(\beta/\alpha)^i \longrightarrow X(\beta'/\alpha')^{i+1}$$

for $i \in \mathbf{Z}$ and $\beta/\alpha,\ \beta'/\alpha' \in \mathbf{Z}^\#_\infty$.
Given $-\infty \le \alpha \le \beta \le \gamma \le \infty$ and $i \in \mathbf{Z}$, we let

$$\begin{array}{rcl}
\Big(X(\beta/\alpha)^i \xrightarrow{x^i} X(\gamma/\alpha)^i\Big) & := & \Big(X(\beta/\alpha)^i \xrightarrow{(1\;0)} X(\beta/\alpha)^i \oplus X(\gamma/\beta)^i\Big) \\
\Big(X(\gamma/\alpha)^i \xrightarrow{x^i} X(\gamma/\beta)^i\Big) & := & \Big(X(\beta/\alpha)^i \oplus X(\gamma/\beta)^i \xrightarrow{\binom{0}{1}} X(\gamma/\beta)^i\Big) \\
\Big(X(\gamma/\beta)^i \xrightarrow{x^i} X(\alpha^{+1}/\beta)^i\Big) & := & \Big(X(\gamma/\beta)^i \xrightarrow{D^i_{\gamma/\beta,\,\beta/\alpha}} X(\beta/\alpha)^{i+1}\Big) .
\end{array}$$

By periodicity up to shift of complexes, this defines $\operatorname{Sp}(X)$. The construction is functorial in $X \in \operatorname{Ob}\operatorname{SFFC}(\mathcal{A})$.

3.3 Spectral sequences

Let $\bar{\mathbf{Z}}^{\#\#}_\infty := \{(\gamma/\alpha, \delta/\beta) \in \bar{\mathbf{Z}}^\#_\infty \times \bar{\mathbf{Z}}^\#_\infty \;:\; \delta^{-1} \le \alpha \le \beta \le \gamma \le \delta \le \alpha^{+1}\}$. Given $(\gamma/\alpha, \delta/\beta) \in \bar{\mathbf{Z}}^{\#\#}_\infty$, we usually write $\delta/\beta/\!\!/\gamma/\alpha := (\gamma/\alpha, \delta/\beta)$. The set $\bar{\mathbf{Z}}^{\#\#}_\infty$ is partially ordered by

$$\delta/\beta/\!\!/\gamma/\alpha \le \delta'/\beta'/\!\!/\gamma'/\alpha' \;\;:\Longleftrightarrow\;\; (\gamma/\alpha \le \gamma'/\alpha' \text{ and } \delta/\beta \le \delta'/\beta') .$$

Define the *spectral sequence* $\mathrm{E}(X) \in \operatorname{Ob} [\![\bar{\mathbf{Z}}^{\#\#}_\infty, \mathcal{A}]\!]$ of X by letting its value on

$$\delta/\beta/\!\!/\gamma/\alpha \;\le\; \delta'/\beta'/\!\!/\gamma'/\alpha'$$

in $\bar{\mathbf{Z}}_\infty^{\#\#}$ be the morphism that appears in the middle column of the diagram

$$\begin{tikzcd}
\mathrm{H}^0\big(X(\gamma/\alpha)\big) \arrow[r, "\shortmid" marking] \arrow[d, "\mathrm{H}^0(x)"'] & \mathrm{E}(\delta/\beta/\!\!/\gamma/\alpha)(X) \arrow[r, "\bullet" marking] \arrow[d, "e"'] & \mathrm{H}^0\big(X(\delta/\beta)\big) \arrow[d, "\mathrm{H}^0(x)"'] \\
\mathrm{H}^0\big(X(\gamma'/\alpha')\big) \arrow[r, "\shortmid" marking] & \mathrm{E}(\delta'/\beta'/\!\!/\gamma'/\alpha')(X) \arrow[r, "\bullet" marking] & \mathrm{H}^0\big(X(\delta'/\beta')\big) .
\end{tikzcd}$$

Given $\delta/\beta/\!\!/\gamma/\alpha \in \bar{\mathbf{Z}}_\infty^{\#\#}$ and $k \in \mathbf{Z}$, we also write

$$\mathrm{E}(\delta/\beta/\!\!/\gamma/\alpha)^{+k}(X) \;:=\; \mathrm{E}\big((\delta/\beta)^{+k}/\!\!/(\gamma/\alpha)^{+k}\big)(X) .$$

Altogether,

$$\begin{array}{ccccccc}
[\![\mathbf{Z}_\infty, \mathrm{C}(\mathcal{A})]\!] & \supseteq & \mathrm{SFFC}(\mathcal{A}) & \longrightarrow & [\![\bar{\mathbf{Z}}_\infty^{\#}, \mathrm{K}(\mathcal{A})]\!] & \longrightarrow & [\![\bar{\mathbf{Z}}_\infty^{\#\#}, \mathcal{A}]\!] \\
 & & X & \longmapsto & \mathrm{Sp}(X) & \longmapsto & \mathrm{E}(X) .
\end{array}$$

3.4 A short exact sequence

Lemma 18 *Given $\varepsilon^{-1} \le \alpha \le \beta \le \gamma \le \delta \le \varepsilon \le \alpha^{+1}$ in $\bar{\mathbf{Z}}_\infty$, we have a short exact sequence*

$$\begin{tikzcd}
\mathrm{E}(\varepsilon/\beta/\!\!/\gamma/\alpha)(X) \arrow[r, "\bullet" marking, "e"] & \mathrm{E}(\varepsilon/\beta/\!\!/\delta/\alpha)(X) \arrow[r, "\shortmid" marking, "e"] & \mathrm{E}(\varepsilon/\gamma/\!\!/\delta/\alpha)(X) .
\end{tikzcd}$$

Proof. See [10, Lem. 3.9]. ▫

Lemma 19 *Given $\varepsilon^{-1} \le \alpha \le \beta \le \gamma \le \delta \le \varepsilon \le \alpha^{+1}$ in $\bar{\mathbf{Z}}_\infty$, we have a short exact sequence*

$$\begin{tikzcd}
\mathrm{E}(\varepsilon/\gamma/\!\!/\delta/\alpha)(X) \arrow[r, "\bullet" marking, "e"] & \mathrm{E}(\varepsilon/\gamma/\!\!/\delta/\beta)(X) \arrow[r, "\shortmid" marking, "e"] & \mathrm{E}(\alpha^{+1}/\gamma/\!\!/\delta/\beta)(X) .
\end{tikzcd}$$

Proof. Apply the functor induced by $\beta/\alpha \longmapsto \alpha^{+1}/\beta$ to $\mathrm{Sp}(X)$. Then apply [10, Lem. 3.9]. ▫

The short exact sequence in Lemma 18 is called a *fundamental short exact sequence (in first notation)*, the short exact sequence in Lemma 19 is called a *fundamental short exact sequence (in second notation).* They will be used without further comment.

3.5 Classical indexing

Let $1 \le r \le \infty$ and let $p,\, q \in \mathbf{Z}$. Denote

$$\mathrm{E}_r^{p,q} \;=\; \mathrm{E}_r^{p,q}(X) \;:=\; \mathrm{E}(-p-1+r/-p-1/\!\!/-p/-p-r)^{+p+q}(X) ,$$

where $i + \infty := \infty$ and $i - \infty := -\infty$ for all $i \in \mathbf{Z}$.

Example 20 The short exact sequences in Lemmata 18, 19 allow to derive the exact couples of Massey. Write

$$\mathrm{D}_r^{i,j} \;=\; \mathrm{D}_r^{i,j}(X) \;:=\; \mathrm{E}(-i/-\infty/\!\!/-i-r+1/-\infty)^{+i+j}(X)$$

for $i,\, j \in \mathbf{Z}$ and $r \geq 1$. We obtain an exact sequence

$$\mathrm{D}_r^{i,\,j} \xrightarrow{e} \mathrm{D}_r^{i-1,\,j+1} \xrightarrow{e} \mathrm{E}_r^{i+r-2,\,j-r+2} \xrightarrow{e} \mathrm{D}_r^{i+r-1,\,j-r+2} \xrightarrow{e} \mathrm{D}_r^{i+r-2,\,j-r+3}$$

by Lemmata 18, 19.

3.6 Comparing proper spectral sequences

Let $X \xrightarrow{f} Y$ be a morphism in $\mathrm{SFFC}(\mathcal{A})$, i.e. a morphism of pointwise split and pointwise finitely filtered complexes with values in $\mathcal{A}$. Write $\mathrm{E}(X) \xrightarrow{\mathrm{E}(f)} \mathrm{E}(Y)$ for the induced morphism on the spectral sequences.
For $\alpha,\, \beta \in \bar{\mathbf{Z}}_\infty$, we write $\alpha \mathrel{\dot{<}} \beta$ if

$$\big(\alpha < \beta\big) \quad \text{or} \quad \big(\alpha = \beta \quad \text{and} \quad \alpha \in \{\infty^{+k} \;:\; k \in \mathbf{Z}\} \cup \{-\infty^{+k} \;:\; k \in \mathbf{Z}\}\big)\;.$$

We write

$$\dot{\bar{\mathbf{Z}}}{}_\infty^{\#\#} \;:=\; \{\delta/\beta/\!\!/\gamma/\alpha \in \bar{\mathbf{Z}}_\infty^{\#\#} \;:\; \delta^{-1} \leq \alpha \mathrel{\dot{<}} \beta \leq \gamma \mathrel{\dot{<}} \delta \leq \alpha^{+1}\}\;.$$

We write

$$\dot{\mathrm{E}} \;=\; \dot{\mathrm{E}}(X) \;:=\; \mathrm{E}(X)|_{\dot{\bar{\mathbf{Z}}}{}_\infty^{\#\#}} \;\in\; \mathrm{Ob}\,[\![\dot{\bar{\mathbf{Z}}}{}_\infty^{\#\#}, \mathcal{A}]\!]$$

for the *proper spectral sequence* of X; analogously for the morphisms.

Lemma 21 *If* $\mathrm{E}(\alpha+1/\alpha-1/\!\!/\alpha/\alpha-2)^{+k}(f)$ *is an isomorphism for all* $\alpha \in \mathbf{Z}$ *and all* $k \in \mathbf{Z}$*, then* $\dot{\mathrm{E}}(f)$ *is an isomorphism.*

Proof. *Claim 1.* We have an isomorphism $\mathrm{E}(\gamma/\beta-1/\!\!/\beta/\beta-2)^{+k}(f)$ for all $k \in \mathbf{Z}$, all $\beta \in \mathbf{Z}$ and all $\gamma \in \mathbf{Z}$ such that $\gamma > \beta$. We have an isomorphism $\mathrm{E}(\beta+1/\beta-1/\!\!/\beta/\alpha-1)^{+k}(f)$ for all $k \in \mathbf{Z}$, all $\beta \in \mathbf{Z}$ and all $\alpha \in \mathbf{Z}$ such that $\alpha < \beta$.
The assertions follow by induction using the exact sequences

$$\begin{array}{rl} \mathrm{E}(\gamma+2/\gamma/\!\!/\gamma+1/\beta)^{+k-1} & \xrightarrow{e} \mathrm{E}(\gamma/\beta-1/\!\!/\beta/\beta-2)^{+k} \\ & \xrightarrow{e} \mathrm{E}(\gamma+1/\beta-1/\!\!/\beta/\beta-2)^{+k} \longrightarrow 0 \end{array}$$

and

$$\begin{array}{rl} 0 \longrightarrow \mathrm{E}(\beta+1/\beta-1/\!\!/\beta/\alpha-2)^{+k} & \xrightarrow{e} \mathrm{E}(\beta+1/\beta-1/\!\!/\beta/\alpha-1)^{+k} \\ & \xrightarrow{e} \mathrm{E}(\beta-1/\alpha-2/\!\!/\alpha-1/\alpha-3)^{+k+1}\;. \end{array}$$

Claim 2. We have an isomorphism $\mathrm{E}(\gamma/\beta-1/\!\!/\beta/\alpha-1)^{+k}(f)$ for all $k \in \mathbf{Z}$ and all $\alpha,\,\beta,\,\gamma \in \mathbf{Z}$ such that $\alpha < \beta < \gamma$.
We proceed by induction on $\gamma - \alpha$. By Claim 1, we may assume that $\alpha < \beta - 1 < \beta + 1 < \gamma$. Consider the image diagram

$$\mathrm{E}(\gamma-1/\beta-1/\!\!/\beta/\alpha-1)^{+k} \overset{e}{\twoheadrightarrow} \mathrm{E}(\gamma/\beta-1/\!\!/\beta/\alpha-1)^{+k} \overset{e}{\rightarrowtail} \mathrm{E}(\gamma/\beta-1/\!\!/\beta/\alpha)^{+k}\,.$$

Claim 3. We have an isomorphism $\mathrm{E}(\delta/\beta/\!\!/\gamma/\alpha)^{+k}(f)$ for all $k \in \mathbf{Z}$ and all $\alpha,\,\beta,\,\gamma,\,\delta \in \mathbf{Z}$ such that $\alpha < \beta \le \gamma < \delta$.
We may assume that $\gamma - \beta \ge 1$, for $\mathrm{E}(\delta/\beta/\!\!/\beta/\alpha)^{+k} = 0$. We proceed by induction on $\gamma - \beta$. By Claim 2, we may assume that $\gamma - \beta \ge 2$. Consider the short exact sequence

$$\mathrm{E}(\delta/\beta/\!\!/\gamma-1/\alpha)^{+k} \overset{e}{\rightarrowtail} \mathrm{E}(\delta/\beta/\!\!/\gamma/\alpha)^{+k} \overset{e}{\twoheadrightarrow} \mathrm{E}(\delta/\gamma-1/\!\!/\gamma/\alpha)^{+k}\,.$$

Claim 4. We have an isomorphism $\mathrm{E}(\delta/\beta/\!\!/\gamma/\alpha)^{+k}(f)$ for all $k \in \mathbf{Z}$ and all $\alpha,\,\beta,\,\gamma,\,\delta \in \mathbf{Z}_\infty$ such that $\alpha < \beta \le \gamma < \delta$.
In view of Claim 3, it suffices to choose $\tilde{\alpha} \in \mathbf{Z}$ small enough such that $\mathrm{E}(\delta/\beta/\!\!/\gamma/\tilde{\alpha})^{+k}(f) = \mathrm{E}(\delta/\beta/\!\!/\gamma/-\infty)^{+k}(f)$; etc.
Claim 5. We have an isomorphism $\mathrm{E}(\delta/\beta/\!\!/\gamma/\alpha)^{+k}(f)$ for all $k \in \mathbf{Z}$ and all $\alpha,\,\beta,\,\gamma,\,\delta \in \mathbf{Z}_\infty$ such that $\alpha \mathrel{\dot{<}} \beta \le \gamma \mathrel{\dot{<}} \delta$.
In view of Claim 4, it suffices to choose $\tilde{\beta} \in \mathbf{Z}$ small enough such that $\mathrm{E}(\delta/\tilde{\beta}/\!\!/\gamma/-\infty)^{+k}(f) = \mathrm{E}(\delta/-\infty/\!\!/\gamma/-\infty)^{+k}(f)$; etc.
Claim 6. We have an isomorphism $\mathrm{E}(\delta/\beta/\!\!/\gamma/\alpha)^{+k}(f)$ for all $k \in \mathbf{Z}$ and all $\alpha,\,\beta,\,\gamma,\,\delta \in \bar{\mathbf{Z}}_\infty$ such that $-\infty \le \delta^{-1} \le \alpha \mathrel{\dot{<}} \beta \le \gamma \le \infty < -\infty^{+1} \le \delta \le \alpha^{+1}$.
In view of Claim 5, it suffices to consider the short exact sequence

$$\mathrm{E}(\infty/\beta/\!\!/\gamma/\delta^{-1})^{+k} \overset{e}{\rightarrowtail} \mathrm{E}(\infty/\beta/\!\!/\gamma/\alpha)^{+k} \overset{e}{\twoheadrightarrow} \mathrm{E}(\delta/\beta/\!\!/\gamma/\alpha)^{+k}\,.$$

Claim 7. The morphism $\dot{\mathrm{E}}(f)$ is an isomorphism.
Suppose given $\alpha,\,\beta,\,\gamma,\,\delta \in \bar{\mathbf{Z}}_\infty$ such that $\delta^{-1} \le \alpha \mathrel{\dot{<}} \beta \le \gamma \mathrel{\dot{<}} \delta \le \alpha^{+1}$. Via a shift, we may assume that we are in the situation of Claim 5 or of Claim 6. ▫

3.7 The first spectral sequence of a double complex

Let $\mathcal{A}$ be an abelian category. Let $X \in \operatorname{Ob}\mathrm{CC}^{\llcorner}(\mathcal{A})$. Given $n \in \mathbf{Z}_\infty$, we write $X^{[n,*}$ for the double complex arising from X by replacing $X^{i,j}$ by 0 for all $i \in [0,n[$. We define a pointwise split and pointwise finitely filtered complex $\mathrm{t_I}X$, called the *first filtration of* $\mathrm{t}X$, by letting $\mathrm{t_I}X(\alpha) := \mathrm{t}X^{[-\alpha,*}$ for $\alpha \in \mathbf{Z}_\infty$; and by letting $\mathrm{t_I}X(\alpha) \longrightarrow \mathrm{t_I}X(\beta)$ be the pointwise split inclusion $\mathrm{t}X^{[-\alpha,*} \longrightarrow \mathrm{t}X^{[-\beta,*}$ for $\alpha,\,\beta \in \mathbf{Z}_\infty$ such that $\alpha \le \beta$. Let $\mathrm{E_I} = \mathrm{E_I}(X) := \mathrm{E}(\mathrm{t_I}X)$. This construction is functorial in $X \in \operatorname{Ob}\mathrm{CC}^{\llcorner}(\mathcal{A})$. Note that $\overline{\mathrm{t_I}X}(\alpha) = X^{-\alpha,k+\alpha}$.

We record the following wellknown lemma in the language we use here.

Lemma 22 *Let $\alpha \in \,]-\infty, 0]$. Let $k \in \mathbf{Z}$ such that $k \geq -\alpha$. We have*

$$\begin{array}{rcl} \mathrm{E}_{\mathrm{I}}(\alpha/\alpha-1/\!\!/\alpha/\alpha-1)^{+k}(X) & = & \mathrm{H}^{k+\alpha}(X^{-\alpha,*}) \\ \mathrm{E}_{\mathrm{I}}(\alpha+1/\alpha-1/\!\!/\alpha/\alpha-2)^{+k}(X) & = & \mathrm{H}^{-\alpha}\big(\mathrm{H}^{k+\alpha}(X^{-,*})\big)\ , \end{array}$$

naturally in $X \in \operatorname{Ob}\mathrm{CC}^{\llcorner}(\mathcal{A})$.

Proof. The first equality follows by $\mathrm{E}_{\mathrm{I}}(\alpha/\alpha-1/\!\!/\alpha/\alpha-1)^{+k} = \mathrm{H}^k \mathrm{t}_{\mathrm{I}} X(\alpha/\alpha-1) = \mathrm{H}^{k+\alpha}(X^{-\alpha,*})$.
The morphism $\mathrm{t}_{\mathrm{I}}X(\alpha/\alpha-1) \longrightarrow \mathrm{t}_{\mathrm{I}}X\big((\alpha-2)^{+1}/\alpha-1\big) = \mathrm{t}_{\mathrm{I}}X\big(\alpha-1/\alpha-2\big)^{\bullet+1}$ from $\mathrm{Sp}(\mathrm{t}_{\mathrm{I}}X)$ is at position $k \geq 0$ given by

$$\overline{\mathrm{t}_{\mathrm{I}}X}(\alpha)^k = X^{-\alpha,k+\alpha} \xrightarrow{\ (-1)^{\alpha}\,\partial\ } X^{-\alpha+1,k+\alpha} = \overline{\mathrm{t}_{\mathrm{I}}X}(\alpha-1)^{k+1}\ ;$$

cf. §1.1.6. In particular, the morphisms

$$\begin{array}{l} \mathrm{E}_{\mathrm{I}}(\alpha+1/\alpha/\!\!/\alpha+1/\alpha)^{+k-1} \xrightarrow{\ e\ } \mathrm{E}_{\mathrm{I}}(\alpha/\alpha-1/\!\!/\alpha/\alpha-1)^{+k} \\ \qquad\qquad\qquad\qquad\qquad\qquad \xrightarrow{\ e\ } \mathrm{E}_{\mathrm{I}}(\alpha-1/\alpha-2/\!\!/\alpha-1/\alpha-2)^{+k+1} \end{array}$$

are given by

$$\begin{array}{l} \mathrm{H}^{k+\alpha}(X^{-\alpha-1,*}) \xrightarrow{\ (-1)^{\alpha+1}\mathrm{H}^{k+\alpha}(\partial)\ } \mathrm{H}^{k+\alpha}(X^{-\alpha,*}) \\ \qquad\qquad\qquad\qquad\qquad\qquad \xrightarrow{\ (-1)^{\alpha}\mathrm{H}^{k+\alpha}(\partial)\ } \mathrm{H}^{k+\alpha}(X^{-\alpha+1,*})\ . \end{array}$$

Now the second equality follows by the diagram

$$\begin{tikzcd}[column sep=tiny] & & \mathrm{E}_{\mathrm{I}}(\alpha{+}1/\alpha{-}1/\!\!/\alpha/\alpha{-}2)^{+k} \arrow[dr, "e"] & & \\ & \mathrm{E}_{\mathrm{I}}(\alpha/\alpha{-}1/\!\!/\alpha/\alpha{-}2)^{+k} \arrow[ur, "e"] \arrow[dr, "e"] & & \mathrm{E}_{\mathrm{I}}(\alpha{+}1/\alpha{-}1/\!\!/\alpha/\alpha{-}1)^{+k} & \\ \mathrm{E}_{\mathrm{I}}(\alpha{+}1/\alpha/\!\!/\alpha{+}1/\alpha)^{+k-1} \arrow[rr, "e"] & & \mathrm{E}_{\mathrm{I}}(\alpha/\alpha{-}1/\!\!/\alpha/\alpha{-}1)^{+k} \arrow[ur, "e"] \arrow[rr, "e"] & & \mathrm{E}_{\mathrm{I}}(\alpha{-}1/\alpha{-}2/\!\!/\alpha{-}1/\alpha{-}2)^{+k+1}\ . \end{tikzcd}$$

□

Remark 23 *Let $X \xrightarrow{f} Y$ be a rowwise quasiisomorphism in $\mathrm{CC}^{\llcorner}(\mathcal{A})$. Then $\mathrm{E}_{\mathrm{I}}(\delta/\beta/\!\!/\gamma/\alpha)^{+k}(f)$ is an isomorphism for $\delta^{-1} \leq \alpha \leq \beta \leq \gamma \leq \delta \leq \alpha^{+1}$ in $\bar{\mathbf{Z}}_{\infty}$ and $k \in \mathbf{Z}$.*

Proof. It suffices to show that the morphism $\mathrm{Sp}(\mathrm{t}_{\mathrm{I}}f)$ in $\llbracket \bar{\mathbf{Z}}_{\infty}^{\#}, \mathrm{K}(\mathcal{A})\rrbracket$ is pointwise a quasiisomorphism. To have this, it suffices to show that $\mathrm{t}f^{[k,*}$ is a quasiisomorphism for $k \geq 0$. But $f^{[k,*}$ is a rowwise quasiisomorphism for $k \geq 0$; cf. §1.1.6. □

Lemma 24 *The functor* $\mathrm{CC}^{\llcorner}(\mathcal{A}) \xrightarrow{\dot{\mathrm{E}}_{\mathrm{I}}} [\![\dot{\mathbf{Z}}_{\infty}^{\#\#}, \mathcal{A}]\!]$ *factors over*

$$\mathrm{KK}^{\llcorner}(\mathcal{A}) \xrightarrow{\dot{\mathrm{E}}_{\mathrm{I}}} [\![\dot{\mathbf{Z}}_{\infty}^{\#\#}, \mathcal{A}]\!] .$$

Proof. By Lemma 1, we have to show that $\dot{\mathrm{E}}_{\mathrm{I}}$ annihilates all elementary horizontally split acyclic double complexes in $\mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathcal{A})$ and all elementary vertically split acyclic double complexes in $\mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathcal{A})$.

Let $U \in \mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathcal{A})$ be an elementary vertically split acyclic double complex concentrated in rows i and $i+1$, where $i \geq 0$. Let $V \in \mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathcal{A})$ be an elementary horizontally split acyclic double complex concentrated in columns j and $j+1$, where $j \geq 0$.

Since V is rowwise acyclic, E_{I} annihilates V by Remark 23, whence so does $\dot{\mathrm{E}}_{\mathrm{I}}$.

Suppose given

$$(*) \qquad -\infty \leq \alpha \mathrel{\dot{<}} \beta \leq \gamma \mathrel{\dot{<}} \delta \leq \infty$$

in $\bar{\mathbf{Z}}_{\infty}$ and $k \in \mathbf{Z}$. We *claim* that the functor $\mathrm{E}_{\mathrm{I}}(\delta/\beta /\!\!/ \gamma/\alpha)^{+k}$ annihilates U. We may assume that $\beta < \gamma$. Note that $\mathrm{E}_{\mathrm{I}}(\delta/\beta /\!\!/ \gamma/\alpha)^{+k}(U)$ is the image of

$$\mathrm{H}^k\big(\mathrm{t}_{\mathrm{I}} U(\gamma/\alpha)\big) \longrightarrow \mathrm{H}^k\big(\mathrm{t}_{\mathrm{I}} U(\delta/\beta)\big) .$$

The double complex $U^{[-\delta,*}/U^{[-\beta,*}$ is columnwise acyclic except possibly if $-\beta = i+1$ or if $-\delta = i+1$. The double complex $U^{[-\gamma,*}/U^{[-\alpha,*}$ is columnwise acyclic except possibly if $-\alpha = i+1$ or if $-\gamma = i+1$. All three remaining combinations of these exceptional cases are excluded by $(*)$, however. Hence $\mathrm{E}_{\mathrm{I}}(\delta/\beta /\!\!/ \gamma/\alpha)^{+k}(U) = 0$. This proves the *claim.*

Suppose given

$$(**) \qquad \delta^{-1} \leq \alpha \mathrel{\dot{<}} \beta \leq \gamma \leq \infty \leq -\infty^{+1} \leq \delta \leq \alpha^{+1} .$$

in $\bar{\mathbf{Z}}_{\infty}$ and $k \in \mathbf{Z}$. We *claim* that the functor $\mathrm{E}_{\mathrm{I}}(\delta/\beta /\!\!/ \gamma/\alpha)^{+k}$ annihilates U. We may assume that $\beta < \gamma$ and that $\delta^{-1} < \alpha$. Note that $\mathrm{E}_{\mathrm{I}}(\delta/\beta /\!\!/ \gamma/\alpha)^{+k}(U)$ is the image of

$$\mathrm{H}^k\big(\mathrm{t}_{\mathrm{I}} U(\gamma/\alpha)\big) \longrightarrow \mathrm{H}^{k+1}\big(\mathrm{t}_{\mathrm{I}} U(\beta/\delta^{-1})\big) .$$

The double complex $U^{[-\beta,*}/U^{[-(\delta^{-1}),*}$ is columnwise acyclic except possibly if $-(\delta^{-1}) = i+1$ or if $-\beta = i+1$. The double complex $U^{[-\gamma,*}/U^{[-\alpha,*}$ is columnwise acyclic except possibly if $-\gamma = i+1$ or if $-\alpha = i+1$. Both remaining combinations of these exceptional cases are excluded by $(**)$, however. Hence $\mathrm{E}_{\mathrm{I}}(\delta/\beta /\!\!/ \gamma/\alpha)^{+k}(U) = 0$. This proves the *claim.*

Both claims taken together show that $\dot{\mathrm{E}}_{\mathrm{I}}$ annihilates U. ◻

4 Grothendieck spectral sequences

4.1 Certain quasiisomorphisms are preserved by a left exact functor

Suppose given abelian categories $\mathcal{A}$, $\mathcal{B}$, and suppose that $\mathcal{A}$ has enough injectives. Let $\mathcal{A} \xrightarrow{F} \mathcal{B}$ be a left exact functor.

Remark 25 *Suppose given an F-acyclic object $X \in \operatorname{Ob}\mathcal{A}$ and an injective resolution $I \in \operatorname{Ob}\mathrm{C}^{[0}(\operatorname{Inj}\mathcal{A})$ of X. Let $\operatorname{Conc} X \xrightarrow{f} I$ be its quasiisomorphism. Then $\operatorname{Conc} FX \xrightarrow{Ff} FI$ is a quasiisomorphism.*

Proof. This follows since F is left exact and since $\mathrm{H}^i(FI) \simeq (\mathrm{R}^i F)X \simeq 0$ for $i \geq 1$. ▫

Remark 26 *Suppose given a complex $U \in \operatorname{Ob}\mathrm{C}^{[0}(\mathcal{A})$ consisting of F-acyclic objects. There exists an injective complex resolution $I \in \operatorname{Ob}\mathrm{C}^{[0}(\operatorname{Inj}\mathcal{A})$ of U such that its quasiisomorphism $U \xrightarrow{f} I$ maps to a quasiisomorphism $FU \xrightarrow{Ff} FI$.*

Proof. Let $J \in \operatorname{Ob}\mathrm{CC}^{\llcorner,\,\mathrm{CE}}(\operatorname{Inj}\mathcal{A})$ be a CE-resolution of U; cf. Remark 9. Since the morphism of double complexes $\operatorname{Conc}_2 U \longrightarrow J$ is a columnwise quasiisomorphism consisting of monomorphisms, taking the total complex, we obtain a quasiisomorphism $U \longrightarrow \mathrm{t}J$ consisting of monomorphisms. By F-acyclicity of the entries of U, the image $\operatorname{Conc}_2 FU \longrightarrow FJ$ under F is a columnwise quasiisomorphism, too; cf. Remark 25. Hence F maps the quasiisomorphism $U \longrightarrow \mathrm{t}J$ to the quasiisomorphism $FU \longrightarrow F\mathrm{t}J$. So we may take $I := \mathrm{t}J$. ▫

Lemma 27 *Suppose given a complex $U \in \operatorname{Ob}\mathrm{C}^{[0}(\mathcal{A})$ consisting of F-acyclic objects and an injective complex resolution $I \in \operatorname{Ob}\mathrm{C}^{[0}(\operatorname{Inj}\mathcal{A})$ of U. Let $U \xrightarrow{f} I$ be its quasiisomorphism. Then $FU \xrightarrow{Ff} FI$ is a quasiisomorphism.*

Proof. Let $U \longrightarrow I'$ be a quasiisomorphism to an injective complex resolution I' that is mapped to a quasiisomorphism by F; cf. Remark 26. Since $U \longrightarrow I'$ is a quasiisomorphism, the induced map ${}_{\mathrm{K}(\mathcal{A})}(U, I) \longleftarrow {}_{\mathrm{K}(\mathcal{A})}(I', I)$ is surjective, so that there exists a morphism $I' \longrightarrow I$ such that $(U \longrightarrow I' \longrightarrow I) = (U \xrightarrow{f} I)$ in $\mathrm{K}(\mathcal{A})$. Since, moreover, $U \xrightarrow{f} I$ is a quasiisomorphism, $I' \longrightarrow I$ is a homotopism. Since $FU \longrightarrow FI'$ is a quasiisomorphism and $FI' \longrightarrow FI$ is a homotopism, we conclude that $FU \longrightarrow FI$ is a quasiisomorphism. ▫

4.2 Definition of the Grothendieck spectral sequence functor

Suppose given abelian categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, and suppose that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. Let $\mathcal{A} \xrightarrow{F} \mathcal{B}$ and $\mathcal{B} \xrightarrow{G} \mathcal{C}$ be left exact functors.
A (F, G)-*acyclic resolution* of $X \in \operatorname{Ob}\mathcal{A}$ is a complex $A \in \operatorname{Ob}\mathrm{C}^{[0}(\mathcal{A})$, together with a quasiisomorphism $\operatorname{Conc} X \longrightarrow A$, such that the following hold.

(A 1) The object A^i is F-acyclic for $i \geq 0$.

(A 2) The object A^i is $(G \circ F)$-acyclic for $i \geq 0$.

(A 3) The object FA^i is G-acyclic for $i \geq 0$.

An object $X \in \operatorname{Ob}\mathcal{A}$ that possesses an (F,G)-acyclic resolution is called (F,G)-*acyclicly resolvable.* The full subcategory of (F,G)-acyclicly resolvable objects in $\mathcal{A}$ is denoted by $\mathcal{A}_{(F,G)}$.
A complex $A \in \operatorname{Ob}\mathrm{C}^{[0}(\mathcal{A})$, together with a quasiisomorphism $\operatorname{Conc} X \longrightarrow A$, is called an F-*acyclic resolution* of $X \in \operatorname{Ob}\mathcal{A}$ if (A 2) holds.

Remark 28 *If F carries injective objects to G-acyclic objects, then* (A 1) *and* (A 3) *imply* (A 2).

Proof. Given $i \geq 0$, we let I be an injective resolution of A^i, and $\tilde{I}$ the acyclic complex obtained by appending A^i to I in position -1. Since A^i is F-acyclic, the complex $F\tilde{I}$ is acyclic; cf. Remark 25. Note that $F\mathrm{B}^0\tilde{I} \simeq FA^i$ is G-acyclic by assumption. Since

$$(\mathrm{R}^k G)F\tilde{I}^j \longrightarrow (\mathrm{R}^k G)F\mathrm{B}^{j+1}\tilde{I} \longrightarrow (\mathrm{R}^{k+1}G)F\mathrm{B}^j\tilde{I}$$

is exact in the middle for $j \geq 0$ and $k \geq 1$, we may conclude by induction on j and by G-acyclicity assumption on $F\tilde{I}^j$ that $F\mathrm{B}^j\tilde{I}$ is G-acyclic for $j \geq 0$. In particular, we have $(\mathrm{R}^1 G)(F\mathrm{B}^j\tilde{I}) \simeq 0$ for $j \geq 0$, whence

$$GF\mathrm{B}^j\tilde{I} \longrightarrow GF\tilde{I}^j \longrightarrow GF\mathrm{B}^{j+1}\tilde{I}$$

is short exact for $j \geq 0$. We conclude that $(G \circ F)\tilde{I}$ is acyclic. Hence A^i is $(G \circ F)$-acyclic. ▫

To see Remark 28, one could also use a Grothendieck spectral sequence, once established.

Remark 29 *Suppose given $X \in \operatorname{Ob}\mathcal{A}$, an injective resolution I of X and an F-acyclic resolution A of X. Then there exists a quasiisomorphism $A \longrightarrow I$ that is mapped to 1_X by H^0. Moreover, any morphism $A \xrightarrow{u} I$ that is mapped to 1_X by H^0 is a quasiisomorphism and is mapped to a quasiisomorphism $FA \xrightarrow{Fu} FI$ by F.*

Proof. Let I' be an injective complex resolution of A such that its quasiisomorphism $A \longrightarrow I'$ is mapped to a quasiisomorphism by F; cf. Remark 26. We use the composite quasiisomorphism $\operatorname{Conc} X \longrightarrow A \longrightarrow I'$ to resolve X by I'. To prove the first assertion, note that there is a homotopism $I' \longrightarrow I$ resolving 1_X; whence the composite $(A \longrightarrow I' \longrightarrow I)$ is a quasiisomorphism resolving 1_X.
To prove the second assertion, note that the induced map ${}_{\mathrm{K}(\mathcal{A})}(A,I) \longleftarrow {}_{\mathrm{K}(\mathcal{A})}(I',I)$ is surjective, whence there is a factorisation $(A \longrightarrow I' \longrightarrow I) = (A \xrightarrow{u} I)$ in $\mathrm{K}(\mathcal{A})$ for some morphism $I' \longrightarrow I$, which, since resolving 1_X as well, is a homotopism. In particular, $A \xrightarrow{u} I$ is a quasiisomorphism. Finally, since $FI' \longrightarrow FI$ is a homotopism, also $FA \xrightarrow{Fu} FI$ is a quasiisomorphism. ▫

Alternatively, in the last step of the preceding proof we could have invoked Lemma 27.

The following construction originates in [5, XVII.§7] and [7, Th. 2.4.1]. In its present form, it has been carried out by Haas in the classical framework [8]. We do not claim any originality.

I do not know whether the use of injectives in $\mathcal{A}$ in the following construction can be avoided; in any case, it would be desirable to do so.

We set out to define the *proper Grothendieck spectral sequence functor*

$$\mathcal{A}_{(F,G)} \xrightarrow{\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G}} [\![\dot{\mathbf{Z}}^{\#\#}_{\infty}, \mathcal{C}]\!] .$$

We define $\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G}$ *on objects.* Suppose given $X \in \operatorname{Ob}\mathcal{A}_{(F,G)}$. Choose an (F,G)-acyclic resolution $A_X \in \operatorname{Ob}\mathrm{C}^{[0}(\mathcal{A})$ of X. Choose a CE-resolution $J_X \in \operatorname{Ob}\mathrm{CC}^{\llcorner}(\operatorname{Inj}\mathcal{B})$ of FA_X. Let $\mathrm{E}^{\mathrm{Gr}}_{F,G}(X) := \mathrm{E}_{\mathrm{I}}(GJ_X) = \mathrm{E}(\mathrm{t_I}GJ_X) \in \operatorname{Ob}[\![\bar{\mathbf{Z}}^{\#\#}_{\infty}, \mathcal{C}]\!]$ be the *Grothendieck spectral sequence* of X with respect to F and G. Accordingly, let

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G}(X) \;:=\; \dot{\mathrm{E}}_{\mathrm{I}}(GJ_X) \;=\; \dot{\mathrm{E}}(\mathrm{t_I}GJ_X) \;\in\; \operatorname{Ob}[\![\dot{\mathbf{Z}}^{\#\#}_{\infty}, \mathcal{C}]\!]$$

be the *proper Grothendieck spectral sequence* of X with respect to F and G.

We define $\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G}$ *on morphisms.* Suppose given $X \in \operatorname{Ob}\mathcal{A}_{(F,G)}$, and let A_X and J_X be as above. Choose an injective resolution $I_X \in \operatorname{Ob}\mathrm{C}^{[0}(\operatorname{Inj}\mathcal{A})$ of X. Choose a quasiisomorphism $A_X \xrightarrow{p_X} I_X$ that is mapped to 1_X by H^0 and to a quasiisomorphism by F; cf. Remark 29. Choose a CE-resolution $K_X \in \operatorname{Ob}\mathrm{CC}^{\llcorner}(\operatorname{Inj}\mathcal{B})$ of FI_X. Choose a morphism $J_X \xrightarrow{q_X} K_X$ in $\mathrm{CC}^{\llcorner}(\operatorname{Inj}\mathcal{B})$ that is mapped to Fp_X by $\mathrm{H}^0\big((-)^{*,-}\big)$; cf. Remark 6.

Note that $J_X \xrightarrow{q_X} K_X$ can be written as a composite in $\mathrm{CC}^{\llcorner,\,\mathrm{CE}}(\operatorname{Inj}\mathcal{B})$ of a rowwise homotopism, followed by a double homotopism; cf. Proposition 17. Hence, so can $GJ_X \xrightarrow{Gq_X} GK_X$. Thus $\dot{\mathrm{E}}_{\mathrm{I}}(GJ_X) \xrightarrow{\dot{\mathrm{E}}_{\mathrm{I}}(Gq_X)} \dot{\mathrm{E}}_{\mathrm{I}}(GK_X)$ is an isomorphism; cf. Remark 23, Lemma 24.

Suppose given $X \xrightarrow{f} Y$ in $\mathcal{A}_{(F,G)}$. Choose a morphism $I_X \xrightarrow{f'} I_Y$ in $\mathrm{C}^{[0}(\mathcal{A})$ that is mapped to f by H^0. Choose a morphism $K_X \xrightarrow{f''} K_Y$ in $\mathrm{CC}^{\llcorner}(\operatorname{Inj}\mathcal{B})$ that is mapped to Ff' by $\mathrm{H}^0\big((-)^{*,-}\big)$; cf. Remark 6. Let

$$\begin{aligned}
&\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G}(X \xrightarrow{f} Y) \;:=\\
&\Big(\dot{\mathrm{E}}_{\mathrm{I}}(GJ_X) \;\underset{\sim}{\xrightarrow{\dot{\mathrm{E}}_{\mathrm{I}}(Gq_X)}}\; \dot{\mathrm{E}}_{\mathrm{I}}(GK_X) \xrightarrow{\dot{\mathrm{E}}_{\mathrm{I}}(Gf'')} \dot{\mathrm{E}}_{\mathrm{I}}(GK_Y) \;\underset{\sim}{\xleftarrow{\dot{\mathrm{E}}_{\mathrm{I}}(Gq_Y)}}\; \dot{\mathrm{E}}_{\mathrm{I}}(GJ_Y)\Big) \;.
\end{aligned}$$

The procedure can be adumbrated as follows.

$$\begin{tikzcd}
& K_X \arrow[rr, "f''"] & & K_Y \\
J_X \arrow[ur, "q_X"] & & J_Y \arrow[ur, "q_Y"] & \\
& I_X \arrow[rr, "f'"] & & I_Y \\
A_X \arrow[ur, "p_X"] & & A_Y \arrow[ur, "p_Y"] & \\
& X \arrow[rr, "f"] & & Y
\end{tikzcd}$$

We show that this defines a functor $\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G} : \mathcal{A}_{(F,G)} \longrightarrow [\![\dot{\mathbf{Z}}^{\#\#}_{\infty}, \mathcal{C}]\!]$. We need to show independence of the construction from the choices of f' and f'', for then functoriality follows by appropriate choices.

Let $I_X \xrightarrow{\tilde{f}'} I_Y$ and $K_X \xrightarrow{\tilde{f}''} K_Y$ be alternative choices. The residue classes of f' and $\tilde{f}'$ in $\mathrm{K}^{[0}(\mathcal{A})$ coincide, whence so do the residue classes of Ff' and $F\tilde{f}'$ in $\mathrm{K}^{[0}(\mathcal{B})$. Therefore, the residue classes of f'' and $\tilde{f}''$ in $\mathrm{KK}^{\llcorner}(\mathcal{B})$ coincide; cf. Proposition 14. Hence, so do the residue classes of Gf'' and $G\tilde{f}''$ in $\mathrm{KK}^{\llcorner}(\mathcal{C})$. Thus $\dot{\mathrm{E}}_{\mathrm{I}}(Gf'') = \dot{\mathrm{E}}_{\mathrm{I}}(G\tilde{f}'')$; cf. Lemma 24.

We show that alternative choices of A_X, I_X and p_X, and of J_X, K_X and q_X, yield isomorphic proper Grothendieck spectral sequence functors.

Let $\tilde{A}_X \xrightarrow{\tilde{p}_X} \tilde{I}_X$ and $\tilde{J}_X \xrightarrow{\tilde{q}_X} \tilde{K}_X$ be alternative choices, where X runs through $\operatorname{Ob}\mathcal{A}_{(F,G)}$.

Suppose given $X \xrightarrow{f} Y$ in $\mathcal{A}_{(F,G)}$. We resolve the commutative quadrangle

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[d, equal] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

in $\mathcal{A}$ to a commutative quadrangle

$$\begin{tikzcd}
I_X \arrow[r, "f'"] \arrow[d, "u_X"'] & I_Y \arrow[d, "u_Y"] \\
\tilde{I}_X \arrow[r, "\tilde{f}'"] & \tilde{I}_Y
\end{tikzcd}$$

in $\mathrm{K}^{[0}(\mathcal{A})$, in which u_X and u_Y are homotopisms; cf. Remark 6. Then we resolve the commutative quadrangle

$$\begin{tikzcd} FI_X \arrow[r, "Ff'"] \arrow[d, "Fu_X"'] & FI_Y \arrow[d, "Fu_Y"] \\ F\tilde{I}_X \arrow[r, "F\tilde{f}'"] & F\tilde{I}_Y \end{tikzcd}$$

in $\mathrm{K}^{[0}(\mathcal{B})$ to a commutative quadrangle

$$\begin{tikzcd} K_X \arrow[r, "f''"] \arrow[d, "v_X"'] & K_Y \arrow[d, "v_Y"] \\ \tilde{K}_X \arrow[r, "\tilde{f}''"] & \tilde{K}_Y \end{tikzcd}$$

in $\mathrm{KK}^{\llcorner}(\mathcal{B})$; cf. Proposition 14. Therein, v_X and v_Y are each composed of a rowwise homotopism, followed by a double homotopism; cf. Proposition 17. So are Gv_X and Gv_Y. An application of $\dot{\mathrm{E}}_{\mathrm{I}}\big(G(-)\big)$ yields the sought isotransformation, viz.

$$\Big(\dot{\mathrm{E}}_{\mathrm{I}}(GJ_X) \xrightarrow[\sim]{\dot{\mathrm{E}}_{\mathrm{I}}(Gq_X)} \dot{\mathrm{E}}_{\mathrm{I}}(GK_X) \xrightarrow[\sim]{\dot{\mathrm{E}}_{\mathrm{I}}(Gv_X)} \dot{\mathrm{E}}_{\mathrm{I}}(G\tilde{K}_X) \xleftarrow[\sim]{\dot{\mathrm{E}}_{\mathrm{I}}(G\tilde{q}_X)} \dot{\mathrm{E}}_{\mathrm{I}}(G\tilde{J}_X)\Big)$$

at $X \in \mathrm{Ob}\,\mathcal{A}_{(F,G)}$; cf. Remark 23, Lemma 24.

Finally, we recall the starting point of the whole enterprise.

Remark 30 ([5, XVII.§7], [7, Th. 2.4.1]) *Suppose given $X \in \mathrm{Ob}\,\mathcal{A}_{(F,G)}$ and $k,\ \ell \in \mathbf{Z}_{\geq 0}$. We have*

$$\begin{array}{lcl} \dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G}(-k+1/-k-1/\!\!/-k/-k-2)^{+k+\ell}(X) & \simeq & (\mathrm{R}^k G)(\mathrm{R}^\ell F)(X) \\ \dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G}(\infty/-\infty/\!\!/\infty/-\infty)^{+k+\ell}(X) & \simeq & \big(\mathrm{R}^{k+\ell}(G\circ F)\big)(X)\ , \end{array}$$

naturally in X.

Proof. Keep the notation of the definition of $\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G}$.
We shall prove the first isomorphism. By Lemma 22, we have

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G}(-k+1/-k-1/\!\!/-k/-k-2)^{+k+\ell}(X) \simeq \mathrm{H}^k(\mathrm{H}^\ell(GJ_X^{-,*}))\ .$$

Since J_X is rowwise split, we have $\mathrm{H}^\ell(GJ_X^{-,*}) \simeq G(\mathrm{H}^\ell J_X^{-,*})$. Note that $\mathrm{H}^\ell J_X^{-,*}$ is an injective resolution of $\mathrm{H}^\ell FA_X$; cf. Remark 8.(1). By Remark 29, $\mathrm{H}^\ell FA_X \xrightarrow[\sim]{\mathrm{H}^\ell Fp_X} \mathrm{H}^\ell FI_X \simeq (\mathrm{R}^\ell F)(X)$. So

$$\mathrm{H}^k(\mathrm{H}^\ell(GJ_X^{-,*})) \simeq \mathrm{H}^k(G(\mathrm{H}^\ell J_X^{-,*})) \simeq (\mathrm{R}^k G)(\mathrm{H}^\ell FA_X) \simeq (\mathrm{R}^k G)(\mathrm{R}^\ell F)(X)\ .$$

We shall prove naturality of the first isomorphism. Suppose given $X \xrightarrow{f} Y$ in $\mathcal{A}_{(F,G)}$. Consider the following commutative diagram. Abbreviate $E := \dot{\mathrm{E}}(-k+1/-k-1/\!\!/-k/-k-2)^{+k+\ell}$.

$$\begin{tikzcd}
E(\mathrm{t}_{\mathrm{I}}GJ_X) \arrow[r, "E(\mathrm{t}_{\mathrm{I}}Gq_X)", "\sim"'] \arrow[d, "\wr"] & E(\mathrm{t}_{\mathrm{I}}GK_X) \arrow[r, "E(\mathrm{t}_{\mathrm{I}}Gf'')"] \arrow[d, "\wr"] & E(\mathrm{t}_{\mathrm{I}}GK_Y) \arrow[d, "\wr"] & E(\mathrm{t}_{\mathrm{I}}GJ_Y) \arrow[l, "E(\mathrm{t}_{\mathrm{I}}Gq_Y)"', "\sim"] \arrow[d, "\wr"] \\
\mathrm{H}^k\mathrm{H}^\ell GJ_X^{-,*} \arrow[r, "\mathrm{H}^k\mathrm{H}^\ell Gq_X^{-,*}", "\sim"'] \arrow[d, "\wr"] & \mathrm{H}^k\mathrm{H}^\ell GK_X^{-,*} \arrow[r, "\mathrm{H}^k\mathrm{H}^\ell Gf''^{-,*}"] \arrow[d, "\wr"] & \mathrm{H}^k\mathrm{H}^\ell GK_Y^{-,*} \arrow[d, "\wr"] & \mathrm{H}^k\mathrm{H}^\ell GJ_Y^{-,*} \arrow[l, "\mathrm{H}^k\mathrm{H}^\ell Gq_Y^{-,*}"', "\sim"] \arrow[d, "\wr"] \\
\mathrm{H}^k G\mathrm{H}^\ell J_X^{-,*} \arrow[r, "\mathrm{H}^k G\mathrm{H}^\ell q_X^{-,*}", "\sim"'] \arrow[d, "\wr"] & \mathrm{H}^k G\mathrm{H}^\ell K_X^{-,*} \arrow[r, "\mathrm{H}^k G\mathrm{H}^\ell f''^{-,*}"] \arrow[d, "\wr"] & \mathrm{H}^k G\mathrm{H}^\ell K_Y^{-,*} \arrow[d, "\wr"] & \mathrm{H}^k G\mathrm{H}^\ell J_Y^{-,*} \arrow[l, "\mathrm{H}^k G\mathrm{H}^\ell q_Y^{-,*}"', "\sim"] \arrow[d, "\wr"] \\
(\mathrm{R}^k G)\mathrm{H}^\ell FA_X \arrow[r, "(\mathrm{R}^k G)\mathrm{H}^\ell Fp_X", "\sim"'] & (\mathrm{R}^k G)\mathrm{H}^\ell FI_X \arrow[r, "(\mathrm{R}^k G)\mathrm{H}^\ell Ff'"] \arrow[d, "\wr"] & (\mathrm{R}^k G)\mathrm{H}^\ell FI_Y \arrow[d, "\wr"] & (\mathrm{R}^k G)\mathrm{H}^\ell FA_Y \arrow[l, "(\mathrm{R}^k G)\mathrm{H}^\ell Fp_Y"', "\sim"] \\
& (\mathrm{R}^k G)(\mathrm{R}^\ell F)(X) \arrow[r, "(\mathrm{R}^k G)(\mathrm{R}^\ell F)(f)"] & (\mathrm{R}^k G)(\mathrm{R}^\ell F)(Y) &
\end{tikzcd}$$

We shall prove the second isomorphism. By Lemma 27, the quasiisomorphism $FA_X \longrightarrow \mathrm{t}J_X$ maps to a quasiisomorphism $GFA_X \longrightarrow \mathrm{t}GJ_X \simeq G\mathrm{t}J_X$. By Lemma 27, the quasiisomorphism $A_X \xrightarrow{p_X} I_X$ maps to a quasiisomorphism $GFA_X \xrightarrow{GFp_X} GFI_X$. So

$$\begin{array}{rcl}
\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G}(\infty/-\infty/\!\!/\infty/-\infty)^{+k+\ell}(X) & \simeq & \mathrm{H}^{k+\ell}(\mathrm{t}GJ_X) \quad \simeq \quad \mathrm{H}^{k+\ell}(G\mathrm{t}J_X) \\
& \simeq & \mathrm{H}^{k+\ell}(GFA_X) \quad \simeq \quad \mathrm{H}^{k+\ell}(GFI_X) \\
& \simeq & \big(\mathrm{R}^{k+\ell}(G\circ F)\big)(X)\ .
\end{array}$$

We shall prove naturality of the second isomorphism. Consider the following diagram. Abbreviate $\tilde{E} := \dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G}(\infty/-\infty/\!\!/\infty/-\infty)^{+k+\ell}$.

$$\begin{tikzcd}
\tilde{E}(\mathrm{t}_{\mathrm{I}}GJ_X) \arrow[r, "\tilde{E}(\mathrm{t}_{\mathrm{I}}Gq_X)", "\sim"'] \arrow[d, "\wr"] & \tilde{E}(\mathrm{t}_{\mathrm{I}}GK_X) \arrow[r, "\tilde{E}(\mathrm{t}_{\mathrm{I}}Gf'')"] \arrow[d, "\wr"] & \tilde{E}(\mathrm{t}_{\mathrm{I}}GK_Y) \arrow[d, "\wr"] & \tilde{E}(\mathrm{t}_{\mathrm{I}}GJ_Y) \arrow[l, "\tilde{E}(\mathrm{t}_{\mathrm{I}}Gq_Y)"', "\sim"] \arrow[d, "\wr"] \\
\mathrm{H}^{k+\ell}\mathrm{t}GJ_X \arrow[r, "\mathrm{H}^{k+\ell}\mathrm{t}Gq_X", "\sim"'] \arrow[d, "\wr"] & \mathrm{H}^{k+\ell}\mathrm{t}GK_X \arrow[r, "\mathrm{H}^{k+\ell}\mathrm{t}Gf''"] \arrow[d, "\wr"] & \mathrm{H}^{k+\ell}\mathrm{t}GK_Y \arrow[d, "\wr"] & \mathrm{H}^{k+\ell}\mathrm{t}GJ_Y \arrow[l, "\mathrm{H}^{k+\ell}\mathrm{t}Gq_Y"', "\sim"] \arrow[d, "\wr"] \\
\mathrm{H}^{k+\ell}G\mathrm{t}J_X \arrow[r, "\mathrm{H}^{k+\ell}G\mathrm{t}q_X", "\sim"'] & \mathrm{H}^{k+\ell}G\mathrm{t}K_X \arrow[r, "\mathrm{H}^{k+\ell}G\mathrm{t}f''"] & \mathrm{H}^{k+\ell}G\mathrm{t}K_Y & \mathrm{H}^{k+\ell}G\mathrm{t}J_Y \arrow[l, "\mathrm{H}^{k+\ell}G\mathrm{t}q_Y"', "\sim"] \\
\mathrm{H}^{k+\ell}GFA_X \arrow[r, "\mathrm{H}^{k+\ell}GFp_X", "\sim"'] \arrow[u, "\wr"] & \mathrm{H}^{k+\ell}GFI_X \arrow[r, "\mathrm{H}^{k+\ell}GFf'"] \arrow[u] \arrow[d, "\wr"] & \mathrm{H}^{k+\ell}GFI_Y \arrow[u] \arrow[d, "\wr"] & \mathrm{H}^{k+\ell}GFA_Y \arrow[l, "\mathrm{H}^{k+\ell}GFp_Y"', "\sim"] \arrow[u, "\wr"] \\
& \big(\mathrm{R}^{k+\ell}(G\circ F)\big)(X) \arrow[r, "(\mathrm{R}^{k+\ell}(G\circ F))(f)"] & \big(\mathrm{R}^{k+\ell}(G\circ F)\big)(Y) &
\end{tikzcd}$$

□

4.3 Haas transformations

The following transformations have been constructed in the classical framework by Haas [8]. We do not claim any originality.

4.3.1 Situation

Consider the following diagram of abelian categories, left exact functors and transformations,

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "U"'] & \mathcal{B} \arrow[r, "G"] \arrow[d, "V"] \arrow[dl, phantom, "\nearrow\!\!\!\nearrow \mu"] & \mathcal{C} \arrow[d, "W"] \arrow[dl, phantom, "\nearrow\!\!\!\nearrow \nu"] \\ \mathcal{A}' \arrow[r, "F'"] & \mathcal{B}' \arrow[r, "G'"] & \mathcal{C}' \end{tikzcd} ,$$

i.e. $F' \circ U \xrightarrow{\mu} V \circ F$ and $G' \circ V \xrightarrow{\nu} W \circ G$. Suppose that the conditions $(1, 2, 3)$ hold.

(1) The categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}'$ and $\mathcal{B}'$ have enough injectives.

(2) The functors U and V carry injectives to injectives.

(3) The functor F carries injective to G-acyclic objects. The functor F' carries injective to G'-acyclic objects.

We have $\mathcal{A}_{(F,G)} = \mathcal{A}$ since an injective resolution is an (F, G)-acyclic resolution. Likewise, we have $\mathcal{A}'_{(F',G')} = \mathcal{A}'$.
Note in particular the case $U = 1_{\mathcal{A}}$, $V = 1_{\mathcal{B}}$ and $W = 1_{\mathcal{C}}$.
We set out to define the *Haas transformations*

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{F',G'}\big(U(-)\big) \xrightarrow{\mathrm{h}^{\mathrm{I}}_{\mu}} \dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G'\circ V}\big(-\big) \xrightarrow{\mathrm{h}^{\mathrm{II}}_{\nu}} \dot{\mathrm{E}}^{\mathrm{Gr}}_{F,W\circ G}\big(-\big) ,$$

where $\mathrm{h}^{\mathrm{I}}_{\mu}$ depends on F, F', G', U, V and μ, and where $\mathrm{h}^{\mathrm{II}}_{\nu}$ depends on F, G, G', V, W and ν.

4.3.2 Construction of the first Haas transformation

Given $T \in \operatorname{Ob}\mathcal{A}$, we let $\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G}(T)$ be defined via an injective resolution I_T of T and via a CE-resolution J_T of FI_T; cf. §4.2.
Given $T' \in \operatorname{Ob}\mathcal{A}'$, we let $\dot{\mathrm{E}}^{\mathrm{Gr}}_{F',G'}(T')$ be defined via an injective resolution $I'_{T'}$ of T' and via a CE-resolution $J'_{T'}$ of $F'I'_{T'}$; cf. §4.2.
We define $\mathrm{h}^{\mathrm{I}}_{\mu}$. Let $X \in \operatorname{Ob}\mathcal{A}$. By Remark 5, there is a unique morphism $I'_{UX} \xrightarrow{h'X} UI_X$ in $\mathrm{K}^{[0}(\mathcal{A}')$ that maps to 1_{UX} under H^0. Let $J'_{UX} \xrightarrow{h''X} VJ_X$

be the unique morphism in $\mathrm{KK}^{\llcorner}(\mathcal{B}')$ that maps to the composite morphism $\Big(F'I'_{UX} \xrightarrow{F'h'X} F'UI_X \xrightarrow{\mu} VFI_X\Big)$ in $\mathrm{K}^{[0}(\mathcal{B}')$ under $\mathrm{H}^0\big((-)^{*,-}\big)$; cf. Lemma 13. Let the *first Haas transformation* be defined by

$$\Big(\dot{\mathrm{E}}^{\mathrm{Gr}}_{F',G'}(UX) \xrightarrow{\mathrm{h}^{\mathrm{I}}_{\mu}X} \dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G'\circ V}(X)\Big) := \Big(\mathrm{E}_{\mathrm{I}}(G'J'_{UX}) \xrightarrow{\mathrm{E}_{\mathrm{I}}(G'h''X)} \mathrm{E}_{\mathrm{I}}(G'VJ_X)\Big)\,.$$

We show that $\mathrm{h}^{\mathrm{I}}_{\mu}$ *is a transformation.* Let $X \xrightarrow{f} Y$ be a morphism in $\mathcal{A}$. Let $I_X \xrightarrow{f'} I_Y$ resolve $X \xrightarrow{f} Y$. Let $J_X \xrightarrow{f''} J_Y$ resolve $FI_X \xrightarrow{f'} FI_Y$. Let $I'_{UX} \xrightarrow{\tilde{f}'} I'_{UY}$ resolve $UX \xrightarrow{Uf} UY$. Let $J'_{UX} \xrightarrow{\tilde{f}''} J_{UY}$ resolve $F'I_{UX} \xrightarrow{F'\tilde{f}'} F'I_{UY}$. The quadrangle

$$\begin{tikzcd} UX \arrow[r, equal] \arrow[d, "Uf"'] & UX \arrow[d, "Uf"] \\ UY \arrow[r, equal] & UY \end{tikzcd}$$

commutes in $\mathcal{A}'$. Hence, by Remark 5, applied to I'_{UX} and UI_Y, the resolved quadrangle

$$\begin{tikzcd} I'_{UX} \arrow[r, "h'X"] \arrow[d, "\tilde{f}'"'] & UI_X \arrow[d, "Uf'"] \\ I'_{UY} \arrow[r, "h'Y"'] & UI_Y \end{tikzcd}$$

commutes in $\mathrm{K}^{[0}(\mathcal{A}')$. Hence both quadrangles in

$$\begin{tikzcd} F'I'_{UX} \arrow[r, "F'h'X"] \arrow[d, "F'\tilde{f}'"'] & F'UI_X \arrow[r, "\mu"] \arrow[d, "F'Uf'"] & VFI_X \arrow[d, "VFf'"] \\ F'I'_{UY} \arrow[r, "F'h'Y"'] & F'UI_Y \arrow[r, "\mu"'] & VFI_Y \end{tikzcd}$$

commute in $\mathrm{K}^{[0}(\mathcal{B}')$. By Lemma 13, applied to J'_{UX} and VJ_Y, the outer quadrangle in the latter diagram can be resolved to the commutative quadrangle

$$\begin{tikzcd} J'_{UX} \arrow[r, "h''X"] \arrow[d, "\tilde{f}''"'] & VJ_X \arrow[d, "Vf''"] \\ J'_{UY} \arrow[r, "h''Y"'] & VJ_Y \end{tikzcd}$$

in $\mathrm{KK}^{\llcorner}(\mathcal{B}')$. Applying $\mathrm{E}_{\mathrm{I}}\big(G'(-)\big)$ and employing the definitions of $\dot{\mathrm{E}}^{\mathrm{Gr}}_{F',G'}$, $\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G'\circ V}$ and $\mathrm{h}^{\mathrm{I}}_{\mu}$, we obtain the sought commutative diagram

$$\begin{tikzcd}
\dot{\mathrm{E}}^{\mathrm{Gr}}_{F',G'}(UX) \arrow[r, "\mathrm{h}^{\mathrm{I}}_{\mu} X"] \arrow[d, "\dot{\mathrm{E}}^{\mathrm{Gr}}_{F',G'}(Uf)"'] & \dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G'\circ V}(X) \arrow[d, "\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G'\circ V}(f)"] \\
\dot{\mathrm{E}}^{\mathrm{Gr}}_{F',G'}(UY) \arrow[r, "\mathrm{h}^{\mathrm{I}}_{\mu} Y"'] & \dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G'\circ V}(Y)
\end{tikzcd}$$

in $[\![\dot{\mathbf{Z}}^{\#\#}_{\infty}, \mathcal{C}']\!]$.

4.3.3 Construction of the second Haas transformation

We maintain the notation of §4.3.2.
Given $X \in \operatorname{Ob}\mathcal{A}$, we let the *second Haas transformation* be defined by

$$\Big(\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G\circ V}\big(X\big) \xrightarrow{\mathrm{h}^{\mathrm{II}}_{\nu} X} \dot{\mathrm{E}}^{\mathrm{Gr}}_{F,W\circ G}\big(X\big)\Big) \quad := \quad \Big(\dot{\mathrm{E}}_{\mathrm{I}}(G'VJ_X) \xrightarrow{\dot{\mathrm{E}}_{\mathrm{I}}(\nu)} \dot{\mathrm{E}}_{\mathrm{I}}(WGJ_X)\Big)\,.$$

It is a transformation since ν is.

5 The first comparison

5.1 The first comparison isomorphism

Suppose given abelian categories $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{B}$ with enough injectives and an abelian category $\mathcal{C}$.
Let $\mathcal{A}\times\mathcal{A}' \xrightarrow{F} \mathcal{B}$ be a biadditive functor. Let $\mathcal{B} \xrightarrow{G} \mathcal{C}$ be an additive functor. Suppose given objects $X \in \operatorname{Ob}\mathcal{A}$ and $X' \in \operatorname{Ob}\mathcal{A}'$. Suppose the following properties to hold.

(a) The functor $F(-,X') : \mathcal{A} \longrightarrow \mathcal{B}$ is left exact.

(a′) The functor $F(X,-) : \mathcal{A}' \longrightarrow \mathcal{B}$ is left exact.

(b) The functor G is left exact.

(c) The object X possesses a $\big(F(-,X'),G\big)$-acyclic resolution $A \in \operatorname{Ob}\mathrm{C}^{[0}(\mathcal{A})$.

(c′) The object X' possesses a $\big(F(X,-),G\big)$-acyclic resolution $A' \in \operatorname{Ob}\mathrm{C}^{[0}(\mathcal{A}')$.

Moreover, the resolutions appearing in (c) and (c′) are stipulated to have the following properties.

(d) For all $k \geq 0$, the quasiisomorphism $\operatorname{Conc} X \longrightarrow A$ is mapped to a quasiisomorphism $\operatorname{Conc} F(X, A'^k) \longrightarrow F(A, A'^k)$ under $F(-, A'^k)$.

(d$'$) For all $k \geq 0$, the quasiisomorphism $\operatorname{Conc} X' \longrightarrow A'$ is mapped to a quasiisomorphism $\operatorname{Conc} F(A^k, X') \longrightarrow F(A^k, A')$ under $F(A^k, -)$.

The conditions $(\mathrm{d}, \mathrm{d}')$ are e.g. satisfied if $F(-, A'^k)$ and $F(A^k, -)$ are exact for all $k \geq 0$.

Theorem 31 (first comparison) *The proper Grothendieck spectral sequence for the functors $F(X, -)$ and G, evaluated at X', is isomorphic to the proper Grothendieck spectral sequence for the functors $F(-, X')$ and G, evaluated at X; i.e.*

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{F(X,-),G}(X') \;\simeq\; \dot{\mathrm{E}}^{\mathrm{Gr}}_{F(-,X'),G}(X)$$

in $[\![\dot{\mathbf{Z}}^{\#\#}_{\infty}, \mathcal{C}]\!]$.

Proof. Let J_A, $J_{A'}$, $J_{A,A'} \in \operatorname{Ob} \mathrm{CC}^{\llcorner}(\operatorname{Inj} \mathcal{B})$ be CE-resolutions of the complexes $F(A, X')$, $F(X, A')$, $\mathrm{t}F(A, A') \in \operatorname{Ob} \mathrm{C}^{[0}(\mathcal{B})$, respectively.
The quasiisomorphism $\operatorname{Conc} X \longrightarrow A$ is mapped to the morphism $F(\operatorname{Conc} X, A') \longrightarrow F(A, A')$, yielding $F(X, A') \longrightarrow \mathrm{t}F(A, A')$, which is a quasiisomorphism since $\operatorname{Conc} F(X, A'^k) \longrightarrow F(A, A'^k)$ is a quasiisomorphism for all $k \geq 0$ by (d).
Choose a CE-resolution $J_{A'} \longrightarrow J_{A,A'}$ of $F(X, A') \longrightarrow \mathrm{t}F(A, A')$; cf. Remark 6. Since the morphism $F(X, A') \longrightarrow \mathrm{t}F(A, A')$ is a quasiisomorphism, $J_{A'} \longrightarrow J_{A,A'}$ is a composite in $\mathrm{CC}^{\llcorner,\,\mathrm{CE}}(\operatorname{Inj} \mathcal{B})$ of a rowwise homotopism and a double homotopism; cf. Proposition 17. So is $GJ_{A'} \longrightarrow GJ_{A,A'}$. Hence, by Remark 23 and by Lemma 24, we obtain an isomorphism of the proper spectral sequences of the first filtrations of the total complexes,

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{F(X,-),G}(X') \;=\; \dot{\mathrm{E}}_{\mathrm{I}}(GJ_{A'}) \xrightarrow{\;\sim\;} \dot{\mathrm{E}}_{\mathrm{I}}(GJ_{A,A'})\,.$$

Likewise, we have an isomorphism

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{F(-,X'),G}(X) \;=\; \dot{\mathrm{E}}_{\mathrm{I}}(GJ_{A}) \xrightarrow{\;\sim\;} \dot{\mathrm{E}}_{\mathrm{I}}(GJ_{A,A'})\,.$$

We compose to an isomorphism $\dot{\mathrm{E}}^{\mathrm{Gr}}_{F(X,-),G}(X') \xrightarrow{\;\sim\;} \dot{\mathrm{E}}^{\mathrm{Gr}}_{F(-,X'),G}(X)$ as sought. □

5.2 Naturality of the first comparison isomorphism

We narrow down the assumptions just as we have done for the introduction of the Haas transformations in §4.3.1 in order to be able to express, in this narrower case, a naturality of the first comparison isomorphism from Theorem 31.

Suppose given abelian categories $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{B}$ with enough injectives and an abelian category $\mathcal{C}$.
Let $\mathcal{A} \times \mathcal{A}' \xrightarrow{F} \mathcal{B}$ be a biadditive functor. Let $\mathcal{B} \xrightarrow{G} \mathcal{C}$ be an additive functor. Suppose that the following properties hold.

(a) The functor $F(-,X') : \mathcal{A} \longrightarrow \mathcal{B}$ is left exact for all $X' \in \operatorname{Ob}\mathcal{A}'$.

(a$'$) The functor $F(X,-) : \mathcal{A}' \longrightarrow \mathcal{B}$ is left exact for all $X \in \operatorname{Ob}\mathcal{A}$.

(b) The functor G is left exact.

(c) For all $X' \in \operatorname{Ob}\mathcal{A}'$, the functor $F(-,X')$ carries injective objects to G-acyclic objects.

(c$'$) For all $X \in \operatorname{Ob}\mathcal{A}$, the functor $F(X,-)$ carries injective objects to G-acyclic objects.

(d) The functor $F(I,-)$ is exact for all $I \in \operatorname{Ob}\operatorname{Inj}\mathcal{A}$.

(d$'$) The functor $F(-,I')$ is exact for all $I' \in \operatorname{Ob}\operatorname{Inj}\mathcal{A}'$.

Proposition 32 *Suppose given $X \xrightarrow{x} \tilde{X}$ in $\mathcal{A}$ and $X' \in \operatorname{Ob}\mathcal{A}'$. Note that we have a transformation $F(x,-) : F(X,-) \longrightarrow F(\tilde{X},-)$. The following quadrangle, whose vertical isomorphisms are given by the construction in the proof of Theorem* 31*, commutes.*

$$\begin{tikzcd}
\dot{\mathrm{E}}^{\mathrm{Gr}}_{F(X,-),G}(X') \arrow[r, "\mathrm{h}^{\mathrm{I}}_{F(x,-)X'}"] \arrow[d, "\wr"'] & \dot{\mathrm{E}}^{\mathrm{Gr}}_{F(\tilde{X},-),G}(X') \arrow[d, "\wr"] \\
\dot{\mathrm{E}}^{\mathrm{Gr}}_{F(-,X'),G}(X) \arrow[r, "\dot{\mathrm{E}}^{\mathrm{Gr}}_{F(-,X'),G}(x)"] & \dot{\mathrm{E}}^{\mathrm{Gr}}_{F(-,X'),G}(\tilde{X})
\end{tikzcd}$$

For the definition of the first Haas transformation $\mathrm{h}^{\mathrm{I}}_{F(x,-)}$, see §4.3.2.
An analogous assertion holds with interchanged roles of $\mathcal{A}$ and $\mathcal{A}'$.

Proof of Proposition 32. Let I resp. $\tilde{I}$ be an injective resolution of X resp. $\tilde{X}$ in $\mathcal{A}$. Let $I \xrightarrow{\hat{x}} \tilde{I}$ be a resolution of $X \xrightarrow{x} \tilde{X}$. Let I' be an injective resolution of X' in $\mathcal{A}'$.
Let $J^{(X)}_{I'}$ resp. $J^{(\tilde{X})}_{I'}$ be a CE-resolution of $F(X,I')$ resp. $F(\tilde{X},I')$.
Let $J_{I,I'}$ resp. $J_{\tilde{I},I'}$ be a CE-resolution of $\mathrm{t}F(I,I')$ resp. $\mathrm{t}F(\tilde{I},I')$.
Let J_I resp. $J_{\tilde{I}}$ be a CE-resolution of $F(I,X')$ resp. $F(\tilde{I},X')$.
We have a commutative diagram

$$\begin{tikzcd}
F(X,I') \arrow[r, "F(x,I')"] \arrow[d] & F(\tilde{X},I') \arrow[d] \\
\mathrm{t}F(I,I') \arrow[r, "\mathrm{t}F(\hat{x},I')"] & \mathrm{t}F(\tilde{I},I') \\
F(I,X') \arrow[r, "F(\hat{x},X')"] \arrow[u] & F(\tilde{I},X') \arrow[u]
\end{tikzcd}$$

in $\mathrm{C}^{[0}(\mathcal{B})$, hence in $\mathrm{K}^{[0}(\mathcal{B})$. By Proposition 14, it can be resolved to a commutative diagram

$$\begin{tikzcd}
J_{I'}^{(X)} \arrow[r] \arrow[d] & J_{I'}^{(\tilde{X})} \arrow[d] \\
J_{I,I'} \arrow[r] & J_{\tilde{I},I'} \\
J_I \arrow[r] \arrow[u] & J_{\tilde{I}} \arrow[u]
\end{tikzcd}$$

in $\mathrm{KK}^{\llcorner}(\mathcal{B})$. Application of $\dot{\mathrm{E}}_{\mathrm{I}}\big(G(-)\big)$ yields the result; cf. Lemma 24.

We refrain from investigating naturality of the first comparison isomorphism in G.

6 The second comparison

6.1 The second comparison isomorphism

Suppose given abelian categories $\mathcal{A}$ and $\mathcal{B}'$ with enough injectives, and abelian categories $\mathcal{B}$ and $\mathcal{C}$.

Let $\mathcal{A} \xrightarrow{F} \mathcal{B}'$ be an additive functor. Let $\mathcal{B} \times \mathcal{B}' \xrightarrow{G} \mathcal{C}$ be a biadditive functor. Suppose given objects $X \in \operatorname{Ob}\mathcal{A}$ and $Y \in \operatorname{Ob}\mathcal{B}$. Let $B \in \operatorname{Ob}\mathrm{C}^{[0}(\mathcal{B})$ be a resolution of Y, i.e. suppose a quasiisomorphism $\operatorname{Conc} Y \longrightarrow B$ to exist. Suppose the following properties to hold.

(a) The functor F is left exact.

(b) The functor $G(Y,-)$ is left exact.

(c) The object X possesses an $(F,\ G(Y,-))$-acyclic resolution $A \in \operatorname{Ob}\mathrm{C}^{[0}(\mathcal{A})$.

(d) The functor $G(B^k,-)$ is exact for all $k \geq 0$.

(e) The functor $G(-,I')$ is exact for all $I' \in \operatorname{Ob}\operatorname{Inj}\mathcal{B}'$.

Remark 33 *Suppose given a morphism $D \xrightarrow{f} D'$ in $\mathrm{CC}^{\llcorner}(\mathcal{C})$. If $\mathrm{H}^{\ell}(f^{-,*})$ is a quasiisomorphism for all $\ell \geq 0$, then f induces an isomorphism*

$$\dot{\mathrm{E}}_{\mathrm{I}}(D) \xrightarrow{\dot{\mathrm{E}}_{\mathrm{I}}(f)} \dot{\mathrm{E}}_{\mathrm{I}}(D')$$

of proper spectral sequences.

Proof. By Lemma 21, it suffices to show that $\mathrm{E}_{\mathrm{I}}(\alpha+1/\alpha-1 /\!\!/ \alpha/\alpha-2)^{+k}(f)$ is an isomorphism for all $\alpha \in \mathbf{Z}$ and all $k \in \mathbf{Z}$. By Lemma 22, this amounts to isomorphisms $\mathrm{H}^k\mathrm{H}^{\ell}(f^{-,*})$ for all $k,\ \ell \geq 0$, i.e. to quasiisomorphisms $\mathrm{H}^{\ell}(f^{-,*})$ for all $\ell \geq 0$. ▫

Consider the double complex $G(B, FA) \in \operatorname{Ob} \mathrm{CC}^{\llcorner}(\mathcal{C})$, where the indices of B count rows and the indices of A count columns. To the first filtration of its total complex, we can associate the proper spectral sequence $\dot{\mathrm{E}}_{\mathrm{I}}(G(B, FA)) \in \operatorname{Ob} [\![\dot{\mathbf{Z}}_{\infty}^{\#\#}, \mathcal{C}]\!]$.

Theorem 34 (second comparison) *The proper Grothendieck spectral sequence for the functors F and $G(Y,-)$, evaluated at X, is isomorphic to $\dot{\mathrm{E}}_{\mathrm{I}}(G(B, FA))$; i.e.*

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G(Y,-)}(X) \;\simeq\; \dot{\mathrm{E}}_{\mathrm{I}}(G(B, FA))$$

in $[\![\dot{\mathbf{Z}}_{\infty}^{\#\#}, \mathcal{C}]\!]$.

Proof. Let $J' \in \operatorname{Ob} \mathrm{CC}^{\llcorner}(\operatorname{Inj} \mathcal{B}')$ be a CE-resolution of FA. By definition, $\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G(Y,-)}(X) = \dot{\mathrm{E}}_{\mathrm{I}}(G(Y, J'))$. By Remark 33, it suffices to find $D \in \operatorname{Ob} \mathrm{CC}^{\llcorner}(\mathcal{C})$ and two morphisms of double complexes

$$G(B, FA) \xrightarrow{\;u\;} D \xleftarrow{\;v\;} G(Y, J')$$

such that $\mathrm{H}^{\ell}(u^{-,*})$ and $\mathrm{H}^{\ell}(v^{-,*})$ are quasiisomorphisms for all $\ell \geq 0$.
Given a complex $U \in \operatorname{Ob} \mathrm{C}^{[0}(\mathcal{B})$, recall that we denote by $\mathrm{Conc}_2 U \in \operatorname{Ob} \mathrm{CC}^{\llcorner}(\mathcal{B})$ the double complex whose row number 0 is given by U, and whose other rows are zero.
We have a diagram

$$G(B, \mathrm{Conc}_2\, FA) \longrightarrow G(B, J') \longleftarrow G(\mathrm{Conc}\, Y, J')$$

in $\mathrm{CCC}^{\llcorner\!\llcorner}(\mathcal{C})$. Let $\ell \geq 0$. Application of $\mathrm{H}^{\ell}\big((-)^{-,=,*}\big)$ yields a diagram

$$(*) \qquad \mathrm{H}^{\ell}\big(G(B, \mathrm{Conc}_2\, FA)^{-,=,*}\big) \longrightarrow \mathrm{H}^{\ell}\big(G(B, J')^{-,=,*}\big) \longleftarrow \mathrm{H}^{\ell}\big(G(\mathrm{Conc}\, Y, J')^{-,=,*}\big)$$

in $\mathrm{CC}^{\llcorner}(\mathcal{C})$. We have

$$\begin{array}{rcl} \mathrm{H}^{\ell}\big(G(B, \mathrm{Conc}_2\, FA)^{-,=,*}\big) & \simeq & G\Big(B\,,\, \mathrm{H}^{\ell}\big((\mathrm{Conc}_2\, FA)^{-,*}\big)\Big) \\ & = & G\big(B, \mathrm{Conc}\, \mathrm{H}^{\ell}(FA)\big) \end{array}$$

and

$$\mathrm{H}^{\ell}\big(G(B, J')^{-,=,*}\big) \;\simeq\; G\big(B, \mathrm{H}^{\ell}(J'^{-,*})\big)\;,$$

since the functor $G(B^k, -)$ is exact for all $k \geq 0$ by (d), or, since the CE-resolution J is rowwise split. Since the CE-resolution J' is rowwise split, we moreover have

$$\mathrm{H}^{\ell}\big(G(\mathrm{Conc}\, Y, J')^{-,=,*}\big) \;\simeq\; G\big(\,\mathrm{Conc}\, Y, \mathrm{H}^{\ell}(J'^{-,*})\big)\;.$$

So the diagram $(*)$ is isomorphic to the diagram

$$(**) \quad G\big(B, \mathrm{Conc}\, \mathrm{H}^{\ell}(FA)\big) \longrightarrow G\big(B, \mathrm{H}^{\ell}(J'^{-,*})\big) \longleftarrow G\big(\,\mathrm{Conc}\, Y,\, \mathrm{H}^{\ell}(J'^{-,*})\big)\;,$$

whose left hand side morphism is induced by the quasiisomorphism $\operatorname{Conc}\mathrm{H}^\ell(FA) \longrightarrow \mathrm{H}^\ell(J'^{-,*})$, and whose right hand side morphism is induced by the quasiisomorphism $\operatorname{Conc} Y \longrightarrow B$.
By exactness of $G(B^k,-)$ for $k \geq 0$, the left hand side morphism of $(**)$ is a rowwise quasiisomorphism. Since $\mathrm{H}^\ell(J'^{k,*})$ is injective, the functor $G(-,\mathrm{H}^\ell(J'^{k,*}))$ is exact by (e), and therefore the right hand side morphism of $(**)$ is a columnwise quasiisomorphism. Thus an application of t to $(**)$ yields two quasiisomorphisms; cf. §1.1.6. Hence, also an application of t to $(*)$ yields two quasiisomorphisms in the diagram

$$\mathrm{tH}^\ell\big(G(B,\operatorname{Conc}_2 FA)^{-,=,*}\big) \longrightarrow \mathrm{tH}^\ell\big(G(B,J')^{-,=,*}\big) \longleftarrow \mathrm{tH}^\ell\big(G(\operatorname{Conc} Y,J')^{-,=,*}\big).$$

Note that $\mathrm{t} \circ \mathrm{H}^\ell\big((-)^{-,=,*}\big) = \mathrm{H}^\ell\big((-)^{-,*}\big) \circ \mathrm{t}_{1,2}$, where $\mathrm{t}_{1,2}$ denotes taking the total complex in the first and the second index of a triple complex; cf. §1.2.2. Hence we have a diagram

$$\mathrm{H}^\ell\big((\mathrm{t}_{1,2}G(B,\operatorname{Conc}_2 FA))^{-,*}\big) \longrightarrow \mathrm{H}^\ell\big((\mathrm{t}_{1,2}G(B,J'))^{-,*}\big) \longleftarrow \mathrm{H}^\ell\big((\mathrm{t}_{1,2}G(\operatorname{Conc} Y,J'))^{-,*}\big)$$

consisting of two quasiisomorphisms. This diagram in turn, is isomorphic to

$$\mathrm{H}^\ell\Big(G(B,FA)^{-,*}\Big) \longrightarrow \mathrm{H}^\ell\Big((\mathrm{t}_{1,2}G(B,J'))^{-,*}\Big) \longleftarrow \mathrm{H}^\ell\Big(\big(G(Y,J')\big)^{-,*}\Big),$$

where the left hand side morphism is obtained by precomposition with the isomorphism $G(B,FA^k) \xrightarrow{\sim} \mathrm{t}\operatorname{Conc}_1 G(B,FA^k) = (\mathrm{t}_{1,2}G(B,\operatorname{Conc}_2 FA))^{-,k}$, where $k \geq 0$; cf. §1.1.6.
Hence we may take

$$\begin{aligned}\big(G(B,FA) \xrightarrow{u} D &\xleftarrow{v} G(B,J')\big)\\ &:= \big(G(B,FA) \longrightarrow \mathrm{t}_{1,2}G(B,J') \longleftarrow G(Y,J')\big).\end{aligned}$$

□

6.2 Naturality of the second comparison isomorphism

Again, we narrow down the assumptions just as we have done for the introduction of the Haas transformations in §4.3.1 to express a naturality of the second comparison isomorphism from Theorem 34.

Suppose given abelian categories $\mathcal{A}$ and $\mathcal{B}'$ with enough injectives, and abelian categories $\mathcal{B}$ and $\mathcal{C}$. Suppose given additive functors $\mathcal{A} \underset{\tilde{F}}{\overset{F}{\rightrightarrows}} \mathcal{B}'$ and a transformation $F \xrightarrow{\phi} \tilde{F}$. Let $\mathcal{B} \times \mathcal{B}' \xrightarrow{G} \mathcal{C}$ be a biadditive functor.
Suppose given a morphism $X \xrightarrow{x} \tilde{X}$ in $\mathcal{A}$ and an object $Y \in \operatorname{Ob}\mathcal{B}$. Let $B \in \operatorname{Ob}\mathrm{C}^{[0}(\mathcal{B})$ be a resolution of Y, i.e. suppose a quasiisomorphism $\operatorname{Conc} Y \longrightarrow B$ to exist. Suppose the following properties to hold.

(a) The functors F and $\tilde{F}$ are left exact and carry injective to $G(Y,-)$-acyclic objects.

(b) The functor $G(Y,-)$ is left exact.

(c) The functor $G(B^k,-)$ is exact for all $k \geq 0$.

(d) The functor $G(-,I')$ is exact for all $I' \in \mathrm{Ob}\,\mathrm{Inj}\,\mathcal{B}'$.

Let $A \xrightarrow{a} \tilde{A}$ in $\mathrm{C}^{[0}(\mathrm{Inj}\,\mathcal{A})$ be an injective resolution of $X \xrightarrow{x} \tilde{X}$ in $\mathcal{A}$. Note that we have a commutative quadrangle

$$\begin{tikzcd} G(B,FA) \arrow[r, "G(B,\phi A)"] \arrow[d, "G(B,Fa)"'] & G(B,\tilde{F}A) \arrow[d, "G(B,\tilde{F}a)"] \\ G(B,F\tilde{A}) \arrow[r, "G(B,\phi\tilde{A})"] & G(B,\tilde{F}\tilde{A}) \end{tikzcd}$$

in $\mathrm{CC}^{\llcorner}(\mathcal{C})$.
Note that once chosen injective resolutions A of X and $\tilde{A}$ of $\tilde{X}$, the image of $G(B,Fa)$ in $\mathrm{KK}^{\llcorner}(\mathcal{C})$ does not depend on the choice of the resolution $A \xrightarrow{a} \tilde{A}$ of $X \xrightarrow{x} \tilde{X}$, for $\mathrm{C}^{[0}(\mathcal{A}) \xrightarrow{G(B,F(-))} \mathrm{CC}^{\llcorner}(\mathcal{C})$ maps an elementary split acyclic complex to an elementary horizontally split acyclic complex.

Lemma 35 *The quadrangle*

$$\begin{tikzcd} \dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G(Y,-)}(X) \arrow[r, "\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G(Y,-)}(x)"] \arrow[d, "\wr"'] & \dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G(Y,-)}(\tilde{X}) \arrow[d, "\wr"] \\ \dot{\mathrm{E}}_{\mathrm{I}}(G(B,FA)) \arrow[r, "\dot{\mathrm{E}}_{\mathrm{I}}(G(B,Fa))"] & \dot{\mathrm{E}}_{\mathrm{I}}(G(B,F\tilde{A})) \end{tikzcd}$$

commutes, where the vertical isomorphisms are those constructed in the proof of Theorem 34.

Proof. Let $J' \xrightarrow{\hat{a}} \tilde{J}'$ be a CE-resolution of $FA \xrightarrow{Fa} F\tilde{A}$. Consider the following commutative diagram in $\mathrm{CC}^{\llcorner}(\mathcal{C})$.

$$\begin{tikzcd} G(Y,J') \arrow[r, "G(Y,\hat{a})"] \arrow[d] & G(Y,\tilde{J}') \arrow[d] \\ \mathrm{t}_{1,2}G(B,J') \arrow[r, "\mathrm{t}_{1,2}G(B,\hat{a})"] & \mathrm{t}_{1,2}G(B,\tilde{J}') \\ G(B,FA) \arrow[r, "G(B,Fa)"] \arrow[u] & G(B,F\tilde{A}) \arrow[u] \end{tikzcd}$$

An application of $\dot{\mathrm{E}}_{\mathrm{I}}$ yields the result. ▫

Lemma 36 *The quadrangle*

$$\begin{tikzcd}
\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G(Y,-)}(X) \arrow[r, "\mathrm{h}^{\mathrm{I}}_{\phi}X"] \arrow[d, "\wr"'] & \dot{\mathrm{E}}^{\mathrm{Gr}}_{\tilde{F},G(Y,-)}(X) \arrow[d, "\wr"] \\
\dot{\mathrm{E}}_{\mathrm{I}}(G(B,FA)) \arrow[r, "\dot{\mathrm{E}}_{\mathrm{I}}(G(B,\phi A))"] & \dot{\mathrm{E}}_{\mathrm{I}}(G(B,\tilde{F}A))
\end{tikzcd}$$

commutes, where the vertical morphisms are those constructed in the proof of Theorem 34.

For the definition of the first Haas transformation $\mathrm{h}^{\mathrm{I}}_{F(x,-)}$, see §4.3.2.

Proof. Let $J' \xrightarrow{\hat{\phi}} \breve{J}'$ be a CE-resolution of $FA \xrightarrow{F\phi} \tilde{F}A$. Consider the following commutative diagram in $\mathrm{CC}^{\llcorner}(\mathcal{C})$.

$$\begin{tikzcd}
G(Y,J') \arrow[r, "G(Y,\hat{\phi})"] \arrow[d] & G(Y,\breve{J}') \arrow[d] \\
\mathrm{t}_{1,2}G(B,J') \arrow[r, "\mathrm{t}_{1,2}G(B,\hat{\phi})"] & \mathrm{t}_{1,2}G(B\breve{J}') \\
G(B,FA) \arrow[r, "G(B,\phi A)"] \arrow[u] & G(B,\tilde{F}A) \arrow[u]
\end{tikzcd}$$

An application of $\dot{\mathrm{E}}_{\mathrm{I}}$ yields the result. □

We refrain from investigating naturality of the second comparison isomorphism in Y.

7 Acyclic CE-resolutions

We record Beyl's Theorem [4, Th. 3.4] (here Theorem 40) in order to document that it fits in our context. The argumentation is entirely due to Beyl [4, Sec. 3], so we do not claim any originality.

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be abelian categories. Suppose $\mathcal{A}$ and $\mathcal{B}$ to have enough injectives. Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ be left exact functors.

7.1 Definition

Let $T \in \operatorname{Ob} \mathrm{C}^{[0}(\mathcal{B})$. In this §7, a CE-resolution of T will synonymously (and not quite correctly) be called an *injective CE-resolution,* to emphasise the fact that its object entries are injective.
We regard $\mathrm{C}^{[0}(\mathcal{B})$ as an exact category as in Remarks 9 and 11.

DEFINITION 37 A double complex $B \in \mathrm{CC}^{\llcorner}(\mathcal{B})$ is called a *G-acyclic CE-resolution* of T if the following conditions are satisfied.

(1) We have $\mathrm{H}^0(B^{*,-}) \simeq T$ and $\mathrm{H}^k(B^{*,-}) \simeq 0$ for all $k \geq 1$.

(2) The morphism of complexes $B^{k,*} \longrightarrow B^{k+1,*}$, consisting of vertical differentials of B, is a pure morphism for all $k \geq 0$.

(3) The object $\mathrm{B}^\ell(B^{k,*})$ is G-acyclic for all $k,\, \ell \geq 0$.

(4) The object $\mathrm{Z}^\ell(B^{k,*})$ is G-acyclic for all $k,\, \ell \geq 0$.

A *G-acyclic CE-resolution* is a G-acyclic CE-resolution of some $T \in \mathrm{Ob}\,\mathrm{C}^{[0}(\mathcal{B})$.

From $(3,4)$ and the short exact sequence $\mathrm{Z}^\ell(B^{k,*}) \longrightarrow B^{k,\ell} \longrightarrow \mathrm{B}^{\ell+1}(B^{k,*})$, we conclude that $B^{k,\ell}$ is G-acyclic for all $k,\, \ell \geq 0$.
From $(3,4)$ and the short exact sequence $\mathrm{B}^\ell(B^{k,*}) \longrightarrow \mathrm{Z}^\ell(B^{k,*}) \longrightarrow \mathrm{H}^\ell(B^{k,*})$, we conclude that $\mathrm{H}^\ell(B^{k,*})$ is G-acyclic for all $k,\, \ell \geq 0$.

EXAMPLE 38 An injective CE-resolution of T is in particular a G-acyclic CE-resolution of T.

Note that given $Y \in \mathrm{Ob}\,\mathrm{C}(\mathcal{B})$ and $\ell \in \mathbf{Z}$, we have $\mathrm{Z}^\ell GY \simeq G\mathrm{Z}^\ell Y$, whence the universal property of the cokernel $\mathrm{H}^\ell GY$ of $GY^{\ell-1} \longrightarrow \mathrm{Z}^\ell GY$ induces a morphism $\mathrm{H}^\ell GY \longrightarrow G\mathrm{H}^\ell Y$. This furnishes a transformation $\mathrm{H}^\ell(GX^{k,*}) \xrightarrow{\theta X} G\mathrm{H}^\ell(X^{k,*})$, natural in $X \in \mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathcal{B})$.

REMARK 39 *If B is a G-acyclic CE-resolution, then the morphism $\mathrm{H}^\ell(GB^{-,*}) \xrightarrow{\theta B} G\mathrm{H}^\ell(B^{-,*})$ is an isomorphism for all $\ell \geq 0$.*

Proof. The sequences

$$\begin{array}{ccccc} G\mathrm{B}^\ell(B^{k,*}) & \longrightarrow & G\mathrm{Z}^\ell(B^{k,*}) & \longrightarrow & G\mathrm{H}^\ell(B^{k,*}) \\ G\mathrm{Z}^{\ell-1}(B^{k,*}) & \longrightarrow & GB^{k,\ell-1} & \longrightarrow & G\mathrm{B}^\ell(B^{k,*}) \end{array}$$

are short exact for $k,\, \ell \geq 0$ by G-acyclicity of $\mathrm{B}^\ell(B^{k,*})$ resp. of $\mathrm{Z}^{\ell-1}(B^{k,*})$. In particular, the cokernel of $GB^{k,\ell-1} \longrightarrow G\mathrm{Z}^\ell(B^{k,*})$ is given by $G\mathrm{H}^\ell(B^{k,*})$. ▫

7.2 A THEOREM OF BEYL

Let $X \in \mathrm{Ob}\,\mathcal{A}_{(F,G)}$. Let $A \in \mathrm{C}^{[0}(\mathcal{A})$ be a (F,G)-acyclic resolution of X. Let $B \in \mathrm{CC}^{\llcorner}(\mathcal{B})$ be a G-acyclic CE-resolution of FA.

THEOREM 40 (BEYL, [4, TH. 3.4]) *We have an isomorphism of proper spectral sequences*

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{F,G}(X) \;\simeq\; \dot{\mathrm{E}}_{\mathrm{I}}(GB)$$

in $[\![\dot{\mathbf{Z}}^{\#\#}_\infty, \mathcal{C}]\!]$.

Proof. Since the proper Grothendieck spectral sequence is, up to isomorphism, independent of the choice of an injective CE-resolution, as pointed out in §4.2, our assertion is equivalent to the existence of an injective CE-resolution J of FA such that $\dot{\mathrm{E}}_{\mathrm{I}}(GJ) \simeq \dot{\mathrm{E}}_{\mathrm{I}}(GB)$. So by Remark 33, it suffices to show that there exists an injective CE-resolution J of FA and a morphism $B \longrightarrow J$ that induces a quasiisomorphism $\mathrm{H}^\ell(GB^{-,*}) \longrightarrow \mathrm{H}^\ell(GJ^{-,*})$ for all $\ell \geq 0$. By Remark 39 and Example 38, it suffices to show that $G\mathrm{H}^\ell(B^{-,*}) \longrightarrow G\mathrm{H}^\ell(J^{-,*})$ is a quasiisomorphism for all $\ell \geq 0$.
By the conditions $(1, 2)$ on B and by G-acyclicity of $\mathrm{H}^\ell(B^{k,*})$ for k, $\ell \geq 0$, the complex $\mathrm{H}^\ell(B^{-,*})$ is a G-acyclic resolution of $\mathrm{H}^\ell(FA)$; cf. Remark 10.
By Remark 4, there exists $J \in \mathrm{Ob}\,\mathrm{CC}^{\llcorner}(\mathrm{Inj}\,\mathcal{B})$ with vertical pure morphisms and split rows, and a morphism $B \longrightarrow J$ consisting rowwise of pure monomorphisms such that $\mathrm{H}^k(B^{*,-}) \longrightarrow \mathrm{H}^k(J^{*,-})$ is an isomorphism of complexes for all $k \geq 0$. In particular, the composite $(\mathrm{Conc}_2\, FA \longrightarrow B \longrightarrow J)$ turns J into an injective CE-resolution of FA.
Let $\ell \geq 0$. Since B is a G-acyclic and J an injective CE-resolution of FA, both $\mathrm{Conc}\,\mathrm{H}^\ell(FA) \longrightarrow \mathrm{H}^\ell(B^{-,*})$ and $\mathrm{Conc}\,\mathrm{H}^\ell(FA) \longrightarrow \mathrm{H}^\ell(J^{-,*})$ are quasiisomorphisms. Hence $\mathrm{H}^\ell(B^{-,*}) \longrightarrow \mathrm{H}^\ell(J^{-,*})$ is a quasiisomorphism, too. Now Lemma 27 shows that $G\mathrm{H}^\ell(B^{-,*}) \longrightarrow G\mathrm{H}^\ell(J^{-,*})$ is a quasiisomorphism as well. ▫

8 Applications

We will apply Theorems 31 and 34 in various algebraic situations. In particular, we will re-prove a theorem of Beyl; viz. Theorem 53 in §8.3.

In several instances below, we will make tacit use of the fact that a left exact functor between abelian categories respects injectivity of objects provided it has an exact left adjoint.

8.1 A Hopf algebra lemma

We will establish Lemma 47 in §8.1.4, needed to prove an acyclicity that enters the proof of the comparison result Theorem 52 in §8.2 for Hopf algebra cohomology, which in turn allows to derive comparison results for group cohomology and Lie algebra cohomology; cf. §§ 8.3, 8.4.

8.1.1 Definition

Let R be a commutative ring. Write $\otimes := \otimes_R$. A *Hopf algebra over* R is an R-algebra H together with R-algebra morphisms $H \xrightarrow{\varepsilon} R$ *(counit)* and $H \xrightarrow{\Delta} H \otimes H$ *(comultiplication)*, and an R-linear map $H \xrightarrow{S} H$ *(antipode)* such that the following conditions (i–iv) hold.
Write $x\Delta = \sum_i xu_i \otimes xv_i$ for $x \in H$, where u_i and v_i are chosen maps from H to H, and where i runs over a suitable indexing set. Note that

$$\textstyle\sum_i (r \cdot x + s \cdot y)u_i \otimes (r \cdot x + s \cdot y)v_i \;=\; r \cdot (\sum_i xu_i \otimes xv_i) + s \cdot (\sum_i yu_i \otimes yv_i)$$

for x, $y \in H$ and r, $s \in R$, whereas u_i and v_i are not necessarily R-linear maps.

> The elegant Sweedler notation [15, §1.2] for the images under $\Delta(\Delta\otimes 1)$ etc. led the author, being new to Hopf algebras, to confusion in a certain case. So we will express them in these more naive terms.

Write $H\otimes H \xrightarrow{\nabla} H$, $x\otimes y \longmapsto x\cdot y$ and $R \xrightarrow{\eta} H$, $r \longmapsto r\cdot 1_H$.
Write $H\otimes H \xrightarrow{\tau} H\otimes H$, $x\otimes y \longmapsto y\otimes x$.

(i) We have $\Delta(\varepsilon\otimes \mathrm{id}_H) = (x \longmapsto 1_R\otimes x)$, i.e. $\sum_i xu_i\varepsilon\cdot xv_i = x$ for $x\in H$.

(i′) We have $\Delta(\mathrm{id}_H\otimes\varepsilon) = (x \longmapsto x\otimes 1_R)$, i.e. $\sum_i xu_i\cdot xv_i\varepsilon = x$ for $x\in H$.

(ii) We have $\Delta(\mathrm{id}_H\otimes\Delta) = \Delta(\Delta\otimes\mathrm{id}_H)$, i.e. $\sum_{i,j} xu_i\otimes xv_iu_j\otimes xv_iv_j = \sum_{i,j} xu_iu_j\otimes xu_iv_j\otimes xv_i$ for $x\in H$.

(iii) We have $\Delta(S\otimes\mathrm{id}_H)\nabla = \varepsilon\eta$, i.e. $\sum_i xu_iS\cdot xv_i = x\varepsilon\cdot 1_H$ for $x\in H$.

(iii′) We have $\Delta(\mathrm{id}_H\otimes S)\nabla = \varepsilon\eta$, i.e. $\sum_i xu_i\cdot xv_iS = x\varepsilon\cdot 1_H$ for $x\in H$.

(iv) We have $S^2 = \mathrm{id}_H$.

In particular, imposing (iv), we stipulate a Hopf algebra to have an *involutive* antipode.

8.1.2 Some basic properties

> In an attempt to be reasonably self-contained, we recall some basic facts on Hopf algebras needed for Lemma 47 below; cf. [15, Ch. IV], [1, §2], [13, §§1-3]. In doing so, we shall use direct arguments.

Suppose given a Hopf algebra H over R.

Remark 41 ([15, Prop. 4.0.1], [1, Th. 2.1.4], [13, 3.4.2])
The following hold.

(1) *We have* $\sum_i (x\cdot y)u_i\otimes(x\cdot y)v_i = \sum_{i,j}(xu_i\cdot yu_j)\otimes(xv_i\cdot yv_j)$ *for* x, $y\in H$.

(2) *We have* $1_HS = 1_H$.

(3) *We have* $(x\cdot y)S = y\cdot x$ *for* x, $y\in H$.

(4) *We have* $S\varepsilon = \varepsilon$.

(5) *We have* $\Delta(S\otimes S)\tau = S\Delta$, *i.e.* $\sum_i xu_iS\otimes xv_iS = \sum_i xSv_i\otimes xSu_i$ *for* $x\in H$.

(6) *We have* $x\cdot y = \sum_i \left(\sum_j (xu_i)u_j\cdot y\cdot(xu_i)v_jS\right)\cdot xv_i$ *for* x, $y\in H$.

(6′) *We have* $y\cdot x = \sum_i xu_i\cdot\left(\sum_j (xv_i)u_jS\cdot y\cdot(xv_i)v_j\right)$ *for* x, $y\in H$.

(7) *We have* $\sum_i xv_i \cdot xu_i S = x\varepsilon \cdot 1_H$ *for* $x \in H$.

(7′) *We have* $\sum_i xv_i S \cdot xu_i = x\varepsilon \cdot 1_H$ *for* $x \in H$.

Proof. Ad (1). Given $x,\, y \in H$, we obtain

$$\textstyle\sum_i (xy)u_i \otimes (xy)v_i \;=\; (xy)\Delta \;=\; x\Delta \cdot y\Delta \;=\; \sum_{i,j} (xu_i \cdot yu_j) \otimes (xv_i \cdot yv_j)\;.$$

Ad (2). Remarking that $1_H\Delta = 1_H \otimes 1_H$, we obtain

$$1_H S \;=\; 1_H\Delta(S \otimes \mathrm{id}_H)\nabla \;\overset{\text{(iii)}}{=}\; 1_H\varepsilon \cdot 1_H \;=\; 1_H\;.$$

Ad (3). Given $x,\, y \in H$, we obtain

$$\begin{array}{rcl}
(x \cdot y)S & \overset{2\,\times\,\text{(i}'\text{)}}{=} & \sum_{i,k}(xu_i \cdot xv_i\varepsilon \cdot yu_k \cdot yv_k\varepsilon)S \\
& \overset{\text{(iii}'\text{)}}{=} & \sum_{i,j,k}(xu_i \cdot yu_k \cdot yv_k\varepsilon)S \cdot xv_iu_j \cdot xv_iv_jS \\
& \overset{\text{(iii}'\text{)}}{=} & \sum_{i,j,k,\ell}(xu_i \cdot yu_k)S \cdot xv_iu_j \cdot yv_ku_\ell \cdot yv_kv_\ell S \cdot xv_iv_jS \\
& \overset{2\,\times\,\text{(ii)}}{=} & \sum_{i,j,k,\ell}(xu_iu_j \cdot yu_ku_\ell)S \cdot xu_iv_j \cdot yu_kv_\ell \cdot yv_kS \cdot xv_iS \\
& \overset{(1)}{=} & \sum_{i,j,k}(xu_i \cdot yu_k)u_jS \cdot (xu_i \cdot yu_k)v_j \cdot yv_kS \cdot xv_iS \\
& \overset{\text{(iii)}}{=} & \sum_{i,k}(xu_i \cdot yu_k)\varepsilon \cdot yv_kS \cdot xv_iS \\
& = & \sum_{i,k}(yu_k\varepsilon \cdot yv_k)S \cdot (xu_i\varepsilon \cdot xv_i)S \\
& \overset{2\,\times\,\text{(i)}}{=} & yS \cdot xS\;.
\end{array}$$

Ad (4). Note that $(y\varepsilon \cdot z)\varepsilon = y\varepsilon \cdot z\varepsilon = (y \cdot z)\varepsilon$ for $y,\, z \in H$. Given $x \in H$, we obtain

$$\begin{array}{rcl}
xS\varepsilon & \overset{\text{(i)}}{=} & (\sum_i xu_i\varepsilon \cdot xv_i)S\varepsilon \;=\; (\sum_i xu_i\varepsilon \cdot xv_iS)\varepsilon \;=\; (\sum_i xu_i \cdot xv_iS)\varepsilon \\
& \overset{\text{(iii}'\text{)}}{=} & (x\varepsilon \cdot 1_H)\varepsilon \;=\; x\varepsilon\;.
\end{array}$$

Ad (5). Given $x \in H$, we obtain

$$\begin{array}{rl}
& x\Delta(S \otimes S)\tau \\
\overset{\text{(i)}}{=} & \sum_i(xu_i\varepsilon \cdot xv_i)\Delta(S \otimes S)\tau \\
= & \sum_i(xu_i\varepsilon \cdot 1_H)\Delta \cdot xv_i\Delta(S \otimes S)\tau \\
\overset{\text{(iii)}}{=} & \sum_{i,j}(xu_iu_jS \cdot xu_iv_j)\Delta \cdot xv_i\Delta(S \otimes S)\tau \\
= & \sum_{i,j} xu_iu_jS\Delta \cdot xu_iv_j\Delta \cdot xv_i\Delta(S \otimes S)\tau \\
\overset{\text{(ii)}}{=} & \sum_{i,j} xu_iS\Delta \cdot xv_iu_j\Delta \cdot xv_iv_j\Delta(S \otimes S)\tau \\
= & \sum_{i,j,k,\ell} xu_iS\Delta \cdot (xv_iu_ju_k \otimes xv_iu_jv_k) \cdot (xv_iv_jv_\ell S \otimes xv_iv_ju_\ell S) \\
= & \sum_{i,j,k,\ell} xu_iS\Delta \cdot (xv_iu_ju_k \cdot xv_iv_jv_\ell S \otimes xv_iu_jv_k \cdot xv_iv_ju_\ell S) \\
\overset{\text{(ii)}}{=} & \sum_{i,j,k,\ell} xu_iS\Delta \cdot (xv_iu_j \cdot xv_iv_jv_kv_\ell S \otimes xv_iv_ju_k \cdot xv_iv_jv_ku_\ell S) \\
\overset{\text{(ii)}}{=} & \sum_{i,j,k,\ell} xu_iS\Delta \cdot (xv_iu_j \cdot xv_iv_jv_kS \otimes xv_iv_ju_ku_\ell \cdot xv_iv_ju_kv_\ell S) \\
\overset{\text{(iii}'\text{)}}{=} & \sum_{i,j,k} xu_iS\Delta \cdot (xv_iu_j \cdot xv_iv_jv_kS \otimes xv_iv_ju_k\varepsilon \cdot 1_H) \\
= & \sum_{i,j,k} xu_iS\Delta \cdot (xv_iu_j \cdot (xv_iv_jv_k \cdot xv_iv_ju_k\varepsilon)S \otimes 1_H) \\
\overset{\text{(i)}}{=} & \sum_{i,j} xu_iS\Delta \cdot (xv_iu_j \cdot xv_iv_jS \otimes 1_H)
\end{array}$$

$$\begin{array}{ll} \overset{\text{(iii}')}{=} & \sum_i xu_i S\Delta \cdot (xv_i\varepsilon \cdot 1_H \otimes 1_H) \\ = & \sum_i (xu_i \cdot xv_i\varepsilon) S\Delta \\ \overset{\text{(i}')}{=} & xS\Delta \, . \end{array}$$

Ad (6). Given x, $y \in H$, we obtain

$$x\cdot y \overset{\text{(i}')}{=} \sum_i xu_i\cdot y\cdot xv_i\varepsilon \overset{\text{(iii)}}{=} \sum_{i,j} xu_i\cdot y\cdot xv_iu_jS\cdot xv_iv_j \overset{\text{(ii)}}{=} \sum_{i,j} xu_iu_j\cdot y\cdot xu_iv_jS\cdot xv_i.$$

Ad (6′). Given $x \in H$, we obtain

$$y\cdot x \overset{\text{(i)}}{=} \sum_i xu_i\varepsilon\cdot y\cdot xv_i \overset{\text{(iii}')}{=} \sum_{i,j} xu_iu_j\cdot xu_iv_jS\cdot y\cdot xv_i \overset{\text{(ii)}}{=} \sum_{i,j} xu_i\cdot xv_iu_jS\cdot y\cdot xv_iv_j.$$

Ad (7). Given $x \in H$, we have

$$\begin{array}{l} \sum_i xv_i \cdot xu_iS \overset{\text{(iv)}}{=} \sum_i xS^2v_i \cdot xS^2u_iS \overset{(5)}{=} \sum_i xSu_iS \cdot xSv_iS^2 \\ \qquad\qquad \overset{\text{(iv)}}{=} \sum_i xSu_iS \cdot xSv_i \overset{\text{(iii)}}{=} xS\varepsilon \cdot 1_H \overset{(4)}{=} x\varepsilon \cdot 1_H \, . \end{array}$$

Ad (7′). Given $x \in H$, we have

$$\begin{array}{l} \sum_i xv_iS \cdot xu_i \overset{\text{(iv)}}{=} \sum_i xS^2v_iS \cdot xS^2u_i \overset{(5)}{=} \sum_i xSu_iS^2 \cdot xSv_iS \\ \qquad\qquad \overset{\text{(iv)}}{=} \sum_i xSu_i \cdot xSv_iS \overset{\text{(iii}')}{=} xS\varepsilon \cdot 1_H \overset{(4)}{=} x\varepsilon \cdot 1_H \, . \end{array}$$

□

In the present §8.1, we shall refer to the assertions Remark 41.(1–7′) just by (1–7′).

8.1.3 Normality

Suppose given a Hopf algebra H over R, and an R-subalgebra $K \subseteq H$. Suppose H and K to be flat as modules over R.

Note that $K \otimes K \longrightarrow H \otimes H$ is injective. We will identify $K \otimes K$ with its image.

The R-subalgebra $K \subseteq H$ is called a *Hopf-subalgebra* if $K\Delta \subseteq K \otimes K$ and $KS \subseteq K$. In this case, we may and will suppose the maps u_i and v_i to restrict to maps from K to K.

Suppose $K \subseteq H$ to be a Hopf-subalgebra. It is called *normal,* if for all $a \in K$ and all $x \in H$, we have

$$\sum_i xu_i \cdot a \cdot xv_iS \in K \qquad \text{and} \qquad \sum_i xu_iS \cdot a \cdot xv_i \in K \, .$$

An ideal $I \subseteq H$ is called a *Hopf ideal* if $I\Delta \subseteq I \otimes H + H \otimes I$ (where we have identified $I \otimes H$ and $H \otimes I$ with their images in $H \otimes H$), $I\varepsilon = 0$ and $IS \subseteq I$. In this case, the quotient H/I carries a Hopf algebra structure via

$$\begin{array}{lllll} H/I & \xrightarrow{\varepsilon} & R\,, & x+I & \longmapsto & x\varepsilon \\ H/I & \xrightarrow{\Delta} & H/I \otimes H/I\,, & x+I & \longmapsto & \sum_i (xu_i + I) \otimes (xv_i + I) \\ H/I & \xrightarrow{S} & H/I\,, & x+I & \longmapsto & xS + I\,. \end{array}$$

Suppose $K \subseteq H$ to be a normal Hopf subalgebra. Write $K^+ := \mathrm{Kern}(H \xrightarrow{\varepsilon} R)$. By $(6, 6', 3, 4)$, $HK^+ = K^+H$ is a Hopf ideal in H.

8.1.4 Some remarks and a lemma

Suppose given a Hopf algebra H over R and a normal Hopf-subalgebra $K \subseteq H$. Suppose H and K to be flat as modules over R.
Write $\bar{H} := H/HK^+$. Given $x \in H$, write $\bar{x} := x + HK^+ \in \bar{H}$ for its residue class.
Let N', N, M, M' and Q be H-modules. Let P be an $\bar{H}$-module, which we also consider as an H-module via $H \longrightarrow \bar{H}$, $x \longmapsto \bar{x}$.
We write ${}_K(N, M) = {}_K(N|_K, M|_K)$ for the R-module of K-linear maps from N to M.

Remark 42 *Given $f \in {}_R(N, M)$ and $x \in H$, we define $x \cdot f \in {}_R(N, M)$ by*

$$[n](x \cdot f) \;:=\; \textstyle\sum_i xu_i \cdot [xv_iS \cdot n]f$$

for $n \in N$. This defines a left H-module structure on ${}_R(N, M)$.

Formally, squared brackets mean the same as parentheses. Informally, squared brackets are to accentuate the arguments of certain maps.

Proof. We *claim* that $x' \cdot (x \cdot f) = (x' \cdot x) \cdot f$ for $x, x' \in H$. Suppose given $n \in N$. We obtain

$$\begin{array}{rcl} [n](x' \cdot (x \cdot f)) & = & \sum_i x'u_i \cdot [x'v_iS \cdot n](x \cdot f) \\ & = & \sum_{i,j} x'u_i \cdot xu_j \cdot [xv_jS \cdot x'v_iS \cdot n]f \\ & \overset{(3)}{=} & \sum_{i,j} (x'u_i \cdot xu_j) \cdot [(x'v_i \cdot xv_j)S \cdot n]f \\ & \overset{(1)}{=} & \sum_i (x' \cdot x)u_i \cdot [(x' \cdot x)v_iS \cdot n]f \\ & = & [n]((x' \cdot x) \cdot f)\ . \end{array}$$

We *claim* that $1_H \cdot f = f$. Suppose given $n \in N$. We obtain

$$[n](1_H \cdot f) \;=\; \textstyle\sum_i 1_H u_i \cdot [1_H v_i S \cdot n]f \;=\; 1_H \cdot [1_H S \cdot n]f \overset{(2)}{=} [n]f\ ,$$

remarking that $1_H\Delta = 1_H \otimes 1_H$. ▫

I owe to G. Hiss the hint to improve a previous weaker version of Corollary 45 below by means of the following Remark 43.

Denote by

$$M^K \;:=\; \{m \in M \;:\; a \cdot m = a\varepsilon \cdot m \text{ for all } a \in K\}$$

the fixed point module of M under K.

Remark 43 *Letting $\bar{x} \cdot m := x \cdot m$ for $x \in H$ and $m \in M^K$, we define an $\bar{H}$-module structure on M^K.*

Proof. The value of the product $\bar{x} \cdot m$ does not depend on the chosen representative x of $\bar{x}$ since, given $y \in H$, $a \in K^+$ and $m \in M^K$, we have

$$y \cdot a \cdot m \;=\; y \cdot a\varepsilon \cdot m \;=\; 0\,.$$

It remains to be shown that given $x \in H$ and $m \in M^K$, the element $x \cdot m$ lies in M^K. In fact, given $a \in K$, we obtain

$$\begin{array}{rcl} a \cdot x \cdot m & \overset{(6')}{=} & \sum_i xu_i \cdot \Big(\sum_j (xv_i)u_j S \cdot a \cdot (xv_i)v_j\Big) \cdot m \\ & = & \sum_i xu_i \cdot \Big(\sum_j (xv_i)u_j S \cdot a \cdot (xv_i)v_j\Big)\varepsilon \cdot m \\ & = & \sum_{i,j} xu_i \cdot xv_iu_jS\varepsilon \cdot a\varepsilon \cdot xv_iv_j\varepsilon \cdot m \\ & \overset{(4)}{=} & \sum_{i,j} xu_i \cdot xv_iu_j\varepsilon \cdot a\varepsilon \cdot xv_iv_j\varepsilon \cdot m \\ & \overset{(\mathrm{ii})}{=} & \sum_{i,j} xu_iu_j \cdot xu_iv_j\varepsilon \cdot a\varepsilon \cdot xv_i\varepsilon \cdot m \\ & \overset{(\mathrm{i}')}{=} & \sum_i xu_i \cdot a\varepsilon \cdot xv_i\varepsilon \cdot m \\ & \overset{(\mathrm{i}')}{=} & a\varepsilon \cdot x \cdot m\,. \end{array}$$

□

Remark 44 *We have $({}_R(N,M))^K = {}_K(N,M)$, as subsets of ${}_R(N,M)$.*

Proof. The module $({}_R(N,M))^K$ consists of the R-linear maps $N \xrightarrow{f} M$ that satisfy

$$\textstyle\sum_i xu_i \cdot [xv_iS \cdot n]f \;=\; x\varepsilon \cdot [n]f\,.$$

for $x \in H$ and $n \in N$. The module ${}_K(N,M)$ consists of the R-linear maps $N \xrightarrow{f} M$ that satisfy

$$[x \cdot n]f \;=\; x \cdot [n]f$$

for $x \in H$ and $n \in N$. By (iii$'$), we have $({}_R(N,M))^K \supseteq {}_K(N,M)$.
It remains to show that $({}_R(N,M))^K \subseteq {}_K(N,M)$. Given $f \in ({}_R(N,M))^K$, $x \in H$ and $n \in N$, we obtain

$$\begin{array}{rcl} x \cdot [n]f & \overset{(\mathrm{i}')}{=} & \sum_i xu_i \cdot xv_i\varepsilon \cdot [n]f \\ & = & \sum_i xu_i \cdot [xv_i\varepsilon \cdot n]f \\ & \overset{(\mathrm{iii})}{=} & \sum_{i,j} xu_i \cdot [xv_iu_jS \cdot xv_iv_j \cdot n]f \\ & \overset{(\mathrm{ii})}{=} & \sum_{i,j} xu_iu_j \cdot [xu_iv_jS \cdot xv_i \cdot n]f \\ & = & \sum_i xu_i\varepsilon \cdot [xv_i \cdot n]f \\ & \overset{(\mathrm{i})}{=} & [x \cdot n]f\,. \end{array}$$

□

Corollary 45 *Given $f \in {}_K(N,M)$ and $x \in H$, we define $\bar{x} \cdot f \in {}_K(N,M)$ by*

$$[n](\bar{x} \cdot f) \;:=\; \textstyle\sum_i xu_i \cdot [xv_iS \cdot n]f$$

for $n \in N$. This defines a left $\bar{H}$-module structure on ${}_K(N,M)$.

Proof. By Remark 42, we may apply Remark 43 to ${}_R(N, M)$. By Remark 44, the assertion follows. ▫

Remark 46 *Given* $f \in {}_K(N, M)$, $x \in H$, *and* H*-linear maps* $N' \xrightarrow{\nu} N$, $M \xrightarrow{\mu} M'$, *we obtain*

$$\nu(\bar{x} \cdot f)\mu \;=\; \bar{x} \cdot (\nu f \mu)\ .$$

Proof. Given $n' \in N'$, we obtain

$$\begin{aligned} [n']\big(\nu(\bar{x} \cdot f)\mu\big) \;&=\; \big(\textstyle\sum_i xu_i \cdot [xv_iS \cdot n'\nu]f\big)\mu \;=\; \textstyle\sum_i xu_i \cdot [xv_iS \cdot n'](\nu f \mu) \\ &=\; [n'](\bar{x} \cdot (\nu f \mu)). \end{aligned}$$

▫

> The following Lemma 47 has been suggested by the referee, and has been achieved with the help of G. Carnovale. It is reminiscent of [16, Cor. 4.3], but easier. It resembles a bit a Fourier inversion.

Note that the right $\bar{H}$-module structure on $\bar{H}$ induces a left $\bar{H}$-module structure on ${}_R(\bar{H}, M)$.

Lemma 47 *We have the following mutually inverse isomorphisms of* $\bar{H}$*-modules.*

$$\begin{array}{rcl} {}_K(H, M) & \xrightarrow[\sim]{\Phi} & {}_R(\bar{H}, M) \\ f & \longmapsto & (\bar{x} \mapsto \sum_i xu_i \cdot [xv_iS]f) \\[2ex] {}_K(H, M) & \xleftarrow[\sim]{\Psi} & {}_R(\bar{H}, M) \\ (x \mapsto \sum_j xv_j \cdot [\,\overline{xu_jS}\,]g) & \longleftarrow & g \end{array}$$

Proof. We *claim* that Φ is a welldefined map. We have to show that $f\Phi$ is welldefined, i.e. that its value at $\bar{x}$ does not depend on the representing element x. Suppose given $y \in H$ and $a \in K^+$. We obtain

$$\begin{array}{rcl} \sum_i (ya)u_i \cdot [(ya)v_iS]f & \overset{(1)}{=} & \sum_{i,j} yu_i \cdot au_j \cdot [(yv_i \cdot av_j)S]f \\ & \overset{(3)}{=} & \sum_{i,j} yu_i \cdot au_j \cdot [av_jS \cdot yv_iS]f \\ & = & \sum_{i,j} yu_i \cdot au_j \cdot av_jS \cdot [yv_iS]f \\ & \overset{(\mathrm{iii}')}{=} & \sum_i yu_i \cdot a\varepsilon \cdot [yv_iS]f \\ & = & 0\ . \end{array}$$

We *claim* that Φ is $\bar{H}$-linear. Suppose given $y \in H$ and $x \in H$. We obtain

$$\begin{array}{rcl} [\bar{x}]((\bar{y}f)\Phi) & = & \sum_i xu_i \cdot [xv_iS](\bar{y}f) \\ & = & \sum_{i,j} xu_i \cdot yu_j \cdot [yv_jS \cdot xv_iS]f \\ & \overset{(3)}{=} & \sum_{i,j} xu_i \cdot yu_j \cdot [(xv_i \cdot yv_j)S]f \\ & \overset{(1)}{=} & \sum_i (x \cdot y)u_i \cdot [(x \cdot y)v_iS]f \\ & = & [\bar{x}](\bar{y}(f\Phi))\ . \end{array}$$

We *claim* that Ψ is a welldefined map. We have to show that $g\Psi$ is K-linear. Suppose given $a \in K$ and $x \in H$. Note that $au_i \in K$ for all i, whence also $au_iS \in K$, and therefore $au_iS \equiv_{HK^+} au_iS\varepsilon \cdot 1_H$. We obtain

$$\begin{array}{rcl}
[a\cdot x](g\Psi) & = & \sum_j (a\cdot x)v_j \cdot [\,\overline{(a\cdot x)u_jS}\,]g \\
& \overset{(1)}{=} & \sum_{i,j} av_i \cdot xv_j \cdot [\,\overline{(au_i\cdot xu_j)S}\,]g \\
& \overset{(3)}{=} & \sum_{i,j} av_i \cdot xv_j \cdot [\,\overline{xu_jS}\cdot\overline{au_iS}\,]g \\
& = & \sum_{i,j} av_i \cdot xv_j \cdot [\,\overline{xu_jS}\cdot\overline{au_iS\varepsilon}\,]g \\
& \overset{(4)}{=} & \sum_{i,j} au_i\varepsilon \cdot av_i \cdot xv_j \cdot [\,\overline{xu_jS}\,]g \\
& \overset{\text{(i)}}{=} & \sum_j a\cdot xv_j \cdot [\,\overline{xu_jS}\,]g \\
& = & a\cdot [x](g\Psi)\;.
\end{array}$$

We *claim* that $\Phi\Psi = \mathrm{id}_{{}_K(H,M)}$. Suppose given $x \in H$. We obtain

$$\begin{array}{rcl}
[x](f\Phi\Psi) & = & \sum_j xv_j \cdot [\,\overline{xu_jS}\,](f\Phi) \\
& = & \sum_{i,j} xv_j \cdot xu_jSu_i \cdot [xu_jSv_iS]f \\
& \overset{(5)}{=} & \sum_{i,j} xv_j \cdot xu_jv_iS \cdot [xu_ju_iS^2]f \\
& \overset{\text{(iv)}}{=} & \sum_{i,j} xv_j \cdot xu_jv_iS \cdot [xu_ju_i]f \\
& \overset{\text{(ii)}}{=} & \sum_{i,j} xv_jv_i \cdot xv_ju_iS \cdot [xu_j]f \\
& \overset{(7)}{=} & \sum_j xv_j\varepsilon \cdot [xu_j]f \\
& \overset{\text{(i)}}{=} & [x]f\;.
\end{array}$$

We *claim* that $\Psi\Phi = \mathrm{id}_{{}_R(\bar H,M)}$. Suppose given $x \in H$. We obtain

$$\begin{array}{rcl}
[\bar x](g\Psi\Phi) & = & \sum_i xu_i \cdot [xv_iS](g\Psi) \\
& = & \sum_{i,j} xu_i \cdot xv_iSv_j \cdot [\,\overline{xv_iSu_jS}\,]g \\
& \overset{(5)}{=} & \sum_{i,j} xu_i \cdot xv_iu_jS \cdot [\,\overline{xv_iv_jS^2}\,]g \\
& \overset{\text{(iv)}}{=} & \sum_{i,j} xu_i \cdot xv_iu_jS \cdot [\,\overline{xv_iv_j}\,]g \\
& \overset{\text{(ii)}}{=} & \sum_{i,j} xu_iu_j \cdot xu_iv_jS \cdot [\,\overline{xv_i}\,]g \\
& \overset{\text{(iii')}}{=} & \sum_i xu_i\varepsilon \cdot [\,\overline{xv_i}\,]g \\
& \overset{\text{(i)}}{=} & [\bar x]g\;.
\end{array}$$

Finally, it follows by $\bar H$-linearity of Φ and by $\Psi = \Phi^{-1}$ that Ψ is $\bar H$-linear. ▫

The tensor product $N\otimes M$ is an H-module via Δ. Note that R is an H-module via ε. Note that $R\otimes M \simeq M \simeq M\otimes R$ as H-modules by (i,i').

Remark 48 (cf. [3, Lem. 3.5.1]) *We have mutually inverse isomorphisms* *of R-modules*

$$\begin{array}{rcl}
{}_{\bar{H}}(P,\ {}_K(Q,M)) & \xrightarrow[\sim]{\alpha} & {}_H(P\otimes Q, M) \\
f & \longmapsto & (p\otimes q \longmapsto [q](pf)) \\
{}_{\bar{H}}(P,\ {}_K(Q,M)) & \xleftarrow[\sim]{\beta} & {}_H(P\otimes Q, M) \\
(p \longmapsto (q \longmapsto [p\otimes q]g)) & \longleftarrow & g\ ,
\end{array}$$

natural in $P \in \operatorname{Ob} \bar{H}\text{-Mod}$*,* $Q \in \operatorname{Ob} H\text{-Mod}$ *and* $M \in \operatorname{Ob} H\text{-Mod}$.

Proof. We *claim* that α is welldefined. We have to show that $f\alpha$ is H-linear. Suppose given $x \in H$. We obtain

$$\begin{array}{rcl}
x\cdot(p\otimes q) & = & \sum_i \overline{xu_i}\cdot p\otimes xv_i\cdot q \\
 & \xmapsto{f\alpha} & \sum_i [xv_i\cdot q]((\overline{xu_i}\cdot p)f) \\
 & = & \sum_i [xv_i\cdot q](\overline{xu_i}\cdot (pf)) \\
 & = & \sum_{i,j} xu_iu_j\cdot [xu_iv_jS\cdot xv_i\cdot q](pf) \\
 & \overset{\text{(ii)}}{=} & \sum_{i,j} xu_i\cdot [xv_iu_jS\cdot xv_iv_j\cdot q](pf) \\
 & \overset{\text{(iii)}}{=} & \sum_i xu_i\cdot [xv_i\varepsilon\cdot q](pf) \\
 & \overset{\text{(i}'\text{)}}{=} & x\cdot [q](pf) \\
 & = & x\cdot [p\otimes q](f\alpha)\ .
\end{array}$$

We *claim* that β is welldefined. First, we have to show that $[p](g\beta)$ is K-linear. Suppose given $a \in K$. We obtain

$$a\cdot q \xmapsto{[p](g\beta)} [p\otimes a\cdot q]g \overset{\text{(i)}}{=} \sum_i [\overline{au_i\varepsilon}\cdot p\otimes av_i\cdot q]g = \sum_i [\overline{au_i}\cdot p\otimes av_i\cdot q]g = a\cdot[p\otimes q]g\ .$$

Second, we have to show that $g\beta$ is $\bar{H}$-linear. Suppose given $x \in H$. We obtain

$$\begin{array}{rcl}
\bar{x}\cdot p & \xmapsto{g\beta} & (q\longmapsto [\bar{x}\cdot p\otimes q]g) \\
 & \overset{\text{(i)}}{=} & (q\longmapsto \sum_i [\overline{xu_i\cdot xv_i\varepsilon}\cdot p\otimes q)]g) \\
 & \overset{\text{(iii}'\text{)}}{=} & (q\longmapsto \sum_{i,j} [\overline{xu_i}\cdot p\otimes xv_iu_j\cdot xv_iv_jS\cdot q)]g) \\
 & \overset{\text{(ii)}}{=} & (q\longmapsto \sum_{i,j} [\overline{xu_iu_j}\cdot p\otimes xu_iv_j\cdot xv_iS\cdot q)]g) \\
 & = & (q\longmapsto \sum_i xu_i\cdot [p\otimes xv_iS\cdot q]g) \\
 & = & \bar{x}\cdot (q\longmapsto [p\otimes q]g)\ .
\end{array}$$

Finally, α and β are mutually inverse. ▫

Corollary 49 *We have* ${}_{\bar{H}}(P, M^K) \simeq {}_{\bar{H}}(P,\ {}_K(R,M)) \simeq {}_H(P,M)$ *as R-modules, natural in P and M.*

Proof. Note that $M \simeq {}_R(R,M)$ as H-modules, whence $M^K \simeq {}_K(R,M)$ as $\bar{H}$-modules by Remarks 43, 44. Now the assertion follows from Remark 48, letting $Q = R$. ▫

8.2 Comparing Hochschild-Serre-Hopf with Grothendieck

Let R be a commutative ring. Suppose given a Hopf algebra H over R (with involutive antipode) and a normal Hopf-subalgebra $K \subseteq R$; cf. §8.1.3. Write $\bar{H} := H/HK^+$. Suppose H, K and $\bar{H}$ to be projective as modules over R. Suppose H to be projective as a module over K.
Let $B \in \operatorname{Ob}\mathrm{C}(H\text{-Mod})$ be a projective resolution of R over H. Let $\bar{B} \in \operatorname{Ob}\mathrm{C}(\bar{H}\text{-Mod})$ be a projective resolution of R over $\bar{H}$. Note that since $\bar{H}$ is projective over R, $\bar{B}|_R \in \operatorname{Ob}\mathrm{C}(R\text{-Mod})$ is a projective resolution of R over R.
Let M be an H-module.
By Corollary 45 and by Remark 46, we have a biadditive functor

$$
\begin{array}{ccccc}
(H\text{-Mod})^\circ & \times & H\text{-Mod} & \xrightarrow{U} & \bar{H}\text{-Mod} \\
(X & , & X') & \longmapsto & U(X,X') := {}_K(X,X') \,.
\end{array}
$$

Write

$$
\begin{array}{ccccc}
(\bar{H}\text{-Mod})^\circ & \times & \bar{H}\text{-Mod} & \xrightarrow{V} & \bar{H}\text{-Mod} \\
(Y & , & Y') & \longmapsto & V(N,M) := {}_{\bar{H}}(N,M)
\end{array}
$$

for the usual Hom-functor.

Lemma 50 *The $\bar{H}$-module $U(H,M)$ is $V(R,-)$-acyclic.*

Proof. By Lemma 47, this amounts to showing that ${}_R(\bar{H},M)$ is $V(R,-)$-acyclic, which in turn amounts to showing that $V\big(\bar{B}\,,\,{}_R(\bar{H},M)\big) = {}_{\bar{H}}\big(\bar{B}\,,\,{}_R(\bar{H},M)\big)$ has vanishing cohomology in degrees ≥ 1. Now,

$$
{}_{\bar{H}}\big(\bar{B}\,,\,{}_R(\bar{H},M)\big) \;\simeq\; {}_R(\bar{H}\otimes_{\bar{H}}\bar{B}\,,\,M) \;\simeq\; {}_R(\bar{B}\,,\,M)\,,
$$

whose cohomology in degree $i \geq 1$ is $\operatorname{Ext}^i_R(R,M) \simeq 0$. ▫
Consider the double complex

$$
D(M) \;=\; D^{-,=}(M) \;:=\; V\big(\bar{B}_-\,,\,U(B_=\,,\,M)\big) \;=\; {}_{\bar{H}}\big(\bar{B}_-\,,\,{}_K(B_=\,,\,M)\big)\,.
$$

Note that $D(M)$ is isomorphic in $\mathrm{CC}^{\llcorner}(R\text{-Mod})$ to ${}_H\big(\bar{B}_-\otimes_R B_=\,,\,M\big)$, naturally in M; cf. Remark 48.
We have functors

$$
\begin{array}{ccccc}
H\text{-Mod} & \xrightarrow{U(R,-)} & \bar{H}\text{-Mod} & \xrightarrow{V(R,-)} & R\text{-Mod}\,. \\
M & \longmapsto & U(R,M)\simeq M^K & & \\
 & & P & \longmapsto & V(R,P)\simeq P^{\bar{H}}
\end{array}
$$

Lemma 51 *Given a projective H-module P, the $\bar{H}$-module $U(P,M)$ is $V(R,-)$-acyclic.*

Proof. It suffices to show that $U(\coprod_\Gamma H,\, M) \;\simeq\; \prod_\Gamma U(H,\, M)$ is $V(R,-)$-acyclic for any indexing set Γ. By Lemma 50, it remains to be shown that $\mathrm{R}^i V(R, \prod_\Gamma Y)$ is isomorphic to $\prod_\Gamma \mathrm{R}^i V(R,Y)$ for a given $\bar{H}$-module Y and for $i \geq 1$. Having chosen an injective resolution J of Y, we may choose the injective resolution $\prod_\Gamma J$ of $\prod_\Gamma Y$. Then

$$\begin{aligned} \mathrm{R}^i V(R, \textstyle\prod_\Gamma Y) \;\simeq\; \mathrm{H}^i V(R, \prod_\Gamma J) \;&\simeq\; \mathrm{H}^i \textstyle\prod_\Gamma V(R,J) \\ &\simeq\; \textstyle\prod_\Gamma \mathrm{H}^i V(R,J) \;\simeq\; \prod_\Gamma \mathrm{R}^i V(R,Y)\;. \end{aligned}$$

□

Theorem 52 *The proper spectral sequences*

$$\dot{\mathrm{E}}_{\mathrm{I}}(D(M)) \qquad \text{and} \qquad \dot{\mathrm{E}}^{\mathrm{Gr}}_{U(R,-),\,V(R,-)}(M)$$

are isomorphic (in $[\![\dot{\mathbf{Z}}^{\#\#}_\infty,\, R\text{-Mod}\,]\!]$*), naturally in* $M \in \mathrm{Ob}\, H\text{-Mod}$.

Proof. To apply Theorem 31 with, in the notation of §5.1,

$$\begin{aligned} &\Big(\mathcal{A}\times\mathcal{A}' \xrightarrow{\;F\;} \mathcal{B} \xrightarrow{\;G\;} \mathcal{C}\Big) \\ &= \Big((H\text{-Mod})^\circ \times H\text{-Mod} \xrightarrow{\;U\;} \bar{H}\text{-Mod} \xrightarrow{V(R,-)} R\text{-Mod}\Big)\;, \end{aligned}$$

and with $X = R$ and $X' = M$, we verify the conditions (a–d′) of loc. cit. in this case.
Ad (c). We *claim* that B is a $\big(U(-,M),\, V(R,-)\big)$-acyclic resolution of R. We have to show that $U(B_i,M)$ is $V(R,-)$-acyclic for $i \geq 0$; cf. §4.2. Since B_i is projective over H, this follows by Lemma 51. This proves the *claim.*
Ad (c′). Let I be an injective resolution of M over H. We *claim* that I is a $\big(U(R,-),\, V(R,-)\big)$-acyclic resolution of M. We have to show that $U(R,I^i)$ is $V(R,-)$-acyclic for $i \geq 0$. In fact, by Corollary 49, $U(R,I^i)$ is an injective $\bar{H}$-module. This proves the *claim.*
Ad (d, d′). We *claim* that $U(B_i,-)$ and $U(-,I^i)$ are exact for $i \geq 0$; cf. §5.1. The former follows from H being projective over K. The latter is a consequence of $I^i|_K$ being injective in K-Mod by exactness of $K\text{-Mod} \xrightarrow{H\otimes_K -} H\text{-Mod}$. This proves the *claim.*
So an application of Theorem 31 yields

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{U(R,-),V(R,-)}(M) \;\simeq\; \dot{\mathrm{E}}^{\mathrm{Gr}}_{U(-,M),V(R,-)}(R)\;.$$

To apply Theorem 34 with, in the notation of §6.1,

$$\begin{aligned} &\Big(\mathcal{A} \xrightarrow{\;F\;} \mathcal{B}'\,,\;\; \mathcal{B}\times\mathcal{B}' \xrightarrow{\;G\;} \mathcal{C}\Big) \\ &= \Big((H\text{-Mod})^\circ \xrightarrow{U(-,M)} \bar{H}\text{-Mod}\,,\;\; (\bar{H}\text{-Mod})^\circ \times \bar{H}\text{-Mod} \xrightarrow{\;V\;} \mathcal{C}\Big)\,, \end{aligned}$$

and with $X = R$ and $Y = R$, we verify the conditions (a–e) of loc. cit. in this case.
Ad (c). We have already remarked that B is a $\big(U(-,M),\, V(R,-)\big)$-acyclic resolution of R.
Ad (d). As a resolution of R over $\bar{H}$, we choose $\bar{B}$.
So an application of Theorem 34 yields

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{U(-,M),V(R,-)}(R) \;\simeq\; \dot{\mathrm{E}}_{\mathrm{I}}\Big(V\big(\bar{B}_{-}\,,\; U(B_{=}\,,\; M)\big)\Big)\;.$$

Naturality in $M \in \mathrm{Ob}\, H\text{-Mod}$ remains to be shown. Suppose given $M \xrightarrow{m} \tilde{M}$ in H-Mod. Note that the requirements of §5.2 are met. By Proposition 32, with roles of $\mathcal{A}$ and $\mathcal{A}'$ interchanged, we have the following commutative quadrangle.

$$\begin{tikzcd}
\dot{\mathrm{E}}^{\mathrm{Gr}}_{U(R,-),V(R,-)}(M) \arrow[r, "\dot{\mathrm{E}}^{\mathrm{Gr}}_{U(R,-),V(R,-)}(m)"] & \dot{\mathrm{E}}^{\mathrm{Gr}}_{U(R,-),V(R,-)}(\tilde{M}) \\
\dot{\mathrm{E}}^{\mathrm{Gr}}_{U(-,M),V(R,-)}(R) \arrow[u, "\wr"] \arrow[r, "\mathrm{h}^{\mathrm{I}}_{U(-,m)}R"] & \dot{\mathrm{E}}^{\mathrm{Gr}}_{U(-,\tilde{M}),V(R,-)}(R) \arrow[u, "\wr"']
\end{tikzcd}$$

Note that the requirements of §6.2 are met. By Lemma 36, we have the following commutative quadrangle.

$$\begin{tikzcd}
\dot{\mathrm{E}}^{\mathrm{Gr}}_{U(-,M),V(R,-)}(R) \arrow[r, "\mathrm{h}^{\mathrm{I}}_{U(-,m)}R"] \arrow[d, "\wr"'] & \dot{\mathrm{E}}^{\mathrm{Gr}}_{U(-,\tilde{M}),V(R,-)}(R) \arrow[d, "\wr"] \\
\dot{\mathrm{E}}_{\mathrm{I}}\Big(V\big(\bar{B}_{-}\,,\; U(B_{=}\,,\; M)\big)\Big) \arrow[r, "\dot{\mathrm{E}}_{\mathrm{I}}(V(\bar{B}_{-}\,,\, U(B_{=},m)))"] & \dot{\mathrm{E}}_{\mathrm{I}}\Big(V\big(\bar{B}_{-}\,,\; U(B_{=}\,,\; \tilde{M})\big)\Big)
\end{tikzcd}$$

□

8.3 COMPARING LYNDON-HOCHSCHILD-SERRE WITH GROTHENDIECK

Let R be a commutative ring. Let G be a group and let $N \trianglelefteq G$ be a normal subgroup. Write $\bar{G} := G/N$. Let M be an RG-module. Write $\mathrm{Bar}_{G;R} \in \mathrm{Ob}\,\mathrm{C}(RG\text{-Mod})$ for the bar resolution of R over RG, having $(\mathrm{Bar}_{G;R})_i = RG^{\otimes(i+1)}$ for $i \geq 0$, the tensor product being taken over R.
Note that RG is a Hopf algebra over R via

$$\begin{array}{lcllcl}
RG & \xrightarrow{\Delta} & RG \otimes RG\,, & g & \longmapsto & g \otimes g \\
RG & \xrightarrow{S} & RG\,, & g & \longmapsto & g^{-1} \\
RG & \xrightarrow{\varepsilon} & R\,, & g & \longmapsto & 1\,,
\end{array}$$

where $g \in G$; cf. §8.1.1. Moreover, RN is a normal Hopf subalgebra of RG such that $RG/(RG)(RN)^{+} \simeq R\bar{G}$; cf. §8.1.3.

Note that RG, RN and $R\bar{G}$ are projective over R, and that RG is projective over RN.
We have functors $RG\text{-Mod} \xrightarrow{(-)^N} R\bar{G}\text{-Mod} \xrightarrow{(-)^{\bar{G}}} R\text{-Mod}$, taking respective fixed points.

Theorem 53 (Beyl, [4, Th. 3.5]) *The proper spectral sequences*

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{(-)^N,\,(-)^{\bar{G}}}(M) \qquad \textit{and} \qquad \dot{\mathrm{E}}_{\mathrm{I}}\Big(\,{}_{RG}\big((\mathrm{Bar}_{\bar{G};R})_{-} \otimes_R (\mathrm{Bar}_{G;R})_{=}\,,\ M\big)\Big)$$

are isomorphic (in $[\![\dot{\mathbf{Z}}^{\#\#}_{\infty},\ R\text{-Mod}\,]\!]$*), naturally in* $M \in \mathrm{Ob}\,RG\text{-Mod}$.

Beyl uses his Theorem 40 to prove Theorem 53. We shall re-derive it from Theorem 52, which in turn relies on the Theorems 31 and 34.

Proof. This follows by Theorem 52. ▫

8.4 Comparing Hochschild-Serre with Grothendieck

Let R be a commutative ring. Let $\mathfrak{g}$ be a Lie algebra over R that is free as an R-module. Let $\mathfrak{n} \trianglelefteq \mathfrak{g}$ be an ideal such that $\mathfrak{n}$ and $\bar{\mathfrak{g}} := \mathfrak{g}/\mathfrak{n}$ are free as R-modules. Let M be a $\mathfrak{g}$-module, i.e. a $\mathcal{U}(\mathfrak{g})$-module. Write $\mathrm{Bar}_{\mathfrak{g};R} \in \mathrm{Ob}\,\mathrm{C}(\mathcal{U}(\mathfrak{g})\text{-Mod})$ for the Chevalley-Eilenberg resolution of R over $\mathcal{U}(\mathfrak{g})$, having $(\mathrm{Bar}_{\mathfrak{g};R})_i = \mathcal{U}(\mathfrak{g}) \otimes_R \wedge^i \mathfrak{g}$ for $i \geq 0$; cf. [5, XIII.§7] or [18, Th. 7.7.2].
Note that $\mathcal{U}(\mathfrak{g})$ is a Hopf algebra over R via

$$\begin{array}{lllllll}
\mathcal{U}(\mathfrak{g}) & \xrightarrow{\Delta} & \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})\,, & g & \longmapsto & g \otimes 1 + 1 \otimes g \\
\mathcal{U}(\mathfrak{g}) & \xrightarrow{S} & \mathcal{U}(\mathfrak{g})\,, & g & \longmapsto & -g \\
\mathcal{U}(\mathfrak{g}) & \xrightarrow{\varepsilon} & R\,, & g & \longmapsto & 0\,,
\end{array}$$

where $g \in \mathfrak{g}$; cf. §8.1.1.
Note that $\mathcal{U}(\mathfrak{g})$, $\mathcal{U}(\mathfrak{n})$ and $\mathcal{U}(\bar{\mathfrak{g}})$ are projective over R, and that $\mathcal{U}(\mathfrak{g})$ is projective over $\mathcal{U}(\mathfrak{n})$; cf. [18, Cor. 7.3.9].
We have functors $\mathcal{U}(\mathfrak{g})\text{-Mod} \xrightarrow{(-)^{\mathfrak{n}}} \mathcal{U}(\bar{\mathfrak{g}})\text{-Mod} \xrightarrow{(-)^{\bar{\mathfrak{g}}}} R\text{-Mod}$, taking respective annihilated submodules; cf. [18, p. 221].

Theorem 54 *The proper spectral sequences*

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{(-)^{\mathfrak{n}},\,(-)^{\bar{\mathfrak{g}}}}(M) \qquad \textit{and} \qquad \dot{\mathrm{E}}_{\mathrm{I}}\Big(\,{}_{\mathcal{U}(\mathfrak{g})}\big((\mathrm{Bar}_{\bar{\mathfrak{g}};R})_{-} \otimes_R (\mathrm{Bar}_{\mathfrak{g};R})_{=}\,,\ M\big)\Big)$$

are isomorphic (in $[\![\dot{\mathbf{Z}}^{\#\#}_{\infty},\ R\text{-Mod}\,]\!]$*), naturally in* $M \in \mathrm{Ob}\,\mathcal{U}(\mathfrak{g})\text{-Mod}$.

Cf. Barnes, [2, Sec. IV.4, Ch. VII].

Proof. This follows by Theorem 52. ▫

8.5 Comparing two spectral sequences for a change of rings

The following application is taken from [5, XVI.§6].

Let R be a commutative ring. Let $A \xrightarrow{\phi} B$ be a morphism of R-algebras. Consider the functors

$$A\text{-Mod} \xrightarrow{{}_A(B,-)} B\text{-Mod}\;, \quad (B\text{-Mod})^\circ \times B\text{-Mod} \xrightarrow{{}_B(-,=)} R\text{-Mod}\;.$$

Let X be an A-module, let Y be a B-module.

We shall compare two spectral sequences with E_2-terms $\mathrm{Ext}^i_B(Y, \mathrm{Ext}^j_A(B,X))$, converging to $\mathrm{Ext}^{i+j}_A(Y,X)$. If one views $X{\Uparrow}_A^B := {}_A(B,X)$ as a way to induce from A-Mod to B-Mod, this measures the failure of the Eckmann-Shapiro-type formula $\mathrm{Ext}^i_B(Y, X{\Uparrow}_A^B) \overset{?}{\simeq} \mathrm{Ext}^i_A(Y,X)$, which holds if B is projective over A.

Let $I \in \operatorname{Ob} \mathrm{C}^{[0}(A\text{-Mod})$ be an injective resolution of X. Let $P \in \operatorname{Ob} \mathrm{C}^{[0}(B\text{-Mod})$ be a projective resolution of Y.

Proposition 55 *The proper spectral sequences*

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{{}_A(B,-),\ {}_B(Y,-)}(X) \qquad \textit{and} \qquad \dot{\mathrm{E}}_{\mathrm{I}}\big({}_B(P_-,\ {}_A(B, I^=))\big)$$

are isomorphic (in $[\![\dot{\mathbf{Z}}^{\#\#}_\infty, R\text{-Mod}]\!]$*).*

Proof. To apply Theorem 34, if suffices to remark that for each injective A-module I', the B-module ${}_A(B,I')$ is injective, and thus ${}_B(Y,-)$-acyclic. ▫

Remark 56 The functor ${}_A(B,-)$ can be replaced by ${}_A(M,-)$, where M is an A-B-bimodule that is flat over B.

8.6 Comparing two spectral sequences for Ext and ⊗

Let R be a commutative ring. Let S be a ring. Let A be an R-algebra. Let M be an R-S-bimodule. Let X and X' be A-modules. Assume that X is flat over R. Assume that $\mathrm{Ext}^i_R(M,X') \simeq 0$ for $i \geq 1$.

Example 57 Let T be a discrete valuation ring, with maximal ideal generated by t. Let $R = T/t^\ell$ for some $\ell \geq 1$. Let $S = T/t^k$, where $1 \leq k \leq \ell$. Let G be a finite group, and let $A = RG$. Let $M = S$. Let X and X' be RG-modules that are both finitely generated and free over R.

Consider the functors

$$(A\text{-Mod})^\circ \times A\text{-Mod} \xrightarrow{{}_A(-,=)} R\text{-Mod} \xrightarrow{{}_R(M,-)} S\text{-Mod}$$

Proposition 58 *The proper Grothendieck spectral sequences*

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{{}_A(X,-),\ {}_R(M,-)}(X') \qquad \textit{and} \qquad \dot{\mathrm{E}}^{\mathrm{Gr}}_{{}_A(-,X'),\ {}_R(M,-)}(X)$$

are isomorphic (in $[\![\dot{\mathbf{Z}}^{\#\#}_{\infty},\, S\text{-Mod}]\!]$*).*

Both have E_2-terms $\mathrm{Ext}^i_R\big(M,\, \mathrm{Ext}^j_A(X,X')\big)$, and both converge to $\mathrm{Ext}^{i+j}_A(X \otimes_R M, X')$. In particular, in the situation of Example 57, both have E_2-terms $\mathrm{Ext}^i_R\big(S,\, \mathrm{Ext}^j_{RG}(X,X')\big)$ and converge to $\mathrm{Ext}^{i+j}_{RG}(X/t^k, X')$.

Proof of Proposition 58. To apply Theorem 31, we comment on the conditions in §5.1.

(c) Given a projective A-module P, we want to show that the R-module ${}_A(P,X')$ is ${}_R(M,-)$-acyclic. We may assume that $P = A$, which is to be viewed as an A-R-bimodule. Now, we have $\mathrm{Ext}^i_R\big(M,\, {}_A(A,X')\big) \simeq \mathrm{Ext}^i_R(M,X') \simeq 0$ for $i \geq 1$ by assumption.

(c′) Given an injective A-module I', the R-module ${}_A(X,I')$ is injective since X is flat over R by assumption. ▫

8.7 Comparing two spectral sequences for $\mathcal{E}xt$ of sheaves

Let $T \xrightarrow{f} S$ be a flat morphism of ringed spaces, i.e. suppose that

$$\mathcal{O}_T \otimes_{f^{-1}\mathcal{O}_S} - \;:\; f^{-1}\mathcal{O}_S\text{-Mod} \longrightarrow \mathcal{O}_T\text{-Mod}$$

is exact. Consequently, $f^* : \mathcal{O}_S\text{-Mod} \longrightarrow \mathcal{O}_T\text{-Mod}$ is exact.
Given $\mathcal{O}_S$-modules $\mathcal{F}$ and $\mathcal{F}'$, we abbreviate ${}_{\mathcal{O}_S}(\mathcal{F},\mathcal{F}') := \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{F},\mathcal{F}') \in \mathrm{Ob}\, R\text{-Mod}$ and ${}_{\mathcal{O}_S}(\!(\mathcal{F},\mathcal{F}')\!) := \mathcal{H}om_{\mathcal{O}_S}(\mathcal{F},\mathcal{F}') \in \mathrm{Ob}\,\mathcal{O}_S\text{-Mod}$.
Let $\mathcal{F}$ be an $\mathcal{O}_S$-module that has a locally free resolution $\mathcal{B} \in \mathrm{Ob}\,\mathrm{C}(\mathcal{O}_S\text{-Mod})$; cf. [9, Prop. III.6.5]. Let $\mathcal{G} \in \mathrm{Ob}\,\mathcal{O}_T\text{-Mod}$. Let $\mathcal{A} \in \mathrm{Ob}\,\mathrm{C}^{[0}(\mathcal{O}_T\text{-Mod})$ be an injective resolution of $\mathcal{G}$.
Consider the functors

$$\mathcal{O}_T\text{-Mod} \xrightarrow{f_*} \mathcal{O}_S\text{-Mod}\,, \quad (\mathcal{O}_S\text{-Mod})^{\circ} \times \mathcal{O}_S\text{-Mod} \xrightarrow{\ {}_{\mathcal{O}_S}(\!(-,=)\!)\ } \mathcal{O}_S\text{-Mod}\ .$$

Proposition 59 *The proper spectral sequences*

$$\dot{\mathrm{E}}^{\mathrm{Gr}}_{f_*,\ {}_{\mathcal{O}_S}(\!(\mathcal{F},-)\!)}(\mathcal{G}) \qquad \textit{and} \qquad \dot{\mathrm{E}}_{\mathrm{I}}\big(\,{}_{\mathcal{O}_S}(\!(\mathcal{B}_-, f_*\mathcal{A}^=)\!)\big)$$

are isomorphic (in $[\![\dot{\mathbf{Z}}^{\#\#}_{\infty},\, \mathcal{O}_S\text{-Mod}]\!]$*).*

In particular, both spectral sequences have E_2-terms $\mathcal{E}xt^i_{\mathcal{O}_S}\big(\mathcal{F},\, (\mathrm{R}^j f_*)(\mathcal{G})\big)$ and converge to $(\mathrm{R}^{i+j}\mathrm{I\!\Gamma}_{\mathcal{F}})(\mathcal{G})$, where $\mathrm{I\!\Gamma}_{\mathcal{F}}(-) := {}_{\mathcal{O}_S}(\!(\mathcal{F}, f_*(-))\!) \simeq f_*\, {}_{\mathcal{O}_T}(\!(f^*\mathcal{F},-)\!)$. For example, if $S = \{*\}$ is a one-point-space and if we write $R := \mathcal{O}_S(S)$, then we can identify $\mathcal{O}_S\text{-Mod} = R\text{-Mod}$. If, in this case, $\mathcal{F} = R/rR$ for some $r \in R$, then $\mathrm{I\!\Gamma}_{R/rR}(\mathcal{G}) \simeq \Gamma(T,\mathcal{G})[r] := \{g \in \mathcal{G}(T) \;:\; rg = 0\}$.

Proof of Proposition 59. To apply Theorem 34, we comment on the conditions in §6.1.

(c) Since f_* maps injective $\mathcal{O}_T$-modules to injective $\mathcal{O}_S$-modules by flatness of $T \xrightarrow{f} S$, the complex $\mathcal{A}$ is an $\big(f_*,\ {}_{\mathcal{O}_S}(\!(\mathcal{F},-)\!)\big)$-acyclic resolution of $\mathcal{G}$.

(e) If $\mathcal{I}$ is an injective $\mathcal{O}_S$-module and $U \subseteq S$ is an open subset, then $\mathcal{I}|_U$ is an injective $\mathcal{O}_U$-module; cf. [9, Lem. III.6.1]. Hence ${}_{\mathcal{O}_S}(\!(-,\mathcal{I})\!)$ turns a short exact sequence of $\mathcal{O}_S$-modules into a sequence that is short exact as a sequence of abelian presheaves, and hence a fortiori short exact as a sequence of $\mathcal{O}_S$-modules. In other words, the functor ${}_{\mathcal{O}_S}(\!(-,\mathcal{I})\!)$ is exact. $\square$

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Arithmetic of Hermitian Forms

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Abstract. We investigate the following two problems on a hermitian form Φ over an algebraic number field: (1) classification of Φ over the ring of algebraic integers; (2) hermitian Diophantine equations. The same types of problems for quadratic forms were treated in the author's previous articles. Here we discuss the hermitian case. Problem (2) concerns an equation $\xi\Phi \cdot {}^t\xi^\rho = \Psi$, where Φ and Ψ represent hermitian forms. We connect the number of such ξ modulo a group of units with the class number and mass of the unitary group of a form Θ such that $\Phi \approx \Psi \oplus \Theta$.

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Introduction

To explain Problems (1) and (2) of the abstract, we take a quadratic extension K of an algebraic number field F, a vector space V over K of dimension n, and a nondegenerate hermitian form $\varphi : V \times V \to K$ with respect to the nontrivial automorphism ρ of K over F. We denote by $d_0(\varphi)$ the coset of $F^\times/N_{K/F}(K^\times)$ represented by $(-1)^{n(n-1)/2}\det(\varphi)$. It is classically known that n, $d_0(\varphi)$, and the indices of φ at certain archimedean primes of F, satisfying a natural consistency condition, determine the isomorphism class of (V, φ), and vice versa. This classification does not answer, however, the question of classification over the ring of integers. To be precise, let $\mathfrak{r}$ denote the ring of algebraic integers in K and $\mathfrak{g} = F \cap \mathfrak{r}$; let $\mathfrak{d}$ be the different of K relative to F. We put

$$\mathfrak{H}_n = \{\Phi \in GL_n(K) \mid \Phi = {}^t\Phi^\rho\}. \tag{0.1}$$

We call a matrix $\Phi = (\varphi_{ij}) \in \mathfrak{H}_n$ *semi-integral* if $\varphi_{ij} \in \mathfrak{d}^{-1}$ and $\varphi_{ii} \in \mathfrak{g}$ for every i and j, which means that $\sum_{i,j} \varphi_{ij} x_i x_j^\rho \in \mathfrak{g}$ for every $(x_i)_{i=1}^n \in \mathfrak{r}^n$. Further we call a semi-integral Φ *reduced* if the following condition is satisfied:

(R) *If* $\Phi = P\Psi \cdot {}^tP^\rho$ *with a semi-integral* Ψ *and* $P = (p_{ij}) \in GL_n(K)$, $p_{ij} \in \mathfrak{r}$, *then* $\det(P) \in \mathfrak{r}^\times$.

These definitions are natural, but cover only a special class of (V, φ), as an $\mathfrak{r}$-lattice in V may not be isomorphic to $\mathfrak{r}^n$. In order to classify all $\mathfrak{g}$-valued hermitian forms, we have to define the genus of a form relative to an isomorphism class of lattices, and study its connection with the isomorphism class of (V, φ). These are nontrivial, and will be treated in §§2.4 and 2.5. We are then able to classify all the genera of $\mathfrak{g}$-valued hermitian forms in terms of matrices (Theorems 2.10 and 2.11). The results can be presented in simpler forms if K is a real or an imaginary quadratic field of odd class number, in which case the above definitions cover all hermitian spaces. Let d be the discriminant of such a K; then $K = \mathbf{Q}(\sqrt{d}\,)$. For a semi-integral Φ with entries in K, put $s(\Phi) = p - q$ when $d < 0$ and Φ as a complex hermitian matrix has p positive and q negative eigenvalues; we do not define $s(\Phi)$ if $d > 0$. Let $\mathfrak{H}_n^0$ be the set of all reduced semi-integral elements of $\mathfrak{H}_n$. Then we can prove (Theorem 2.14):

(A) *Let three integers* n, σ, *and* e *be given as follows:* $0 < n \in 2\mathbf{Z}$, $\sigma \in 2\mathbf{Z}$, $|\sigma| \le n$; $\sigma = 0$ *if* $d > 0$; e *is positive and squarefree. Let* r *be the number of prime factors of* e. *Suppose that* $\sigma - 2r \in 4\mathbf{Z}$ *and no prime factor of* e *splits in* K. *Then there exists an element* Φ *of* $\mathfrak{H}_n^0$ *such that*

$$\det(\sqrt{d}\,\Phi) = (-1)^{\sigma/2}e \quad \textit{and} \quad s(\Phi) = \sigma \qquad \textit{if } d < 0,$$
$$\det(\sqrt{d}\,\Phi) = \tau e \quad \textit{with} \quad \tau = 1 \textit{ or } -1 \qquad \textit{if } d > 0.$$

Moreover, every element of $\mathfrak{H}_n^0$ *is of this type. Its genus is determined by* (σ, e) *if* $d < 0$, *and by* e *if* $d > 0$. *If* $d > 0$ *and* $-1 \in N_{K/\mathbf{Q}}(K^\times)$, *then both* e *and* $-e$ *can occur as* $\det(\sqrt{d}\,\Phi)$ *for* Φ *in the same genus. If* $d > 0$ *and* $-1 \notin N_{K/\mathbf{Q}}(K^\times)$, *then* τ *is uniquely determined by the condition that a prime number* p *divides* e *if and only if* $\tau e \notin N_{K/\mathbf{Q}}(K_p^\times)$, *where* $K_p = K \otimes_\mathbf{Q} \mathbf{Q}_p$.

This concerns the case of even n. We have similar but somewhat different results for odd n (Theorem 2.15). In fact we discussed in [S5] and [S6] semi-integral and reduced quadratic forms and obtained results of the same type. If K is imaginary, the hermitian case is almost parallel to the case of quadratic forms over $\mathbf{Q}$, but the theory for real K is more subtle, as can be seen from the above statement.

Let us now turn to the second problem. Before explaining the principal results, let us first discuss natural problems which are more basic and which must be settled before investigating the main question. Given (V, φ) as before, let $U^\varphi(V)$ and $SU^\varphi(V)$ denote the unitary group and the special unitary group of φ, defined as subgroups of $GL(V, K)$. Take an $\mathfrak{r}$-lattice L in V and put

$$(0.2) \qquad \Gamma(L) = \{\alpha \in U^\varphi(V) \mid L\alpha = L\}, \qquad \Gamma^1(L) = \Gamma(L) \cap SU^\varphi(V).$$

Then we ask, for a fixed $q \in F^\times$, whether the set $\{h \in V \mid \varphi[h] = q\}$ modulo $\Gamma^1(L)$ is a finite set. A similar question can be asked by replacing F, K, (V, φ) and L by their localizations at a nonarchimedean prime and by defining an obvious analogue of $\Gamma^1(L)$. We will prove that the answer is affirmative in both global and local cases, provided $n > 1$ (Theorems 3.3 and 4.2). The same is true for the problem about the solutions ξ of the equation $\xi\Phi \cdot {}^t\xi^\rho = \Psi$, where Ψ is of size m, and ξ belongs to an $\mathfrak{r}$-lattice in the space of $(m \times n)$-matrices with entries in K, where m is a positive integer $< n$ (Theorems 5.2 and 5.3). We already proved in [S3] the analogues of these facts for quadratic forms and orthogonal groups.

In order to go beyond the mere finiteness, we consider the adelizations $U^\varphi(V)_\mathbf{A}$ and $SU^\varphi(V)_\mathbf{A}$, and define their open subgroups C and C^1 by

$$C = \{\gamma \in U^\varphi(V)_\mathbf{A} \mid L\gamma = L\}, \quad C^1 = C \cap SU^\varphi(V)_\mathbf{A}, \tag{0.3}$$

where L is a fixed $\mathfrak{r}$-lattice in V. Given two solutions ξ_0 and ξ_1 of the equation $\xi\Phi \cdot {}^t\xi^\rho = \Psi$, we say that they belong to the same *genus* (with respect to C) if $\xi_0\gamma_v = \xi_1$ for every nonarchimedean prime v with an element $(\gamma_v)_v \in C$. Naturally they are said to belong to the same *class* if $\xi_0\gamma = \xi_1$ with $\gamma \in \Gamma(L)$. Now to explain our principal ideas in the simplest case, put $G = U^\varphi(V)$ and $H = \{\alpha \in G \mid \xi_0\alpha = \xi_0\}$; also assume for the moment that $G_\mathbf{A} = GC$. Then there is a bijection of $H\backslash H_\mathbf{A}/(H_\mathbf{A} \cap C)$ onto the set of classes in the genus of ξ_0, and so

$$\#\{H\backslash H_\mathbf{A}/(H_\mathbf{A} \cap C)\} = \text{the number of classes in the genus of } \xi_0. \tag{B}$$

Here and henceforth $\#\{X\}$ denotes the number of elements in a set X. If $G_\mathbf{A} \neq GC$, the right-hand side becomes a finite sum of the class numbers of several genera (Theorem 5.4). Since the left-hand side is *the class number* of the unitary group H with respect to $H_\mathbf{A} \cap C$, equality (B) connects it to the solutions ξ of $\xi\Phi \cdot {}^t\xi^\rho = \Psi$.

If $m = 1$, the results can be stated in a more transparent way. Returning to the hermitian form $\varphi : V \times V \to K$, put $\varphi[h] = \varphi(h, h)$ for $h \in V$. Then the equation $\xi\Phi \cdot {}^t\xi^\rho = \Psi$ can be written $\varphi[h] = q$ with $h \in V$ and $q \in F^\times$; thus h and q replace ξ and Ψ. Given a fractional ideal $\mathfrak{b}$ in K, put

$$L[q, \mathfrak{b}] = \{h \in V \mid \varphi[h] = q, \varphi(h, L) = \mathfrak{b}\}. \tag{0.4}$$

We call L *integral* if $\varphi[x] \in \mathfrak{g}$ for every $x \in L$ and call L *maximal* if it is maximal among the integral $\mathfrak{r}$-lattices. The point of considering $L[q, \mathfrak{b}]$ is that $L[q, \mathfrak{b}]$, *if nonempty, consists of a single genus with respect to* C in the above sense. This is clearly a result of local nature; unfortunately, its proof given in Section 3 is not short. In this case $H = U^\varphi(W)$, where W is the orthogonal complement of Kh in V. Now we can prove (Theorem 4.4, Corollary 5.8):

(C) *For every* $y \in G_{\mathbf{A}}$, *there is a bijection of* $H\backslash(H_{\mathbf{A}} \cap GyC)/(H_{\mathbf{A}} \cap C)$ *onto* $(Ly^{-1})[q,\, \mathfrak{b}]/\varDelta_y$, *where* $\varDelta_y = G \cap yCy^{-1}$.

(D) *Take* $\{y_i\}_{i\in I} \subset G_{\mathbf{A}}$ *so that* $G_{\mathbf{A}} = \bigsqcup_{i\in I} Gy_iC$ *and put* $L_i = Ly_i^{-1}$. *Then*

$$\sum_{i\in I} \#\bigl\{L_i[q,\, \mathfrak{b}]/\varGamma(L_i)\bigr\} = \#\bigl\{H\backslash H_{\mathbf{A}}/(H_{\mathbf{A}} \cap C)\bigr\}.$$

(E) *If the archimedean factor of* $G_{\mathbf{A}}$ *is compact, then*

$$\sum_{i\in I} \#\bigl\{L_i[q,\, \mathfrak{b}]\bigr\}/\#\bigl\{\varGamma_i\bigr\} = \mathfrak{m}(H,\, H_{\mathbf{A}} \cap C),$$

where the right-hand side is the mass of H *with respect to the open subgroup* $H_{\mathbf{A}} \cap C$ *of* $H_{\mathbf{A}}$ *in the sense of* [S2].

These assertions are true with $SU^{\varphi}(V)$, $SU^{\varphi}(W)$ and C^1 in place of G, H, and C, if $\dim(V) > 2$ and we impose a certain condition on $(q,\, \mathfrak{b})$. Notice that (E) gives the mass of H by means of the number of solutions h of $\varphi[h] = q$ under the condition $\varphi(h,\, L) = \mathfrak{b}$, while (D) gives the class number of H.

In our recent book [S3, Chapter III] we developed a theory of a Diophantine equation $\varphi[h] = q$ for a quadratic form φ defined on a vector space over an algebraic number field. The principal result is that to each "primitive solution" h of this equation for a fixed q, considered modulo the group of units $\varGamma$, one can associate a "class" of lattices with respect to the orthogonal group H of the restriction of φ to a subspace of codimension 1. Consequently the class number of H equals the number of such h modulo $\varGamma$. This includes as a special case the result of Gauss that the number of primitive representations of q as the sum of three squares equals an elementary factor times the class number of primitive binary quadratic forms of discriminant $-q$. Also, formulas of type (B) and (E) were proved in [S3] and [S6] for quadratic forms. The reader is referred to [S7] for some more historical and technical comments on this subject. Now (B), (C), (D), and (E) are hermitian analogues of these results. In order to develop the theory for hermitian forms, we are naturally guided by the formulation in the case of quadratic forms, but we need new ideas and technique, and it is wrong to say that we can do things "in the same way." This is especially so when we consider the problem with respect to the special unitary group instead of the unitary group. Thus there are two theories with respect to these two types of groups, and one, that for the special unitary group, is more complex than the other, and in a sense more interesting.

1. Generalities on hermitian forms and unitary groups

1.1. For an associative ring A with an identity element we denote by $A^{\times}$ the group of all invertible elements of A, and for positive integers m and n we denote by A_n^m the A-module of all $(m \times n)$-matrices with entries in A. We put $M_n(A) = A_n^n$ when we view it as a ring, and denote by 1_n its identity element.

We take a basic field F and a couple (K, ρ) consisting of an F-algebra K and a nontrivial F-linear automorphism ρ of K belonging to the following two types:

(I) K is a separable quadratic extension of F and ρ generates $\mathrm{Gal}(K/F)$;
(II) $K = F \times F$ and $(x, y)^\rho = (y, x)$ for $(x, y) \in F \times F$.

In our later discussion, K of type (II) will appear as a localization of a global K of type (I). For a matrix $\alpha = (a_{ij})$ with entries in K we denote by ${}^t\alpha$ the transpose of α, and put $\alpha^\rho = (a_{ij}^\rho)$ and $\alpha^* = {}^t\alpha^\rho$; we put also $\alpha^{-\rho} = (\alpha^\rho)^{-1}$ when α is invertible. For a subring S of K we write $\alpha \prec S$ if all the entries of α are contained in S. Given a left K-module V, we denote by $\mathrm{End}(V, K)$ the ring of all K-linear endomorphisms of V and put $GL(V, K) = \mathrm{End}(V, K)^\times$. We let $\mathrm{End}(V, K)$ act on V on the right; namely we denote by $w\alpha$ the image of $w \in V$ under $\alpha \in \mathrm{End}(V, K)$.

Let V be a left K-module isomorphic to K_n^1; we put then $n = \dim(V)$. By a hermitian space we mean a structure (V, φ), where φ is a hermitian form on V, that is, an F-bilinear map $\varphi : V \times V \to K$ such that

$$\varphi(x, y)^\rho = \varphi(y, x), \tag{1.1}$$

$$\varphi(ax, by) = ab^\rho\varphi(x, y) \quad \textit{for every} \quad a, b \in K. \tag{1.2}$$

Whenever we speak of a hermitian space (V, φ), we assume that φ is nondegenerate, and put $\varphi[x] = \varphi(x, x)$ for $x \in V$. We define groups $U^\varphi(V)$ and $SU^\varphi(V)$ by

$$U^\varphi = U^\varphi(V) = \{\alpha \in GL(V, K) \mid \varphi[x\alpha] = \varphi[x] \quad \text{for every} \ x \in V\}, \tag{1.3a}$$

$$SU^\varphi = SU^\varphi(V) = \{\alpha \in U^\varphi(V) \mid \det(\alpha) = 1\}. \tag{1.3b}$$

For every free K-submodule X of V on which φ is nondegenerate, we put

$$X^\perp = \{y \in V \mid \varphi(y, X) = 0\}, \tag{1.4}$$

and define $U^\varphi(X)$ and $SU^\varphi(X)$ by (1.3a, b) with X in place of V; namely we use φ for its restriction to X. We always identify $U^\varphi(X)$ with the subgroup of $U^\varphi(V)$ consisting of the elements α such that $y\alpha = y$ for every $y \in X^\perp$. Similarly we view $SU^\varphi(X)$ as a subgroup of $SU^\varphi(V)$.

Let h be an element of V such that $\varphi[h] \neq 0$. Then

$$\{x \in V \mid \varphi[x] = \varphi[h]\} = \begin{cases} h \cdot U^\varphi & \text{if} \quad \dim(V) = 1, \\ h \cdot SU^\varphi & \text{if} \quad \dim(V) > 1. \end{cases} \tag{1.5}$$

This follows easily from the generalized Witt theorem; see [S2, Lemma 1.3], for example. The case $K = F \times F$ is not included in that theorem, but the structure of (V, φ) for such a K is determined by $\dim(V)$ as shown in [S2, §2.13], and so the fact corresponding to the Witt theorem is trivially true; see also §1.8 below.

1.2. Let φ_0 be the matrix that represents φ with respect to a K-basis of V; then we denote by $d_0(V, \varphi)$ the element of $F^\times/N_{K/F}(K^\times)$ represented by $(-1)^{n/2}\det(\varphi_0)$ or $(-1)^{(n-1)/2}\det(\varphi_0)$ according as n is even or odd. This does not depend on the choice of a K-basis of V. We denote $d_0(V, \varphi)$ simply by $d_0(V)$ or $d_0(\varphi)$ when there is no fear of confusion.

Given $s \in F^\times$, we denote by $\{K, s\}$ the quaternion algebra B over F in which K is embedded and which is given by

$$B = K + K\omega, \quad \omega^2 = s, \quad \omega a = a^\rho \omega \text{ for every } a \in K. \tag{1.6}$$

Since B is determined by K and $sN_{K/F}(K^\times)$, for a coset $\varepsilon \in F^\times/N_{K/F}(K^\times)$ we denote by $\{K, \varepsilon\}$ the algebra $\{K, s\}$ with any $s \in \varepsilon$. In particular, we can associate with (V, φ) a quaternion algebra $\{K, d_0(\varphi)\}$.

Lemma 1.3. *Given (V, φ) as in §1.1, suppose $\dim(V) = 2$ and put $B = \{K, d_0(\varphi)\}$. Then there is a ring-injection j of B into $\mathrm{End}(V, K)$ and an element ℓ of V such that $\varphi[\ell] \neq 0$, $\ell j(B) = V$, $\ell j(a) = a\ell$ for every $a \in K$, and $\varphi[\ell j(\xi)] = \varphi[\ell]\xi\xi^\iota$ for every $\xi \in B$, where ι is the main involution of B. Moreover, $\mathrm{Tr}_{K/F}\big(\varphi(\ell\alpha, \ell\beta)\big) = \varphi[\ell]\mathrm{Tr}_{B/F}(\alpha\beta^\iota)$ for every $\alpha, \beta \in B$, and*

$$U^\varphi(V) = \big\{z^{-1}\alpha \,\big|\, z \in K^\times,\, \alpha \in B^\times,\, zz^\rho = \alpha\alpha^\iota\big\}, \tag{1.7a}$$

$$SU^\varphi(V) = \big\{\alpha \in B^\times \,\big|\, \alpha\alpha^\iota = 1\big\}, \tag{1.7b}$$

where we identify α with $j(\alpha)$ for $\alpha \in B$.

Proof. Identify V with K_2^1 so that $\varphi(x, y) = x\varphi_0 y^*$ for $x, y \in K_2^1$ with $\varphi_0 = \mathrm{diag}[c, -cs]$, where $c, s \in F^\times$. Then $d_0(\varphi) = sN_{K/F}(K^\times)$, and so B is given by (1.6). Define $j : B \to M_2(K)$ by $j(a+b\omega) = \begin{bmatrix} a & b \\ sb^\rho & a^\rho \end{bmatrix}$ and put $\ell = (1, 0)$. Then it is an easy exercise to verify all the statements; cf. [S2, Lemmas 4.3, 4.4, and (4.3.2)]. Notice that $j(B) = \big\{\alpha \in M_2(K) \,\big|\, \alpha^\iota\varphi_0 = \varphi_0\alpha^*\big\}$.

1.4. When K is a field, by a *weak Witt decomposition of* V we mean a direct sum decomposition of V with $2r$ elements e_i, f_i, and a subspace Z of V such that

$$V = \textstyle\sum_{i=1}^r (Ke_i + Kf_i) + Z, \qquad Z = \big(\sum_{i=1}^r (Ke_i + Kf_i)\big)^\perp, \tag{1.8a}$$

$$\varphi(e_i, e_j) = \varphi(f_i, f_j) = 0, \quad \varphi(e_i, f_j) = \delta_{ij} \quad \text{for every } i \text{ and } j. \tag{1.8b}$$

Clearly $\sum_{i=1}^r (Ke_i + Kf_i)$ is a subspace of dimension $2r$. We call this a *Witt decomposition* if $\varphi[x] \neq 0$ for every $x \in Z$, $\neq 0$, in which case we call Z a *core subspace* of (V, φ) and $\dim(Z)$ the *core dimension* of (V, φ). If ζ is the restriction of φ to Z, then clearly $d_0(\varphi) = d_0(\zeta)$.

1.5. In this paper a *global field* means an algebraic number field of finite degree, and a *local field* the completion of a global field at a nonarchimedean prime. For a global field F we denote by $\mathfrak{g}$ the ring of algebraic integers in F;

for a local F we denote by $\mathfrak{g}$ the ring of local integers in F in the standard sense. An archimedean completion of a global field will not be called a local field. In both local and global cases, by a $\mathfrak{g}$-*lattice* in a finite-dimensional vector space V over F, we mean a finitely generated $\mathfrak{g}$-submodule of V that spans V over F.

Let (K, ρ) be as in §1.1 with a local or global F. We then denote by $\mathfrak{r}$ the ring of all elements of K integral over $\mathfrak{g}$, and by $\mathfrak{d}$ the different of K relative to F. We have $\mathfrak{r} = \mathfrak{d} = \mathfrak{g} \times \mathfrak{g}$ if $K = F \times F$. By a $\mathfrak{g}$-*ideal* we mean a fractional ideal in F, and similarly by an $\mathfrak{r}$-*ideal* we mean a fractional ideal in K if K is a field. If $K = F \times F$, an $\mathfrak{r}$-*ideal* means a subset of K of the form $\mathfrak{a} \times \mathfrak{b}$ with $\mathfrak{g}$-ideals $\mathfrak{a}$ and $\mathfrak{b}$. In both local and global cases, by an $\mathfrak{r}$-*lattice* in a K-module V as in §1.1 we mean a $\mathfrak{g}$-lattice in V stable under multiplication by the elements of $\mathfrak{r}$. Given two $\mathfrak{r}$-lattices L and M in V, we denote by $[L/M]$ the $\mathfrak{r}$-ideal generated by $\det(\alpha)$ for all $\alpha \in GL(V)$ such that $L\alpha \subset M$. Thus $[L/L\alpha] = \det(\alpha)\mathfrak{r}$. In particular, if $\alpha \in U^{\varphi}$ and K is a field in the local case, then $[L/L\alpha] = \mathfrak{r}$. If $K = F \times F$, however, $[L/L\alpha] = \mathfrak{a} \times \mathfrak{a}^{-1}$ with a $\mathfrak{g}$-ideal $\mathfrak{a}$ for $\alpha \in U^{\varphi}$.

Lemma 1.6. *Two hermitian spaces* (V, φ) *and* (V', φ') *in the local case are isomorphic if and only if* $\dim(V) = \dim(V')$ *and* $d_0(\varphi) = d_0(\varphi')$.

This is well known. For the proof, see [S2, Proposition 5.3], for example.

1.7. Let (V, φ) be defined with a local or global F. For a $\mathfrak{g}$-lattice L in V we denote by $\mu(L)$ the $\mathfrak{g}$-ideal generated by $\varphi[x]$ for all $x \in L$. We call a $\mathfrak{g}$-lattice L in V *integral* if $\mu(L) \subset \mathfrak{g}$; we call an $\mathfrak{r}$-lattice L *maximal* if L is maximal among the integral $\mathfrak{r}$-lattices. (This is what we called $\mathfrak{g}$-maximal in [S2].) For basic properties of maximal lattices in V the reader is referred to [S1] or Sections 4 and 5 of [S2]. For example, we note ([S2, (4.7.1)])

$$\varphi(L, L) \subset \mathfrak{d}^{-1} \text{ if } L \text{ is integral.} \tag{1.9}$$

If $n > 1$ and L is maximal, then $\mu(L) = \mathfrak{g}$. This fact in the global case follows from the local case, which can be seen from [S2, Lemmas 5.4 and 5.6].

Given an $\mathfrak{r}$-lattice L in V, $q \in F^{\times}$, and an $\mathfrak{r}$-ideal $\mathfrak{b}$, we put

$$\widehat{L} = \{x \in V \mid \varphi(x, L) \subset \mathfrak{d}^{-1}\}, \tag{1.10a}$$

$$L[q] = \{x \in L \mid \varphi[x] = q\}, \tag{1.10b}$$

$$L[q, \mathfrak{b}] = \{x \in V \mid \varphi[x] = q,\ \varphi(x, L) = \mathfrak{b}\}. \tag{1.10c}$$

By (1.9), we have $L \subset \widehat{L}$ if L is integral. The set $L[q, \mathfrak{b}]$ is not necessarily contained in $L[q]$. If M is another $\mathfrak{r}$-lattice in V, then we easily see that $[L/M]^{\rho} = [\widehat{M}/\widehat{L}]$. If $L_1 = L\alpha$ with $\alpha \in U^{\varphi}$, then $\widehat{L}_1 = \widehat{L}\alpha$, and so $[\widehat{L}_1/L_1] = [\widehat{L}/L]$.

The notation being as in (1.8a, b), take a maximal $\mathfrak{r}$-lattice M in Z and put

$$L = \textstyle\sum_{i=1}^{r}(\mathfrak{r}e_i + \mathfrak{d}^{-1}f_i) + M. \tag{1.11}$$

Then L is maximal; see [S2, Lemma 4.9 (2)]. We can easily verify that

$$\widehat{L} = \textstyle\sum_{i=1}^{r}(\mathfrak{r}e_i + \mathfrak{d}^{-1}f_i) + \widehat{M}, \quad \widehat{M} = \bigl\{x \in Z \bigm| \varphi(x,\, M) \subset \mathfrak{d}^{-1}\bigr\}. \tag{1.12}$$

1.8. Let us now consider the case $K = F \times F$; then we define the core dimension of $(V,\, \varphi)$ to be 0. We can write an element of $V = K_n^1$ in the form $(x,\, y)$ with $x,\, y \in F_n^1$. Taking a suitable coordinate system, we can assume that

$$\varphi\bigl((x,\, y),\, (z,\, w)\bigr) = (x \cdot {}^t w,\, y \cdot {}^t z) \qquad (x,\, y,\, z,\, w \in F_n^1). \tag{1.13}$$

This is shown in [S2, §2.13]. We have then $\varphi\bigl[(x,\, y)\bigr] = x \cdot {}^t y$ and

$$U^\varphi = \bigl\{(\xi,\, \widetilde{\xi}) \bigm| \xi \in GL_n(F)\bigr\}, \quad SU^\varphi = \bigl\{(\xi,\, \widetilde{\xi}) \bigm| \xi \in SL_n(F)\bigr\}, \tag{1.14}$$

where $\widetilde{\xi} = {}^t\xi^{-1}$; also $\mathfrak{g}_n^1 \times \mathfrak{g}_n^1$ is a maximal lattice. It should be noted that if $h \in V$ and $\varphi[h] \neq 0$, then Kh is isomorphic to K.

LEMMA 1.9. *Let L be a maximal lattice in V and t the core dimension of $(V,\, \varphi)$; suppose K is a local field; put*

$$E = \bigl\{e \in \mathfrak{r}^\times \bigm| ee^\rho = 1\bigr\}, \quad E_0 = \bigl\{e^\rho/e \bigm| e \in \mathfrak{r}^\times\bigr\}, \quad E_L = \det\bigl(C(L)\bigr),$$

where $C(L) = \bigl\{\alpha \in U^\varphi(V) \bigm| L\alpha = L\bigr\}$. Then the following assertions hold:

(i) *$[E : E_0] = 1$ or 2 according as K is unramified or ramified over F.*

(ii) *$E_L = E_0$ if $t = 0$; $E_L = E$ if $t > 0$.*

This is included in [S1, Lemma 4.16 and Proposition 4.18] and [S2, Lemma 5.11].

Suppose K is a local field ramified over F; let L and M be maximal lattices in V. Then there exists an element $\alpha \in U^\varphi$ such that $M = L\alpha$ as shown in (i) of the following lemma. We then denote by $e(L/M)$ the element of E/E_0 represented by $\det(\alpha)$. This is well defined in view of (ii) above.

LEMMA 1.10. *Let L and M be maximal lattices in V in the local case, and let t be the core dimension of $(V,\, \varphi)$. Then the following assertions hold:*

(i) *There exists an element α of U^φ such that $M = L\alpha$.*

(ii) *We can take such an α from SU^φ if $t > 0$ or K is a field unramified over F.*

(iii) *Suppose K is ramified over F and $t = 0$; then $M = L\alpha$ with $\alpha \in SU^\varphi$ if and only if $e(L/M) = 1$.*

(iv) *Suppose $K = F \times F$; then $M = L\alpha$ with $\alpha \in SU^\varphi$ if and only if $[L/M] = \mathfrak{r}$.*

Proof. The first assertion is included in [S1, Propositions 3.3 and 4.13] and also in [S2, Lemmas 4.12 and 5.9]. Next, suppose K is a field; given M, take

$\alpha \in U^{\varphi}$ so that $L\alpha = M$ and put $c = \det(\alpha)$. Then $c \in E$. If K is unramified over F or $t > 0$, then Lemma 1.9 guarantees an element β of U^{φ} such that $L\beta = L$ and $\det(\beta) = c^{-1}$. Then $\beta\alpha \in SU^{\varphi}$ and $L\beta\alpha = M$. This proves (ii). Assertion (iii) can be proved in a similar way. Assertion (iv) is included in [S1, Proposition 3.3].

1.11. Let us now consider the case with $n = 2$ and a local F. Using the symbols B, j, and ℓ of Lemma 1.3, we first observe that

$$B \cong M_2(F) \iff t = 0 \iff 1 \in d_0(\varphi). \tag{1.15}$$

Let $\mathfrak{O}$ be a maximal order in B containing $\mathfrak{r}$. We can find an element $\gamma \in B^{\times}$ such that $\varphi[\ell]^{-1} = \gamma\gamma^{\iota}$. Put $M = \ell j(\mathfrak{O}\gamma)$ and $\widehat{\mathfrak{O}} = \bigl\{\alpha \in B \bigm| \mathrm{Tr}_{B/F}(\alpha\mathfrak{O}) \subset \mathfrak{g}\bigr\}$. Identifying B with $j(B)$, for $\alpha \in B$ we see that

$$\begin{aligned}\ell\alpha\gamma \in \widehat{M} \iff \varphi(\ell\alpha\gamma,\, M) \subset \mathfrak{d}^{-1} &\iff \mathrm{Tr}_{K/F}\bigl(\mathfrak{r}\varphi(\ell\alpha\gamma,\, M)\bigr) \subset \mathfrak{g}\\ &\iff \mathrm{Tr}_{B/F}(\alpha\mathfrak{O}) \subset \mathfrak{g} \iff \alpha \in \widehat{\mathfrak{O}}.\end{aligned}$$

Thus $\widehat{M} = \ell j(\widehat{\mathfrak{O}}\gamma)$. If B is not a division algebra, then $\widehat{\mathfrak{O}} = \mathfrak{O}$, so that $\widehat{M} = M$, which means that M is maximal. Suppose B is a division algebra; then $\mathfrak{O} = \bigl\{\alpha \in B \bigm| \alpha\alpha^{\iota} \in \mathfrak{g}\bigr\}$ as noted in [S3, Theorem 5.13], and so $M = \bigl\{x \in V \bigm| \varphi[x] \in \mathfrak{g}\bigr\}$, which is a unique maximal lattice in V by [S2, Lemma 5.4]. Since $\widehat{\mathfrak{O}} = \mathfrak{P}^{-1}$ with the maximal ideal $\mathfrak{P}$ of $\mathfrak{O}$, we have $\widehat{M} = \ell j(\mathfrak{P}^{-1}\gamma)$. Thus $[\widehat{M}/M] = \mathfrak{p}\mathfrak{r}$ with the maximal ideal $\mathfrak{p}$ of $\mathfrak{g}$.

2. Classification of hermitian forms over a global field

2.1. Throughout this section we assume that F is a global field and K is a quadratic extension of F. We denote by $\mathbf{a}$ and $\mathbf{h}$ the sets of archimedean primes and nonarchimedean primes of F respectively, and put $\mathbf{v} = \mathbf{a} \cup \mathbf{h}$. Given an algebraic group G defined over F, we define G_v for each $v \in \mathbf{v}$ and the adelization $G_{\mathbf{A}}$ as usual, and view G and G_v as subgroups of $G_{\mathbf{A}}$. We then denote by $G_{\mathbf{a}}$ and $G_{\mathbf{h}}$ the archimedean and nonarchimedean factors of $G_{\mathbf{A}}$, respectively. In particular, the adelization of the multiplicative group $F^{\times}$ is denoted by $F_{\mathbf{A}}^{\times}$, which is the idele group of F. For $x \in G_{\mathbf{A}}$ and $v \in \mathbf{v}$ we denote by x_v the v-component of x.

Given $(V,\, \varphi)$ over F, for each $v \in \mathbf{v}$ we can define the v-localization $(V,\, \varphi)_v = (V_v,\, \varphi_v)$ with $\varphi_v : V_v \times V_v \to K_v$ in a natural way. For $v \in \mathbf{h}$ let t_v be the core dimension of $(V,\, \varphi)_v$. Since $x \mapsto \varphi_v[x]$ for $x \in V_v$ can be viewed as an F_v-valued quadratic form, we have $2t_v \le 4$ by a well known principle, and so $t_v \le 2$. Let $\mathbf{r}_0$ denote the set of all real archimedean primes of F that do not split in K. If $v \in \mathbf{a}$ and $v \notin \mathbf{r}_0$, then there is only one isomorphism class of $(V,\, \varphi)_v$ for each n. For each fixed $v \in \mathbf{r}_0$ we have a pair of nonnegative integers $(p_v,\, q_v)$ such that φ_v is represented by $\mathrm{diag}[1_{p_v},\, -1_{q_v}]$ when F_v and K_v are identified with $\mathbf{R}$ and $\mathbf{C}$. We put then $s_v(\varphi) = p_v - q_v$, and call $s_v(\varphi)$ the *index* of φ at v. Clearly $|s_v(\varphi)|$ is the core dimension of φ_v, $s_v(\varphi) - n \in 2\mathbf{Z}$, and $|s_v(\varphi)| \le n$; also, n and $s_v(\varphi)$ determine (p_v, q_v), and vice versa.

For an $\mathfrak{r}$-lattice L in V and $v \in \mathbf{h}$ we denote by L_v the $\mathfrak{r}_v$-linear span of L in V_v. Also, for $\xi \in GL(V, K)_{\mathbf{A}}$ we denote by $L\xi$ the lattice in V such that $(L\xi)_v = L_v\xi_v$ for every $v \in \mathbf{h}$. By the $U^\varphi(V)$-genus (resp. $SU^\varphi(V)$*-genus*) of L we understand the set of all lattices of the form $L\xi$ with $\xi \in U^\varphi(V)_{\mathbf{A}}$ (resp. $\xi \in SU^\varphi(V)_{\mathbf{A}}$). Also, by the $U^\varphi(V)$-class (resp. $SU^\varphi(V)$*-class*) of L we understand the set of all lattices of the form $L\alpha$ with $\alpha \in U^\varphi(V)$ (resp. $\alpha \in SU^\varphi(V)$).

The classification of (V, φ) over a number field was done by Landherr in [L]. We formulate the results in the form that suits our later purposes, and give a proof for the reader's convenience. To be precise, we are going to show that the isomorphism classes of hermitian spaces correspond bijectively to the sets of data consisting of the following objects:

(2.1a) $0 < n \in \mathbf{Z}$; $\varepsilon \in F^\times$; *an integer* σ_v, *given for each* $v \in \mathbf{r}_0$, *such that* $|\sigma_v| \le n$ *and* $\sigma_v - n \in 2\mathbf{Z}$.

We look for (V, φ) such that $\dim(V) = n$, $d_0(\varphi)$ is represented by ε, and $s_v(\varphi) = \sigma_v$ for every $v \in \mathbf{r}_0$. Clearly the following condition is necessary:

(2.1b) $(-1)^{\sigma_v/2}\varepsilon > 0$ *for every* $v \in \mathbf{r}_0$ *if* $n \in 2\mathbf{Z}$ *and* $(-1)^{(\sigma_v-1)/2}\varepsilon > 0$ *for every* $v \in \mathbf{r}_0$ *if* $n - 1 \in 2\mathbf{Z}$.

Theorem 2.2. (i) *The isomorphism class of* (V, φ) *is determined by* n, $\{\sigma_v\}$, *and* $d_0(\varphi)$.

(ii) *Given* n, ε, *and* $\{\sigma_v\}$ *satisfying* (2.1a, b), *there exists a hermitian space* (V, φ) *such that* $\dim(V) = n$, $\varepsilon \in d_0(\varphi)$, *and* $s_v(\varphi) = \sigma_v$ *for every* $v \in \mathbf{r}_0$.

Proof. Clearly n and $\{\sigma_v\}$ determine $(V, \varphi)_v$ for every $v \in \mathbf{a}$, and n and $d_0(\varphi)$ determine $(V, \varphi)_v$ for every $v \in \mathbf{h}$ by Lemma 1.6. Therefore we obtain (i) in view of the Hasse principle. We prove (ii) by induction on n. The case $n = 1$ is trivial, and so we assume $n > 1$. We first prove the case in which $\sigma_v \ge 0$ for every $v \in \mathbf{r}_0$. Let $\tau_v = \sigma_v - 1$. Then the set $\bigl(n-1, (-1)^{n-1}\varepsilon, \{\tau_v\}\bigr)$ satisfies (2.1a, b), and therefore by induction we can find a hermitian space (W, ψ) such that $\dim(W) = n - 1$, $\tau_v = s_v(\psi)$ for every $v \in \mathbf{r}_0$, and $(-1)^{n-1}\varepsilon \in d_0(\psi)$. Put $V = K \oplus W$ and define φ on V by $\varphi[a \oplus y] = aa^\rho + \psi[y]$ for $a \in K$ and $y \in W$. Then clearly $\varepsilon \in d_0(\varphi)$ and $s_v(\varphi) = \sigma_v$ for every $v \in \mathbf{r}_0$.

Now, given $\{\sigma_v\}$ with possibly negative σ_v, take $c \in F^\times$ so that $c < 0$ or $c > 0$ at $v \in \mathbf{r}_0$ according as $\sigma_v < 0$ or $\sigma_v \ge 0$. Then the set $\bigl(n, c^n\varepsilon, \{|\sigma_v|\}\bigr)$ satisfies (2.1a, b). Therefore we can find a hermitian space (V_1, φ_1) such that $\dim(V_1) = n$, $c^n\varepsilon \in d_0(\varphi_1)$, and $s_v(\varphi_1) = |\sigma_v|$ for every $v \in \mathbf{r}_0$. Put $\varphi = c\varphi_1$. Then $\varepsilon \in d_0(\varphi)$ and $s_v(\varphi) = \sigma_v$ for every $v \in \mathbf{r}_0$. This completes the proof.

Theorem 2.3. *Given* (V, φ), *put* $B = \{K, d_0(\varphi)\}$ *using the notation of* §*1.2. Let* $\mathfrak{e}$ *be the product of the prime ideals of* F *ramified in* B; *also let* L *be a maximal lattice in* V. *Then the following assertions hold.*

(i) $[\widehat{L}/L] = \mathfrak{e}\mathfrak{r}$ *if* n *is even.*

(ii) *When* n *is odd, put* $d_0(\varphi)\mathfrak{g} = \mathfrak{a}N_{K/F}(\mathfrak{b})$ *with an* $\mathfrak{r}$*-ideal* $\mathfrak{b}$ *and a squarefree integral* $\mathfrak{g}$*-ideal* $\mathfrak{a}$ *whose prime factors remain prime in* K. *Then* $[\widehat{L}/L] = \mathfrak{a}\mathfrak{d}$, *where* $\mathfrak{d}$ *is the different of* K *relative to* F.

Proof. For $v \in \mathbf{h}$ let t_v be the core dimension of $(V, \varphi)_v$. Suppose $n \in 2\mathbf{Z}$; then $t_v = 0$ if and only if $d_0(\varphi)$ is represented by an element of $N_{K/F}(K_v^\times)$, that is, if and only if v does not divide $\mathfrak{e}$. If n is odd, the isomorphism class of $(V, \varphi)_v$ depends on $\mathfrak{a}_v$ and $\mathfrak{d}_v$. Thus our assertions can be reduced to the question about $[\widehat{L}_v/L_v]$ for $v \in \mathbf{h}$. In fact, suppressing the subscript v, we have, in the local case,

(2.2) $[\widehat{L}/L] = \mathfrak{r}$ *if* $t = 0$; $[\widehat{L}/L] = \mathfrak{p}\mathfrak{r}$ *if* $t = 2$; $[\widehat{L}/L] = \mathfrak{d}$ *if* $t = 1$ *and* $d_0(\varphi) \cap \mathfrak{g}^\times \neq \emptyset$; $[\widehat{L}/L] = \mathfrak{p}\mathfrak{d}$ *if* $t = 1$ *and* $d_0(\varphi) \cap \mathfrak{g}^\times = \emptyset$.

Here $\mathfrak{p}$ is the maximal ideal of $\mathfrak{g}$. In view of Lemma 1.10 (i), it is sufficient to prove this for a special choice of L. If $K = F \times F$, then we can put $L = \mathfrak{g}_n^1 \times \mathfrak{g}_n^1$ as noted in §1.8, and so $\widehat{L} = L$. Thus we assume that K is a field. By (1.12), $L = \widehat{L}$ if $t = 0$. Let $M = \{x \in Z \mid \varphi[x] \in \mathfrak{g}\}$. By (1.12), $[\widehat{L}/L] = [\widehat{M}/M]$. We have seen that $[\widehat{M}/M] = \mathfrak{p}\mathfrak{r}$ in §1.11 if $t = 2$. If $t = 1$, then $M = \mathfrak{r}\ell$ with an element ℓ such that $\varphi[\ell]\mathfrak{g}$ is $\mathfrak{g}$ or $\mathfrak{p}\mathfrak{g}$. Thus $\widehat{M} = \mathfrak{d}^{-1}\varphi[\ell]^{-1}\ell$, and so $[\widehat{L}/L] = [\widehat{M}/M] = \varphi[\ell]\mathfrak{d}$, which completes the proof of (2.2). Combining the results on $[\widehat{L}_v/L_v]$ for all $v \in \mathbf{h}$, we obtain our theorem.

2.4. To illustrate Theorem 2.3 in terms of matrices, we have to define the genus and class of a hermitian matrix. We put

$$\mathfrak{G} = GL_n(K), \qquad \mathfrak{H}_n = \{\Phi \in \mathfrak{G} \mid \Phi^* = \Phi\}, \qquad L_0 = \mathfrak{r}_n^1, \tag{2.3}$$

$$E = \mathfrak{G}_{\mathbf{a}} \prod_{v \in \mathbf{h}} GL_n(\mathfrak{r}_v), \qquad E_\xi = \xi^{-1}E\xi \quad (\xi \in \mathfrak{G}_{\mathbf{A}}), \tag{2.4}$$

$$\Delta_\xi = E_\xi \cap \mathfrak{G}, \qquad \Delta_\xi^1 = E_\xi \cap SL_n(K). \tag{2.5}$$

Every $\mathfrak{r}$-lattice L in K_n^1 can be given as $L = L_0\xi$ with $\xi \in \mathfrak{G}_{\mathbf{A}}$, and $E_\xi = \{y \in \mathfrak{G}_{\mathbf{A}} \mid Ly = L\}$. We denote by $\mathfrak{H}_n(\xi)$ the set of all $\Phi \in \mathfrak{H}_n$ such that $x\Phi x^* \in \mathfrak{g}$ for every $x \in L_0\xi$. We call such a Φ *reduced* (relative to ξ) if the following condition is satisfied:

$$\Phi \in \mathfrak{H}_n(\zeta^{-1}\xi) \text{ with } \zeta \in \mathfrak{G}_{\mathbf{h}} \cap \prod_{v \in \mathbf{h}} M_n(\mathfrak{r}_v) \Longrightarrow \zeta \in E. \tag{2.6}$$

We denote by $\mathfrak{H}_n^0(\xi)$ the set of all reduced elements of $\mathfrak{H}_n(\xi)$.

We say that two elements Φ and Ψ of $\mathfrak{H}_n(\xi)$ belong to the same *genus* (relative to ξ) if there exists an element ε of E_ξ such that $\varepsilon\Phi\varepsilon^* = \Psi$; they are said to belong to the same U-*class* (resp. SU-*class*) if $\alpha\Phi\alpha^* = \Psi$ for some $\alpha \in \Delta_\xi$ (resp. $\alpha \in \Delta_\xi^1$). These depend on the choice of $L = L_0\xi$.

Given $\Phi \in \mathfrak{H}_n$, put $V = K_n^1$ and $\varphi[x] = x\Phi x^*$ for $x \in V$. Then we obtain a hermitian space (V, φ), which we denote by $[\Phi]$, and we write $U(\Phi)$ and $SU(\Phi)$ for $U^\varphi(V)$ and $SU^\varphi(V)$ as subgroups of $\mathfrak{G}$. Put $L = L_0\xi$ with $\xi \in \mathfrak{G}_\mathbf{A}$. Clearly L is integral if $\Phi \in \mathfrak{H}_n(\xi)$, in which case L is maximal if and only if $\Phi \in \mathfrak{H}_n^0(\xi)$. Thus an element of $\mathfrak{H}_n^0(\xi)$ determines a hermitian space and a maximal lattice.

2.5. To parametrize all genera of $\Phi \in \mathfrak{H}_n^0(\xi)$, we need a few more symbols:

$$\mathfrak{t} = K_\mathbf{A}^\times \cap \left(K_\mathbf{a}^\times \prod_{v \in \mathbf{h}} \mathfrak{t}_v\right), \qquad \mathfrak{t}_v = \left\{y \in K_v^\times \,\middle|\, yy^\rho \in \mathfrak{g}_v^\times\right\}, \tag{2.7}$$

$$T = \left\{x \in \mathfrak{G}_\mathbf{A} \,\middle|\, \det(x) \in \mathfrak{t}\right\}. \tag{2.8}$$

Notice that $\mathfrak{t}_v = \mathfrak{r}_v^\times \cdot \left\{y \in K_v^\times \,\middle|\, yy^\rho = 1\right\}$ for every $v \in \mathbf{h}$, and $\mathfrak{t}_v = \mathfrak{r}_v^\times$ if v does not split in K. Let $\mathfrak{I}_K$ denote the ideal group of K and $\mathfrak{I}_{K/F}^0$ the subgroup of $\mathfrak{I}_K$ generated by the ideals $\mathfrak{a}$ such that $N_{K/F}(\mathfrak{a}) = \mathfrak{g}$ and the principal ideals. Now there is a sequence of isomorphisms:

$$\mathfrak{G}_\mathbf{A}/T\mathfrak{G} \;\cong\; K_\mathbf{A}^\times/K^\times\mathfrak{t} \;\cong\; \mathfrak{I}_K/\mathfrak{I}_{K/F}^0. \tag{2.9}$$

The last isomorphism can be obtained by the map $y \mapsto y\mathfrak{r}$ for $y \in K_\mathbf{A}^\times$. As for the first isomorphism, we first note, for every $\xi \in \mathfrak{G}_\mathbf{A}$ and $\Phi \in \mathfrak{H}_n$,

$$T\mathfrak{G}\xi = E\xi U(\Phi)_\mathbf{A}\mathfrak{G} = \left\{x \in \mathfrak{G}_\mathbf{A} \,\middle|\, \det(\xi^{-1}x) \in K^\times\mathfrak{t}\right\}. \tag{2.10}$$

Clearly the last set contains the second set. Conversely, suppose $x \in \mathfrak{G}_\mathbf{A}$ and $\det(\xi^{-1}x) = by$ with $b \in K^\times$ and $y \in \mathfrak{t}$. We can find $z, w \in K_\mathbf{A}^\times$ such that $z_v \in \mathfrak{r}_v^\times$ and $w_v w_v^\rho = 1$ for every $v \in \mathbf{h}$ and $y = zw$. We can find $\varepsilon \in E$, $\alpha \in \mathfrak{G}$, and $\gamma \in U(\Phi)_\mathbf{A}$ such that $\det(\varepsilon) = z$, $\det(\alpha) = b$, and $\det(\gamma) = w$. Then $\det(x^{-1}\varepsilon\xi\gamma\alpha) = 1$. By strong approximation in $SL_n(K)$ we see that $x^{-1}\varepsilon\xi\gamma\alpha \in x^{-1}ExSL_n(K)$, and so $x^{-1}\varepsilon\xi\gamma\alpha = x^{-1}\varepsilon' x\beta$ with $\varepsilon' \in E$ and $\beta \in SL_n(K)$. Then $x = (\varepsilon')^{-1}\varepsilon\xi\gamma\alpha\beta^{-1} \in E\xi U(\Phi)_\mathbf{A}\mathfrak{G}$, which proves the last equality of (2.10). That $T\mathfrak{G}$ equals the last set of (2.10) for $\xi = 1$ can be proved in the same way. Thus $T\mathfrak{G}$ is the inverse image of $K^\times\mathfrak{t}$ under the map $x \mapsto \det(x)$, and so $T\mathfrak{G}$ is a normal subgroup of $\mathfrak{G}_\mathbf{A}$. Then we obtain the first isomorphism of (2.9) and also the first equality of (2.10) for every $\xi \in \mathfrak{G}_\mathbf{A}$.

PROPOSITION 2.6. (i) *For $\Phi, \Psi \in \mathfrak{H}_n^0(\xi)$, $\xi \in \mathfrak{G}_\mathbf{A}$, the spaces $[\Phi]$ and $[\Psi]$ are isomorphic if and only if they belong to the same genus.*

(ii) *Let X be a complete set of representatives for $\mathfrak{G}_\mathbf{A}/T\mathfrak{G}$, and for each $\xi \in \mathfrak{G}_\mathbf{A}$ let Y_ξ be a complete set of representatives for the genera of the elements of $\mathfrak{H}_n^0(\xi)$. Then the hermitian spaces $[\Phi]$ obtained from $\Phi \in Y_\xi$ for all $\xi \in X$ exhaust all isomorphism classes of n-dimensional hermitian spaces without overlapping.*

Proof. Let Φ and Ψ be elements of $\mathfrak{H}_n^0(\xi)$ belonging to the same genus. Then there exists an element $\varepsilon \in E_\xi$ such that $\varepsilon\Phi\varepsilon^* = \Psi$, and the Hasse principle guarantees an element α of $\mathfrak{G}$ such that $\Psi = \alpha\Phi\alpha^*$. Thus $[\Psi]$ is isomorphic to $[\Phi]$. Conversely, suppose $[\Phi]$ and $[\Psi]$ are isomorphic for $\Phi, \Psi \in \mathfrak{H}_n^0(\xi)$. Then $\Phi = \beta\Psi\beta^*$ for some $\beta \in \mathfrak{G}$. Now $L_0\xi$ is maximal in both $[\Phi]$ and $[\Psi]$, and $L_0\xi\beta$ is maximal in $[\Psi]$. Thus $L_0\xi\beta = L_0\xi\gamma$ with $\gamma \in U(\Psi)_{\mathbf{A}}$ by Lemma 1.10. Put $\zeta = \beta\gamma^{-1}$. Then $\zeta \in E_\xi$, and $\zeta\Psi\zeta^* = \Phi$. Therefore Ψ belongs to the genus of Φ. This proves (i). Clearly every n-dimensional hermitian space is isomorphic to $[\Psi]$ for some $\Psi \in \mathfrak{H}_n$. Take a maximal lattice L in $K_n^1 = V$ and put $L = L_0\eta$ with $\eta \in \mathfrak{G}_{\mathbf{A}}$. We have then $\eta \in T\mathfrak{G}\xi$ with some $\xi \in X$. By (2.10) we can put $\xi = \varepsilon\eta\gamma\alpha^{-1}$ with $\varepsilon \in E$, $\gamma \in U(\Psi)_{\mathbf{A}}$, and $\alpha \in \mathfrak{G}$. Put $\Phi = \alpha\Psi\alpha^*$. Then α gives an isomorphism of $[\Phi]$ onto $[\Psi]$. Now $L_0\eta\gamma$ is a maximal lattice in $[\Psi]$ and $L_0\xi\alpha = L_0\eta\gamma$, and so $L_0\xi$ is a maximal lattice in $[\Phi]$. Thus $\Phi \in \mathfrak{H}_n^0(\xi)$. By (i), Φ can be replaced by a member of Y_ξ. This shows that every (V, φ) can be obtained as described in (ii). Now suppose $[\Phi_1]$ and $[\Phi_2]$ are isomorphic for $\Phi_i \in Y_{\xi_i}$ with $\xi_1, \xi_2 \in X$. Then $\Phi_1 = \alpha\Phi_2\alpha^*$ with $\alpha \in \mathfrak{G}$. Since $L_0\xi_i$ is maximal, we have $L_0\xi_1\alpha = L_0\xi_2\zeta$ with $\zeta \in U(\Phi_2)_{\mathbf{A}}$. Then $\xi_1\alpha\zeta^{-1}\xi_2^{-1} \in E$, and so $\det(\xi_1\xi_2^{-1}) \in K^\times\mathfrak{t}$. We are taking the ξ_i from X, and therefore $\xi_1 = \xi_2$ by (2.10). By (i), Φ_2 belongs to the genus of Φ_1, and so $\Phi_2 = \Phi_1$. This completes the proof.

2.7. The connection of a class of hermitian matrices with a class of lattices is not so simple in general. Given $\Phi \in \mathfrak{H}_n^0(\xi)$ with a fixed ξ, in order to exhaust all classes in the $U(\Phi)$-genus of $L_0\xi$, we have to consider the genera of elements in $\mathfrak{H}_n^0(\xi\zeta)$ for all $\zeta \in T\mathfrak{G}/E_\xi\mathfrak{G}$. Thus $\mathfrak{H}_n^0(\xi)$ is sufficient if and only if $K^\times\mathfrak{t} = K^\times\det(E)$. We will not go into details, as we do not need the result in our later treatment.

The case of SU-class is simpler. Fix $\xi \in \mathfrak{G}$ and put $L = L_0\xi$. For $\Phi \in \mathfrak{H}_n(\xi)$ we define the *SU-genus* (relative to ξ) of Φ to be the set of all $\Psi \in \mathfrak{H}_n^0(\xi)$ such that $\Psi = \varepsilon\Phi\varepsilon^*$ with $\varepsilon \in E_\xi$ such that $\det(\varepsilon) = 1$. Clearly $\det(\Psi) = \det(\Phi)$. Given such Ψ and ε, the Hasse principle guarantees an element $\alpha \in \mathfrak{G}$ such that $\Psi = \alpha\Phi\alpha^*$. Then $\det(\alpha)\det(\alpha)^\rho = 1$. Changing α for $\alpha\gamma$ with a suitable $\gamma \in U(\Phi)$, we may assume that $\det(\alpha) = 1$. Since $L\alpha = L\varepsilon^{-1}\alpha$ and $\varepsilon^{-1}\alpha \in SU(\Phi)_{\mathbf{A}}$, we see that $L\alpha$ belongs to the $SU(\Phi)$-genus of L. We then associate the $SU(\Phi)$-class of $L\alpha$ to Ψ. We can easily verify that the set of all SU-classes in the SU-genus of Φ contained in $\mathfrak{H}_n(\xi)$ are in one-to-one correspondence with the set of $SU(\Phi)$-classes in the $SU(\Phi)$-genus of L.

2.8. Define (V, φ) by $V = K_n^1$ and $\varphi[x] = x\Phi x^*$ as above with any $\Phi \in \mathfrak{H}_n$. Put $L = L_0\xi$ with $\xi \in \mathfrak{G}_{\mathbf{A}}$. We easily see that $\widehat{L} = \mathfrak{d}^{-1}L_0(\Phi\xi^*)^{-1}$, and so

$$[\widehat{L}/L] = \det(\Phi\xi\xi^\rho)\mathfrak{d}^n \quad \text{if} \quad L = L_0\xi. \tag{2.11}$$

We need a few more symbols. First, we put $\mathfrak{d}_0 = \mathfrak{d}^2 \cap F$. For $v \in \mathbf{r}_0$ and (V, φ)

isomorphic to $[\Phi]$ with $\Phi \in \mathfrak{H}_n$ we put $s_v(\Phi) = s_v(\varphi)$ and $d_0(\Phi) = d_0(\varphi)$.

Lemma 2.9. *Let B be a quaternion algebra over F and K a quadratic extension of F contained in B; let $\mathfrak{r}$ be the maximal order of K and $\mathfrak{O}$ a maximal order in B containing $\mathfrak{r}$; further let $\mathfrak{e}$ be the product of the prime ideals in F ramified in B and $\mathfrak{d}$ the different of K relative to F. Then there exists a $\mathfrak{g}$-ideal $\mathfrak{a}$ such that $\mathfrak{O}$ is isomorphic as a left $\mathfrak{r}$-module to $\mathfrak{r} \oplus \mathfrak{a}$ and $N_{K/F}(\mathfrak{d}\mathfrak{a}) = s\mathfrak{e}$ with an element s such that B is isomorphic to $\{K, s\}$. Moreover, the coset $\mathfrak{a}\mathfrak{I}^0_{K/F}$ is independent of the choice of $\mathfrak{O}$, and $\mathfrak{O}$ is isomorphic as a right $\mathfrak{r}$-module to $\mathfrak{r} \oplus \mathfrak{a}^\iota$, where ι is the main involution of B.*

Proof. Take $\varepsilon \in F^\times$ so that $B = \{K, \varepsilon\}$, and consider $(V, \varphi) = [\Phi]$ over K with $V = K^1_2$ and φ such that $\varepsilon \in d_0(\varphi)$. Using the symbols y, j of Lemma 1.3, identify $j(\alpha)$ with α for $\alpha \in B$, and put $M = y\mathfrak{O}\gamma$ with $\gamma \in B^\times_{\mathbf{h}}$ such that $\varphi[y]^{-1} = \gamma_v\gamma_v^\iota$ for every $v \in \mathbf{h}$. Applying the local result of §1.11 to M_v, we see that M is maximal and $[\widehat{M}/M] = \mathfrak{e}\mathfrak{r}$. Put $M = \mathfrak{r}^1_2\xi$ with some $\xi \in GL_2(K)_{\mathbf{A}}$ and $\mathfrak{a} = \det(\xi)\mathfrak{r}$. Then by a well-known principle M is $\mathfrak{r}$-isomorphic to $\mathfrak{r} \oplus \mathfrak{a}$. By (2.11) we have $\det(\Phi)N_{K/F}(\mathfrak{d}\mathfrak{a}) = \mathfrak{e}$. Then we obtain the first assertion of our lemma by taking $s = -\det(\Phi)^{-1}$. Let $\mathfrak{O}'$ be another maximal order in B containing $\mathfrak{r}$. By the Chevalley-Hasse-Noether theorem (see [E, Satz 7]) there exists an $\mathfrak{r}$-ideal $\mathfrak{b}$ such that $\mathfrak{b}\mathfrak{O}' = \mathfrak{O}\mathfrak{b}$. Take $c \in K^\times_{\mathbf{A}}$ so that $\mathfrak{b} = c\mathfrak{r}$. Then for each $v \in \mathbf{h}$ we can find $\eta_v \in GL_2(K_v)$ such that $yc_v^{-1}xc_v = yx\eta_v$ for every $x \in B_v$. Then $y\mathfrak{O}'_v = y\mathfrak{O}_v\eta_v$, and so $y\mathfrak{O}' = M\gamma^{-1}\eta$ with $\eta = (\eta_v)_{v\in\mathbf{h}}$. Using the map j in the proof of lemma 1.3, we find that $\eta_v = \mathrm{diag}[1, c_v^\rho/c_v]$, and so $\det(\gamma^{-1}\eta)\mathfrak{r} = \varphi[y]\mathfrak{b}^{-1}\mathfrak{b}^\rho \in \mathfrak{I}^0_{K/F}$. Thus $\det(\xi\gamma^{-1}\eta)\mathfrak{r} \in \mathfrak{a}\mathfrak{I}^0_{K/F}$, which proves the second assertion. We can put $\mathfrak{O} = \mathfrak{r}z + \mathfrak{a}w$ with elements z and w. Applying ι to this, we obtain the last assertion.

We call the coset $\mathfrak{a}\mathfrak{I}^0_{K/F}$ in the above lemma *the characteristic coset of K relative to B*. Using this notion, we now reformulate Theorems 2.2 and 2.3 in terms of the matrices Φ in $\mathfrak{H}^0_n(\xi)$.

Theorem 2.10 (The case of even n). *Let the symbols n, $\{\sigma_v\}_{v\in\mathbf{r}_0}$, ε, and ξ be given as follows: $0 < n \in 2\mathbf{Z}$, $\sigma_v \in 2\mathbf{Z}$, $|\sigma_v| \le n$; $\varepsilon \in F^\times$, $\xi \in \mathfrak{G}_{\mathbf{A}}$. Let $B = \{K, \varepsilon\}$; let $\mathfrak{e}$ be the product of the prime ideals in F ramified in B, and $\mathfrak{k}$ the characteristic coset of K relative to B. Suppose that $(-1)^{\sigma_v/2}\varepsilon > 0$ at each $v \in \mathbf{r}_0$ and*

$$\det(\xi)\mathfrak{d}^{(n-2)/2} \in \mathfrak{k}. \tag{2.12}$$

Then there exists an element Φ of $\mathfrak{H}^0_n(\xi)$ such that

$$\varepsilon \in d_0(\Phi), \quad \det(\Phi\xi\xi^\rho)\mathfrak{d}_0^{n/2} = \mathfrak{e}, \quad s_v(\Phi) = \sigma_v \text{ for every } v \in \mathbf{r}_0. \tag{2.13}$$

Moreover, every element of $\mathfrak{H}^0_n(\xi)$ is of this type, and the coset $T\mathfrak{G}\xi$ and the genus of Φ are determined by $(\varepsilon N_{K/F}(K^\times), \{\sigma_v\}_{v\in\mathbf{r}_0})$.

Proof. Let the symbols n, $\{\sigma_v\}_{v\in\mathbf{r}_0}$, ε, and ξ be given as in our theorem. Then Theorem 2.2 combined with Proposition 2.6 guarantees an element Ψ of $\mathfrak{H}_n^0(\eta)$ with some $\eta \in \mathfrak{G}_{\mathbf{A}}$ such that $\varepsilon \in d_0(\Psi)$ and $\sigma_v = s_v(\Psi)$ for every $v \in \mathbf{r}_0$. Put $\mathfrak{y} = \det(\eta)\mathfrak{r}$. By Theorem 2.3 (i), (2.11), and Lemma 2.9 we see that $\mathfrak{y}\mathfrak{d}^{(n-2)/2} \in \mathfrak{k}$. Combining this with our condition (2.12), we see that $\mathfrak{y} \in \det(\xi)\mathfrak{I}_{K/F}^0$, and so $\det(\eta^{-1}\xi) \in K^\times\mathfrak{t}$, which implies $\eta \in T\mathfrak{G}\xi$; see (2.9). By Proposition 2.6 (ii), $[\Psi]$ is isomorphic to $[\Phi]$ with some $\Phi \in \mathfrak{H}_n^0(\xi)$. Replacing (Ψ, η) by (Φ, ξ), we obtain (2.13). This proves the first part of our theorem. Conversely, given $\Phi \in \mathfrak{H}_n^0(\xi)$, put $L = L_0\xi$. Let $\mathfrak{e}$ be the product of the prime ideals in F ramified in $\{K, d_0(\varphi)\}$. By Theorem 2.2 (i) and (2.11), $\det(\Phi\xi\xi^\rho)\mathfrak{d}_0^{n/2} = \mathfrak{e}$, which together with Lemma 2.9 implies condition (2.12). This proves the second part. The last part follows from Proposition 2.6.

Theorem 2.11 (The case of odd n). *Let the symbols n, $\{\sigma_v\}_{v\in\mathbf{r}}$, ε, and ξ be given as follows: $0 < n-1 \in 2\mathbf{Z}$, $\sigma_v - 1 \in 2\mathbf{Z}$, $|\sigma_v| \le n$; $\varepsilon \in F^\times$ and $\xi \in \mathfrak{G}_{\mathbf{A}}$. Let $\varepsilon\mathfrak{g} = \mathfrak{a}N_{K/F}(\mathfrak{b})$ with an $\mathfrak{r}$-ideal $\mathfrak{b}$ and a squarefree integral $\mathfrak{g}$-ideal $\mathfrak{a}$ whose prime factors remain prime in K. Suppose $(-1)^{(\sigma_v-1)/2}\varepsilon > 0$ at each $v \in \mathbf{r}_0$ and*

$$\det(\xi)\mathfrak{d}^{(n-1)/2}\mathfrak{b} \in \mathfrak{I}_{K/F}^0. \tag{2.14}$$

Then there exists an element Φ of $\mathfrak{H}_n^0(\xi)$ such that

$$\varepsilon \in d_0(\Phi), \quad \det(\Phi\xi\xi^\rho)\mathfrak{d}_0^{(n-1)/2} = \mathfrak{a}, \quad s_v(\Phi) = \sigma_v \text{ for every } v \in \mathbf{r}_0. \tag{2.15}$$

Moreover, every element of $\mathfrak{H}_n^0(\xi)$ is of this type, and the coset $T\mathfrak{G}\xi$ and the genus of Φ are determined by $(\varepsilon N_{K/F}(K^\times), \{\sigma_v\}_{v\in\mathbf{r}_0})$.

This can be proved in exactly the same fashion as for Theorem 2.10.

Lemma 2.12. *Suppose F has class number 1. Then the class number of K is odd if and only if $\mathfrak{I}_K = \mathfrak{I}_{K/F}^0$, in which case $-1 \in N_{K/\mathbf{Q}}(K^\times)$ if and only if $-1 \in N_{K/\mathbf{Q}}(\mathfrak{r}^\times)$.*

Proof. Suppose the class number of K is odd. Then every $\mathfrak{r}$-ideal $\mathfrak{a}$ is of the form $\mathfrak{a} = c\mathfrak{b}^2$ with an $\mathfrak{r}$-ideal $\mathfrak{b}$ and $c \in K^\times$. Thus $\mathfrak{a} = c\mathfrak{b}\mathfrak{b}^\rho\mathfrak{b}(\mathfrak{b}^\rho)^{-1} \in \mathfrak{I}_{K/F}^0$ as $\mathfrak{b}\mathfrak{b}^\rho$ is principal, and so $\mathfrak{I}_K = \mathfrak{I}_{K/F}^0$. Suppose the class number of K is even. Then there exists an $\mathfrak{r}$-ideal $\mathfrak{x}$ whose ideal class is not a square. Suppose $\mathfrak{x} \in \mathfrak{I}_{K/F}^0$. Then $\mathfrak{x} = z\mathfrak{y}^{-1}\mathfrak{y}^\rho$ with $z \in K^\times$ and an $\mathfrak{r}$-ideal $\mathfrak{y}$. Thus $\mathfrak{x} = z\mathfrak{y}\mathfrak{y}^\rho\mathfrak{y}^{-2}$, a contradiction, as $\mathfrak{y}\mathfrak{y}^\rho$ is principal. This proves the first part. To prove the second part, suppose $-1 = \alpha\alpha^\rho$ with $\alpha \in K^\times$; put $\alpha\mathfrak{r} = \mathfrak{b}\mathfrak{c}^{-1}$ with integral $\mathfrak{r}$-deals $\mathfrak{b}$ and $\mathfrak{c}$ that are relatively prime. Then $\mathfrak{b}\mathfrak{b}^\rho = \mathfrak{c}\mathfrak{c}^\rho$ and we easily see that $\mathfrak{b} = \mathfrak{c}^\rho$, and so $\mathfrak{c}^2 = \alpha^{-1}\mathfrak{c}\mathfrak{c}^\rho$, which is principal. If the class number of K is odd, then $\mathfrak{c} = c\mathfrak{r}$ with $c \in \mathfrak{r}$. Thus $\alpha c = \varepsilon c^\rho$ with $\varepsilon \in \mathfrak{r}^\times$. Then $\varepsilon\varepsilon^\rho = \alpha\alpha^\rho = -1$. This completes the proof.

The last statement of the above lemma is false if the class number of K is even. For example, let $K = \mathbf{Q}(\sqrt{34}\,)$. Then the class number is 2 and $-1 = \alpha\alpha^\rho$ with $\alpha = (3+\sqrt{34}\,)/5$, but $-1 \notin N_{K/\mathbf{Q}}(\mathfrak{r}^\times)$.

2.13. Let us now take K to be a real or an imaginary quadratic field whose class number is odd. We denote by d the discriminant of K. Thus $F = \mathbf{Q}$ and $K = \mathbf{Q}(\sqrt{d}\,)$. By Lemma 2.12 we have $\mathfrak{I}_K = \mathfrak{I}^0_{K/F}$, and so $\mathfrak{G}_\mathbf{A} = T\mathfrak{G}$ by (2.9). Therefore by Proposition 2.6, every hermitian space over K is isomorphic to $[\Phi]$ with $\Phi \in \mathfrak{H}^0_n(1_n)$, and $\mathfrak{r}^1_n$ is a maximal lattice in it. For simplicity we put $\mathfrak{H}^1_n = \mathfrak{H}_n(1_n)$ and $\mathfrak{H}^0_n = \mathfrak{H}^0_n(1_n)$. Then $\mathfrak{H}^1_n$ consists of all $\Phi = (c_{ij}) \in \mathfrak{H}_n$ such that $\sqrt{d}\,c_{ij} \in \mathfrak{r}$ and $c_{ii} \in \mathbf{Z}$ for every i and j; $\mathfrak{H}^0_n$ consists of all $\Phi \in \mathfrak{H}^1_n$ satisfying the following condition:

(2.16) *If* $\Phi = P\Psi P^*$, $\Psi \in \mathfrak{H}^1_n$, *and* $P \in GL_n(K) \cap M_n(\mathfrak{r})$, *then* $\det(P) \in \mathfrak{r}^\times$.

For $\Phi \in \mathfrak{H}_n$ we put $s(\Phi) = p - q$ if K is imaginary and Φ as a complex hermitian matrix has p positive and q negative eigenvalues; we put $s(\Phi) = 0$ if $d > 0$ and $n \in 2\mathbf{Z}$; we do not define $s(\Phi)$ if $d > 0$ and $n \notin 2\mathbf{Z}$, and so the symbol $s(\Phi)$ in that case must be ignored. Clearly two elements Φ_1 and Φ_2 of $\mathfrak{H}^1_n$ belong to the same genus if $s(\Phi_1) = s(\Phi_2)$ and $\Phi_1 = P_v\Phi_2P_v^*$ with some $P_v \in GL_n(\mathfrak{r}_v)$ for every $v \in \mathbf{h}$. Now, for $L = \mathfrak{r}^1_n$ we have $[\widehat{L}/L] = \det(\sqrt{d}\,\Phi)\mathfrak{r}$ by (2.11). For $d > 0$ we fix an embedding of K into $\mathbf{R}$, and take $\sqrt{d} > 0$.

THEOREM 2.14 (The case of even n). *Let $K = \mathbf{Q}(\sqrt{d}\,)$ as in §2.13, and let three integers n, σ, and e be given as follows: $0 < n \in 2\mathbf{Z}$, $\sigma \in 2\mathbf{Z}$, $|\sigma| \le n$; $\sigma = 0$ if $d > 0$; e is positive and squarefree. Let r be the number of prime factors of e. Suppose that $\sigma - 2r \in 4\mathbf{Z}$ and no prime factor of e splits in K. Then there exists an element Φ of $\mathfrak{H}^0_n$ such that*

$$\det(\sqrt{d}\,\Phi) = (-1)^{\sigma/2}e \quad \textit{and} \quad s(\Phi) = \sigma \quad \textit{if } d < 0, \tag{2.17a}$$

$$\det(\sqrt{d}\,\Phi) = \tau e \quad \textit{with} \quad \tau = 1 \textit{ or } -1 \quad \textit{if } d > 0. \tag{2.17b}$$

Moreover, every element of $\mathfrak{H}^0_n$ is of this type. Its genus is determined by (σ, e) if $d < 0$, and by e if $d > 0$. If $d > 0$ and $-1 \in N_{K/\mathbf{Q}}(K^\times)$, then both e and $-e$ can occur as $\det(\sqrt{d}\,\Phi)$ for Φ in the same genus. If $d > 0$ and $-1 \notin N_{K/\mathbf{Q}}(K^\times)$, then τ is uniquely determined by the condition that a prime number p divides e if and only if $\tau e \notin N_{K/\mathbf{Q}}(K_p^\times)$.

Proof. Given (n, σ, e) as in our theorem, we can find a quaternion algebra B over $\mathbf{Q}$ which is ramified at p if and only if $p|e$. Then B is definite if and only if r is odd. Since $\sigma - 2r \in 4\mathbf{Z}$, we see that $d < 0$ if r is odd. Our assumption on the prime factors of e allows us to put $B = \{K, \varepsilon\}$ with $\varepsilon \in \mathbf{Q}^\times$. Then $(-1)^{\sigma/2}\varepsilon > 0$ if $d < 0$. By Theorem 2.10, we obtain $\Phi \in \mathfrak{H}^0_n$ satisfying (2.13) with $\xi = 1_n$, as (2.12) can be ignored. Then $\mathfrak{e} = e\mathbf{Z}$ and $\det(\sqrt{d}\,\Phi) = \tau e$ with $\tau = \pm 1$. Since $s(\Phi) = \sigma$, we see that $\tau = (-1)^{\sigma/2}$ if $d < 0$. The same theorem says that every element of $\mathfrak{H}_n^0$ is of this type, and its genus is determined by $\varepsilon N_{K/F}(K^\times)$ and σ. We easily see that (e, σ) determines $(\varepsilon N_{K/F}(K^\times), \sigma)$, and vice versa. If $-1 \in N_{K/\mathbf{Q}}(K^\times)$, then by Lemma 2.12, $\mathfrak{r}^\times$ contains an element ζ such that $\zeta\zeta^\rho = -1$. Then $\det(P\Phi P^*) = -\det(\Phi)$ for $P = \mathrm{diag}[\zeta, 1_{n-1}]$. Thus both e and $-e$ can happen. Suppose $d > 0$ and $-1 \notin N_{K/\mathbf{Q}}(K^\times)$. Then $\{K, e\}$ is not isomorphic to $\{K, -e\}$. Since $-d \in N_{K/\mathbf{Q}}(K^\times)$, $d_0(\varphi)$ is represented by $\det(\sqrt{d}\Phi)$. If $\det(\sqrt{d}\Phi) = \tau e$, then $B = \{K, \tau e\}$. Thus τ is uniquely determined by the condition that a prime number p divides e if and only if $\tau e \notin N_{K/\mathbf{Q}}(K_p^\times)$.

Theorem 2.15 (The case of odd n). *Let $K = \mathbf{Q}(\sqrt{d}\,)$ as in §2.13 and let four integers n, σ, τ, and e be given as follows: $0 < n-1 \in 2\mathbf{Z}$; σ is necessary only if $d < 0$, $\sigma - 1 \in 2\mathbf{Z}$, and $|\sigma| \le n$; τ is necessary only if $d > 0$, and $\tau = 1$ or -1; e is positive and squarefree, and every prime factor of e remains prime in K. Then there exists an element Φ of $\mathfrak{H}_n^0$ such that*

$$\det(\sqrt{d}\,\Phi) = (-1)^{(\sigma-1)/2} e\sqrt{d} \quad \text{and} \quad s(\Phi) = \sigma \quad \text{if } d < 0, \tag{2.18a}$$

$$\det(\sqrt{d}\,\Phi) = \tau e\sqrt{d} \quad \text{if } d > 0. \tag{2.18b}$$

Moreover, every element of $\mathfrak{H}_n^0$ is of this type, and its genus is determined by (σ, e) if $d < 0$ and by (τ, e) if $d > 0$. For $d > 0$ the sets $(1, e)$ and $(-1, e)$ determine the same genus if and only if $-1 \in N_{K/\mathbf{Q}}(K^\times)$.

Proof. Given (n, σ, τ, e) as in our theorem, take $\varepsilon = (-1)^{(\sigma-1)/2}e$ if $d < 0$ and $\varepsilon = (-1)^{(n-1)/2}\tau e$ if $d > 0$. Then Theorem 2.11 with $\xi = 1_n$ and $\mathfrak{a} = e\mathbf{Z}$ guarantees an element Φ of $\mathfrak{H}_n^0$ satisfying (2.15). We can easily verify that (2.18a, b) hold. That every $\Phi \in \mathfrak{H}_n^0$ is of this type also follows from Theorem 2.11, as $d_0(\Phi)$ can be represented by e or $-e$ with a positive integer e as in our theorem. Since the last assertion is obvious, our proof is complete.

Corollary 2.16. *Let $K = \mathbf{Q}(\sqrt{d}\,)$ and $\mathfrak{H}_n^1$ be as in §2.13; let $0 < n \in \mathbf{Z}$ and $\sigma \in \mathbf{Z}$.*

(i) *If $d < 0$, there exists an element Φ of $\mathfrak{H}_n^1$ such that $\det(\sqrt{d}\,\Phi) = 1$ and $s(\Phi) = \sigma$ exactly when $n \in 2\mathbf{Z}$ and $\sigma \in 4\mathbf{Z}$.*

(ii) *Suppose $d > 0$; then there exists an element Φ of $\mathfrak{H}_n^1$ such that $\det(\sqrt{d}\,\Phi) = 1$ if and only if $n \in 2\mathbf{Z}$. Moreover, there exists an element Φ' of $\mathfrak{H}_n^1$ such that $\det(\sqrt{d}\,\Phi') = -1$ if and only if $n \in 2\mathbf{Z}$ and $-1 \in N_{K/\mathbf{Q}}(K^\times)$.*

Proof. From Theorem 2.15 we see that $\det(\sqrt{d}\,\Phi) = \pm 1$ for $\Phi \in \mathfrak{H}_n^1$ cannot happen if n is odd. Take $e = 1$ in Theorem 2.14. Then $r = 0$, and we obtain our results immediately from that theorem. Notice that if $d > 0$ and $-1 \notin N_{K/\mathbf{Q}}(K^\times)$, then $\{K, -1\}$ is a division algebra, and so $-1 \notin N_{K/\mathbf{Q}}(K_p^\times)$ for some prime number p.

The above corollary is a natural analogue of a well-known fact on unimodular quadratic form over $\mathbf{Q}$.

2.17. EXAMPLES. (2) Take $n = 2$, $d = 21$, and $e_1 = 11 \cdot 13$; then the class number of K is 1 and $-1 \notin N_{K/\mathbf{Q}}(K^\times)$. For $\Phi_1 = \begin{bmatrix} 7 & 2/\sqrt{21} \\ -2/\sqrt{21} & -1 \end{bmatrix}$ we have $\det(\sqrt{21}\,\Phi_1) = -e_1$ and $\Phi_1 \in \mathfrak{H}_2^0$. But we cannot have $\det(\sqrt{21}\,\Phi) = e_1$ for $\Phi \in \mathfrak{H}_2^0$.

Next take $e_2 = 3 \cdot 7 \cdot 11 \cdot 13$. Then $\det(\sqrt{21}\,\Phi)$ for $\Phi \in \mathfrak{H}_2^0$ can be $-e_2$ but cannot be e_2. Also, $\{K, 11 \cdot 13\}$ is ramified at $p = 3, 7$, but $\{K, -11 \cdot 13\}$ is not. From this we can derive that $\mathrm{diag}[11, -13]$ is reduced, but $\mathrm{diag}[11, 13]$ is not.

3. HERMITIAN DIOPHANTINE EQUATIONS OVER A LOCAL FIELD

3.1. Throughout this section we fix (V, φ) in the local case, and put $n = \dim(V)$. We denote by $\mathfrak{p}$ the maximal ideal of $\mathfrak{g}$, and by t the core dimension of (V, φ). Then $t \le 2$ as observed in §2.1. For an $\mathfrak{r}$-lattice L in V we put

$$C(L) = \{\alpha \in U^\varphi(V) \,|\, L\alpha = L\}, \qquad C^1(L) = C(L) \cap SU^\varphi(V). \tag{3.1}$$

Define $L[q, \mathfrak{b}]$ and $L[q]$ by (1.10b, c). Clearly $L[q, \mathfrak{b}]$ and $L[q]$ are stable under right multiplication by the elements of $C(L)$, and so the four orbit sets $L[q, \mathfrak{b}]/C(L)$, $L[q]/C(L)$, $L[q, \mathfrak{b}]/C^1(L)$, and $L[q]/C^1(L)$ are meaningful. Now our principal results of this section are the following two theorems.

THEOREM 3.2. *Suppose that F is local and $n > 1$. Let L be a maximal $\mathfrak{r}$-lattice in V. Then for every $q \in F^\times$ and every $\mathfrak{r}$-ideal $\mathfrak{b}$ the following assertions hold:*

(i) $\#\{L[q, \mathfrak{b}]/C(L)\} \le 1$.

(ii) $\#\{L[q, \mathfrak{b}]/C^1(L)\} < \infty$.

(iii) $\#\{L[q, \mathfrak{b}]/C^1(L)\} \le 1$ *if we exclude the following two cases:* (a) $n = 2$ *and* $t = 0$; (b) $t = 1$, $q\mathfrak{r} = \mathfrak{b}\mathfrak{b}^\rho$, *and* $\mathfrak{d} \ne \mathfrak{r}$.

(iv) *If* $n = 2$ *and* $t = 0$, *then*

$$\#\{L[q, \mathfrak{d}^{-1}]/C^1(L)\} = \begin{cases} 1 & \text{if } q \in \mathfrak{g}^\times, \\ N(q\mathfrak{g})\bigl[1 - \{K/F\}N(\mathfrak{p})^{-1}\bigr] & \text{if } q \in \mathfrak{p}, \end{cases} \tag{3.2}$$

where $N(\mathfrak{a}) = \#(\mathfrak{g}/\mathfrak{a})$, *and* $\{K/F\} = 1, -1,$ *or* 0, *according as* $K = F \times F$, K *is an unramified quadratic extension of* F, *or* K *is ramified over* F.

(v) $L[q, \mathfrak{d}^{-1}] \ne \emptyset$ *exactly in the following cases:* (a) $t = 0$ *and* $q \in \mathfrak{g}$; (b) $t = 1$, $\mathfrak{d} = \mathfrak{r}$, $d_0(\varphi) = N_{K/F}(K^\times)$, *and* $q \in \mathfrak{g}$; (c) $t = 1$, $d_0(\varphi) \not\subset \mathfrak{g}^\times N_{K/F}(K^\times)$, *and* $q \in \mathfrak{p}^{-1}$; (d) $t = 1$, $\mathfrak{d} \ne \mathfrak{r}$, *and* $q \in \mathfrak{p}\mathfrak{d}^{-2} \cup \bigl(\mathfrak{d}^{-2} \cap d_0(\varphi)\bigr)$; (e) $n = t = 2$ *and* $\mathfrak{g} \subset q\mathfrak{g} \subset \mathfrak{p}^{-1}$ *if* $\mathfrak{d} = \mathfrak{r}$; $q\mathfrak{g} = \mathfrak{p}^{-1}$ *if* $\mathfrak{d} \ne \mathfrak{r}$; (f) $n > 2 = t$ *and* $q \in \mathfrak{p}^{-1}$.

The proof will be given in §3.6 through §3.12.

The quantity $\#\{L[q, \mathfrak{b}]/C^1(L)\}$ in Case (b) of (iii) is not so simple. We will discuss that case in Lemma 3.13 (ii).

For every $\alpha \in U^\varphi$ we have $C(L\alpha) = \alpha^{-1}C(L)\alpha$, $C^1(L\alpha) = \alpha^{-1}C^1(L)\alpha$, and $(L\alpha)[q,\, \mathfrak{b}] = L[q,\, \mathfrak{b}]\alpha$. Therefore, in view of Lemma 1.10 (i), it is sufficient to prove Theorem 3.2 for a special choice of L. Also, since $c \cdot L[q,\, \mathfrak{b}] = L[cc^\rho q,\, c\mathfrak{b}]$ for every $c \in K^\times$, it is sufficient to prove our theorem when $\mathfrak{b} = \mathfrak{d}^{-1}$ or $\mathfrak{b} = \mathfrak{r}$.

Theorem 3.3. *If* $n > 1$, *we have* $\#\bigl\{\Lambda[q]/C^1(\Lambda)\bigr\} < \infty$ *for every* $q \in F^\times$ *and every* $\mathfrak{r}$*-lattice* Λ *in* V.

Proof. Let L be a maximal lattice in V. Given Λ, we can find $c \in F^\times$ such that $c\Lambda \subset L$. Then $c\Lambda[q] \subset L[c^2q]$. For any two open compact subgroups D and E of SU^φ we have $[D : D \cap E] < \infty$. Therefore it is sufficient to prove that $\#\bigl\{L[q]/C^1(L)\bigr\} < \infty$. Given $h \in L[q]$, put $\mathfrak{b} = \varphi(h,\, L)$. Then $q\mathfrak{r} \subset \mathfrak{b} \subset \mathfrak{d}^{-1}$, and hence $L[q] \subset \bigcup_{\mathfrak{b}} L[q,\, \mathfrak{b}]$, where $\mathfrak{b}$ runs over the $\mathfrak{r}$-ideals $\mathfrak{b}$ such that $q\mathfrak{r} \subset \mathfrak{b} \subset \mathfrak{d}^{-1}$. Therefore the desired fact follows from Theorem 3.2 (ii).

Lemma 3.4. *Let* L *be a maximal lattice in* V. *Suppose* $\dim(V) > 1$ *and* $q \in \mathfrak{g}^\times$; *then* $\#\bigl\{L[q,\, \mathfrak{r}]/C(L)\bigr\} \le 1$. *Moreover,* $\#\bigl\{L[q,\, \mathfrak{r}]/C^1(L)\bigr\} \le 1$ *if* K *is a field unramified over* F, *or the core dimension of* $(Kh)^\perp$ *is not* 0 *for some* $h \in L[q,\, \mathfrak{r}]$.

Proof. Let $h,\, k \in L[q,\, \mathfrak{r}]$. We see that $L + \mathfrak{r}h$ is integral, and so $h \in L$, as L is maximal. Given $x \in L$, put $y = x - \varphi[h]^{-1}\varphi(x,\, h)h$. Then $y \in L \cap (Kh)^\perp$. From this we can derive that $L = \mathfrak{r}h \oplus M$ with $M = L \cap (Kh)^\perp$. Similarly $L = \mathfrak{r}k \oplus M'$ with $M' = L \cap (Kk)^\perp$. Since L is maximal, M and M' must be maximal. By (1.5) we can find an element α of $SU^\varphi(V)$ such that $k = h\alpha$. Then $M\alpha$ is a maximal lattice in $(Kk)^\perp$. By Lemma 1.10 (i) we can find an element β of $U^\varphi\bigl((Kk)^\perp\bigr)$ such that $M\alpha\beta = M'$; by Lemma 1.10 (ii) such a β can be taken from $SU^\varphi\bigl((Kk)^\perp\bigr)$ if K is a field unramified over F, or the core dimension of $(Kk)^\perp$ is not 0. Extend β to V by putting $k\beta = k$. Then $\alpha\beta \in C(L)$ and $h\alpha\beta = k$. We have $\alpha\beta \in C^1(L)$ under the said conditions on K or on Kk. This proves our lemma. Notice that the core dimension of $(Kh)^\perp$ depends only on $hU(\varphi)$.

3.5. We call an element x of $\mathfrak{r}^1_n$ *primitive* if $x\mathfrak{r}^n_1 = \mathfrak{r}$. Replacing $\mathfrak{r}$ by $\mathfrak{g}$, we can similarly define the primitive elements of $\mathfrak{g}^1_n$. Given an integral $\mathfrak{r}$-lattice L in V, identify V and L with K^1_n and $\mathfrak{r}^1_n$ with respect to an $\mathfrak{r}$-basis $\{z_i\}_{i=1}^n$ of L; also let $\varphi_0 = \bigl(\varphi(z_i,\, z_j)\bigr)_{i,j=1}^n$. By (1.9) we have $\delta\varphi_0 \prec \mathfrak{r}$ for any element δ of $\mathfrak{r}$ such that $\delta\mathfrak{r} = \mathfrak{d}$. Moreover, $\varphi(x,\, L) = \mathfrak{d}^{-1}$ for $x \in L = \mathfrak{r}^1_n$ if and only if $\delta x\varphi_0$ is primitive.

To prove Theorem 3.2, we fix a maximal lattice L in V, and hereafter write simply C and C^1 for $C(L)$ and $C^1(L)$; we always assume $n > 1$.

3.6. In this subsection we consider the case $K = F \times F$, using the notation of §1.8. We can put $L = \mathfrak{g}^1_n \times \mathfrak{g}^1_n$.

Let $h = (a, b) \in L[q, \mathfrak{r}]$ with $a, b \in F_n^1$. Then $\varphi(h, L) = (a\mathfrak{g}_1^n, b\mathfrak{g}_1^n)$, and so both a and b are primitive. Let $\{e_i\}_{i=1}^n$ be the standard basis of F_n^1. Take $\alpha \in SL_n(\mathfrak{g})$ so that $a\alpha = e_1$ and put $c = b \cdot {}^t\alpha^{-1} = (c_i)_{i=1}^n$. Then $c_1 = q$. Suppose $q \in \mathfrak{g}^\times$; then define $\beta \in SL_n(\mathfrak{g})$ by $\beta = \begin{bmatrix} 1 & x \\ 0 & 1_{n-1} \end{bmatrix}$ with $x = (q^{-1}c_2, \dots, q^{-1}c_n)$ and put $\gamma = \alpha \cdot {}^t\beta$. Then $a\gamma = e_1$ and $b \cdot {}^t\gamma^{-1} = c\beta^{-1} = qe_1$. Thus $L[q, \mathfrak{r}]/C^1$ is represented by (e_1, qe_1) if $q \in \mathfrak{g}^\times$.

Next suppose $q \in \mathfrak{p}$; then $(c_2, \dots, c_n)$ is primitive, and so we can find $\beta \in GL_{n-1}(\mathfrak{g})$ such that $(c_2, \dots, c_n)\beta = (1, 0, \dots, 0)$. Put $\gamma = \alpha \cdot \mathrm{diag}[1, {}^t\beta^{-1}]$. Then $a\gamma = e_1$ and $b \cdot {}^t\gamma^{-1} = (q, 1, 0, \dots, 0)$. This shows that $\#\bigl\{L[q, \mathfrak{r}]/C\bigr\} = 1$. If $n > 2$, we can take $\beta \in SL_{n-1}(\mathfrak{g})$, and hence $\#\bigl\{L[q, \mathfrak{r}]/C^1\bigr\} = 1$.

Finally suppose $n = 2$ and $q \in \mathfrak{p}$; then we have shown that $L[q, \mathfrak{r}]/C^1$ can be represented by the elements of the form $\bigl(e_1, (q, s)\bigr)$ with $s \in \mathfrak{g}^\times$. Suppose $\bigl(e_1, (q, s)\bigr)\alpha = \bigl(e_1, (q, t)\bigr)$ with $\alpha = (\gamma, {}^t\gamma^{-1}) \in C^1$, $\gamma \in SL_2(\mathfrak{g})$, and $s, t \in \mathfrak{g}^\times$. Clearly $\gamma = \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix}$ with $v \in \mathfrak{g}$, and so $t = s - qv$. Since the procedure is reversible, we see that $\#\bigl\{L[q, \mathfrak{r}]/C^1\bigr\} = \#(\mathfrak{g}/q\mathfrak{g})^\times$, which gives (3.2) for $K = F \times F$. This completes the proof of Theorem 3.2 in the case $K = F \times F$.

Hereafter from §3.7 through §3.12 we assume that K is a field.

3.7. Let us consider the case $n = t = 2$. Let the symbols be as in Lemma 1.3 and §1.11; we identify B with $j(B)$. In view of Lemma 1.6 we can take $\varphi[\ell] = c = 1$ in the proof of Lemma 1.3, and so we can take $\gamma = 1$ in §1.11. Thus $M = \ell\mathfrak{O}$. Since φ is anisotropic, $SU^\varphi(V) = C^1(M) = \bigl\{\alpha \in \mathfrak{O}^\times \bigm| \alpha\alpha^\iota = 1\bigr\}$. From (1.5) we see that $\#\bigl\{M[q, \mathfrak{d}^{-1}]/C^1\bigr\} = 1$ if $M[q, \mathfrak{d}^{-1}] \neq \emptyset$. Since $\mathrm{Tr}_{K/F}\bigl(\varphi(\ell\alpha, \ell\beta)\bigr) = \mathrm{Tr}_{B/F}(\alpha\beta^\iota)$ for $\alpha, \beta \in B$, we have $\varphi(\ell\alpha, M) = \mathfrak{d}^{-1}$ only if $\mathrm{Tr}_{B/F}(\alpha\mathfrak{O}) = \mathfrak{g}$, which is so if and only if $\alpha\mathfrak{O} = \mathfrak{O}$ or $\alpha\mathfrak{O} = \mathfrak{P}^{-1}$. Take such an α and assume that K is unramified over F. Then $\ell\alpha \in M[\alpha\alpha^\iota, \mathfrak{r}]$. Thus $M[q, \mathfrak{d}^{-1}] \neq \emptyset$ if and only if $q\mathfrak{g}$ is $\mathfrak{g}$ or $\mathfrak{p}^{-1}$. Next suppose that K is ramified over F. Clearly $M[q, \mathfrak{d}^{-1}] \neq \emptyset$ for some q, and $q\mathfrak{g}$ is $\mathfrak{g}$ or $\mathfrak{p}^{-1}$ for the same reason as above. Suppose $q \in \mathfrak{g}^\times$. Then we can find an element $\xi \in \mathfrak{O}^\times$ such that $\xi\xi^\iota = q$. Then $\varphi[x\xi] = q\varphi[x]$ for every $x \in V$, $M\xi = M$, and $M[1, \mathfrak{b}]\xi = M[q, \mathfrak{b}]$. Let $d_0(\varphi) = sN_{K/F}(K^\times)$ as in the proof of Lemma 1.3. We may assume that $s \in \mathfrak{g}^\times$, $s \notin N_{K/F}(\mathfrak{r}^\times)$, and $\varphi_0 = \mathrm{diag}[1, -s]$. Observe that $\mathfrak{O}$ consists of the elements $a + b\omega$ with $a, b \in K$ such that $a + a^\rho \in \mathfrak{g}$ and $aa^\rho - sbb^\rho \in \mathfrak{g}$. Now let $\ell\alpha \in M[1, \mathfrak{d}^{-1}]$ with $\alpha \in B$. Then $\alpha\alpha^\iota = 1$, and so $\ell \in M[1, \mathfrak{d}^{-1}]$. Thus $\mathfrak{d}^{-1} = \varphi(\ell, M) = \varphi(\ell, \ell\mathfrak{O})$. For $a + b\omega \in \mathfrak{O}$ as above, we have $\varphi(\ell, \ell(a + b\omega)) = a^\rho$, and so $\mathfrak{O}$ contains an element $a + b\omega$ such that $a\mathfrak{r} = \mathfrak{d}^{-1}$. Put $N_{K/F}(\mathfrak{d}) = \mathfrak{p}^\kappa$ with $0 < \kappa \in \mathbf{Z}$. Then $aa^\rho\mathfrak{g} = \mathfrak{p}^{-\kappa}$, and $bb^\rho\mathfrak{g} = \mathfrak{p}^{-\kappa}$ as $aa^\rho - sbb^\rho \in \mathfrak{g}$. Putting $a^{-1}b = c$, we obtain $scc^\rho - 1 \in \mathfrak{p}^\kappa$, a contradiction, as $s \notin N_{K/F}(\mathfrak{r}^\times)$. Thus $M[q, \mathfrak{d}^{-1}] \neq \emptyset$ only if $q\mathfrak{g} = \mathfrak{p}^{-1}$. This proves Theorem 3.2 when $n = t = 2$.

3.8. Take an element $u \in \mathfrak{r}$ so that $\mathfrak{r} = \mathfrak{g}[u]$ and put $\delta = u - u^\rho$. Then $\mathfrak{d} = \delta\mathfrak{r}$ and $\delta^\rho = -\delta$. We will often use these u and δ in our later treatment.

Take a decomposition of V as in (1.8a, b), and assume that it is a Witt decomposition; thus $t = \dim(Z)$. Put $g_i = \delta^{-1} f_i$ and

$$L = \sum_{i=1}^{r} (\mathfrak{r}e_i + \mathfrak{r}g_i) + M, \qquad M = \{y \in Z \,|\, \varphi[y] \in \mathfrak{g}\}. \tag{3.3}$$

Then L is a maximal lattice in V as noted in §1.7. With an $\mathfrak{r}$-basis $\{m_i\}_{i=1}^t$ of M, consider matrix representation with respect to $\{e_1, \dots, e_r, m_1, \dots, m_t, g_1, \dots, g_r\}$. Then φ is represented by

$$\varphi_0 = \begin{bmatrix} 0 & 0 & -\delta^{-1}1_r \\ 0 & \zeta & 0 \\ \delta^{-1}1_r & 0 & 0 \end{bmatrix}, \tag{3.4}$$

where $\zeta = \big(\varphi(m_i, m_j)\big)_{i,j=1}^t$. Now write an element of $GL(V)$ as a matrix with 9 matrix blocks corresponding to the blocks of (3.4), and let P be the group consisting of the elements of U^φ whose lower left 3 blocks under the diagonal blocks are all 0; also let $P^1 = P \cap SU^\varphi$. Then

$$U^\varphi = PC(L) \quad \text{and} \quad SU^\varphi = P^1 C^1(L); \tag{3.5}$$

see [S2, Proposition 5.16]. If $r = 1$, P consists of the matrices of the form

$$\begin{bmatrix} a & b & a^{-\rho}(s + u^\rho b\zeta b^*) \\ 0 & e & -\delta a^{-\rho} e\zeta b^* \\ 0 & 0 & a^{-\rho} \end{bmatrix}, \tag{3.6}$$

where $a \in K^\times$, $b \in K_t^1$, $e \in U^\zeta(Z)$, and $s \in F$.

3.9. The case $t = 0$. Represent the elements of V by row vectors in $K_r^1 \times K_r^1$ with respect to the basis $\{e_1, \dots, e_r, g_1, \dots, g_r\}$, and $GL(V)$ by $GL_{2r}(K)$, acting on the right. Then for $h = (y, z) \in K_r^1 \times K_r^1$ we have $\varphi[h] = \delta^{-1}(zy^* - yz^*)$ and $\varphi(h, L) = \delta^{-1}\sum_{i=1}^r (\mathfrak{r}y_i + \mathfrak{r}z_i)$. Now φ is represented by $\delta^{-1}\eta$, where

$$\eta = \begin{bmatrix} 0 & -1_r \\ 1_r & 0 \end{bmatrix}.$$

Therefore $\mathrm{diag}[\alpha, \widetilde{\alpha}] \in U^\varphi$ for every $\alpha \in GL_r(K)$, where $\widetilde{\alpha} = (\alpha^*)^{-1}$. Suppose $\varphi(h, L) = \mathfrak{d}^{-1}$ for $h = (y, z)$; then $\sum_{i=1}^r (\mathfrak{r}y_i + \mathfrak{r}z_i) = \mathfrak{r}$ and $\varphi[h] \in \mathfrak{g}$. Putting $k = (e_1, qug_1)$ with $q \in F$ (not necessarily $\neq 0$), let us prove

$$\varphi[h] = q \ \textit{and}\ \varphi(h, L) = \mathfrak{d}^{-1} \Longrightarrow h \in kC(L);\ h \in kC^1(L)\ \textit{if}\ r > 1. \tag{3.7}$$

Since $\eta \in C^1$, changing h for $h\eta$ if necessary, we may assume that $\sum_{i=1}^r \mathfrak{r}y_i = \mathfrak{r}$. We can find an element $\alpha \in GL_r(\mathfrak{r})$ such that $y\alpha = e_1$; we can even take α from $SL_r(\mathfrak{r})$ if $r > 1$. Put $w = z\widetilde{\alpha}$. Then $(y, z)\mathrm{diag}[\alpha, \widetilde{\alpha}] = (e_1, w)$, and so $w_1 - w_1^\rho = q\delta$. Thus we can put $w_1 = p + qu$ with $p \in \mathfrak{g}$. Define an element $s = s^* \in M_r(\mathfrak{r})$ so that $s_{11} = p$ and $s_{1j} = w_j$ for $j > 1$, and put

$\beta = \begin{bmatrix} 1_r & s \\ 0 & 1_r \end{bmatrix}$. Then $\beta \in C^1$ and $k\beta = (e_1, w)$. If $r > 1$, then we see that $h \in kC^1$, and so $L[q, \mathfrak{d}^{-1}] = kC^1$. This proves (3.7) and also Theorem 3.2 (i), (iii), (v) when $t = 0$.

Suppose $r = 1$; then $y \in \mathfrak{r}^\times$. Since C^1 is a normal subgroup of C, the above argument shows that $h \in \bigcup_t (t, t^{-\rho}qu)C^1$, where t runs over $\mathfrak{r}^\times$. Define B as in Lemma 1.3. From (1.7b), (1.15), and the last equality in the proof of Lemma 1.3 we obtain $B = M_2(F)$, $SU^\varphi = SL_2(F)$, and $C^1 = SL_2(\mathfrak{g})$. Let $t \in \mathfrak{r}^\times \cap \mathfrak{g}[qu]$. Then $t = a + cqu$ with $a, c \in \mathfrak{g}$, and we can find an element $\gamma \in SL_2(\mathfrak{g})$ of the form $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We have $(1, qu)\gamma = (t, t^{-\rho}v)$ with $v \in \mathfrak{r}$. Then $v - v^\rho = q\delta$, so that $v = qu - s$ with $s \in \mathfrak{g}$. Since $tt^\rho \in \mathfrak{g}^\times$, we can put $s = tt^\rho s'$ with $s' \in \mathfrak{g}$. Put $\sigma = \begin{bmatrix} 1 & s' \\ 0 & 1 \end{bmatrix}$. Then $\sigma \in C^1$ and $(t, t^{-\rho}v)\sigma = (t, t^{-\rho}qu)$, which shows that $(t, t^{-\rho}qu) \in (1, qu)C^1$ if $t \in \mathfrak{g}[qu]$. Since the converse is obvious, we thus obtain $L[q, \mathfrak{d}^{-1}] = \bigsqcup_{t\in\tau}(t, t^{-\rho}qu)C^1$, where $\tau = \mathfrak{r}^\times/\mathfrak{g}[qu]^\times$. This proves (3.2) and completes the proof of Theorem 3.2 when $t = 0$.

3.10. The notation being as in (3.3), put $H_j = \sum_{i=1}^{j}(Ke_i + Kg_i)$. Let us now show that given $h \in V$, there exists an element α of C^1 such that $h\alpha \in H_1 + Z$. This is obvious if $h \in Z$ or $r = 1$. So assume that $h \notin Z$ and $r > 1$. Put $h = w + k$ with $k \in Z$ and $w = \sum_{i=1}^{r}(y_ie_i + z_ig_i) \in H_r$, $y_i, z_i \in K$. Then we can put $\sum_{i=1}^{r}(y_i\mathfrak{r} + z_i\mathfrak{r}) = d\mathfrak{r}$ with $d \in K^\times$. Taking $d^{-1}w$ as h of (3.7), we can find an element $\gamma \in C^1(L \cap H_r)$ such that $d^{-1}w\gamma \in H_1$. Extend γ to an element of $C^1(L)$ by defining $x\gamma = x$ for every $x \in Z$. Then we obtain the desired fact. This means that if $n > t > 0$, then it is sufficient to prove Theorem 3.2 when $r = 1$.

3.11. Case $t > 0$, $r = 1$. Writing simply e and g for e_1 and g_1, we have $V = Ke + Z + Kg$ and $L = \mathfrak{r}e + M + \mathfrak{r}g$ with a maximal lattice M in Z. Let $h = ye + x + zg$ with $y, z \in K$ and $x \in Z$. Then $\varphi[h] = \delta^{-1}(zy^\rho - yz^\rho) + \zeta[x]$, where ζ is the restriction of φ to Z, and $\varphi(h, L) = \delta^{-1}(\mathfrak{r}y + \mathfrak{r}z) + \zeta(x, M)$. Suppose $h \in L[q, \mathfrak{d}^{-1}]$. Then $\mathfrak{r}y + \mathfrak{r}z + \mathfrak{d}\zeta(x, M) = \mathfrak{r}$, and hence $y, z \in \mathfrak{r}$, $x \in \widehat{M}$, and $\varphi[h] - \zeta[x] \in \mathfrak{g}$. We identify an element of Z with a row vector of K_t^1 with respect to an $\mathfrak{r}$-basis of M. Then an element $ae + b + cg$ of V with $a, c \in K$ and $b \in Z$ can be identified with a row vector $[a \quad b \quad c]$ of K_{t+2}^1. If $t = 2$ and M corresponds to $\mathfrak{O}$ as in §1.11, then $\widehat{M}$ corresponds to $\mathfrak{P}^{-1}$, and so $\zeta[x] \in \mathfrak{p}^{-1}$; consequently $\varphi[h] \in \mathfrak{p}^{-1}$. Since $e + qug \in L[q, \mathfrak{d}^{-1}]$ if $q \in \mathfrak{g}$, we see that $L[q, \mathfrak{d}^{-1}] \neq \emptyset$ for every $q \in \mathfrak{g}$.

(a) First suppose $t = 2$ and $q \in \mathfrak{p}$. Suppose also that $\mathfrak{r}y + \mathfrak{r}z \neq \mathfrak{r}$. Then $yz^\rho \in \mathfrak{p}\mathfrak{r}$, and so $\delta^{-1}(zy^\rho - yz^\rho) \in \mathfrak{p}$. Thus $\zeta[x] \in \mathfrak{p}$, and hence $\zeta(x, M) \neq \mathfrak{d}^{-1}$. (In §3.7 we showed that $\zeta(x, M) = \mathfrak{d}^{-1}$ only if $\mathfrak{g} \subset \varphi[x]\mathfrak{g} \subset \mathfrak{p}^{-1}$.) Therefore

$\delta\varphi(h, L) \neq \mathfrak{r}$, a contradiction. This shows that (y, z) must be primitive. By (3.7) there is an element α of $C(\mathfrak{r}e + \mathfrak{r}g)$ such that $(ye + zg)\alpha = e + aug$ with $a \in \mathfrak{g}$. Extend α to an element γ of $SU^{\varphi}(V)$ by defining $w\gamma = w\sigma$ for $w \in Z$ with $\sigma \in U^{\varphi}(Z)$ such that $\det(\sigma) = \det(\alpha)^{-1}$. Since $M\sigma = M$, we see that $\gamma \in C^1$, and $h\gamma = [1 \quad k \quad au]$ with $k \in M$ such that $\zeta[k] = q - a$. Given another $h' \in L[q, \mathfrak{d}^{-1}]$, we can similarly find an element γ' of C^1 such that $h'\gamma' = [1 \quad k' \quad a'u]$ with $a' \in \mathfrak{g}$ and $k' \in M$ such that $\zeta[k'] = q - a'$. Put $b = k' - k$ and

$$\tau = \begin{bmatrix} 1 & b & s + u^{\rho}b\zeta b^* \\ 0 & 1 & -\delta\zeta b^* \\ 0 & 0 & 1 \end{bmatrix} \tag{3.8}$$

with some $s \in \mathfrak{g}$. This is a special case of (3.6) and belongs to $SU^{\varphi}(V)$. Since $b \prec \mathfrak{r}$ and $\delta\zeta \prec \mathfrak{r}$, we see that $\tau \in C^1$. We choose s so that $h\gamma\tau = [1 \quad k' \quad a'u]$, which is so if and only if $a'u = s + u^{\rho}b\zeta b^* - \delta k\zeta b^* + au$. This can be achieved by taking $s = \mathrm{Tr}_{K/F}\big(u\zeta(k, k')\big) - (u + u^{\rho})\zeta[k']$. Then $h'\gamma' = h\gamma\tau$, and so $h' \in hC^1$ as expected.

(b) Next suppose $t = 2$ and $q \notin \mathfrak{p}$. Then $q \in \mathfrak{g}^{\times}$ or $q\mathfrak{g} = \mathfrak{p}^{-1}$. If $\mathfrak{d} \neq \mathfrak{r}$ and $q \in \mathfrak{g}$, then $\zeta[x] \in \mathfrak{g}$, and so $\delta\zeta(x, M) \neq \mathfrak{r}$ as shown in §3.7. Consequently $\mathfrak{r}y + \mathfrak{r}z = \mathfrak{r}$ in such a case, and the argument of case (a) is applicable. Therefore we may assume that $q\mathfrak{g} = \mathfrak{p}^{-1}$ if $\mathfrak{d} \neq \mathfrak{r}$. Then as observed in §3.7, we can find an element v of $M[q, \mathfrak{d}^{-1}]$. The same can be said for both cases $q \in \mathfrak{g}^{\times}$ and $q\mathfrak{g} = \mathfrak{p}^{-1}$ if $\mathfrak{d} = \mathfrak{r}$. We identify v with the row vector $[0 \quad v \quad 0]$, which can be viewed as an element of $L[q, \mathfrak{d}^{-1}]$, and so $L[q, \mathfrak{d}^{-1}] \neq \emptyset$ in such cases. Combining (1.5) and (3.5), we have $h \in vP^1C^1$, and hence $h = v\pi\alpha$ with $\pi \in P^1$ and $\alpha \in C^1$. Write π in the form (3.6) and focus our attention on the element e of $U^{\zeta}(Z)$ there. Since $U^{\zeta}(Z) = C(M)$, we see that $ve \in M[q, \mathfrak{d}^{-1}]$, and hence $ve = v\varepsilon$ with $\varepsilon \in C^1(M) = SU^{\zeta}(Z)$ as shown in §3.7. Let $\beta = \mathrm{diag}[1, \varepsilon, 1]$ and $\pi_1 = \pi\beta^{-1}$. Then $\beta \in C^1, \pi_1 \in P^1$, and $v\pi_1 = h\alpha^{-1}\beta^{-1} \in L[q, \mathfrak{d}^{-1}]$. Our choice of ε shows that $v\pi_1 = [0 \quad v \quad p]$ with $p \in \mathfrak{r}$. Since $v \in M[q, \mathfrak{d}^{-1}]$, $\delta v\zeta$ is primitive (see §3.5), and so we can find an element b of $\mathfrak{r}_2^1$ such that $\delta v\zeta b^* = -p$. Define τ by (3.8) with this b and $s = 0$. Then $\tau \in C^1$ and $v\tau = v\pi_1 = h\alpha^{-1}\beta^{-1}$. This shows that $h \in vC^1$. Thus we obtain Theorem 3.2 when $n > t = 2$.

(c) Finally suppose $t = 1$; let M and ζ be as in (3.3) and (3.4). Then $\zeta = \varphi[m]$ and $M = \mathfrak{r}m$ with an element m. Clearly $\zeta \in \mathfrak{g}^{\times}$ or $\zeta\mathfrak{g} = \mathfrak{p}$; the latter case occurs only when $\mathfrak{d} = \mathfrak{r}$. We first treat the case where $\zeta \in \mathfrak{g}^{\times}$; the other case will be treated in §3.12. Thus $h = ye + zg + sm$ with $y, z, s \in K$ such that $y\mathfrak{r} + z\mathfrak{r} + s\mathfrak{d} = \mathfrak{r}$ and $\delta^{-1}(zy^{\rho} - yz^{\rho}) + \zeta ss^{\rho} = q$. Suppose $\mathfrak{d} = \mathfrak{r}$ and $y\mathfrak{r} + z\mathfrak{r} \neq \mathfrak{r}$. Then $s \in \mathfrak{r}^{\times}$ and we see that $q = \varphi[h] \in \mathfrak{g}^{\times}$. Then $\#\big\{L[q, \mathfrak{d}^{-1}]/C^1(L)\big\} \leq 1$ by Lemma 3.4.

(c1) Let us now prove the case in which $q \in \mathfrak{g}$ and $y\mathfrak{r} + z\mathfrak{r} = \mathfrak{r}$ for both ramified and unramified K. Putting $p = \delta^{-1}(zy^\rho - yz^\rho)$ and applying (3.7) to $ye + zg$, we find $\gamma \in C(L)$ such that $M\gamma = M$ and $(y, z)\gamma = (1, pu)$ with $p \in \mathfrak{g}$. Replacing γ by $\gamma\alpha$ with $\alpha \in C$ such that $m\alpha = \det(\gamma)^{-1}m$ and α is the identity map on $Ke+Kg$, we may assume that $\gamma \in C^1$. We have $h\gamma = [1 \quad x \quad pu]$ with $x \in \mathfrak{r}$. We consider

$$\sigma = \begin{bmatrix} 1 & -x & u^\rho \zeta x x^\rho \\ 0 & 1 & \delta\zeta x^\rho \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.9}$$

which is similar to (3.8) and belongs to C^1. We have then $h\gamma\sigma = [1 \quad 0 \quad s]$ with $s \in \mathfrak{g}$. Taking $(1, s)$ as (y, z) above, we find $\gamma' \in C^1$ such that $h\gamma\sigma\gamma' = [1 \quad 0 \quad qu]$. Thus $h \in kC^1$ with $k = e + qug$; also Theorem 3.2 (v), (b) is valid.

(c2) Suppose K is ramified over F. If $q \in \mathfrak{g}$, then $s \in \mathfrak{r}$, so that $y\mathfrak{r} + z\mathfrak{r} = \mathfrak{r}$. and (c1) covers this case. Thus we assume that $q \notin \mathfrak{g}$. Then $s \notin \mathfrak{r}$ and $q\mathfrak{g} = ss^\rho\mathfrak{g}$. Put $\mathfrak{d} = \mathfrak{q}^\kappa$ with the maximal ideal $\mathfrak{q}$ of $\mathfrak{r}$ and $0 < \kappa \in \mathbf{Z}$. Since $s\mathfrak{d} \subset \mathfrak{r}$, we can put $s^{-1}\mathfrak{r} = \mathfrak{q}^a$ with $0 < a \le \kappa$; then $q\mathfrak{g} = \mathfrak{p}^{-a}$. Thus a is determined by q. Suppose $a < \kappa$; then $s\mathfrak{d} \neq \mathfrak{r}$, so that $y\mathfrak{r} + z\mathfrak{r} = \mathfrak{r}$. By the same technique as in (c1), we can find $\gamma \in C^1$ such that $h\gamma = [1 \quad x \quad pu]$ with $p \in \mathfrak{g}$ and $x \in K$. Then $x\mathfrak{q}^a = \mathfrak{r}$. Let $k = [1 \quad x_1 \quad p_1u] \in L[q, \mathfrak{d}^{-1}]$ with $p_1 \in \mathfrak{g}$ and x_1 such that $x_1\mathfrak{q}^a = \mathfrak{r}$. Our task is to show that $k \in hC$. For simplicity put $N(w) = ww^\rho$ for $w \in K^\times$. We have $\zeta N(x) + p = q = \zeta N(x_1) + p_1$, and so $N(x^{-1}x_1) - 1 \in \mathfrak{p}^a$. Thus $N(x^{-1}x_1) \in N(\mathfrak{r}^\times) \cap (1 + \mathfrak{p}^a) = N(1 + \mathfrak{q}^a)$ by [S2, Lemma 17.6 (2)]. We can therefore put $N(x^{-1}x_1) = N(d)$ with $d \in 1 + \mathfrak{q}^a$. Put $\alpha = \mathrm{diag}[1_r, dx_1^{-1}x, 1_r]$. Then $\alpha \in C^1$ and $k\alpha = [1 \quad dx \quad p_1u]$. Put $b = dx - x$ and consider τ of (3.8) with this b and any $s \in \mathfrak{g}$. Then $\tau \in C^1$ and $h\gamma\tau = [1 \quad dx \quad c]$ with $c \in \mathfrak{g}$ such that $c - c^\rho = \delta p_1$. Choosing s suitably, we obtain $c = p_1u$. Then $k \in hC^1$.

(c3) It remains to treat the case $a = \kappa$. Then $s\mathfrak{d} = \mathfrak{r}$ and $q\mathfrak{r} = \mathfrak{d}^{-2}$. Put $h_0 = \delta h$ and $q_0 = -\delta^2 q$. Then $h_0 \in L[q_0, \mathfrak{r}]$. Since $q_0 \in \mathfrak{g}^\times$, by Lemma 3.4 we see that $\#\big\{L[q, \mathfrak{d}^{-1}]/C\big\} = \#\big\{L[q_0, \mathfrak{r}]/C\big\} \le 1$.

(c4) As for (d) of Theorem 3.2 (v), we have seen that $L[q, \mathfrak{d}^{-1}] \neq \emptyset$ only if $q \in \mathfrak{d}^{-2}$; also $e + qug \in L[q, \mathfrak{d}^{-1}]$ if $q \in \mathfrak{g}$. Thus it remains to consider the case where $\mathfrak{d} \neq \mathfrak{r}$ and $q \notin \mathfrak{g}$. Suppose $q\mathfrak{g} = \mathfrak{p}^{-a}$ with $0 < a < \kappa$. We can find $c \in \mathfrak{g}^\times$ such that $c - 1 \in \mathfrak{p}^a$ and $c \notin N(\mathfrak{r}^\times)$. Then q or $c^{-1}q$ represents $d_0(\varphi)$. If $q \in d_0(\varphi)$, then $q = \zeta ss^\rho$ with $s \in K$, and $e + sm \in L[q, \mathfrak{d}^{-1}]$. If $q \in cd_0(\varphi)$, put $q = c\zeta ss^\rho$ with $s \in K$ and $p = \zeta ss^\rho(c-1)$. Then $p \in \mathfrak{g}$ and $e + pu\mathfrak{g} + sm \in L[q, \mathfrak{d}^{-1}]$. Thus $L[q, \mathfrak{d}^{-1}] \neq \emptyset$ if $q \in \mathfrak{p}\mathfrak{d}^{-2}$. The case $q\mathfrak{r} = \mathfrak{d}^{-2}$ will be settled in Lemma 3.13.

3.12. Let us now treat the case where $t = 1$ and $\zeta\mathfrak{g} = \mathfrak{p}$. Then $\mathfrak{d} = \mathfrak{r}$ and $\delta \in \mathfrak{r}^\times$. To avoid possible confusion, we use the letter π instead of ζ; thus $\varphi[m] = \pi$. Let $h = ae + bf + cm \in L[q, \mathfrak{r}]$. Then $\mathfrak{r}a + \mathfrak{r}b + \mathfrak{r}\pi c = \mathfrak{r}$ and $ab^\rho + a^\rho b + \pi cc^\rho = q$. Clearly $q \in \mathfrak{p}^{-1}$; also $c \in \mathfrak{r}$ if and only if $q \in \mathfrak{g}$. Given

$q \in \mathfrak{g}$, we can find $b \in \mathfrak{r}$ such that $b + b^\rho = q$. Then $e + bf \in L[q, \mathfrak{r}]$. Given $q \in \mathfrak{p}^{-1}$, $\notin \mathfrak{g}$, we can find $d \in \mathfrak{r}^\times$ such that $dd^\rho = \pi q$. Then $\pi^{-1}dm \in L[q, \mathfrak{r}]$. Thus $L[q, \mathfrak{r}] \neq \emptyset$ if and only if $q \in \mathfrak{p}^{-1}$.

(1) Let us first assume that $q \in \mathfrak{g}$; then $c \in \mathfrak{r}$ and $\mathfrak{r}a + \mathfrak{r}b = \mathfrak{r}$. By (3.7) there exists an element β of $C(\mathfrak{r}e + \mathfrak{r}f)$ such that $k\beta = e + sf$ with $s \in \mathfrak{r}$. Extend β to an element α of $C^1(L)$ by defining $m\alpha = \det(\beta)^{-1}m$. Then $h\alpha = e + sf + c_1 m$ with $c_1 \in \mathfrak{r}$. Now represent the elements of $U^\varphi(V)$ by matrices with respect to $\{e, m, f\}$. Then (3.5) holds with the subgroup P of U^φ consisting of the upper triangular matrices. Observe that P contains every matrix of the form

$$\begin{bmatrix} 1 & -x & y \\ 0 & 1 & \pi x^\rho \\ 0 & 0 & 1 \end{bmatrix} \tag{3.10}$$

with $x, y \in K$ such that $y + y^\rho = -\pi x x^\rho$. Since $\mathrm{Tr}_{K/F}(\mathfrak{r}) = \mathfrak{g}$, for any $x \in \mathfrak{r}$ we can take such a y from $\mathfrak{r}$. Let γ be the matrix of (3.10) with $x = c_1$ and $y \in \mathfrak{r}$. Then $\gamma \in C^1(L)$ and $h\alpha\gamma = e + zf$ with $z \in \mathfrak{r}$ such that $z + z^\rho = q$. Take $z_1 \in \mathfrak{r}$ so that $z_1 + z_1^\rho = q$. Denote by ε the matrix of (3.10) with $x = 0$ and $y = z_1 - z$. Then $\varepsilon \in C^1(L)$ and $(e+zf)\varepsilon = e + z_1 f$. This gives the desired result when $q \in \mathfrak{g}$.

(2) Next we consider the case $c \notin \mathfrak{r}$; put $d = \pi c$. Then $d \in \mathfrak{r}^\times$ and $q = ab^\rho + a^\rho b + \pi^{-1}dd^\rho$. Thus $\pi q \in \mathfrak{g}^\times$, and so we can find an element $d_0 \in \mathfrak{r}^\times$ such that $d_0 d_0^\rho = \pi q$. Put $k = \pi^{-1}d_0 m$. Then $k \in L[q, \mathfrak{r}]$. By (1.5), we have $h = k\alpha$ with $\alpha \in SU^\varphi$, and by (3.5) we can put $\alpha = \beta\gamma$ with $\beta \in P^1$ and $\gamma \in C^1$. Replacing β by $\beta\xi$ with a suitable diagonal matrix ξ belonging to C^1, we may assume that the center entry of β is 1. (Here we need Lemma 1.9 (i).) Let $[0 \;\; 1 \;\; j]$ be the second row of β. Then $[0 \;\; \pi^{-1}d_0 \;\; \pi^{-1}d_0 j] = k\beta = h\gamma^{-1} \in L[q, \mathfrak{r}]$, and so $\pi^{-1}d_0 j \in \mathfrak{r}$. Put $x = \pi^{-1}j^\rho$; then $x \in \mathfrak{r}$. Let ε be the matrix of (3.10) with this x and y such that $y + y^\rho = -\pi x x^\rho$. Then $\varepsilon \in C^1$ and $k\varepsilon = k\beta = h\gamma^{-1}$, which gives the desired fact.

It only remains to discuss $L[q, \mathfrak{d}^{-1}]/C^1$ when $\mathfrak{d} \neq \mathfrak{r}$ and $q\mathfrak{r} = \mathfrak{d}^{-2}$. (In (c3) we treated $L[q, \mathfrak{d}^{-1}]/C$.) The problem is settled by (ii) of the following Lemma.

Lemma 3.13. *If K is a field ramified over F, then the following assertions hold:*

(i) *Let $W = (Kh)^\perp$ with $h \in L[q, \mathfrak{b}]$ and let $E_1 = \det\bigl(C \cap U^\varphi(W)\bigr)$, where we view $U^\varphi(W)$ as the subgroup of $U^\varphi(V)$ consisting of the elements ξ such that $h\xi = h$. Then $\#\bigl\{L[q, \mathfrak{b}]/C^1\bigr\} = [E_L : E_1] < \infty$, where E_L is as in Lemma 1.9.*

(ii) *If $1 < n - 1 \in 2\mathbf{Z}$ and $q\mathfrak{r} = \mathfrak{d}^{-2}$, then $\#\bigl\{L[q, \mathfrak{d}^{-1}]/C^1\bigr\} = 2$ or 0 according as q represents $d_0(\varphi)$ or not.*

Proof. Clearly $C \cap U^\varphi(W)$ is an open subgroup of $U^\varphi(W)$. Now $\det : U^\varphi(W) \to E$ is a continuous surjective map, and so it is an open map by virtue of a well know principle; see [S3, Lemma 8.0]. Thus E_1 is an open subgroup of E, and so $[E : E_1] < \infty$, as E is compact. Therefore $[E_L : E_1] < \infty$. Next, take a finite subset $B \subset C$ so that $\{\det(\beta)\}_{\beta \in B}$ gives E_L/E_1. Let $k \in L[q, \mathfrak{b}]$. By Theorem 3.2 (i), $k = h\alpha$ with $\alpha \in C$. Then $\det(\alpha\beta^{-1}) \in E_1$ for some $\beta \in B$, so that $\det(\alpha\beta^{-1}) = \det(\gamma)$ with $\gamma \in C \cap U^{\varphi}(W)$. Put $\xi = \alpha^{-1}\gamma\beta$. Then $\xi \in C^1$ and $h\beta = h\gamma\beta = k\xi$, that is, $k \in h\beta C^1$. Thus $L[q, \mathfrak{b}] = \bigcup_{\beta \in B} h\beta C^1$. We easily see that the last union is disjoint, and so we obtain (i). To prove (ii), let the notation be as in (3.3) and (3.4); let $M = \mathfrak{r}m$, $\varphi[m] = \zeta \in \mathfrak{g}^{\times}$, and $\mathfrak{d} = \mathfrak{q}^{\kappa}$ as in (c2) above. Then ζ represents $d_0(\varphi)$. Suppose $q\mathfrak{r} = \mathfrak{d}^{-2}$ and $h \in L[q, \mathfrak{d}^{-1}]$; put $h = \sum_{i=1}^{r}(y_i e_i + z_i g_i) + sm$ with $y_i, z_i, s \in K$. Then $y_i, z_i \in \mathfrak{r}$, and $q - \zeta s s^{\rho} \in \mathfrak{g}$, so that $s\mathfrak{r} = \mathfrak{d}^{-1}$ and $(\zeta s s^{\rho})^{-1} q \in 1 + \mathfrak{p}^{\kappa} \in N_{K/F}(\mathfrak{r}^{\times})$; thus q represents $d_0(\varphi)$. This shows that $L[q, \mathfrak{d}^{-1}] \neq \emptyset$ only for such a q. Taking such a q, we can put $q = \zeta x x^{\rho}$ with $x \in K^{\times}$. Let $k = xm$ and $W = (Kk)^{\perp}$. Then $k \in L[q, \mathfrak{d}^{-1}]$, $W = \sum_{i=1}^{r}(Ke_i + Kg_i)$, and $C \cap U^{\varphi}(W) = C(\Lambda)$ with $\Lambda = \sum_{i=1}^{r}(\mathfrak{r}e_i + \mathfrak{r}g_i)$; thus $E_1 = E_{\Lambda}$. By Lemma 1.9, $E_L = E$, $E_{\Lambda} = E_0$, and $[E : E_0] = 2$, which together with (i) shows that $\#\{L[q, \mathfrak{d}^{-1}]/C^1\} = 2$. This completes the proof.

LEMMA 3.14. *Let $W = (Kh)^{\perp}$ with $h \in L[q, \mathfrak{b}]$ and let $\Lambda = L \cap W$. Suppose that $0 < n - 1 \in 2\mathbf{Z}$. Then there is a unique maximal lattice in W containing Λ at least in the following two cases: (1) $q(\mathfrak{b}\mathfrak{b}^{\rho})^{-1} = \mathfrak{r}$; (2) $q(\mathfrak{b}\mathfrak{b}^{\rho})^{-1} = \mathfrak{p}\mathfrak{r}$ and $\{K, qd_0(\varphi)\}$ is a division algebra. Moreover, Λ is maximal and $C(\Lambda) = C(L) \cap U^{\varphi}(W)$ in Case (1); Λ is maximal and $[C(\Lambda) : C(L) \cap U^{\varphi}(W)] = N(\mathfrak{p}) + 1$ in Case (2) if K is unramified over F and $d_0(\varphi) = N_{K/F}(K^{\times})$. These assertions are true with C^1 instead of C.*

Proof. Changing h for ch with some $c \in K^{\times}$, we may assume that $\mathfrak{b} = \mathfrak{r}$. This does not change the ideal $q(\mathfrak{b}\mathfrak{b}^{\rho})^{-1}$ nor $\{K, qd_0(\varphi)\}$. Thus $q \in \mathfrak{g}^{\times}$ in Case (1) and $q\mathfrak{g} = \mathfrak{p}$ in Case (2). Suppose $q \in \mathfrak{g}^{\times}$. Then $L = \mathfrak{r}h \oplus \Lambda$ as shown in the proof of Lemma 3.4, and Λ is maximal as noted there; clearly $C(\Lambda) = C(L) \cap U^{\varphi}(W)$ and $C^1(\Lambda) = C^1(L) \cap U^{\varphi}(W)$. Next suppose $q\mathfrak{g} = \mathfrak{p}$ and $\{K, qd_0(\varphi)\}$ is a division algebra. Let u, δ, and $\{e_i, g_i\}$ be as in §3.8. Since $n \notin 2\mathbf{Z}$, we have $L = \mathfrak{r}m + \sum_{i=1}^{r}(\mathfrak{r}e_i + \mathfrak{r}g_i)$ with an element m such that $\varphi[m] = \zeta$. Theorem 3.2 (i) allows us to replace h by any element in $L[q, \mathfrak{r}]$. Thus we can put $h = e_1 + qug_1$. Put now $k = e_1 + qu^{\rho}g_1$, $Y = Kk + Km$, and $N = \mathfrak{r}k + \mathfrak{r}m$. Then $\varphi[k] = -q$, $W = Y \oplus \sum_{i=2}^{r}(Ke_i + Kg_i)$, $\Lambda = N + \sum_{i=2}^{r}(\mathfrak{r}e_i + \mathfrak{r}g_i)$, and $d_0(Y) = qd_0(\varphi)$. Since $\{K, qd_0(\varphi)\}$ is a division algebra, (Y, φ) is anisotropic, and so has a unique maximal lattice M as noted in §3.7. Put $\Lambda' = M + \sum_{i=2}^{r}(\mathfrak{r}e_i + \mathfrak{r}g_i)$. Then clearly Λ' is the unique maximal lattice in W containing Λ.

To prove the remaining part, we assume that $q\mathfrak{g} = \mathfrak{p}$, $\mathfrak{d} = \mathfrak{r}$, and $d_0(\varphi) = N_{K/F}(K^{\times})$. Since $N \subset M$ and $[\widehat{M} : M] = N(\mathfrak{p})^2 = [\widehat{N} : N]$, we obtain $M = N$, and so Λ is maximal. We easily see that $C(L) \cap U^{\varphi}(W) \subset C(\Lambda)$. Let $\gamma \in C(\Lambda)$.

Then $L\gamma$ is a maximal lattice in V containing Λ, so that $\sum_{i=2}^{r}(\mathfrak{r}e_i+\mathfrak{r}g_i)\subset L\gamma$. By [S2, Lemma 4.9 (i)] we have $L\gamma = J+\sum_{i=2}^{r}(\mathfrak{r}e_i+\mathfrak{r}g_i)$ with a maximal lattice J in $Km+Ke_1+Kg_1$. Since $m\in J$, from (1.9) we see that $\varphi(m,\,J)=\mathfrak{r}$, and so $J=\mathfrak{r}m+H$ with a maximal lattice H in Ke_1+Kg_1 as shown in the proof of Lemma 3.4. Now $h=h\gamma\in L\gamma$ and $k\in\Lambda=\Lambda\gamma$; thus $h,\,k\in J$. Put $H_0=\mathfrak{r}e_1+\mathfrak{r}g_1$. By Lemma 1.10 (ii), $H=H_0\alpha$ with $\alpha\in SU^{\varphi}(Ke_1+Kg_1)$. By (3.5) we can take α in the form (3.6). Identify $GL(Ke_1+Kg_1)$ with $GL_2(K)$ with respect to the basis $\{e_1,\,g_1\}$. Replacing α by an element of $C(H_0)\alpha$, we can put $\alpha=\begin{bmatrix}a^{-1} & b\\ 0 & a\end{bmatrix}$ with $a\in F^{\times}$ and $b\in F$. Put $z=a^{-1}e_1+bg_1$ and $w=ag_1$; then $H=\mathfrak{r}z+\mathfrak{r}w$. Since $\delta e_1=uk-u^{\rho}h\in H$ and $q\delta g_1=h-k\in H$, we see that $\mathfrak{p}\subset a\mathfrak{g}\subset\mathfrak{g}$, and so we may assume that $a=1$ or $a=q$. Now $\varphi(z,\,e_1)\in\mathfrak{r}$, and so $b\in\mathfrak{g}$. Thus if $a=1$, then $H=H_0$. Suppose $a=q$; put $\alpha_b=\begin{bmatrix}a^{-1} & b\\ 0 & a\end{bmatrix}$. Then $H_0\alpha_b=H_0\alpha_{b'}$ if and only if $b-b'\in\mathfrak{p}$. Thus there exist exactly $N(\mathfrak{p})+1$ maximal lattices in Ke_1+Kg_1 containing h and k, and so there exist at most $N(\mathfrak{p})+1$ lattices of the form $L\gamma$ with $\gamma\in C(\Lambda)$. This shows that

$$\big[C(\Lambda):C(L)\cap U^{\varphi}(W)\big]\leq N(\mathfrak{p})+1. \tag{3.11}$$

To show that this is actually an equality, define the symbols $\ell,\,j,\,\omega$, and $\mathfrak{O}$ as in Lemma 1.3, (1.6), and §3.7, with Y as the space V there. In the proof of Lemma 1.3 take $\varphi_0=\mathrm{diag}[\zeta,\,-q]$ and $s=\zeta^{-1}q$. Then the identification of V with K_2^1 in the proof of Lemma 1.3 (which is unrelated to the above identification of $GL(Ke_1+Kg_1)$ with K_2^2) identifies m with $(1,\,0)$ and k with $(0,\,1)$, so that m here equals ℓ of Lemma 1.3; also, $M=\ell j(\mathfrak{O})=mj(\mathfrak{O})$. Let β be an element of $\mathfrak{O}$ such that $\beta\beta^{\iota}=1$; define $\xi\in SU^{\varphi}(V)$ by $\xi=j(\beta)$ on Y and $x\xi=x$ for every $x\in Kh+\sum_{i=2}^{r}(Ke_i+Kg_i)$. Then $\xi\in C^1(\Lambda)$. Put $\beta=c+d\omega$ with $c,\,d\in\mathfrak{r}$. Then $\ell\xi=c\ell+dk$ and $k\xi=sd^{\rho}\ell+c^{\rho}k$. Since $\delta e_1=uk-u^{\rho}h$ and $q\delta g_1=h-k$, we have $q\delta g_1\xi=(1-c^{\rho})e_1-sd^{\rho}\ell+q(u-c^{\rho}u^{\rho})g_1$. Thus $\xi\in C^1(L)$ if and only if $c-1\in\mathfrak{p}\mathfrak{r}$, which is so if and only if $\beta-1\in\mathfrak{P}$. Since $\beta\beta^{\iota}=1$, this shows that there exist at least $N(\mathfrak{p})+1$ different $L\gamma$ with $\gamma\in C^1(\Lambda)$, and so

$$N(\mathfrak{p})+1\leq\big[C^1(\Lambda):C^1(L)\cap U^{\varphi}(W)\big].$$

This combined with (3.11) proves that

$$\big[C^1(\Lambda):C^1(L)\cap U^{\varphi}(W)\big]=\big[C(\Lambda):C(L)\cap U^{\varphi}(W)\big]=N(\mathfrak{p})+1,$$

which completes the proof.

We insert here the classification of the structures $(V,\,s\varphi)$ with $s\in F^{\times}$. If $K=F\times F$, the matter is settled in §1.8.

Proposition 3.15. (i) *If F is a local field and* $\dim(V)$ *is even, then* (V, φ) *is isomorphic to* $(V, s\varphi)$ *for every* $s \in F^\times$.

(ii) *Suppose F is an arbitrary field and* $\dim(V)$ *is odd; let* $s \in F^\times$; *then* (V, φ) *is isomorphic to* $(V, s\varphi)$ *if and only if* $s \in N_{K/F}(K^\times)$.

Proof. The first assertion is included in Lemma 1.6. The second assertion can be proved easily in the same manner as for [S3, Theorem 7.13] in the even-dimensional case.

4. Hermitian Diophantine equations over a global field

4.1. Throughout this section we assume that F is an algebraic number field and K is a quadratic extension of F; we fix a hermitian space (V, φ) and use the notation of §2.1. For an $\mathfrak{r}$-lattice L in V we put

$$\varGamma(L) = \{\alpha \in U^\varphi(V) \mid L\alpha = L\}, \qquad \varGamma^1(L) = \varGamma(L) \cap SU^\varphi(V). \tag{4.1}$$

Given $h \in V$ such that $\varphi[h] \neq 0$, put $W = (Kh)^\perp$. We view $U^\varphi(W)$ as a subgroup of $U^\varphi(V)$, and $SU^\varphi(W)$ as a subgroup of $SU^\varphi(V)$ as explained in §1.1. For $\sigma \in U^\varphi(V)_\mathbf{A}$, the symbol $h\sigma$ is meaningful as an element of $V_\mathbf{A}$.

Theorem 4.2. *If* $\dim(V) > 1$, *then* $\#\{\Lambda[q]/\varGamma^1(\Lambda)\} < \infty$ *for every* $q \in F^\times$ *and every* $\mathfrak{r}$*-lattice* Λ *in* V.

Proof. Assuming $\dim(V) > 1$ and $\Lambda[q] \neq \emptyset$, take $h \in \Lambda[q]$ and define W as above; put $G = SU^\varphi(V)$, $H = SU^\varphi(W)$, $D = \{x \in G_\mathbf{A} \mid \Lambda x = \Lambda\}$, and $D_v = G_v \cap D$ for $v \in \mathbf{h}$. Then $\varGamma^1(\Lambda) = G \cap D$ and $D_v = C^1(\Lambda_v)$. By Theorem 3.3 and (1.5) we have $\Lambda_v[q] = \bigcup_{\alpha \in X_v} h\alpha D_v$ with a finite subset X_v of G_v. By Theorem 3.2 (iii) and (iv) we can take $X_v = \{1\}$ if Λ_v is maximal, v is not ramified in K, and $q \in \mathfrak{g}_v^\times$. Thus $X_v = \{1\}$ for almost all $v \in \mathbf{h}$. Put $X = \prod_{v \in \mathbf{h}} X_v$. This is a finite subset of $G_\mathbf{h}$. For each $\xi \in X$ we can find a finite subset E_ξ of $H_\mathbf{h}$ such that $H_\mathbf{A} = \bigcup_{\varepsilon \in E_\xi} H\varepsilon(H_\mathbf{A} \cap \xi D \xi^{-1})$. Then $H_\mathbf{A}\xi D = \bigcup_{\varepsilon \in E_\xi} H\varepsilon\xi D$. For each (ε, ξ) such that $G \cap \varepsilon\xi D \neq \emptyset$ pick $\beta_{\varepsilon,\xi} \in G \cap \varepsilon\xi D$. Now let $k \in \Lambda[q]$. Then $k = h\xi\zeta$ for some $\xi \in X$ and $\zeta \in D$. On the other hand $k = h\alpha$ with $\alpha \in G$ by (1.5). Then $\alpha\zeta^{-1}\xi^{-1} \in H_\mathbf{A}$, so that $\alpha \in H_\mathbf{A}\xi D$. Thus $\alpha \in H\varepsilon\xi D$ for some $\varepsilon \in E_\xi$. Then $\alpha \in H\beta_{\varepsilon,\xi} D \cap G = H\beta_{\varepsilon,\xi}\varGamma^1(\Lambda)$, and hence $k = h\alpha \in hb_{\varepsilon,\xi}\varGamma^1(\Lambda)$. Since the $b_{\varepsilon,\xi}$ form a finite set, we obtain our theorem.

4.3. We now fix a maximal lattice L in V, and put

$$C = \{\gamma \in U^\varphi(V)_\mathbf{A} \mid L\gamma = L\}, \quad C^1 = C \cap SU^\varphi(V)_\mathbf{A}. \tag{4.2}$$

We are going to state our main theorem with respect to a pair (G, H) belonging to the following two types of objects:

Type U: $G = U^\varphi(V)$ and $H = U^\varphi(W)$;

Type SU: $G = SU^\varphi(V)$ and $H = SU^\varphi(W)$.

Here $W = (Kh)^{\perp}$ with a fixed $h \in V$. For a subset S of $U^{\varphi}(V)_{\mathbf{A}}$ the symbol hS is meaningful as a subset of $V_{\mathbf{A}}$. Therefore $V \cap hS$ is a well-defined subset of V.

Theorem 4.4. *Suppose* $\dim(V) > 1$. *For a fixed* $h \in V$ *such that* $\varphi[h] \neq 0$ *put* $W = (Kh)^{\perp}$, *and take* (G, H) *of Type U or SU as above. Let* $D = D_0 G_{\mathbf{a}}$ *with an open compact subgroup* D_0 *of* $G_{\mathbf{h}}$. *Then the following assertions hold.*

(i) *For* $y \in G_{\mathbf{A}}$ *we have* $H_{\mathbf{A}} \cap GyD \neq \emptyset$ *if and only if* $V \cap hDy^{-1} \neq \emptyset$.

(ii) *Fixing* $y \in G_{\mathbf{A}}$, *for every* $\varepsilon \in H_{\mathbf{A}} \cap GyD$ *take* $\alpha \in G$ *so that* $\varepsilon \in \alpha yD$. *Then the map* $\varepsilon \mapsto h\alpha$ *gives a bijection of* $H\backslash(H_{\mathbf{A}} \cap GyD)/(H_{\mathbf{A}} \cap D)$ *onto* $(V \cap hDy^{-1})/\Delta_y$, *where* $\Delta_y = G \cap yDy^{-1}$.

(iii) *Take* $\{y_i\}_{i \in I} \subset G_{\mathbf{A}}$ *so that* $G_{\mathbf{A}} = \bigsqcup_{i \in I} Gy_iD$, *and put* $\Gamma_i = G \cap y_iDy_i^{-1}$. *Then*

$$\#\bigl\{H\backslash H_{\mathbf{A}}/(H_{\mathbf{A}} \cap D)\bigr\} = \sum_{i \in I} \#\bigl\{(V \cap hDy_i^{-1})/\Gamma_i\bigr\}. \tag{4.3}$$

(iv) *Let* $q = \varphi[h]$ *and* $\mathfrak{b} = \varphi(h, L)$. *Then for every* $y \in U^{\varphi}(V)_{\mathbf{A}}$, *we have*

$$V \cap hCy^{-1} = \bigl(Ly^{-1}\bigr)[q, \mathfrak{b}]. \tag{4.4}$$

(v) *Suppose moreover that* $\dim(V) > 2$ *and the following condition is satisfied:*

(4.5) *If* n *is odd, then* $q_v \mathfrak{r}_v \neq \mathfrak{b}_v \mathfrak{b}_v^{\rho}$ *for every* $v \in \mathbf{h}$ *ramified in* K.

Then for every $y \in SU^{\varphi}(V)_{\mathbf{A}}$ *we have*

$$V \cap hC^1y^{-1} = \bigl(Ly^{-1}\bigr)[q, \mathfrak{b}]. \tag{4.6}$$

Proof. Let y, ε, and α be as in (ii); then clearly $h\alpha \in V \cap hDy^{-1}$. If $\eta\varepsilon\zeta \in \beta yD$ with $\eta \in H$, $\zeta \in H_{\mathbf{A}} \cap D$, and $\beta \in G$, then $\beta^{-1}\eta\alpha \in G \cap yDy^{-1} = \Delta_y$, and hence $h\alpha = h\eta\alpha \in h\beta\Delta_y$. Thus our map is well defined. Next let $k \in V \cap hDy^{-1}$. Then $k = h\delta y^{-1}$ with $\delta \in D$, and moreover, by (1.5), $k = h\xi$ with $\xi \in G$. Then $h = h\xi y\delta^{-1}$, so that $\xi y\delta^{-1} \in H_{\mathbf{A}}$. Thus $\xi y\delta^{-1} \in H_{\mathbf{A}} \cap GyD$. This shows that k is the image of an element of $H_{\mathbf{A}} \cap GyD$. To prove that the map is injective, suppose $\varepsilon \in \alpha yD \cap H_{\mathbf{A}}$ and $\delta \in \beta yD \cap H_{\mathbf{A}}$ with α, $\beta \in G$, and $h\alpha = h\beta\sigma$ with $\sigma \in \Delta_y$. Put $\omega = \beta\sigma\alpha^{-1}$. Then $h\omega = h$, so that $\omega \in H$. Since $\sigma \in yDy^{-1}$, we have $\beta yD = \beta\sigma yD = \omega\alpha yD$, and hence $\delta \in \beta yD \cap H_{\mathbf{A}} = \omega\alpha yD \cap H_{\mathbf{A}} = \omega(\alpha yD \cap H_{\mathbf{A}}) = \omega(\varepsilon D \cap H_{\mathbf{A}}) = \omega\varepsilon(D \cap H_{\mathbf{A}}) \subset H\varepsilon(D \cap H_{\mathbf{A}})$. This proves the injectivity, and completes the proof of (ii). At the same time we obtain (i).

Since $H_{\mathbf{A}} = \bigsqcup_{i \in I}(H_{\mathbf{A}} \cap Gy_iD)$, we can derive (iii) immediately from (ii).

As for (v), clearly $V \cap hC^1 \subset L[q, \mathfrak{b}]$. Conversely, if $x \in L[q, \mathfrak{b}]$, then $x \in hC^1$ by Theorem 3.2 (iii). Thus

$$V \cap hC^1 = L[q, \mathfrak{b}]. \tag{4.7}$$

If $k \in V \cap hC^1y^{-1}$, then $hC^1y^{-1} = kyC^1y^{-1}$. Now $\varphi\bigl(k, Ly^{-1}\bigr) = \varphi(h, L) = \mathfrak{b}$. Taking k, yC^1y^{-1}, and Ly^{-1} in place of h, C^1, and L in (4.7), we obtain $V \cap kyC^1y^{-1} = \bigl(Ly^{-1}\bigr)[q, \mathfrak{b}]$. This proves (4.6) when $V \cap hC^1y^{-1} \neq \emptyset$. To prove the remaining case, suppose $\ell \in \bigl(Ly^{-1}\bigr)[q, \mathfrak{b}]$; then $\varphi\bigl(\ell_v y_v, L_v\bigr) = \mathfrak{b}_v = \varphi(h, L)_v$ for every $v \in \mathbf{h}$, and so, by Theorem 3.2 (iii), $\ell y \in hC^1$. Taking ℓ, yC^1y^{-1}, and Ly^{-1} in place of h, C^1, and L in (4.7), we obtain $\emptyset \neq \bigl(Ly^{-1}\bigr)[q, \mathfrak{b}] = V \cap \ell yC^1y^{-1} = V \cap hC^1y^{-1}$. This shows that $\bigl(Ly^{-1}\bigr)[q, \mathfrak{b}] = \emptyset$ if $V \cap hC^1y^{-1} = \emptyset$, and hence (4.6) holds for every $y \in G_{\mathbf{A}}$. This proves (v). Assertion (iv) can be proved in the same way.

In view of (i) we can restrict the indices i on the right-hand side of (4.3) to those for which $H_{\mathbf{A}} \cap Gy_iD \neq \emptyset$. If I' denotes the set of all such i's, then $H_{\mathbf{A}} = \bigsqcup_{j \in I'}(H_{\mathbf{A}} \cap Gy_jD)$.

Combining (4.3) and (4.4), we obtain, for (G, H) of type U, an equality

$$\#\bigl\{H\backslash H_{\mathbf{A}}/(H_{\mathbf{A}} \cap C)\bigr\} = \sum_{i \in I} \#\bigl\{(Ly_i^{-1})[q, \mathfrak{b}]/\varGamma_i\bigr\}, \tag{4.8}$$

where $\{y_i\}$ is such that $G_{\mathbf{A}} = \bigsqcup_{i \in I} Gy_iC$ and $\varGamma_i = G \cap y_iCy_i^{-1}$. We can state a similar formula for (G, H) of type SU when $n > 2$ and (4.5) is satisfied. Formula (4.8) connects the class number of H with respect to $H_{\mathbf{A}} \cap C$ to the solutions h of the equation $\varphi[h] = q$ under the condition $\varphi(h, Ly_i^{-1}) = \mathfrak{b}$.

5. Nonscalar hermitian Diophantine equations

5.1. So far we discussed the equation $\varphi[h] = q$ with a scalar q. We can formulate a similar problem with nonscalar q, which can be stated in terms of matrices as follows. We take F to be local or global. Given $q^* = q \in GL_m(K)$ and $\varphi^* = \varphi \in GL_n(K)$, we consider the solutions $h \in K_n^m$ of the equation $h\varphi h^* = q$. Here and throughout this section we assume $n > m > 0$. More intrinsically, take (V, φ) as before and take also (X, q) with a nondegenerate hermitian form q on a free K-module X of dimension m. We consider $h \in \mathrm{Hom}(X, V)$ such that $\varphi[xh] = q[x]$ for every $x \in X$. Since q is nondegenerate, h must be injective. To simplify our notation, for every $k \in \mathrm{Hom}(X, V)$ we denote by $\varphi[k]$ the hermitian form on X defined by $\varphi[k][x] = \varphi[xk]$ for every $x \in X$. Then our problem concerns the solutions $h \in \mathrm{Hom}(X, V)$ of the equation $\varphi[h] = q$ for a fixed q. If $m = 1$ and $X = K$, then $q \in F^\times$, and an element h of V defines an element of $\mathrm{Hom}(K, V)$ that sends c to ch for $c \in K$, and every element of $\mathrm{Hom}(K, V)$ is of this type. Thus the problem about $\varphi[h] = q$ with $q \in F^\times$ is the one-dimensional special case. Let h be an element of $\mathrm{Hom}(X, V)$ such that $\mathrm{rank}(\varphi[h]) = m$. Then

$$\bigl\{x \in \mathrm{Hom}(X, V) \bigm| \varphi[x] = \varphi[h]\bigr\} = h \cdot SU^\varphi. \tag{5.1}$$

This is similar to (1.5), and follows easily from the Witt theorem in the unitary case. Though we take V to be coordinate-free, it is practical to take X to be

K_m^1, and so take q to be a hermitian element of $GL_m(K)$, and $q[x] = xqx^*$ for $x \in K_m^1$. For $h \in \mathrm{Hom}(X, V)$ and $1 \le i \le m$ we define "the i-th row" of h to be the element h_i of V determined by $ah = \sum_{i=1}^m a_i h_i$ for $a = (a_i)_{i=1}^m \in K_m^1 = X$.

We first prove a local finiteness result that generalizes Theorem 3.3.

Theorem 5.2. *Suppose F is a local field; let Λ be an $\mathfrak{r}$-lattice in $\mathrm{Hom}(X, V)$ and let $D = \{\gamma \in SU^\varphi(V) \mid \Lambda\gamma = \Lambda\}$. Then, given $h \in \mathrm{Hom}(X, V) \cap \Lambda$ such that $\varphi[h]$ is nondegenerate, there exists a finite subset A of $SU^\varphi(V)$ such that*

$$\{x \in \Lambda \mid \varphi[x] = \varphi[h]\} = \textstyle\bigsqcup_{\alpha \in A} h\alpha D. \tag{5.2}$$

Moreover, suppose $K = F \times F$ or K is a field unramified over F; suppose also that $\varphi[h] \in GL_m(\mathfrak{r})$ and $\Lambda = \{\lambda \in \mathrm{Hom}(X, V) \mid \mathfrak{r}_m^1\lambda \subset L\}$ with a maximal lattice L in V. Then we can take $A = \{1\}$.

Proof. The first part is Theorem 3.3 if $m = 1$, and so we assume $m > 1$ and prove (5.2) by induction on m. Put $q = \varphi[h]$. Changing (h, q, Λ) for $(ch, cqc^*, c\Lambda)$ with a suitable $c \in GL_m(K)$, we may assume that $q = \mathrm{diag}[a, \eta]$ with $a \in F^\times$ and $\eta^* = \eta \in GL_{m-1}(K)$. Also we may assume that $\Lambda = \{\kappa \in \mathrm{Hom}(X, V) \mid M\kappa \subset L\}$ with $M = \mathfrak{r}_m^1$ and an $\mathfrak{r}$-lattice L in V. Then $D = \{\alpha \in SU^\varphi(V) \mid L\alpha = L\}$. If $x \in \Lambda$ and $\varphi[x] = q$, then $x_1, h_1 \in L$ and $\varphi[x_1] = \varphi[h_1] = a$, and hence by Theorem 3.3 there exists a finite subset B of L such that such an x_1 belongs to $\bigcup_{b\in B} bD$ and $\varphi[b] = a$ for every $b \in B$. Suppose $x_1 = b\gamma$ with $b \in B$ and $\gamma \in D$. Put $W_b = (Kb)^\perp$, $y = x\gamma^{-1}$ and $z = [y_i]_{i=2}^m$. Then $y_1 = b$ and $\varphi[y] = \varphi[h]$, so that $\varphi[z] = \eta$, and $y_i \in W_b$ for $i > 1$. We can view z as an element of $\mathrm{Hom}(K_{m-1}^1, W_b)$. Then $\mathfrak{r}_{m-1}^1 z \subset L \cap W_b$. Put $E = \{\varepsilon \in SU^\varphi(W_b) \mid (L \cap W_b)\varepsilon = L \cap W_b\}$. By induction there exists a finite subset U_b of $\mathrm{Hom}(K_{m-1}^1, W_b)$ such that

$$\{z \in \mathrm{Hom}(K_{m-1}^1, W_b) \mid \mathfrak{r}_{m-1}^1 z \subset L \cap W_b,\ \varphi[z] = \eta\} = \textstyle\bigsqcup_{u \in U_b} uE.$$

We can find a finite subset S of E such that $E = \bigsqcup_{\sigma\in S}\sigma(D \cap E)$. Then $y = \begin{bmatrix} b \\ z \end{bmatrix} = \begin{bmatrix} b \\ u\sigma\tau \end{bmatrix}$ with $u \in U_b$, $\sigma \in S$, and $\tau \in D \cap E$. Thus $x = \begin{bmatrix} b \\ u\sigma \end{bmatrix}\tau\gamma$, and $\tau\gamma \in D$. This shows that $x \in \bigsqcup_{k\in P} kD$ with a finite subset P of the left-hand side of (5.2), as the elements (b, u, σ) form a finite set. By (5.1), for each $k \in P$ there exists an element α of $SU^\varphi(V)$ such that $k = h\alpha$. This proves the first assertion.

Next suppose that the conditions on K, q, Λ, and φ as in the second assertion are satisfied. Take an $\mathfrak{r}$-basis of L and identify V, L, and φ with K_n^1, $\mathfrak{r}_n^1$, and a hermitian matrix with respect to that basis. Given $\ell \in L = \mathfrak{r}_n^1$, put $z = q^{-1}h\varphi\ell^*$ and $y = \ell - z^*h$. Since $\varphi \prec \mathfrak{r}$ and $q \in GL_m(\mathfrak{r})$, we see that $y \in L$, and for every $w \in X$ we have $wh\varphi y^* = 0$, so that $y \in (Xh)^\perp$. Put $M = \mathfrak{r}_m^1$ and $Y = (Xh)^\perp$. Then $V = Xh \oplus Y$ and $L = Mh \oplus (L \cap Y)$. Suppose $\varphi[k] = q$ with $k \in \Lambda$. Then similarly $L = Mk \oplus (L \cap Z)$ with $Z = (Xk)^\perp$. Since L

is maximal, Mh resp. Mk is maximal in Xh resp. Xk, and $L \cap Y$ resp. $L \cap Z$ is maximal in Y resp. Z. By (5.1) there exists an element $\gamma \in SU^{\varphi}(V)$ such that $h\gamma = k$. Then $Mh\gamma = Mk$, $Y\gamma = Z$, and $(L \cap Y)\gamma$ is $\mathfrak{r}$-maximal in Z, so that by Lemma 3.4 (i), $(L \cap Y)\gamma\varepsilon = L \cap Z$ with some $\varepsilon \in U^{\varphi}(Z)$. Define $\alpha \in GL(V)$ by $\alpha = \gamma$ on Xh and $\alpha = \gamma\varepsilon$ on Y. Then $\alpha \in U^{\varphi}(V)$, $L\alpha = L$, and $h\alpha = k$. Since $\det(\alpha) \in \mathfrak{r}^{\times}$, we can find an element ξ of $U^{\varphi}(Z)$ such that $(L \cap Z)\xi = L \cap Z$ and $\det(\xi) = \det(\alpha)^{-1}$. This is clear if $K = F \times F$; see §3.6. If K is a field unramified over F, then the fact is included in Lemma 1.9. Extend ξ to an element of $U^{\varphi}(V)$ by putting $xk\xi = xk$ for $x \in X$. Then $\alpha\xi \in SU^{\varphi}(V)$, $h\alpha\xi = k$, and $L\alpha\xi = L$. Clearly $\Lambda\alpha\xi = \Lambda$, and hence we obtain (5.2) with $A = \{1\}$. This completes the proof.

Next we prove a generalization of Theorem 4.2, which is a global version of the above theorem.

Theorem 5.3. *Suppose that F is an algebraic number field; let Λ be an $\mathfrak{r}$-lattice in* $\mathrm{Hom}(X, V)$, $\mathit{\Gamma} = \{ \gamma \in SU^{\varphi}(V) \mid \Lambda\gamma = \Lambda \}$, *and* $T_q = \{ x \in \Lambda \mid \varphi[x] = q \}$ *with* $q^* = q \in GL_m(K)$. *Then* $T_q/\mathit{\Gamma}$ *is a finite set.*

Proof. We assume the existence of $h \in T_q$. Put $W = (Xh)^{\perp}$, $G = SU^{\varphi}(V)$, $H = SU^{\varphi}(W)$, $M = \mathfrak{r}_m^1$, $D = \{\gamma \in G_{\mathbf{A}} \mid \Lambda\gamma = \Lambda\}$, and $D_v = D \cap G_v$ for $v \in \mathbf{h}$. We identify H with $\{\alpha \in G \mid h\alpha = h\}$. Fix a maximal lattice L in V. By Theorem 5.2, for each $v \in \mathbf{h}$ there exists a finite subset E_v of G_v such that

$$\{x \in \Lambda_v \mid \varphi[x] = q \} = \textstyle\bigsqcup_{\varepsilon \in E_v} h\varepsilon D_v.$$

Now for almost all $v \in \mathbf{h}$ we have $\Lambda_v = \{\gamma \in \mathrm{Hom}(X_v, V_v) \mid M_v\gamma \subset L_v\}$, L_v is maximal, v is unramified in K, and $q \in GL_m(\mathfrak{r}_v)$. Therefore, by Theorem 5.2, we can take $E_v = \{1\}$ for almost all $v \in \mathbf{h}$. Consequently we can find a finite subset E of $G_{\mathbf{h}}$ such that $T_q \subset \bigcup_{\eta \in E} h\eta D$. If $x \in T_q$, then $x \in hG$ by (5.1). Thus $x = h\alpha = h\eta\delta$ with $\alpha \in G$, $\eta \in E$, and $\delta \in D$. We have $\alpha\delta^{-1}\eta^{-1} \in H_{\mathbf{A}}$, and hence $\alpha \in H_{\mathbf{A}}\eta D$. For each $\eta \in E$ we can find a finite subset Z_η of $H_{\mathbf{h}}$ such that $H_{\mathbf{A}} = \bigsqcup_{\zeta \in Z_\eta} H\zeta(H_{\mathbf{A}} \cap \eta D\eta^{-1})$. Then $H_{\mathbf{A}}\eta D = \bigcup_{\zeta \in Z_\eta} H\zeta\eta D$, and hence $\alpha \in \bigcup_{\eta,\zeta}(G \cap H\zeta\eta D) = \bigcup_{\eta,\zeta} H(G \cap \zeta\eta D)$. For each (ζ, η) such that $G \cap \zeta\eta D \neq \emptyset$, pick any $\beta \in G \cap \zeta\eta D$. Then $G \cap \zeta\eta D = G \cap \beta D = \beta\mathit{\Gamma}$. Let B be the set of such β's chosen for each (ζ, η). Then $\alpha \in \bigcup_{\beta \in B} H\beta\mathit{\Gamma}$, and thus $h\alpha \in \bigcup_{\beta \in B} h\beta\mathit{\Gamma}$, which proves our theorem.

Theorem 5.4. *Suppose that F is an algebraic number field. With a fixed* $h \in \mathrm{Hom}(X, V)$ *such that* $\mathrm{rank}(\varphi[h]) = m$, *put* $q = \varphi[h]$, $W = (Xh)^{\perp}$, $G = U^{\varphi}(V)$, $H = U^{\varphi}(W)$, *and* $\mathscr{V} = \mathrm{Hom}(X, V)$. *Let D be an open subgroup of $G_{\mathbf{A}}$ containing $G_{\mathbf{a}}$ such that $D \cap G_{\mathbf{h}}$ is compact, and let* $G_{\mathbf{A}} = \bigsqcup_{i \in I} Gy_iD$. *Then assertions (i), (ii), and (iii) of Theorem 4.4 are valid if we take the symbols h, G, H, and D there to be those of the present setting, and replace V there by $\mathscr{V}$. In particular we have*

(5.3) $$\#\{H\backslash H_{\mathbf{A}}/(H_{\mathbf{A}}\cap D)\} = \sum_{i\in I}\#\{(\mathscr{V}\cap hDy_i^{-1})/\Gamma_i\},$$

where $\Gamma_i = G\cap y_iDy_i^{-1}$. *The same is true with* $G = SU^{\varphi}(V)$ *and* $H = SU^{\varphi}(W)$.

Proof. We can repeat the proof of Theorem 4.4 with obvious modifications.

Theorem 5.5. *In the setting of Theorem 5.4 with* $G = SU^{\varphi}(V)$ *and* $H = SU^{\varphi}(W)$ *suppose that* $n-m>1$ *and neither* $G_{\mathbf{a}}$ *nor* $H_{\mathbf{a}}$ *is compact. Let* k *be an element of* $\mathrm{Hom}(X, V)$ *such that* $k = h\gamma_v$ *for every* $v\in\mathbf{h}$ *with some* $(\gamma_v)_{v\in\mathbf{h}}\in D\cap G_{\mathbf{h}}$. *Then there exists an element* $\alpha\in G\cap D$ *such that* $k = h\alpha$. *In particular, if* $m=1$ *and (4.5) is satisfied, then* $\#\{L[q, \mathfrak{b}]/\Gamma^1(L)\}\le 1$ *for every maximal lattice* L *in* V.

Proof. By our assumptions, strong approximation holds on G and H, and so we have $G_{\mathbf{A}} = GD$ and $H_{\mathbf{A}} = H(H_{\mathbf{A}}\cap D)$. Thus we can take $\{y_i\}_{i\in I} = \{1\}$. Therefore (5.3) implies that $\#\{(\mathscr{V}\cap hD)/(G\cap D)\} = 1$, which gives the first assertion. This combined with (4.6) proves the second assertion.

5.6. Before proceeding further, let us recall the notion of the mass of an algebraic group G with respect to an open subgroup D of $G_{\mathbf{A}}$ containing $G_{\mathbf{a}}$ and such that $G_{\mathbf{h}}\cap D$ is compact. For simplicity here we take G to be U^{φ} or SU^{φ} and assume that $U^{\varphi}_{\mathbf{a}}$ is compact. For $x\in G_{\mathbf{A}}$ put $\Delta_x = G\cap xDx^{-1}$ and $\nu(\Delta_x) = [\Delta_x : 1]^{-1}$. Then the *the mass of* G *with respect to* D is defined by

(5.4) $$\mathfrak{m}(G, D) = \sum_{b\in\mathscr{B}}\nu(\Delta_b),\quad \mathscr{B} = G\backslash G_{\mathbf{A}}/D.$$

For this the reader is referred to [S2, (10.9.4), (24.1.1), (24.1.2)]. If D' is a subgroup of $G_{\mathbf{A}}$ of the same type as D, then from [S2, Lemma 24.2] we obtain

(5.5) $$[D : D\cap D']\mathfrak{m}(G, D) = \mathfrak{m}(G, D\cap D') = [D' : D\cap D']\mathfrak{m}(G, D').$$

Theorem 5.7. *In the setting of Theorem 5.4, suppose that* $G_{\mathbf{a}}$ *is compact. Then for every* $y\in G_{\mathbf{A}}$ *we have*

(5.6) $$\nu(\Delta_y)\#\{\mathscr{V}\cap hDy^{-1}\} = \sum_{\varepsilon\in\mathscr{E}}\nu(\Delta_\varepsilon),$$

where $\mathscr{E} = H\backslash(H_{\mathbf{A}}\cap GyD)/(H_{\mathbf{A}}\cap D)$ *and* $\Delta_x = H\cap xDx^{-1}$. *Moreover, let* $G_{\mathbf{A}} = \bigsqcup_{i\in I}Gy_iD$ *and* $\Gamma_i = G\cap y_iDy_i^{-1}$; *then*

(5.7) $$\sum_{i\in I}\nu(\Gamma_i)\#\{\mathscr{V}\cap hDy_i^{-1}\} = \mathfrak{m}(H, H_{\mathbf{A}}\cap D).$$

Proof. To prove (5.6), we may assume that $H_{\mathbf{A}}\cap GyD\ne\emptyset$. For $\varepsilon\in\mathscr{E}$ take $\alpha_\varepsilon\in G$ so that $\varepsilon\in\alpha_\varepsilon yD$. Then $H\cap\alpha_\varepsilon\Delta_y\alpha_\varepsilon^{-1} = H\cap\alpha_\varepsilon yDy^{-1}\alpha_\varepsilon^{-1} = H\cap\varepsilon D\varepsilon^{-1} = \Delta_\varepsilon$. Now $\mathscr{V}\cap hDy^{-1} = \bigsqcup_{\varepsilon\in\mathscr{E}}h\alpha_\varepsilon\Delta_y$ by the part of Theorem 5.4 corresponding to Theorem 4.4 (ii). For $\gamma, \gamma'\in\Gamma(\Lambda)$ we have $h\alpha_\varepsilon\gamma = h\alpha_\varepsilon\gamma'$ if and only if $\alpha_\varepsilon\gamma'\gamma^{-1}\alpha_\varepsilon^{-1}\in H$, that is, $\gamma'\gamma^{-1}\in\alpha_\varepsilon^{-1}H\alpha_\varepsilon\cap\Delta_y = \alpha_\varepsilon^{-1}\Delta_\varepsilon\alpha_\varepsilon$, so that

$$\#\{h\alpha_\varepsilon\Delta_y\} = [\Delta_y : \alpha_\varepsilon^{-1}\Delta_\varepsilon\alpha_\varepsilon] = \nu(\Delta_\varepsilon)/\nu(\Delta_y).$$

Therefore we obtain (5.6). Next, let $\mathscr{E}_i = H\backslash(H_{\mathbf{A}} \cap Gy_iD)/(H_{\mathbf{A}} \cap D)$. Then $H_{\mathbf{A}} = \bigsqcup_{i\in I}(H_{\mathbf{A}} \cap Gy_iD) = \bigsqcup_{i\in I}\bigsqcup_{\varepsilon\in\mathscr{E}_i} H\varepsilon(H_{\mathbf{A}} \cap D)$, and so $\mathfrak{m}(H,\, H_{\mathbf{A}} \cap D) = \sum_{i\in I}\sum_{\varepsilon\in\mathscr{E}_i}\nu(\Delta_\varepsilon)$, which combined with (5.6) proves (5.7).

Corollary 5.8. *Define C and C^1 by (4.2) with a maximal lattice L in V; take $(G,\, H)$ of type U as in Theorem 4.4; suppose that $G_{\mathbf{a}}$ is compact. Let $G_{\mathbf{A}} = \bigsqcup_{i\in I} Gy_iC$, $L_i = Ly_i^{-1}$, and $\Gamma_i = \Gamma(L_i)$. Then*

$$\sum_{i\in I}\nu(\Gamma_i)\#\{L_i[q,\, \mathfrak{b}]\} = \mathfrak{m}(H,\, H_{\mathbf{A}} \cap C), \tag{5.8}$$

where $q = \varphi[h]$ and $\mathfrak{b} = \varphi(h,\, L)$. This is valid for $(G,\, H)$ of type SU if we replace C and $\Gamma(\cdot\cdot)$ by C^1 and $\Gamma^1(\cdot\cdot)$, provided $n > 2$ and (4.5) is satisfied.

Proof. Take $m = 1$ and $D = C$ in Theorem 5.7. Combining (5.7) with (4.4), we obtain (5.8). The case of SU^φ follows similarly from (4.6).

5.9. Formulas (5.7) and (5.8) are similar to, but different from, the formula of Siegel about $\sum_i \nu(\Gamma_i)\#\{L_i[q]\}$. We already explained in [S3, §13.13] the main differences between our formulas in the orthogonal case given in that book and that of Siegel. In principle, our comments there apply to the present unitary case.

Now in [S2, Theorem 24.4] we gave an exact formula for $\mathfrak{m}(G,\, D)$ for $G = U^\varphi$ and a certain type of D, under the condition that if n is odd, then $d_0(\varphi)$ is represented by an element of $\mathfrak{g}^\times$. The group $H_{\mathbf{A}}\cap C$ in (5.8) does not necessarily belong to the types of D there, but we can compute $[D : H_{\mathbf{A}} \cap C]$ by means of Lemma 3.14 under some conditions on $(q,\, \mathfrak{b})$. Then we obtain $\mathfrak{m}(H,\, H_{\mathbf{A}} \cap C)$ from [S2, Theorem 24.4] by (5.5).

Proposition 5.10. *In the setting of Theorem 5.4 suppose that $n - m$ is odd. Then the structure $(W,\, \det(q)\varphi)$ depends only on φ and the indices of q at the real archimedean primes of F ramified in K.*

Proof. Let ψ be the restriction of φ to W. Then we can easily verify that $d_0\big(\det(q)\psi\big) = \det(q)^{n-m}d_0(\psi) = (-1)^{n-1}d_0(\varphi)$ as $m - n$ is odd. This combined with Theorem 2.2 (i) proves our proposition.

This is an analogue of the fact concerning a quadratic form in even dimension with square discriminant given in [S4, Theorem 1.12].

We insert here some results about the relationship between various invariants associated with U^φ and those with SU^φ.

Proposition 5.11. *Let D be an open subgroup of $U^\varphi_{\mathbf{A}}$ containing $U^\varphi_{\mathbf{a}}$ and such that $U^\varphi_{\mathbf{h}}\cap D$ is compact; put $P = \{x \in K^\times \mid xx^\rho = 1\}$ and $D^1 = D\cap SU^\varphi_{\mathbf{A}}$. Then $U^\varphi SU^\varphi_{\mathbf{A}}D$ is a normal subgroup of $U^\varphi_{\mathbf{A}}$ and*

(5.9a) $$[U^\varphi_{\mathbf{A}} : U^\varphi SU^\varphi_{\mathbf{A}} D] = [P_{\mathbf{A}} : P\det(D)],$$

(5.9b) $$\#(U^\varphi\backslash U^\varphi_{\mathbf{A}}/D) \le \sum_{x\in\Xi} \#(SU^\varphi\backslash SU^\varphi_{\mathbf{A}}/xD^1x^{-1}),$$

where $\Xi = U^\varphi_{\mathbf{A}}/U^\varphi SU^\varphi_{\mathbf{A}} D$*. Moreover, if* $U^\varphi_{\mathbf{a}}$ *is compact, then*

(5.9c) $$\mathfrak{m}(U^\varphi,\, D) \le [P_{\mathbf{A}} : P\det(D)]\cdot \mathfrak{m}(SU^\varphi,\, D^1).$$

Furthermore, if $P\cap\det(D) = \det(U^\varphi\cap yDy^{-1})$ *for every* $y\in U^\varphi_{\mathbf{A}}$*, then the equality holds in (5.9b), and*

(5.9d) $$\#\bigl(P\cap\det(D)\bigr)\mathfrak{m}(U^\varphi,\, D) = [P_{\mathbf{A}} : P\det(D)]\mathfrak{m}(SU^\varphi,\, D^1).$$

Proof. Since $P = \det(U^\varphi)$, we can easily show that

(5.10) $$U^\varphi SU^\varphi_{\mathbf{A}} Dx = \bigl\{y\in U^\varphi_{\mathbf{A}} \bigm| \det(y)\in P\det(Dx)\bigr\}$$

for every $x\in U^\varphi_{\mathbf{A}}$. This shows that $U^\varphi SU^\varphi_{\mathbf{A}} D$ is a normal subgroup of $U^\varphi_{\mathbf{A}}$ and $U^\varphi_{\mathbf{A}}/U^\varphi SU^\varphi_{\mathbf{A}} D$ is isomorphic to $P_{\mathbf{A}}/[P\det(D)]$, as $P_{\mathbf{A}} = \det(U^\varphi_{\mathbf{A}})$. Thus we obtain (5.9a); we also see that $U^\varphi SU^\varphi_{\mathbf{A}}\backslash U^\varphi_{\mathbf{A}}/D$ can be identified with $U^\varphi_{\mathbf{A}}/U^\varphi SU^\varphi_{\mathbf{A}} D$. Given $x\in U^\varphi_{\mathbf{A}}$, take $B_x\subset SU^\varphi_{\mathbf{A}}$ so that $SU^\varphi_{\mathbf{A}} = \bigsqcup_{b\in B_x} SU^\varphi bxD^1x^{-1}$. Then we have $U^\varphi SU^\varphi_{\mathbf{A}} xD = \bigcup_{b\in B_x} U^\varphi bxD$, and hence $U^\varphi_{\mathbf{A}} = \bigcup_{x\in\Xi}\bigcup_{b\in B_x} U^\varphi bxD$. From this we obtain (5.9b). To prove (5.9c), put $\Gamma_x = U^\varphi\cap xDx^{-1}$ and $\Gamma^1_x = SU^\varphi\cap xD^1x^{-1}$ for $x\in U^\varphi_{\mathbf{A}}$. Then $\mathfrak{m}(U^\varphi,\, D)\le \sum_{x\in\Xi}\sum_{b\in B_x}\nu(\Gamma_{bx})\le \sum_{x\in\Xi}\sum_{b\in B_x}\nu(\Gamma^1_{bx}) = \sum_{x\in\Xi}\mathfrak{m}(SU^\varphi,\, xD^1x^{-1})$. Now formula (5.5) shows that $\mathfrak{m}(SU^\varphi,\, D^1)$ depends only on the measure of D^1. (If $U^\varphi_{\mathbf{a}}$ is not compact, we have to consider the measure of $D^1_{\mathbf{h}}$.) Since $\mathfrak{m}(SU^\varphi,\, xD^1x^{-1}) = \mathfrak{m}(SU^\varphi, D^1)$, we obtain (5.9c).

Suppose $P\cap\det(D) = \det(\Gamma_y)$ for every $y\in U^\varphi_{\mathbf{A}}$. Suppose also that $b'x = abxd$ for $a\in U^\varphi$, $d\in D$, and $b, b'\in B_x$. Then $\det(a) = \det(d^{-1})\in P\cap\det(D) = \det(\Gamma_{bx})$, and so $\det(a) = \det(c)$ with $c\in\Gamma_{bx}$. Put $e = x^{-1}b^{-1}cbx$. Then $e\in D$, $\det(ed) = 1$, and $b'x = abxd = ac^{-1}bxed\in SU^\varphi bxD^1$. Thus $b' = b$. This shows that $U^\varphi SU^\varphi_{\mathbf{A}} xD = \bigsqcup_{b\in B_x} SU^\varphi bxD$, from which we obtain the equality in (5.9b). Also, $\nu(\Gamma^1_{bx})/\nu(\Gamma_{bx}) = [\Gamma_{bx} : \Gamma^1_{bx}] = \#\bigl(\det(\Gamma_{bx})\bigr) = \#\bigl(P\cap\det(D)\bigr)$, and so

$$\begin{aligned}\#\bigl(P\cap\det(D)\bigr)\mathfrak{m}(U^\varphi,\, D) &= \sum_{x\in\Xi}\sum_{b\in B_x}\nu(\Gamma^1_{bx})\\ &= \sum_{x\in\Xi}\mathfrak{m}(SU^\varphi,\, xD^1x^{-1}) = \#(\Xi)\cdot\mathfrak{m}(SU^\varphi,\, D^1),\end{aligned}$$

which is (5.9d).

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The Critical Values of Certain Dirichlet Series

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Abstract. We investigate the values of several types of Dirichlet series $D(s)$ for certain integer values of s, and give explicit formulas for the value $D(s)$ in many cases. The easiest types of D are Dirichlet L-functions and their variations; a somewhat more complex case involves elliptic functions. There is one new type that includes $\sum_{n=1}^{\infty}(n^2+1)^{-s}$ for which such values have not been studied previously.

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Introduction

By a Dirichlet character modulo a positive integer d we mean as usual a $\mathbf{C}$-valued function χ on $\mathbf{Z}$ such that $\chi(x) = 0$ if x is not prime to d, and χ induces a character on $(\mathbf{Z}/d\mathbf{Z})^\times$. In this paper we always assume that χ is primitive and nontrivial, and so $d > 1$. For such a χ we put

$$L(s,\,\chi) = \sum_{n=1}^{\infty}\chi(n)n^{-s}. \tag{0.1}$$

It is well known that if k is a positive integer such that $\chi(-1) = (-1)^k$, then $L(k,\,\chi)$ is π^k times an algebraic number, or equivalently, $L(1-k,\,\chi)$ is an algebraic number. In fact, there is a well-known formula, first proved by Hecke in [3]:

$$kd^{1-k}L(1-k,\,\chi) = -\sum_{a=1}^{d-1}\chi(a)B_k(a/d), \tag{0.2}$$

where $B_k(t)$ is the Bernoulli polynomial of degree k. Actually Hecke gave the result in terms of $L(k,\,\chi)$, but here we state it in the above form. Hecke's proof is based on a classical formula

$$B_k(t) = -k!(2\pi i)^{-k}\sum_{0\neq h\in\mathbf{Z}}h^{-k}\mathbf{e}(ht) \qquad (0<k\in\mathbf{Z},\ 0<t<1). \tag{0.3}$$

There is also a well-known proof of (0.2), which is essentially the functional equation of $L(s,\,\chi)$ combined with a proof of (0.3). We will not discuss it in the present paper, as it is not particularly inspiring.

In [9] we gave many formulas for $L(1-k,\,\chi)$ different from (0.2). The primary purpose of the present paper is to give elementary proofs for some of them, as well as (0.2), and discuss similar values of a few more types of Dirichlet series. The point of our new proofs can be condensed to the following statement: *We find infinite sum expressions for $L(s,\,\chi)$, which are valid for all $s \in \mathbf{C}$ and so can be evaluated at $s = 1-k$, whereas the old proof of Hecke and our proofs in [9]* employ calculations at $s = k$ and involve the Gauss sum of $\overline{\chi}$.

To make our exposition smooth we put

$$\mathbf{e}(z) = \exp(2\pi i z) \qquad (z \in \mathbf{C}), \tag{0.4}$$

$$H = \bigl\{ z \in \mathbf{C} \,\big|\, \mathrm{Im}(z) > 0 \bigr\}. \tag{0.5}$$

The three additional types of Dirichlet series we consider naturally involve a complex variable s, and are defined as follows:

$$D^{\nu}(s;\,a,\,p) = \sum_{-a \neq n \in \mathbf{Z}} (n+a)^{\nu}|n+a|^{-\nu-s}\mathbf{e}\bigl(p(n+a)\bigr), \tag{0.6}$$

where $a \in \mathbf{R}$, $p \in \mathbf{R}$, and $\nu = 0$ or 1,

$$\mathcal{L}_k(s,\,z) = \sum_{m \in \mathbf{Z}} \mathbf{e}(mr)(z+m)^{-k}|z+m|^{-2s} \qquad (k \in \mathbf{Z},\ r \in \mathbf{Q},\ z \in H), \tag{0.7}$$

$$\varphi_{\nu}(u,\,s;\,L) = \sum_{\alpha \in L} (u+\alpha)^{-\nu}|u+\alpha|^{\nu-2s}, \tag{0.8}$$

where L is a lattice in $\mathbf{C}$, $0 \leq \nu \in \mathbf{Z}$, and $u \in \mathbf{C},\notin L$. We should also note

$$\mathfrak{E}(z,\,s) = \mathrm{Im}(z)^{s} \sum_{(m,\,n)} (mz+n)^{-k}|mz+n|^{-2s} \qquad (0 \leq k \in \mathbf{Z},\, z \in H), \tag{0.9}$$

where $(m,\,n)$ runs over the nonzero elements of $\mathbf{Z}^2$. The value $\mathfrak{E}(z,\,\mu)$ for an integer μ such that $1-k \leq \mu \leq 0$ was already discussed in [9], and so it is not the main object of study in this paper, but we mention it because (0.8) is a natural analogue of (0.9). We will determine in Section 3 the value $\varphi_{\nu}(u,\,\kappa/2;\,L)$ for an integer κ such that $2-\nu \leq \kappa \leq \nu$, which may be called a *nearly holomorphic elliptic function.* Now (0.6) is closely connected with $L(s,\,\chi)$. In [9, Theorem 4.2] we showed that $D^{\nu}(k;\,a,\,p)$ for $0 < k \in \mathbf{Z}$ is elementary factors times the value of a generalized Euler polynomial $E_{c,k-1}(t)$ at $t = p$. In Section 2 we will reformulate this in terms of $D^{\nu}(1-k;\,a,\,p)$. Finally, the nature of the series of (0.7) is quite different from the other types. We will show in Section 4 that $i^k\mathcal{L}_k(\beta,\,z)$ is a $\mathbf{Q}$-rational expression in π, $\mathbf{e}(z/N)$, and $\mathrm{Im}(z)$, if $\beta \in \mathbf{Z}$ and $-k < \beta \leq 0$, where N is the smallest positive integer such that $Nr \in \mathbf{Z}$. Similar results will also be given under other conditions on β. In the final section we will make some comments in the case where the base field is an algebraic number field.

1. $L(1-k,\,\chi)$

1.1. We start with an elementary proof of (0.2). Strange as it may sound, the main idea is the binomial theorem. We first note

$$B_n(t)=\sum_{\nu=0}^{n}\binom{n}{\nu}B_\nu t^{n-\nu}\qquad(0\le n\in\mathbf{Z}),\tag{1.1}$$

$$B_0=1,\quad \zeta(0)=-1/2=B_1,\tag{1.2a}$$

$$n\zeta(1-n)=-B_n\qquad(1<n\in\mathbf{Z}),\tag{1.2b}$$

where B_n is the nth Bernoulli number. Formulas (1.1) and (1.2a) are well-known; (1.2b) is usually given only for even n, but actually true also for odd n, since $\zeta(-2m)=0=B_{2m+1}$ for $0<m\in\mathbf{Z}$.

To prove (0.2), we first make a trivial calculation:

$$\begin{aligned}L(s,\,\chi)-\sum_{a=1}^{d-1}\chi(a)a^{-s}&=\sum_{m=1}^{\infty}\sum_{a=1}^{d-1}\chi(dm+a)(dm+a)^{-s}\\&=\sum_{m=1}^{\infty}\sum_{a=1}^{d-1}\chi(a)(dm)^{-s}\left(1+\frac{a}{dm}\right)^{-s}.\end{aligned}$$

Now we apply the binomial theorem to $(1+X)^{-s}$. Thus the last double sum equals

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{a=1}^{d-1}\chi(a)(dm)^{-s}\sum_{r=0}^{\infty}\binom{-s}{r}\left(\frac{a}{dm}\right)^{r}\\&=\sum_{r=0}^{\infty}\binom{-s}{r}\sum_{m=1}^{\infty}m^{-s-r}d^{-s}\sum_{a=1}^{d-1}\chi(a)(a/d)^{r},\end{aligned}\tag{1.3}$$

where

$$\binom{\tau}{r}=\frac{\tau(\tau-1)\ldots(\tau-r+1)}{r!},$$

which is of course understood to be 1 if $r=0$. So far our calculation is formal, but can be justified at least for $\mathrm{Re}(s)>1$. Indeed, put $\mathrm{Re}(s)=\sigma$ and $|s|=\alpha$. Then

$$\left|\binom{-s}{r}\right|\le\frac{\alpha(\alpha+1)\ldots(\alpha+r-1)}{r!}=(-1)^r\binom{-\alpha}{r}.\tag{1.4}$$

Therefore the triple sum obtained from (1.3) by taking the absolute value of each term is majorized by

$$\begin{aligned}&\sum_{a=1}^{d-1}\sum_{r=0}^{\infty}\sum_{m=1}^{\infty}m^{-\sigma-r}d^{-\sigma}\binom{-\alpha}{r}\left(\frac{-a}{d}\right)^{r}\\&\le\zeta(\sigma)d^{-\sigma}\sum_{a=1}^{d-1}\sum_{r=0}^{\infty}\binom{-\alpha}{r}\left(\frac{-a}{d}\right)^{r}=\zeta(\sigma)d^{-\sigma}\sum_{a=1}^{d-1}\left(1-\frac{a}{d}\right)^{-\alpha}\end{aligned}$$

if $\sigma > 1$. Thus, for $\mathrm{Re}(s) > 1$, (1.3) can be justified, and so

$$L(s,\,\chi) - \sum_{a=1}^{d-1}\chi(a)a^{-s} = \sum_{r=0}^{\infty}\binom{-s}{r}\zeta(s+r)d^{-s}\sum_{a=1}^{d-1}\chi(a)(a/d)^r. \tag{1.5}$$

We can show that the last sum $\sum_{r=0}^{\infty}$ defines a meromorphic function in s on the whole $\mathbf{C}$. For that purpose given $s \in \mathbf{C}$, take a positive integer μ so that $\mathrm{Re}(s) > -\mu$ and decompose the sum into $\sum_{r=0}^{\mu+1}$ and $\sum_{r=\mu+2}^{\infty}$. There is no problem about the first sum, as it is finite. As for the latter, we have $|\zeta(s+r)| \le \zeta(2)$ for $r \ge \mu+2$. Putting $\varepsilon = (d-1)/d$, we have $|\sum_{a=1}^{d-1}\chi(a)(a/d)^r| \le (d-1)\varepsilon^r$. Therefore for $\mathrm{Re}(s) > -\mu$ the infinite sum $\sum_{r=\mu+2}^{\infty}$ can be majorized by

$$d^{\mu}(d-1)\zeta(2)\sum_{r=0}^{\infty}\binom{-\alpha}{r}(-\varepsilon)^r = d^{\mu}(d-1)\zeta(2)(1-\varepsilon)^{-\alpha}.$$

This proves the desired meromorphy of the right-hand side of (1.5).

Now, for $0 < k \in \mathbf{Z}$, without assuming that $\chi(-1) = (-1)^k$, we evaluate (1.5) at $s = 1-k$. We easily see that $\binom{-s}{r} = 0$ and $\zeta(s+r)$ is finite for $s = 1-k$ if $r > k$. We have to be careful about the term for $r = k$, as $\zeta(s+k)$ has a pole at $s = 1-k$. Since

$$\lim_{s\to 1-k}\binom{-s}{k}\zeta(s+k) = \lim_{s\to 1-k}\binom{-s}{k}\frac{1}{s-1+k} = \frac{-1}{k}, \tag{1.6}$$

the term for $r = k$ at $s = 1-k$ produces $-(d^{k-1}/k)\sum_{a=1}^{d-1}\chi(a)(a/d)^k$. Thus the evaluation of (1.5) at $s = 1-k$ gives

$$\begin{aligned} kd^{1-k}L(1-k,\,\chi) &= k\sum_{a=1}^{d-1}\chi(a)(a/d)^{k-1} \\ &\quad - \sum_{a=1}^{d-1}\chi(a)(a/d)^k + \sum_{r=0}^{k-1}k\binom{k-1}{r}\zeta(1+r-k)\sum_{a=1}^{d-1}\chi(a)(a/d)^r. \end{aligned} \tag{1.7}$$

By (1.2b) we have, for $0 \le r < k-1$,

$$k\binom{k-1}{r}\zeta(1+r-k) = \frac{-k}{k-r}\binom{k-1}{r}B_{k-r} = -\binom{k}{r}B_{k-r}.$$

The term for $r = k-1$ produces $k\zeta(0)\sum_{a=1}^{d-1}\chi(a)(a/d)^{k-1}$, which combined with the first term on the right-hand side of (1.7) gives $-kB_1\sum_{a=1}^{d-1}\chi(a)(a/d)^{k-1}$. Thus we obtain

$$kd^{1-k}L(1-k,\,\chi) = -\sum_{r=0}^{k}\binom{k}{r}B_{k-r}\sum_{a=1}^{d-1}\chi(a)(a/d)^r,$$

which together with (1.1) proves (0.2). Notice that we did not assume that $\chi(-1) = (-1)^k$, and so we proved (0.2) for every positive integer k. If $\chi(-1) = (-1)^{k-1}$, we have $L(1-k,\,\chi) = 0$, which means that the right-hand side of (0.2) is 0 if $\chi(-1) = (-1)^{k-1}$. This can be proved more directly; see [9, (4.28)].

In the above calculation the term for $r = 0$ actually vanishes, as $\sum_{a=1}^{d-1} \chi(a) = 0$. However, we included the term for the following reason. In later subsections we will consider similar infinite sums with r ranging from 0 to ∞, of which the terms for $r = 0$ are not necessarily zero.

1.2. By the same technique as in §1.1 (that is, employing the binomial theorem) we will express $L(1-k, \chi)$ explicitly in terms of a polynomial Φ_{k-1} of degree $k-1$. Writing n for $k-1$, the polynomial is defined by

$$\Phi_n(t) = t^n - \sum_{\nu=1}^{[(n+1)/2]} \binom{n}{2\nu-1} (2^{2\nu}-1)\frac{B_{2\nu}}{\nu} t^{n+1-2\nu} \quad (0 \le n \in \mathbf{Z}), \tag{1.8}$$

where B_ν denotes the Bernoulli number as before. We understand that $\Phi_0(t) = 1$. We will eventually show that Φ_n is the classical Euler polynomial of degree n, but we prove Theorem 1.4 below with this definition of Φ_n, with no knowledge of the Euler polynomial. We first prove:

LEMMA 1.3. *Let χ be a primitive Dirichlet character of conductor $4d_0$ with $0 < d_0 \in \mathbf{Z}$. Then $\chi(a - 2d_0) = -\chi(a)$ for every $a \in \mathbf{Z}$.*

Proof. We may assume that a is prime to $2d_0$, as the desired equality is trivial otherwise. Then we can find an integer b such that $ab - 1 \in 4d_0\mathbf{Z}$, and we have $\chi(a - 2d_0) = \chi(a)\chi(1 - 2d_0 b)$. Since $(1 - 2d_o b)^2 - 1 \in 4d_0\mathbf{Z}$, we have $\chi(1 - 2d_o b) = \pm 1$. Suppose $\chi(1 - 2d_o b) = 1$; let $x = 1 - 2d_0 y$ with $y \in \mathbf{Z}$. Then $x^b - (1 - 2d_0 b)^y \in 4d_0\mathbf{Z}$, and so $\chi(x)^b = 1$. Thus $\chi(x) = 1$, as b is odd. This shows that the conductor of χ is a divisor of $2d_0$, a contradiction. Therefore $\chi(1 - 2d_0 b) = -1$, which proves the desired fact.

THEOREM 1.4. *Let χ be a nontrivial primitive Dirichlet character modulo d, and let k be a positive integer such that $\chi(-1) = (-1)^k$.*

(i) *If $d = 2q + 1$ with $0 < q \in \mathbf{Z}$, then*

$$L(1-k, \chi) = \frac{d^{k-1}}{2^k \chi(2) - 1} \sum_{b=1}^{q} (-1)^b \chi(b) \Phi_{k-1}(b/d). \tag{1.9}$$

(ii) *If $d = 4d_0$ with $1 < d_0 \in \mathbf{Z}$, then*

$$L(1-k, \chi) = (2d_0)^{k-1} \sum_{a=1}^{d_0-1} \chi(a) \Phi_{k-1}(2a/d). \tag{1.10}$$

Before proving these, we note that these formulas are *better than* (0.2) in the sense that $\Phi_{k-1}(t)$ is of degree $k-1$, whereas $B_k(t)$ is of degree k.

Proof. We first put

$$Z(s) = \sum_{n=1}^{\infty} (-1)^n n^{-s}, \qquad \Lambda(s) = \sum_{n=1}^{\infty} (-1)^n \chi(n) n^{-s}.$$

We easily see that

$$\Lambda(s) + L(s,\,\chi) = 2\sum_{n=1}^{\infty}\chi(2n)(2n)^{-s} = \chi(2)2^{1-s}L(s,\,\chi),$$

and a similar equality holds for $Z(s)$. Thus

$$Z(s) = \zeta(s)(2^{1-s} - 1), \qquad \Lambda(s) = L(s,\chi)\bigl\{\chi(2)2^{1-s} - 1\bigr\}.$$

We prove (i) by computing $\Lambda(1-k)$ for a given k in the same elementary way as we did in §1.1. With q as in (i) we observe that every positive integer m not divisible by d can be written uniquely $m = nd + a$ with $0 \le n \in \mathbf{Z}$ or $m = nd - a$ with $0 < n \in \mathbf{Z}$, where in either case a is in the range $0 < a \le q$. Therefore

$$\begin{aligned}\Lambda(s) &= \sum_{a=1}^{q}(-1)^a\chi(a)a^{-s}\\ &+ \sum_{a=1}^{q}\sum_{n=1}^{\infty}\Bigl\{(-1)^{nd+a}\chi(nd+a)(nd+a)^{-s} + (-1)^{nd-a}\chi(nd-a)(nd-a)^{-s}\Bigr\}.\end{aligned}$$

The last double sum can be written

$$\sum_{a=1}^{q}\sum_{n=1}^{\infty}(-1)^{n+a}d^{-s}n^{-s}\left\{\chi(a)\left(1+\frac{a}{nd}\right)^{-s} + \chi(-a)\left(1-\frac{a}{nd}\right)^{-s}\right\}.$$

Applying the binomial theorem to $(1 \pm X)^{-s}$, we obtain

$$\begin{aligned}\Lambda(s) &- \sum_{a=1}^{q}(-1)^a\chi(a)a^{-s}\\ &= \sum_{n=1}^{\infty}\sum_{r=0}^{\infty}(-1)^n(dn)^{-r-s}\binom{-s}{r}\bigl\{1+(-1)^{r+k}\bigr\}\sum_{a=1}^{q}(-1)^a\chi(a)a^r\\ &= \sum_{r=0}^{\infty}d^{-s}\binom{-s}{r}Z(s+r)\bigl\{1+(-1)^{r+k}\bigr\}\sum_{a=1}^{q}(-1)^a\chi(a)(a/d)^r.\end{aligned}$$

By the same technique as in §1.1, we can justify this for $\mathrm{Re}(s) > 1$. We can even show that the last sum $\sum_{r=0}^{\infty}$ is absolutely convergent for every $s \in \mathbf{C}$ as follows. We first note that Z is an entire function. Take a positive integer μ and s so that $\mathrm{Re}(s) > -\mu$. Then for $r \ge \mu+2$ we have $|Z(s+r)| \le \zeta(2)$. Put $|s| = \alpha$. We have also $|\sum_{a=1}^{q}(-1)^a\chi(a)(a/d)^r| \le 2^{-r}q$. Therefore for $\mathrm{Re}(s) > -\mu$ the infinite sum $\sum_{r=\mu+2}^{\infty}$ can be majorized by

$$2d^{\mu}\zeta(2)q\sum_{r=0}^{\infty}\binom{-\alpha}{r}(-2)^{-r} = 2d^{\mu}\zeta(2)q(1-2^{-1})^{-\alpha}.$$

This proves the desired convergence of $\sum_{r=0}^{\infty}$. Substituting $1-k$ for s in the above equality, we obtain $\Lambda(1-k)$ as an infinite sum, which is actually a finite sum, because $\binom{k-1}{r} = 0$ if $r \ge k$. (This time, the term $r = k$ causes no problem.) Also, we need only those r such that $k-r \in 2\mathbf{Z}$. Putting $k-r = 2\nu$, we find that

$$d^{1-k}L(1-k,\,\chi)\{\chi(2)2^k-1\} = d^{1-k}\Lambda(1-k)$$
$$= \sum_{a=1}^{q}(-1)^a\chi(a)(a/d)^{k-1} + 2\sum_{\nu=1}^{[k/2]}\binom{k-1}{2\nu-1}Z(1-2\nu)\sum_{a=1}^{q}(-1)^a\chi(a)(a/d)^{k-2\nu}.$$

From (1.2b) we obtain

$$2\nu Z(1-2\nu) = 2\nu(2^{2\nu}-1)\zeta(1-2\nu) = (1-2^{2\nu})B_{2\nu}. \tag{1.11}$$

Using this expression for $Z(1-2\nu)$, we obtain a formula for $L(1-k,\,\chi)$. Then comparison of it with our definition of Φ_n proves (1.9).

Next, let $d = 4d_0$ with $1 < d_0 \in \mathbf{Z}$ as in (ii). Observe that the set of all positive integers greater than d_0 and not divisible by d_0 is the disjoint union of the sets

$$\{4\nu d_0 \pm a \mid 0 < a < d_0,\, 0 < \nu \in \mathbf{Z}\} \sqcup \{(4\nu+2)d_0 \pm a \mid 0 < a < d_0,\, 0 \le \nu \in \mathbf{Z}\}.$$

Clearly $\chi(4\nu d_0 \pm a) = \chi(\pm a)$; also $\chi\big((4\nu+2)d_0 \pm a\big) = -\chi(\pm a)$ by Lemma 1.3. Therefore we have

$$\begin{aligned} L(s,\,\chi) &= \sum_{a=1}^{d_0-1}\chi(a)a^{-s} \\ &+ \sum_{\nu=1}^{\infty}\sum_{a=1}^{d_0-1}\{\chi(a)(4\nu d_0+a)^{-s} + \chi(-a)(4\nu d_0-a)^{-s}\} \\ &- \sum_{\nu=0}^{\infty}\sum_{a=1}^{d_0-1}\{\chi(a)\big((4\nu+2)d_0+a\big)^{-s} + \chi(-a)\big((4\nu+2)d_0-a\big)^{-s}\}. \end{aligned}$$

Employing the binomial theorem in the same manner as before, we have

$$\begin{aligned} L(s,\,\chi) &- \sum_{a=1}^{d_0-1}\chi(a)a^{-s} \\ &= \sum_{\nu=1}^{\infty}\sum_{r=0}^{\infty}\binom{-s}{r}(4\nu d_0)^{-s-r}\{1+(-1)^{k+r}\}\sum_{a=1}^{d_0-1}\chi(a)a^r \\ &- \sum_{\nu=0}^{\infty}\sum_{r=0}^{\infty}\binom{-s}{r}\big((4\nu+2)d_0\big)^{-s-r}\{1+(-1)^{k+r}\}\sum_{a=1}^{d_0-1}\chi(a)a^r. \end{aligned}$$

Notice that $\sum_{\nu=1}^{\infty}(4\nu)^{-s} - \sum_{\nu=0}^{\infty}(4\nu+2)^{-s} = 2^{-s}Z(s)$. Therefore

$$L(s,\,\chi) = \sum_{a=1}^{d_0-1}\chi(a)a^{-s} + \sum_{r=0}^{\infty}\binom{-s}{r}(2d_0)^{-s-r}Z(s+r)\{1+(-1)^{k+r}\}\sum_{a=1}^{d_0-1}\chi(a)a^r.$$

The validity of this formula for all $s \in \mathbf{C}$ can be proved in the same way as in the previous case. The last infinite sum $\sum_{r=0}^{\infty}$ evaluated at $s = 1-k$ becomes a finite sum $\sum_{r=0}^{k-1}$, which is actually extended only over those r such that $k - r = 2\nu$ with $\nu \in \mathbf{Z}$. Therefore, using (1.11), we obtain (1.10).

1.5. Let us now show that Φ_n coincides with the classical Euler polynomial. In [9, (4.2)] we defined polynomials $E_{c,\,n}(t)$ for $c = -\mathbf{e}(\alpha)$ with $\alpha \in \mathbf{R}, \notin \mathbf{Z}$, by

$$\frac{(1+c)e^{tz}}{e^z+c} = \sum_{n=0}^{\infty} \frac{E_{c,n}(t)}{n!} z^n. \tag{1.12}$$

If $c = 1$, the polynomial $E_{1,n}(t)$ is the classical Euler polynomial of degree n. Our task is to prove

$$E_{1,n} = \Phi_n. \tag{1.13}$$

We first note here some basic formulas:

$$E_{c,n}(t) = (1+c^{-1})n!(2\pi i)^{-n-1}\sum_{h\in\mathbf{Z}}(h+\alpha)^{-n-1}\mathbf{e}\big((h+\alpha)t\big) \tag{1.14}$$

$$(c = -\mathbf{e}(\alpha),\ \alpha \in \mathbf{R},\ \notin \mathbf{Z};\ 0 < t < 1 \text{ if } n = 0;\ 0 \le t \le 1 \text{ if } 0 < n \in \mathbf{Z}),$$

$$E_{c,n}(t+r) = \sum_{k=0}^{n}\binom{n}{k}E_{c,k}(r)t^{n-k} \qquad (0 \le n \in \mathbf{Z}), \tag{1.15}$$

$$E_{1,0}(0) = 1, \quad E_{1,n}(0) = 2(1-2^{n+1})(n+1)^{-1}B_{n+1} \quad (0 < n \in \mathbf{Z}). \tag{1.16}$$

Formula (1.14) was given in [9, (4.5)]; the sum $\sum_{h\in\mathbf{Z}}$ means $\lim_{m\to\infty}\sum_{|h|\le m}$ if $n = 0$. Replacing t in (1.12) by $t+r$ and making an obvious calculation, we obtain (1.15). We have $E_{c,0}(t) = 1$ as noted in [9, (4.3h)]. Clearly $E_{1,n}(0) = 0$ if n is even. Assuming n to be odd, take $t = 0$ and $\alpha = 1/2$ in (1.14), and recall that $2 \cdot m!(2\pi i)^{-m}\zeta(m) = -B_m$ if $0 < m \in 2\mathbf{Z}$. Then we obtain $E_{1,n}(0)$ as stated in (1.16). Taking $r = 0$ in (1.15) and using (1.16), we obtain (1.13). The value $E_{c,n}(0)$ for an arbitrary c is given in [9, (4.6)].

1.6. In [9, Theorem 4.14] we proved, for χ, d, and k as in Theorem 1.4,

$$L(1-k,\,\chi) = \frac{d^{k-1}}{2^k - \overline{\chi}(2)}\sum_{a=1}^{q}\chi(a)E_{1,k-1}(2a/d), \tag{1.17}$$

where $q = [(d-1)/2]$, and derived (i) and (ii) above, with $E_{1,k-1}$ in place of Φ_{k-1}, from (1.17). In fact, (i) and (ii) combined are equivalent to (1.17). Though this is essentially explained in [9, p. 36], here let us show that (1.17) for even d follows from (ii). With $d = 4d_0$ as before, we have $[(d-1)/2] = 2d_0 - 1$ and

$$\begin{aligned}&\sum_{a=1}^{2d_0-1}\chi(a)\Phi_{k-1}(2a/d)\\ &\qquad = \sum_{a=1}^{d_0-1}\Big\{\chi(a)\Phi_{k-1}(2a/d) + \chi(2d_0-a)\Phi_{k-1}\big(2(2d_0-a)/d\big)\Big\}.\end{aligned}$$

We have $E_{1,n}(1-t) = (-1)^nE_{1,n}(t)$ as noted in [9, (4.3f)]. This combined with (1.13) shows that $\Phi_{k-1}(1-t) = (-1)^{k-1}\Phi_{k-1}(t)$. By Lemma 1.3, we have $\chi(2d_0 - a) = -\chi(-a) = (-1)^{k+1}\chi(a)$, and so the last sum equals

$$2\sum_{a=1}^{d_0-1}\chi(a)\varPhi_{k-1}(2a/d).$$

Therefore (1.17) follows from (1.10) if $d = 4d_0$. Similarly we can derive(1.17) for odd d from (1.9), which, in substance, is shown in the last paragraph of [9, p. 36].

1.7. Our technique is applicable even to $\zeta(1-k)$. Instead of $\zeta(s)$ we consider $W(s) = \sum_{m=0}^{\infty}(2m+1)^{-s}$. We have clearly

$$\begin{aligned} W(s) &= 1 + \sum_{m=1}^{\infty}(2m+1)^{-s} = 1 + \sum_{m=1}^{\infty}(2m)^{-s}\left(1+\frac{1}{2m}\right)^{-s} \\ &= 1 + \sum_{m=1}^{\infty}(2m)^{-s}\sum_{r=0}^{\infty}\binom{-s}{r}(2m)^{-r} = 1 + \sum_{r=0}^{\infty}\zeta(s+r)\binom{-s}{r}2^{-s-r}. \end{aligned}$$

We evaluate this at $s = 1-k$ with $0 < k \in \mathbf{Z}$. Our calculation is similar to that of §1.1; we use (1.6) for determining the term for $r = k$, which produces $-(2k)^{-1}$. Thus

$$(1-2^{k-1})\zeta(1-k) = W(1-k) = 1 - \frac{1}{2k} + \sum_{r=0}^{k-1}\binom{k-1}{r}2^{k-1-r}\zeta(1-k+r).$$

Taking $k = 1$, we find a well-known fact $\zeta(0) = -1/2$. Also, $\zeta(1-k)$ appears on both sides. Therefore, putting $k-r = t$ and rearranging our sum, we obtain

$$(1-2^{k})\zeta(1-k) = \frac{k-1}{2k} + \sum_{t=2}^{k-1}\binom{k-1}{t-1}2^{t-1}\zeta(1-t).$$

This holds for every even or odd integer $k > 1$. Recall that $\zeta(-m) = 0$ for $0 < m \in 2\mathbf{Z}$. Thus, taking $k = 2n$ with $0 < n \in \mathbf{Z}$, we obtain a formula for $\zeta(1-2n)$ as a linear combination of $\zeta(1-2\nu)$ for $1 \le \nu < n$ (which is 0 if $n = 1$) plus a constant as follows:

$$(1.18)\qquad (1-2^{2n})\zeta(1-2n) = \frac{2n-1}{4n} + \sum_{\nu=1}^{n-1}\binom{2n-1}{2\nu-1}2^{2\nu-1}\zeta(1-2\nu).$$

Similarly, taking $k = 2n+1$ and putting $t = 2\nu$, we obtain

$$(1.19)\qquad \sum_{\nu=1}^{n}\binom{2n}{2\nu-1}2^{2\nu-1}\zeta(1-2\nu) = \frac{-n}{2n+1}.$$

Either of these equalities (1.18) and (1.19) expresses $\zeta(1-2n)$ as a $\mathbf{Q}$-linear combination of $\zeta(1-2\nu)$ for $1 \le \nu < n$ plus a constant. The two expressions are different, as can easily be seen.

In [9, (11.8)] we gave a similar recurrence formula which can be written

$$(1.20)\quad 4(1-2^{n+1})\zeta(-n) = 1 + 2\sum_{k=2}^{n}\binom{n}{k-1}(2^k-1)\zeta(1-k) \qquad (0 < n \in \mathbf{Z}).$$

Taking n to be even or odd, we again obtain two different recurrence formulas for $\zeta(1-2n)$. It should be noted that the technique of using the binomial theorem is already in §68 of Landau [5], in which $(s-1)\zeta(s)$ is discussed, while we employ $W(s)$.

2. Extending the parameters c and n in $E_{c,n}$

2.1. The function $E_{c,n}(t)$ is a polynomial in t of degree n, and involves $c=-\mathbf{e}(\alpha)$ with $\alpha \in \mathbf{R}$. We now extend this in two ways: first, we take $\alpha \in \mathbf{C}, \notin \mathbf{Z}$; second, we consider $(h+\alpha)^{-s}$ instead of $(h+\alpha)^{-n-1}$. The first case is simpler. Since $E_{c,n}(t)$ is a polynomial in t and $(1+c)^{-1}$ as noted in [9, p. 26], we can define a function $\mathcal{E}_n(\alpha, t)$ by

$$\mathcal{E}_n(\alpha,\, t) = E_{c,n}(t), \quad c=-\mathbf{e}(\alpha),\ \alpha \in \mathbf{C}, \notin \mathbf{Z},\ 0 \le n \in \mathbf{Z}. \tag{2.1}$$

This is a polynomial in t, whose coefficients are holomorphic functions in $\alpha \in \mathbf{C}, \notin \mathbf{Z}$. Now equality (1.14) can be extended to

$$\mathcal{E}_n(\alpha,\, t) = \big(1-\mathbf{e}(-\alpha)\big)n!(2\pi i)^{-n-1}\sum_{h\in\mathbf{Z}}(h+\alpha)^{-n-1}\mathbf{e}\big((h+\alpha)t\big) \tag{2.2}$$

for all $\alpha \in \mathbf{C}, \notin \mathbf{Z}$, where $0<t<1$ if $n=0$, and $0 \le t \le 1$ if $n>0$. Indeed, if $n>0$, the right-hand side is absolutely convergent, and defines a holomorphic function. Since (2.2) holds for $\alpha \in \mathbf{R}, \notin \mathbf{Z}$, we obtain (2.2) as expected. If $n=0$, we have to consider $\lim_{m\to\infty}\sum_{|h|\le m}(h+\alpha)^{-1}\mathbf{e}\big((h+\alpha)t\big)$. Clearly

$$\sum_{h=-m}^{m}\frac{\mathbf{e}(ht)}{\alpha+h} = \frac{1}{\alpha}+\sum_{h=1}^{m}\frac{2\alpha\cdot\cos(2\pi ht)}{\alpha^2-h^2}+2i\sum_{h=1}^{m}\frac{h\cdot\sin(2\pi ht)}{h^2-\alpha^2}.$$

The last sum on the right-hand side equals

$$\sum_{h=1}^{m}\frac{\sin(2\pi ht)}{h}+\sum_{h=1}^{m}\frac{\sin(2\pi ht)\alpha^2}{h(h^2-\alpha^2)}.$$

It is well-known that the first sum tends to a finite value as $m\to\infty$. Obviously the last sum converges to a holomorphic function in $\alpha \in \mathbf{C}, \notin \mathbf{Z}$ as $m \to \infty$. Thus we can justify (2.2) for $n=0$.

Formula (2.2) for $n=0$ (with $-\alpha$ in place of α) can be written

$$\frac{\mathbf{e}(t\alpha)}{1-\mathbf{e}(\alpha)} = \frac{1}{2\pi i}\sum_{h\in\mathbf{Z}}\frac{\mathbf{e}(th)}{h-\alpha} \qquad (\alpha \in \mathbf{C}, \notin \mathbf{Z},\ 0<t<1). \tag{2.3}$$

This was first given by Kronecker [4].

2.2. We next ask if the power $(h+a)^{-n-1}$ in (1.14) can be replaced by $(h+a)^{-s}$ with a complex parameter s. Since $h+a$ can be negative, $(h+a)^{-s}$ is not suitable. Thus, for $s\in\mathbf{C}$, $a\in\mathbf{R}$, $p\in\mathbf{R}$, and $\nu=0$ or 1 we put

$$D^{\nu}(s;\, a,\, p) = \sum_{-a\ne n\in\mathbf{Z}}(n+a)^{\nu}|n+a|^{-\nu-s}\mathbf{e}\big(p(n+a)\big), \tag{2.4}$$

$$T^\nu(s;\,a,\,p) = \Gamma\bigl((s+\nu)/2\bigr)\pi^{-(s+\nu)/2}D^\nu(s;\,a,\,p). \tag{2.5}$$

Clearly the infinite series of (2.4) is absolutely convergent for $\mathrm{Re}(s) > 1$, and defines a holomorphic function of s there. Notice that if $k - \nu \in 2\mathbf{Z}$, then $D^\nu(k;\,0,\,t) = \sum_{0\neq n\in\mathbf{Z}} n^{-k}\mathbf{e}(nt)$, which is the infinite sum of (0.3). Thus the Bernoulli polynomials are included in our discussion.

THEOREM 2.3. *The function $T^\nu(s;\,a,\,p)$ can be continued as a meromorphic function of s to the whole $\mathbf{C}$. It is entire if $\nu = 1$. If $\nu = 0$, then $T^0(s;\,a,\,p)$ is*

$$\frac{-2\delta(a)}{s} + \frac{2\mathbf{e}(ap)\delta(p)}{s-1}$$

plus an entire function, where $\delta(x) = 1$ if $x \in \mathbf{Z}$ and $\delta(x) = 0$ if $x \notin \mathbf{Z}$. Moreover,

$$T^\nu(1-s;\,a,\,p) = i^{-\nu}\mathbf{e}(ap)T^\nu(s;\,-p,\,a). \tag{2.6}$$

Proof. Put $\varphi(x) = x^\nu\mathbf{e}(-x^2z^{-1}/2 + px)$ for $x \in \mathbf{R}$ and $z \in H$. Denote by $\widehat{\varphi}$ the Fourier transform of φ. Then from [9, (2.25)] we easily obtain $\widehat{\varphi}(x) = i^{-\nu}(-iz)^\kappa(x-p)^\nu\mathbf{e}\bigl((x-p)^2z/2\bigr)$, where $\kappa = \nu + 1/2$. Put also

$$f(z) = \sum_{n\in\mathbf{Z}}(n+a)^\nu\mathbf{e}\bigl((n+a)^2z/2 + p(n+a)\bigr),$$

and $f^\#(z) = (-iz)^{-\kappa}f(-z^{-1})$. Then $f(-z^{-1}) = \sum_{n\in\mathbf{Z}}\varphi(n+a)$, which equals $\sum_{m\in\mathbf{Z}}\mathbf{e}(ma)\widehat{\varphi}(m)$ by virtue of the Poisson summation formula. In this way we obtain

$$f^\#(z) = i^{-\nu}\sum_{m\in\mathbf{Z}}\mathbf{e}(ma)(m-p)^\nu\mathbf{e}\bigl((m-p)^2z/2\bigr).$$

Now $T^\nu(2s-\nu;\,a,\,p)$ is the Mellin transform of $f(iy)$, and so we obtain our theorem by the general principle of Hecke, which is given as Theorem 3.2 in [9].

THEOREM 2.4. *For $\nu = 0$ or 1, $0 \le a \le 1$, and a positive integer k such that $k - \nu \in 2\mathbf{Z}$ we have*

$$D^0(0;\,a,\,p) = -\delta(a), \tag{2.7}$$

$$D^\nu(\nu - 2m;\,a,\,p) = 0 \quad \text{if} \quad 0 < m \in \mathbf{Z}, \tag{2.8}$$

$$D^\nu(1-k;\,a,\,p) = 2(2\pi i)^{-k}(k-1)!\mathbf{e}(ap)D^\nu(k;\,-p,\,a), \tag{2.9}$$

$$D^\nu(1-k;\,a,\,p) = -2\mathbf{e}(ap)B_k(a)/k \quad \text{if} \quad p \in \mathbf{Z}, \tag{2.10}$$

$$D^\nu(1-k;\,a,\,p) = \frac{2\mathbf{e}(ap)}{1-\mathbf{e}(p)}E_{c,k-1}(a) \quad \text{if} \quad p \notin \mathbf{Z}, \tag{2.11}$$

where $c = -\mathbf{e}(-p)$, and we have to assume that $0 < a < 1$ in (2.10) and (2.11) if $k = 1$.

Proof. By Theorem 2.3, $\bigl[sT^0(s;\,a,\,p)\bigr]_{s=0} = -2\delta(a)$, from which we obtain (2.7). Next, let $0 < m \in \mathbf{Z}$. Since $\Gamma\bigl((s+\nu)/2\bigr)D^\nu(s;\,a,\,p)$ is finite and

$\Gamma\bigl((s+\nu)/2\bigr)$ has a pole at $s = \nu - 2m$, we obtain (2.8). We easily see that $\Gamma(1/2-m) = \pi^{1/2}(-2)^m \prod_{t=1}^m (2t-1)^{-1}$. Therefore from (2.6) we obtain (2.9). If $p \in \mathbf{Z}$, then $D^\nu(k;\, -p,\, a) = \sum_{0\neq n\in\mathbf{Z}} n^{-k}\mathbf{e}(an)$. The well-known classical formula, stated in [9, (4.9)] (and also as (0.3)), shows that the last sum equals $-(2\pi i)^k B_k(a)/k!$ for $0 \le a \le 1$ if $k > 1$, and for $0 < a < 1$ if $k = 1$. If $p \notin \mathbf{Z}$, then $D^\nu(k;\, -p,\, a) = \sum_{n\in\mathbf{Z}} (n-p)^{-k}\mathbf{e}\bigl(a(n-p)\bigr)$. By (1.14), this equals $(1+c^{-1})^{-1}(2\pi i)^k E_{c,k-1}(a)/(k-1)!$, where $c = -\mathbf{e}(-p)$, under the same condition on a. Combining these with (2.9), we obtain (2.10) and (2.11).

We note here a special case of (2.10):

$$D^\nu(1-k;\, 0,\, 0) = \begin{cases} -2B_k/k & \text{if } k > 1, \\ 0 & \text{if } k = 1. \end{cases} \tag{2.12}$$

It should be noted that $D^1(s;\, 0,\, 0) = 0$.

3. Nearly holomorphic elliptic functions

3.1. Let L be a lattice in $\mathbf{C}$. As an analogue of (2.4) we put

$$\varphi_\nu(u,\, s;\, L) = \sum_{\alpha\in L} (u+\alpha)^{-\nu}|u+\alpha|^{\nu-2s} \tag{3.1}$$

for $0 \le \nu \in \mathbf{Z}$, $u \in \mathbf{C}, \notin L$, and $s \in \mathbf{C}$. Clearly

$$\varphi_\nu(\lambda u,\, s;\, \lambda L) = \lambda^{-\nu}|\lambda|^{\nu-2s}\varphi_\nu(u,\, s;\, L) \quad \text{for every} \quad \lambda \in \mathbf{C}^\times, \tag{3.2a}$$

$$\varphi_\nu(u+\alpha,\, s;\, L) = \varphi_\nu(u,\, s;\, L) \quad \text{for every} \quad \alpha \in L. \tag{3.2b}$$

If L is a $\mathbf{Z}$-lattice in an imaginary quadratic field K and $u \in K$, (3.1) is the same as the series of [9, (7.1)]. The analytic properties of the series that we proved there can easily be extended to the case of (3.1). First of all, the right-hand side of (3.1) is absolutely convergent for $\mathrm{Re}(s) > 1$, and defines a holomorphic function of s there.

Theorem 3.2. *Put $\Phi(u,\, s) = \pi^{-s}\Gamma(s+\nu/2)\varphi_\nu(u,\, s;\, L)$. Then $\Phi(u,\, s)$ can be continued to the whole s-plane as a meromorphic function in s, which is entire if $\nu > 0$. If $\nu = 0$, then $\Phi(u,\, s)$ is an entire function of s plus $v(L)^{-1}/(s-1)$, where $v(L) = \mathrm{vol}(\mathbf{C}/L)$. Moreover, $\Phi(u,\, s)$ is a C^∞ function in u, except when $\nu = 0$ and $s = 1$, and each derivative $(\partial/\partial u)^a(\partial/\partial\overline{u})^b\Phi(u,\, s)$ is meromorphic in s on the whole $\mathbf{C}$.*

Proof. This can be proved by the same argument as in [9, §7.2], except for the differentiability with respect to u and the last statement about the derivatives, which can be shown as follows. As shown in the proof of [9, Theorem 3.2], the product $\pi^{-s}\Gamma(s)\varphi_\nu(u,\, s-\nu/2;\, L)$ minus the pole part can be written

$$\int_p^\infty F(u,\,y)y^{s-1}dy + \int_p^\infty G(u,\,y)y^{\nu-s}dy,$$

where

$$F(u,\,y) = \sum_{\alpha\in L}(\overline{u}+\alpha)^\nu \exp\bigl(-\pi|u+\alpha|^2y\bigr),$$

$$G(u,\,y) = A\sum_{\beta\in B}\exp\bigl(\pi i(\beta\overline{u}+\overline{\beta}u)\bigr)\sum_{\xi-\beta\in M}\xi^\nu\exp\bigl(-\pi|\xi|^2y\bigr)$$

with a constant A, a finite subset B of $\mathbf{C}$, a positive constant p, and lattices L and M in $\mathbf{C}$. Therefore the differentiability and the last statement follow from the standard fact on differentiation under the integral sign.

3.3. Before stating the next theorem, we note a few elementary facts. Take L in the form $L = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$ with complex numbers ω_1 and ω_2 such that $\omega_1/\omega_2 \in H$. We put then $v(\omega_1,\,\omega_2) = v(L)$. It can easily be seen that

$$v(\omega_1,\,\omega_2) = |\omega_2|^2\mathrm{Im}(\omega_1/\omega_2) = (2i)^{-1}(\omega_1\overline{\omega}_2 - \overline{\omega}_1\omega_2), \tag{3.3}$$

and in particular, $v(z,\,1) = \mathrm{Im}(z)$. We also recall the function ζ of Weierstrass defined by

$$\zeta(u) = \zeta(u;\,\omega_1,\,\omega_2) = \frac{1}{u} + \sum_{0\neq\alpha\in L}\left\{\frac{1}{u-\alpha}+\frac{1}{\alpha}+\frac{u}{\alpha^2}\right\}, \tag{3.4}$$

where $L = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$. It is well known that

$$\zeta(-u) = -\zeta(u), \qquad (\partial/\partial u)\zeta(u;\,\omega_1,\,\omega_2) = -\wp(u;\,\omega_1,\,\omega_2) \tag{3.5}$$

with the Weierstrass function $\wp$. We put as usual

$$\eta_\mu(\omega_1,\,\omega_2) = 2\zeta(\omega_\mu/2) \qquad (\mu = 1,\,2). \tag{3.6a}$$

Then

$$\zeta(u+\omega_\mu) = \zeta(u) + \eta_\mu(\omega_1,\,\omega_2). \tag{3.6b}$$

We also need the classical nonholomorphic Eisenstein series E_2 of weight 2, which can be given by

$$E_2(z) = \frac{1}{8\pi y} - \frac{1}{24} + \sum_{n=1}^\infty\Biggl(\sum_{0<d|n} d\Biggr)\mathbf{e}(nz). \tag{3.7}$$

We are interested in the value of $\varphi_\nu(u,\,s;\,L)$ at $s = \nu/2$, which is meaningful for every $\nu\in\mathbf{Z},\,>0$, by Theorem 3.2. The results can be given as follows.

Theorem 3.4. *For $L = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$ with $\omega_1/\omega_2\in H$ we have*

$$\varphi_\nu(u,\,\nu/2;\,L) = \frac{(-1)^\nu}{(\nu-1)!}\frac{\partial^{\nu-2}}{\partial u^{\nu-2}}\wp(u;\,\omega_1,\,\omega_2) \qquad (2<\nu\in\mathbf{Z}), \tag{3.8}$$

$$\varphi_2(u,\,1;\,L) = \wp(u;\,\omega_1,\,\omega_2) - 8\pi^2\omega_2^{-2}E_2(\omega_1/\omega_2), \tag{3.9}$$

$$\varphi_1(u,\,1/2;\,L) = \zeta(u) + 8\pi^2\omega_2^{-2}E_2(\omega_1/\omega_2)u - \pi v(L)^{-1}\overline{u}, \tag{3.10}$$

$$\eta_\mu(\omega_1,\,\omega_2) = \pi\overline{\omega}_\mu v(L)^{-1} - 8\pi^2\omega_2^{-2}E_2(\omega_1/\omega_2)\omega_\mu \qquad (\mu = 1, 2). \tag{3.11}$$

Proof. If $\nu > 2$, then clearly $\varphi_\nu(u,\,\nu/2;\,L) = \sum_{\alpha\in L}(u+\alpha)^{-\nu}$, from which we obtain (3.8). The cases $\nu = 2$ and $\nu = 1$ are more interesting. We first note that

$$(\partial/\partial u)\varphi_\nu(u,\,s;\,L) = (-s-\nu/2)\varphi_{\nu+1}(u,\,s+1/2;\,L), \tag{3.12a}$$

$$(\partial/\partial\overline{u})\varphi_\nu(u,\,s;\,L) = (-s+\nu/2)\varphi_{\nu-1}(u,\,s+1/2;\,L), \tag{3.12b}$$

at least for sufficiently large $\mathrm{Re}(s)$. Since both sides of (3.12a, b) are meromorphic in s on the whole $\mathbf{C}$, we obtain (3.12a, b) for every s. The first formula with $\nu = 2$ produces

$$(\partial/\partial u)\varphi_2(u,\,1;\,L) = -2\varphi_3(u,\,3/2;\,L) = (\partial/\partial u)\wp(u;\,\omega_1,\,\omega_2),$$

from which we obtain $\varphi_2(u,\,1;\,L) = \wp(u;\,\omega_1,\,\omega_2) + c(\overline{u})$ with an anti-holomorphic function $c(\overline{u})$. Since (3.12b) shows that $\varphi_2(u,\,1;\,L)$ is holomorphic in u, we see that $c(\overline{u})$ does not involve u or $\overline{u}$, that is, it is a constant depending only on L. Suppose $L = \mathbf{Z}z + \mathbf{Z}$ with $z \in H$. For $0 < N \in \mathbf{Z}$ and $(p,\,q) \in \mathbf{Z}^2,\notin N\mathbf{Z}^2$ define a standard Eisenstein series $\mathfrak{E}_\nu^N(z,\,s;\,p,\,q)$ of level N by [9, (9.1)]. Then we easily see that

$$\varphi_\nu\big((pz+q)/N,\,s;\,L\big) = N^{2s}y^{\nu/2-s}\mathfrak{E}_\nu^N(z,\,s-\nu/2;\,p,\,q),$$
$$\varphi_\nu\big((pz+q)/N,\,\nu/2;\,L\big) = N^{\nu}\mathfrak{E}_\nu^N(z,\,0;\,p,\,q).$$

Define F_ν and $\mathcal{F}_2$ as in [9, (10.10b, c, d)]. Taking $\nu = 2$, we obtain

$$\varphi_2\big((pz+q)/N,\,1;\,L\big) = N^2\mathfrak{E}_2^N(z,\,0;\,p,\,q) = (2\pi i)^2\mathcal{F}_2(z;\,p/N,\,q/N).$$

By [9, (10.13)], $\mathcal{F}_2(z;\,a,\,b) = (2\pi i)^{-2}\wp\big(az+b;\,z,\,1\big) + 2E_2(z)$ with E_2 of (3.7). Therefore we can conclude that

$$\varphi_2(u,\,1;\,\mathbf{Z}z+\mathbf{Z}) = \wp(u;\,z,\,1) - 8\pi^2E_2(z). \tag{3.13}$$

More generally, using (3.2a) we obtain (3.9).

We next consider the case $\nu = 1$. Since $(\partial/\partial u)\zeta(u;\,\omega_1,\,\omega_2) = -\wp(u;\,\omega_1,\,\omega_2)$, from (3.9) and (3.12a) we obtain

$$\begin{aligned}(\partial/\partial u)\varphi_1(u,\,1/2;\,L) &= -\varphi_2(u,\,1;\,L)\\ &= (\partial/\partial u)\zeta(u;\,\omega_1,\,\omega_2) + 8\pi^2\omega_2^{-2}E_2(\omega_1/\omega_2).\end{aligned}$$

We have also

$$(\partial/\partial\overline{u})\varphi_1(u,\,1/2;\,L) = \lim_{\sigma\to 1}(1-\sigma)\varphi_0(u,\,\sigma;\,L) = -\pi/v(L),$$

since the residue of $\pi^{-s}\Gamma(s)\varphi_0(u,\,s;\,L)$ at $s = 1$ is $v(L)^{-1}$ as shown in Theorem 3.2. Therefore $\varphi_1(u,\,1/2;\,L) = -\pi\overline{u}/v(L) + g(u)$ with a function g holomorphic in u. Clearly $\partial g/\partial u = (\partial/\partial u)\varphi_1(u,\,1/2;\,L)$, and so we can conclude that

$$\varphi_1(u,\,1/2;\,L) = \zeta(u) + 8\pi^2\omega_2^{-2}E_2(\omega_1/\omega_2)u - \pi v(L)^{-1}\overline{u} + \xi(L) \tag{3.14}$$

with a constant $\xi(L)$ independent of u. From (3.2a) we obtain $\varphi_1(-u,\,s;\,L) = -\varphi_1(u,\,s;\,L)$. Also $\zeta(-u) = -\zeta(u)$. Thus $\xi(L) = 0$, and consequently we obtain (3.10). Since $\varphi_1(u,\,1/2;\,L)$ is invariant under $u \mapsto u + \omega_\mu$, we obtain (3.11) from (3.10) and (3.6b).

3.5. In [9] we discussed the value of an Eisenstein series $E(z,\,s)$ of weight k at $s = -m$ for an integer m such that $0 \le m \le k-1$, and observed that it is nearly holomorphic in the sense that it is a polynomial in y^{-1} with holomorphic functions as coefficients; for a precise statement, see [9, Theorem 9.6]. As an analogue we investigate $\varphi_\nu(u,\,\kappa/2;\,L)$ for an integer κ such that $2-\nu \le \kappa \le \nu$ and $\kappa - \nu \in 2\mathbf{Z}$. From (3.12b) we obtain, for $0 \le a \in \mathbf{Z}$,

$$(\partial/\partial\overline{u})^a\varphi_\nu(u,\,(\nu/2)-a;\,L) = a!\cdot\varphi_{\nu-a}(u,\,(\nu-a)/2;L). \tag{3.15}$$

Theorem 3.6. *Let κ be an integer such that $2-\nu \le \kappa \le \nu$ and $\kappa-\nu \in 2\mathbf{Z}$. Then $\varphi_\nu(u,\,\kappa/2;L)$ is a polynomial in $\overline{u}$ of degree d with holomorphic functions in u as coefficients, where $d = (\nu-\kappa)/2$ if $\nu+\kappa \ge 4$ and $d = (\nu-\kappa+2)/2$ if $\nu+\kappa = 2$. The leading term is $\overline{u}^d\varphi_{(\nu+\kappa)/2}(u,\,(\nu+\kappa)/4;\,L)$ or $-\pi d^{-1}v(L)^{-1}\overline{u}^d$ according as $\nu+\kappa \ge 4$ or $\nu+\kappa = 2$.*

Proof. Given κ as in the theorem, put $a = (\nu-\kappa)/2$. Then $(\nu/2)-a = \kappa/2$ and $\nu-a = (\nu+\kappa)/2 \ge 1$. If $\nu-a \ge 2$, then by Theorem 3.4, $\varphi_{\nu-a}(u,\,(\nu-a)/2;L)$ is holomorphic in u, and so (3.15) shows that $\varphi_\nu(u,\,\kappa/2;\,L)$ is a polynomial in $\overline{u}$ of degree a with holomorphic functions in u as coefficients. If $\nu-a = 1$, the function $\varphi_1(u,\,1/2;\,L)$ is linear in $\overline{u}$ as given in (3.10). Therefore we obtain our theorem.

Thus, we may call $\varphi_\nu(u,\,\kappa/2;\,L)$ a *nearly holomorphic elliptic function.* In the higher-dimensional case it is natural to consider theta functions instead of periodic functions. For details of the basic ideas and results on this the reader is referred to [6] and [7].

4. The series with a parameter in H

4.1. To state the following lemma, we first define a confluent hypergeometric function $\tau(y;\,\alpha,\,\beta)$ for $y>0$ and $(\alpha,\,\beta) \in \mathbf{C}^2$ by

$$\tau(y;\,\alpha,\,\beta) = \int_0^\infty e^{-yt}(1+t)^{\alpha-1}t^{\beta-1}dt. \tag{4.1}$$

This is convergent for $\mathrm{Re}(\beta) > 0$. It can be shown that $\Gamma(\beta)^{-1}\tau(y;\,\alpha,\,\beta)$ can be continued to a holomorphic function in $(\alpha,\,\beta)$ on the whole $\mathbf{C}^2$; see [9, Section A3], for example. Also, for $v \in \mathbf{C}^\times$ and $\alpha \in \mathbf{C}$ we define v^α by

$$v^\alpha = \exp(\alpha\,\log(v)), \qquad -\pi < \mathrm{Im}[\log(v)] \le \pi. \tag{4.2}$$

LEMMA 4.2. *For* $\alpha, \beta \in \mathbf{C}$ *such that* $\mathrm{Re}(\alpha+\beta) > 1$, $0 \le r < 1$, *and* $z = x+iy \in H$ *we have*

$$
\begin{aligned}
i^{\alpha-\beta}(2\pi)^{-\alpha-\beta}\Gamma(\alpha)\Gamma(\beta)&\sum_{m\in\mathbf{Z}}\mathbf{e}(mr)(z+m)^{-\alpha}(\overline{z}+m)^{-\beta}\\
&=\sum_{n=1}^{\infty}\mathbf{e}\big((n-r)z\big)(n-r)^{\alpha+\beta-1}\tau\big(4\pi(n-r)y;\,\alpha,\,\beta\big)\\
&+\sum_{n=1}^{\infty}\mathbf{e}\big(-(n+r)\overline{z}\big)(n+r)^{\alpha+\beta-1}\tau\big(4\pi(n+r)y;\,\beta,\,\alpha\big)\\
&+\begin{cases}(4\pi y)^{1-\alpha-\beta}\Gamma(\alpha+\beta-1) & \text{if} \quad r=0,\\ \mathbf{e}(-r\overline{z})r^{\alpha+\beta-1}\tau\big(4\pi ry;\,\beta,\,\alpha\big) & \text{if} \quad r\neq 0.\end{cases}
\end{aligned}
$$

Proof. If $r = 0$, this is Lemma A3.4 of [9]. The case with nontrivial r can be proved in the same way as follows. Define two functions $f(x)$ and $f_1(x)$ of $x \in \mathbf{R}$ by $f(x) = (x+iy)^{-\alpha}(x-iy)^{-\beta}$ with a fixed $y > 0$ and $f_1(x) = \mathbf{e}(rx)f(x)$. Then $\hat{f}_1(x) = \hat{f}(x-r)$, and so the Poisson summation formula (see [9, (2.9)]) shows that

$$
\mathbf{e}(-rx)\sum_{m\in\mathbf{Z}}f_1(x+m)=\mathbf{e}(-rx)\sum_{n\in\mathbf{Z}}\hat{f}_1(n)\mathbf{e}(nx)=\sum_{n\in\mathbf{Z}}\mathbf{e}\big((n-r)x\big)\hat{f}(n-r).
$$

In [9, p. 133] we determined $\hat{f}$ explicitly in terms of τ as follows:

$$
i^{\alpha-\beta}(2\pi)^{-\alpha-\beta}\Gamma(\alpha)\Gamma(\beta)\hat{f}(t)=\begin{cases}\mathbf{e}(ity)t^{\alpha+\beta-1}\tau(4\pi ty;\,\alpha,\,\beta) & (t>0),\\ \mathbf{e}(-ity)|t|^{\alpha+\beta-1}\tau(4\pi|t|y;\,\beta,\,\alpha) & (t<0),\\ (4\pi y)^{1-\alpha-\beta}\Gamma(\alpha+\beta-1) & (t=0).\end{cases}
$$

Therefore we obtain our lemma.

4.3. We now need an elementary result:

$$
\sum_{n=1}^{\infty}n^{k-1}x^n=\frac{xP_k(x)}{(1-x)^k}\qquad(1\le k\in\mathbf{Z}). \tag{4.3}
$$

Here x is an indeterminate and P_k is a polynomial. We have $P_1 = P_2 = 1$ and $P_{k+1} = (kx-x+1)P_k - (x^2-x)P_k'$ for $k \ge 2$. Thus P_k is of degree $k-2$ for $k \ge 2$. These are easy; see [9, p. 17]. We also need two formulas and an estimate given as (A3.11), (A3.14), and Lemma A3.2 in [9]:

$$
\tau(y;\,n,\,\beta)=\sum_{\mu=0}^{n-1}\binom{n-1}{\mu}\Gamma(\beta+\mu)y^{-\mu-\beta}\qquad(0<n\in\mathbf{Z}), \tag{4.4}
$$

$$
\big[\tau(y;\,\alpha,\,\beta)/\Gamma(\beta)\big]_{\beta=0}=1, \tag{4.5}
$$

(4.6) $\Gamma(\beta)^{-1}y^{\beta}\tau(y;\,\alpha,\,\beta)$ *is bounded when* (α, β) *belongs to a compact subset of* $\mathbf{C}^2$ *and* $y > c$ *with a positive constant* c.

Our principal aim of this section is to study the nature of the series

$$\mathcal{L}_k(s,\,z) = \sum_{m\in\mathbf{Z}} \mathbf{e}(mr)(z+m)^{-k}|z+m|^{-2s}, \tag{4.7}$$

for certain integer values of s. Here $k \in \mathbf{Z}$, $r \in \mathbf{R}$, $s \in \mathbf{C}$, and $z \in H$. The sum depends only on r modulo $\mathbf{Z}$, and so we may assume that $0 \le r < 1$. Clearly this series is absolutely convergent for $\mathrm{Re}(2s+k) > 1$, and defines a holomorphic function of s there.

Theorem 4.4. *The function $\mathcal{L}_k(s,\,z)$ can be continued as a meromorphic function of s to the whole $\mathbf{C}$, which is entire if $r \notin \mathbf{Z}$. If $r \in \mathbf{Z}$, the locations of the poles of $\mathcal{L}_k(s,\,z)$ are the same as those of $\Gamma(2s+k-1)/\{\Gamma(s+k)\Gamma(s)\}$.*

Proof. Our function is the infinite series of Lemma 4.2 defined with $\alpha = s+k$ and $\beta = s$. Therefore our assertion can easily be verified by means of the formula of Lemma 4.2 and the estimate given by (4.6).

Theorem 4.5. *Assuming that $r \in \mathbf{Q}$, let N be the smallest positive integer such that $Nr \in \mathbf{Z}$ and let $\beta \in \mathbf{Z}$. Then the following assertions hold:*

(i) *If $\beta > 0$ or $\beta + k > 0$, then $\mathcal{L}_k(s,\,z)$ is finite at $s = \beta$ and $i^k\mathcal{L}_k(\beta,\,z)$ is a rational function in π, $\mathbf{e}(z/N)$, $\mathbf{e}(-\overline{z}/N)$, and $\mathrm{Im}(z)$ with coefficients in $\mathbf{Q}$.*

(ii) *If $-k < \beta \le 0$, then $i^k\{1-\mathbf{e}(z)\}^{k+\beta}\mathcal{L}_k(\beta,\,z)$ is a polynomial in π, $\mathbf{e}(z/N)$, $\mathrm{Im}(z)$, and $\mathrm{Im}(z)^{-1}$ with coefficients in $\mathbf{Q}$.*

(iii) *If $0 < \beta \le -k$, then $i^k\{1-\mathbf{e}(-\overline{z})\}^{\beta}\mathcal{L}_k(\beta,\,z)$ is a polynomial in π, $\mathbf{e}(-\overline{z}/N)$, $\mathrm{Im}(z)$, and $\mathrm{Im}(z)^{-1}$ with coefficients in $\mathbf{Q}$.*

Proof. As we already said, we may assume that $0 \le r < 1$. Put $\alpha = \beta + k$. We first have to study the nature of $\Gamma(2s+k-1)/\{\Gamma(s+k)\Gamma(s)\}$ at $s = \beta$. This is clearly finite at $s = \beta$ if $\alpha + \beta > 1$. Suppose $\alpha + \beta \le 1$; then $\alpha \le 0$ if $\beta > 0$, and $\beta \le 0$ if $\alpha > 0$. In all cases the value is finite, and in fact is a rational number. We now evaluate the formula of Lemma 4.2 divided by $\Gamma(\alpha)\Gamma(\beta)$. If $\alpha > 0$, we have, by (4.4),

$$\tau\big(4\pi(n-r)y;\,\alpha,\,\beta\big)/\Gamma(\beta) = \sum_{\mu=0}^{\alpha-1}\binom{\alpha-1}{\mu}(n-r)^{-\mu-\beta}(4\pi y)^{-\mu-\beta}\prod_{\kappa=0}^{\mu-1}(\beta+\kappa).$$

Thus an infinite sum of the form $\sum_{n=1}^{\infty}\mathbf{e}\big((n-r)z\big)(n-r)^{\alpha-\mu-1}$ appears. Applying the binomial theorem to the power of $n-r$, we see that the sum is a $\mathbf{Q}$-linear combination of $\mathbf{e}(-rz)\sum_{n=1}^{\infty}\mathbf{e}(nz)n^{\nu}$ for $0 \le \nu \le \alpha-\mu-1$. We can handle $\tau\big(4\pi(n+r)y;\,\beta,\,\alpha\big)/\Gamma(\alpha)$ in a similar way if $\beta > 0$. Put $\mathbf{q} = \mathbf{e}(z)$ and $\mathbf{q}_r = \mathbf{e}(rz)$. Then, assuming that $0 < r < 1$, $\alpha > 0$, and $\beta > 0$, we have

$$\begin{aligned} i^k\mathcal{L}_k(\beta,\,z) &= \mathbf{q}_r^{-1}\sum_{\mu=0}^{\alpha-1}\sum_{\nu=0}^{\alpha-\mu-1} a_{\mu\nu}\pi^{\alpha-\mu}y^{-\mu-\beta}\sum_{n=1}^{\infty}n^{\nu}\mathbf{q}^n \\ &\quad + \overline{\mathbf{q}}_r\sum_{\mu=0}^{\beta-1}\sum_{\nu=0}^{\beta-\mu-1} b_{\mu\nu}\pi^{\beta-\mu}y^{-\mu-\alpha}\sum_{n=0}^{\infty}n^{\nu}\overline{\mathbf{q}}^n, \end{aligned}$$

where $a_{\mu\nu}$ and $b_{\mu\nu}$ are rational numbers depending on β, k, and r. Applying (4.3) to $\sum_{n=1}^{\infty} n^{\nu}X^n$ with $X = \mathbf{q}$ and $X = \overline{\mathbf{q}}$, we obtain (i). Suppose $\beta \le 0$ and $\beta + k > 0$; then the sum involving $\tau\bigl(4\pi(n+r)y;\, \beta,\, \alpha\bigr)/\bigl\{\Gamma(\alpha)\Gamma(\beta)\bigr\}$ vanishes and we obtain (ii). The case in which $\beta > 0$ and $\beta + k \le 0$ is similar and produces (iii). If $r = 0$, the constant term of $i^k\mathcal{L}_k(\beta,\, z)$ is $2\pi(2y)^{1-\alpha-\beta}\Gamma(\alpha+\beta-1)/[\Gamma(a)\Gamma(\beta)]$, which causes no problem. This completes the proof.

One special case is worthy of attention. Taking $\beta = 0$ and $1 < k = \alpha \in \mathbf{Z}$, and using (4.5), we obtain, for $0 \le r < 1$,

$$\sum_{m\in\mathbf{Z}} \frac{\mathbf{e}\bigl(r(z+m)\bigr)}{(z+m)^k} = \frac{(2\pi i)^k}{(k-1)!}\sum_{\nu=1}^{k}\binom{k-1}{\nu-1} r^{k-\nu}\frac{\mathbf{q}P_\nu(\mathbf{q})}{(\mathbf{q}-1)^\nu}, \tag{4.8}$$

where $\mathbf{q} = \mathbf{e}(z)$. We assume $0 < r < 1$ and $\sum_{m\in\mathbf{Z}} = \lim_{h\to\infty}\sum_{|m|\le h}$ when $k = 1$. In (4.7) we take z in H, but in (4.8) we can take $z \in \mathbf{C}, \notin \mathbf{Z}$, since both sides of (4.8) are meaningful for such z. If $k = 1$, the result is the same as (2.3).

We can mention another special case. Namely, take $z = ia$ with a positive rational number a. Then we see that the values

$$\sum_{m\in\mathbf{Z}} (a^2 + m^2)^{-\beta} \tag{4.9}$$

for $0 < \beta \in \mathbf{Z}$ belong to the field generated by π and $e^{-2\pi a}$ over $\mathbf{Q}$, and therefore any three such values satisfy a nontrivial algebraic equation over $\mathbf{Q}$.

5. The rationality over a totally real base field

5.1. Throughout this section we fix a totally real algebraic number field F. The algebraicity of $\pi^{-k}L(k,\, \chi)$ can be generalized to the case of L-functions over F, but there is no known formulas similar to (0.2), (1.9), (1.10), except that Siegel proved some such formulas in [10] and [11] when $[F:\mathbf{Q}] = 2$. The paper [1] of Hecke may be mentioned in this connection. In this section we merely consider a generalization of (2.4) and prove an algebraicity result on its critical values, without producing explicit expressions.

We denote by $\mathfrak{g}$, D_F, and $\mathbf{a}$ the maximal order of F, the discriminant of F, and the set of archimedean primes of F. We also put $\mathrm{Tr}(x) = \mathrm{Tr}_{F/\mathbf{Q}}(x)$ for $x \in F$ and $[F:\mathbf{Q}] = g$. For $\alpha \in F$ and a fractional ideal $\mathfrak{a}$ in F we put $\alpha + \mathfrak{a} = \bigl\{\, \alpha + x \,\big|\, x \in \mathfrak{a} \,\bigr\}$ and $\widetilde{\mathfrak{a}} = \bigl\{\xi \in F \,|\, \mathrm{Tr}(\xi\mathfrak{a}) \subset \mathbf{Z}\bigr\}$.

Given α and $\mathfrak{a}$ as above, $\xi \in F$, $0 < \mu \in \mathbf{Z}$, and a (sufficiently small) subgroup U of $\mathfrak{g}^\times$ of finite index, we put

$$D_\mu(s;\, \xi,\, \alpha,\, \mathfrak{a}) = r_U \sum_{0\ne h\in U\backslash(\alpha+\mathfrak{a})} \mathbf{e}_{\mathbf{a}}(h\xi)h^{-\mu\mathbf{a}}|h|^{(\mu-s)\mathbf{a}}, \tag{5.1}$$

$$r_U = [\mathfrak{g}^\times : U]^{-1}, \tag{5.1a}$$

where $\mathbf{e_a}(\xi) = \mathbf{e}\big(\sum_{v\in\mathbf{a}}(\xi_v)\big)$ for $\xi \in F$ and $x^{t\mathbf{a}} = \prod_{v\in\mathbf{a}} x_v^t$ for $x \in \mathbf{C}^{\mathbf{a}}$ and $U\backslash X$ means a complete set of representatives for X modulo multiplication by the elements of U. We have to take U so small that the sum of (5.1) is meaningful. For instance, it is sufficient to take

$$U \subset \big\{u \in \mathfrak{g}^\times \,\big|\, u^{\mathbf{a}} = 1,\, u\xi - \xi \in \alpha^{-1}\widetilde{\mathfrak{g}} \cap \widetilde{\mathfrak{a}}\big\}.$$

The factor r_U makes the quantity of (5.1) independent of the choice of U. Clearly the sum is convergent for $\mathrm{Re}(s) > 1$. Now $D_\mu(s;\, \xi,\, \alpha,\, \mathfrak{a})$ is a special case of the series of [8, (18.1)], and so from Lemma 18.2 of [8] we see that it can be continued as a holomorphic function in s to the whole $\mathbf{C}$.

Theorem 5.2. *For $0 < \mu \in \mathbf{Z}$ we have*

$$(2\pi i)^{-\mu g} D_F^{1/2} D_\mu(\mu;\, \xi,\, 0,\, \mathfrak{a}) \in \mathbf{Q}, \tag{5.2}$$

$$D_\mu(1-\mu;\, 0,\, \alpha,\, \mathfrak{a}) \in \mathbf{Q}. \tag{5.3}$$

Proof. The last formula is a restatement of Proposition 18.10(2) of [8]. To prove (5.2), let $\mathfrak{b} = \big\{x \in \mathfrak{a} \,\big|\, \mathbf{e_a}(x\xi) = 1\big\}$ and let R be a complete set of representatives for $\mathfrak{a}/\mathfrak{b}$. Then

$$D_\mu(s;\, \xi,\, 0,\, \mathfrak{a}) = \sum_{\beta\in R} \mathbf{e_a}(\beta\xi) D_\mu(s; 0,\, \beta,\, \mathfrak{b}).$$

Put $Q_\mu(\beta,\, \mathfrak{b}) = (2\pi i)^{-\mu g} D_F^{1/2} D_\mu(\mu;\, 0,\, \beta,\, \mathfrak{b})$. Then the quantity of (5.2) equals $\sum_{\beta\in R} \mathbf{e_a}(\beta\xi) Q_\mu(\beta,\, \mathfrak{b})$. Let $t \in \prod_p \mathbf{Z}_p^\times$ and let σ be the image of t under the canonical homomorphism of $\mathbf{Q}_{\mathbf{A}}^\times$ onto $\mathrm{Gal}(\mathbf{Q}_{\mathrm{ab}}/\mathbf{Q})$. Our task is to show that the last sum is invariant under σ. By [8, Proposition 18.10(1)] we have $Q_\mu(\beta,\, \mathfrak{b})^\sigma = Q_\mu(\beta_1,\, \mathfrak{b})$ with $\beta_1 \in F$ such that $(t\beta_1 - \beta)_v \in \mathfrak{b}_v$ for every nonarchimedean prime v of F. For $\beta \in R$ there is a unique $\beta_1 \in R$ with that property. Now $\mathbf{e}(c)^\sigma = \mathbf{e}(t^{-1}c)$ for every $c \in \mathbf{Q}/\mathbf{Z} = \prod_p(\mathbf{Q}_p/\mathbf{Z}_p)$; see [8, (8.2)]. Since $\mathbf{e_a}(\beta\xi) = \mathbf{e}\big(\mathrm{Tr}(\beta\xi)\big)$, we easily see that $\mathbf{e_a}(\beta\alpha)^\sigma = \mathbf{e_a}(\beta_1\alpha)$, which gives the desired fact.

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Divisibility of the Dirac Magnetic Monopole as a Two-Vector Bundle over the Three-Sphere

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Abstract. We show that when the gerbe μ representing a magnetic monopole is viewed as a virtual 2-vector bundle, then it decomposes, modulo torsion, as two times a virtual 2-vector bundle ς. We therefore interpret ς as representing half a magnetic monopole, or a semipole.

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§1. Introduction

Let A be a connective S-algebra, where S is the sphere spectrum, and let $K(A) = K_0(\pi_0(A)) \times BGL_\infty(A)^+$ be its algebraic K-theory space. The natural map $w\colon BGL_1(A) \to K(A)$ is given by the inclusion of 1×1 matrices $BGL_1(A) \to BGL_\infty(A)$, followed by the canonical map into the plus construction. Let ku denote the connective complex K-theory spectrum, with $\pi_* ku = \mathbb{Z}[u]$, $|u| = 2$, and let $\pi\colon ku \to H\mathbb{Z}$ be the unique 2-connected S-algebra map to the integral Eilenberg–Mac Lane spectrum. Its homotopy fiber is bu, with $\pi_* bu = (u) \subset \mathbb{Z}[u]$. We define $BSL_1(ku)$ and $K(\pi)$ as the homotopy fibers of the induced maps $\pi\colon BGL_1(ku) \to BGL_1(\mathbb{Z})$ and $\pi\colon K(ku) \to K(\mathbb{Z})$,

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respectively, so that we have the following commutative diagram of horizontal homotopy fiber sequences

$$
\begin{tikzcd}
BSL_1(ku) \arrow[r] \arrow[d] & BGL_1(ku) \arrow[r, "\pi"] \arrow[d, "w"] & BGL_1(\mathbb{Z}) \arrow[d, "w"] \\
K(\pi) \arrow[r] & K(ku) \arrow[r, "\pi"] & K(\mathbb{Z})\,.
\end{tikzcd} \tag{1.1}
$$

We have a map from the Eilenberg–Mac Lane complex $K(\mathbb{Z},3)$ to the upper left hand corner of this diagram, induced by the infinite loop space inclusion $BU(1) \to BU_\otimes$ and the equivalences $K(\mathbb{Z},3) \simeq BBU(1)$ and $BBU_\otimes \simeq BSL_1(ku)$. Recall that the space $K(\mathbb{Z},3)$ represents gerbes with band $U(1)$ [Br93, Ch. V], whereas $K(ku)$ represents virtual 2-vector bundles [BDR04, Thm. 4.10], [BDRR]. A 2-vector bundle of rank 1 is the same as a gerbe, and the composite map

$$K(\mathbb{Z},3) \to K(ku) \tag{1.2}$$

represents the construction that views a gerbe as a virtual 2-vector bundle.

We now consider gerbes and 2-vector bundles over the base space S^3. There is a map $\mu\colon S^3 \to K(\mathbb{Z},3)$ representing a generator of $H^3(S^3) = \mathbb{Z}$, or dually, corepresenting a generator of $\pi_3 K(\mathbb{Z},3) = \mathbb{Z}$. It also represents a $U(1)$-gerbe over S^3, which is interpreted in [Br93, Ch. VII] as a mathematical model for a magnetic monopole, stationary in time and localized at one point.

Parallel transport in this gerbe, around closed loops in S^3, defines a holonomy line bundle over the free loop space of S^3 [Br93, Ch. VI]. Its complex 1-dimensional fibers can be interpreted as the state spaces of these free loops, viewed as strings in S^3. Parallel transport over compact surfaces in S^3, between tensor products of copies of this line bundle, defines an action functional that makes these state spaces part of a field theory. Here the compact surfaces are viewed as world sheets in S^3. In a quantized theory one would consider Hilbert spaces of sections in the holonomy line bundle, rather than its individual fibers, as the state spaces.

Following the point of view explained in [AR, §5.5], we also view 2-vector bundles over a base space as data defining (virtual) state spaces and action functionals for strings in that base. More field theories arise this way, since the state spaces are no longer restricted to being 1-dimensional, hence it is also possible to model more kinds of particles by 2-vector bundles than those arising from gerbes.

In particular we may ask, as the second author did, how the magnetic monopole μ over S^3 decomposes when viewed as a virtual 2-vector bundle. Does it remain a single particle?

The addition in the abelian group $\pi_3 K(ku)$ is induced by the H-group multiplication of $K(ku)$, which represents the direct sum of virtual 2-vector bundles, or in the above terms, the superposition of two particles. Therefore, a mathematical formulation of the question stated above is: "What is the structure of $\pi_3 K(ku) = K_3(ku)$, and what is the image of $\mu \in \pi_3 K(\mathbb{Z}, 3)$ in that group?" The surprising answer, which the title of this paper refers to, is that modulo torsion, μ becomes divisible by two as a virtual 2-vector bundle. In more detail, there are virtual 2-vector bundles ς and ν over S^3, with $24\nu = 0$, such that

$$\mu + \nu = 2\varsigma \tag{1.3}$$

in $K_3(ku)$. Modulo torsion, ς is therefore half a magnetic monopole. Ignoring torsion is justified in the physical interpretation, since the numerical invariants of a field theory traditionally take torsion-free values, and will send ν to zero. On the other hand, both μ and ς have infinite order in $K_3(ku)$.

§2. Statement of results

Let $i\colon S \to K(ku)$ be the unit map, and recall that $\pi_3(S) = \mathbb{Z}/24\{\nu\}$ and $K_3(\mathbb{Z}) \cong \mathbb{Z}/48\{\lambda\}$ [LS76]. The composite map $\pi i\colon S \to K(\mathbb{Z})$ induces the injection $\pi_3(S) \to K_3(\mathbb{Z})$ that takes ν to 2λ.

By [Wa78, Prop. 1.2], as generalized in [BM94, Prop. 10.9], the homotopy fiber $K(\pi)$ is 2-connected. Hence $K_i(ku) \to K_i(\mathbb{Z})$ is an isomorphism for $i \le 2$. Here is what happens in dimension three:

Theorem 2.1. *(a) The maps $K(\mathbb{Z}, 3) \to BSL_1(ku) \to K(\pi)$ induce isomorphisms*

$$\mathbb{Z}\{\mu\} = \pi_3 K(\mathbb{Z}, 3) \xrightarrow{\cong} \pi_3 BSL_1(ku) \xrightarrow{\cong} K_3(\pi)\,.$$

(b) The unit map $i\colon S \to K(ku)$ induces a homomorphism

$$\mathbb{Z}/24\{\nu\} = \pi_3(S) \xrightarrow{i_*} K_3(ku)$$

that identifies the source with the torsion subgroup in the target. We abbreviate $i_*(\nu)$ to $\nu \in K_3(ku)$.

(c) The homotopy fiber sequence $K(\pi) \to K(ku) \to K(\mathbb{Z})$ induces a short exact sequence

$$0 \to K_3(\pi) \to K_3(ku) \xrightarrow{\pi_*} K_3(\mathbb{Z}) \to 0$$

which is isomorphic to the nontrivial extension

$$0 \to \mathbb{Z}\{\mu\} \to \mathbb{Z}\{\varsigma\} \oplus \mathbb{Z}/24\{\nu\} \xrightarrow{\pi_*} \mathbb{Z}/48\{\lambda\} \to 0\,.$$

Here the first map takes μ to $2\varsigma - \nu$, and the second map takes ς to λ and ν to 2λ.

Corollary 2.2. *The map $K(\mathbb{Z}, 3) \to K(ku)$ that represents viewing $U(1)$-gerbes as virtual 2-vector bundles induces the homomorphism*

$$\mathbb{Z}\{\mu\} = \pi_3 K(\mathbb{Z}, 3) \to K_3(ku) = \mathbb{Z}\{\varsigma\} \oplus \mathbb{Z}/24\{\nu\}$$

that takes μ to $2\varsigma - \nu$, where $24\nu = 0$. The image of $\varsigma \in K_3(ku)$ in $K_3(\mathbb{Z})$ is the generating element λ of order forty-eight.

Corollary 2.3. *There is no "determinant map"*

$$\det\colon K(ku) \to BGL_1(ku)$$

such that the composite $\det \circ w$ *is an equivalence.*

Corollary 2.2 is readily extracted from Theorem 2.1(a) and (c). Corollary 2.3 follows, since $\det\colon K_3(ku) \to \pi_3 BGL_1(ku) \cong \mathbb{Z}\{\mu\}$ cannot map $2\varsigma - \nu$ to μ.

Remark 2.4. For commutative rings R there is a determinant map $\det\colon K(R) \to BGL_1(R)$, which is left inverse to w. On the other hand, it follows from [Wa82, Cor. 3.7] that $w\colon BGL_1(S) \to K(S)$ admits no such retraction up to homotopy. In [AR, §5.2], the first and third authors used the existence of a rational determinant map $\det_{\mathbb{Q}}\colon K(ku) \to BSL_1(ku)_{\mathbb{Q}} \simeq (BBU_{\otimes})_{\mathbb{Q}}$ to define the rational anomaly bundle of a 2-vector bundle, generalizing the definition of the anomaly line bundle of a gerbe. Corollary 2.3 shows that no such generalization can be integrally defined on all of $K(ku)$. This suggests that an integral anomaly bundle will only be defined on a space covering $K(ku)$, classifying 2-vector bundles with some form of higher orientation.

§3. Proofs

Proof of Thm. 2.1(a). In view of the infinite loop space splitting $BU_{\otimes} \simeq BU(1) \times BSU_{\otimes}$ it is clear that $K(\mathbb{Z},3) \simeq BBU(1) \to BBU_{\otimes} \simeq BSL_1(ku)$ is 4-connected. For the second part, we refer to the proof of [BM94, Prop. 10.9] to see that there is an isomorphism

$$\operatorname*{colim}_n M_n(\pi_2(bu))/[GL_n(\pi_0(ku)), M_n(\pi_2(bu))] \cong K_3(\pi)\,. \tag{3.1}$$

Here M_n denotes the ring of $n \times n$ matrices, and GL_n acts on M_n by conjugation. Furthermore, under the isomorphism (3.1), $\pi_3 BSL_1(ku) \to K_3(\pi)$ factors as

$$\pi_3 BSL_1(ku) \cong \pi_2(bu) = M_1(\pi_2(bu))/[GL_1(\pi_0(ku)), M_1(\pi_2(bu))]\,, \tag{3.2}$$

followed by the canonical map from the term $n = 1$ into the colimit in (3.1). For each $n \geq 1$ the matrix trace induces an isomorphism [Ka83, Prop. 1.3]

$$M_n(\pi_2(bu))/[GL_n(\pi_0(ku)), M_n(\pi_2(bu))] \xrightarrow{\cong} \pi_2(bu)/[\pi_0(ku), \pi_2(bu)] = \pi_2(bu)\,,$$

hence each structure map in the colimit is an isomorphism, and therefore the canonical map from (3.2) to $K_3(\pi)$ is also an isomorphism. □

To proceed, we make use of the natural trace map $tr\colon K(A) \to THH(A)$ to topological Hochschild homology [BHM93]. We define $THH(\pi)$ as the homotopy fiber of $\pi\colon THH(ku) \to THH(\mathbb{Z})$, so as to obtain the following commutative diagram of horizontal homotopy fiber sequences

$$\begin{array}{ccccc} K(\pi) & \longrightarrow & K(ku) & \xrightarrow{\pi} & K(\mathbb{Z}) \\ \downarrow & & \downarrow{\scriptstyle tr} & & \downarrow{\scriptstyle tr} \\ THH(\pi) & \longrightarrow & THH(ku) & \xrightarrow{\pi} & THH(\mathbb{Z})\,. \end{array} \tag{3.3}$$

Proof of Thm. 2.1(b) and (c). Passing to homotopy groups, we get the following vertical map of short exact sequences

$$
\begin{tikzcd}
0 \arrow[r] & K_3(\pi) \arrow[r] \arrow[d, "(\cong)"] & K_3(ku) \arrow[r, "\pi_*"] \arrow[d, "tr_*"] & K_3(\mathbb{Z}) \arrow[r] \arrow[d, "tr_*"] & 0 \\
0 \arrow[r] & THH_3(\pi) \arrow[r] & THH_3(ku) \arrow[r, "\pi_*"] & THH_3(\mathbb{Z}) \arrow[r] & 0\,.
\end{tikzcd}
\tag{3.4}
$$

Here $K_3(\pi) \to K_3(ku)$ is injective because $K_4(\mathbb{Z}) = 0$ [Ro00], and $K_3(ku) \to K_3(\mathbb{Z})$ is surjective because $K_2(\pi) = 0$. Furthermore, $THH_3(\pi) \to THH_3(ku)$ is injective because $THH_4(\mathbb{Z}) = 0$ [Bö], [FP98, Cor. 3.2] and $THH_3(ku) \to THH_3(\mathbb{Z})$ is surjective because

$$\mathbb{Z}/48\{\lambda\} = K_3(\mathbb{Z}) \xrightarrow{tr_*} THH_3(\mathbb{Z}) = \mathbb{Z}/2\{e\}$$

takes λ to e [BM94, Thm. 10.14], [Ro98, Thm. 1.1] and the right hand square commutes. The left hand vertical map $K_3(\pi) \to THH_3(\pi)$ is split injective, by [BM94, Thm. 10.12]. We shall soon see that it is in fact an isomorphism.
The 2-primary homotopy of $THH(ku)$ is fully computed in [AHL], but in low dimensions the following direct argument suffices. The homotopy cofiber ku/S of $S \to ku$ is 1-connected, with $\pi_2(ku/S) \cong \mathbb{Z}$. By construction, $THH(ku)$ is the geometric realization of a simplicial spectrum, and the map from the $(n-1)$-skeleton to the n-skeleton has cofiber $\Sigma^n ku \wedge (ku/S) \wedge \cdots \wedge (ku/S)$, with n copies of ku/S, which is $(3n-1)$-connected. By induction, the map from the 1-skeleton to all of $THH(ku)$ is 5-connected. Furthermore, the 0-simplices ku split off from the 1-skeleton of $THH(ku)$ since ku is commutative, so $THH_3(ku) \cong \pi_3(ku) \oplus \pi_3(\Sigma ku \wedge (ku/S)) \cong \mathbb{Z}\{\epsilon\}$, for some choice of generator ϵ.
Diagram (3.4) is therefore isomorphic to

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}\{\mu\} \arrow[r] \arrow[d, "\cong"] & K_3(ku) \arrow[r, "\pi_*"] \arrow[d, "tr_*"] & \mathbb{Z}/48\{\lambda\} \arrow[r] \arrow[d, "tr_*"] & 0 \\
0 \arrow[r] & \mathbb{Z}\{2\epsilon\} \arrow[r] & \mathbb{Z}\{\epsilon\} \arrow[r, "\pi_*"] & \mathbb{Z}/2\{e\} \arrow[r] & 0\,,
\end{tikzcd}
\tag{3.5}
$$

where the split injection $\mathbb{Z}\{\mu\} \to \mathbb{Z}\{2\epsilon\}$ must be an isomorphism. (We assume that we have chosen our orientations so that μ maps to 2ϵ, rather than -2ϵ.) The right hand square is a pullback, so there is a short exact sequence

$$0 \to \mathbb{Z}/24\{\nu\} \xrightarrow{i_*} K_3(ku) \xrightarrow{tr_*} \mathbb{Z}\{\epsilon\} \to 0\,, \tag{3.6}$$

where the image of the injective homomorphism $i_* \colon \pi_3(S) \to K_3(ku)$ is identified under $\pi_* \colon K_3(ku) \to K_3(\mathbb{Z})$ with the kernel of $tr_* \colon K_3(\mathbb{Z}) \to THH_3(\mathbb{Z})$. Hence the image of i_* equals the kernel of $tr_* \colon K_3(ku) \to THH_3(ku)$.

To fix a splitting of (3.6), we let $\varsigma \in K_3(ku)$ be the class mapping to ϵ in $THH_3(ku)$ and to λ in $K_3(\mathbb{Z})$. This is admissible, since both classes map to e in $THH_3(\mathbb{Z})$. Then

$$K_3(ku) \cong \mathbb{Z}\{\varsigma\} \oplus \mathbb{Z}/24\{\nu\}\,,$$

and $\mu \in K_3(\pi)$ maps to 2ς in $K_3(ku)$ modulo the image of i_*. Since μ continues to 0 in $K_3(\mathbb{Z})$, the exact formula must be $2\varsigma - \nu$. $\square$

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Natural G-Constellation Families

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Abstract. Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$. G-constellations are a scheme-theoretic generalization of orbits of G in $\mathbb{C}^n$. We study flat families of G-constellations parametrised by an arbitrary resolution of the quotient space $\mathbb{C}^n/G$. We develop a geometrical naturality criterion for such families, and show that, for an abelian G, the number of equivalence classes of these natural families is finite.

The main intended application is the derived McKay correspondence.

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0 Introduction

Let $G \subseteq \mathrm{GL}_n(\mathbb{C})$ be a finite subgroup. In this paper we classify flat families of G-*constellations* parametrised by a given resolution Y of the singular quotient space $X = \mathbb{C}^n/G$.

$$\begin{tikzcd} Y \arrow[dr, "\pi"] & & \mathbb{C}^n \arrow[dl, "q"'] \\ & X & \end{tikzcd}$$

A G-constellation is a scheme-theoretical generalization of a set-theoretical orbit of G in $\mathbb{C}^n$. They first arose in the context of moduli space constructions of crepant resolutions of X. Interpreting G-constellations in terms of representations of the McKay quiver of G, it is possible to use the methods of [Kin94] to consruct via GIT fine moduli spaces of stable G-constellations. The main irreducible component of such a moduli space turns out to be a projective crepant resolution of X. By varying the stability parameter θ it is possible to obtain

different resolutions M_θ. In case of $n = 3$ and G abelian, it is possible to obtain all projective crepant resolutions in this way [CI04]. For further details see [Cra01], [CI04], [CMT07a], [CMT07b].
The formal definition of a G-constellation is:

Definition 0.1 ([Cra01]). A G-constellation is a G-equivariant coherent sheaf $\mathcal{F}$ whose global sections $\Gamma(\mathbb{C}^n, \mathcal{F})$, as a representation of G, are isomorphic to the regular representation.

Families of G-constellations also occur naturally as objects defining *Fourier-Mukai transforms* (cf. [BKR01],[CI04], [BO95] and [Bri99]) which give a category equivalence $D(Y) \xrightarrow{\sim} D^G(\mathbb{C}^n)$ between the bounded derived categories of coherent sheaves on Y and of G-equivariant coherent sheaves on $\mathbb{C}^n$, respectively. This equivalence is known as the *derived McKay correspondence* (cf. [Rei97], [BKR01], [Kaw05], [Kal08]). It is the derived category interpretation of the classical McKay correspondence between the representation theory of G and the geometry of crepant resolutions of $\mathbb{C}^n/G$. It was conjectured by Reid in [Rei97] to hold for any finite subgroup G of $\mathrm{SL}_n(\mathbb{C})$ and any crepant resolution Y of $\mathbb{C}^n/G$.
In this paper we take an arbitrary resolution $Y \to \mathbb{C}^n/G$ and prove that it can support only a finite number (up to a twist by a line bundle) of flat families of G-constellations. We give a complete classification of these families which allows one to explicitly compute them. For the precise statement of the classification see the end of this introduction.
A motivation for this study is the fact that if a flat family of G-constellations on a crepant resolution Y of $\mathbb{C}^n/G$ is sufficiently orthogonal, then it defines an equivalence $D(Y) \to D^G(\mathbb{C}^n)$ ([Log08], Theorem 1.1), i.e. the derived McKay correspondence conjecture holds for Y. For an example of a specific application of this see [Log08], §4, where the first known example of a derived McKay correspondence for a non-projective crepant resolution is explicitly constructed.
This paper is laid out as follows. At the outset we allow Y to be any normal scheme birational to the quotient space X and first of all we move from the category $\mathbf{Coh}^G(\mathbb{C}^n)$ to the equivalent category $\mathbf{Mod}^{\mathrm{fg}}\text{-}R \rtimes G$ of the finitely-generated modules for the cross product algebra $R \rtimes G$, where R denotes the coordinate ring $\mathbb{C}[x_1, \dots, x_n]$ of $\mathbb{C}^n$. This makes a family of G-constellations into a vector bundle on Y. In Section 1 we develop a geometrical naturality criterion for such families: mimicking the moduli spaces M_θ of θ-stable G-constellations and their tautological families, we demand for a G-constellation parametrised in a family $\mathcal{F}$ by a point $p \in Y$ to be supported precisely on the G-orbit corresponding to the point $\pi(p)$ in the quotient space X. In other words, the support of the corresponding sheaf on $Y \times \mathbb{C}^n$ must lie within the fibre product $Y \times_X \mathbb{C}^n$. We call the families which satisfy this condition *gnat*-families (short for a *geometrically natural*) and demonstrate (Proposition 1.5) that they enjoy a number of other natural properties, including being equivalent (locally isomorphic) to the natural family $\pi^* q_* \mathcal{O}_{\mathbb{C}^n}$ on the open set of Y which lies over the free orbits in X. In this natural family a G-constellation which

lies over a free orbit is the unique G-constellation supported on that orbit - its reduced subscheme structure. Thus, in a sense, *gnat*-families can be viewed as flat deformations of free orbits of G.
Another property which characterises *gnat*-families is that it is possible to embed them into $K(\mathbb{C}^n)$, considered as a constant sheaf on Y. This leads us to study G-equivariant locally free sub-$\mathcal{O}_Y$-modules of $K(\mathbb{C}^n)$. In Section 2, we study the rank one case. A G-invariant invertible sub-$\mathcal{O}_Y$-module of $K(\mathbb{C}^n)$ is just a Cartier divisor, and we define $G\text{-}\operatorname{Car}(Y)$, a group of G-Cartier divisors on Y, as a natural extension of the group of Cartier divisors which fits into a short exact sequence

$$1 \to \operatorname{Car}(Y) \to G\text{-}\operatorname{Car}(Y) \xrightarrow{\rho} G^\vee \to 1$$

where $G^\vee$ is the group of 1-dimensional irreducible representations of G.
We then define $\mathbb{Q}$-valued valuations of these G-Cartier divisors at prime Weil divisors of Y and define $G\text{-}\operatorname{Div} Y$, the group of G-Weil divisors of Y, as a torsion-free subgroup of $\mathbb{Q}$-Weil divisors which fits into a following exact sequence:

$$\begin{tikzcd}
1 \arrow[r] & \operatorname{Car} Y \arrow[r] \arrow[d, hook, "val_K"'] & G\text{-}\operatorname{Car} Y \arrow[r, "\rho"] \arrow[d, "val_{K_G}"'] & G^\vee \arrow[r] \arrow[d, two heads, "val_{G^\vee}"'] & 1 \\
0 \arrow[r] & \operatorname{Div} Y \arrow[r] & G\text{-}\operatorname{Div} Y \arrow[r] & val_{G^\vee}(G^\vee) \arrow[r] & 0
\end{tikzcd}$$

We then show that the three vertical maps in this diagram, val_K, the ordinary $\mathbb{Z}$-valued valuation of Cartier divisors, val_{K_G}, the $\mathbb{Q}$-valued valuation of G-Cartier divisors, and their quotient $val_{G^\vee}$, a $\mathbb{Q}/\mathbb{Z}$-valued valuation of $G^\vee$, are all isomorphisms when Y is smooth and proper over X.
Then, in Section 3, we observe that when our group G is abelian all its irreducible representations are of rank 1, so any *gnat*-family splits into invertible G-eigensheaves. Thus G-Weil divisors are all that we need to classify it after an embedding into $K(\mathbb{C}^n)$. We further show that, since any *gnat*-family $\mathcal{F}$ embedded into $K(\mathbb{C}^n)$ must be closed under the natural action of R on the latter, all the G-eigensheaves into which $\mathcal{F}$ decomposes must be, in a certain sense, close to each other inside $K(\mathbb{C}^n)$. Up to a twist by a line bundle, this leaves only a finite number of possibilities for the corresponding G-Weil divisors. Thus, surprisingly, the number of equivalence classes of *gnat*-families on any Y is finite.
Our main result (Theorem 4.1) is:

Theorem (Classification of *gnat*-families). *Let G be a finite abelian subgroup of $\mathrm{GL}_n(\mathbb{C})$, X the quotient of $\mathbb{C}^n$ by the action of G and Y a resolution of X. Then isomorphism classes of gnat-families on Y are in 1-to-1 correspondence with linear equivalence classes of G-divisor sets $\{D_\chi\}_{\chi \in G^\vee}$, each D_χ a χ-Weil divisor, which satisfy the inequalities*

$$D_\chi + (f) - D_{\chi\rho(f)} \geq 0 \quad \forall\, \chi \in G^\vee, G\text{-}homogeneous\ f \in R$$

Here $\rho(f) \in G^\vee$ is the homogeneous weight of f. Such a divisor set $\{D_\chi\}$ corresponds then to a gnat-family $\bigoplus \mathcal{L}(-D_\chi)$.
This correspondence descends to a 1-to-1 correspondence between equivalence classes of gnat-families and sets $\{D_\chi\}$ as above and with $D_{\chi_0} = 0$, where χ_0 is the trivial character. Furthermore, each divisor D_χ in such a set satisfies

$$M_\chi \geq D_\chi \geq -M_{\chi^{-1}}$$

where $\{M_\chi\}$ is a fixed divisor set defined by

$$M_\chi = \sum_P (\min_{f \in R_\chi} v_P(f))P$$

As a consequence, the number of equivalence classes of gnat-families on Y is finite.

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1 *gnat*-FAMILIES

1.1 FAMILIES OF G-CONSTELLATIONS

Let G be a finite abelian group and let V_{giv} be an n-dimensional faithful representation of G. We identify the symmetric algebra $S(V_{\text{giv}}{}^\vee)$ with the coordinate ring R of $\mathbb{C}^n$ via a choice of such an isomorphism that the induced action of G on $\mathbb{C}^n$ is diagonal. The (left) action of G on V_{giv} induces a (left) action of G on R, where we adopt the convention that

$$g.f(\mathbf{v}) = f(g^{-1}.\mathbf{v}) \qquad \forall\, g \in G, f \in R, \mathbf{v} \in V_{\text{giv}}, \tag{1.1}$$

When we consider the induced scheme morphisms $g : \mathbb{C}^n \to \mathbb{C}^n$ and the induced sheaf morphisms $g : \mathcal{O}_{\mathbb{C}^n} \to g_*^{-1}\mathcal{O}_{\mathbb{C}_n}$, the convention above ensures that for any point $x \in \mathbb{C}^n$ and any function f in the stalk $\mathcal{O}_{\mathbb{C}^n,x}$ at x, the function $g.f$ is, naturally, an element of the stalk $\mathcal{O}_{\mathbb{C}^n,g.x}$ at $g.x$
Corresponding to the inclusion $R^G \subset R$ of the subring of G-invariant functions we have the quotient map $q : \mathbb{C}^n \to X$, where $X = \operatorname{Spec} R^G$ is the quotient space. This space is generally singular.
We first wish to establish a notion of a family of G-*Constellations* parametrised by an arbitrary scheme.

DEFINITION 1.1 ([CI04]). A G-`constellation` is a G-equivariant coherent sheaf $\mathcal{F}$ on $\mathbb{C}^n$ such that $H^0(\mathcal{F})$ is isomorphic, as a $\mathbb{C}[G]$-module, to the regular representation V_{reg}.

We would like for a family of G-constellations to be a locally free sheaf on Y, whose restriction to any point of Y would give us the respective G-constellation. We'd like this restriction to be a finite-dimensional vector-space, and for this purpose, it would be better to consider, instead of the whole G-constellation $\mathcal{F}$, just its space of global sections $\Gamma(\mathcal{F})$. It is a vector space with G and R actions, satisfying

$$g.(f.\mathbf{v}) = (g.f).(g.\mathbf{v}) \tag{1.2}$$

On the other hand, for any vector space V with G and R actions satisfying (1.2), we can define maps $g : \tilde{V} \to g_*^{-1}\tilde{V}$ to give the sheaf $\tilde{V} = V \otimes_R \mathcal{O}_{\mathbb{C}^n}$ a G-equivariant structure. It is convenient to view such vector spaces as modules for the following non-commutative algebra:

Definition 1.2. A cross-product algebra $R \rtimes G$ is an algebra, which has the vector space structure of $R \otimes_{\mathbb{C}} \mathbb{C}[G]$ and the product defined by setting, for all $g_1, g_2 \in G$ and $f_1, f_2 \in R$,

$$(f_1 \otimes g_1) \times (f_2 \otimes g_2) = (f_1(g_1.f_2)) \otimes (g_1 g_2) \tag{1.3}$$

This is not a pure formalism - $R \rtimes G$ is one of the *non-commutative crepant resolutions* of $\mathbb{C}^n/G$, a certain class of non-commutative algebras introduced by Michel van den Bergh in [dB02] as an analogue of a commutative crepant resolution for an arbitrary non-quotient Gorenstein singularity. For three-dimensional terminal singularities, van den Bergh shows ([dB02], Theorem 6.3.1) that if a non-commutative crepant resolution Q exists, then it is possible to construct commutative crepant resolutions as moduli spaces of certain stable Q-modules.
Functors $\Gamma(\bullet)$ and $\tilde{\bullet} = (\bullet) \otimes_R \mathcal{O}_{\mathbb{C}^n}$ give an equivalence (compare to [Har77], p. 113, Corollary 5.5) between the categories of quasi-coherent G-equivariant sheaves on $\mathbb{C}^n$ and of $R \rtimes G$-modules. G-constellations then correspond to $R \rtimes G$-modules, whose underlying G-representation is V_{reg}. As an abuse of notation, we shall use the term 'G-constellation' to refer to both the equivariant sheaf and the corresponding $R \rtimes G$-module.

Definition 1.3. A `family of` G`-constellations parametrised by a scheme` S is a sheaf $\mathcal{F}$ of $(R \rtimes G) \otimes_{\mathbb{C}} \mathcal{O}_S$-modules, locally free as an $\mathcal{O}_S$-module, and such that for any point $\iota_p :$ Spec $\mathbb{C} \hookrightarrow S$, its fiber $\mathcal{F}_{|p} = \iota_p^* \mathcal{F}$ is a G-constellation.
We shall say that two families $\mathcal{F}$ and $\mathcal{F}'$ are equivalent if they are locally isomorphic as $(R \rtimes G) \otimes_{\mathbb{C}} \mathcal{O}_S$-modules.

1.2 *gnat*-Families

Let Y be a normal scheme and $\pi : Y \to X$ be a birational map.

$$\begin{tikzcd} Y \arrow[dr, "\pi"] & & \mathbb{C}^n \arrow[dl, "q"'] \\ & X & \end{tikzcd}$$

We wish to refine the definition (1.3) above and develop a notion of a geometrically natural family of G-constellations parametrised by Y.

Any free G-orbit supports a unique G-cluster $Z \subset \mathbb{C}^n$: the reduced induced closed subscheme structure. Let U be an open subset of Y such that $\pi(U)$ consists of free orbits of G and consider the sheaf $\pi^* q_* \mathcal{O}_{\mathbb{C}^n}$ restricted to U. It has a natural $(R \rtimes G)$-module structure induced from $\mathcal{O}_{\mathbb{C}^n}$. It is locally free as an $\mathcal{O}_U$ module, since the quotient map q is flat wherever G acts freely. Its fiber at a point $p \in Y$ is $\Gamma(\mathcal{O}_Z)$, where Z is the G-cluster corresponding to the free orbit $q^{-1}\pi(p)$. Thus $\pi^* q_* \mathcal{O}_{\mathbb{C}^n}$ is a natural family of G-constellations, indeed of G-clusters, on $U \subset Y$.

Its fiber at the generic point of Y is $K(\mathbb{C}^n)$. The Normal Basis Theorem from Galois theory ([Gar86], Theorem 19.6) gives an isomorphism from $K(\mathbb{C}^n)$ to the generic fiber of any G-constellation family on Y, which we can write as $K(Y) \otimes_{\mathbb{C}} V_{\mathrm{reg}}$, but this isomorphism is only $K(Y)$ and G, but not necessarily R, equivariant.

On the other hand, for any G-constellation in a sense of G-equivariant sheaf, we can consider its support in $\mathbb{C}^n$. For instance, in the natural family $\pi^* q_* \mathcal{O}_{\mathbb{C}^n}$ discussed above the support of the G-constellation parametrised by a point $p \in U$ is precisely the G-orbit $q^{-1}\pi(p)$. This turns out to be the criterion we seek and we shall show that any family satisfying it is generically equivalent to the natural one.

Definition 1.4. A *gnat*-`family` $\mathcal{F}$ (short for *geometrically natural* family) is a family of G-constellations parametrised by Y such that for any $p \in Y$

$$q\left(\operatorname{Supp}_{\mathbb{C}^n} \mathcal{F}_{|p}\right) = \pi(p) \tag{1.4}$$

Proposition 1.5. *Let Y be a normal scheme and $\pi : Y \to X$ be a birational map. Let $\mathcal{F}$ be a family of G-constellations on Y. Then the following are equivalent:*

1. *On any $U \subset Y$, such that πU consists of free orbits, $\mathcal{F}$ is equivalent to $\pi^* q_* \mathcal{O}_{\mathbb{C}^n}$.*

2. *There exists an $(R \rtimes G) \otimes_{\mathbb{C}} K(Y)$-module isomorphism:*

$$\mathcal{F}_{|p_Y} \xrightarrow{\sim} (\pi^* q_* \mathcal{O}_{\mathbb{C}^n})_{p_Y}$$

where p_Y is the generic point of Y.

3. *There exists an* $(R \rtimes G) \otimes_{\mathbb{C}} \mathcal{O}_Y$*-module embedding*

$$F \hookrightarrow K(\mathbb{C}^n)$$

where $K(\mathbb{C}^n)$ *is viewed as a constant sheaf on* Y *and given a* $\mathcal{O}_Y$*-module structure via the birational map* $\pi : Y \to X$*.*

4. $\mathcal{F}$ *is a gnat-family.*

5. *The action of* $(R \rtimes G) \otimes_{\mathbb{C}} \mathcal{O}_Y$ *on* $\mathcal{F}$ *descends to the action of* $(R \rtimes G) \otimes_{R^G} \mathcal{O}_Y$*, where* R^G*-module structure on* $\mathcal{O}_Y$ *is induced by the map* $\pi : Y \to X$*.*

Proof. 1 $\Rightarrow$ 2 is restricting any of the local isomorphisms to the stalk at the generic point p_Y of Y. 2 $\Rightarrow$ 3: the embedding is given by the natural map $\mathcal{F} \hookrightarrow \mathcal{F} \otimes K(Y)$. As Y is irreducible and $\mathcal{F}$ is locally free, $\mathcal{F} \otimes K(Y)$ is isomorphic to $\mathcal{F}_{p_Y}$, and hence to $K(\mathbb{C}^n)$. 3 $\Rightarrow$ 5 is immediate by inspecting the natural $R \rtimes G \otimes_{\mathbb{C}} \mathcal{O}_Y$-module structure on $K(\mathbb{C}^n)$. 5 $\Rightarrow$ 4 is also immediate, as the descent of the action of $R \rtimes G \otimes_{\mathbb{C}} \mathcal{O}_Y$ to that of $R \rtimes G \otimes_{R^G} \mathcal{O}_Y$ implies that for any $p \in Y$ we have $\mathfrak{m}_{\pi(p)} \subset \operatorname{Ann}_R \mathcal{F}_{|p}$, where $\mathfrak{m}_{\pi(p)} \subset R^G$ is the maximal ideal of $\pi(p)$. Therefore $\mathfrak{m}_{\pi(p)} = (\operatorname{Ann}_R \mathcal{F}_{|p})^G$, which is equivalent to (1.4).
4 $\Rightarrow$ 5: Consider the following composition of algebra morphisms:

$$R \rtimes G \otimes_{\mathbb{C}} \mathcal{O}_Y \xrightarrow{\alpha} \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{F}) \xrightarrow{\beta_p} \operatorname{End}_{\mathbb{C}}(\mathcal{F}_{|p})$$

where α is the action map of $R \rtimes G \otimes_{\mathbb{C}} \mathcal{O}_Y$ on $\mathcal{F}$ and β_p is restriction to the fiber at a point $p \in Y$.
To show that α filters through $R \rtimes G \otimes_{R^G} \mathcal{O}_Y$ it suffices to show that for any $f \in R^G$ we have $f \otimes 1 - 1 \otimes f \in \ker(\alpha)$. From (1.4) we have $\mathfrak{m}_{\pi(p)} = (\operatorname{Ann}_R \mathcal{F}_{|p})^G$, and therefore

$$\beta_p \alpha((f - f(p)) \otimes 1) = 0$$

Observe that $\beta_p \alpha(f(p) \otimes 1) = f(p)\, 1_{\operatorname{End}_{\mathbb{C}} \mathcal{F}_{|p}} = \beta_p \alpha(1 \otimes f)$, and therefore

$$\beta_p \alpha(f \otimes 1 - 1 \otimes f) = 0 \tag{1.5}$$

As $\mathcal{E}nd_{\mathcal{O}_Y} \mathcal{F}$ is locally free, (1.5) holding $\forall\, p \in Y$ implies $\alpha(f \otimes 1 - 1 \otimes f) = 0$, as required.
5 $\Rightarrow$ 1: We have the $R \rtimes G \otimes_{R^G} \mathcal{O}_Y$-action on $\mathcal{F}$:

$$R \rtimes G \otimes_{R^G} \mathcal{O}_Y \xrightarrow{\alpha} \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{F})$$

LHS is isomorphic to $\pi^* \mathcal{E}nd_{\mathcal{O}_X}(q_* \mathcal{O}_{\mathbb{C}^n})$. Over U, since q is flat over $\pi(U)$, LHS is further isomorphic to $\mathcal{E}nd_{\mathcal{O}_U}(\pi^* q_* \mathcal{O}_{\mathbb{C}_n})$. Thus we have:

$$\mathcal{E}nd_{\mathcal{O}_U}(\pi^* q_* \mathcal{O}_{\mathbb{C}_n}) \xrightarrow{\alpha'} \mathcal{E}nd_{\mathcal{O}_U}(\mathcal{F}) \tag{1.6}$$

This map (1.6) is an $\mathcal{O}_U$-algebra homomorphism of (split) Azumaya algebras over U of the same rank. By a general result on Azumaya algebras any such is an isomorphism (see [ACvdE05], Theorem 5.3, for full generality, but the original result in [AG60], Corollary 3.4 will also suffice here). Now Skolem-Noether theorem for Azumaya algebras ([Mil80], IV, §2, Proposition 2.3) implies that locally α' must be induced by isomorphisms $\pi^* q_* \mathcal{O}_{\mathbb{C}^n} \xrightarrow{\sim} \mathcal{F}$. □

2 G-CARTIER AND G-WEIL DIVISORS

If $\mathcal{F}$ is a *gnat*-family, by Proposition 1.5 we can embed it into $K(\mathbb{C}^n)$. We need, therefore, to study G-subsheaves of $K(\mathbb{C}^n)$ which are locally free on Y. In this section we treat the rank 1 case, i.e. the invertible sheaves. Now, on an arbitrary scheme S, an invertible sheaf together with its embedding into $K(S)$ defines a unique Cartier divisor on S. But here we embed not into $K(Y)$ but into its Galois extension $K(\mathbb{C}^n)$. Recall that we identify $K(Y)$ with $K(\mathbb{C}^n)^G$ via the birational map $Y \xrightarrow{\pi} X$. We therefore seek to extend the familiar construction of Cartier divisors to accommodate for this fact.

2.1 G-CARTIER DIVISORS

We write $G^\vee$ for $\mathrm{Hom}(G, \mathbb{C}^*)$, the group of irreducible representations of G of rank 1.

DEFINITION 2.1. We shall say that a rational function $f \in K(\mathbb{C}^n)$ is `G-homogeneous of weight` $\chi \in G^\vee$ if

$$g.f = \chi(g^{-1})f \quad \forall\, g \in G \tag{2.1}$$

We shall denote by $K_\chi(\mathbb{C}^n)$ the subset of $K(\mathbb{C}^n)$ of homogeneous elements of a specific weight χ and by $K_G(\mathbb{C}^n)$ the subset of $K(\mathbb{C}^n)$ of all the G-homogeneous elements. We shall use R_χ and R_G to mean $R \cap K_\chi(\mathbb{C}^n)$ and $R \cap K_G(\mathbb{C}^n)$ respectively.

NB: The choice of a sign is dictated by wanting $f \in R$ to be homogeneous of weight $\chi \in G^\vee$ if $f(g.v) = \chi(g)f(v)$ for all $g \in G$ and $v \in \mathbb{C}^n$.
The invertible elements of $K_G(\mathbb{C}^n)$ form a multiplicative group which we shall denote by $K^*_G(\mathbb{C}^n)$. We have a short exact sequence:

$$1 \to K^*(Y) \to K^*_G(\mathbb{C}^n) \xrightarrow{\rho} G^\vee \to 1 \tag{2.2}$$

The following replicates, almost word-for-word, the definition of a Cartier divisor in [Har77], pp. 140-141.

DEFINITION 2.2. A group of `G-Cartier divisors` on Y, denoted by G-$\mathrm{Car}(Y)$ is the group of global sections of the sheaf of multiplicative groups $K^*_G(\mathbb{C}^n)/\mathcal{O}^*_Y$, i.e. the quotient of the constant sheaf $K^*_G(\mathbb{C}^n)$ on Y by the sheaf $\mathcal{O}^*_Y$ of invertible regular functions.

Observe that (2.2) gives a well-defined short exact sequence:

$$1 \to \operatorname{Car}(Y) \to G\text{-}\operatorname{Car}(Y) \xrightarrow{\rho} G^\vee \to 1 \tag{2.3}$$

Given a G-Cartier divisor, we call its image $\chi \in G^\vee$ under ρ `the weight` of the divisor and say, further, that the divisor is χ-`Cartier`.

A G-Cartier divisor can be specified by a choice of an open cover $\{U_i\}$ of Y and functions $\{f_i\} \subseteq K^*_G(\mathbb{C}^n)$ such that $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}^*_Y)$. In such case, the weight of the divisor is the weight of any one of f_i.

As with ordinary Cartier divisors, we say that a G-Cartier divisor is principal if it lies in the image of the natural map $K^*_G(\mathbb{C}^n) \to K^*_G(\mathbb{C}^n)/\mathcal{O}^*_Y$ and call two divisors linearly equivalent if their difference is principal.

Consider now a χ-Cartier divisor D on Y specified by a collection $\{(U_i, f_i)\}$ where U_i form an open cover of Y and $f_i \in K^*_\chi(\mathbb{C}^n)$. We define an invertible sheaf $\mathcal{L}(D)$ on Y as the sub-$\mathcal{O}_Y$-module of $K(\mathbb{C}^n)$ generated by f_i^{-1} on U_i. Observe that G acts on $\mathcal{L}(D)$, the action being the restriction of the one on $K(\mathbb{C}^n)$, and that it acts on every section by the character χ.

Proposition 2.3. *The map $D \to \mathcal{L}(D)$ gives an isomorphism between G-$\operatorname{Car} Y$ and the group of invertible G-subsheaves of $K(\mathbb{C}^n)$. Furthermore, it descends to an isomorphism of the group G-Cl of G-Cartier divisors up to linear equivalence and the group G-Pic of invertible G-sheaves on Y.*

Proof. A standard argument in [Har77], Proposition 6.13, shows everything claimed, apart from the fact we can embed any invertible G-sheaf $\mathcal{L}$, with G acting by some $\chi \in G^\vee$, as a sub-$\mathcal{O}_Y$-module into $K(\mathbb{C}^n)$.

Given such $\mathcal{L}$, we consider the sheaf $\mathcal{L} \otimes_{\mathcal{O}_Y} K(Y)$. On every open set U_i where $\mathcal{L}$ is trivial, it is G-equivariantly isomorphic to the constant sheaf $K_\chi(\mathbb{C}^n)$. On an irreducible scheme a sheaf constant on an open cover is constant itself, so as Y is irreducible we have $\mathcal{L} \otimes_{\mathcal{O}_Y} K(Y) \simeq K_\chi(\mathbb{C}^n)$ and a particular choice of this isomorphism gives the necessary embedding as

$$\mathcal{L} \to \mathcal{L} \otimes_{\mathcal{O}_Y} K(Y) \simeq K_\chi(\mathbb{C}^n) \subset K(\mathbb{C}^n)$$

□

2.2 Homogeneous valuations

We now aim to develop a matching notion of G-Weil divisors. Recall that the homomorphism from ordinary Cartier to ordinary Weil divisors is defined in terms of valuations of rational functions at prime Weil divisors of Y.

Valuations at prime divisors of Y define a unique group homomorphism val_K from $K^*(Y)$ to $\operatorname{Div} Y$, the group of Weil divisors. Looking at the short exact sequence (2.2), we see that val_K must extend uniquely to a homomorphism val_{K_G} from $K^*_G(\mathbb{C}^n)$ to $\mathbb{Q}$-$\operatorname{Div} Y$, as $G^\vee$ is finite and $\mathbb{Q}$ is injective. We further obtain a quotient homomorphism $val_{G^\vee}$ from $G^\vee$ to $\mathbb{Q}/\mathbb{Z}$-$\operatorname{Div} Y$.

Explicitly, we set:

DEFINITION 2.4. Let P be a prime Weil divisor on Y.
For any $f \in K^*_G(\mathbb{C}^n)$, observe that $f^{|G|}$ is necessarily of trivial weight and hence lies in $K(Y)$. We define `valuation of` f `at` P to be

$$v_P(f) = \frac{1}{|G|} v_P(f^{|G|}) \in \mathbb{Q} \tag{2.4}$$

where $v_P(f^{|G|})$ is the ordinary valuation in the local ring of P.
For any $\chi \in G^\vee$, observe that for any f, f' homogeneous of weight χ their ratio f/f' is of trivial character and therefore has integer valuation. We define `valuation of` χ `at` P to be

$$v_P(\chi) = \operatorname{frac}(v_P(f)) \in \mathbb{Q}/\mathbb{Z} \tag{2.5}$$

where f is any homogeneous function of weight χ and frac(-) denotes the fractional part.

It can be readily verified that $val_{K_G} = \sum v_P(\text{-})P$ and $val_{G^\vee} = \sum v_P(\text{-})P$. Furthermore, the short exact sequence (2.3) becomes a commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \operatorname{Car} Y \arrow[r] \arrow[d, "val_K"'] & G\text{-}\operatorname{Car} Y \arrow[r, "\rho"] \arrow[d, "val_{K_G}"'] & G^\vee \arrow[r] \arrow[d, "val_{G^\vee}"'] & 1 \\
0 \arrow[r] & \operatorname{Div} Y \arrow[r] & \mathbb{Q}\text{-}\operatorname{Div} Y \arrow[r] & \mathbb{Q}/\mathbb{Z}\text{-}\operatorname{Div} Y \arrow[r] & 0
\end{tikzcd} \tag{2.6}$$

2.3 G-WEIL DIVISORS

Aiming to have a short exact sequence similar to (2.3), we now define the group G-Div Y of G-Weil divisors to be the subgroup of $\mathbb{Q}$-Div Y, which consists of the pre-images of $val_{G^\vee}(G^\vee) \subset \mathbb{Q}/\mathbb{Z}$-Div Y.

DEFINITION 2.5. We say that a $\mathbb{Q}$-Weil divisor $\sum q_P P$ on Y is `a` G`-Weil divisor` if there exists $\chi \in G^\vee$ such that

$$\operatorname{frac}(q_P) = v_P(\chi) \qquad \text{for all prime Weil } P \tag{2.7}$$

We call a G-Weil divisor principal if it is an image of a single function $f \in K^*_G(\mathbb{C}^n)$ under val_{K^G}, call two G-Weil divisors linearly equivalent if their difference is principal and call a divisor $\sum q_i D_i$ effective if all $q_i \geq 0$.
We now have a following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \operatorname{Car} Y \arrow[r] \arrow[d, hook, "val_K"'] & G\text{-}\operatorname{Car} Y \arrow[r, "\rho"] \arrow[d, "val_{K_G}"'] & G^\vee \arrow[r] \arrow[d, two heads, "val_{G^\vee}"'] & 1 \\
0 \arrow[r] & \operatorname{Div} Y \arrow[r] & G\text{-}\operatorname{Div} Y \arrow[r] & val_{G^\vee}(G^\vee) \arrow[r] & 0
\end{tikzcd} \tag{2.8}$$

A warning: for general Y, even a smooth one, G-Cartier and G-Weil divisors may not be very well behaved. For an example let Y be the smooth locus of X. It can be shown, that while val_K is an isomorphism, val_{K_G} is not even injective as $G\text{-}\operatorname{Car} Y$ has torsion. And $val_{G^\vee}$ is the zero map, thus $G\text{-}\operatorname{Div} Y$ is just $\operatorname{Div} Y$.

Proposition 2.6. *If Y is smooth and proper over X, then val_K, val_{K_G} and $val_{G^\vee}$ in (2.8) are isomorphisms.*

Proof. If Y is smooth, or at least locally factorial, val_K is well-known to be an isomorphism ([Har77], Proposition 6.11). It therefore suffices to show that $val_{G^\vee}$ is injective and hence an isomorphism. As diagram (2.8) commutes, val_{K_G} will then also have to be an isomorphism.

Fix $\chi \in G^\vee$. Let Y_χ denote the normalisation of $Y \times_X (\mathbb{C}^n/\ker\chi)$. It is a Galois covering of Y whose Galois group is $\chi(G)$. By Zariski-Nagata's purity of the branch locus theorem ([Zar58], Proposition 2), the ramification locus of $Y_\chi \to Y$ is either empty or of pure codimension one. As Y is smooth, $Y_\chi \to Y$ being finite and unramified would make it an étale cover. Which is impossible, since a resolution of a quotient singularity is well known to be simply-connected (see, for instance, [Ver00], Theorem 4.1).

Thus, we can assume there exists a ramification divisor $P \subset Y_\chi$. Let Q be its image in Y. Let $\operatorname{Ram}(P)$ be the subgroup of G which fixes P pointwise. Then $n_{\text{ram}} = |\operatorname{Ram}(P)/\ker\chi|$ is the ramification index of P. We can take ordinary integer valuations of $K^*_\chi(\mathbb{C}^n)$ on prime divisors of Y_χ as $K^*_\chi(\mathbb{C}^n) \subset K(\mathbb{C}^n)^{\ker\chi}$. It is easy to see that for any $f \in K^*_\chi(\mathbb{C}^n)$

$$v_Q(f) = \frac{1}{n_{\text{ram}}} v_P(f) \tag{2.9}$$

where LHS is a rational valuation in sense of Definition 2.4.

If $v_Q(\chi) = 0$, then $v_Q(K^*_\chi(\mathbb{C}^n)) \subset \mathbb{Z}$. Then necessarily $v_Q(K^*_\chi(\mathbb{C}^n)) = \mathbb{Z}$, as $K^*_\chi(\mathbb{C}^n)$ is a coset of $K(Y)$ in $K^*_G(\mathbb{C}^n)$. In particular, there would exist $f_\chi \in K^*_\chi(\mathbb{C}^n)$, such that $v_Q(f_\chi) = 0$, i.e. f_χ is a unit in $\mathcal{O}_{Y_\chi,P}$. Which is impossible: any $g \in \operatorname{Ram}(P)$ fixes P pointwise, in particular $f - g.f \in \mathfrak{m}_{Y,P}$ for any $f \in \mathcal{O}_{Y,P}$. As $\operatorname{Ram}(P)/\ker\chi$ is non-trivial we can choose g such that $\chi(g) \neq 1$ and then $f_\chi = \frac{1}{1-\chi(g)}(f_\chi - g.f_\chi)$ must lie in $\mathfrak{m}_{Y,P}$. This finishes the proof.

For abelian G, this all can be seen very explicitly by exploiting the toric structure of the singularity: even though we do not assume the resolution Y to be toric, it has been proven by Bouvier ([Bou98], Theorem 1.1) and by Ishii and Kollár ([KI03], Corollary 3.17, in a more general context of Nash problem) that every essential divisor over X (i.e. a divisor which must appear on every resolution) is toric. The set of essential toric divisors is well understood - it can be identified with the Hilbert basis of the positive octant of the toric lattice of weights, and then with a subset of $\operatorname{Ext}^1(G^\vee, \mathbb{Z}) = \operatorname{Hom}(G^\vee, \mathbb{Q}/\mathbb{Z})$. This correspondence sends each divisor precisely to the valuation of $G^\vee$ at it, see [Log04], Section 4.3 for more detail. $\square$

We also show that, away from a finite number of prime divisors on Y, all G-Weil divisors are ordinary Weil.

Proposition 2.7. *Unless a prime divisor $P \subset Y$ is exceptional or its image in X is a branch divisor of $\mathbb{C}^n \to X$, the valuation $v_P : G^\vee \to \mathbb{Q}/\mathbb{Z}$ is the zero-map.*

Proof. If P is not exceptional, let Q be its image in X. The valuations at P and Q are the same, so it suffices to prove the statement about v_Q. Let P' be any divisor in $\mathbb{C}^n$ which lies above Q. As in Proposition 2.6, for any $f \in K^*_G(\mathbb{C}^n)$ we have $v_Q(f) = \frac{1}{n_{\text{ram}}} v_{P'}(f)$ where n_{ram} is the ramification index of P'. Unless Q is a branch divisor, $n_{\text{ram}} = 1$ and $v_Q = v_{P'}$. Which makes v_Q integer-valued on $K^*_G(\mathbb{C}^n)$ and makes the quotient homomorphism $G^\vee \to \mathbb{Q}/\mathbb{Z}$ the zero map. □

3 Classification of *gnat*-families

3.1 Reductor Sets

From now on, in addition to assuming that G is a finite group acting faithfully on V_{giv}, we also assume that G is abelian. We further assume that Y is smooth and $\pi : Y \to X$ is proper.

Let $\mathcal{F}$ be a gnat-family on Y. Write the decomposition of $\mathcal{F}$ into G-eigensheaves as $\bigoplus_{\chi \in G^\vee} \mathcal{F}_\chi$. By Proposition 1.5 we can embed $\mathcal{F}$ into $K(\mathbb{C}^n)$ and, as was demonstrated in Proposition 2.3, the image of each $\mathcal{F}_\chi$ defines a χ-Cartier divisor. Hence $\mathcal{F} \simeq \bigoplus_\chi \mathcal{L}(-D_\chi)$ for some set $\{D_\chi\}_{\chi \in G^\vee}$ of G-Weil divisors.

Definition 3.1. Let $\{D_\chi\}_{\chi \in G^\vee}$ be a set of G-Weil divisors on Y. We call it a `reductor set` if each D_χ is a χ-Weil divisor and $\oplus \mathcal{L}(-D_\chi)$ is a gnat-family on Y. We call a reductor set `normalised` if $D_{\chi_0} = 0$. We say that two reductor sets $\{D_\chi\}$ and $\{D'_\chi\}$ are linearly equivalent if there exists $f \in K(Y)$ such that $D_\chi - D'_\chi = \operatorname{Div} f$ for all $\chi \in G^\vee$.

Lemma 3.2. *Let $\{D_\chi\}$ and $\{D'_\chi\}$ be two reductor sets. Any $(R \rtimes G) \otimes \mathcal{O}_Y$-module morphism $\phi : \bigoplus \mathcal{L}(-D_\chi) \to \bigoplus \mathcal{L}(-D'_\chi)$ is necessarily a multiplication inside $K(\mathbb{C}^n)$ by some $f \in K(Y)$.*

Proof. Because of G-equivariance ϕ decomposes as $\bigoplus_{\chi \in G^\vee} \phi_\chi$ with ϕ_χ a morphism $\mathcal{L}(-D_\chi) \to \mathcal{L}(-D'_\chi)$. Each ϕ_χ is a morphism of invertible sub-$\mathcal{O}_Y$-modules of $K(\mathbb{C}^n)$ and so is necessarily a multiplication by some $f_\chi \in K(Y)$: consider the induced map $\mathcal{O}_Y \to \mathcal{L}(-D_\chi + D'_\chi)$ and take f_χ to be the image of 1 under this map.

It remains to show that all f_χ are equal. Fix any $\chi \in G^\vee$ and consider any G-homogeneous $m \in R$ of weight χ. Take any $s \in \mathcal{L}(-D_{\chi_0})$. Then $ms \in \mathcal{L}(-D_\chi)$ and by R-equivariance of ϕ

$$\phi_\chi(ms) = m\phi_{\chi_0}(s) = f_{\chi_0} ms \tag{3.1}$$

and hence $f_\chi = f_{\chi_0}$ for all $\chi \in G^\vee$. □

Corollary 3.3. *Isomorphism classes of gnat-families on Y are in 1-to-1 correspondence with linear equivalence classes of reductor sets.*

Proof. If in the proof of Lemma 3.2 each ϕ_χ is an isomorphism, then f, by construction, globally generates each $\mathcal{L}(-D'_\chi + D_\chi)$. Thus $D_\chi - D'_\chi = \mathrm{Div}(f)$. ☐

Proposition 3.4. *Let $\{D_\chi\}$ and $\{D'_\chi\}$ be two reductor sets. Then $\bigoplus \mathcal{L}(-D_\chi)$ and $\bigoplus \mathcal{L}(-D'_\chi)$ are equivalent (locally isomorphic) if and only if there exists a Weil divisor N such that $D_\chi - D'_\chi = N$ for all $\chi \in G^\vee$.*

Proof. The 'if' direction is immediate.
Conversely, if the families are equivalent, then by applying Lemma 3.2 to each local isomorphism, we obtain the data $\{U_i, f_i\}$, where U_i are an open cover of Y and on each U_i multiplication by f_i is an isomorphism $\bigoplus \mathcal{L}(-D_\chi) \xrightarrow{\sim} \bigoplus \mathcal{L}(-D'_\chi)$. One can readily check that such $\{U_i, f_i\}$ must define a Cartier divisor and that the corresponding Weil divisor is the requisite divisor N. ☐

Corollary 3.5. *In each equivalence classes of gnat-families there is precisely one family whose reductor set is normalised.*

3.2 Reductor Condition

We now investigate when is a set $\{D_\chi\}$ of G-divisors a reductor set.
This issue is the issue of $\bigoplus \mathcal{L}(-D_\chi)$ actually being $(R \rtimes G) \otimes \mathcal{O}_Y$-module. By definition it is a sub-$\mathcal{O}_Y$-module of $K(\mathbb{C}^n)$, but there is no a priori reason for it to also be closed under the natural $R \rtimes G$-action on $K(\mathbb{C}^n)$. If it is closed, it can be checked that it trivially satisfies all the other requirements in Proposition 1.5, item 3, which makes it a *gnat*-family. We further observe that $\bigoplus \mathcal{L}(-D_\chi)$ is always closed under the action of G, so it all boils down to the closure under the action of R.
Recall, that we write R_G for $R \cap K^*_G(\mathbb{C}^n)$, the G-homogeneous regular polynomials, and R_χ for $R \cap K^*_\chi(\mathbb{C}^n)$, the G-homogeneous regular polynomials of weight $\chi \in G^\vee$.

Proposition 3.6 (Reductor Condition). *Let $\{D_\chi\}_{\chi \in G^\vee}$ be a set with each D_χ a χ-Weil divisor. Then it is a reductor set if and only if, for any $f \in R_G$, the divisor*

$$D_\chi + (f) - D_{\chi\rho(f)} \geq 0 \tag{3.2}$$

i.e. it is effective.

Remarks:

1. If we choose a G-eigenbasis of V_{giv}, then its dual basis, a set of basic monomials $x_1, \ldots, x_n$, generates R_G as a semi-group. As condition (3.2) is multiplicative on f, it is sufficient to check it only for f being one of x_i. This leaves us with a finite number of inequalities to check.

2. Numerically, if we write each D_χ as $\sum q_{\chi,P}P$, inequalities (3.2) subdivide into independent sets of inequalities

$$q_{\chi,P} + v_P(f) - q_{\chi\rho(f),P} \geq 0 \qquad \forall\, \chi \in G^\vee \tag{3.3}$$

a set for each prime divisor P on Y. This shows that a *gnat*-family can be specified independently at each prime divisor of Y: we can construct reductor sets $\{D_\chi\}$ by independently choosing for each prime divisor P any of the sets of numbers $\{q_{\chi,P}\}_{\chi\in G^\vee}$ which satisfy (3.3).

3. There is an interesting link here with the work of Craw, Maclagan and Thomas in [CMT07a] which bears further investigation. In a toric context, they have rediscovered these inequalities as dual, in a certain sense, to the defining equations of the coherent component Y_θ of the moduli space M_θ of θ-semistable G-constellations. They then use them to compute the distinguished θ-semistable G-constellations parametrised by torus orbits of Y_θ. In particular, their Theorem 7.2 allows them to explicitly write down the tautological gnat-family on Y_θ and suggests that, up to a reflection, it is the gnat family which minimizes $\theta.\{D_\chi\}$. We shall see an example of that for the case of $Y_\theta = \mathrm{Hilb}^G$ in our Proposition 3.17.

Proof. Take an open cover U_i on which all $\mathcal{L}(-D_\chi)$ are trivialised and write $g_{\chi,i}$ for the generator of $\mathcal{L}(-D_\chi)$ on U_i. As R is a direct sum of its G-homogeneous parts, it is sufficient to check the closure under the action of just the homogeneous functions. Thus it suffices to establish that for each $f \in R_G$, each U_i and each $\chi \in G^\vee$

$$f g_{\chi,i} \in \mathcal{O}_Y(U_i) g_{\chi\rho(f),i}$$

On the other hand, with the notation above, G-Cartier divisor $D_\chi + (f) - D_{\chi\rho(f)}$ is given on U_i by $\frac{f g_{\chi,i}}{g_{\chi\rho(f),i}}$ and it being effective is equivalent to

$$\frac{f g_{\chi,i}}{g_{\chi\rho(f),i}} \in \mathcal{O}_Y(U_i)$$

for all U_i's. The result follows. □

3.3 CANONICAL FAMILY

We have not yet given any evidence of any gnat-families actually existing on an arbitrary resolution Y of X.

PROPOSITION 3.7 (Canonical family). *Let Y be a resolution of $X = \mathbb{C}^n/G$. Define the set $\{C_\chi\}_{\chi\in G^\vee}$ of G-Weil divisors by*

$$C_\chi = \sum v(P,\chi)P$$

where P runs over all prime Weil divisors on Y and $v(P,\chi)$ are the numbers introduced in Definition 2.4 (lifted to $[0,1) \subset \mathbb{Q}$).
Then $\{C_\chi\}_{\chi \in G^\vee}$ is a reductor set.
We call the corresponding family `the canonical gnat-family on` Y.

Proof. We must show that $\{C_\chi\}$ satisfies the inequalities (3.2). Choose any $\chi \in G^\vee$, any $f \in R_G$ and any prime divisor P on Y. Observe that $0 \leq v_P(\chi), v_P(\chi\rho(f)) < 1$ by definition, while $v_P(f) \geq 0$ since $f^{|G|}$ is regular on all of Y. So we must have

$$v_P(\chi) + v_P(f) - v_P(\chi\rho(f)) > -1$$

As the above expression must be integer-valued, we further have

$$v_P(\chi) + v_P(f) - v_P(\chi\rho(f)) \geq 0$$

as required. $\square$

This family has a following geometrical description:

Proposition 3.8. *On any resolution Y, the canonical family is isomorphic to the pushdown to Y of the structure sheaf $\mathcal{N}$ of the normalisation of the reduced fiber product $Y \times_X \mathbb{C}^n$.*

Proof. First we construct a $(R \rtimes G) \otimes \mathcal{O}_Y$-module embedding of $\mathcal{N}$ into $K(\mathbb{C}^n)$. Let α be the map $\mathcal{O}_Y \otimes_{R^G} R \to K(\mathbb{C}^n)$ which sends $a \otimes b$ to ab. It is $R \rtimes G \otimes \mathcal{O}_Y$-equivariant. If we show that $\ker \alpha$ is the nilradical of $\mathcal{O}_Y \otimes_{R^G} R$, then $\mathcal{N}$ can be identified with the integral closure of the image of α in $K(\mathbb{C}^n)$. Due to G-equivariance α decomposes as $\bigoplus_{\chi \in G^\vee} \alpha_\chi$ with each α_χ a morphism $\mathcal{O}_Y \otimes_{R^G} R_\chi \to K_\chi(\mathbb{C}^n)$. Observe that $(\mathcal{O}_Y \otimes_{R^G} R_\chi)^{|G|} \subset \mathcal{O}_Y \otimes_{R^G} R_{\chi_0} = \mathcal{O}_Y$ as a product of $|G|$ homogeneous functions is invariant. Hence $(\ker \alpha_\chi)^{|G|} \subset \ker \alpha_{\chi_0} = 0$ as required.
Write $\bigoplus_{\chi \in G^\vee} \mathcal{N}_\chi$ for the decomposition of $\mathcal{N}$ into G-eigensheaves. Fix a point $p \in Y$ and observe that $f \in K_\chi(\mathbb{C}^n)$ is integral over the local ring $\mathcal{N}_p$ if and only if $f^{|G|} \in (\mathcal{N}_{\chi_0})_p = \mathcal{O}_{Y,p}$. Therefore

$$(\mathcal{N}_\chi)_p = \{f \in K_\chi(\mathbb{C}^n) \mid G\text{-Weil divisor } \operatorname{Div}(f) \text{ is effective at } p\}$$

In particular, the generator c_χ of C_χ at p lies in $(\mathcal{N}_\chi)_p$. Observe further that for any $f \in (\mathcal{N}_\chi)_p$ the Weil divisor $\operatorname{Div}(f) - C_\chi$ is effective at p as the coefficients of C_χ are just the fractional parts of those of $\operatorname{Div}(f)$ and the latter is effective. Therefore c_χ generates $(\mathcal{N}_\chi)_p$ as $\mathcal{O}_{Y,p}$-module, giving $\mathcal{N}_\chi = \mathcal{L}(-C_\chi)$ as required. $\square$

3.4 Symmetries

Having demonstrated that the set of equivalence classes of gnat-families is always non-empty, we now establish two types of symmetries which this set possesses. It is worth noting that from the description of the symmetries of the chambers in the parameter space of the stability conditions for G-constellations described in [CI04], Section 2.5, it follows that all the symmetries described below take the subset of *gnat*-families on Y consisting of universal families of stable G-constellations into itself.

Proposition 3.9 (Character Shift). *Let $\{D_\chi\}$ be a normalised reductor set. Then for any χ in $G^\vee$*

$$D'_{\chi\lambda} = D_\chi - D_{\lambda^{-1}} \tag{3.4}$$

is also a normalised reductor set. We call it `the χ-shift` of $\{D_\chi\}$.

Proof. Writing out the reductor condition (3.2) for the new divisor set $\{D'_\chi\}$ we get:

$$(D_\chi - D_{\lambda^{-1}}) + (m) - (D_{\chi\rho(m)} - D_{\lambda^{-1}}) \geq 0$$

Cancelling out D_λ^{-1}, we obtain precisely the reductor condition for the original set $\{D_\chi\}$. And since

$$D'_{\chi_0} = D'_{\lambda^{-1}\lambda} = D_{\lambda^{-1}} - D_{\lambda^{-1}} = 0$$

we see that the new reductor set is normalised. □

NB: Observe, that for a reductor set $\{D_\chi\}$ and for any χ-Weil divisor N, the set $\{D_\chi + N\}$ is linearly equivalent to the χ-shift of $\{D_\chi\}$.

Proposition 3.10 (Reflection). *Let $\{D_\chi\}$ be a normalised reductor set. Then the set $\{-D_{\chi^{-1}}\}$ is also a normalised reductor set, which we call* `the reflection` of $\{D_\chi\}$.

Proof. We need to show that

$$-D_{\chi^{-1}} + (m) - (-D_{\chi^{-1}\rho(m)^{-1}}) \geq 0$$

Rearranging we get

$$D_{\chi^{-1}\rho(m)^{-1}} + (m) - D_{\chi^{-1}\rho(m)^{-1}\rho(m)} \geq 0$$

which is one of the reductor equations the original set $\{D_\chi\}$ must satisfy. As $D'_{\chi_0} = -D_{\chi_0} = 0$, the new set is normalised. □

3.5 Maximal shift family and finiteness

We now examine the individual line bundles $\mathcal{L}(-D_\chi)$ in a *gnat*-family and show that the reductor condition imposes a restriction on how far apart from each other they can be.

Lemma 3.11. *Let $\{D_\chi\}$ be a reductor set. Write each D_χ as $\sum q_{\chi,P}P$, where P ranges over all the prime Weil divisors on Y. For any $\chi_1, \chi_2 \in G^\vee$ and for any prime Weil divisor P, we necessarily have*

$$\min_{f\in R_{\chi_1/\chi_2}} v_P(f) \quad \geq \quad q_{\chi_1,P} - q_{\chi_2,P} \quad \geq \quad -\min_{f\in R_{\chi_2/\chi_1}} v_P(f) \tag{3.5}$$

Proof. Both inequalities follow directly from the reductor condition (3.2): the right inequality by setting $\chi = \chi_1 \in G^\vee$, $\rho(f) = \frac{\chi_2}{\chi_1}$ and letting f vary within $R_{\rho(f)}$; the left inequality by setting $\chi = \chi_2$ and $\rho(f) = \frac{\chi_1}{\chi_2}$. □

This suggests the following definition:

Definition 3.12. For each character $\chi \in G^\vee$, we define the `maximal shift` χ`-divisor` M_χ *to be*

$$M_\chi = \sum_P (\min_{f\in R_\chi} v_P(f))P \tag{3.6}$$

where P ranges over all prime Weil divisors on Y.

Lemma 3.13. *The G-Weil divisor set $\{M_\chi\}$ is a normalised reductor set. We call the corresponding family* `the maximal shift` *gnat*`-family` on Y.

Proof. We need to show that for any $f \in R_G$ and any prime divisor P

$$v_P(m_\chi) + v_P(f) - v_P(m_{\chi\rho(f)}) \geq 0$$

where m_χ and $m_{\chi\rho(f)}$ are chosen to achieve the minimality in (3.6).
Observe that $m_\chi f$ is also a G-homogeneous element of R, therefore by the minimality of $v_P(m_{\chi\rho(f)})$ we have

$$v_P(m_\chi f) \geq v_P(m_{\chi\rho(f)})$$

as required.
To establish that $M_{\chi_0} = 0$ we observe that for any G-homogeneous $f \in R$ we have $v_P(f) \geq 0$ on any prime Weil divisor P as $f^{|G|}$ is globally regular. Moreover for f in $R_{\chi_0} = R^G$ this lower bound is achieved by $f = 1$. □

Observe that with Lemma 3.13 we have established another gnat-family which always exists on any resolution Y. While sometimes it coincides with the canonical family, generally the two are distinct.

Proposition 3.14 (Maximal Shifts). *Let $\{D_\chi\}$ be a normalised reductor set. Then for any $\chi \in G^\vee$*

$$M_\chi \geq D_\chi \geq -M_{\chi^{-1}} \tag{3.7}$$

Moreover both the bounds are achieved.

Proof. To establish that (3.7) holds set $\chi_2 = \chi_0$ in Lemma 3.11. Lemma 3.13 shows that the bounds are achieved. $\square$

PROPOSITION 3.15. *If the coefficient of a maximal shift divisor M_χ at a prime divisor $P \subset Y$ is non-zero, then either P is an exceptional divisor or the image of P in X is a branch divisor of $\mathbb{C}^n \xrightarrow{q} X$.*

Proof. Let P be a prime divisor on X which is not a branch divisor of q. Let $\chi \in G^\vee$. By the defining formula (3.6) it suffices to find $f \in R_\chi$ such that $v_P(f) = 0$.
As R is a PID, there exist $t_1, \dots, t_k \in R$ such that $(t_1), \dots, (t_k)$ are all the distinct prime divisors lying over P in $\mathbb{C}^n$. Observe that the product $t_1 \dots t_n$ must be G-homogeneous. Since P is not a branch divisor, there exists $u \in R$ such that $t_1 \dots t_k u$ is invariant and $u \notin (t_i)$ for all i. Then $u' = u^{|G|-1}$ is a G-homogeneous function of the same weight as $t_1 \dots t_k$ and $v_P(u') = 0$. Now take any $f \in R_\chi$ and consider its factorization into irreducibles. G-homogeneity of f implies that all t_i occur with the same power k. Now replacing $(t_1 \dots t_n)^k$ in the factorization by $(u')^k$ we obtain an element of R_χ whose valuation at P is zero. $\square$

COROLLARY 3.16. *The number of equivalence classes of gnat-families on Y is finite.*

Proof. Let $\{D_\chi\}$ be a normalised reductor set. Coefficients of D_χ at prime divisors P of Y have fixed fractional parts (Definition 2.5), are bound above and below (Proposition 3.14) and are zero at all but finite number of P (Proposition 3.15). This leaves only a finite number of possibilities. $\square$

For one particular resolution Y the family provided by the maximal shift divisors has a nice geometrical description.

PROPOSITION 3.17. *Let $Y = \operatorname{Hilb}^G \mathbb{C}^n$, the coherent component of the moduli space of G-clusters in $\mathbb{C}^n$. If Y is smooth, then $\bigoplus \mathcal{L}(-M_\chi)$ is the universal family $\mathcal{F}$ of G-clusters parametrised by Y, up to the usual equivalence of families.*

Proof. Firstly $\mathcal{F}$ is a gnat-family, as over any set $U \subset X$ such that G acts freely on $q^{-1}(U)$ we have $\mathcal{F}|_U \simeq \pi^* q_* \mathcal{O}_{\mathbb{C}^n}|_U$. Write $\mathcal{F}$ as $\oplus \mathcal{L}(-D_\chi)$ for some reductor set $\{D_\chi\}$. Take an open cover $\{U_i\}$ of Y and consider the generators $\{f_{\chi,i}\}$ of D_χ on each U_i. Working up to equivalence, we can consider $\{D_\chi\}$ to be normalised and so $f_{\chi_0,i} = 1$ for all U_i.
Now any G-cluster Z is given by some invariant ideal $I \subset R$ and so the corresponding G-constellation $H^0(\mathcal{O}_Z)$ is given by R/I. In particular note that R/I is generated by R-action on the generator of χ_0-eigenspace. Therefore any $f_{\chi,i}$ is generated from $f_{\chi_0,i} = 1$ by R-action, which means that all $f_{\chi,i}$ lie in R.
But this means that for any prime Weil divisor P on Y we have

$$v_P(f_{\chi,i}) \geq \min_{f \in R_\chi} v_P(f)$$

and therefore $D_\chi \geq M_\chi$. Now Proposition 3.14 forces the equality. □

4 Conclusion

We summarise the results achieved in the following theorem:

Theorem 4.1 (Classification of *gnat*-families). *Let G be a finite abelian subgroup of $\mathrm{GL}_n(\mathbb{C})$, X the quotient of $\mathbb{C}^n$ by the action of G, Y nonsingular and $\pi : Y \to X$ a proper birational map. Then isomorphism classes of gnat-families on Y are in 1-to-1 correspondence with linear equivalence classes of G-divisor sets $\{D_\chi\}_{\chi \in G^\vee}$, each D_χ a χ-Weil divisor, which satisfy the inequalities*

$$D_\chi + (f) - D_{\chi\rho(f)} \geq 0 \quad \forall\, \chi \in G^\vee, G\text{-homogeneous } f \in R$$

Such a divisor set $\{D_\chi\}$ corresponds then to a gnat-family $\bigoplus \mathcal{L}(-D_\chi)$.
This correspondence descends to a 1-to-1 correspondence between equivalence classes of gnat-families and sets $\{D_\chi\}$ as above and with $D_{\chi_0} = 0$. Furthermore, each divisor D_χ in such a set satisfies inequality

$$M_\chi \geq D_\chi \geq -M_{\chi^{-1}}$$

where $\{M_\chi\}$ is a fixed divisor set defined by

$$M_\chi = \sum_P (\min_{f \in R_\chi} v_P(f))P$$

As a consequence, the number of equivalence classes of gnat-families is finite.

Proof. Corollary 3.3 establishes the correspondence between isomorphism classes of *gnat*-families and linear equivalence classes of reductor sets. Proposition 3.6 gives description of reductor sets as the divisor sets satisfying the reductor condition inequalities.
Corollary 3.5 gives the correspondence on the level of equivalence classes of *gnat*-families and normalised reductor sets. Proposition 3.14 establishes the bounds on the set of all normalised reductor sets and Corollary 3.16 uses it to show that the set of all normalised reductor sets is finite. □

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The Global Structure of Moduli Spaces of Polarized p-Divisible Groups

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Abstract. We study the global structure of moduli spaces of quasi-isogenies of polarized p-divisible groups introduced by Rapoport and Zink. Using the corresponding results for non-polarized p-divisible groups from a previous paper, we determine their dimensions and their sets of connected components and of irreducible components.

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1 Introduction

Let k be an algebraically closed field of characteristic $p > 2$. Let $W = W(k)$ be its ring of Witt vectors and $L = \mathrm{Quot}(W)$. Let σ be the Frobenius automorphism on k as well as on W. By Nilp_W we denote the category of schemes S over $\mathrm{Spec}(W)$ such that p is locally nilpotent on S. Let $\overline{S}$ be the closed subscheme of S that is defined by the ideal sheaf $p\mathcal{O}_S$. Let $(\mathbb{X}, \lambda_{\mathbb{X}})$ be a principally polarized p-divisible group over k. If X is a p-divisible group, we denote its dual by $\hat{X}$. Then the polarization $\lambda_{\mathbb{X}}$ is an isomorphism $\mathbb{X} \rightarrow \hat{\mathbb{X}}$.
We consider the functor

$$\mathcal{M} : \mathrm{Nilp}_W \rightarrow \mathrm{Sets},$$

which assigns to $S \in \mathrm{Nilp}_W$ the set of isomorphism classes of pairs (X, ρ), where X is a p-divisible group over S and $\rho : \mathbb{X}_{\overline{S}} = \mathbb{X} \times_{\mathrm{Spec}(k)} \overline{S} \rightarrow X \times_S \overline{S}$ is a quasi-isogeny such that the following condition holds. There exists a principal polarization $\lambda : X \rightarrow \hat{X}$ such that $\rho^\vee \circ \lambda_{\overline{S}} \circ \rho$ and $\lambda_{\mathbb{X},\overline{S}}$ coincide up to a scalar. Two pairs (X_1, ρ_1) and (X_2, ρ_2) are isomorphic if $\rho_1 \circ \rho_2^{-1}$ lifts to an isomorphism $X_2 \rightarrow X_1$. This functor is representable by a formal scheme $\mathcal{M}$

which is locally formally of finite type over $\mathrm{Spf}(W)$ (see [RaZ], Thm. 3.25). Let $\mathcal{M}_{\mathrm{red}}$ be its underlying reduced subscheme, that is the reduced subscheme of $\mathcal{M}$ defined by the maximal ideal of definition. Then $\mathcal{M}_{\mathrm{red}}$ is a scheme over $\mathrm{Spec}(k)$.

The analogues of these moduli spaces for p-divisible groups without polarization have been studied by de Jong and Oort in [JO] for the case that the rational Dieudonné module of $\mathbb{X}$ is simple and in [V1] without making this additional assumption. There, the sets of connected components and of irreducible components, as well as the dimensions, are determined. In the polarized case, the moduli spaces $\mathcal{M}_{\mathrm{red}}$ have been examined in several low-dimensional cases. For example, Kaiser ([Kai]) proves a twisted fundamental lemma for GSp_4 by giving a complete description in the case that $\mathbb{X}$ is two-dimensional and supersingular. An independent description of this case is given by Kudla and Rapoport in [KR]. In [Ri], Richartz describes the moduli space in the case of three-dimensional supersingular $\mathbb{X}$. In this paper we derive corresponding results on the global structure of the mo duli space $\mathcal{M}_{\mathrm{red}}$ for arbitrary $\mathbb{X}$.

The first main result of this paper concerns the set of connected components of $\mathcal{M}_{\mathrm{red}}$.

Theorem 1. *Let $\mathbb{X}$ be nontrivial and let $\mathbb{X}_{\mathrm{m}} \times \mathbb{X}_{\mathrm{bi}} \times \mathbb{X}_{\mathrm{et}}$ be the decomposition into its multiplicative, bi-infinitesimal, and étale part. Then*

$$\pi_0(\mathcal{M}_{\mathrm{red}}) \cong \big(GL_{\mathrm{ht}(\mathbb{X}_{\mathrm{m}})}(\mathbb{Q}_p)/GL_{\mathrm{ht}(\mathbb{X}_{\mathrm{m}})}(\mathbb{Z}_p)\big) \times \mathbb{Z}.$$

Next we consider the set of irreducible components of $\mathcal{M}_{\mathrm{red}}$. Let (N,F) be the rational Dieudonné module of $\mathbb{X}$. Here, N is an L-vector space of dimension $\mathrm{ht}(\mathbb{X})$ and $F : N \to N$ is a σ-linear isomorphism. The polarization $\lambda_{\mathbb{X}}$ induces an anti-symmetric bilinear perfect pairing $\langle \cdot, \cdot \rangle$ on N. Let G be the corresponding general symplectic group of automorphisms of N respecting $\langle \cdot, \cdot \rangle$ up to a scalar. Let

$$J = \{g \in G(L) \mid g \circ F = F \circ g\}.$$

It is the set of $\mathbb{Q}_p$-valued points of an algebraic group over $\mathbb{Q}_p$ (see [RaZ], Prop. 1.12). There is an action of J on $\mathcal{M}_{\mathrm{red}}$.

Theorem 2. *The action of J on the set of irreducible components of $\mathcal{M}_{\mathrm{red}}$ is transitive.*

We choose a decomposition $N = \bigoplus_{j=1}^{l} N^j$ with N^j simple of slope $\lambda_j = m_j/(m_j+n_j)$ with $(m_j,n_j)=1$ and $\lambda_j \leq \lambda_{j'}$ for $j<j'$. Let

$$m = \left\lfloor \frac{1}{2} \sum_j \min\{m_j, n_j\} \right\rfloor,$$

where $\lfloor x \rfloor$ is the greatest integer less or equal x. As N is the isocrystal of a polarized p-divisible group, its Newton polygon is symmetric, i. e. $\lambda_{l+1-j} =$

$1-\lambda_j$. Hence we obtain

$$m = \left\lfloor \sum_{\{j \mid m_j < n_j\}} m_j + \frac{1}{2} \mid \{j \mid m_j = n_j = 1\} \mid \right\rfloor . \tag{1.1}$$

Theorem 3. *$\mathcal{M}_{\rm red}$ is equidimensional of dimension*

$$\dim \mathcal{M}_{\rm red} = \frac{1}{2} \left(\sum_j \frac{(m_j-1)(n_j-1)}{2} + \sum_{j<j'} m_j n_{j'} + m \right) . \tag{1.2}$$

Note that the equidimensionality is already a consequence of Theorem 2. However, it also follows from the proof of the dimension formula without requiring additional work.

Our results on the set of connected components and on the dimension of $\mathcal{M}_{\rm red}$ are analogous to those for other affine Deligne-Lusztig sets for split groups where a scheme structure is known. We now define these affine Deligne-Lusztig varieties and give a brief overview over the general results in comparison to the results for the case treated in this paper.

Let $\mathcal{O}$ be a finite unramified extension of $\mathbb{Z}_p$ or $\mathbb{F}_p[[t]]$ and let G be a split connected reductive group over $\mathcal{O}$. Let $\boldsymbol{F}$ be the quotient field of $\mathcal{O}$. Let $K = G(\mathcal{O})$. Let L be the completion of the maximal unramified extension of $\boldsymbol{F}$ and let σ be the Frobenius of L over $\boldsymbol{F}$. Let A be a maximal torus and B a Borel subgroup containing A. Let $\mu \in X_*(A)$ be dominant and let $b \in G(L)$. Let ε^μ be the image of p or $t \in \boldsymbol{F}^\times$ under μ. Let

$$X_\mu(b) = \{g \in G(L)/K \mid g^{-1} b \sigma(g) \in K \varepsilon^\mu K\} \tag{1.3}$$

be the generalized affine Deligne-Lusztig set associated to μ and b. We assume that $b \in B(G,\mu)$ to have that $X_\mu(b)$ is nonempty (compare [Ra], 5). There are two cases where it is known that $X_\mu(b)$ is the set of k-valued points of a scheme. Here, k denotes the residue field of $\mathcal{O}_L$. The first case is that $\boldsymbol{F} = \mathbb{Q}_p$ and that $X_\mu(b)$ is the set of k-valued points of a Rapoport-Zink space of type (EL) or (PEL). In that case μ is always minuscule. Rapoport-Zink spaces without polarization were considered in [V1], in that case $G = GL_h$. For the moduli spaces considered in this paper let $G = GSp_{2h}$. We choose a basis $\{e_i, f_i \mid 1 \leq i \leq h\}$ identifying N with L^{2h} and the symplectic form on N with the symplectic form on L^{2h} defined by requiring that $\langle e_i, e_j\rangle = \langle f_i, f_j\rangle = 0$ and $\langle e_i, f_j\rangle = \delta_{i,h+1-j}$. Let B be the Borel subgroup of $G = GSp_{2h}$ fixing the complete isotropic flag $(e_1) \subset (e_1, e_2) \subset \cdots \subset (e_1, \dots, e_h)$. We choose A to be the diagonal torus. Let $\pi_1(G)$ be the quotient of $X_*(A)$ by the coroot lattice of G. Then the multiplier $G \to \mathbb{G}_{\rm m}$ induces an isomorphism $\pi_1(G) \to \pi_1(\mathbb{G}_{\rm m}) \cong \mathbb{Z}$. Let $\mu \in X_*(A)$ be the unique minuscule element whose image in $\pi_1(G)$ is 1. Then p^μ is a diagonal matrix with diagonal entries 1 and p, each with multiplicity h. We write $F = b\sigma$ with $b \in G$. Note that there is a bijection between $\mathcal{M}_{\text{red}}(k)$ and the set of Dieudonné lattices in N. Using the above notation, we have the bijection

$$\begin{array}{rcl} X_\mu(b) & \to & \mathcal{M}_{\text{red}}(k) \\ g & \mapsto & g(W(k)^{2h}). \end{array}$$

The second case is that $\boldsymbol{F}$ is a function field. Here $X_\mu(b)$ obtains its scheme structure by considering it as a subset of the affine Grassmannian $G(L)/G(\mathcal{O}_L)$. In this case we do not have to assume μ to be minuscule. The $X_\mu(b)$ are locally closed subschemes of the affine Grassmannian. The closed affine Deligne-Lusztig varieties $X_{\preceq\mu}(b)$ are defined to be the closed reduced subschemes of $G(L)/G(\mathcal{O}_L)$ given by $X_{\preceq\mu}(b) = \bigcup_{\mu'\preceq\mu} X_{\mu'}(b)$. Here $\mu' \preceq \mu$ if $\mu - \mu'$ is a non-negative linear combination of positive coroots. Note that the two schemes $X_\mu(b)$ and $X_{\preceq\mu}(b)$ coincide if μ is minuscule.

The sets of connected components of the moduli spaces of non-polarized p-divisible groups are given by a formula completely analogous to Theorem 1 (compare [V1], Thm. A). For closed affine Deligne-Lusztig varieties in the function field case, the set of connected components is also given by a generalization of the formula in Theorem 1 (see [V3], Thm. 1). The sets of connected components of the non-closed $X_\mu(b)$ are not known in general. There are examples (compare [V3], Section 3) which show that a result analogous to Theorem 1 cannot hold for all non-closed $X_\mu(b)$.

The only further general case where the set of irreducible components is known are the reduced subspaces of moduli spaces of p-divisible groups without polarization. Here, the group J also acts transitively on the set of irreducible components. There are examples of affine Deligne-Lusztig varieties in the function field case associated to non-minuscule μ where this is no longer true (compare [V2], Ex. 6.2).

To discuss the formula for the dimension let us first reformulate Theorem 3. Let $G = GSp_{2h}$ and μ be as above. Let $\nu = (\lambda_i) \in \mathbb{Q}^h \cong X_*(A)_{\mathbb{Q}}$ be the (dominant) Newton vector associated to (N, F) as defined by Kottwitz, see [Ko1]. Let ρ be the half-sum of the positive roots of G and ω_i the fundamental weights of the adjoint group G_{ad}. Then one can reformulate (1.2) as

$$\dim \mathcal{M}_{\text{red}} = \langle 2\rho, \mu - \nu\rangle + \sum_i \lfloor \langle \omega_i, \nu - \mu\rangle \rfloor. \tag{1.4}$$

In this form, the dimension formula proves a special case of a conjecture by Rapoport (see [Ra], Conjecture 5.10) for the dimension of affine Deligne-Lusztig varieties. Denote by $\mathrm{rk}_{\mathbb{Q}_p}$ the dimension of a maximal $\mathbb{Q}_p$-split subtorus and let $\mathrm{def}_G(F) = \mathrm{rk}_{\mathbb{Q}_p} G - \mathrm{rk}_{\mathbb{Q}_p} J$. Note that $\mathrm{def}_G(F)$ only depends on the conjugacy class of F or equivalently on the σ-conjugacy class of b if we write $F = b\sigma$ for some $b \in G$. In our case, it is equal to $h - \lceil l/2 \rceil$ where l is the number of simple summands of N. Using Kottwitz's reformulation of the right hand side of (1.4) in [Ko2], we obtain

$$\dim \mathcal{M}_{\text{red}} = \langle \rho, \mu - \nu\rangle - \frac{1}{2}\mathrm{def}_G(F). \tag{1.5}$$

For the case of moduli of p-divisible groups for $G = GL_h$, the analogous formula for the dimension is shown in [V1]. In the function field case, the dimension of the generalized affine Deligne-Lusztig variety has been determined in [V2], [GHKR]. The formula for the dimension is also in this case the analogue of (1.5).

The dimension of the moduli spaces $\mathcal{M}_{\mathrm{red}}$ is also studied by Oort and by Chai using a different approach. In [O2], Oort defines an almost product structure (that is, up to a finite morphism) on Newton strata of moduli spaces of polarized abelian varieties. It is given by an isogeny leaf and a central leaf for the p-divisible group. The dimension of the isogeny leaf is the same as that of the corresponding $\mathcal{M}_{\mathrm{red}}$. The dimension of the central leaf is determined by Chai in [C] and also by Oort in [O4]. The dimension of the Newton polygon stratum itself is known from [O1]. Then the dimension of $\mathcal{M}_{\mathrm{red}}$ can also be computed as the difference of the dimensions of the Newton polygon stratum and the central leaf.

We outline the content of the different sections of the paper. In Section 2 we introduce the necessary background and notation, and reduce the problem to the case of bi-infinitesimal groups. In the third and fourth section, we define the open dense subscheme $\mathcal{S}_1$ where the a-invariant of the p-divisible group is 1 and describe its set of closed points. This description is refined in Sections 5 and 6 to prove the theorems on the set of irreducible components and on the dimension, respectively. In the last section we determine the set of connected components.

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2 Notation and preliminary reductions

2.1 A decomposition of the rational Dieudonné module

The principal polarization $\lambda_{\mathbb{X}}$ equips the rational Dieudonné module (N, F) of $\mathbb{X}$ with a nondegenerate anti-symmetric bilinear pairing $\langle\cdot,\cdot\rangle$. It satisfies

$$\langle v, Fw\rangle = \sigma(\langle Vv, w\rangle) \tag{2.1}$$

for all $x, y \in N$.

We assumed k to be algebraically closed. Then the classification of isocrystals shows that N has a decomposition into subisocrystals N_i of one of the following types. Let l be the number of supersingular summands in a decomposition of N into simple isocrystals. Then

$$N = \begin{cases} N_0 \oplus N_1 & \text{if } l \text{ is even} \\ N_0 \oplus N_{\frac{1}{2}} \oplus N_1 & \text{otherwise,} \end{cases} \tag{2.2}$$

satisfying the following three properties.

1. The slopes of N_0 are smaller or equal to $\frac{1}{2}$.
2. The summand $N_{\frac{1}{2}}$ is simple and supersingular.
3. N_1 is the isocrystal dual to N_0, i.e.

$$\langle N_0, N_0\rangle = \langle N_1, N_1\rangle = \langle N_0, N_{\frac{1}{2}}\rangle = \langle N_1, N_{\frac{1}{2}}\rangle = 0.$$

Note that if $l > 1$, then this decomposition is not unique and N_0 and N_1 also contain supersingular summands. For $i \in \{0, \frac{1}{2}, 1\}$ we denote by p_i the canonical projection $N \to N_i$.

The moduli spaces $\mathcal{M}_{\mathrm{red}}$ for different $(\mathbb{X}, \lambda_{\mathbb{X}})$ in the same isogeny class are isomorphic. Replacing $\mathbb{X}$ by an isogenous group we may assume that

$$\mathbb{X} = \begin{cases} \mathbb{X}_0 \times \mathbb{X}_1 & \text{if } l \text{ is even} \\ \mathbb{X}_0 \times \mathbb{X}_{\frac{1}{2}} \times \mathbb{X}_1 & \text{otherwise.} \end{cases} \tag{2.3}$$

Here, $\mathbb{X}_i$ is such that its rational Dieudonné module is N_i.

Mapping $(X, \rho) \in \mathcal{M}_{\mathrm{red}}(k)$ to the Dieudonné module of X defines a bijection between $\mathcal{M}_{\mathrm{red}}(k)$ and the set of Dieudonné lattices in N that are self-dual up to a scalar. Here a sublattice Λ of N is called a Dieudonné lattice if $\varphi(\Lambda) \subseteq \Lambda$ for all φ in the Dieudonné ring of k,

$$\mathcal{D} = \mathcal{D}(k) = W(k)[F, V]/(FV = VF = p, aV = V\sigma(a), Fa = \sigma(a)F). \tag{2.4}$$

All lattices considered in this paper are Dieudonné lattices. A lattice $\Lambda \subset N$ is self-dual up to a scalar if the dual lattice $\Lambda^\vee$ satisfies $\Lambda^\vee = c\Lambda$ with $c \in L^\times$. The following notion is introduced by Oort in [O3].

Definition 2.1. Let X be a p-divisible group over k and $\Lambda_{\min}$ be its Dieudonné module. Then X is a minimal p-divisible group if $\mathrm{End}(\Lambda_{\min})$ is a maximal order in $\mathrm{End}(\Lambda_{\min}) \otimes_W L$.

Remark 2.2. By Morita equivalence X is minimal if and only if $\Lambda_{\min}$ is the direct sum of submodules $\Lambda^i_{\min}$ such that $N^i = \Lambda^i_{\min} \otimes_W L$ is simple and that $\mathrm{End}(\Lambda^i_{\min})$ is a maximal order in $\mathrm{End}(N^i)$, which is Oort's original definition. Note that in every isogeny class of p-divisible groups over k there is exactly one isomorphism class of minimal p-divisible groups (compare [O3], 1.1).

Lemma 2.3. *There is a k-valued point (X, ρ) of $\mathcal{M}_{\mathrm{red}}$ such that X is minimal.*

Proof. Let N_0 and N_1 as in the decomposition above. Let $\Lambda_{\min,0} \subset N_0$ be the lattice of a minimal p-divisible group and let $\Lambda_{\min,\frac{1}{2}} \subset N_{\frac{1}{2}}$ be the Dieudonné module of $\mathbb{X}_{\frac{1}{2}}$. There is only one isomorphism class of one-dimensional supersingular p-divisible groups and it consists of minimal p-divisible groups. Let $c \in L^\times$ with $\Lambda^\vee_{\min,\frac{1}{2}} = c\Lambda_{\min,\frac{1}{2}}$. Let

$$\Lambda_{\min,1} = \{x \in N_1 \mid \langle x, cy\rangle \in W \text{ for all } y \in \Lambda_{\min,0}\}.$$

Then $\Lambda_{\min,1}$ is also the Dieudonné module of a minimal p-divisible group. Furthermore, $\Lambda_{\min} = \Lambda_{\min,0} \oplus \Lambda_{\min,\frac{1}{2}} \oplus \Lambda_{\min,1}$ satisfies $\Lambda_{\min}^\vee = c\Lambda_{\min}$. Thus $\Lambda_{\min}$ corresponds to an element of $\mathcal{M}_{\mathrm{red}}(k)$ and to a minimal p-divisible group. $\square$

Remark 2.4. There is the following explicit description of the Dieudonné module of a minimal p-divisible group: Let $N = \bigoplus_j N^j$ be a decomposition of N into simple isocrystals. For each j we write the slope of N^j as $m_j/(m_j+n_j)$ with $(m_j, n_j) = 1$. Then there is a basis $e_1^j, \dots, e_{m_j+n_j}^j$ of N^j with $F(e_i^j) = e_{i+m_j}^j$ for all i, j. Here we use the notation $e_{i+m_j+n_j}^j = pe_i^j$. For the existence compare for example [V1], 4.1. Furthermore, these bases may be chosen such that $\langle e_i^j, e_{i'}^{j'} \rangle = \delta_{j,l+1-j'} \cdot \delta_{i,m_j+n_j+1-i'}$ for $1 \leq i, i' \leq m_j + n_j = m_{l+1-j} + n_{l+1-j}$. Then we can take the lattice $\Lambda_{\min}$ to be the lattice generated by these basis elements e_i^j.

2.2 Moduli of non-polarized p-divisible groups

For the moment let $\mathbb{X}$ be a p-divisible group without polarization. Then associated to $\mathbb{X}$ there is an analogous moduli problem of quasi-isogenies of p-divisible groups without polarization. If $\mathbb{X}$ is polarized, we thus obtain two functors which are closely related. In this section we recall the definition of the moduli spaces of non-polarized p-divisible groups and relate them to $\mathcal{M}_{\mathrm{red}}$. Besides, we provide a technical result on lattices in isocrystals which we need in the following section.

Let $\mathcal{M}_{\mathbb{X}}^{\mathrm{np}}$ be the functor associating to a scheme $S \in \mathrm{Nilp}_W$ the set of pairs (X, ρ) where X is a p-divisible group over S and ρ a quasi-isogeny $\mathbb{X}_{\overline{S}} \rightarrow X_{\overline{S}}$. Two such pairs (X_1, ρ_1) and (X_2, ρ_2) are identified in this set if $\rho_1 \circ \rho_2^{-1}$ lifts to an isomorphism $X_2 \rightarrow X_1$ over S. This functor is representable by a formal scheme which is locally formally of finite type over $\mathrm{Spf}(W)$ (see [RaZ], Theorem 2.16). Let $\mathcal{M}_{\mathbb{X},\mathrm{red}}^{\mathrm{np}}$ be its reduced subscheme. We always include $\mathbb{X}$ in this notation, because we compare $\mathcal{M}_{\mathrm{red}}$ to the two moduli spaces $\mathcal{M}_{\mathbb{X},\mathrm{red}}^{\mathrm{np}}$ and $\mathcal{M}_{\mathbb{X}_0,\mathrm{red}}^{\mathrm{np}}$.

Let $J^{\mathrm{np}} = \{g \in GL(N) \mid g \circ F = F \circ g\}$. Then $J \subseteq J^{\mathrm{np}}$.

If $\mathbb{X}$ is a principally polarized p-divisible group, then forgetting the polarization induces a natural inclusion as a closed subscheme

$$\mathcal{M}_{\mathrm{red}} \hookrightarrow \mathcal{M}_{\mathbb{X},\mathrm{red}}^{\mathrm{np}}.$$

Furthermore, there is a natural inclusion as a closed subscheme

$$\mathcal{M}_{\mathbb{X}_0,\mathrm{red}}^{\mathrm{np}} \hookrightarrow \mathcal{M}_{\mathrm{red}} \tag{2.5}$$

mapping an S-valued point (X_0, ρ_0) to $(X_0 \times X_0^\vee, (\rho, \rho^\vee))$ if the number of supersingular summands of N is even and to $(X_0 \times X_{\frac{1}{2}} \times X_0^\vee, (\rho, \rho_{\frac{1}{2}}, \rho^\vee))$ otherwise. Here $X_{\frac{1}{2}} = \mathbb{X}_{\frac{1}{2},S}$ is the base-change of the unique one-dimensional supersingular p-divisible group over k and $\rho_{\frac{1}{2}} = \mathrm{id}$.

Let $\tilde{v}$ be the valuation on the Dieudonné ring $\mathcal{D}$ determined by

$$\tilde{v}(aF^iV^j) = 2v_p(a) + i + j \tag{2.6}$$

for every $a \in W(k)$.

Lemma 2.5. *One can decompose each $B \in \mathcal{D}$ uniquely as $B = \mathrm{LT}(B) + B'$ with $\tilde{v}(B') > \tilde{v}(B)$ and*

$$\mathrm{LT}(B) = \sum_{0 \leq i \leq \tilde{v}(B), 2\alpha + i = \tilde{v}(B)} p^{\alpha}([a_i]V^i + [b_i]F^i).$$

Here $[a_i]$ and $[b_i]$ are Teichmüller representatives of elements of $k^{\times}$ or 0 and $[b_0] = 0$.

Proof. The V^i with $i \geq 0$ and the F^i with $i > 0$ together form a basis of the $W(k)$-module $\mathcal{D}$. Besides, as k is perfect, every element of $W(k)$ can be written in a unique way as $x = \sum_{\alpha \geq 0} p^{\alpha}[x_{\alpha}]$. Hence we can write $B = \sum_{i \geq 0} x_i V^i + \sum_{i > 0} y_i F^i = \sum_{i \geq 0} \sum_{\alpha \geq 0} p^{\alpha}[x_{i,\alpha}]V^i + \sum_{i>0} \sum_{\alpha \geq 0} p^{\alpha}[y_{i,\alpha}]F^i$ where x_i, y_i are 0 for all but finitely many i. By the definition of $\tilde{v}(B)$, all $x_{i,\alpha}, y_{i,\alpha}$ with $i + 2\alpha < \tilde{v}(B)$ vanish. Let $\mathrm{LT}(B)$ be the sum of all terms $p^{\alpha}[x_{i,\alpha}]V^i$ and $p^{\alpha}[y_{i,\alpha}]F^i$ on which $\tilde{v}$ takes the value $\tilde{v}(B)$, i.e. those with $2\alpha + i = \tilde{v}(B)$. Then $\mathrm{LT}(B)$ is as in the lemma and $\tilde{v}(B - \mathrm{LT}(B)) > \tilde{v}(B)$. □

Lemma 2.6. *Let $(N_0, b_0\sigma)$ be the rational Dieudonné module of some p-divisible group over k. Let $m = v_p(\det b_0)$ and $n = \dim_L(N_0) - m$. Let $v \in N_0$ be not contained in any proper sub-isocrystal of N_0.*

1. *$\mathrm{Ann}(v) = \{\varphi \in \mathcal{D} \mid \varphi(v) = 0\}$ is a principal left ideal of $\mathcal{D}$. There is a generating element of the form*

$$A = aF^n + bV^m + \sum_{i=0}^{n-1} a_i F^i + \sum_{i=1}^{m-1} b_i V^i.$$

with $a, b \in W^{\times}$ and $a_i, b_i \in W$.

2. *If N_0 is simple (and thus of slope $m/(m+n)$), we have*

$$\mathrm{LT}(A) = \begin{cases} [a]F^n & \text{if } n < m \\ [b]V^m & \text{if } m < n \\ [a]F + [b]V & \text{if } m = n = 1 \end{cases} \tag{2.7}$$

for some $a, b \in k^{\times}$.

3. *Let $N_0 = \oplus_j N^j$ be a decomposition of N_0 into simple summands. Then $\mathrm{LT}(A) = \mathrm{LT}(\prod_j L_j)$. Here each L_j is of the form (2.7) associated to some nonzero element in N^j.*

Proof. We use induction on the number of summands in a decomposition of N_0 into simple isocrystals. If N_0 is simple, the lemma follows immediately from [V1], Lemma 4.12. For the induction step write $N_0 = N' \oplus N''$ where N' is simple. Let A' be as in the lemma and associated to N' and $p_{N'}(v)$ where $p_{N'}: N \to N'$ is the projection. Note that an element of an isocrystal is not contained in any proper sub-isocrystal if and only if the Dieudonné module generated by the element is a lattice. Let Λ be the lattice generated by v. The Dieudonné module generated by $A'(v)$ is equal to $\Lambda \cap N''$, and hence also a lattice. We may therefore apply the induction hypothesis to $A'(v)$ and N'' and obtain some A'' generating $\mathrm{Ann}(A'(v))$. Thus $\mathrm{Ann}(v)$ is a principal left ideal generated by $A''A'$. Multiplying the corresponding expressions for A'' and A', the lemma follows. $\square$

2.3 Reduction to the bi-infinitesimal case

Let $\mathbb{X} = \mathbb{X}_{\mathrm{et}} \times \mathbb{X}_{\mathrm{bi}} \times \mathbb{X}_{\mathrm{m}}$ be the decomposition of $\mathbb{X}$ into its étale, bi-infinitesimal, and multiplicative parts.

Lemma 2.7. *We have*

$$\mathcal{M}_{\mathbb{X},\mathrm{red}} \cong \begin{cases} \mathcal{M}^{\mathrm{np}}_{\mathbb{X}_{\mathrm{et}},\mathrm{red}} \times \mathcal{M}_{\mathbb{X}_{\mathrm{bi}},\mathrm{red}} & \text{if } \mathbb{X}_{\mathrm{bi}} \text{ is nontrivial} \\ \mathcal{M}^{\mathrm{np}}_{\mathbb{X}_{\mathrm{et}},\mathrm{red}} \times \mathbb{Z} & \text{otherwise.} \end{cases}$$

and

$$\mathcal{M}^{\mathrm{np}}_{\mathbb{X}_{\mathrm{et}},\mathrm{red}} \cong GL_{\mathrm{ht}(\mathbb{X}_{\mathrm{et}})}(\mathbb{Q}_p)/GL_{\mathrm{ht}(\mathbb{X}_{\mathrm{et}})}(\mathbb{Z}_p).$$

Proof. Consider the following morphism ι from the right to the left hand side of the first isomorphism. In the first case, an S-valued point $\big((X_{\mathrm{et}}, \rho_{\mathrm{et}}), (X_{\mathrm{bi}}, \rho_{\mathrm{bi}})\big)$ is mapped to $(X_{\mathrm{et}} \times X_{\mathrm{bi}} \times X_{\mathrm{m}}, (\rho_{\mathrm{et}}, \rho_{\mathrm{bi}}, \rho_{\mathrm{m}}))$ where $X_{\mathrm{m}} = \hat{X}_{\mathrm{et}}$. Furthermore, ρ_{m} is the dual isogeny of $c \cdot \rho_{\mathrm{et}}$ and c is the scalar determined by the duality condition for ρ_{bi}. In the second case $((X_{\mathrm{et}}, \rho_{\mathrm{et}}), l)$ is mapped to $(X_{\mathrm{et}} \times X_{\mathrm{m}}, (\rho_{\mathrm{et}}, \rho_{\mathrm{m}}))$ with $X_{\mathrm{m}} = \hat{X}_{\mathrm{et}}$ and $\rho_{\mathrm{m}} = (p^l \cdot \rho_{\mathrm{et}})^{\vee}$. In both cases ι is a monomorphism, and to check that it is a closed immersion we verify the valuation criterion for properness. Let (X, ρ) be a $k[[t]]$-valued point of $\mathcal{M}_{\mathbb{X},\mathrm{red}}$ such that the generic point is in the image of ι. Let $\pi_X : X \twoheadrightarrow X_{\mathrm{et}}$ with X_{et} étale over $\mathrm{Spec}(k[[t]])$ and X inf initesimal over X_{et}, as in [M], Lemma II.4.8. Our assumption implies that this map has a right inverse $X_{\mathrm{et},k((t))} \to X_{k((t))}$ after base change to $\mathrm{Spec}(k((t)))$. By [J1], Corollary 1.2, this morphism lifts to a morphism $X_{\mathrm{et}} \to X$ over $k[[t]]$. Together with the inclusion of the kernel of π_X in X we obtain a morphism of the product of an étale and an infinitesimal p-divisible group over $\mathrm{Spec}(k[[t]])$ to X. Its inverse is constructed similarly by lifting the projection morphism of $X_{k((t))}$ to the kernel of π_X from $k((t))$ to $k[[t]]$. Hence X can be written as a product of an étale and an infinitesimal p-divisible group. As X is selfdual, it is then also the product of an étale, a bi-infinitesimal, and a multiplicative p-divisible group, thus of the form $X_{\mathrm{et}} \times X_{\mathrm{bi}} \times X_{\mathrm{m}}$ with $X_{\mathrm{m}} = \hat{X}_{\mathrm{et}}$. The quasi-isogeny ρ is compatible with this decompos ition, and the compatibility with the polarizations shows that the induced quasi-isogenies $(\rho_{\mathrm{et}}, \rho_{\mathrm{bi}}, \rho_{\mathrm{m}})$ have the property that ρ_{bi} is selfdual up to some scalar c and ρ_{m} is the dual isogeny of $c \cdot \rho_{\mathrm{et}}$. Hence (X, ρ) is in the image of ι. This finishes the proof that ι is proper, hence a closed immersion.

To show that ι is an isomorphism it is thus enough to show that each k-valued point of the left hand side is contained in its image. From the Hodge-Newton decomposition (see [Kat], Thm. 1.6.1) we obtain for each k-valued point (X,ρ) a decomposition $X = X_{\rm et}\times X_{\rm bi}\times X_{\rm m}$ and $\rho=\rho_{\rm et}\times\rho_{\rm bi}\times\rho_{\rm m}$ into the étale, bi-infinitesimal, and multiplicative parts. The compatibility with the polarization then yields that up to some scalar p^l, the quasi- isogenies $\rho_{\rm m}$ and $\rho_{\rm et}$ are dual. From this the first isomorphism follows. The second isomorphism is shown by an easy calculation (compare [V1], Section 3). $\square$

The lemma reduces the questions after the global structure of $\mathcal{M}_{\rm red}$ to the same questions for $\mathcal{M}_{\mathbb{X}_{\rm bi},{\rm red}}$. Thus from now on we assume that $\mathbb{X}$ is bi-infinitesimal.

3 The dense subscheme $\mathcal{S}_1$

In [V1], 4.2 we define an open dense subscheme $\mathcal{S}^{\rm np}_{\mathbb{X},1}$ of $\mathcal{M}^{\rm np}_{\mathbb{X},{\rm red}}$. Let $\Lambda\subset N$ be the lattice associated to $x\in\mathcal{M}^{\rm np}_{\mathbb{X},{\rm red}}(k)$. Then $x\in\mathcal{S}^{\rm np}_{\mathbb{X},1}$ if and only if $a(\Lambda)=\dim_k(\Lambda/(F\Lambda+V\Lambda))=1$. As F and V are topologically nilpotent on Λ, this is equivalent to the existence of some $v\in\Lambda$ such that Λ is the Dieudonné submodule of N generated by v. Note that $a(\Lambda)$ can also be defined as $\dim_k(\mathrm{Hom}(\alpha_p,X))$ where X is the p-divisible group associated to Λ.
Let

$$\mathcal{S}_1=\mathcal{S}^{\rm np}_{\mathbb{X},1}\cap\mathcal{M}_{\rm red}\subseteq\mathcal{M}_{\rm red}.$$

Then $\mathcal{S}_1$ is open in $\mathcal{M}_{\rm red}$.

Lemma 3.1. *The open subscheme $\mathcal{S}_1$ is dense in $\mathcal{M}_{\rm red}$.*

Proof. Recall that we assume $\mathbb{X}$ to be bi-infinitesimal. Let $(\overline{X},\overline{\rho})\in\mathcal{M}_{\rm red}(k)$ and let $\overline{\lambda}$ be a corresponding polarization of $\overline{X}$. Note that by [M], Lemma II.4.16 (or by [J2], Lemma 2.4.4) there is an equivalence of categories between p-divisible groups over an adic, locally noetherian affine formal scheme $\mathrm{Spf}(A)$ and over $\mathrm{Spec}(A)$. From [O1], Corollary 3.11 we obtain a deformation (X,λ) of $(\overline{X},\overline{\lambda})$ over $\mathrm{Spec}(k[[t]])$ such that the generic fiber satisfies $a=1$. Next we show that after a base change we may also deform $\overline{\rho}$ to a quasi-isogeny ρ between (X,λ) and the constant p-divisible group $(\mathbb{X},\lambda_{\mathbb{X}})$ that is compatible with the polarizations. From [OZ], Corollary 3.2 we obtain a deformation of $\overline{\rho}$ to a quasi-isogeny between X and a constant p-divisible group Y after a base-change to the perfect hull of $k[[t]]$. As Y is constant it is quasi-isogenous to the base change $\mathbb{X}_{\mathrm{Spec}(k[[t]]^{\rm perf})}$ of $\mathbb{X}$. After composing the deformation of $\overline{\rho}$ with a quasi-isogeny between Y and $\mathbb{X}_{\mathrm{Spec}(k[[t]]^{\rm perf})}$ we may assume that Y is already equal to $\mathbb{X}_{\mathrm{Spec}(k[[t]]^{\rm perf})}$. Let x be the point of $\mathrm{Spec}(k[[t]]^{\rm perf})$ over the generic point of $\mathrm{Spec}(k[[t]])$. Then we may further compose the quasi-isogeny with a self-quasi-isogeny of $\mathbb{X}_{\mathrm{Spec}(k[[t]]^{\rm perf})}$ such that in x it is compatible with the polarizations of the two groups in this point. Thus we obtain a $k[[t]]^{\rm perf}$-valued point of $\mathcal{M}^{\rm np}_{\mathbb{X},{\rm red}}$ such that the image of x is in $\mathcal{M}_{\rm red}$. As $\mathcal{M}_{\rm red}$ is closed, this has to be a $k[[t]]^{\rm perf}$-valued point of $\mathcal{M}_{\rm red}$. Modifying the point by a suitable elemen t of J, we may assume that the special point is mapped to $(\overline{X},\overline{\rho})$. In the generic point, the a-invariant of the p-divisible group X is 1. Thus this provides the desired deformation of $(\overline{X},\overline{\rho})$ to a point of $\mathcal{S}_1$. $\square$

To determine the dimension and the set of irreducible components of $\mathcal{M}_{\mathrm{red}}$ it is thus sufficient to consider $\mathcal{S}_1$. We proceed in the same way as for the moduli spaces of p-divisible groups without polarization. In contrast to the non-polarized case it turns out to be useful to use two slightly different systems of coordinates to prove the assertions on the dimension and on the set of irreducible components of $\mathcal{M}_{\mathrm{red}}$.
Let us briefly recall the main steps for the moduli spaces $\mathcal{M}^{\mathrm{np}}_{\mathbb{X},\mathrm{red}}$ of non-polarized p-divisible groups. Their sets of irreducible components and their dimension are determined by studying $\mathcal{S}^{\mathrm{np}}_{\mathbb{X},1}$. In [V1], 4 it is shown that the connected components of $\mathcal{S}^{\mathrm{np}}_{\mathbb{X},1}$ are irreducible and that $J^{\mathrm{np}} = \{j \in GL(N) | j\circ F = F\circ j\}$ acts transitively on them. The first step to prove this is to give a description of $\mathcal{S}^{\mathrm{np}}_{\mathbb{X},1}(k)$. One uses that each such point corresponds to a lattice in N with a-invariant 1. As Dieudonné modules, these lattices are generated by a single element and the description of the set of points is given by classifying these elements generating the lattices. The second step consists in the construction of a family in $\mathcal{M}^{\mathrm{np}}_{\mathbb{X},\mathrm{red}}$ to show that a set of points which seems to parametrize an irreducible compo nent of $\mathcal{S}^{\mathrm{np}}_{\mathbb{X},1}$ indeed comes from an irreducible subscheme. More precisely, a slight reformulation of the results in [V1], Section 4 yields the following proposition.

Proposition 3.2. *Let (N,F) be the isocrystal of a p-divisible group $\mathbb{X}$ over k. Let $m = v_p(\det F)$. Let $S = \mathrm{Spec}(R) \in \mathrm{Nilp}_W$ be a reduced affine scheme with $pR = 0$ and let $j \in J^{\mathrm{np}}$. Let $v \in N_R = N\otimes_L W(R)[\frac{1}{p}]$ such that in every $x \in S(k)$, the reduction v_x of v in x satisfies that*

$$v_x \in j\Lambda_{\min}$$

and

$$v_p(\det j) = \max\{v_p(\det j') \mid j' \in J^{\mathrm{np}} \text{ and } v_x \in j'\Lambda_{\min}\}.$$

Here, $\Lambda_{\min} \subset N$ is the lattice of the minimal p-divisible group in Remark 2.4. Let $\tilde{R} = \sigma^{-m}(R)$ be the unique reduced extension of R such that $\sigma^m : \tilde{R} \to \tilde{R}$ has image R. Let $\tilde{v} \in N_{\tilde{R}}$ with $\sigma^m(\tilde{v}) = v$. Then there is a morphism $\varphi : \mathrm{Spec}(\tilde{R}) \to \mathcal{M}^{\mathrm{np}}_{\mathbb{X},\mathrm{red}}$ such that for every $x \in \mathrm{Spec}(\tilde{R})(k)$, the image $\varphi(x)$ corresponds to the Dieudonné module Λ_x in N generated by v_x.
Assume in addition that $\mathbb{X}$ is principally polarized and that for every x, the Dieudonné module Λ_x corresponds to a point of $\mathcal{M}_{\mathrm{red}}$. Then φ factors through $\mathcal{M}_{\mathrm{red}}$.

Note that the second condition on v_x (or more precisely the existence of the maximum) implies that the Dieudonné submodule of N generated by v_x is a lattice.

Proof. To prove the first assertion we may assume that $j = \mathrm{id}$. Note that $\tilde{v}$ satisfies the same conditions as v. The conditions on the $\tilde{v}_x$ are reformulated in [V1], Lemma 4.8. The condition given there is exactly the condition needed in [V1], Section 4.4 to construct a display over S leading to the claimed morphism φ. It maps x to the Dieudonné lattice generated by $\sigma^m(\tilde{v}_x) = v_x$. The second assertion is trivial as S is reduced and $\mathcal{M}_{\mathrm{red}}$ a closed subscheme of $\mathcal{M}^{\mathrm{np}}_{\mathbb{X},\mathrm{red}}$. $\square$

Remark 3.3. We use the same notation as in the proposition. From [V1], 4.4 we also obtain that under the conditions of Proposition 3.2, the elements

$$v, Vv, \dots, V^{v_p(\det F)}v, Fv, \dots, F^{\dim N - v_p(\det F)-1}v$$

are a basis of the free $W(\tilde{R})[\frac{1}{p}]$-module $N_{\tilde{R}}$. They are the images of the standard basis of N under some element of $GL(N_{\tilde{R}})$.
We apply the preceding to the situation of an isocrystal $N = N_0$ and its dual, N_1. Then $GL(N_0) \times \mathbb{G}_{\mathrm{m}}$ is isomorphic to the Siegel Levi subgroup of $GSp(N_0 \oplus N_1)$. Let $v \in N_0 = N$ be as in Proposition 3.2. Then there are elements $y_i \in (N_1)_{\tilde{R}}$ which form a basis of $(N_1)_{\tilde{R}}$ which is dual to the basis

$$(x_1, \dots, x_{\dim N_0}) = (v, Vv, \dots, V^{v_p(\det F)}v, Fv, \dots, F^{\dim N_0 - v_p(\det F)-1}v)$$

with respect to $\langle \cdot, \cdot \rangle$. In other words, the $y_i \in (N_1)_{\tilde{R}}$ are such that $\langle x_i, y_j \rangle = \delta_{ij}$.

4 Geometric points of $\mathcal{S}_1$

4.1 N with an even number of supersingular summands

In this subsection we consider the case that N has an even number of supersingular summands. By (2.2) we have a decomposition $N = N_0 \oplus N_1$.
Recall that by a lattice we always mean a Dieudonné lattice. Let $\Lambda \subset N$ be the lattice corresponding to a k-valued point of $\mathcal{M}_{\mathrm{red}}$. Then $\Lambda^\vee = c\Lambda$ for some $c \in L^\times$. Let $\Lambda_0 = p_0(\Lambda)$ and $\Lambda_1 = \Lambda \cap N_1$. For a subset M of N and $\delta \in \{0, 1\}$ let

$$(M)^\vee_\delta = \{x \in N_\delta \mid \langle x, x' \rangle \in W \text{ for all } x' \in M\}. \tag{4.1}$$

Then $c\Lambda_1 = (\Lambda_0)^\vee_1$. Hence Λ_0 and Λ_1 correspond to dual p-divisible groups, which implies $a(\Lambda_0) = a(\Lambda_1)$.
The geometric points of $\mathcal{S}_1$ correspond to lattices Λ that in addition satisfy $a(\Lambda) = 1$. Especially, $a(\Lambda_0) = a(\Lambda_1) = 1$. In this subsection we classify a slightly larger class of lattices. We fix a lattice $\Lambda_0 \subset N_0$ with $a(\Lambda_0) = 1$ and $c \in L^\times$. Then we consider all lattices $\Lambda \subset N$ with

$$p_0(\Lambda) = \Lambda_0 \text{ and } \Lambda^\vee = c\Lambda. \tag{4.2}$$

Note that we have a description of the set of lattices $\Lambda_0 \subset N_0$ with $a(\Lambda_0) = 1$ from [V1], see also Section 2.2.
The considerations above show that $\Lambda \cap N_1 = \Lambda_1 = c^{-1}(\Lambda_0)^\vee_1$ is determined by Λ_0 and c. Let v_0 be an element generating Λ_0 as a Dieudonné module. If

$v \in \Lambda$ with $p_0(v) = v_0$, then Λ is generated by v and Λ_1. Let A be a generator of $\mathrm{Ann}(v_0)$ as in Lemma 2.6. We write $v = v_0 + v_1$ for some $v_1 \in N_1$. Then $Av = Av_1 \in \Lambda_1$.

Remark 4.1. Let Λ_0 and c be as above, and let $\Lambda_1 = c^{-1}(\Lambda_0)_1^\vee$. Let Λ be a Dieudonné lattice with $p_0(\Lambda) = \Lambda_0$, $\Lambda \cap N_1 \supseteq \Lambda_1$ and

$$\Lambda^\vee \supseteq c\Lambda. \tag{4.3}$$

Let $\mathrm{vol}(\cdot)$ denote the volume of a lattice, normalized in such a way that the lattice corresponding to the basepoint $(\mathbb{X}, \mathrm{id})$ of $\mathcal{M}_{\mathrm{red}}$ has volume 0. The conditions imply above that $\mathrm{vol}(\Lambda^\vee) \leq \mathrm{vol}(c\Lambda) \leq \mathrm{vol}(c(\Lambda_0 \oplus \Lambda_1)) = \mathrm{vol}((\Lambda_0 \oplus \Lambda_1)^\vee)$. Dualizing the inequality for the first and last term, we see that all terms must be equal. Thus Λ satisfies (4.2) and $\Lambda \cap N_1 = \Lambda_1$.

The next step in the description of lattices with (4.2) is to reformulate (4.3). To do so, we fix a generator v_0 of Λ_0 and describe the set of all $v_1 \in N_1$ such that the lattice Λ generated by $v = v_0 + v_1$ and Λ_1 as a Dieudonné lattice satisfies (4.3) and $\Lambda \cap N_1 = \Lambda_1$. Generators for Λ as a W-module are given by Λ_1, and all $F^i v$ with $i \geq 0$ and $V^i v$ with $i > 0$. Let m, n be as in Lemma 2.6 (associated to the given N_0). Note that as N_0 contains all simple summands of N with slope $< 1/2$ and half of the supersingular summands, m is the same as in (1.1) and $n = h - m \geq m$. By Lemma 2.6, 1. applied to $v_0 \in N_0$, the $F^i v$ with $i > n$, and the $V^i v$ with $i \geq m$ can be written as a linear combination of the $F^i v$ with $i \leq n$ and the $V^i v$ with $i < m$, and a summand in $\mathrm{Ann}(v_0) \cdot v \subset \Lambda \cap N_1 = \Lambda_1$. Hence Λ is already generated by Λ_1, the $F^i v$ with $0 \leq i \leq n$ and the $V^i v$ with $0 < i < m$. The inclusion (4.3) is equivalent to $\langle x, y \rangle \in c^{-1}W$ for all $x, y \in \Lambda$. This is equivalent to the same condition for pairs (x, y) where x and y are among the generators of Λ described above. From the definition of Λ_1 we see that the values on pairs of elements of Λ automatically satisfy this if one of the elements is in Λ_1. By (2.1) it is enough to consider the products of v with all other generators. Thus (4.3) is equivalent to

$$\langle v, F^i v \rangle \in c^{-1}W$$

and

$$\langle v, V^i v \rangle \in c^{-1}W \tag{4.4}$$

for $n \geq i > 0$. Furthermore, the equations for V^i together with (2.1) imply those for F^i.

If x and y are elements of the same of the summands N_0 or N_1, then $\langle x, y \rangle = 0$. Hence the decomposition of v together with (2.1) shows that (4.4) is equivalent to

$$\langle v_0, V^i v_1 \rangle - \langle V^i v_0, v_1 \rangle = \langle F^i v_0, v_1 \rangle^{\sigma^{-i}} - \langle V^i v_0, v_1 \rangle \in c^{-1}W. \tag{4.5}$$

For $\phi \in \mathcal{D}$ let

$$\xi_{v_1}(\phi) = \langle \phi v_0, v_1 \rangle. \tag{4.6}$$

Then ξ_{v_1} is left-W-linear in ϕ. Let A be a generator of $\mathrm{Ann}(v_0)$ as in Lemma 2.6. Then

$$\xi_{v_1}(\psi A) = 0 \tag{4.7}$$

for all $\psi \in \mathcal{D}$. Note that an element $v_1 \in N_1$ is uniquely determined by $\langle v_1, F^i v_0\rangle$ for $i \in \{0, \dots, n-1\}$ and $\langle v_1, V^i v_0\rangle$ for $i \in \{1, \dots, m\}$. We are looking for the set of v_1 satisfying (4.5). In terms of ξ_{v_1}, this is

$$\xi_{v_1}(F^i)^{\sigma^{-i}} - \xi_{v_1}(V^i) \in c^{-1}W. \tag{4.8}$$

Lemma 4.2. *1. Let M be the set of W-linear functions $\xi : \mathcal{D} \to L$ with (4.7) and (4.8) for $i \leq n$. Then (4.6) defines a bijection between M and the set of elements $v_1 \in N_1$ as above.*

2. Let $\overline{M}$ be the set of functions $\xi : \mathcal{D} \to L/c^{-1}W$ with the same properties as in 1. Then (4.6) defines a bijection between $\overline{M}$ and the set of equivalence classes of elements v_1 as above. Here two such elements are called equivalent if their difference is in $c^{-1}(\Lambda_0)_1^\vee$.

Proof. Let $\xi : \mathcal{D} \to L$ be given. An element v_1 of N_1 is uniquely determined by the value of $\langle \cdot, v_1\rangle$ on $v_0, Fv_0, \dots, F^{h-m-1}v_0, Vv_0, \dots, V^m v_0$. These h values may be chosen arbitrarily. For the values of $\langle \cdot, v_1\rangle$ on the other elements of $\mathcal{D}v_0$, a complete set of relations is given by $\langle \psi A v_0, v_1\rangle = 0$ for all $\psi \in \mathcal{D}$. This is equivalent to (4.7). Furthermore, (4.8) is equivalent to the condition that the lattice generated by Λ_1 and $v_0 + v_1$ satisfies all required duality properties. To prove 2., we want to lift $\xi : \mathcal{D} \to L/c^{-1}W$ to a function with values in L. We lift the values of ξ at $\phi \in \{V^m, V^{m-1}, \dots, 1, \dots, F^{h-m-1}\}$ arbitrarily. Then the lifts of the remaining values are uniquely determined by (4.7). As (4.8) was satisfied before, it still holds (as a relation modulo $c^{-1}W$) for the lifted functions. Then 1. implies the existence of v_1. Let now w_1 be a second element inducing ξ (mod $c^{-1}W$). Then $\langle \phi v_0, w_1 - v_1\rangle \in c^{-1}W$ for all $\phi \in \mathcal{D}$. Hence $w_1 - v_1 \in c^{-1}(\Lambda_0)_1^\vee$. □

4.2 N with an odd number of supersingular summands

As parts of this case are similar to the previous one, we mainly describe the differences. By (2.2) we have a decomposition $N = N_0 \oplus N_{\frac{1}{2}} \oplus N_1$.
We want to classify the lattices $\Lambda \subset N$ corresponding to k-valued points of $\mathcal{S}_1$. As before let $\Lambda_0 = p_0(\Lambda)$ and $\Lambda_1 = \Lambda \cap N_1$. Let $c \in L^\times$ with $\Lambda^\vee = c\Lambda$. Then $c\Lambda_1 = (\Lambda_0)_1^\vee$. Besides,

$$c\Lambda \cap N_{\frac{1}{2}} = (p_{\frac{1}{2}}(\Lambda))_{\frac{1}{2}}^\vee. \tag{4.9}$$

Here we use $(\cdot)_{\frac{1}{2}}^\vee$ analogously to (4.1).
Again we use the description of the Dieudonné lattices $\Lambda_0 \subset N_0$ with $a(\Lambda_0) = 1$. We have to classify the Λ corresponding to some fixed Λ_0 and c, and begin by describing and normalizing the possible images under the projection to $N_0 \oplus N_{\frac{1}{2}}$. Let $v \in \Lambda$ with $\mathcal{D}v = \Lambda$ and write $v = v_0 + v_{\frac{1}{2}} + v_1$ with $v_i \in N_i$. Let A with $\tilde{v}(A) = m$ be a generator of $\mathrm{Ann}(v_0)$ as in Lemma 2.6.

Proposition 4.3. *1. There is a $j \in J$ such that $j(v)$ is of the form $v_0 + \tilde{v}_{\frac{1}{2}} + \tilde{v}_1$ with $\tilde{v}_i \in N_i$ and $A\mathcal{D}\tilde{v}_{\frac{1}{2}} = \mathcal{D}A\tilde{v}_{\frac{1}{2}}$.*

2. Let j be as in the previous statement. Then $p_{\frac{1}{2}}(j\Lambda)$ is the unique Dieudonné lattice in $N_{\frac{1}{2}}$ with $p_{\frac{1}{2}}(j\Lambda)^\vee = (cp^m)p_{\frac{1}{2}}(j\Lambda)$. Besides, $(j\Lambda) \cap N_{\frac{1}{2}} = p^m p_{\frac{1}{2}}(j\Lambda)$.

Proof. To prove 1. let $\tilde{v}_{\frac{1}{2}} \in N_{\frac{1}{2}}$ be such that $v'_{\frac{1}{2}} = v_{\frac{1}{2}} - \tilde{v}_{\frac{1}{2}}$ is in the kernel of A and $A\mathcal{D}\tilde{v}_{\frac{1}{2}} = \mathcal{D}A\tilde{v}_{\frac{1}{2}}$.
We first reduce the assertion of 1. to the case where N_0 and N_1 are simple of slope $\frac{1}{2}$. Let $A_{\frac{1}{2}}$ be a generator of $\mathrm{Ann}(v'_{\frac{1}{2}})$ as in Lemma 2.6. As $A \in \mathrm{Ann}(v'_{\frac{1}{2}})$, we can write $A = \tilde{A}A_{\frac{1}{2}}$ with $\tilde{A} \in \mathcal{D}$. Then $\tilde{A}$ generates $\mathrm{Ann}(A_{\frac{1}{2}}v_0)$. From the description of annihilators of elements of N_0 in Lemma 2.6 we see that we may write $v_0 = v'_0 + \tilde{v}_0$ with $A_{\frac{1}{2}}v'_0 = 0$ and $\tilde{v}_0$ lying in a proper subisocrystal $\tilde{N}_0$ of N_0. Then v'_0 generates a simple subisocrystal N'_0 of N_0 of slope $\frac{1}{2}$ and $N_0 = N'_0 \oplus \tilde{N}_0$. Let N'_1 be the subisocrystal of N_1 which is dual to N'_0. Then we want to show that the assertion of the proposition holds for some $j \in J \cap \mathrm{End}(N'_0 \oplus N'_{\frac{1}{2}} \oplus N'_1)$. To simplify the notation, we may assume that N only consists of these three summa nds.
We construct the inverse of the claimed element $j \in J$. Let $\tilde{j} \in \{g \in GL(N_0 \oplus N_1) \mid g \circ F = F \circ g\}$ be in the unipotent radical of the parabolic subgroup stabilizing the subspace N_1. We assume that $\tilde{j} \notin J$, i. e. that $\tilde{j}$ is not compatible with the pairing. Let $v_0 + v_1$ with $v_1 \in N_1$ be the image of v_0. Then $\mathrm{Ann}(v_1) = \mathrm{Ann}(v_0)$ and $f = \langle v_0 + v_1, F(v_0 + v_1)\rangle \neq 0$. Let A be a generator of $\mathrm{Ann}(v_0)$ as in Lemma 2.6. Then $A = aF + a_0 + bV$ for some $a, b \in W^\times$ and $a_0 \in W$. We obtain

$$0 = \langle v_0 + v_1, A(v_0 + v_1)\rangle^\sigma = a^\sigma f^\sigma - bf.$$

This is a $\mathbb{Q}_p$-linear equation of degree p, thus the set of solutions is a one-dimensional $\mathbb{Q}_p$-vector space in L. As A also generates $\mathrm{Ann}(v'_{\frac{1}{2}})$, the number $\langle v'_{\frac{1}{2}}, F(v'_{\frac{1}{2}})\rangle$ is also in this vector space. Hence there is an $\alpha \in \mathbb{Q}_p^\times$ wit h $\alpha f = \langle v'_{\frac{1}{2}}, F(v'_{\frac{1}{2}})\rangle$. By multiplying v_1 by a suitable factor, we may assume that $\alpha = -1$. Note that this does not change $\mathrm{Ann}(v_1)$. This implies that

$$\langle v_0 + v'_{\frac{1}{2}} + v_1, F(v_0 + v'_{\frac{1}{2}} + v_1)\rangle = 0.$$

Besides, we have $\mathrm{Ann}(v_0 + v'_{\frac{1}{2}} + v_1) = \mathrm{Ann}(v_0)$. The element j^{-1} we are constructing will map v_0 to $v_0 + v'_{\frac{1}{2}} + v_1$. Let $\tilde{N}_0 = \mathcal{D}(v_0 + v'_{\frac{1}{2}} + v_1)$. Then we can extend j^{-1} uniquely to a linear map from N_0 to $\tilde{N}_0$ which is compatible with F. On N_1, we define j^{-1} to be the identity. Then one easily checks that $j^{-1} : N_0 \oplus N_1 \to \tilde{N}_0 \oplus N_1$ respects the pairing. It remains to define j^{-1} on $N_{\frac{1}{2}}$. Let $\tilde{N}_{\frac{1}{2}}$ be the orthogonal complement of $\tilde{N}_0 \oplus N_1$. Then $\tilde{N}_{\frac{1}{2}} \subseteq N_{\frac{1}{2}} \oplus N_1$

Let $j^{-1}(v'_{\frac{1}{2}}) = u$ where $u \in \tilde{N}_{\frac{1}{2}}$ is such that $p_{N_{\frac{1}{2}}}(u) = v'_{\frac{1}{2}}$. Then $\mathrm{Ann}(u) \subseteq \mathrm{Ann}(v'_{\frac{1}{2}})$. As u is contained in a simple isocrystal of slope $\frac{1}{2}$, this inclusion has to be an equality. As $\langle N_{\frac{1}{2}} \oplus N_1, N_1\rangle = 0$, we have $\langle u, Fu\rangle = \langle v'_{\frac{1}{2}}, Fv'_{\frac{1}{2}}\rangle$. Hence we can extend j^{-1} to an element of J. Then $p_{\frac{1}{2}}(j(v_0+v_{\frac{1}{2}}+v_1)) = v_{\frac{1}{2}}-v'_{\frac{1}{2}} = \tilde{v}_{\frac{1}{2}}$. Thus j satisfies all properties of 1.

It remains to prove 2. We may assume that $j = 1$. Note that there is exactly one Dieudonné lattice of each volume in $N_{\frac{1}{2}}$. Equivalently, for each $\alpha \in L^\times$ there is exactly one $\Lambda \subset N_{\frac{1}{2}}$ with $\Lambda^\vee = \alpha\Lambda$. (For the rest of the proof all dual lattices are the dual lattices inside the selfdual isocrystal $N_{\frac{1}{2}}$.) We have $\Lambda \cap N_{\frac{1}{2}} = c^{-1}(p_{\frac{1}{2}}(\Lambda))^\vee \subseteq p_{\frac{1}{2}}(\Lambda)$. Let $\Lambda_{\frac{1}{2}}$ be the lattice with $c^{-1}(\Lambda_{\frac{1}{2}})^\vee = \Lambda_{\frac{1}{2}}$. Then

$$\Lambda \cap N_{\frac{1}{2}} \subseteq \Lambda_{\frac{1}{2}} \subseteq p_{\frac{1}{2}}(\Lambda) \tag{4.10}$$

and the lengths of the two inclusions are equal. We have to show that the length of the inclusions are both equal to m. The lattice $p_{\frac{1}{2}}(\Lambda)$ also contains $A(p_{\frac{1}{2}}(\Lambda))$. As $\tilde{v}(A) = m$, the length of this inclusion is m. Furthermore,

$$\Lambda \cap N_{\frac{1}{2}} = \mathrm{Ann}(Av_1)Av = \mathrm{Ann}(Av_1)Av_{\frac{1}{2}} \subseteq \mathcal{D}Av_{\frac{1}{2}} = A\mathcal{D}v_{\frac{1}{2}} = Ap_{\frac{1}{2}}(\Lambda).$$

Note that here we only know that the length of the inclusion is $\geq m = \tilde{v}(A_1)$ where A_1 is a generator of $\mathrm{Ann}(Av_1)$. Thus we obtain a second chain of inclusions

$$\Lambda \cap N_{\frac{1}{2}} \subseteq Ap_{\frac{1}{2}}(\Lambda) \subseteq p_{\frac{1}{2}}(\Lambda).$$

We compare this to (4.10). To show that the length of the first inclusion of this chain is not greater than the length of the second inclusion, we have to show that $Ap_{\frac{1}{2}}(\Lambda) \subseteq \Lambda_{\frac{1}{2}}$. By definition of $\Lambda_{\frac{1}{2}}$ this is equivalent to $Ap_{\frac{1}{2}}(\Lambda) \subseteq c^{-1}(Ap_{\frac{1}{2}}(\Lambda))^\vee$. To prove this last inclusion we use again the duality relation for Λ. Note that $Ap_{\frac{1}{2}}(\Lambda) = \mathcal{D}Av_{\frac{1}{2}} = p_{\frac{1}{2}}(\Lambda \cap (N_{\frac{1}{2}} \oplus N_1))$. Let $x, y \in N_{\frac{1}{2}} \oplus N_1$. Then $\langle x, y\rangle = \langle p_{\frac{1}{2}}(x), p_{\frac{1}{2}}(y)\rangle$. Thus the duality relation for Λ implies that $Ap_{\frac{1}{2}}(\Lambda) \subseteq c^{-1}(Ap_{\frac{1}{2}}(\Lambda))^\vee$. □

For both Theorem 2 and Theorem 3 it is enough to describe a locally closed subset of $\mathcal{S}_1$ whose image under the action of J is all of $\mathcal{S}_1$. Thus we may assume that $j = 1$ and that v itself already satisfies the property of the proposition. Especially, $p_{\frac{1}{2}}(\Lambda)$ is then determined by c.

The element $v_{\frac{1}{2}}$ may be modified by arbitrary elements in $p_{\frac{1}{2}}(\Lambda \cap (N_{\frac{1}{2}} \oplus N_1))$ without changing Λ. Indeed, for each such element there is an element in Λ whose projection to $N_0 \oplus N_{\frac{1}{2}}$ is the given element. Thus for fixed v_0, the projection of Λ to $N_0 \oplus N_{\frac{1}{2}}$ is described by the element $v_{\frac{1}{2}}$ varying in the W-module

$$p_{\frac{1}{2}}(\Lambda)/p_{\frac{1}{2}}(\Lambda \cap (N_{\frac{1}{2}} \oplus N_1)) = p_{\frac{1}{2}}(\Lambda)/A(p_{\frac{1}{2}}(\Lambda))$$

of length m which is independent of Λ. To choose coordinates for $v_{\frac{1}{2}}$ we use that this module is isomorphic to $W/p^{\lfloor m/2\rfloor}W \oplus W/p^{\lceil m/2\rceil}W$. Under this isomorphism, the element $v_{\frac{1}{2}}$ is mapped to an element of the form

$$\sum_{i=1}^{\lfloor m/2\rfloor}[y_i]p^{i-1} \oplus \sum_{i=\lfloor m/2\rfloor+1}^{m}[y_i]p^{i-\lfloor m/2\rfloor-1}. \tag{4.11}$$

Here we use that k is perfect, and $[y_i]$ is the Teichmüller representative of an element y_i of k.
Note that $a(\Lambda)=1$ (or the condition that $j=1$) implies that $A(v_{\frac{1}{2}})$ is a generator of $\Lambda\cap N_{\frac{1}{2}}$ and not only an arbitrary element. This is an open condition on $p_{\frac{1}{2}}(\Lambda)/p_{\frac{1}{2}}(\Lambda\cap(N_{\frac{1}{2}}\oplus N_1))$. More precisely, it excludes a finite number of hyperplanes (compare [V1], Lemma 4.8).
Let now $v_{\frac{1}{2}}$ also be fixed. It remains to determine the set of possible v_1 such that $\Lambda=\mathcal{D}(v_0+v_{\frac{1}{2}}+v_1)$ is a lattice with $\Lambda^\vee=c\Lambda$. The same arguments as in the previous case show that v_1 can be chosen in an open subset of the set of v_1 with

$$\langle v_0,\phi v_1\rangle+\langle v_1,\phi v_0\rangle\equiv-\langle v_{\frac{1}{2}},\phi v_{\frac{1}{2}}\rangle \pmod{c^{-1}W}. \tag{4.12}$$

for all $\phi\in\mathcal{D}$.

Remark 4.4. Let $\phi\in\mathcal{D}$ with $\tilde{v}(\phi)=2m$. Then $\phi v_{\frac{1}{2}}\in p^m p_{\frac{1}{2}}(\Lambda)\subset c^{-1}\Lambda^\vee$. Especially, $\langle v_{\frac{1}{2}},\phi v_{\frac{1}{2}}\rangle$ is in $c^{-1}W$. This is later used in the form that $a_i=-\langle v_{\frac{1}{2}},F^i v_{\frac{1}{2}}\rangle$ satisfies (6.3).

Analogously to the previous case we use (4.6) to define ξ_v. Then we also obtain the analogue of Lemma 4.2.

Lemma 4.5. *1. Let M be the set of W-linear functions $\xi:\mathcal{D}\to L$ with (4.7) and (4.12) for $i\le n$. Then (4.6) defines a bijection between M and the set of elements $v_1\in N_1$ as above.*

2. Let $\overline{M}$ be the set of functions $\xi:\mathcal{D}\to L/c^{-1}W$ with the same properties as in 1. Then (4.6) defines a bijection between $\overline{M}$ and the set of equivalence classes of elements v_1 as above. Here two such elements are called equivalent if their difference is in $c^{-1}(\Lambda_0)_1^\vee$.

5 The set of irreducible components

Lemma 5.1. *Let $\Lambda\subset N_0\oplus N_1$ be a lattice generated by a sublattice $\Lambda_1\subset N_1$ and an element v with $v=v_0+v_1$ for some $v_0\in N_0$ and $v_1\in N_1$. Let $\tilde{\Lambda}$ be generated by Λ_1 and $v_0+\tilde{v}_1$ for some $\tilde{v}_1\in N_1$. If $\xi_{\tilde{v}_1}(F^i)^{\sigma^{-i}}-\xi_{\tilde{v}_1}(V^i)=\xi_{v_1}(F^i)^{\sigma^{-i}}-\xi_{v_1}(V^i)$ for every $i\in\{1,\dots,h\}$ then there is a $j\in J$ with $j(\Lambda)=\tilde{\Lambda}$.*

Proof. The assumption implies that $\langle v_0+\tilde{v}_1-v_1,\varphi(v_0+\tilde{v}_1-v_1)\rangle=0$ for $\varphi\in\{1,V,\dots,V^h\}$ (see the reformulation of (4.4) in Section 4.1). By (2.1), the same holds for $\varphi \in \{F, \ldots, F^h\}$. As $\dim N = 2h$, the $\varphi(v_0 + \tilde{v}_1 - v_1)$ for these elements $\varphi \in \mathcal{D}$ generate $N' = (\mathcal{D}(v_0 + \tilde{v}_1 - v_1))[1/p] \subseteq N$ as an L-vector space. Especially,

$$\langle v_0 + \tilde{v}_1 - v_1, \varphi(v_0 + \tilde{v}_1 - v_1)\rangle = 0 \tag{5.1}$$

for all $\varphi \in \mathcal{D}$. Let A be a generator of $\mathrm{Ann}(v_0)$ as in Lemma 2.6. Then (5.1) for $\varphi = \varphi' A$ implies that $\langle v_0, \varphi' A(\tilde{v}_1 - v_1)\rangle = 0$ for all $\varphi' \in \mathcal{D}$. Thus $A(\tilde{v}_1 - v_1) = 0$. Let $j \in GL(N)$ be defined by $v_0 \mapsto v_0 + \tilde{v}_1 - v_1$, $j|_{N_1} = \mathrm{id}$, and $j \circ F = F \circ j$. To check that this is well-defined we have to verify that $Aj(v_0) = j(Av_0) = 0$. But $A(j(v_0)) = A(v_0 + \tilde{v}_1 - v_1) = 0$. By definition j commutes with F. Furthermore, (5.1) implies that $j \in G(L)$. Hence $j \in J$. $\square$

For v_1 as above and $i \in \{1, \ldots, h\}$ let

$$\psi_i(v_1) = \xi_{v_1}(V^i) - \xi_{v_1}(F^i)^{\sigma^{-i}}. \tag{5.2}$$

Then the lemma yields the following corollary.

Corollary 5.2. *Let Λ and $\tilde{\Lambda}$ be two extensions of Λ_0 and Λ_1 as described in the previous section (or, in the case of an odd number of supersingular summands, two extensions of Λ_0 and Λ_1 associated to the same $v_{\frac{1}{2}}$) and let $v = v_0 + v_1$ and $\tilde{v} = v_0 + \tilde{v}_1$ (resp. $v = v_0 + v_{\frac{1}{2}} + v_1$ and $\tilde{v} = v_0 + v_{\frac{1}{2}} + \tilde{v}_1$) be the generators. Then $\psi_i(v_1) = \psi_i(\tilde{v}_1)$ for all i implies that Λ and $\tilde{\Lambda}$ are in one J-orbit.*

Let $v_0 \in N_0$ such that $\mathcal{D}v_0$ is a lattice in N_0. Then the next proposition implies that for each $(c_1, \ldots, c_h) \in L^h$ there is a $v_1 \in N_1$ with $\psi_i(v_1) = c_i$ for all i.
We fix an irreducible component of $\mathcal{S}^{\mathrm{np}}_{\mathbb{X}_0,1}$. Then [V1], 4 describes a morphism from a complement of hyperplanes in an affine space to this irreducible component that is a bijection on k-valued points. Let $\mathrm{Spec}(R_0)$ be this open subscheme of the affine space. One first defines a suitable element $v_{0,R_0} \in N_0 \otimes_W W(R_0)$. The morphism is then constructed in such a way that each k-valued point x of $\mathrm{Spec}(R_0)$ is mapped to the lattice in N_0 generated by the reduction of $\sigma^m(v_{0,R_0})$ at x.

Proposition 5.3. *Let R be a reduced k-algebra containing $\sigma^m(R_0)$. Let $c_1, \ldots, c_h \in W(R)[1/p]$. Then there is a morphism $R \to R'$ where R' is a limit of étale extensions of R and a $v_1 \in N_{1,R'}$ with $\psi_i(v_1) = c_i$ for all i. Here, the ψ_i are defined with respect to the universal element $\sigma^m(v_{0,R_0}) \in (N_0)_{\sigma^m(R_0)}$.*

For the proof we need the following lemma to simplify the occurring system of equations.

Lemma 5.4. *Let R be an $\mathbb{F}_p$-algebra and let $m, n \in \mathbb{N}$ with $m \leq n$. For $0 \leq i \leq m$ and $0 \leq j \leq n$ let $P_{ij}(x) \in (W(R)[\frac{1}{p}])[x]$ be a linear combination of* *the* $\sigma^l(x) = x^{p^l}$ *with* $l \geq 0$*. Assume that the coefficient of* x *is zero for* $j < i$ *and in* $W(R)^\times$ *for* $i = j$*. Consider the system of equations*

$$\sum_{j=0}^{n} P_{ij}(x_j) = a_i$$

with $a_i \in W(R)[\frac{1}{p}]$ *and* $i = 0, \dots, m$*. It is equivalent to a system of equations of the form* $\sum_j Q_{ij}(x_j) = b_i$ *with* $b_i \in W(R)[\frac{1}{p}]$ *such that the* Q_{ij} *satisfy the same conditions as the* P_{ij} *and in addition* $Q_{ij} = 0$ *if* $j < i$*.*

Proof. We use a modification of the Gauss algorithm to show by induction on λ that the system is equivalent to a system of relations of the form $\sum_j Q^\lambda_{ij}(x_j) = b^\lambda_i$ with $b^\lambda_i \in L$ such that the Q^λ_{ij} satisfy the same conditions as the P_{ij} and in addition $Q^\lambda_{ij} = 0$ if $j < i$ and $j \leq \lambda$. For the induction step we have to carry out the following set of modifications for $j = \lambda + 1$ and each $i > \lambda + 1$. If Q^λ_{ij} vanishes, we do not make any modification. We now assume Q^λ_{ij} to be nontrivial. Let $\sigma^{l_i}(x)$ and $\sigma^{l_j}(x)$ be the highest powers of x occurring in Q^λ_{ii} and Q^λ_{ij}. If $l_i < l_j$, we modify the jth equation by a suitable multiple of $\sigma^{l_j - l_i}$ applied to the ith equation to lower l_j. Else we modify the ith equation by a suitable multiple of $\sigma^{l_i - l_j}$ applied to the jth equation to lower l_i. We proceed in this way as long as none of the two polynomials Q^λ_{ii} and Q^λ_{ij} becomes trivial. Note that the defining properties of the P_{ij} are preserved by these modifications. As (by induction) Q^λ_{ij} does not have a linear term, the linear term of Q^λ_{ii} remains unchanged. Thus this process of modifications ends after a finite number of steps with equations $\sum_j Q^{\lambda+1}_{ij}(x_j) = b^{\lambda+1}_i$ which satisfy $Q^{\lambda+1}_{ij} = 0$ for $j < i$ and $j \leq \lambda + 1$. For $\lambda + 1 = n$, this is what we wanted. □

Proof of Proposition 5.3. An element $v_1 \in N_{1,R'}$ is determined by the values of ξ_{v_1} at any h consecutive elements of $\dots, F^2, F, 1, V, V^2, \dots$. The other values of ξ are then determined by $\xi_{v_1}(\phi A) = 0$ for all $\phi \in \mathcal{D}$. Here $A \in \mathrm{Ann}(v_0)$ is as in Lemma 2.6. Indeed, each of these equations for $\phi = F^i$ or V^i for some i gives a linear equation with coefficients in L between the values of ξ_{v_1} at $h+1$ consecutive elements of $\dots, F^2, F, 1, V, V^2, \dots$. For the proof of the proposition we take the values $\xi_{v_1}(F^i)$ for $i \in \{1, \dots, h\}$ as values determining v_1. Then all other values are linear combinations of these $\xi_{v_1}(F^i)$.
The definition of ψ_{v_1} in (5.2) yields

$$\xi_{v_1}(V^i)^{\sigma^i} = \xi_{v_1}(F^i) + \psi_i(v_1)^{\sigma^i}$$

for $i \in \{1, \dots, h\}$. On the other hand, $\xi_{v_1}(V^i)^{\sigma^i}$ is a linear combination of the $\xi_{v_1}(F^j)^{\sigma^i}$ for $j \in \{1, \dots, h\}$. From this we obtain a system of h equations for the $\xi_{v_1}(F^i)$ with $1 \leq i \leq h$ of the same form as in Lemma 5.4. The resulting equations $\sum_j Q_{ij}(\xi_{v_1}(F^j)) = b_i$ may be reformulated as $Q_{ii}(\xi_{v_1}(F^i)) = c_i$ where c_i also contains the summands corresponding to powers of F larger than i. We can then consider these equations by decreasing induction on i. For each i, the polynomial $Q_{ii}(x)$ is a linear combination of powers of x of the form x^{σ^l}, and its linear term does not vanish. Thus there is a limit R' of étale extensions of R and $\xi_{v_1}(F^i) \in W(R') \otimes \mathbb{Q}$ with $v_p(\xi_{v_1}(F^i)) \geq v_p(c_i)$ satisfying these equations. Note that R' is in general an infinite extension of R, because the equations are between elements of $W(R) \otimes \mathbb{Q}$ and not over R itself. Given ξ_{v_1}, Remark 3.3 shows that there is an element $v_1 \in (N_1)_{R'}$ which induces ξ_{v_1}. Indeed, choose v_1 to be a suitable linear combination of the dual basis defined there. $\square$

5.1 Proof of Theorem 2

We begin by constructing an irreducible subscheme of the subscheme of $\mathcal{S}_1$ where the height of the universal quasi-isogeny is 0. The k-valued points of this subscheme correspond to lattices Λ with $a(\Lambda) = 1$ and $\Lambda^\vee = \Lambda$. There is a $d \in \mathbb{N}$ such that for each $(c_1, \dots, c_h) \in (p^d W)^h$, the v_1 constructed in Proposition 5.3 lies in the lattice $\Lambda_1 \subset N_1$. Let R_0 as above. In the case of an even number of supersingular summands let $R_1 = \sigma^m(R_0)$. Otherwise let $\sigma^m(v_{0,R_0}) + v_{\frac{1}{2}} \in N_{\sigma^m(R_0)[y_1,\dots,y_m]}$ where v_{0,R_0} is as above and where $v_{\frac{1}{2}} \in (N_{\frac{1}{2}})_{\sigma^m(R_0)[y_1,\dots,y_m]}$ is identified with the element in (4.11). The open condition on $\operatorname{Spec}(\sigma^m(R_0)[y_1,\dots,y_m])$ that $Av_{\frac{1}{2}}$ is a generator and not only an element of $p_{\frac{1}{2}}(\Lambda \cap N_{\frac{1}{2}} \oplus N_1)$ is equivalent to $\mathcal{D}Av_{\frac{1}{2}} = A\mathcal{D}v_{\frac{1}{2}}$. This condition is satisfied by all y_1 that do not lie in some finite-dimensional $\mathbb{Q}_p$-subvector space of k determined by the kernel of A (compare the proof of Proposition 4.3 1.). In this case let R_1 be the extension of $\sigma^m(R_0)$ corresponding to this affine open subscheme. Let in both cases

$$R = R_1[x_{i,j} \mid i \in \{1, \dots, h\}, j \in \{0, \dots, d-1\}].$$

For $i \in \{1, \dots, h\}$ let $c_i = \sum_{j=0}^{d-1} [x_{i,j}] p^j \in W(R)$. Let $\operatorname{Spec}(R')$ and $v_1 \in N_{R'}$ be as in Proposition 5.3. Let $v = \sigma^m(v_{0,R_0}) + v_1$, resp. $v = \sigma^m(v_{0,R_0}) + v_{\frac{1}{2}} + v_1$. Let $S = \operatorname{Spec}(R)$ be an irreducible component of the affine open subscheme of $\operatorname{Spec}(R')$ consisting of the points x with $v_{1,x} \in (\mathcal{D}v_x)_1^\vee \setminus (F(\mathcal{D}v_x)_1^\vee + V(\mathcal{D}v_x)_1^\vee)$. We denote the image of v in N_R also by v. A s we already know that $\mathcal{S}_1$ is dense, this open subset is nonempty. Let $\tilde{R}$ be the inverse image of R under σ^h as in Proposition 3.2. Note that $v_p(\det F) = h$, whereas $v_p(\det F|_{N_0}) = m$. Let $\tilde{S} = \operatorname{Spec}(\tilde{R})$. The next step is to define an associated morphism $\varphi : \tilde{S} \to \mathcal{M}_{\text{red}}$ such that in each k-valued point x of $\tilde{S}$, the image in $\mathcal{M}_{\text{red}}(k)$ corresponds to the lattice generated by the reduction v_x of v at x. By Proposition 3.2 it is enough to show that there is a $j \in J$ such that for each $x \in \tilde{S}(k)$, we have $v_x \in j\Lambda_{\min}$ and $v_p(\det j) = \max\{v_p(\det j') \mid v_x \in j'\Lambda_{\min}\}$. Let η be the generic point of $\tilde{S}$ and let $j_\eta \in J$ be such a maximizing element for η. Then the same holds for each k-valued point in an open and thus dense subscheme of $\tilde{S}$. As the property $v_x \in j_\eta\Lambda_{\min}$ is closed, it is true for each $x \in \tilde{S}(k)$. In [V1], 4 it is shown that for lattices $\Lambda \subset N$ with $a(\Lambda) = 1$, the difference $\operatorname{vol}(\Lambda) - \max\{v_p(\det j') \mid \Lambda \subset eqj'\Lambda_{\min}\}$ is a constant only depending on N. In our case, the duality condition shows that $\mathrm{vol}(\mathcal{D}v_x)$ is constant on $\tilde{S}$ and only depending on c and N. Thus the maximum is also constant. Hence in every k-valued point, $v_p(\det j_\eta)$ is equal to this maximum, which is what we wanted for the existence of $\varphi : \tilde{S} \to \mathcal{M}_{\mathrm{red}}$. We obtain an irreducible subscheme $\varphi(\tilde{S})$ of $\mathcal{S}_1 \subseteq \mathcal{M}_{\mathrm{red}}$.

To show that J acts transitively on the set of irreducible components we have to show that for each $x \in \mathcal{S}_1(k)$ there is an element $j \in J$ such that jx lies in the image of φ. Let $\Lambda \subset N$ be the lattice corresponding to x. The first step is to show that there is a $j \in J$ such that $j(\Lambda)$ is selfdual (and not only up to a scalar $c(\Lambda)$). It is enough to show that there is a $j \in J$ such that $v_p(c(\Lambda)) = v_p(c(j\Lambda)) + 1$. Such an element is given by taking the identity on N_1, multiplication by p on N_0, and the map $e_i^{\frac{1}{2}} \mapsto e_{i+1}^{\frac{1}{2}}$ on $N_{\frac{1}{2}}$. Here we use the notation of Remark 2.4 for the basis of N. Next we want to apply an element of J modifying Λ_0. We have $a(p_0(\Lambda)) = a(\Lambda \cap N_1) = 1$. From the classification of lattices with $a = 1$ in [V1], 4 we obtain that $J_{\mathbb{X}_0}^{\mathrm{np}}$ (which may be considered as a subgroup of J by mapping $j \in J_{\mathbb{X}_0}^{\mathrm{np}}$ to the map consisting of j and its dual on N_1) is acting transitively on the set of irreducible components of $\mathcal{M}_{\mathbb{X}_0,\mathrm{red}}^{\mathrm{np}}$. Thus by possibly multiplying with such an element we assume that Λ_0 lies in the fixed irreducible component chosen for Proposition 5.3. Recall from Section 4.2 that in the case of an odd number of supersingular summands, there is a $j \in J$ mapping the element $v_{\frac{1}{2}}$ to the irreducible family described there. It remains to study the possible extensions of the lattices Λ_0 and Λ_1 (or in the second case of the sublattice of $N_0 \oplus N_{\frac{1}{2}}$ determined by Λ_0 and $v_{\frac{1}{2}}$ and of Λ_1). They are given by the associated elements v_1. Fix a generating element $\sigma^m(v_0)$ of Λ_0 (in the second case also an element $v_{\frac{1}{2}}$) and let v_1 be an element associated to the e xtension Λ with $a(\Lambda) = 1$. Then Lemma 5.1 and the construction of S show that there is an element of J mapping Λ to a lattice associated to a point of S inducing the same ψ_i as Λ. Thus the image of $\varphi(\tilde{S})$ under J is $\mathcal{S}_1$, which proves the theorem. ☐

6 Dimension

We use the same notation as before, namely Λ is the lattice corresponding to a point of $\mathcal{S}_1$, generated by an element $v = v_0 + v_{\frac{1}{2}} + v_1$ with $v_i \in N_i$. Again, A is a generator of $\mathrm{Ann}(v_0)$ and $\Lambda_0 = p_0(\Lambda)$ and $\Lambda_1 = \Lambda \cap N_1$.

To determine the dimension of $\mathcal{S}_1$ and of $\mathcal{M}_{\mathrm{red}}$ we have to classify the elements v_1 of Section 4 up to elements in $c^{-1}\Lambda_1$ and not up to the (locally finite) action of J which we used in Section 5. To do so, it is more useful to use the values of ξ_{v_1} as coordinates instead of the values of ψ_{v_1}.

We investigate the set of possible values $\xi(\phi) \in L/c^{-1}W$ for $\phi \in \mathcal{D}$ using decreasing induction on $\tilde{v}(\phi) \geq 0$. Here, $\tilde{v}$ is as in (2.6). Recall from Lemma 4.2 2. that the use of functions ξ with values in $L/c^{-1}W$ instead of L corresponds to considering v_1 as an element of N_1/Λ_1. But as v_1 and $v_1 + \delta$ with $\delta \in \Lambda_1$ lead to the same lattice Λ, this is sufficient to determine the set of possible extensions of Λ_0 and Λ_1.

Instead of equations (4.7) and (4.8) we consider the following slightly more general problem to treat at the same time the case of an odd number of supersingular summands. There, (4.8) is replaced by (4.12). We want to consider W-linear functions $\xi : \mathcal{D} \to L/c^{-1}W$ with

$$\xi(F^i) - \xi(V^i)^{\sigma^i} \equiv a_i \pmod{c^{-1}W} \tag{6.1}$$

$$\xi(\psi A) \equiv 0 \pmod{c^{-1}W} \tag{6.2}$$

for all $\psi \in \mathcal{D}$. Here $a_i \in L$ are given elements satisfying

$$a_i p^{j_i} \in c^{-1}W \quad \text{if} \quad 2j_i + i \geq 2m. \tag{6.3}$$

Let $\mathcal{D}^i = \{\phi \in \mathcal{D} \mid \tilde{v}(\phi) \geq i\}$. We call a W-linear function

$$\xi^{i_0} : \mathcal{D}^{i_0} \to L/(c^{-1}W)$$

satisfying (6.1) and (6.2) a partial solution of level i_0. Then the induction step consists in determining the possible partial solutions ξ^{i_0} of level i_0 leading to a fixed solution of level $i_0 + 1$. Note that the assumption on a_i implies that there exists the trivial partial solution $\xi^{2m} \equiv 0$ of level $2m$ inducing partial solutions of all higher levels. Recall that we assumed F and V to be elementwise topologically nilpotent on N. Thus for each function ξ with (6.1) and (6.2) there is a level i such that ξ induces the trivial partial solution of level i.

Assume that we already know the $\xi(\phi)$ for $\tilde{v}(\phi) > i_0$ and want to determine its possible values for $\tilde{v}(\phi) = i_0$. Then we know in particular $\xi(p\phi) = p\xi(\phi) \in L/c^{-1}W$, or $\xi(\phi) \in L/p^{-1}c^{-1}W$. We want to determine the possible liftings modulo $c^{-1}W$.

A basis of the k-vector space $\mathcal{D}^{i_0}/\mathcal{D}^{i_0+1}$ is given by the $i_0 + 1$ monomials

$$F^{i_0}, VF^{i_0-1} = pF^{i_0-2}, \ldots, V^{i_0-1}F = pV^{i_0-2}, V^{i_0}.$$

Equation (6.1) leads to $\lfloor i_0/2 \rfloor$ relations between the values of ξ on these monomials. Recall that $\tilde{v}(A) = m$. Thus if $\tilde{v}(\phi) = i_0 - m$ for some $\phi \in \mathcal{D}$, (6.2) leads to a relation between the value of ξ on $\mathrm{LT}(\phi A) \in \mathcal{D}^{i_0}$ and values on $\mathcal{D}^{i_0+1}$. As the ξ are linear, it is sufficient to consider the $\max\{0, i_0 - m + 1\}$ relations for $\phi \in \{F^{i_0-m}, pF^{i_0-m-2}, \ldots, V^{i_0-m}\} \cap \mathcal{D}^{i_0-m}$. This count of relations leads to the notation

$$r(i_0) = \lfloor i_0/2 \rfloor + \max\{0, i_0 - m + 1\}.$$

Then $i_0 + 1 \leq r(i_0)$ is equivalent to $i_0 \geq 2m$.

The following proposition is the main tool to prove Theorem 3 on the dimension of the moduli spaces.

Proposition 6.1. *1. Let $i_0 \geq 2m$. Then there is a partial solution ξ^{i_0} of (6.1) and (6.2) of level i_0. If we fix ξ^{i_0} and an $l \in \mathbb{N}$ with $l \geq i_0$, there are only finitely many other partial solutions $\tilde{\xi}^{i_0}$ of level i_0 such that $\xi^{i_0} - \tilde{\xi}^{i_0}$ induces the trivial partial solution of level l of the associated homogenous system of equations.*

2. *Let $i_0 + 1 > r(i_0)$ and let ξ^{i_0+1} be a partial solution of (6.1) and (6.2) of level $i_0 + 1$. Then to obtain a partial solution ξ^{i_0} of level i_0 inducing ξ^{i_0+1}, one may choose the lifts to $L/c^{-1}W$ of the values of ξ^{i_0} at the first $i_0+1-r(i_0)$ monomials $p^\alpha V^\beta$ with $2\alpha+\beta = i_0$ and $\beta \leq 2(i_0-r(i_0))+1$ arbitrarily. Each of the remaining values lies in some finite nonempty set depending polynomially on the values on the previous monomials.*

Proof. Note that the existence statement in the first assertion is satisfied as the condition on the a_i yields that there is the trivial solution of level $2m$. We show the two assertions simultaneously. Let ξ^{i_0+1} be a fixed partial solution of level $i_0 + 1$ for any i_0. It is enough to show that for a lift ξ^{i_0}, the values of the first $\max\{0, i_0 + 1 - r(i_0)\}$ variables can be chosen arbitrarily, and that the remaining values then lie in some finite set depending polynomially on the values on the previous variables. If $i_0 + 1 > r(i_0)$, we have to show that this finite set is nonempty. We investigate the relations (6.1) and (6.2) more closely. The first set of relations shows that $\xi^{i_0}(p^aF^b)$ with $2a + b = i_0$ is determined by $\xi^{i_0}(p^aV^b)$. Thus it is sufficient to consider this latter set of values. Besides, we have to consider (6.2) for $\psi \in \{V^{i_0-m}, pV^{i_0-m-2}, \dots, F^{i_0-m}\}$. For $B \in \mathcal{D}$ let $\mathrm{LT}(B)$ as in Lemma 2.6. Then the equations for the values of ξ^{i_0} relate $\xi^{i_0}(\mathrm{LT}(\psi A))$ to something which is known by the induction hypothesis. Let us recall the description of $\mathrm{LT}(A)$ from Lemma 2.6 3. Let h' be the number of supersingular summands of N_0. Let $j \geq 0$ with $i_0 - m - j \geq 0$. Then $\mathrm{LT}(V^{i_0-m-j}F^jA)$ is a linear combination of $V^{i_0-j}F^j, \dots, V^{i_0-j-h'}F^{j+h'}$ whose coefficients are Teichmüller representatives of elements of k. Furthermore, the coefficients of $\xi^{i_0}(V^{i_0-j}F^j)$ and $\xi^{i_0}(V^{i_0-j-h'}F^{j+h'})$ are units in W. Using (6.1) we may replace values of ξ^{i_0} at monomials in F by σ-powers of the values of the corresponding monomials in V. We thus obtain a relation between a polynomial in the remaining $\lceil (i_0+1)/2 \rceil$ values of ξ^{i_0} and an expression which is known by induction. For $2j \leq i_0$, the first summand $\xi^{i_0}(V^{i_0-j}F^j)$ remains the variable associated to the highest power of V which occurs linearly in this polynomial. In the following we ignore all equations for $2j > i_0$. They only occur for $i_0 > 2m$, a case where we only want to prove the finiteness of the set of solutions. The system of equations with $2j \leq i_0$ is of the form considered in Lemma 5.4. The proof of this Lemma for coefficients in $L/c^{-1}W$ is the same as for coefficients in L. Thus we obtain that the lifts of the values at the $i_0+1-r(i_0)$ variables associated to the largest values of j can be chosen freely and the other ones have to satisfy some relation of the form $Q_{ii}(x) \equiv b_i$ for some given b_i. As the Q_{ii} have a linear term they are nontrivial. This implies that the set of solutions of these equations is nonempty and finite and dep ends polynomially on the previous values. □

6.1 Proof of Theorem 3

By Lemma 3.1 it is enough to show that $\mathcal{S}_1$ is equidimensional of the claimed dimension. From [V1], 4 we obtain that the connected components of $\mathcal{S}^{\mathrm{np}}_{\mathbb{X},1}$ are irreducible. The discrete invariant with values in $J^{\mathrm{np}}/(J^{\mathrm{np}} \cap \mathrm{Stab}(\Lambda_{\min}))$

distinguishing the components is given by $\Lambda \mapsto j_\Lambda$ with $\Lambda \subseteq j_\Lambda \Lambda_{\min}$ and

$$v_p(\det j_\Lambda) = \max\{v_p(\det j) \mid j \in J^{\mathrm{np}}, \Lambda \subseteq j\Lambda_{\min}\}.$$

Especially, j_Λ is constant on each connected component of $\mathcal{S}_1 \subseteq \mathcal{S}^{\mathrm{np}}_{\mathbb{X},1}$. Besides, $p_0(j_\Lambda \Lambda_{\min})$ determines the connected component of $p_0(\Lambda) = \Lambda_0$ inside $\mathcal{S}_{\mathbb{X}_0,1}$. Thus we may fix an irreducible component of $\mathcal{S}_{\mathbb{X}_0,1}$ and determine the dimension of the union of connected components of $\mathcal{S}_1$ such that Λ_0 is in this fixed component. Let R_0 and R_1, v_0 and $v_{\frac{1}{2}}$ be as in the proof of Theorem 2. Again we use the functions ξ defined with respect to $\sigma^m(v_0)$ instead of v_0. Fix an arbitrary partial solution ξ^{2m} of (6.1) and (6.2) of level $2m$. Let

$$R_2 = R_1[x_{i\beta} \mid i \geq 0, 1 \leq \beta \leq i+1-r(i)].$$

We use decreasing induction on i to lift ξ^{2m} to a partial solution of level i over an étale extension R_2^i of R_2. Let $R_2^{2m} = R_2$. Assume that a lift ξ^{i+1} is given. Then Proposition 6.1 shows that the values at $i+1-r(i)$ monomials with $\tilde{v} = i$ may be lifted arbitrarily to a value of ξ^i. If $p^\alpha V^\beta$ with $2\alpha+\beta = i$ and $\beta \leq i+1-r(i)$ is such a monomial we write (using the induction hypothesis) $\xi^{i+1}(p^{\alpha+1}V^\beta) = \sum_{i<v_p(c^{-1})}[a_i]p^i$ with $a_i \in R_2^{i+1}$. Then we choose

$$\xi^i(p^\alpha V^\beta) = \sum_{i<v_p(c^{-1})} [a_i]p^{i-1} + [x_{i\beta}]p^{v_p(c^{-1})-1}.$$

Let R_2^i be the extension of R_2^{i+1} given by adjoining further variables $x_{i\beta}$ for larger β parametrizing the other values of the lift of ξ^{i+1} to ξ^i and with relations as in Proposition 6.1, 2. and its proof. More precisely, R_2^i is obtained from R_2^{i+1} by a finite number of extensions given by equations of the form $Q_{jj}(x) \equiv b_j \pmod{c^{-1}W}$ where $Q_{jj}(x)$ is a polynomial that is a finite linear combination of the monomials x^{p^l} with $l \geq 0$ such that the coefficient of x is in $W(R_2^{i+1})^\times$. This implies that R_2^i is a finite étale extension of R_2^{i+1}. Let $R_3 = R_2^0$. Let $v_{1,R_3} \in N_{1,R_3}$ be such that $\xi_{v_{1,R_3}} = \xi^0$. Its existence follows again from the existence of the dual basis in Remark 3.3. Let $v = \sigma^m(v_{0,R_0}) + v_{1,R_3}$, or $v = \sigma^m(v_{0,R_0}) + v_{\frac{1}{2}} + v_{1,R_3}$. As in the proof of Theorem 2 let $S = \S pec(R)$ be an irreducible component of the affine open subscheme of $\mathrm{Spec}(R_3)$ over which $\mathcal{D}v$ contains $(\mathcal{D}v)_1^\vee$. As we want to compute the dimension of $\mathcal{S}_1$, we only have to consider these subschemes. Let $\tilde{R} = \sigma^{-m}(R)$ as in Proposition 3.2. The same argument as in the proof of Theorem 2 shows that there is a morphism $\varphi : \mathrm{Spec}(\tilde{R}) \to \mathcal{M}_{\mathrm{red}}$ mapping $x \in \mathrm{Spec}(\tilde{R})(k)$ to the lattice generated by v_x. The finiteness statements in Proposition 6.1 imply that for each given $y \in \mathcal{S}_1$ (and thus given ξ) there is an open neighborhood in $\mathcal{S}_1$ which only contains points of $\varphi(\mathrm{Spec}(\tilde{R}))$ for a finite number of choices of an irreducible component of $\mathcal{S}_{\mathbb{X}_0,1}$ and a corresponding component S. Besides, the construction of R_3 together with the description of the k-valued points of $\mathcal{S}_1$ shows that for each $y \in \mathcal{S}_1(k)$ there is exactly one irreducible component of $\mathcal{S}_{\mathbb{X}_0,1}$, one corresponding component S, and one point $x \in \mathrm{Spec}(\tilde{R})(k)$ such that $\varphi(x) = y$. Thus $\dim \mathcal{M}_{\mathrm{red}} = \dim \mathcal{S}_1$ is the maximum of $\dim \operatorname{Spec}(\tilde{R})$ for all irreducible components S. It remains to show that this is equal to the right hand side of (1.2). Note that R_i is equidimensional for $i = 0,1,2,3$. From the construction of S we see that in case of an even number of supersingular summands,

$$\begin{aligned}\dim \operatorname{Spec}(\tilde{R}) &= \dim S = \dim R_3 = \dim R_2 \\ &= \dim \mathcal{M}^{\mathrm{np}}_{\mathbb{X}_0,\mathrm{red}} + \sum_{i\geq 0} \max\{0, i+1-r(i)\}. \end{aligned} \tag{6.4}$$

In the other case,

$$\begin{aligned}\dim \operatorname{Spec}(\tilde{R}) &= \dim S = \dim R_3 = \dim R_2 \\ &= \dim \mathcal{M}^{\mathrm{np}}_{\mathbb{X}_0,\mathrm{red}} + \sum_{i\geq 0} \max\{0, i+1-r(i)\} + m. \end{aligned} \tag{6.5}$$

The last summand corresponds to the choice of $v_{\frac{1}{2}}$.

From the decomposition of N into l simple summands N^j we obtain a decomposition $N_0 = \bigoplus_{j=1}^{l_0} N^j$ with $l_0 = \lfloor l/2 \rfloor$. Let again $\lambda_j = m_j/(m_j+n_j)$ be the slope of N^j. Recall from [V1], Theorem B that

$$\dim \mathcal{M}^{\mathrm{np}}_{\mathbb{X}_0,\mathrm{red}} = \sum_{j=1}^{l_0} \frac{(m_j-1)(n_j-1)}{2} + \sum_{\{j,j' \mid j<j'\leq l_0\}} m_j n_{j'}.$$

We denote the right hand side of (1.2) by D. Let us first consider the case of an even number of supersingular summands. Then by the symmetry of the Newton polygon we obtain

$$D - \dim \mathcal{M}^{\mathrm{np}}_{\mathbb{X}_0,\mathrm{red}} = \frac{1}{2} \sum_{j<j'\leq l} m_j n_{j'} + \frac{m}{2} - \sum_{j<j'\leq l_0} m_j n_{j'}.$$

Again by the symmetry of the Newton polygon this is equal to

$$\begin{aligned} &= \sum_{j=1}^{l_0} \sum_{j'=l_0+1}^{l} \frac{m_j n_{j'}}{2} + \frac{m}{2} \\ &= \sum_{j,j'=1}^{l_0} \frac{m_j m_{j'}}{2} + \frac{m}{2} \\ &= \frac{m(m+1)}{2}. \end{aligned}$$

In the other case, the same calculation shows that

$$\begin{aligned} D - \dim \mathcal{M}^{\mathrm{np}}_{\mathbb{X}_0,\mathrm{red}} &= \frac{m(m+1)}{2} + 2\sum_{j=1}^{l_0} \frac{m_j n_{l_0+1}}{2} \\ &= \frac{m(m+1)}{2} + m. \end{aligned}$$

In the last step we used that N^{l_0+1} is supersingular, hence $n_{l_0+1} = 1$.
On the other hand (6.4) implies that in the case of an even number of supersingular summands,

$$\begin{aligned}
\dim \mathcal{M}_{\mathrm{red}} - \dim \mathcal{M}^{\mathrm{np}}_{\mathbb{X}_0,\mathrm{red}} &= \sum_{i\geq 0} \max\{0, i+1-r(i)\} \\
&= \sum_{i=0}^{m-1} \left(\left\lfloor \frac{i}{2} \right\rfloor + 1\right) + \sum_{i=m}^{2m-1} \left(\left\lfloor \frac{i}{2} \right\rfloor - i + m\right) \\
&= m + \sum_{i=0}^{2m-1} \left\lfloor \frac{i}{2} \right\rfloor - \sum_{i=0}^{m-1} i \\
&= \frac{m(m+1)}{2}.
\end{aligned}$$

The same calculation with (6.5) shows that for an odd number of supersingular summands

$$\dim \mathcal{M}_{\mathrm{red}} - \dim \mathcal{M}^{\mathrm{np}}_{\mathbb{X}_0,\mathrm{red}} = \frac{m(m+1)}{2} + m.$$

Together with the calculation of $D - \dim \mathcal{M}^{\mathrm{np}}_{\mathbb{X}_0,\mathrm{red}}$, this implies Theorem 3. $\square$

7 Connected components

In this section we determine the set of connected components of $\mathcal{M}_{\mathrm{red}}$. The reduction to the bi-infinitesimal case in Section 2.3 shows that Theorem 1 follows from the next theorem.

Theorem 7.1. *Let $\mathbb{X}$ be bi-infinitesimal and non-trivial. Then*

$$\begin{aligned}
\kappa : \mathcal{M}_{\mathrm{red}}(k) &\to \mathbb{Z} \\
\Lambda &\mapsto v_p(c(\Lambda)),
\end{aligned}$$

where $\Lambda^\vee = c(\Lambda)\cdot\Lambda$, induces a bijection

$$\pi_0(\mathcal{M}_{\mathrm{red}}) \cong \mathbb{Z}.$$

Proof. From Theorem 2 we obtain a J-equivariant surjection $\pi : J \twoheadrightarrow \pi_0(\mathcal{M}_{\mathrm{red}})$. Besides, the map κ induces a surjection $\pi_0(\mathcal{M}_{\mathrm{red}}) \to \mathbb{Z}$. We choose the base point of $\mathcal{M}$ to be a minimal p-divisible group. Let $\Lambda_{\min}$ be the corresponding lattice in N. An element $j\Lambda_{\min}$ with $j \in J$ is in the kernel of κ if and only if $(j\Lambda_{\min})^\vee = j\Lambda_{\min}$. This is equivalent to $j\Lambda_{\min} = j'\Lambda_{\min}$ for some $j' \in J \cap Sp_{2h}(L)$. Thus we have to show that $J \cap Sp_{2h}(L)$ is mapped to a single connected component of $\mathcal{M}_{\mathrm{red}}$. Our choice of the base point implies that $\mathrm{Stab}(\Lambda_{\min}) = K$. Thus the surjection π maps $J \cap K$ to the component of the identity. Note that $J \cap Sp_{2h}(L) = J_{\mathrm{der}}(\mathbb{Q}_p)$ where J_{der} is the derived group of J. Hence the elements of $(J \cap Sp_{2h}(L))/(J \cap K)$ correspond to vertices in the building of J_{der}. The building of J_{der} is connected. To show that all vertices correspond to points in one connected component of $\mathcal{M}_{\mathrm{red}}$, it is thus enough to show that if Λ_0, Λ_1 are the lattices corresponding to two such vertices such that $\Lambda_0 \cap \Lambda_1 = \Lambda$ is of colength 1 in Λ_0 and Λ_1, then the two lattices correspond to points in the same connected component of $\mathcal{M}_{\mathrm{red}}$. As a W-module Λ_0 is generated by Λ and v_0 for some $v_0 \in N$. As the slopes of F are in $(0,1)$ we have $Fv_0, Vv_0 \in \Lambda$. Similarly Λ_1 is generated by Λ and some v_1 with $Fv_1, Vv_1 \in \Lambda$. For $a \in \mathbb{A}^1(k)$ let $\Lambda_a = \langle \Lambda, v_0 + a(v_1 - v_0)\rangle$. As Λ_0 and Λ_1 are selfdual one easily sees that Λ_a is selfdual for each a. By [V1], Lemma 3.4 there is a morphism $\mathbb{A}^1 \to \mathcal{M}^{\mathrm{np}}_{\mathrm{red}}(\mathbb{X})$ mapping each point a as above to the point of $\mathcal{M}^{\mathrm{np}}_{\mathrm{red}}(\mathbb{X})$ corresponding to Λ_a. As all Λ_a are selfdual, this induces a corresponding morphism $f : \mathbb{A}^1 \to \mathcal{M}_{\mathrm{red}}$. Hence $f(0)$ and $f(1)$, the points corresponding to Λ_0 and Λ_1, are in the same connected component of $\mathcal{M}_{\mathrm{red}}$. □

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