Bloch and Kato's Exponential Map: Three Explicit Formulas

To Kazuya Kato on the occasion of his fiftieth birthday

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Abstract. The purpose of this article is to give formulas for Bloch-Kato's exponential map and its dual for an absolutely crystalline p-adic representation V, in terms of the (φ, Γ)-module associated to V. As a corollary of these computations, we can give a very simple and slightly improved description of Perrin-Riou's exponential map, which interpolates Bloch-Kato's exponentials for the twists of V. This new description directly implies Perrin-Riou's reciprocity formula.

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Introduction

In his article [Ka93] on L-functions and rings of p-adic periods, K. Kato wrote:

> *I believe that there exist explicit reciprocity laws for all p-adic representations of* $\mathrm{Gal}(\overline{K}/K)$*, though I can not formulate them. For a de Rham representation* V*, this law should be some explicit description of the relationship between* $\mathbf{D}_{\mathrm{dR}}(V)$ *and the Galois cohomology of* V*, or more precisely, some explicit descriptions of the maps* exp *and* exp* *of* V*.*

In this paper, we explain how results of Benois, Cherbonnier-Colmez, Colmez, Fontaine, Kato, Kato-Kurihara-Tsuji, Perrin-Riou, Wach and the author give such an explicit description when V is a crystalline representation of an unramified field.

Let p be a prime number, and let V be a p-adic representation of $G_K = \mathrm{Gal}(\overline{K}/K)$ where K is a finite extension of $\mathbf{Q}_p$. Such objects arise (for example) as the étale cohomology of algebraic varieties, hence their interest in arithmetic algebraic geometry.

Let $\mathbf{B}_{\mathrm{cris}}$ and $\mathbf{B}_{\mathrm{dR}}$ be the rings of periods of Fontaine, and let $\mathbf{D}_{\mathrm{cris}}(V)$ and $\mathbf{D}_{\mathrm{dR}}(V)$ be the invariants attached to V by Fontaine's construction. Bloch and Kato have defined in [BK91, §3], for a de Rham representation V, an "exponential" map,

$$\exp_{K,V} : \mathbf{D}_{\mathrm{dR}}(V)/\operatorname{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V) \to H^1(K,V).$$

It is obtained by tensoring the so-called fundamental exact sequence:

$$0 \to \mathbf{Q}_p \to \mathbf{B}_{\mathrm{cris}}^{\varphi=1} \to \mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^+ \to 0$$

with V and taking the invariants under the action of G_K. The exponential map is then the connecting homomorphism $\mathbf{D}_{\mathrm{dR}}(V)/\operatorname{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V) \to H^1(K,V)$.

The reason for their terminology is the following (cf. [BK91, 3.10.1]): if G is a formal Lie group of finite height over $\mathcal{O}_K$, and $V = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T$ where T is the p-adic Tate module of G, then V is a de Rham representation and $\mathbf{D}_{\mathrm{dR}}(V)/\operatorname{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V)$ is identified with the tangent space $\tan(G(K))$ of $G(K)$. In this case, we have a commutative diagram:

$$\begin{array}{ccc} \tan(G(K)) & \xrightarrow{\exp_G} & \mathbf{Q} \otimes_{\mathbf{Z}} G(\mathcal{O}_K) \\ {\scriptstyle =}\downarrow & & {\scriptstyle \delta_G}\downarrow \\ \mathbf{D}_{\mathrm{dR}}(V)/\operatorname{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V) & \xrightarrow{\exp_{K,V}} & H^1(K,V), \end{array}$$

where δ_G is the Kummer map, the upper $\exp_G$ is the usual exponential map, and the lower $\exp_{K,V}$ is Bloch-Kato's exponential map.

The cup product $\cup : H^1(K,V) \times H^1(K,V^*(1)) \to H^2(K,\mathbf{Q}_p(1)) \simeq \mathbf{Q}_p$ defines a perfect pairing, which we can use (by dualizing twice) to define Bloch and Kato's dual exponential map $\exp^*_{K,V^*(1)} : H^1(K,V) \to \mathrm{Fil}^0\,\mathbf{D}_{\mathrm{dR}}(V)$. Kato has given in [Ka93] a very simple formula for $\exp^*_{K,V^*(1)}$, see proposition II.5 below.

When K is an unramified extension of $\mathbf{Q}_p$ and V is a crystalline representation of G_K, Perrin-Riou has constructed in [Per94] a period map $\Omega_{V,h}$ which interpolates the $\exp_{K,V(k)}$ as k runs over the positive integers. It is a crucial ingredient in the construction of p-adic L functions, and is a vast generalization of Coleman's map. Perrin-Riou's constructions were further generalized by Colmez in [Col98].

Let us recall the main properties of her map. For that purpose we need to introduce some notation which will be useful throughout the article. Let $H_K = \mathrm{Gal}(\overline{K}/K(\mu_{p^\infty}))$, let Δ_K be the torsion subgroup of $\Gamma_K = G_K/H_K = \mathrm{Gal}(K(\mu_{p^\infty})/K)$ and let $\Gamma_K^1 = \mathrm{Gal}(K(\mu_{p^\infty})/K(\mu_p))$ so that $\Gamma_K \simeq \Delta_K \times \Gamma_K^1$. Let $\Lambda_K = \mathbf{Z}_p[[\Gamma_K]]$ and $\mathcal{H}(\Gamma_K) = \mathbf{Q}_p[\Delta_K] \otimes_{\mathbf{Q}_p} \mathcal{H}(\Gamma_K^1)$ where $\mathcal{H}(\Gamma_K^1)$ is the set of $f(\gamma_1 - 1)$ with $\gamma_1 \in \Gamma_K^1$ and where $f(T) \in \mathbf{Q}_p[[T]]$ is a power series which converges on the p-adic open unit disk.

Recall that the Iwasawa cohomology groups of V are the projective limits for the corestriction maps of the $H^i(K_n,V)$ where $K_n = K(\mu_{p^n})$. More precisely, if T is any lattice of V then $H^i_{\mathrm{Iw}}(K,V) = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} H^i_{\mathrm{Iw}}(K,T)$ where $H^i_{\mathrm{Iw}}(K,T) = \varprojlim_n H^i(K_n,T)$ so that $H^i_{\mathrm{Iw}}(K,V)$ has the structure of a $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \Lambda_K$-module (see §II.4 for more details). Roughly speaking, these cohomology groups are where Euler systems live (at least locally).

The main result of [Per94] is the construction, for a crystalline representation V of G_K of a family of maps (parameterized by $h \in \mathbf{Z}$):

$$\Omega_{V,h} : \mathcal{H}(\Gamma_K) \otimes_{\mathbf{Q}_p} \mathbf{D}_{\mathrm{cris}}(V) \to \mathcal{H}(\Gamma_K) \otimes_{\Lambda_K} H^1_{\mathrm{Iw}}(K,V)/V^{H_K},$$

whose main property is that they interpolate Bloch and Kato's exponential map. More precisely, if $h, j \gg 0$, then the diagram:

$$\begin{tikzcd}
\left(\mathcal{H}(\Gamma_K) \otimes_{\mathbf{Q}_p} \mathbf{D}_{\mathrm{cris}}(V(j))\right)^{\Delta=0} \arrow[r, "\Omega_{V(j),h}"] \arrow[d, "\Xi_{n,V(j)}"'] & \mathcal{H}(\Gamma_K) \otimes_{\Lambda_K} H^1_{\mathrm{Iw}}(K,V(j))/V(j)^{H_K} \arrow[d, "\mathrm{pr}_{K_n,V(j)}"'] \\
K_n \otimes_K \mathbf{D}_{\mathrm{cris}}(V) \arrow[r, "(h+j-1)!\times", "\exp_{K_n,V(j)}"'] & H^1(K_n,V(j))
\end{tikzcd}$$

is commutative where Δ and $\Xi_{n,V}$ are two maps whose definition is rather technical. Let us just say that the image of Δ is finite-dimensional over $\mathbf{Q}_p$ and that $\Xi_{n,V}$ is a kind of evaluation-at-$(\varepsilon^{(n)}-1)$ map (see §II.5 for a precise definition).

Using the inverse of Perrin-Riou's map, one can then associate to an Euler system a p-adic L-function (see for example [Per95]). For an enlightening survey of this, see [Col00]. If one starts with $V = \mathbf{Q}_p(1)$, then Perrin-Riou's map is the inverse of the Coleman isomorphism and one recovers Kubota-Leopoldt's p-adic L-functions. It is therefore important to be able to construct the maps $\Omega_{V,h}$ as explicitly as possible.

The goal of this article is to give formulas for $\exp_{K,V}$, $\exp^*_{K,V^*(1)}$, and $\Omega_{V,h}$ in terms of the (φ,Γ)-module associated to V by Fontaine. As a corollary, we recover a theorem of Colmez which states that Perrin-Riou's map interpolates the $\exp^*_{K,V^*(1-k)}$ as k runs over the negative integers. This is equivalent to Perrin-Riou's conjectured reciprocity formula (proved by Benois, Colmez and Kato-Kurihara-Tsuji). Our construction of $\Omega_{V,h}$ is actually a slight improvement over Perrin-Riou's (one does not have to kill the Λ_K-torsion, see remark II.14). In addition, our construction should generalize to the case of de Rham representations, to families and to settings other than cyclotomic.

We refer the reader to the text itself for a statement of the actual formulas (theorems II.3, II.6 and II.13) which are rather technical.

This article does not really contain any new results, and it is mostly a reinterpretation of formulas of Cherbonnier-Colmez (for the dual exponential map), and of Benois and Colmez and Kato-Kurihara-Tsuji (for Perrin-Riou's map) in the language of the author's article [Ber02] on p-adic representations and differential equations.

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Finally, it is a pleasure to dedicate this article to Kazuya Kato on the occasion of his fiftieth birthday.

I. Periods of p-adic representations

Throughout this article, k will denote a finite field of characteristic $p > 0$, so that if $W(k)$ denotes the ring of Witt vectors over k, then $F = W(k)[1/p]$ is a finite unramified extension of $\mathbf{Q}_p$. Let $\overline{\mathbf{Q}}_p$ be the algebraic closure of $\mathbf{Q}_p$, let K be a finite totally ramified extension of F, and let $G_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$ be the absolute Galois group of K. Let μ_{p^n} be the group of p^n-th roots of unity; for every n, we will choose a generator $\varepsilon^{(n)}$ of μ_{p^n}, with the additional requirement that $(\varepsilon^{(n)})^p = \varepsilon^{(n-1)}$. This makes $\varprojlim_n \varepsilon^{(n)}$ into a generator of $\varprojlim_n \mu_{p^n} \simeq \mathbf{Z}_p(1)$. We set $K_n = K(\mu_{p^n})$ and $K_\infty = \cup_{n=0}^{+\infty} K_n$. Recall that the cyclotomic character $\chi : G_K \to \mathbf{Z}_p^*$ is defined by the relation: $g(\varepsilon^{(n)}) = (\varepsilon^{(n)})^{\chi(g)}$ for all

$g \in G_K$. The kernel of the cyclotomic character is $H_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K_\infty)$, and χ therefore identifies $\Gamma_K = G_K/H_K$ with an open subgroup of $\mathbf{Z}_p^*$.

A *p-adic representation* V is a finite dimensional $\mathbf{Q}_p$-vector space with a continuous linear action of G_K. It is easy to see that there is always a $\mathbf{Z}_p$-lattice of V which is stable by the action of G_K, and such lattices will be denoted by T. The main strategy (due to Fontaine, see for example [Fo88b]) for studying p-adic representations of a group G is to construct topological $\mathbf{Q}_p$-algebras B (*rings of periods*), endowed with an action of G and some additional structures so that if V is a p-adic representation, then

$$D_B(V) = (B \otimes_{\mathbf{Q}_p} V)^G$$

is a B^G-module which inherits these structures, and so that the functor $V \mapsto D_B(V)$ gives interesting invariants of V. We say that a p-adic representation V of G is *B-admissible* if we have $B \otimes_{\mathbf{Q}_p} V \simeq B^d$ as $B[G]$-modules.

In the next two paragraphs, we will recall the construction of a number of rings of periods. The relations between these rings are mapped in appendix C.

I.1. p-adic Hodge theory. In this paragraph, we will recall the definitions of Fontaine's rings of periods. One can find some of these constructions in [Fo88a] and most of what we will need is proved in [Col98, III] to which the reader should refer in case of need. He is also invited to turn to appendix C.

Let $\mathbf{C}_p$ be the completion of $\overline{\mathbf{Q}}_p$ for the p-adic topology and let

$$\widetilde{\mathbf{E}} = \varprojlim_{x \mapsto x^p} \mathbf{C}_p = \{(x^{(0)}, x^{(1)}, \cdots) \mid (x^{(i+1)})^p = x^{(i)}\},$$

and let $\widetilde{\mathbf{E}}^+$ be the set of $x \in \widetilde{\mathbf{E}}$ such that $x^{(0)} \in \mathcal{O}_{\mathbf{C}_p}$. If $x = (x^{(i)})$ and $y = (y^{(i)})$ are two elements of $\widetilde{\mathbf{E}}$, we define their sum $x + y$ and their product xy by:

$$(x+y)^{(i)} = \lim_{j \to +\infty} (x^{(i+j)} + y^{(i+j)})^{p^j} \quad \text{and} \quad (xy)^{(i)} = x^{(i)} y^{(i)},$$

which makes $\widetilde{\mathbf{E}}$ into an algebraically closed field of characteristic p. If $x = (x^{(n)})_{n \geq 0} \in \widetilde{\mathbf{E}}$, let $v_{\mathbf{E}}(x) = v_p(x^{(0)})$. This is a valuation on $\widetilde{\mathbf{E}}$ for which $\widetilde{\mathbf{E}}$ is complete; the ring of integers of $\widetilde{\mathbf{E}}$ is $\widetilde{\mathbf{E}}^+$. Let $\widetilde{\mathbf{A}}^+$ be the ring $W(\widetilde{\mathbf{E}}^+)$ of Witt vectors with coefficients in $\widetilde{\mathbf{E}}^+$ and let

$$\widetilde{\mathbf{B}}^+ = \widetilde{\mathbf{A}}^+[1/p] = \{\sum_{k \gg -\infty} p^k [x_k],\ x_k \in \widetilde{\mathbf{E}}^+\}$$

where $[x] \in \widetilde{\mathbf{A}}^+$ is the Teichmüller lift of $x \in \widetilde{\mathbf{E}}^+$. This ring is endowed with a map $\theta : \widetilde{\mathbf{B}}^+ \to \mathbf{C}_p$ defined by the formula

$$\theta\left(\sum_{k \gg -\infty} p^k [x_k]\right) = \sum_{k \gg -\infty} p^k x_k^{(0)}.$$

The absolute Frobenius $\varphi : \widetilde{\mathbf{E}}^+ \to \widetilde{\mathbf{E}}^+$ lifts by functoriality of Witt vectors to a map $\varphi : \widetilde{\mathbf{B}}^+ \to \widetilde{\mathbf{B}}^+$. It's easy to see that $\varphi(\sum p^k[x_k]) = \sum p^k[x_k^p]$ and that φ is bijective.

Let $\varepsilon = (\varepsilon^{(i)})_{i\geq 0} \in \widetilde{\mathbf{E}}^+$ where $\varepsilon^{(n)}$ is defined above, and define $\pi = [\varepsilon] - 1$, $\pi_1 = [\varepsilon^{1/p}] - 1$, $\omega = \pi/\pi_1$ and $q = \varphi(\omega) = \varphi(\pi)/\pi$. One can easily show that $\ker(\theta : \widetilde{\mathbf{A}}^+ \to \mathcal{O}_{\mathbf{C}_p})$ is the principal ideal generated by ω.

The ring $\mathbf{B}_{\mathrm{dR}}^+$ is defined to be the completion of $\widetilde{\mathbf{B}}^+$ for the $\ker(\theta)$-adic topology:

$$\mathbf{B}_{\mathrm{dR}}^+ = \varprojlim_{n\geq 0} \widetilde{\mathbf{B}}^+/(\ker(\theta)^n).$$

It is a discrete valuation ring, whose maximal ideal is generated by ω; the series which defines $\log([\varepsilon])$ converges in $\mathbf{B}_{\mathrm{dR}}^+$ to an element t, which is also a generator of the maximal ideal, so that $\mathbf{B}_{\mathrm{dR}} = \mathbf{B}_{\mathrm{dR}}^+[1/t]$ is a field, endowed with an action of G_K and a filtration defined by $\mathrm{Fil}^i(\mathbf{B}_{\mathrm{dR}}) = t^i\mathbf{B}_{\mathrm{dR}}^+$ for $i \in \mathbf{Z}$.

We say that a representation V of G_K is *de Rham* if it is $\mathbf{B}_{\mathrm{dR}}$-admissible which is equivalent to the fact that the filtered K-vector space

$$\mathbf{D}_{\mathrm{dR}}(V) = (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_K}$$

is of dimension $d = \dim_{\mathbf{Q}_p}(V)$.

Recall that the topology of $\widetilde{\mathbf{B}}^+$ is defined by taking the collection of open sets $\{([\overline{\pi}]^k, p^n)\widetilde{\mathbf{A}}^+\}_{k,n\geq 0}$ as a family of neighborhoods of 0. The ring $\mathbf{B}_{\max}^+$ is defined as being

$$\mathbf{B}_{\max}^+ = \{\sum_{n\geq 0} a_n \frac{\omega^n}{p^n} \text{ where } a_n \in \widetilde{\mathbf{B}}^+ \text{ is sequence converging to } 0\},$$

and $\mathbf{B}_{\max} = \mathbf{B}_{\max}^+[1/t]$. The ring $\mathbf{B}_{\max}$ was defined in [Col98, III.2] where a number of its properties are established. It is closely related to $\mathbf{B}_{\mathrm{cris}}$ but tends to be more amenable (loc. cit.). One could replace ω by any generator of $\ker(\theta)$ in $\widetilde{\mathbf{A}}^+$. The ring $\mathbf{B}_{\max}$ injects canonically into $\mathbf{B}_{\mathrm{dR}}$ and, in particular, it is endowed with the induced Galois action and filtration, as well as with a continuous Frobenius φ, extending the map $\varphi : \widetilde{\mathbf{B}}^+ \to \widetilde{\mathbf{B}}^+$. Let us point out that φ does not extend continuously to $\mathbf{B}_{\mathrm{dR}}$. One also sets $\widetilde{\mathbf{B}}_{\mathrm{rig}}^+ = \cap_{n=0}^{+\infty}\varphi^n(\mathbf{B}_{\max}^+)$.

We say that a representation V of G_K is *crystalline* if it is $\mathbf{B}_{\max}$-admissible or (which is the same) $\widetilde{\mathbf{B}}_{\mathrm{rig}}^+[1/t]$-admissible (the periods of crystalline representations live in finite dimensional F-vector subspaces of $\mathbf{B}_{\max}$, stable by φ, and so in fact in $\cap_{n=0}^{+\infty}\varphi^n(\mathbf{B}_{\max}^+)[1/t]$); this is equivalent to requiring that the F-vector space

$$\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{B}_{\max} \otimes_{\mathbf{Q}_p} V)^{G_K} = (\widetilde{\mathbf{B}}_{\mathrm{rig}}^+[1/t] \otimes_{\mathbf{Q}_p} V)^{G_K}$$

be of dimension $d = \dim_{\mathbf{Q}_p}(V)$. Then $\mathbf{D}_{\mathrm{cris}}(V)$ is endowed with a Frobenius φ induced by that of $\mathbf{B}_{\max}$ and $(\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_K} = \mathbf{D}_{\mathrm{dR}}(V) = K \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$ so that a crystalline representation is also de Rham and $K \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$ is a filtered

K-vector space. Note that this definition of $\mathbf{D}_{\mathrm{cris}}(V)$ is compatible with the "usual" one (via $\mathbf{B}_{\mathrm{cris}}$) because $\cap_{n=0}^{+\infty}\varphi^n(\mathbf{B}^+_{\max}) = \cap_{n=0}^{+\infty}\varphi^n(\mathbf{B}^+_{\mathrm{cris}})$.

If V is a p-adic representation, we say that V is *Hodge-Tate*, with Hodge-Tate weights $h_1, \cdots, h_d$, if we have a decomposition $\mathbf{C}_p \otimes_{\mathbf{Q}_p} V \simeq \oplus_{j=1}^d \mathbf{C}_p(h_j)$. We will say that V is positive if its Hodge-Tate weights are negative (the definition of the sign of the Hodge-Tate weights is unfortunate; some people change the sign and talk about geometrical weights). By using the map $\theta : \mathbf{B}^+_{\mathrm{dR}} \to \mathbf{C}_p$, it is easy to see that a de Rham representation is Hodge-Tate and that the Hodge-Tate weights of V are those integers h such that $\mathrm{Fil}^{-h}\,\mathbf{D}_{\mathrm{dR}}(V) \neq \mathrm{Fil}^{-h+1}\,\mathbf{D}_{\mathrm{dR}}(V)$.

To summarize, let us recall that crystalline implies de Rham implies Hodge-Tate. Of course, the significance of these definitions is to be found in geometrical applications. For example, if V is the Tate module of an abelian variety A, then V is de Rham and it is crystalline if and only if A has good reduction.

I.2. (φ,Γ)-modules. The results recalled in this paragraph can be found in [Fo91], and the version which we use here is described in [CC98] and [CC99].

Let $\widetilde{\mathbf{A}}$ be the ring of Witt vectors with coefficients in $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{B}} = \widetilde{\mathbf{A}}[1/p]$. Let $\mathbf{A}_F$ be the completion of $\mathcal{O}_F[\pi,\pi^{-1}]$ in $\widetilde{\mathbf{A}}$ for this ring's topology, which is also the completion of $\mathcal{O}_F[[\pi]][\pi^{-1}]$ for the p-adic topology (π being small in $\widetilde{\mathbf{A}}$). This is a discrete valuation ring whose residue field is $k((\varepsilon-1))$. Let $\mathbf{B}$ be the completion for the p-adic topology of the maximal unramified extension of $\mathbf{B}_F = \mathbf{A}_F[1/p]$ in $\widetilde{\mathbf{B}}$. We then define $\mathbf{A} = \mathbf{B} \cap \widetilde{\mathbf{A}}$, $\mathbf{B}^+ = \mathbf{B} \cap \widetilde{\mathbf{B}}^+$ and $\mathbf{A}^+ = \mathbf{A} \cap \widetilde{\mathbf{A}}^+$. These rings are endowed with an action of Galois and a Frobenius deduced from those on $\widetilde{\mathbf{E}}$. We set $\mathbf{A}_K = \mathbf{A}^{H_K}$ and $\mathbf{B}_K = \mathbf{A}_K[1/p]$. When $K = F$, the two definitions are the same. Let $\mathbf{B}^+_F = (\mathbf{B}^+)^{H_F}$ as well as $\mathbf{A}^+_F = (\mathbf{A}^+)^{H_F}$ (those rings are not so interesting if $K \neq F$). One can show that $\mathbf{A}^+_F = \mathcal{O}_F[[\pi]]$ and that $\mathbf{B}^+_F = \mathbf{A}^+_F[1/p]$.

If V is a p-adic representation of G_K, let $\mathbf{D}(V) = (\mathbf{B} \otimes_{\mathbf{Q}_p} V)^{H_K}$. We know by [Fo91] that $\mathbf{D}(V)$ is a d-dimensional $\mathbf{B}_K$-vector space with a slope 0 Frobenius and a residual action of Γ_K which commute (it is an étale (φ,Γ_K)-*module*) and that one can recover V by the formula $V = (\mathbf{B} \otimes_{\mathbf{B}_K} \mathbf{D}(V))^{\varphi=1}$.

If T is a lattice of V, we get analogous statements with $\mathbf{A}$ instead of $\mathbf{B}$: $\mathbf{D}(T) = (\mathbf{A} \otimes_{\mathbf{Z}_p} T)^{H_K}$ is a free $\mathbf{A}_K$-module of rank d and $T = (\mathbf{A} \otimes_{\mathbf{A}_K} \mathbf{D}(T))^{\varphi=1}$.

The field $\mathbf{B}$ is a totally ramified extension (because the residual extension is purely inseparable) of degree p of $\varphi(\mathbf{B})$. The Frobenius map $\varphi : \mathbf{B} \to \mathbf{B}$ is injective but therefore not surjective, but we can define a left inverse for φ, which will play a major role in the sequel. We set: $\psi(x) = \varphi^{-1}(p^{-1}\,\mathrm{Tr}_{\mathbf{B}/\varphi(\mathbf{B})}(x))$.

Let us now set $K=F$ (i.e. we are now working in an unramified extension of $\mathbf{Q}_p$). We say that a p-adic representation V of G_F is of *finite height* if $\mathbf{D}(V)$ has a basis over $\mathbf{B}_F$ made up of elements of $\mathbf{D}^+(V)=(\mathbf{B}^+\otimes_{\mathbf{Q}_p} V)^{H_F}$. A result of Fontaine ([Fo91] or [Col99, III.2]) shows that V is of finite height if and only if $\mathbf{D}(V)$ has a sub-$\mathbf{B}_F^+$-module which is free of finite rank d, and stable by φ. Let us recall the main result (due to Colmez, see [Col99, théorème 1] or also [Ber02, théorème 3.10]) regarding crystalline representations of G_F:

Theorem I.1. *If V is a crystalline representation of G_F, then V is of finite height.*

If $K\neq F$ or if V is no longer crystalline, then it is no longer true in general that V is of finite height, but it is still possible to say something about the periods of V. Every element $x\in\widetilde{\mathbf{B}}$ can be written in a unique way as $x=\sum_{k\gg-\infty}p^k[x_k]$, with $x_k\in\widetilde{\mathbf{E}}$. For $r>0$, let us set:

$$\widetilde{\mathbf{B}}^{\dagger,r}=\left\{x\in\widetilde{\mathbf{B}},\ \lim_{k\to+\infty}v_{\mathbf{E}}(x_k)+\frac{pr}{p-1}k=+\infty\right\}.$$

This makes $\widetilde{\mathbf{B}}^{\dagger,r}$ into an intermediate ring between $\widetilde{\mathbf{B}}^+$ and $\widetilde{\mathbf{B}}$. Let us set $\mathbf{B}^{\dagger,r}=\mathbf{B}\cap\widetilde{\mathbf{B}}^{\dagger,r}$, $\widetilde{\mathbf{B}}^{\dagger}=\cup_{r\geq0}\widetilde{\mathbf{B}}^{\dagger,r}$, and $\mathbf{B}^{\dagger}=\cup_{r\geq0}\mathbf{B}^{\dagger,r}$. If R is any of the above rings, then by definition $R_K=R^{H_K}$.

We say that a p-adic representation V is *overconvergent* if $\mathbf{D}(V)$ has a basis over $\mathbf{B}_K$ made up of elements of $\mathbf{D}^\dagger(V)=(\mathbf{B}^\dagger\otimes_{\mathbf{Q}_p}V)^{H_K}$. The main result on the overconvergence of p-adic representations of G_K is the following (cf [CC98, corollaire III.5.2]):

Theorem I.2. *Every p-adic representation V of G_K is overconvergent, that is there exists $r=r(V)$ such that $\mathbf{D}(V)=\mathbf{B}_K\otimes_{\mathbf{B}_K^{\dagger,r}}\mathbf{D}^{\dagger,r}(V)$.*

The terminology "overconvergent" can be explained by the following proposition, which describes the rings $\mathbf{B}_K^{\dagger,r}$. Let e_K be the ramification index of K_∞/F_∞ and let F' be the maximal unramified extension of F contained in K_∞ (note that F' can be larger than F):

Proposition I.3. *Let $\mathcal{B}^\alpha_{F'}$ be the set of power series $f(X)=\sum_{k\in\mathbf{Z}}a_kX^k$ such that a_k is a bounded sequence of elements of F', and such that $f(X)$ is holomorphic on the p-adic annulus $\{p^{-1/\alpha}\leq|T|<1\}$.*

There exist $r(K)$ and $\pi_K\in\mathbf{B}_K^{\dagger,r(K)}$ such that if $r\geq r(K)$, then the map $f\mapsto f(\pi_K)$ from $\mathcal{B}^{e_Kr}_{F'}$ to $\mathbf{B}_K^{\dagger,r}$ is an isomorphism. If $K=F$, then $F'=F$ and one can take $\pi_F=\pi$.

I.3. p-adic representations and differential equations. We shall now recall some of the results of [Ber02], which allow us to recover $\mathbf{D}_{\mathrm{cris}}(V)$ from the (φ,Γ)-module associated to V. Let $\mathcal{H}^\alpha_{F'}$ be the set of power series $f(X)=$

$\sum_{k\in\mathbf{Z}} a_k X^k$ such that a_k is a sequence (not necessarily bounded) of elements of F', and such that $f(X)$ is holomorphic on the p-adic annulus $\{p^{-1/\alpha} \leq |T| < 1\}$.

For $r \geq r(K)$, define $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$ as the set of $f(\pi_K)$ where $f(X) \in \mathcal{H}^{e_K r}_{F'}$. Obviously, $\mathbf{B}^{\dagger,r}_K \subset \mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$ and the second ring is the completion of the first one for the natural Fréchet topology. If V is a p-adic representation, let

$$\mathbf{D}^{\dagger,r}_{\mathrm{rig}}(V) = \mathbf{B}^{\dagger,r}_{\mathrm{rig},K} \otimes_{\mathbf{B}^{\dagger,r}_K} \mathbf{D}^{\dagger,r}(V).$$

One of the main technical tools of [Ber02] is the construction of a large ring $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$, which contains $\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}$ and $\widetilde{\mathbf{B}}^{\dagger}$. This ring is a bridge between p-adic Hodge theory and the theory of (φ,Γ)-modules.

As a consequence of the two above inclusions, we have:

$$\mathbf{D}_{\mathrm{cris}}(V) \subset (\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}[1/t] \otimes_{\mathbf{Q}_p} V)^{G_K} \quad \text{and} \quad \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)[1/t] \subset (\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}[1/t] \otimes_{\mathbf{Q}_p} V)^{H_K}.$$

One of the main results of [Ber02] is then (cf. [Ber02, theorem 3.6]):

Theorem I.4. *If V is a p-adic representation of G_K then:* $\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)[1/t])^{\Gamma_K}$. *If V is positive, then* $\mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)^{\Gamma_K}$.

Note that one does not need to know what $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ looks like in order to state the above theorem. We will not give the rather technical construction of that ring, but recall that $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$ is the completion of $\mathbf{B}^{\dagger,r}_K$ for that ring's natural Fréchet topology and that $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ is the union of the $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$. Similarly, there is a natural Fréchet topology on $\widetilde{\mathbf{B}}^{\dagger,r}$, $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ is the completion of $\widetilde{\mathbf{B}}^{\dagger,r}$ for that topology, and $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} = \cup_{r\geq 0}\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$. Actually, one can show that $\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}} \subset \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ for any r and there is an exact sequence (see [Ber02, lemme 2.18]):

$$0 \to \widetilde{\mathbf{B}}^{+} \to \widetilde{\mathbf{B}}^{+}_{\mathrm{rig}} \oplus \widetilde{\mathbf{B}}^{\dagger,r} \to \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}} \to 0,$$

which the reader can take as providing a definition of $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$.

Recall that if $n \geq 0$ and $r_n = p^{n-1}(p-1)$, then there is a well-defined injective map $\varphi^{-n} : \widetilde{\mathbf{B}}^{\dagger,r_n} \to \mathbf{B}^{+}_{\mathrm{dR}}$, and this map extends (see for example [Ber02, §2.2]) to an injective map $\varphi^{-n} : \widetilde{\mathbf{B}}^{\dagger,r_n}_{\mathrm{rig}} \to \mathbf{B}^{+}_{\mathrm{dR}}$.

The reader who feels that he needs to know more about those constructions and theorem I.4 above is invited to read either [Ber02] or the expository paper [Col01] by Colmez. See also appendix C.

Let us now return to the case when $K = F$ and V is a crystalline representation of G_F. In this case, Colmez's theorem tells us that V is of finite height so that one can write $\mathbf{D}^{\dagger,r}_{\mathrm{rig}}(V) = \mathbf{B}^{\dagger,r}_{\mathrm{rig},F} \otimes_{\mathbf{B}^{+}_F} \mathbf{D}^{+}(V)$ and theorem I.4 above therefore says that $\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{B}^{\dagger,r}_{\mathrm{rig},F}[1/t] \otimes_{\mathbf{B}^{+}_F} \mathbf{D}^{+}(V))^{\Gamma_F}$.

One can give a more precise result. Let $\mathbf{B}^+_{\mathrm{rig},F}$ be the set of $f(\pi)$ where $f(X) = \sum_{k\geq 0} a_k X^k$ with $a_k \in F$, and such that $f(X)$ is holomorphic on the p-adic open unit disk. Set $\mathbf{D}^+_{\mathrm{rig}}(V) = \mathbf{B}^+_{\mathrm{rig},F} \otimes_{\mathbf{B}^+_F} \mathbf{D}^+(V)$. One can then show (see [Ber03, §II.2]) the following refinement of theorem I.4:

Proposition I.5. *We have* $\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{D}^+_{\mathrm{rig}}(V)[1/t])^{\Gamma_F}$ *and if* V *is positive then* $\mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}^+_{\mathrm{rig}}(V)^{\Gamma_F}$.

Indeed if $\mathbf{N}(V)$ denotes, in the terminology of [loc. cit.], the Wach module associated to V, then $\mathbf{N}(V) \subset \mathbf{D}^+(V)$ when V is positive and it is shown in [loc. cit., §II.2] that under that hypothesis, $\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{B}^+_{\mathrm{rig},F} \otimes_{\mathbf{B}^+_F} \mathbf{N}(V))^{\Gamma_F}$.

I.4. Construction of cocycles. The purpose of this paragraph is to recall the constructions of [CC99, §I.5] and extend them a little bit. In this paragraph, V will be an arbitrary p-adic representation of G_K. Recall that in loc. cit., a map $h^1_{K,V} : \mathbf{D}(V)^{\psi=1} \to H^1(K,V)$ was constructed, and that (when Γ_K is torsion free at least) it gives rise to an exact sequence:

$$0 \longrightarrow \mathbf{D}(V)^{\psi=1}_{\Gamma_K} \xrightarrow{h^1_{K,V}} H^1(K,V) \longrightarrow \left(\frac{\mathbf{D}(V)}{\psi-1}\right)^{\Gamma_K} \longrightarrow 0.$$

We shall extend $h^1_{K,V}$ to a map $h^1_{K,V} : \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1} \to H^1(K,V)$. We will first need a few facts about the ring of periods $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ and the modules $\mathbf{D}^{\dagger,r}_{\mathrm{rig}}(V)$.

Lemma I.6. *If* r *is large enough and* $\gamma \in \Gamma_K$ *then*

$$1-\gamma : \mathbf{D}^{\dagger,r}_{\mathrm{rig}}(V)^{\psi=0} \to \mathbf{D}^{\dagger,r}_{\mathrm{rig}}(V)^{\psi=0}$$

is an isomorphism.

Proof. We will first show that $1-\gamma$ is injective. By theorem I.4, an element in the kernel of $1-\gamma$ would have to be in $\mathbf{D}_{\mathrm{cris}}(V)$ and therefore in $\mathbf{D}_{\mathrm{cris}}(V)^{\psi=0}$ which is obviously 0.

We will now prove surjectivity. Recall that by [CC98, II.6.1], if r is large enough and $\gamma \in \Gamma_K$ then $1-\gamma : \mathbf{D}^{\dagger,r}(V)^{\psi=0} \to \mathbf{D}^{\dagger,r}(V)^{\psi=0}$ is an isomorphism whose inverse is uniformly continuous for the Fréchet topology of $\mathbf{D}^{\dagger,r}(V)$.

In order to show the surjectivity of $1-\gamma$ it is therefore enough to show that $\mathbf{D}^{\dagger,r}(V)^{\psi=0}$ is dense in $\mathbf{D}^{\dagger,r}_{\mathrm{rig}}(V)^{\psi=0}$ for the Fréchet topology. For r large enough, $\mathbf{D}^{\dagger,r}(V)$ has a basis in $\varphi(\mathbf{D}^{\dagger,r/p}(V))$ so that

$$\mathbf{D}^{\dagger,r}(V)^{\psi=0} = (\mathbf{B}^{\dagger,r}_K)^{\psi=0} \cdot \varphi(\mathbf{D}^{\dagger,r/p}(V))$$
$$\mathbf{D}^{\dagger,r}_{\mathrm{rig}}(V)^{\psi=0} = (\mathbf{B}^{\dagger,r}_{\mathrm{rig},K})^{\psi=0} \cdot \varphi(\mathbf{D}^{\dagger,r/p}(V)).$$

The fact that $\mathbf{D}^{\dagger,r}(V)^{\psi=0}$ is dense in $\mathbf{D}^{\dagger,r}_{\mathrm{rig}}(V)^{\psi=0}$ for the Fréchet topology will therefore follow from the density of $(\mathbf{B}^{\dagger,r}_K)^{\psi=0}$ in $(\mathbf{B}^{\dagger,r}_{\mathrm{rig},K})^{\psi=0}$. This last statement follows from the facts that by definition $\mathbf{B}_K^{\dagger,r/p}$ is dense in $\mathbf{B}_{\mathrm{rig},K}^{\dagger,r/p}$ and that:

$$(\mathbf{B}_K^{\dagger,r})^{\psi=0} = \oplus_{i=1}^{p-1}[\varepsilon]^i\varphi(\mathbf{B}_K^{\dagger,r/p}) \quad \text{and} \quad (\mathbf{B}_{\mathrm{rig},K}^{\dagger,r})^{\psi=0} = \oplus_{i=1}^{p-1}[\varepsilon]^i\varphi(\mathbf{B}_{\mathrm{rig},K}^{\dagger,r/p}).$$

□

Lemma I.7. *The following maps are all surjective and their kernel is* $\mathbf{Q}_p$*:*

$$1-\varphi : \widetilde{\mathbf{B}}^\dagger \to \widetilde{\mathbf{B}}^\dagger, \quad 1-\varphi : \widetilde{\mathbf{B}}^+_{\mathrm{rig}} \to \widetilde{\mathbf{B}}^+_{\mathrm{rig}} \quad \textit{and} \quad 1-\varphi : \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \to \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}.$$

Proof. We'll start with the assertion on the kernel of $1-\varphi$. Since $\widetilde{\mathbf{B}}^+_{\mathrm{rig}} \subset \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ and $\widetilde{\mathbf{B}}^\dagger \subset \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ it is enough to show that $(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}})^{\varphi=1} = \mathbf{Q}_p$. If $x \in (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}})^{\varphi=1}$, then [Ber02, prop 3.2] shows that actually $x \in (\widetilde{\mathbf{B}}^+_{\mathrm{rig}})^{\varphi=1}$, and therefore $x \in (\widetilde{\mathbf{B}}^+_{\mathrm{rig}})^{\varphi=1} = (\mathbf{B}^+_{\max})^{\varphi=1} = \mathbf{Q}_p$ by [Col98, III.3].

The surjectivity of $1-\varphi : \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \to \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ results from the surjectivity of $1-\varphi$ on the first two spaces since by [Ber02, lemme 2.18], one can write $\alpha \in \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ as $\alpha = \alpha^+ + \alpha^-$ with $\alpha^+ \in \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ and $\alpha^- \in \widetilde{\mathbf{B}}^\dagger$.

The surjectivity of $1-\varphi : \widetilde{\mathbf{B}}^+_{\mathrm{rig}} \to \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ follows from the facts that $1-\varphi : \mathbf{B}^+_{\max} \to \mathbf{B}^+_{\max}$ is surjective (see [Col98, III.3]) and that $\widetilde{\mathbf{B}}^+_{\mathrm{rig}} = \cap_{n=0}^{+\infty}\varphi^n(\mathbf{B}^+_{\max})$.

The surjectivity of $1-\varphi : \widetilde{\mathbf{B}}^\dagger \to \widetilde{\mathbf{B}}^\dagger$ follows from the facts that $1-\varphi : \widetilde{\mathbf{B}} \to \widetilde{\mathbf{B}}$ is surjective (it is surjective on $\widetilde{\mathbf{A}}$ as can be seen by reducing modulo p and using the fact that $\widetilde{\mathbf{E}}$ is algebraically closed) and that if $\beta \in \widetilde{\mathbf{B}}$ is such that $(1-\varphi)\beta \in \widetilde{\mathbf{B}}^\dagger$, then $\beta \in \widetilde{\mathbf{B}}^\dagger$ as we shall see presently.

If $x = \sum_{i=0}^{+\infty} p^i[x_i] \in \widetilde{\mathbf{A}}$, let us set $w_k(x) = \inf_{i \leq k} v_{\mathbf{E}}(x_i) \in \mathbf{R} \cup \{+\infty\}$. The definition of $\widetilde{\mathbf{B}}^{\dagger,r}$ shows that $x \in \widetilde{\mathbf{B}}^{\dagger,r}$ if and only if $\lim_{k\to+\infty} w_k(x) + \frac{pr}{p-1}k = +\infty$. A short computation also shows that $w_k(\varphi(x)) = pw_k(x)$ and that $w_k(x+y) \geq \inf(w_k(x), w_k(y))$ with equality if $w_k(x) \neq w_k(y)$.

It is then clear that

$$\lim_{k\to+\infty} w_k((1-\varphi)x) + \frac{pr}{p-1}k = +\infty \implies \lim_{k\to+\infty} w_k(x) + \frac{p(r/p)}{p-1}k = +\infty$$

and so if $x \in \widetilde{\mathbf{A}}$ is such that $(1-\varphi)x \in \widetilde{\mathbf{B}}^{\dagger,r}$ then $x \in \widetilde{\mathbf{B}}^{\dagger,r/p}$ and likewise for $x \in \widetilde{\mathbf{B}}$ by multiplying by a suitable power of p. □

The torsion subgroup of Γ_K will be denoted by Δ_K. We also set $\Gamma_K^n = \mathrm{Gal}(K_\infty/K_n)$. When $p \neq 2$ and $n \geq 1$ (or $p = 2$ and $n \geq 2$), Γ_K^n is torsion free. If $x \in 1 + p\mathbf{Z}_p$, then there exists $k \geq 1$ such that $\log_p(x) \in p^k\mathbf{Z}_p^*$ and we'll write $\log_p^0(x) = \log_p(x)/p^k$.

If K and n are such that Γ_K^n is torsion-free, then we will construct maps $h^1_{K_n,V}$ such that $\mathrm{cor}_{K_{n+1}/K_n} \circ h^1_{K_{n+1},V} = h^1_{K_n,V}$. If Γ_K^n is no longer torsion free, we'll therefore define $h^1_{K_n,V}$ by the relation $h^1_{K_n,V} = \mathrm{cor}_{K_{n+1}/K_n} \circ h^1_{K_{n+1},V}$. In the following proposition, we therefore assume that Γ_K is torsion free (and therefore procyclic), and we let γ denote a topological generator of Γ_K. Recall that if M is a Γ_K-module, it is customary to write M_{Γ_K} for $M/\operatorname{im}(\gamma-1)$.

PROPOSITION I.8. *If $y \in \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1}$, then there exists $b \in \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V$ such that $(\gamma-1)(\varphi-1)b = (\varphi-1)y$ and the formula*

$$h^1_{K,V}(y) = \log_p^0(\chi(\gamma)) \left[\sigma \mapsto \frac{\sigma-1}{\gamma-1} y - (\sigma-1)b\right]$$

then defines a map $h^1_{K,V} : \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1}_{\Gamma_K} \to H^1(K,V)$ which does not depend either on the choice of a generator γ of Γ_K or on the choice of a particular solution b, and if $y \in \mathbf{D}(V)^{\psi=1} \subset \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1}$, then $h^1_{K,V}(y)$ coincides with the cocycle constructed in [CC99, I.5].

Proof. Our construction closely follows [CC99, I.5]; to simplify the notations, we can assume that $\log_p^0(\chi(\gamma)) = 1$. The fact that if we start from a different γ, then the two $h^1_{K,V}$ we get are the same is left as an easy exercise for the reader.

Let us start by showing the existence of $b \in \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V$. If $y \in \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1}$, then $(\varphi-1)y \in \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=0}$. By lemma I.6, there exists $x \in \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=0}$ such that $(\gamma-1)x = (\varphi-1)y$. By lemma I.7, there exists $b \in \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V$ such that $(\varphi-1)b = x$.

Recall that we define $h^1_{K,V}(y) \in H^1(K,V)$ by the formula:

$$h^1_{K,V}(y)(\sigma) = \frac{\sigma-1}{\gamma-1} y - (\sigma-1)b.$$

Notice that, a priori, $h^1_{K,V}(y) \in H^1(K, \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V)$, but

$$\begin{aligned}(\varphi-1)h^1_{K,V}(y)(\sigma) &= \frac{\sigma-1}{\gamma-1}(\varphi-1)y - (\sigma-1)(\varphi-1)b \\ &= \frac{\sigma-1}{\gamma-1}(\gamma-1)x - (\sigma-1)x \\ &= 0,\end{aligned}$$

so that $h^1_{K,V}(y)(\sigma) \in (\mathbf{B}^\dagger_{\mathrm{rig}})^{\varphi=1} \otimes_{\mathbf{Q}_p} V = V$. In addition, two different choices of b differ by an element of $(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}})^{\varphi=1} \otimes_{\mathbf{Q}_p} V = V$, and therefore give rise to two cohomologous cocycles.

It is clear that if $y \in \mathbf{D}(V)^{\psi=1} \subset \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)^{\psi=1}$, then $h^1_{K,V}(y)$ coincides with the cocycle constructed in [CC99, I.5], as can be seen by their identical construction, and it is immediate that if $y \in (\gamma-1)\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$, then $h^1_{K,V}(y) = 0$. □

Lemma I.9. *We have* $\mathrm{cor}_{K_{n+1}/K_n} \circ h^1_{K_{n+1},V} = h^1_{K_n,V}$.

Proof. The proof is exactly the same as that of [CC99, §II.2] and in any case it is rather easy. □

II. Explicit formulas for exponential maps

Recall that $\exp_{K,V} : \mathbf{D}_{\mathrm{dR}}(V)/\operatorname{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V) \to H^1(K,V)$ is obtained by tensoring the fundamental exact sequence (see [Col98, III.3]):

$$0 \to \mathbf{Q}_p \to \mathbf{B}^{\varphi=1}_{\max} \to \mathbf{B}_{\mathrm{dR}}/\mathbf{B}^+_{\mathrm{dR}} \to 0$$

with V and taking the invariants under the action of G_K (note once again that $\mathbf{B}^{\varphi=1}_{\mathrm{cris}} = \mathbf{B}^{\varphi=1}_{\max}$). The exponential map is then the connecting homomorphism $\mathbf{D}_{\mathrm{dR}}(V)/\operatorname{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V) \to H^1(K,V)$.

The cup product $\cup : H^1(K,V) \times H^1(K,V^*(1)) \to H^2(K,\mathbf{Q}_p(1)) \simeq \mathbf{Q}_p$ defines a perfect pairing, which we use (by dualizing twice) to define Bloch and Kato's dual exponential map $\exp^*_{K,V^*(1)} : H^1(K,V) \to \operatorname{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V)$.

The goal of this chapter is to give explicit formulas for Bloch-Kato's maps for a p-adic representation V, in terms of the (φ,Γ)-module $\mathbf{D}(V)$ attached to V. Throughout this chapter, V will be assumed to be a crystalline representation of G_F.

II.1. Preliminaries on some Iwasawa algebras.

Recall that (cf [CC99, III.2] or [Ber02, §2.4] for example) we have maps $\varphi^{-n} : \widetilde{\mathbf{B}}^{\dagger,r_n}_{\mathrm{rig}} \to \mathbf{B}^+_{\mathrm{dR}}$ whose restriction to $\mathbf{B}^+_{\mathrm{rig},F}$ satisfy $\varphi^{-n}(\mathbf{B}^+_{\mathrm{rig},F}) \subset F_n[[t]]$ and which can then characterized by the fact that π maps to $\varepsilon^{(n)}\exp(t/p^n) - 1$.

If $z \in F_n((t)) \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$, then the constant coefficient (i.e. the coefficient of t^0) of z will be denoted by $\partial_V(z) \in F_n \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$. This notation should not be confused with that for the derivation map ∂ defined below.

We will make frequent use of the following fact:

Lemma II.1. *If $y \in (\mathbf{B}^+_{\mathrm{rig},F}[1/t] \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}$, then for any $m \geq n \geq 0$, the element $p^{-m}\operatorname{Tr}_{F_m/F_n} \partial_V(\varphi^{-m}(y)) \in F_n \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$ does not depend on m and we have:*

$$p^{-m}\operatorname{Tr}_{F_m/F_n} \partial_V(\varphi^{-m}(y)) = \begin{cases} p^{-n}\partial_V(\varphi^{-n}(y)) & \text{if } n \geq 1 \\ (1-p^{-1}\varphi^{-1})\partial_V(y) & \text{if } n = 0. \end{cases}$$

Proof. Recall that if $y = t^{-\ell}\sum_{k=0}^{+\infty} a_k\pi^k \in \mathbf{B}^+_{\mathrm{rig},F}[1/t] \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$, then

$$\varphi^{-m}(y) = p^{m\ell}t^{-\ell}\sum_{k=0}^{+\infty}\varphi^{-m}(a_k)(\varepsilon^{(m)}\exp(t/p^m)-1)^k,$$

and that by the definition of ψ, $\psi(y) = y$ means that:

$$\varphi(y) = \frac{1}{p}\sum_{\eta^p=1} y(\eta(1+T)-1).$$

The lemma then follows from the fact that if $m \geq 2$, then the conjugates of $\varepsilon^{(m)}$ under $\mathrm{Gal}(F_m/F_{m-1})$ are the $\eta\varepsilon^{(m)}$, where $\eta^p = 1$, while if $m = 1$, then the conjugates of $\varepsilon^{(1)}$ under $\mathrm{Gal}(F_1/F)$ are the η, where $\eta^p = 1$ but $\eta \neq 1$. $\square$

We will also need some facts about the Iwasawa algebra of Γ_F and some differential operators which it contains. Recall that since F is an unramified extension of $\mathbf{Q}_p$, $\Gamma_F \simeq \mathbf{Z}_p^*$ and that $\Gamma_F^n = \mathrm{Gal}(F_\infty/F_n)$ is the set of elements $\gamma \in \Gamma_F$ such that $\chi(\gamma) \in 1 + p^n\mathbf{Z}_p$.

The completed group algebra of Γ_F is $\Lambda_F = \mathbf{Z}_p[[\Gamma_F]] \simeq \mathbf{Z}_p[\Delta_F] \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[[\Gamma_F^1]]$, and we set $\mathcal{H}(\Gamma_F) = \mathbf{Q}_p[\Delta_F] \otimes_{\mathbf{Q}_p} \mathcal{H}(\Gamma_F^1)$ where $\mathcal{H}(\Gamma_F^1)$ is the set of $f(\gamma-1)$ with $\gamma \in \Gamma_F^1$ and where $f(X) \in \mathbf{Q}_p[[X]]$ is convergent on the p-adic open unit disk. Examples of elements of $\mathcal{H}(\Gamma_F)$ are the ∇_i (which are Perrin-Riou's ℓ_i's), defined by

$$\nabla_i = \ell_i = \frac{\log(\gamma)}{\log_p(\chi(\gamma))} - i.$$

We will also use the operator $\nabla_0/(\gamma_n - 1)$, where γ_n is a topological generator of Γ_F^n. It is defined (see [Ber02, §4.1]) by the formula:

$$\frac{\nabla_0}{\gamma_n-1} = \frac{\log(\gamma_n)}{\log_p(\chi(\gamma_n))(\gamma_n-1)} = \frac{1}{\log_p(\chi(\gamma_n))}\sum_{i\geq 1}\frac{(1-\gamma_n)^{i-1}}{i},$$

or equivalently by

$$\frac{\nabla_0}{\gamma_n-1} = \lim_{\substack{\eta\in\Gamma_F^n\\ \eta\to 1}}\frac{\eta-1}{\gamma_n-1}\frac{1}{\log_p(\chi(\eta))}.$$

It is easy to see that $\nabla_0/(\gamma_n-1)$ acts on F_n by $1/\log_p(\chi(\gamma_n))$.

Note that "$\nabla_0/(\gamma_n-1)$" is a suggestive notation for this operator but it is not defined as a (meaningless) quotient of two operators.

The algebra $\mathcal{H}(\Gamma_F)$ acts on $\mathbf{B}^+_{\mathrm{rig},F}$ and one can easily check that:

$$\nabla_i = t\frac{d}{dt} - i = \log(1+\pi)\partial - i, \quad \text{where} \quad \partial = (1+\pi)\frac{d}{d\pi}.$$

In particular, $\nabla_0\mathbf{B}^+_{\mathrm{rig},F} \subset t\mathbf{B}^+_{\mathrm{rig},F}$ and if $i \geq 1$, then

$$\nabla_{i-1}\circ\cdots\circ\nabla_0\mathbf{B}^+_{\mathrm{rig},F} \subset t^i\mathbf{B}^+_{\mathrm{rig},F}.$$

Lemma II.2. *If $n \geq 1$, then $\nabla_0/(\gamma_n - 1)(\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \subset (t/\varphi^n(\pi))(\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0}$ so that if $i \geq 1$, then:*

$$\nabla_{i-1} \circ \cdots \circ \nabla_1 \circ \frac{\nabla_0}{\gamma_n - 1}(\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \subset \left(\frac{t}{\varphi^n(\pi)}\right)^i (\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0}.$$

Proof. Since $\nabla_i = t \cdot d/dt - i$, the second claim follows easily from the first one, which we will now show. By the standard properties of p-adic holomorphic functions, what we need to do is to show that if $x \in (\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0}$, then

$$\frac{\nabla_0}{\gamma_n - 1} x(\varepsilon^{(m)} - 1) = 0$$

for all $m \geq n+1$.

On the one hand, up to a scalar factor, one has for $m \geq n+1$:

$$\frac{\nabla_0}{\gamma_n - 1} x(\varepsilon^{(m)} - 1) = \mathrm{Tr}_{F_m/F_n}\, x(\varepsilon^{(m)} - 1)$$

as can be seen from the fact that

$$\frac{\nabla_0}{\gamma_n - 1} = \lim_{\substack{\eta \in \Gamma_F^n \\ \eta \to 1}} \frac{\eta - 1}{\gamma_n - 1} \cdot \frac{1}{\log_p(\chi(\eta))}.$$

On the other hand, the fact that $\psi(x) = 0$ implies that for every $m \geq 2$, $\mathrm{Tr}_{F_m/F_{m-1}}\, x(\varepsilon^{(m)} - 1) = 0$. This completes the proof. □

Finally, let us point out that the actions of any element of $\mathcal{H}(\Gamma_F)$ and of φ commute. Since $\varphi(t) = pt$, we also see that $\partial \circ \varphi = p\varphi \circ \partial$.

We will henceforth assume that $\log_p(\chi(\gamma_n)) = p^n$, so that $\log^0_p(\chi(\gamma_n)) = 1$, and in addition $\nabla_0/(\gamma_n - 1)$ acts on F_n by p^{-n}.

II.2. Bloch-Kato's exponential map. The goal of this paragraph is to show how to compute Bloch-Kato's map in terms of the (φ, Γ)-module of V. Let $h \geq 1$ be an integer such that $\mathrm{Fil}^{-h}\, \mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}_{\mathrm{cris}}(V)$.

Recall that we have seen that $\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{D}^+_{\mathrm{rig}}(V)[1/t])^{\Gamma_F}$ and that by [Ber03, §II.3] there is an isomorphism:

$$\mathbf{B}^+_{\mathrm{rig},F}[1/t] \otimes_F \mathbf{D}_{\mathrm{cris}}(V) = \mathbf{B}^+_{\mathrm{rig},F}[1/t] \otimes_F \mathbf{D}^+_{\mathrm{rig}}(V).$$

If $y \in \mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$, then the fact that $\mathrm{Fil}^{-h}\, \mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}_{\mathrm{cris}}(V)$ implies by the results of [Ber03, §II.3] that $t^h y \in \mathbf{D}^+_{\mathrm{rig}}(V)$, so that if

$$y = \sum_{i=0}^{d} y_i \otimes d_i \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=1},$$

then

$$\nabla_{h-1} \circ \cdots \circ \nabla_0(y) = \sum_{i=0}^{d} t^h \partial^h y_i \otimes d_i \in \mathbf{D}^+_{\mathrm{rig}}(V)^{\psi=1}.$$

One can then apply the operator $h^1_{F_n,V}$ to $\nabla_{h-1} \circ \cdots \circ \nabla_0(y)$, and the main result of this paragraph is:

Theorem II.3. *If $y \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}$, then*

$$h^1_{F_n,V}(\nabla_{h-1} \circ \cdots \circ \nabla_0(y)) = (-1)^{h-1}(h-1)! \begin{cases} \exp_{F_n,V}(p^{-n}\partial_V(\varphi^{-n}(y))) & \text{if } n \geq 1 \\ \exp_{F,V}((1-p^{-1}\varphi^{-1})\partial_V(y)) & \text{if } n = 0. \end{cases}$$

Proof. Because the diagram

$$\begin{array}{ccc} F_{n+1} \otimes_F \mathbf{D}_{\mathrm{cris}}(V) & \xrightarrow{\exp_{F_{n+1},V}} & H^1(F_{n+1}, V) \\ {\scriptstyle \mathrm{Tr}_{F_{n+1}/F_n}} \downarrow & & {\scriptstyle \mathrm{cor}_{F_{n+1}/F_n}} \downarrow \\ F_n \otimes_F \mathbf{D}_{\mathrm{cris}}(V) & \xrightarrow{\exp_{F_n,V}} & H^1(F_n, V) \end{array}$$

is commutative, it is enough to prove the theorem under the further assumption that Γ^n_F is torsion free. Let us then set $y_h = \nabla_{h-1} \circ \cdots \circ \nabla_0(y)$. Since we are assuming for simplicity that $\log^0_p(\chi(\gamma_n)) = 1$, the cocycle $h^1_{F_n,V}(y_h)$ is defined by:

$$h^1_{F_n,V}(y_h)(\sigma) = \frac{\sigma - 1}{\gamma_n - 1} y_h - (\sigma - 1) b_{n,h}$$

where $b_{n,h}$ is a solution of the equation $(\gamma_n - 1)(\varphi - 1) b_{n,h} = (\varphi - 1) y_h$. In lemma II.2 above, we proved that:

$$\nabla_{i-1} \circ \cdots \circ \nabla_1 \circ \frac{\nabla_0}{\gamma_n - 1} (\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \subset \left(\frac{t}{\varphi^n(\pi)}\right)^i (\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0}.$$

It is then clear that if one sets

$$z_{n,h} = \nabla_{h-1} \circ \cdots \circ \frac{\nabla_0}{\gamma_n - 1}(\varphi - 1) y,$$

then

$$\begin{aligned} z_{n,h} &\in \left(\frac{t}{\varphi^n(\pi)}\right)^h (\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \otimes_F \mathbf{D}_{\mathrm{cris}}(V) \\ &\subset \varphi^n(\pi^{-h}) \mathbf{D}^+_{\mathrm{rig}}(V)^{\psi=0} \\ &\subset \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=0}. \end{aligned}$$

Recall that $q = \varphi(\pi)/\pi$. By lemma II.4 (which will be stated and proved below), there exists an element $b_{n,h} \in \varphi^{n-1}(\pi^{-h}) \widetilde{\mathbf{B}}^+_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V$ such that

$$(\varphi - \varphi^{n-1}(q^h))(\varphi^{n-1}(\pi^h) b_{n,h}) = \varphi^n(\pi^h) z_{n,h},$$

so that $(1-\varphi)b_{n,h} = z_{n,h}$ with $b_{n,h} \in \varphi^{n-1}(\pi^{-h})\widetilde{\mathbf{B}}^+_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V$.

If we set

$$w_{n,h} = \nabla_{h-1} \circ \cdots \circ \frac{\nabla_0}{\gamma_n - 1} y,$$

then $w_{n,h}$ and $b_{n,h} \in \mathbf{B}_{\max} \otimes_{\mathbf{Q}_p} V$ and the cocycle $h^1_{F_n,V}(y_h)$ is then given by the formula $h^1_{F_n,V}(y_h)(\sigma) = (\sigma-1)(w_{n,h} - b_{n,h})$. Now $(\varphi-1)b_{n,h} = z_{n,h}$ and $(\varphi-1)w_{n,h} = z_{n,h}$ as well, so that $w_{n,h} - b_{n,h} \in \mathbf{B}^{\varphi=1}_{\max} \otimes_{\mathbf{Q}_p} V$.

We can also write $h^1_{F_n,V}(y_h)(\sigma) = (\sigma-1)(\varphi^{-n}(w_{n,h}) - \varphi^{-n}(b_{n,h}))$. Since we know that $b_{n,h} \in \varphi^{n-1}(\pi^{-h})\mathbf{B}^+_{\max} \otimes_{\mathbf{Q}_p} V$, we have $\varphi^{-n}(b_{n,h}) \in \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$.

The definition of the Bloch-Kato exponential gives rise to the following construction: if $x \in \mathbf{D}_{\mathrm{dR}}(V)$ and $\widetilde{x} \in \mathbf{B}^{\varphi=1}_{\max} \otimes_{\mathbf{Q}_p} V$ is such that $x - \widetilde{x} \in \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$ then $\exp_{K,V}(x)$ is the class of the cocyle $g \mapsto g(\widetilde{x}) - \widetilde{x}$.

The theorem will therefore follow from the fact that:

$$\varphi^{-n}(w_{n,h}) - (-1)^{h-1}(h-1)!p^{-n}\partial_V(\varphi^{-n}(y)) \in \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V,$$

since we already know that $\varphi^{-n}(b_{n,h}) \in \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$.

In order to show this, first notice that

$$\varphi^{-n}(y) - \partial_V(\varphi^{-n}(y)) \in tF_n[[t]] \otimes_F \mathbf{D}_{\mathrm{cris}}(V).$$

We can therefore write

$$\frac{\nabla_0}{\gamma_n - 1}\varphi^{-n}(y) = p^{-n}\partial_V(\varphi^{-n}(y)) + tz_1$$

and a simple recurrence shows that

$$\nabla_{i-1} \circ \cdots \circ \frac{\nabla_0}{1-\gamma_n}\varphi^{-n}(y) = (-1)^{i-1}(i-1)!p^{-n}\partial_V(\varphi^{-n}(y)) + t^i z_i,$$

with $z_i \in F_n[[t]] \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$. By taking $i = h$, we see that

$$\varphi^{-n}(w_{n,h}) - (-1)^{h-1}(h-1)!p^{-n}\partial_V(\varphi^{-n}(y)) \in \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V,$$

since we chose h such that $t^h\mathbf{D}_{\mathrm{cris}}(V) \subset \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$. □

We will now prove the technical lemma which was used above:

Lemma II.4. *If $\alpha \in \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$, then there exists $\beta \in \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ such that*

$$(\varphi - \varphi^{n-1}(q^h))\beta = \alpha.$$

Proof. By [Ber02, prop 2.19] applied to the case $r = 0$, the ring $\widetilde{\mathbf{B}}^+$ is dense in $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ for the Fréchet topology. Hence, if $\alpha \in \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$, then there exists $\alpha_0 \in \widetilde{\mathbf{B}}^+$ such that $\alpha - \alpha_0 = \varphi^n(\pi^h)\alpha_1$ with $\alpha_1 \in \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ (one may also show this directly; the point is that when one completes all the localizations are the same).

The map $\varphi - \varphi^{n-1}(q^h) : \widetilde{\mathbf{B}}^+ \to \widetilde{\mathbf{B}}^+$ is surjective, because $\varphi - \varphi^{n-1}(q^h) : \widetilde{\mathbf{A}}^+ \to \widetilde{\mathbf{A}}^+$ is surjective, as can be seen by reducing modulo p and using the fact that $\widetilde{\mathbf{E}}$ is algebraically closed and that $\widetilde{\mathbf{E}}^+$ is its ring of integers.

One can therefore write $\alpha_0 = (\varphi - \varphi^{n-1}(q^h))\beta_0$. Finally by lemma I.7, there exists $\beta_1 \in \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ such that $\alpha_1 = (\varphi - 1)\beta_1$, so that $\varphi^n(\pi^h)\alpha_1 = (\varphi - \varphi^{n-1}(q^h))(\varphi^{n-1}(\pi^h)\beta_1)$. □

II.3. Bloch-Kato's dual exponential map. In the previous paragraph, we showed how to compute Bloch-Kato's exponential map for V. We will now do the same for the dual exponential map. The starting point is Kato's formula [Ka93, §II.1], which we recall below (it is valid for any field K):

Proposition II.5. *If V is a de Rham representation, then the map from $\mathbf{D}_{\mathrm{dR}}(V)$ to $H^1(K, \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)$ defined by $x \mapsto [g \mapsto \log(\chi(\overline{g}))x]$ is an isomorphism, and the dual exponential map $\exp^*_{V^*(1)} : H^1(K,V) \to \mathbf{D}_{\mathrm{dR}}(V)$ is equal to the composition of the map $H^1(K,V) \to H^1(K, \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)$ with the inverse of this isomorphism.*

Let us point out that the image of $\exp^*_{V^*(1)}$ is included in $\mathrm{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V)$ and that its kernel is $H^1_g(K,V)$, the subgroup of $H^1(K,V)$ corresponding to classes of de Rham extensions of $\mathbf{Q}_p$ by V.

Let us now return to a crystalline representation V of G_F. We then have the following formula, which is proved in much more generality (i.e. for de Rham representations) in [CC99, IV.2.1]:

Theorem II.6. *If $y \in \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1}$ and $y \in \mathbf{D}^+_{\mathrm{rig}}(V)[1/t]$ (so that in particular $y \in (\mathbf{B}^+_{\mathrm{rig},F}[1/t] \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}$), then*

$$\exp^*_{F_n,V^*(1)}(h^1_{F_n,V}(y)) = \begin{cases} p^{-n}\partial_V(\varphi^{-n}(y)) & \text{if } n \geq 1 \\ (1-p^{-1}\varphi^{-1})\partial_V(y) & \text{if } n = 0. \end{cases}$$

Note that by theorem A.3, we know that $\mathbf{D}^\dagger(V)^{\psi=1} \subset \mathbf{D}^+_{\mathrm{rig}}(V)[1/t]$.

Proof. Since the following diagram

$$\begin{array}{ccc} H^1(F_{n+1},V) & \xrightarrow{\exp^*_{F_{n+1},V^*(1)}} & F_{n+1} \otimes_F \mathbf{D}_{\mathrm{cris}}(V) \\ {\scriptstyle \mathrm{cor}_{F_{n+1}/F_n}} \downarrow & & \downarrow {\scriptstyle \mathrm{Tr}_{F_{n+1}/F_n}} \\ H^1(F_n,V) & \xrightarrow{\exp^*_{F_n,V^*(1)}} & F_n \otimes_F \mathbf{D}_{\mathrm{cris}}(V) \end{array}$$

is commutative, we only need to prove the theorem when Γ^n_F is torsion free. We then have (bearing in mind that we are assuming that $\log^0_p(\chi(\gamma_n)) = 1$ for simplicity):

$$h^1_{F_n,V}(y)(\sigma) = \frac{\sigma - 1}{\gamma_n - 1} y - (\sigma - 1)b,$$

where $(\gamma_n - 1)(\varphi - 1)b = (\varphi - 1)y$. Recall that $\widetilde{\mathbf{B}}^\dagger_{\text{rig}} = \cup_{r>0}\widetilde{\mathbf{B}}^{\dagger,r}_{\text{rig}}$. Since $b \in \widetilde{\mathbf{B}}^\dagger_{\text{rig}} \otimes_{\mathbf{Q}_p} V$, there exists $m \gg 0$ such that $b \in \widetilde{\mathbf{B}}^{\dagger,r_m}_{\text{rig}} \otimes_{\mathbf{Q}_p} V$. Recall also that we have seen in I.3 that the map φ^{-m} embeds $\widetilde{\mathbf{B}}^{\dagger,r_m}_{\text{rig}}$ into $\mathbf{B}^+_{\text{dR}}$. We can then write

$$h^1(y)(\sigma) = \frac{\sigma - 1}{\gamma_n - 1} \varphi^{-m}(y) - (\sigma - 1)\varphi^{-m}(b),$$

and $\varphi^{-m}(b) \in \mathbf{B}^+_{\text{dR}} \otimes_{\mathbf{Q}_p} V$. In addition, $\varphi^{-m}(y) \in F_m((t)) \otimes_F \mathbf{D}_{\text{cris}}(V)$ and $\gamma_n - 1$ is invertible on $t^k F_m \otimes_F \mathbf{D}_{\text{cris}}(V)$ for every $k \neq 0$. This shows that the cocycle $h^1_{F_n,V}(y)$ is cohomologous in $H^1(F_n, \mathbf{B}_{\text{dR}} \otimes_{\mathbf{Q}_p} V)$ to

$$\sigma \mapsto \frac{\sigma - 1}{\gamma_n - 1}(\partial_V(\varphi^{-m}(y)))$$

which is itself cohomologous (since $\gamma_n - 1$ is invertible on $F_m^{\text{Tr}_{F_m/F_n}=0}$) to

$$\begin{aligned}\sigma \mapsto \frac{\sigma - 1}{\gamma_n - 1}&\left(p^{n-m} \operatorname{Tr}_{F_m/F_n} \partial_V(\varphi^{-m}(y))\right) \\ &= \sigma \mapsto p^{-n} \log(\chi(\overline{\sigma}))p^{n-m} \operatorname{Tr}_{F_m/F_n} \partial_V(\varphi^{-m}(y)).\end{aligned}$$

It follows from this and Kato's formula (proposition II.5) that

$$\begin{aligned}\exp^*_{F_n,V^*(1)}(h^1_{F_n,V}(y)) &= p^{-m} \operatorname{Tr}_{F_m/F_n} \partial_V(\varphi^{-m}(y)) \\ &= \begin{cases} p^{-n}\partial_V(\varphi^{-n}(y)) & \text{if } n \geq 1 \\ (1 - p^{-1}\varphi^{-1})\partial_V(y) & \text{if } n = 0. \end{cases}\end{aligned}$$

□

II.4. Iwasawa theory for p-adic representations. In this specific paragraph, V can be taken to be an arbitrary representation of G_K. Recall that the Iwasawa cohomology groups $H^i_{\text{Iw}}(K,V)$ are defined by $H^i_{\text{Iw}}(K,V) = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} H^i_{\text{Iw}}(K,T)$ where T is any G_K-stable lattice of V, and where

$$H^i_{\text{Iw}}(K,T) = \varprojlim_{\operatorname{cor}_{K_{n+1}/K_n}} H^i(K_n,T).$$

Each of the $H^i(K_n,T)$ is a $\mathbf{Z}_p[\Gamma_K/\Gamma^n_K]$-module, and $H^i_{\text{Iw}}(K_n,T)$ is then endowed with the structure of a Λ_K-module where

$$\Lambda_K = \mathbf{Z}_p[[\Gamma_K]] = \mathbf{Z}_p[\Delta_K] \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[[\Gamma^1_K]].$$

The $H^i_{\text{Iw}}(K,V)$ have been studied in detail by Perrin-Riou, who proved the following (see for example [Per94, §3.2]):

Proposition II.7. *If V is a p-adic representation of G_K, then $H^i_{\text{Iw}}(K,V) = 0$ whenever $i \neq 1$, 2. In addition:*

(1) *the torsion sub-module of* $H^1_{\mathrm{Iw}}(K,V)$ *is a* $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \Lambda_K$*-module isomorphic to* V^{H_K} *and* $H^1_{\mathrm{Iw}}(K,V)/V^{H_K}$ *is a free* $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \Lambda_K$*-module whose rank is* $[K:\mathbf{Q}_p]d$*;*
(2) $H^2_{\mathrm{Iw}}(K,V) = \left(V^*(1)^{H_K}\right)^*$.

If $y \in \mathbf{D}(T)^{\psi=1}$ (where T is still a lattice of V), then the sequence of $\{h^1_{F_n,V}(y)\}_n$ is compatible for the corestriction maps, and therefore defines an element of $H^1_{\mathrm{Iw}}(K,T)$. The following theorem is due to Fontaine and is proved in [CC99, §II.1]:

THEOREM II.8. *The map* $y \mapsto \varprojlim_n h^1_{K_n,V}(y)$ *defines an isomorphism from* $\mathbf{D}(T)^{\psi=1}$ *to* $H^1_{\mathrm{Iw}}(K,T)$ *and from* $\mathbf{D}(V)^{\psi=1}$ *to* $H^1_{\mathrm{Iw}}(K,V)$.

Notice that $V^{H_K} \subset \mathbf{D}(V)^{\psi=1}$, and it is its $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \Lambda_K$-torsion submodule. In addition, it is shown in [CC99, §II.3] that the modules $\mathbf{D}(V)/(\psi-1)$ and $H^2_{\mathrm{Iw}}(K,V)$ are naturally isomorphic. One can summarize the above results as follows:

COROLLARY II.9. *The complex of* $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \Lambda_K$*-modules*

$$0 \longrightarrow \mathbf{D}(V) \xrightarrow{1-\psi} \mathbf{D}(V) \longrightarrow 0$$

computes the Iwasawa cohomology of V.

There is a natural projection map $\mathrm{pr}_{K_n,V} : H^i_{\mathrm{Iw}}(K,V) \to H^i(K_n,V)$ and when $i=1$ it is of course equal to the composition of:

$$H^1_{\mathrm{Iw}}(K,V) \longrightarrow \mathbf{D}(V)^{\psi=1} \xrightarrow{h^1_{K_n,V}} H^1(K_n,V).$$

II.5. PERRIN-RIOU'S EXPONENTIAL MAP. By using the results of the previous paragraphs, we can give a "uniform" formula for the image of an element $y \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}$ in $H^1(F_n, V(j))$ under the composition of the following maps:

$$\left(\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V)\right)^{\psi=1} \xrightarrow{\nabla_{h-1}\circ\cdots\circ\nabla_0} \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1} \xrightarrow{\otimes e_j}$$
$$\mathbf{D}^\dagger_{\mathrm{rig}}(V(j))^{\psi=1} \xrightarrow{h^1_{F_n,V(j)}} H^1(F_n,V(j)).$$

Here e_j is a basis of $\mathbf{Q}_p(j)$ such that $e_{j+k} = e_j \otimes e_k$ so that if V is a p-adic representation, then we have compatible isomorphisms of $\mathbf{Q}_p$-vector spaces $V \to V(j)$ given by $v \mapsto v \otimes e_j$.

Theorem II.10. *If $y \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}$, and $h \geq 1$ is such that $\mathrm{Fil}^{-h}\, \mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}_{\mathrm{cris}}(V)$, then for all j with $h+j \geq 1$, we have:*

$$h^1_{F_n,V(j)}(\nabla_{h-1} \circ \cdots \circ \nabla_0(y) \otimes e_j) = (-1)^{h+j-1}(h+j-1)! \\ \times \begin{cases} \exp_{F_n,V(j)}(p^{-n}\partial_{V(j)}(\varphi^{-n}(\partial^{-j}y \otimes t^{-j}e_j))) & \text{if } n \geq 1 \\ \exp_{F,V(j)}((1-p^{-1}\varphi^{-1})\partial_{V(j)}(\partial^{-j}y \otimes t^{-j}e_j)) & \text{if } n = 0, \end{cases}$$

while if $h+j \leq 0$, then we have:

$$\exp^*_{F_n,V^*(1-j)}(h^1_{F_n,V(j)}(\nabla_{h-1} \circ \cdots \circ \nabla_0(y) \otimes e_j)) = \\ \frac{1}{(-h-j)!} \begin{cases} p^{-n}\partial_{V(j)}(\varphi^{-n}(\partial^{-j}y \otimes t^{-j}e_j)) & \text{if } n \geq 1 \\ (1-p^{-1}\varphi^{-1})\partial_{V(j)}(\partial^{-j}y \otimes t^{-j}e_j) & \text{if } n = 0. \end{cases}$$

Proof. If $h+j \geq 1$, then the following diagram is commutative:

$$\begin{array}{ccc} \mathbf{D}^+_{\mathrm{rig}}(V)^{\psi=1} & \xrightarrow{\otimes e_j} & \mathbf{D}^+_{\mathrm{rig}}(V(j))^{\psi=1} \\ {\scriptstyle \nabla_{h-1}\circ\cdots\circ\nabla_0}\uparrow & & {\scriptstyle \nabla_{h+j-1}\circ\cdots\circ\nabla_0}\uparrow \\ \left(\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V)\right)^{\psi=1} & \xrightarrow{\partial^{-j}\otimes t^{-j}e_j} & \left(\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V(j))\right)^{\psi=1} \end{array}.$$

and the theorem is then a straightforward consequence of theorem II.3 applied to $\partial^{-j}y \otimes t^{-j}e_j$, $h+j$ and $V(j)$ (which are the j-th twists of y, h and V).

If on the other hand $h+j \leq 0$, and Γ^n_F is torsion free, then theorem II.6 shows that

$$\exp^*_{F_n,V^*(1-j)}(h^1_{F_n,V(j)}(\nabla_{h-1} \circ \cdots \circ \nabla_0(y) \otimes e_j)) = \\ p^{-n}\partial_{V(j)}(\varphi^{-n}(\nabla_{h-1} \circ \cdots \circ \nabla_0(y) \otimes e_j))$$

in $\mathbf{D}_{\mathrm{cris}}(V(j))$, and a short computation involving Taylor series shows that

$$p^{-n}\partial_{V(j)}(\varphi^{-n}(\nabla_{h-1} \circ \cdots \circ \nabla_0(y) \otimes e_j)) = \\ (-h-j)!^{-1} p^{-n}\partial_{V(j)}(\varphi^{-n}(\partial^{-j}y \otimes t^{-j}e_j)).$$

Finally, to get the case $n = 0$, one just needs to use the corresponding statement of theorem II.6 or equivalently to corestrict. □

Remark II.11. The notation ∂^{-j} is somewhat abusive if $j \geq 1$ as ∂ is not injective on $\mathbf{B}^+_{\mathrm{rig},F}$ (it is surjective as can be seen by "integrating" directly a power series) but the reader can check for himself that this leads to no ambiguity in the formulas of theorem II.10 above.

We will now use the above result to give a construction of Perrin-Riou's exponential map. If $f \in \mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$, we define $\Delta(f)$ to be the image of

$\oplus_{k=0}^h \partial^k(f)(0)$ in $\oplus_{k=0}^h(\mathbf{D}_{\mathrm{cris}}(V)/(1-p^k\varphi))(k)$. There is then an exact sequence of $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \Lambda_F$-modules (see [Per94, §2.2] for a proof):

$$0 \longrightarrow \oplus_{k=0}^h t^k \mathbf{D}_{\mathrm{cris}}(V)^{\varphi=p^{-k}} \longrightarrow \left(\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V)\right)^{\psi=1} \xrightarrow{1-\varphi}$$
$$(\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \otimes_F \mathbf{D}_{\mathrm{cris}}(V) \xrightarrow{\Delta} \oplus_{k=0}^h \left(\frac{\mathbf{D}_{\mathrm{cris}}(V)}{1-p^k\varphi}\right)(k) \longrightarrow 0.$$

If $f \in ((\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\Delta=0}$, then by the above exact sequence there exists

$$y \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}$$

such that $f = (1-\varphi)y$, and since $\nabla_{h-1} \circ \cdots \circ \nabla_0$ kills $\oplus_{k=0}^{h-1} t^k \mathbf{D}_{\mathrm{cris}}(V)^{\varphi=p^{-k}}$ we see that $\nabla_{h-1} \circ \cdots \circ \nabla_0(y)$ does not depend upon the choice of such a y unless $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=p^{-h}} \neq 0$.

Definition II.12. Let $h \geq 1$ be an integer such that $\mathrm{Fil}^{-h}\,\mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}_{\mathrm{cris}}(V)$ and such that $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=p^{-h}} = 0$. One deduces from the above construction a well-defined map:

$$\Omega_{V,h} : ((\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\Delta=0} \to \mathbf{D}^+_{\mathrm{rig}}(V)^{\psi=1},$$

given by $\Omega_{V,h}(f) = \nabla_{h-1} \circ \cdots \circ \nabla_0(y)$ where $y \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}$ is such that $f = (1-\varphi)y$.

If $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=p^{-h}} \neq 0$ then we get a map:

$$\Omega_{V,h} : ((\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\Delta=0} \to \mathbf{D}^+_{\mathrm{rig}}(V)^{\psi=1}/V^{G_F=\chi^h}.$$

Theorem II.13. *If V is a crystalline representation and $h \geq 1$ is such that we have* $\mathrm{Fil}^{-h}\,\mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}_{\mathrm{cris}}(V)$*, then the map*

$$\Omega_{V,h} : ((\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\Delta=0} \to \mathbf{D}^+_{\mathrm{rig}}(V)^{\psi=1}/V^{H_F}$$

which takes $f \in ((\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\Delta=0}$ to $\nabla_{h-1} \circ \cdots \circ \nabla_0((1-\varphi)^{-1}f)$ is well-defined and coincides with Perrin-Riou's exponential map.

Proof. The map $\Omega_{V,h}$ is well defined because as we have seen above, the kernel of $1-\varphi$ is killed by $\nabla_{h-1} \circ \cdots \circ \nabla_0$, except for $t^h \mathbf{D}_{\mathrm{cris}}(V)^{\varphi=p^{-h}}$, which is mapped to copies of $\mathbf{Q}_p(h) \subset V^{H_F}$.

The fact that $\Omega_{V,h}$ coincides with Perrin-Riou's exponential map follows directly from theorem II.10 above applied to those j's for which $h+j \geq 1$, and the fact that by Perrin-Riou's [Per94, théorème 3.2.3], the $\Omega_{V,h}$ are uniquely determined by the requirement that they satisfy the following diagram for $h, j \gg 0$

(see remark II.17 about the signs however):

$$\begin{array}{ccc} \big(\mathcal{H}(\Gamma_F)\otimes_{\mathbf{Q}_p}\mathbf{D}_{\mathrm{cris}}(V(j))\big)^{\Delta=0} & \xrightarrow{\Omega_{V(j),h}} & \mathcal{H}(\Gamma_F)\otimes_{\Lambda_F} H^1_{\mathrm{Iw}}(F,V(j))/V(j)^{H_F} \\ \Xi_{n,V(j)}\Big\downarrow & & \mathrm{pr}_{F_n,V(j)}\Big\downarrow \\ F_n\otimes_F \mathbf{D}_{\mathrm{cris}}(V) & \xrightarrow[\exp_{F_n,V(j)}]{(h+j-1)!\times} & H^1(F_n,V(j)). \end{array}$$

Here $\Xi_{n,V(j)}(g) = p^{-n}(\varphi\otimes\varphi)^{-n}(f)(\varepsilon^{(n)}-1)$ where f is such that

$$(1-\varphi)f = g(\gamma-1)(1+\pi) \in (\mathbf{B}^+_{\mathrm{rig},F}\otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=0}$$

and the φ on the left of $\varphi\otimes\varphi$ is the Frobenius on $\mathbf{B}^+_{\mathrm{rig},F}$ while the φ on the right is the Frobenius on $\mathbf{D}_{\mathrm{cris}}(V)$. Our F_n is Perrin-Riou's H_{n-1}.

Note that by theorem II.8, we have an isomorphism $\mathbf{D}(V)^{\psi=1}\simeq H^1_{\mathrm{Iw}}(F,V)$ and therefore we get a map $\mathcal{H}(\Gamma_F)\otimes_{\Lambda_F} H^1_{\mathrm{Iw}}(F,V)\to \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\psi=1}$. On the other hand, there is a map

$$\mathcal{H}(\Gamma_F)\otimes_{\mathbf{Q}_p}\mathbf{D}_{\mathrm{cris}}(V(j))\to(\mathbf{B}^+_{\mathrm{rig},F}\otimes_F\mathbf{D}_{\mathrm{cris}}(V))^{\psi=0}$$

which sends $\sum f_i(\gamma-1)\otimes d_i$ to $\sum f_i(\gamma-1)(1+\pi)\otimes d_i$. These two maps allow us to compare the diagram above with the formulas given by theorem II.10. □

Remark II.14. By the above remarks, if V is a crystalline representation and $h\geq 1$ is such that we have $\mathrm{Fil}^{-h}\,\mathbf{D}_{\mathrm{cris}}(V)=\mathbf{D}_{\mathrm{cris}}(V)$ and $\mathbf{Q}_p(h)\not\subset V$, then the map

$$\Omega_{V,h}:((\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0}\otimes_F\mathbf{D}_{\mathrm{cris}}(V))^{\Delta=0}\to\mathbf{D}^+_{\mathrm{rig}}(V)^{\psi=1}$$

which takes $f\in((\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0}\otimes_F\mathbf{D}_{\mathrm{cris}}(V))^{\Delta=0}$ to $\nabla_{h-1}\circ\cdots\circ\nabla_0((1-\varphi)^{-1}f)$ is well-defined, without having to kill the Λ_F-torsion of $H^1_{\mathrm{Iw}}(F,V)$ which improves upon Perrin-Riou's construction.

Remark II.15. It is clear from theorem II.10 that we have:

$$\Omega_{V,h}(x)\otimes e_j=\Omega_{V(j),h+j}(\partial^{-j}x\otimes t^{-j}e_j)\quad\text{and}\quad\nabla_h\circ\Omega_{V,h}(x)=\Omega_{V,h+1}(x),$$

and following Perrin-Riou, one can use these formulas to extend the definition of $\Omega_{V,h}$ to all $h\in\mathbf{Z}$ by tensoring all $\mathcal{H}(\Gamma_F)$-modules with the field of fractions of $\mathcal{H}(\Gamma_F)$.

II.6. The explicit reciprocity formula. In this paragraph, we shall recall Perrin-Riou's explicit reciprocity formula and show that it follows easily from theorem II.10 above.

There is a map $\mathcal{H}(\Gamma_F)\to(\mathbf{B}^+_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}$ which sends $f(\gamma-1)$ to $f(\gamma-1)(1+\pi)$. This map is a bijection and its inverse is the Mellin transform so that if $g(\pi)\in(\mathbf{B}^+_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}$, then $g(\pi)=\mathrm{Mel}(g)(1+\pi)$. See [Per00, B.2.8] for a reference, where Perrin-Riou has also extended Mel to $(\mathbf{B}^\dagger_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}$. If $f,g\in(\mathbf{B}^\dagger_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}$ then we define $f*g$ by the formula $\mathrm{Mel}(f*g)=\mathrm{Mel}(f)\,\mathrm{Mel}(g)$. Let

$[-1] \in \Gamma_F$ be the element such that $\chi([-1]) = -1$, and let ι be the involution of Γ_F which sends γ to γ^{-1}. The operator ∂^j on $(\mathbf{B}^+_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}$ corresponds to Tw_j on Γ_F (Tw_j is defined by $\mathrm{Tw}_j(\gamma) = \chi(\gamma)^j\gamma$). For instance, it is a bijection. We will make use of the facts that $\iota \circ \partial^j = \partial^{-j} \circ \iota$ and that $[-1] \circ \partial^j = (-1)^j \partial^j \circ [-1]$.

If V is a crystalline representation, then the natural maps

$$\mathbf{D}_{\mathrm{cris}}(V) \otimes_F \mathbf{D}_{\mathrm{cris}}(V^*(1)) \longrightarrow \mathbf{D}_{\mathrm{cris}}(\mathbf{Q}_p(1)) \xrightarrow{\mathrm{Tr}_{F/\mathbf{Q}_p}} \mathbf{Q}_p$$

allow us to define a perfect pairing $[\cdot,\cdot]_V : \mathbf{D}_{\mathrm{cris}}(V) \times \mathbf{D}_{\mathrm{cris}}(V^*(1))$ which we extend by linearity to

$$[\cdot,\cdot]_V : (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=0} \times (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V^*(1)))^{\psi=0} \to (\mathbf{B}^+_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}$$

by the formula $[f(\pi) \otimes d_1, g(\pi) \otimes d_2]_V = (f * g)(\pi)[d_1, d_2]_V$.

We can also define a semi-linear (with respect to ι) pairing

$$\langle\cdot,\cdot\rangle_V : \mathbf{D}^+_{\mathrm{rig}}(V)^{\psi=1} \times \mathbf{D}^+_{\mathrm{rig}}(V^*(1))^{\psi=1} \to (\mathbf{B}^+_{\mathrm{rig},\mathbf{Q}_p})^{\psi=0}$$

by the formula

$$\langle y_1, y_2\rangle_V = \varprojlim_n \sum_{\tau \in \Gamma_F/\Gamma_F^n} \langle \tau^{-1}(h^1_{F_n,V}(y_1)), h^1_{F_n,V^*(1)}(y_2)\rangle_{F_n,V} \cdot \tau(1+\pi)$$

where the pairing $\langle\cdot,\cdot\rangle_{F_n,V}$ is given by the cup product:

$$\langle\cdot,\cdot\rangle_{F_n,V} : H^1(F_n, V) \times H^1(F_n, V^*(1)) \to H^2(F_n, \mathbf{Q}_p(1)) \simeq \mathbf{Q}_p.$$

The pairing $\langle\cdot,\cdot\rangle_V$ satisfies the relation $\langle \gamma_1 x_1, \gamma_2 x_2\rangle_V = \gamma_1 \iota(\gamma_2)\langle x_1, x_2\rangle_V$. Perrin-Riou's explicit reciprocity formula (proved by Colmez [Col98], Benois [Ben00] and Kato-Kurihara-Tsuji [KKT96]) is then:

THEOREM II.16. *If $x_1 \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=0}$ and $x_2 \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V^*(1)))^{\psi=0}$, then for every h, we have:*

$$(-1)^h \langle \Omega_{V,h}(x_1), [-1] \cdot \Omega_{V^*(1),1-h}(x_2)\rangle_V = -[x_1, \iota(x_2)]_V.$$

Proof. By the theory of p-adic interpolation, it is enough to prove that if $x_i = (1-\varphi)y_i$ with $y_1 \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\psi=1}$ and $y_2 \in (\mathbf{B}^+_{\mathrm{rig},F} \otimes_F \mathbf{D}_{\mathrm{cris}}(V^*(1)))^{\psi=1}$ then for all $j \gg 0$:

$$\left(\partial^{-j}(-1)^h \langle \Omega_{V,h}(x_1), [-1] \cdot \Omega_{V^*(1),1-h}(x_2)\rangle_V\right)(0) = -\left(\partial^{-j}[x_1, \iota(x_2)]_V\right)(0).$$

The above formula is equivalent to:

$$\begin{aligned}(1)\quad (-1)^{h+j} \langle & h^1_{F,V(j)} \Omega_{V(j),h+j}(\partial^{-j}x_1 \otimes t^{-j}e_j), \\ & h^1_{F,V^*(1-j)} \Omega_{V^*(1-j),1-h-j}(\partial^j x_2 \otimes t^j e_{-j})\rangle_{F,V(j)} \\ & \quad = -[\partial_{V(j)}(\partial^{-j}x_1 \otimes t^{-j}e_j), \partial_{V^*(1-j)}(\partial^j x_2 \otimes t^j e_{-j})]_{V(j)}.\end{aligned}$$

By combining theorems II.10 and II.13 with remark II.15 we see that for $j \gg 0$:

$$\begin{aligned}&h^1_{F,V(j)}\Omega_{V(j),h+j}(\partial^{-j}x_1 \otimes t^{-j}e_j)\\ &\quad = (-1)^{h+j-1}\exp_{F,V(j)}((h+j-1)!(1-p^{-1}\varphi^{-1})\partial_{V(j)}(\partial^{-j}y_1 \otimes t^{-j}e_j)),\end{aligned}$$

and that

$$\begin{aligned}&h^1_{F,V^*(1-j)}\Omega_{V^*(1-j),1-h-j}(\partial^{j}x_2 \otimes t^{j}e_{-j})\\ &\quad = (\exp^*_{F,V^*(1-j)})^{-1}(h+j-1)!^{-1}((1-p^{-1}\varphi^{-1})\partial_{V^*(1-j)}(\partial^{j}y_2 \otimes t^{j}e_{-j})).\end{aligned}$$

Using the fact that by definition, if $x \in \mathbf{D}_{\mathrm{cris}}(V(j))$ and $y \in H^1(F,V(j))$ then

$$[x, \exp^*_{F,V^*(1-j)} y]_{V(j)} = \langle \exp_{F,V(j)} x, y\rangle_{F,V(j)},$$

we see that:

$$\begin{aligned}(2)\quad &\langle h^1_{F,V(j)}\Omega_{V(j),h+j}(\partial^{-j}x_1 \otimes t^{-j}e_j),\\ &\qquad h^1_{F,V^*(1-j)}\Omega_{V^*(1-j),1-h-j}(\partial^{j}x_2 \otimes t^{j}e_{-j})\rangle_{F,V(j)}\\ &\qquad = (-1)^{h+j-1}[(1-p^{-1}\varphi^{-1})\partial_{V(j)}(\partial^{-j}y_1 \otimes t^{-j}e_j),\\ &\qquad\qquad (1-p^{-1}\varphi^{-1})\partial_{V^*(1-j)}(\partial^{j}y_2 \otimes t^{j}e_{-j})]_{V(j)}.\end{aligned}$$

It is easy to see that under $[\cdot,\cdot]$, the adjoint of $(1-p^{-1}\varphi^{-1})$ is $1-\varphi$, and that if $x_i = (1-\varphi)y_i$, then

$$\begin{aligned}\partial_{V(j)}(\partial^{-j}x_1 \otimes t^{-j}e_j) &= (1-\varphi)\partial_{V(j)}(\partial^{-j}y_1 \otimes t^{-j}e_j),\\ \partial_{V^*(1-j)}(\partial^{j}x_2 \otimes t^{j}e_{-j}) &= (1-\varphi)\partial_{V^*(1-j)}(\partial^{j}y_2 \otimes t^{j}e_{-j}),\end{aligned}$$

so that (2) implies (1), and this proves the theorem. □

Note that as I. Fesenko pointed out it is better to call the above statement an "explicit reciprocity formula" rather than an "explicit reciprocity law" as the latter terminology is reserved for statements of a more global nature.

Remark II.17. One should be careful with all the signs involved in those formulas. Perrin-Riou has changed the definition of the ℓ_i operators from [Per94] to [Per99] (the new ℓ_i is minus the old ℓ_i). The reciprocity formula which is stated in [Per99, 4.2.3] does not seem (to me) to have the correct sign. On the other hand, the formulas of [Ben00, Col98] do seem to give the correct signs, but one should be careful that [Col98, IX.4.5] uses a different definition for one of the pairings, and that the signs in [CC99, IV.3.1] and [Col98, VII.1.1] disagree. Our definitions of $\Omega_{V,h}$ and of the pairing agree with Perrin-Riou's ones (as they are given in [Per99]).

APPENDIX A. THE STRUCTURE OF $\mathbf{D}(T)^{\psi=1}$

The goal of this paragraph is to prove a theorem which says that for a crystalline representation V, $\mathbf{D}(V)^{\psi=1}$ is quite "small". See theorem A.3 for a precise statement.

Let V be a crystalline representation of G_F and let T denote a G_F-stable lattice of V. The following proposition, which improves slightly upon the results of N. Wach [Wa96], is proved in detail in [Ber03, §II.1]:

PROPOSITION A.1. *If T is a lattice in a positive crystalline representation V, then there exists a unique sub $\mathbf{A}_F^+$-module $\mathbf{N}(T)$ of $\mathbf{D}^+(T)$, which satisfies the following conditions:*

1. *$\mathbf{N}(T)$ is an $\mathbf{A}_F^+$-module free of rank $d = \dim_{\mathbf{Q}_p}(V)$;*
2. *the action of Γ_F preserves $\mathbf{N}(T)$ and is trivial on $\mathbf{N}(T)/\pi\mathbf{N}(T)$;*
3. *there exists an integer $r \geq 0$ such that $\pi^r \mathbf{D}^+(T) \subset \mathbf{N}(T)$.*

Furthermore, $\mathbf{N}(T)$ is stable by φ, and the $\mathbf{B}_F^+$-module $\mathbf{N}(V) = \mathbf{B}_F^+ \otimes_{\mathbf{A}_F^+} \mathbf{N}(T)$ is the unique sub-$\mathbf{B}_F^+$-module of $\mathbf{D}^+(V)$ satisfying the corresponding conditions.

The $\mathbf{A}_F^+$-module $\mathbf{N}(T)$ is called the *Wach module* associated to T.

Notice that $\mathbf{N}(T(-1)) = \pi\mathbf{N}(T) \otimes e_{-1}$. When V is no longer positive, we can therefore define $\mathbf{N}(T)$ as $\pi^{-h}\mathbf{N}(T(-h)) \otimes e_h$, for h large enough so that $V(-h)$ is positive. Using the results of [Ber03, §III.4], one can show that:

PROPOSITION A.2. *If T is a lattice in a crystalline representation V of G_F, whose Hodge-Tate weights are in $[a;b]$, then $\mathbf{N}(T)$ is the unique sub-$\mathbf{A}_F^+$-module of $\mathbf{D}^+(T)[1/\pi]$ which is free of rank d, stable by Γ_F with the action of Γ_F being trivial on $\mathbf{N}(T)/\pi\mathbf{N}(T)$, and such that $\mathbf{N}(T)[1/\pi] = \mathbf{D}^+(T)[1/\pi]$.*

Finally, we have $\varphi(\pi^b\mathbf{N}(T)) \subset \pi^b\mathbf{N}(T)$ and $\pi^b\mathbf{N}(T)/\varphi^*(\pi^b\mathbf{N}(T))$ is killed by q^{b-a}. The construction $T \mapsto \mathbf{N}(T)$ gives a bijection between Wach modules over $\mathbf{A}_F^+$ which are lattices in $\mathbf{N}(V)$ and Galois lattices T in V.

We shall now show that $\mathbf{D}(V)^{\psi=1}$ is not very far from being included in $\mathbf{N}(V)$. Indeed:

THEOREM A.3. *If V is a crystalline representation of G_F, whose Hodge-Tate weights are in $[a;b]$, then $\mathbf{D}(V)^{\psi=1} \subset \pi^{a-1}\mathbf{N}(V)$.*

If in addition V has no quotient isomorphic to $\mathbf{Q}_p(a)$, then actually $\mathbf{D}(V)^{\psi=1} \subset \pi^a\mathbf{N}(V)$.

Before we prove the above statement, we will need a few results concerning the action of ψ on $\mathbf{D}(T)$. In lemmas A.5 through A.7, we will assume that the Hodge-Tate weights of V are ≥ 0. In particular, $\mathbf{N}(T) \subset \varphi^*\mathbf{N}(T)$ so that $\psi(\mathbf{N}(T)) \subset \mathbf{N}(T)$.

Lemma A.4. *If $m \geq 1$, then there exists a polynomial $Q_m(X) \in \mathbf{Z}_p[X]$ such that $\psi(\pi^{-m}) = \pi^{-m}(p^{m-1} + \pi Q_m(\pi))$.*

Proof. By the definition of ψ, it is enough to show that if $m \geq 1$, there exists a polynomial $Q_m(X) \in \mathbf{Z}[X]$ such that

$$\frac{1}{p}\sum_{\eta^p=1} \frac{1}{(\eta(1+X)-1)^m} = \frac{p^{m-1} + ((1+X)^p-1)Q_m((1+X)^p-1)}{((1+X)^p-1)^m},$$

which is left as an exercise for the reader (or his students). □

Lemma A.5. *If $k \geq 1$, then $\psi(p\mathbf{D}(T) + \pi^{-(k+1)}\mathbf{N}(T)) \subset p\mathbf{D}(T) + \pi^{-k}\mathbf{N}(T)$. In addition, $\psi(p\mathbf{D}(T) + \pi^{-1}\mathbf{N}(T)) \subset p\mathbf{D}(T) + \pi^{-1}\mathbf{N}(T)$.*

Proof. If $x \in \mathbf{N}(T)$, then one can write $x = \sum \lambda_i \varphi(x_i)$ with $\lambda_i \in \mathbf{A}_F^+$ and $x_i \in \mathbf{N}(T)$, so that $\psi(\pi^{-(k+1)}x) = \sum \psi(\pi^{-(k+1)}\lambda_i)x_i$. By the preceding lemma, $\psi(\pi^{-(k+1)}\lambda_i) \in p\mathbf{A}_F + \pi^{-k}\mathbf{A}_F^+$ whenever $k \geq 1$. The lemma follows easily, and the second claim is proved in the same way. □

Lemma A.6. *If $k \geq 1$ and $x \in \mathbf{D}(T)$ has the property that $\psi(x) - x \in p\mathbf{D}(T) + \pi^{-k}\mathbf{N}(T)$, then $x \in p\mathbf{D}(T) + \pi^{-k}\mathbf{N}(T)$.*

Proof. Let ℓ be the smallest integer ≥ 0 such that $x \in p\mathbf{D}(T) + \pi^{-\ell}\mathbf{N}(T)$. If $\ell \leq k$, then we are done and otherwise lemma A.5 shows that

$$\psi(x) \in p\mathbf{D}(T) + \pi^{-(\ell-1)}\mathbf{N}(T),$$

so that $\psi(x) - x$ would be in $p\mathbf{D}(T) + \pi^{-\ell}\mathbf{N}(T)$ but not $p\mathbf{D}(T) + \pi^{-(\ell-1)}\mathbf{N}(T)$, a contradiction if $\ell > k$. □

Lemma A.7. *We have $\mathbf{D}(T)^{\psi=1} \subset \pi^{-1}\mathbf{N}(T)$.*

Proof. We shall prove by induction that $\mathbf{D}(T)^{\psi=1} \subset p^k\mathbf{D}(T) + \pi^{-1}\mathbf{N}(T)$ for $k \geq 1$. Let us start with the case $k = 1$. If $x \in \mathbf{D}(T)^{\psi=1}$, then there exists some $j \geq 1$ such that $x \in p\mathbf{D}(T) + \pi^{-j}\mathbf{N}(T)$. If $j = 1$ we are done and otherwise the fact that $\psi(x) = x$ combined with lemma A.5 shows that j can be decreased by 1. This proves our claim for $k = 1$.

We will now assume our claim to be true for k and prove it for $k + 1$. If $x \in \mathbf{D}(T)^{\psi=1}$, we can therefore write $x = p^k y + n$ where $y \in \mathbf{D}(T)$ and $n \in \pi^{-1}\mathbf{N}(T)$. Since $\psi(x) = x$, we have $\psi(n) - n = p^k(\psi(y) - y)$ so that $\psi(y) - y \in \pi^{-1}\mathbf{N}(T)$ (this is because $p^k\mathbf{D}(T) \cap \mathbf{N}(T) = p^k\mathbf{N}(T)$). By lemma A.6, this implies that $y \in p\mathbf{D}(T) + \pi^{-1}\mathbf{N}(T)$, so that we can write $x = p^k(py' + n') + n = p^{k+1}y' + (p^k n' + n)$, and this proves our claim.

Finally, it is clear that our claim implies the lemma: if one can write $x = p^k y_k + n_k$, then the n_k will converge for the p-adic topology to a $n \in \pi^{-1}\mathbf{N}(T)$ such that $x = n$. □

Proof of theorem A.3. Clearly, it is enough to show that if T is a G_F-stable lattice of V, then $\mathbf{D}(T)^{\psi=1} \subset \pi^{a-1}\mathbf{N}(T)$. It is also clear that we can twist V as we wish, and we shall now assume that the Hodge-Tate weights of V are in $[0; h]$. In this case, the theorem says that $\mathbf{D}(T)^{\psi=1} \subset \pi^{-1}\mathbf{N}(T)$, which is the content of lemma A.7 above.

Let us now prove that if a positive V has no quotient isomorphic to $\mathbf{Q}_p$, then actually $\mathbf{D}(T)^{\psi=1} \subset \mathbf{N}(T)$. Recall that $\mathbf{N}(T) \subset \varphi^*(\mathbf{N}(T))$, since the Hodge-Tate weights of V are ≥ 0, so that if $e_1, \cdots, e_d$ is a basis of $\mathbf{N}(T)$, then there exists $q_{ij} \in \mathbf{A}_F^+$ such that $e_i = \sum_{j=1}^d q_{ij}\varphi(e_j)$. If $\psi(\sum_{i=1}^d \alpha_i e_i) = \sum_{i=1}^d \alpha_i e_i$, with $\alpha_i \in \pi^{-1}\mathbf{A}_F^+$, then this translates into $\psi(\sum_{i=1}^d \alpha_i q_{ij}) = \alpha_j$ for $1 \leq j \leq d$.

Let $\alpha_{i,n}$ be the coefficient of π^n in α_i, and likewise for $q_{ij,n}$. Since $\psi(1/\pi) = 1/\pi$, the equations $\psi(\sum_{i=1}^d \alpha_i q_{ij}) = \alpha_j$ then tell us that for $1 \leq j \leq d$:

$$\sum_{i=1}^d \alpha_{i,-1} q_{ij,0} = \varphi(\alpha_{j,-1}).$$

Since $\mathbf{N}(V)/\pi\mathbf{N}(V) \simeq \mathbf{D}_{\mathrm{cris}}(V)$ as φ-modules by [Ber03, §III.4], the above equations say that 1 is an eigenvalue of φ on $\mathbf{D}_{\mathrm{cris}}(V)$. It is easy to see that if a representation has positive Hodge-Tate weights and $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi=1} \neq 0$, then V has a quotient isomorphic to $\mathbf{Q}_p$. □

Remark A.8. It is proved in [Ber03, III.2] that $\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{B}^+_{\mathrm{rig},F} \otimes_{\mathbf{B}^+_F} \mathbf{N}(V))^{G_F}$ and that if $\mathrm{Fil}^{-h}\,\mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}_{\mathrm{cris}}(V)$, then

$$\left(\frac{t}{\pi}\right)^h \mathbf{B}^+_{\mathrm{rig},F} \otimes_{\mathbf{B}^+_F} \mathbf{D}_{\mathrm{cris}}(V) \subset \mathbf{B}^+_{\mathrm{rig},F} \otimes_{\mathbf{B}^+_F} \mathbf{N}(V).$$

In all the above constructions, one could therefore replace $\mathbf{D}^+_{\mathrm{rig}}(V)$ by $\mathbf{B}^+_{\mathrm{rig},F} \otimes_{\mathbf{B}^+_F} \pi^h\mathbf{N}(V)$. For example, the image of the map $\Omega_{V,h}$ is included in $(\pi^h\mathbf{B}^+_{\mathrm{rig},F} \otimes_{\mathbf{B}^+_F} \mathbf{N}(V))^{\psi=1}$ so that we really get a map:

$$\Omega_{V,h} : ((\mathbf{B}^+_{\mathrm{rig},F})^{\psi=0} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\Delta=0} \to (\pi^h\mathbf{B}^+_{\mathrm{rig},F} \otimes_{\mathbf{B}^+_F} \mathbf{N}(V))^{\psi=1}.$$

This slight refinement may be useful in order to prove Perrin-Riou's $\delta_{\mathbf{Z}_p}$ conjecture.

Appendix B. List of notations

Here is a list of the main notations in the order in which they occur:

I: p, k, $W(k)$, F, K, G_K, μ_{p^n}, $\varepsilon^{(n)}$, K_n, K_∞, H_K, Γ_K, χ, V, T.

I.1: $\mathbf{C}_p$, $\widetilde{\mathbf{E}}$, $\widetilde{\mathbf{E}}^+$, $v_{\mathbf{E}}$, $\widetilde{\mathbf{A}}^+$, $\widetilde{\mathbf{B}}^+$, θ, φ, ε, π, π_1, ω, q, $\mathbf{B}_{\mathrm{dR}}^+$, $\mathbf{B}_{\mathrm{dR}}$, $\mathbf{D}_{\mathrm{dR}}(V)$, $\mathbf{B}_{\max}^+$, $\mathbf{B}_{\max}$, $\mathbf{B}_{\mathrm{cris}}$, $\widetilde{\mathbf{B}}_{\mathrm{rig}}^+$, $\mathbf{D}_{\mathrm{cris}}(V)$, h.

I.2: $\widetilde{\mathbf{A}}$, $\widetilde{\mathbf{B}}$, $\mathbf{A}_F$, $\mathbf{B}$, $\mathbf{B}_F$, $\mathbf{A}$, $\mathbf{B}^+$, $\mathbf{A}^+$, $\mathbf{A}_K$, $\mathbf{B}_K$, $\mathbf{A}_F^+$, $\mathbf{B}_F^+$, $\mathbf{D}(V)$, ψ, $\mathbf{D}^+(V)$, $\widetilde{\mathbf{B}}^{\dagger,r}$, $\mathbf{B}^{\dagger,r}$, $\widetilde{\mathbf{B}}^\dagger$, $\mathbf{B}^\dagger$, $\mathbf{D}^\dagger(V)$, $\mathbf{D}^{\dagger,r}(V)$, e_K, F', π_K.

I.3: $\mathbf{B}_{\mathrm{rig},K}^{\dagger,r}$, $\mathbf{D}_{\mathrm{rig}}^{\dagger,r}(V)$, $\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger$, $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$, r_n, φ^{-n}, $\mathbf{B}_{\mathrm{rig},F}^+$, $\mathbf{D}_{\mathrm{rig}}^+(V)$.

I.4: $h_{K,V}^1$, w_k, Δ_K, Γ_K^n, $\log_p^0$, γ, M_Γ.

II: $\exp_{K,V}$, $\exp_{K,V^*(1)}^*$.

II.1: ∂_V, Λ_F, $\mathcal{H}(\Gamma_F)$, ∇_i, $\nabla_0/(\gamma_n-1)$, ∂.

II.4: T, $H_{\mathrm{Iw}}^i(K,V)$, $\mathrm{pr}_{K,V}$.

II.5: e_j, Δ, $\Omega_{V,h}$, $\Xi_{n,V}$.

II.6: Mel, Tw_j, $[-1]$, ι, $[\cdot,\cdot]_V$, $\langle\cdot,\cdot\rangle_V$, ℓ_i.

A: T, $\mathbf{N}(V)$.

Appendix C. Diagram of the rings of periods

The following diagram summarizes the relationships between the different rings of periods. The arrows ending with $\twoheadrightarrow$ are surjective, the dotted arrow $\dashrightarrow$ is an inductive limit of maps defined on subrings ($\varphi^{-n}:\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r_n}\to\mathbf{B}_{\mathrm{dR}}^+$), and all the other ones are injective.

$$\begin{tikzcd}
 & & \mathbf{B}_{\max}^+ \arrow[r, two heads] & \mathbf{B}_{\mathrm{dR}}^+ \arrow[dd, two heads, "\theta"] \\
 & \widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger \arrow[urr, dotted] & \widetilde{\mathbf{B}}_{\mathrm{rig}}^+ \arrow[l] \arrow[u] & \\
\widetilde{\mathbf{B}} & \widetilde{\mathbf{B}}^\dagger \arrow[l] \arrow[u] & \widetilde{\mathbf{B}}^+ \arrow[l] \arrow[u] \arrow[r, two heads, "\theta"] & \mathbf{C}_p \\
\widetilde{\mathbf{A}} \arrow[u] \arrow[d, two heads] & \widetilde{\mathbf{A}}^\dagger \arrow[l] \arrow[u] & \widetilde{\mathbf{A}}^+ \arrow[l] \arrow[u] \arrow[r, two heads, "\theta"] \arrow[d, two heads] & \mathcal{O}_{\mathbf{C}_p} \arrow[u] \arrow[d, two heads] \\
\widetilde{\mathbf{E}} & & \widetilde{\mathbf{E}}^+ \arrow[ll] \arrow[r, two heads, "\theta"] & \mathcal{O}_{\mathbf{C}_p}/p
\end{tikzcd}$$

All the rings with tildes also have versions without a tilde: one goes from the latter to the former by making Frobenius invertible and completing.

The three rings in the leftmost column (at least their tilde-free versions) are related to the theory of (φ, Γ)-modules. The two rings on the top line are related to p-adic Hodge theory. To go from one theory to the other, one goes from one place to the other through all the intermediate rings but as the reader will notice, one has to go "upstream".

Let us finally review the different rings of power series which occur in this article; let $C[r;1[$ be the annulus$\{z \in \mathbf{C}_p,\ p^{-1/r} \leq |z|_p < 1\}$. We then have:

$\mathbf{A}_F^+$	$\mathcal{O}_F[[\pi]]$	$\mathbf{A}_F$	$\mathcal{O}_F\widehat{[[\pi]][\pi^{-1}]}$
$\mathbf{B}_F^+$	$F \otimes_{\mathcal{O}_F} \mathcal{O}_F[[\pi]]$	$\mathbf{B}_F$	$F \otimes_{\mathcal{O}_F} \mathcal{O}_F\widehat{[[\pi]][\pi^{-1}]}$

$\mathbf{A}_F^{\dagger,r}$	Laurent series $f(\pi)$, convergent on $C[r;1[$, and bounded by 1
$\mathbf{B}_F^{\dagger,r}$	Laurent series $f(\pi)$, convergent on $C[r;1[$, and bounded

$\mathbf{B}_{\mathrm{rig},F}^{\dagger,r}$	Laurent series $f(\pi)$, convergent on $C[r;1[$
$\mathbf{B}_{\mathrm{rig},F}^{+}$	$f(\pi) \in F[[\pi]],\ f(\pi)$ converges on the open unit disk $D[0;1[$

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