Analysis on Arithmetic Schemes. I

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Dedicated to Kazuya Kato

Abstract. A shift invariant measure on a two dimensional local field, taking values in formal power series over reals, is introduced and discussed. Relevant elements of analysis, including analytic duality, are developed. As a two dimensional local generalization of the works of Tate and Iwasawa a local zeta integral on the topological Milnor K_2^t-group of the field is introduced and its properties are studied.

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The functional equation of the twisted zeta function of algebraic number fields was first proved by E. Hecke (for a recent exposition see [N, Ch. VII]). J. Tate [T] and, for unramified characters without local theory, K. Iwasawa [I1–I2] lifted the zeta function to a zeta integral defined on an adelic space. Their method of proving the functional equation and deriving finiteness of several number theoretical objects is a generalization of one of the proofs of the functional equation by B. Riemann. The latter in the case of rational numbers uses an appropriate theta formula derived from a summation formula which itself follows from properties of Fourier transform. Hence the functional equation of the zeta integral reflects symmetries of the Fourier transform on the adelic object and its quotients, and the right mixture of their multiplicative and additive structures. With slight modification the approach of Tate and Iwasawa for characteristic zero can be extended to a uniform treatment of any characteristic, e.g. [W2]. Earlier, the functional field case was treated similarly to Hecke's method by E. Witt in 1936 (cf. [Rq, sect. 7.4]), and it was also established using essentially harmonic analysis on finite rings by H.L. Schmid and O. Teichmüller [ST] in 1943 (for a modern exposition see e.g. [M,3.5]); those works were not widely known.

Two components are important for a two dimensional generalization of the works of Iwasawa and Tate (both locally and globally). The first component is appropriate objects from the right type of higher class field theory. In the local case these are so called topological Milnor K^t-groups ([P2–P4], [F1–F3], [IHLF]), for the local class field theory see [Ka1], [P3–P4], [F1–F3], [IHLF]. The second component is an appropriate theory of measure, integration and generalized Fourier transform on objects associated to arithmetic schemes of higher dimensions. In the local case such objects are a higher local field and its K^t-group. It is crucial that the additive and multiplicative groups of the field are not locally compact in dimension > 1. The bare minimum of the theory for two dimensional local fields is described in part 1 of this work. Unlike the dimension one case, this theory on the additive group of the field is not enough to immediately define a local zeta integral: for the zeta integral one uses the topological K_2^t-group of the field whose closed subgroups correspond (via class field theory) to abelian extensions of the field. A local zeta integral on them with its properties is introduced and discussed in part 2.

We concentrate on the dimension two case, but as usual, the two dimensional case should lead relatively straightforward to the general case.

We briefly sketch the contents of each part, see also their introductions. Of course, throughout the text we use ideas and constructions of the one dimensional theory, whose knowledge is assumed.

Two dimensional local fields are self dual objects in appropriate sense, c.f. section 3. In part 1, in the absence of integration theory on non locally compact abelian groups and harmonic analysis on them, we define a new shift invariant measure

$$\mu: \mathcal{A} \longrightarrow \mathbb{R}((X))$$

on appropriate ring $\mathcal{A}$ of subsets of a two dimensional local field. This measure is not countably additive, but very close to such. The variable X can be viewed as an infinitesimally small positive element of $\mathbb{R}((X))$ with respect to the standard two dimensional topology, see section 1. The main motivation for this measure comes from nonstandard mathematics, but the exposition in this work does not use the latter. Elements of measure theory and integration theory are developed in sections 4–7 to the extent required for this work. We have basically all analogues of the one dimensional theory. Section 9 presents a generalization of the Fourier transform and a proof of a double transform formula (or transform inverse formula) for functions in a certain space. We comment on the case of formal power series over archimedean local fields in section 11.

In part 2, using a covering of a so called topological Milnor K_2^t-group of a two dimensional local field (which is the central object in explicit local class field theory) by the product of the group of units with itself we introduce integrals over the K_2^t-group. In this part we use three maps

$$\mathfrak{o}: T = \mathcal{O}^\times \times \mathcal{O}^\times \longrightarrow F \times F, \qquad \mathfrak{r}, \mathfrak{t}: T \longrightarrow K_2^t(F),$$

which in some sense generalize the one dimensional map $E \setminus \{0\} \longrightarrow E$ on the module, additive and multiplicative level structures respectively. In section 17 we define the main object – a local zeta integral $\zeta(f, \chi)$ associated to a function $f\colon F \times F \longrightarrow \mathbb{C}$ continuous on T and quasi-character $\chi\colon K_2^t(F) \longrightarrow \mathbb{C}^{\times}$

$$\zeta(f, \chi) = \int_T f(\alpha)\, \chi_{\mathfrak{t}}(\alpha)\, |\mathfrak{t}(\alpha)|_2^{-2}\, d\mu_T(\alpha),$$

for the notations see section 17. The definition of the zeta integral is the result of several trials which included global tests. Its several first properties in analogy with the one dimensional case are discussed and proved. A local functional equation for appropriate class of functions is proved in section 19. In the ramified case one meets new difficulties in comparison to the dimension one case, they are discussed in section 21. Zeta integrals for formal power series fields over archimedean local fields are introduced in section 23, their values are not really new in comparison to the dimension one case; this agrees with the well known fact that the class field theory for such fields degenerate and does not really use Milnor K_2.

For an extension of this work to adelic and K-delic theory on arithmetic schemes and applications to zeta integral and zeta functions of arithmetic schemes see [F4].

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1. Measure and integration on higher local fields

We start with discussion of self duality of higher local fields which distinguishes them among other higher dimensional local objects. Then we introduce a nontrivial shift invariant measure on higher dimensional local fields, which may be viewed as a higher dimensional generalization of Haar measure on one dimensional fields. We define and study properties of integrals of a certain class of functions on the field. Self duality leads to a higher dimensional transform on appropriate space of functions. One of the key properties, a double transform formula for a class of functions in a space which generalizes familiar one dimensional spaces, is proved in section 9.

For various results about higher local fields see sections of [IHLF].

1. In higher dimensions sequential aspects of the topology are fundamental: for example, in dimensions greater than one the multiplication is not always continuous but is sequentially continuous. Certainly, in the one dimensional case one does not see the difference between the two.

In a two dimensional local field $\mathbb{R}((X))$ a sequence of series $\left(\sum a_{i,n}X^i\right)_n$ is a fundamental sequence if there is i_0 such that for all n coefficients $a_{i,n}$ are zero for all $i < i_0$ and for every i the sequence $(a_{i,n})_n$ is a fundamental sequence; similarly one defines convergence of sequences. Every fundamental sequence converges. Now consider on $\mathbb{R}((X))$ the so called sequential saturation topology (sequential topology for short): a set is open if and only if every sequence which converges to any element of it has almost all its elements in the set.

The sequence $(aX)^n$, $n \in \mathbb{N}$, tends to zero for every $a \in \mathbb{R}$. Hence, if a norm on $\mathbb{R}((X))$ takes values in an extension of $\mathbb{R}$ and is the usual module on the coefficients, then one deduces that $|aX| < 1$ and so $|X|$ is smaller than any positive real number. In other words, for two dimensional fields the local parameter X plays the role of an infinitesimally small element, and therefore it is very natural to use hyperconstructions of nonstandard mathematics. Nevertheless, to simplify the reading the following text does not contain anything nonstandard.

2. Let F be a two dimensional local field with local parameters t_1, t_2 (t_2 is a local parameter with respect to the discrete valuation of rank 1) with finite last residue field having q elements. Denote its ring of integers by O and the group of units by U; denote by R the set of multiplicative representatives of the last finite residue field. The multiplicative group $F^\times$ is the product of infinite cyclic groups generated by t_2 and t_1 and the group of units U.

Denote by $\mathcal{O}$ the ring of integers of F with respect to the discrete valuation of rank one (so t_2 generates the maximal ideal $\mathcal{M}$ of $\mathcal{O}$). There is a projection $p\colon \mathcal{O} \longrightarrow \mathcal{O}/\mathcal{M} = E$ where E is the (first) residue field of F.

Fractional ideals of F are of two types: principal $t_2^i t_1^j O$ and nonprincipal $\mathcal{M}^i$.

For the definition of the topology on F see [sect. 1 part I of IHLF]. We shall view F endowed with the sequential saturation topology (see [sect. 6 part I of IHLF]), so sequentially continuous maps from F are the same as continuous maps. If the reader prefers to use the former topology on F, then the word “continuous” should everywhere below be replaced with “sequentially continuous”.

3. For a field $L((t))$ denote by $\mathrm{res}_i = \mathrm{res}_{t^i}\colon L((t)) \longrightarrow L$ the linear map $\sum a_j t^j \to a_i$. Similarly define $\mathrm{res} = \mathrm{res}_{-1}\colon L\{\{t\}\} \longrightarrow L$ in the case where L is a complete discrete valuation field of characteristic zero (for the definition of $L\{\{t\}\}$ see [Z2]).

For a one dimensional local field L denote by ψ_L a character of the additive group L with conductor O_L. Introduce

$$\psi' = \psi_L \circ \mathrm{res}_0\colon L((t)) \longrightarrow \mathbb{C}^\times, \quad \psi' = \psi_L \circ (\pi_L^{-1}\,\mathrm{res})\colon L\{\{t\}\} \longrightarrow \mathbb{C}^\times,$$

where π_L is a prime element of L. The conductor of ψ' is the ring O of the corresponding field: i.e. $\psi'(O) = 1 \neq \psi'(t_1^{-1}O)$. An arbitrary two dimensional local field F of mixed characteristic has a finite extension of type $L\{\{t\}\}$ which is the compositum of F and a finite extension of $\mathbb{Q}_p$ [sect. 3 of Z2], so the restriction of ψ' on F has conductor O_F.

Thus, in each case we have a character

$$\psi_0 \colon F \longrightarrow \mathbb{C}^\times$$

with conductor O_F.

The additive group F is self-dual:

Lemma. *The group of continuous characters on F consists of characters of the form $\alpha \mapsto \psi(a\alpha)$ where a runs through all elements of F.*

This assertion is easy, at least for the usual topology on F in the positive characteristic case where F is dual to appropriately topologized $\Omega^2_{F/\mathbb{F}_q}$ (c.f. [P1], [Y]) which itself is noncanonically dual to F. Since we are working with a stronger, sequential saturation topology, we indicate a short proof of duality.

Proof. Given a continuous character ψ', there are i, j such that $\psi'(t_2^i t_1^j O) = 1$, and so we may assume that the conductor of ψ' is O. In the equal characteristic case the restriction of ψ' on $\mathcal{O}$ induces a continuous character on E which by the one dimensional theory is a "shift" of ψ_E, hence there is $a_0 \in \mathcal{O}$ such that $\psi_1(\alpha) = \psi'(\alpha) - \psi(a_0\alpha)$ is trivial on $\mathcal{O}$. Similarly by induction, there is $a_i \in t_2^i\mathcal{O}$ such that $\psi_{i+1}(\alpha) = \psi_i(\alpha) - \psi(a_i\alpha)$ is trivial on $t_2^{-i}\mathcal{O}$. Then $\psi'(\alpha) = \psi(a\alpha)$ with $a = \sum_0^{+\infty} a_i$.

In the mixed characteristic case it suffices to consider the case of $\mathbb{Q}_p\{\{t\}\}$. The restriction of ψ' on $t^i\mathbb{Q}_p$ is of the form $\alpha \mapsto \psi(a_i\alpha)$ with $a_i \in t^{-i-1}p\mathbb{Z}_p$, and $a_i \to 0$ when $i \to +\infty$; hence $\psi'(\alpha) = \psi(a\alpha)$ for $a = \sum_{-\infty}^{+\infty} a_i$.

Remark. Equip characters of F with the topology of uniform convergence on compact subsets (with respect to the sequential topology) of F; an example of such a set in the equal characteristic case is $\{\sum_{i \geqslant i_0} a_i t_2^i : a_i \in W_i\}$ where W_i are compact subsets of E. It is easy to verify that the map $a \mapsto (\alpha \mapsto \psi(a\alpha))$ is a homeomorphism between F and its continuous characters.

4. To introduce a measure on F we first specify a nice class of measurable sets.

Definition. A distinguished subset is of the form $\alpha + t_2^i t_1^j O$. Denote by $\mathcal{A}$ the minimal ring in the sense of [H] containing all distinguished sets.

It is easy to see that if the intersection of two distinguished sets is nonempty, then it equals to one of them. This implies that an element A of $\mathcal{A}$ can be written as $\bigcup_i A_i \setminus (\bigcup_j B_j)$ with distinguished disjoint sets A_i and distinguished disjoint sets B_j such that $\bigcup B_j \subset \bigcup A_i$ (moreover, one can even arrange that each B_j is a subset of some A_i). One easily checks that if A is also of the similar form $\bigcup_l C_l \setminus (\bigcup_k D_k)$ then $A = \bigcup(A_i \bigcap C_l) \setminus \big(\bigcup(A_i \bigcap B_j \bigcap C_l \bigcap D_k)\big)$.

Every element of $\mathcal{A}$ is a disjoint (maybe infinite countable) union of some distinguished subsets. Distinguished sets are closed but not open. For example, sets $O \setminus t_2\mathcal{O} = \bigcup_{j\geqslant 0} t_1^j U$ and $\mathcal{O}^\times = \bigcup_{j\in\mathbb{Z}} t_1^j U$ do not belong to $\mathcal{A}$.

Alternatively, $\mathcal{A}$ is the minimal ring which contains sets $\alpha + t_2^i p^{-1}(S)$, $i \in \mathbb{Z}$, where S is a compact open subset of E.

Definition–Lemma. There is a unique measure μ on F with values in $\mathbb{R}((X))$ which is a shift invariant finitely additive measure on $\mathcal{A}$ such that $\mu(\emptyset) = 0$,

$$\mu(t_2^i t_1^j O) = q^{-j} X^i.$$

The proof immediately follows from the properties of the distinguished sets. The measure μ depends on the choice of t_2 but not on the choice of t_1.

We get $\mu(t_2^i p^{-1}(S)) = X^i \mu_E(S)$ where μ_E is the normalized Haar measure on E such that $\mu_E(O_E) = 1$.

Remarks.

1. This measure can be viewed as induced (in appropriate sense) from a measure which takes values in hyperreals ${}^*\mathbb{R}$. A field of hyperreal numbers ${}^*\mathbb{R}$ introduced by Robinson [Ro1] (for a modern exposition see e.g. [Go1]) is a minimal field extension of $\mathbb{R}$ which contains infinitesimally small elements and on which one has all reasonable analogues of analytic constructions. If one fixes a positive infinitesimal $\omega^{-1} \in {}^*\mathbb{R}$, then a surjective homomorphism from the fraction field of approachable polynomials $\mathbb{R}[X]^{ap}$ to $\mathbb{R}((X))$, and $\omega^{-1} \mapsto X$ determines an isomorphism of a subquotient of ${}^*\mathbb{R}$ onto $\mathbb{R}((X))$.

The first variant of this work employed hyperobjects, and its conceptual value ought to be emphasized.

2. The measure μ takes values in $\mathbb{R}((X))$ (for its topology see section 1) which has the total ordering: $\sum_{n\geqslant n_0} a_n X^n > 0$, $a_{n_0} \in \mathbb{R}^\times$, iff $a_{n_0} > 0$. Notice two unusual properties: $a_n = 1 - q^{-n}$ is smaller than $1 - X$, but the limit of (a_n) is 1; not every subset bounded from below has an infimum. Thus, the standard concepts in real valued (or Banach spaces valued) measure theory, e.g. the outer measure, do not seem to be useful here. In particular, the integral to be defined below will possess some unusual properties like those in section 8.

3. The set $O \in \mathcal{A}$ of measure 1 is the disjoint union of $t_2 O \in \mathcal{A}$ of measure X, $t_2 t_1^{-j} O \setminus t_2 t_1^{-j+1} O$ of measure $q^j(1-q^{-1})X$ for $j > 0$, and $t_1^l O \setminus t_1^{l+1} O$ of measure $q^{-l}(1-q^{-1})$ for $l \geqslant 0$. Since $\sum_{j>0} q^j$ diverges, the measure μ is not σ-additive. It is σ-additive for those sets of countably many disjoint sets A_n in the algebra $\mathcal{A}$ which "don't accumulate at break points" from one horizontal line $t_2^m O$ to $t_2^{m+1} O$, i.e. for which the series $\mu(A_n)$ absolutely converges (see section 6).

5. For $A \in \mathcal{A}$, $\alpha \in F^{\times}$ one has $\alpha A \in \mathcal{A}$ and $\mu(\alpha A) = |\alpha|\mu(A)$, where $|\ |$ is a two dimensional module: $|0| = 0$, $|t_2^i t_1^j u| = q^{-j} X^i$ for $u \in U$. The module is a generalization of the usual module on locally compact fields. For example, every $\alpha \in F$ can be written as a convergent series $\sum \alpha_{i,j}$ with $\alpha_{i,j} \in t_2^i t_1^j O$ and $|\alpha_{i,j}| \to 0$. This simplifies convergence conditions in previous use, e.g. [Z2].

6. We introduce a space R_F of complex valued functions on F and their integrals against the measure μ.

Definition. Call a series $\sum c_n$, $c_n \in \mathbb{C}((X))$, absolutely convergent if it converges and if $\sum |\operatorname{res}_{X^i}(c_n)|$ converges for every i.

For an absolutely convergent series $\sum c_n$ and every subsequence n_m the series $\sum c_{n_m}$ absolutely converges and the limit does not depend on the terms order.

Lemma. *Suppose that a function $f\colon F \longrightarrow \mathbb{C}$ can be written as $\sum c_n\, char_{A_n}$ with countably many disjoint distinguished A_n, $c_n \in \mathbb{C}$, where $char_C$ is the characteristic function of C, and suppose that the series $\sum c_n \mu(A_n)$ absolutely converges. If f has a second presentation of the same type $\sum d_m\, char_{B_m}$, then $\sum c_n \mu(A_n) = \sum d_m \mu(B_m)$.*

Proof. Notice that if $\bigcup A_n = \bigcup B_m$ for distinguished disjoint sets, then for every n either A_n equals to the union of some of B_m, or the union of A_n and possibly several other A's equals to one of B_m. It remains to use the following property: if a distinguished set C is the disjoint union of distinguished sets C_n and $\sum \mu(C_n)$ absolutely converges, then for every $a \in F$ and integer i condition $C \supset a + t_2^i \mathcal{O}$ implies $C_n \supset a + t_2^i \mathcal{O}$ for all n; therefore $\mu(C) = \sum \mu(C_n)$.

Definition. Define R_F as the vector space generated by functions f as in the previous lemma and by functions which are zero outside finitely many points. For an f as in the previous lemma define its integral

$$\int f d\mu = \sum c_n \mu(A_n),$$

and for f which are zero outside finitely many points define its integral as zero.

The previous definition implies that the integral is an additive function.

Example. Let a function f satisfy the following conditions:

there is i_0 such that $f(\beta) = 0$ for all $\beta \in t_2^i t_1^j U$, all $i \neq i_0$;

there is $k(j)$ such that for every $\alpha \in t_2^{i_0} t_1^j U$ $f(\alpha) = f(\alpha + \beta)$ for all $\beta \in t_2^{i_0} t_1^{k(j)} O$.

For every $\theta \in R^{\times}$ write the set $\theta t_2^{i_0} t_1^j + t_2^{i_0} t_1^{j+1} O$ as a disjoint union of finitely many $c_{\theta,j,l} + t_2^{i_0} t_1^{k(j)} O$, $l \in L_{\theta,j}$; then

$$\int f d\mu = \Big(\sum_{j \in \mathbb{Z}} \sum_{\theta \in R^{\times}} \sum_{l \in L_{\theta,j}} f(c_{\theta,j,l})\, q^{-k(j)} \Big) X^{i_0}.$$

If the series in the brackets absolutely converges, then $f \in R_F$.
Redenote f as f_{i_0}. Then $\sum_{i\geqslant i_1} f_i$ also belongs to R_F.

For another example see section 8.

7. Some important for harmonic analysis functions do not belong to R_F, for example, the function $\alpha \mapsto \psi(a\alpha)\, char_A(\alpha)$ for, say, $A = O \in \mathcal{A}$, $a \notin \mathcal{O}$. In the one dimensional case all such functions do belong to the analogue of R_F.

Recall that in the one dimensional case the integral over an open compact subgroup of a nontrivial character is zero. This leads to the following natural

Definition. Denote $\psi(C) = 0$ is ψ takes more than one value on a distinguished set C and $=$ the value of ψ if it is constant on C. For a distinguished set A and $a \in F^{\times}$ define

$$\int \psi(a\alpha)\, char_A(\alpha)\, d\mu(\alpha) = \mu(A)\, \psi(aA).$$

So, if $A \subset \mathcal{O}$ and is the preimage of a shift of a compact open subgroup of the residue field E with respect to p, and if $a \in \mathcal{O} \setminus t_2\mathcal{O}$, then the previous definition agrees with the property of a character on E, mentioned above.

Lemma. *For a function $f = \sum c_n\, \psi(a_n\alpha)\, char_{A_n}(\alpha)$ with finite set $\{a_n\}$ and with countably many disjoint distinguished A_n such that the series $\sum c_n\mu(A_n)$ absolutely converges the sum $\sum c_n \int \psi(a_n\alpha)\, char_{A_n}(\alpha)\, d\mu(\alpha)$ does not depend on the choice of c_n, a_n, A_n.*

Proof. To show correctness, one can reduce to sets on which $|f|$ is constant, then use a simple fact that if a distinguished set C is the disjoint union of distinguished sets C_n, and $\psi(a\alpha)\, char_C(\alpha) = \sum d_n\psi(b_n\alpha)\, char_{C_n}(\alpha)$ with $|d_n| = 1$, absolutely convergent series $\sum d_n\, \mu(C_n)$ and finitely many distinct b_n, then $\psi(aC) = \sum d_n\, \psi(b_nC_n)\, \mu(C_n)$.

Definition. Put

$$\int f\, d\mu = \sum c_n \int \psi(a_n\alpha)\, char_{A_n}(\alpha)\, d\mu(\alpha).$$

Denote by R'_F the space generated by functions $f(\alpha)\, \psi(a\alpha)$ with $f \in R_F$, $a \in F$, and functions which are zero outside a point; all of them are integrable. This space will be enough for the purposes of this work.

Remark. Here is a slightly different approach to the extension of R_F: Suppose that a function $f\colon F \to \mathbb{C}$ is zero outside a distinguished subgroup A of F. Suppose that there are finitely many $a_1, \dots, a_m \in A$ such that the function $g(x) = \sum_i f(a_i + x)$ belongs to R_F. Then *define*

$$\int f = \frac{1}{m} \int g\, d\mu.$$

First of all, this is well defined: if $h(x) = \sum_j f(b_j + x)$ belongs to R_F for $b_1, \dots, b_n \in A$, then $\sum_{i=1}^m h(a_i+x) = \sum_{j=1}^n g(b_j+x)$, and from $h(a_i+x) \in R_F$ and shift invariant property, and the similar property for shifts of g one gets $m \int h\, d\mu = n \int g\, d\mu$.

Second, for two functions $f_1, f_2 \colon A \to \mathbb{C}$ if $\sum_i f_1(a_i+x), \sum_j f_2(b_j+x) \in R_F$, then $\sum_{i,j} f_l(a_i + b_j + x) \in R_F$ for $l = 1, 2$ and so $\int f_1 + f_2 = \int f_1 + \int f_2$. Denote by R''_F the space of all such f.

Third, this definition is compatible with all the properties of R_F in the previous section: if f already belongs to R_F then $\int f = \int f\, d\mu$. Also, $\int f(x) = \int f(a+x)$ for $a \in A$.

Finally, this definition is compatible with the preceding definitions of this section: of course, for nontrivial characters on a distinguished subgroup one has $g = 0$. The space R'_F is a subspace of R''_F: for a function $f = \sum c_n\, \psi(a\alpha)\, char_{A_n}(\alpha)$, such that $\psi(aA_n) = 0$, i.e. $A_n + a^{-1}t_1^{-1}O = A_n$, for every n, choose a_i in $a^{-1}t_1^{-1}O$.

For $f \in R'_F$ we have

$$\int f(\alpha + a)\, d\mu(\alpha) = \int f(\alpha)\, d\mu(\alpha)$$

and using section 5

$$\int f(\alpha)\, d\mu(\alpha) = |a| \int f(a\alpha)\, d\mu(\alpha).$$

For a subset S of F put $\int_S f d\mu = \int f\, char_S d\mu$.

Example. We have $\int_{t_2^i t_1^j O} \psi(a\alpha)\, d\mu(\alpha) = q^{-j}X^i$ if $a \in t_2^{-i}t_1^{-j}O$ and $= 0$ otherwise (since then $\psi(a\alpha)$ is a nontrivial character on $t_2^i t_1^j O$). Hence

$$\int_{t_2^i t_1^j U} \psi(a\alpha)\, d\mu(\alpha) = \begin{cases} 0 & \text{if } a \notin t_2^{-i}t_1^{-j-1}O, \\ -q^{-1-j}X^i & \text{if } a \in t_2^{-i}t_1^{-j-1}U, \\ q^{-j}(1-q^{-1})X^i & \text{if } a \in t_2^{-i}t_1^{-j}O. \end{cases}$$

8. Example. If two functions $f, h \colon \mathcal{O} \longrightarrow \mathbb{C}$ are constant on $t_2\mathcal{O} \setminus \{0\}$ and their restriction to $\mathcal{O} \setminus t_2\mathcal{O}$ coincide, then

$$\int_{\mathcal{O}} f\, d\mu = \int_{\mathcal{O}} h\, d\mu.$$

Indeed, if $(f-h)|_{t_2\mathcal{O}\setminus\{0\}} = c$, then

$$\int_{\mathcal{O}} (f-h) d\mu = \int_O (f-h) d\mu = \int_O c d\mu - \int_{O \setminus t_2\mathcal{O}} c d\mu = c - c\,(1-q^{-1}) \sum_{j \geqslant 0} q^{-j} = 0.$$

From the previous we deduce $\int_{t_2\mathcal{O}} c\, d\mu = 0$, and therefore, similarly,

$$\int_F c\, char_{\mathcal{O}}\, d\mu = \int_{\mathcal{O}} c\, d\mu = 0, \quad \int_{\mathcal{O}\setminus t_2\mathcal{O}} c\, d\mu = 0.$$

Of course, the sets $\mathcal{O}, \mathcal{O} \setminus t_2\mathcal{O}$ are not in $\mathcal{A}$.

Remark. One has $\int_{Z_l} d\mu = \sum_{-l\leqslant j\leqslant l} q^{-j}$ where $Z_l = \bigcup_{-l\leqslant j\leqslant l} t_1^j U$, and $\mathcal{O}\backslash t_2\mathcal{O}$ is the "limit" of Z_l when $l \to \infty$. Compare with "equality" $\sum_{n\in\mathbb{Z}} z^n = 0$ used by L. Euler. One of interpretations of it is to view z as a complex variable, then for every integer m the sum of analytic continuations of two rational functions $\sum_{n\in\mathbb{Z},n<m} z^n$ and $\sum_{n\in\mathbb{Z},n\geqslant m} z^n$ is zero. Euler's equality can be applied to show equivalence of the Riemann–Roch theorem and the functional equation of the zeta function of a one dimensional global field of positive characteristic (cf. [Rq, sect. 4.3.3]).

From the definitions we immediately get

Lemma. *Suppose that $g\colon E \longrightarrow \mathbb{C}$ is integrable over E with respect to the normalized Haar measure μ_E as in section 4. Then the function $g\circ p$ extended by zero outside $\mathcal{O}$ is in R_F and*

$$\int_{\mathcal{O}} g\circ p\, d\mu = \int_E g\, d\mu_E.$$

One can say that the Haar measure μ_E equals $p_*(\mu')$ where the measure μ' on $\mathcal{O}$ is induced by μ by extending functions on $\mathcal{O}$ by zero to F.

9. Definition. For $f \in R_F$ introduce the transform function

$$\mathcal{F}(f)(\beta) = \int_F f(\alpha)\,\psi(\alpha\beta)\,d\mu(\alpha).$$

In particular, $\mathcal{F}(char_O) = char_O$.

Definition. Denote by Q_F the subspace of R_F consisting of functions f with support in $\mathcal{O}$ and such that $f|_{\mathcal{O}} = g\circ p|_{\mathcal{O}}$ for a function $g\colon E \longrightarrow \mathbb{C}$ which belongs to the Bruhat space generated by characteristic functions of shifts of proper fractional ideals of E.

Theorem. *Given $f \in Q_F$, the function $\mathcal{F}(f)$ belongs to Q_F and we have a double transform formula*

$$\mathcal{F}^2(f)(\alpha) = f(-\alpha).$$

Proof. First note that $\psi(\alpha) = \psi_E \circ p(\alpha)$ for all $\alpha \in \mathcal{O}$ where ψ_E is an appropriate character on E with conductor O_E.

Using sections 7 and 8 we shall verify that

if $\beta \notin \mathcal{O}$ then $\mathcal{F}(f)(\beta) = 0$;

if $\beta \in \mathcal{O}$ then $\mathcal{F}(f)(\beta) = \mathcal{F}(g)\circ p\,(\beta)$ (where $\mathcal{F}(g)$ denotes the Fourier transform of g with respect to ψ_E and μ_E).

For $\beta \notin \mathcal{O}$ the definitions imply

$$\mathcal{F}(f)(\beta) = \int_{\mathcal{O}} f(\alpha)\,\psi(\alpha\beta)\,d\mu(\alpha) = 0.$$

For $\beta \in \mathcal{O} \setminus \mathcal{O}^\times$

$$\begin{aligned}\mathcal{F}(f)(\beta) &= \int_{\mathcal{O}} f(\alpha)\, d\mu(\alpha) = \int_{\mathcal{O}} g \circ p(\alpha)\, d\mu(\alpha)\\ &= \int_E g(\overline{\alpha})\, d\mu_E(\overline{\alpha}) = \mathcal{F}(g)(0) = \mathcal{F}(f)(0).\end{aligned}$$

For $\beta \in \mathcal{O}^\times$

$$\begin{aligned}&\mathcal{F}(f)(\beta) = \int_F f(\alpha)\, \psi(\alpha\beta)\, d\mu(\alpha) = \int_{\mathcal{O}} f(\alpha)\, \psi(\alpha\beta)\, d\mu(\alpha)\\ &= \int_{\mathcal{O}} g \circ p(\alpha)\, \psi_E \circ p(\alpha\beta)\, d\mu(\alpha) = \int_E g(\overline{\alpha})\, \psi_E(\overline{\alpha}\overline{\beta})\, d\mu_E(\,\overline{\alpha}) = \mathcal{F}(g)(\,p(\beta)).\end{aligned}$$

It remains to use the one dimensional double transform formula.

As a corollary, we deduce that if a function $f\colon F \longrightarrow \mathbb{C}$ with support in $\mathcal{O}^\times$ coincides on $\mathcal{O}^\times$ with a function $h \in Q_F$, then

$$\mathcal{F}^2(f)(\alpha) = \begin{cases} f(-\alpha) & \text{if } \alpha \in \mathcal{O}^\times \bigcup (F \setminus \mathcal{O}),\\ h(0) & \text{if } \alpha \in \mathcal{O} \setminus \mathcal{O}^\times.\end{cases}$$

Remark. It is more natural to integrate $\mathbb{C}((X))$-valued functions on F: for a function $f = \sum_{i \geqslant i_0} f_i X^i \colon F \longrightarrow \mathbb{C}\,((X))$, with $f_i \in R'_F$ define

$$\int_F f d\mu = \sum X^i \int_F f_i d\mu.$$

Similarly to the previous text one checks the correctness of the definition and properties of the integral. In particular, the previous theorem remains true for functions $f = \sum_{i \geqslant i_0} f_i X^i$, f_i in the space $Q'_F = \{\alpha \mapsto \sum_{n \geqslant n_0} g_n(t_2^{-n}\alpha)\}$ where g_n belong to Q_F (or even just "lifts" of functions on E for which the double transform formula holds).

10. Introduce the product measure $\mu_{F\times F} = \mu_F \otimes \mu_F$ on $F \times F$ and define spaces $R_{F\times F} = R_F \otimes R_F$, $R'_{F\times F} = R'_F \otimes R'_F$ and $Q_{F\times F} = Q_F \otimes Q_F$.

For a function $f \in Q_{F\times F}$ define its transform

$$\mathcal{F}(f)(\beta_1, \beta_2) = \int_{F\times F} f(\alpha_1, \alpha_2)\, \psi(\alpha_1\beta_1 + \alpha_2\beta_2)\, d\mu(\alpha_1)\, d\mu(\alpha_2).$$

From section 9 we deduce $\mathcal{F}^2(f)(\alpha_1, \alpha_2) = f(-\alpha_1, -\alpha_2)$ for $f \in Q_{F\times F}$.

11. In the case of two dimensional local fields of type $F = K((t))$, where K is an archimedean local field, O, U are not defined. Define a character $\psi: K((t)) \longrightarrow \mathbb{C}^\times$ to be the composite of the $\mathrm{res}_0: K((t)) \longrightarrow K$ and the archimedean character $\psi_K(\alpha) = \exp(2\pi i \operatorname{Tr}_{K/\mathbb{R}}(\alpha))$ on K. The role of distinguished sets is played by $A = a + t^i D + t^{i+1} K[[t]]$ where D is an open ball in K. In this case if the intersection of two distinguished sets is nonempty then it equals either to one of them, or to a smaller distinguished set. The measure is the shift invariant additive measure μ on the ring generated by distinguished sets and such that $\mu(A) = \mu_K(D) X^i$. It can be extended to $\mu(a + t^i C + t^{i+1} K[[t]]) = \mu_K(C) X^i$, where $C \subset K$ is a Lebesgue measurable set.

The module is $|\sum_{i \geqslant i_0} a_i t^i| = |a_{i_0}|_K X^{i_0}$, where the module on the real K is the absolute value and on the complex K is the square of the absolute value.

The fields of this type are not used in the global theory, but we briefly sketch the analogues of the previous constructions.

The definitions of spaces R_F and R'_F follow the general pattern of sections 6 and 7: for disjoint distinguished sets A_n such that $\sum c_n \, \mu(A_n)$ absolutely converges put

$$\int \sum c_n \, char_{A_n} \, d\mu = \sum c_n \, \mu(A_n).$$

For a function $f = \sum c_n \, \psi(a_n \alpha) \, char_{A_n}(\alpha)$ with finite set $\{a_n\}$ and with countably many disjoint distinguished A_n such that the series $\sum c_n \, \mu(A_n)$ absolutely converges put

$$\int f \, d\mu = \sum c_n \int_{A_n} \psi(a_n \alpha) \, d\mu(\alpha),$$

where

$$\int_A \psi(c\alpha) \, d\mu(\alpha) = \psi(ca) \, X^i \int_D \psi_K(c_{-i}\beta) \, d\mu_K(\beta), \quad c_{-i} = \mathrm{res}_{t^{-i}} \, c,$$

A, a, D, i are defined at the beginning of this section.

The analogues of the definitions and results of sections 8 and 9 hold. The transform of a function $f \in R_F$ is defined by the same formula

$$\mathcal{F}(f)(\beta) = \int_F f(\alpha) \, \psi(\alpha\beta) \, d\mu(\alpha).$$

The space Q_F consists of functions $f \in R_F$ such that $f = g \circ p$ for a function g in the Schwartz space on K, where $p = \mathrm{res}_{t^0}$. The double transform formula is $\mathcal{F}^2(f)(\alpha) = f(-\alpha)$ for $f \in Q_F$.

Remark. It is interesting to investigate if further extensions of this sort of measure theory and harmonic analysis (they seem to be more powerful than applications of Wiener's measure) would be useful for mathematical description of Feynman integrals and integration on loop spaces.

12. In general, for the global theory, we need to work with normalized measures corresponding to shifted characters. Similar to the one dimensional case, if we start with a character ψ with conductor aO then the dual to μ measure μ' on F is normalized such that $|a|\mu(O)\mu'(O) = 1$. The double transform formula of section 9 holds.

For example, if $\mu' = \mu$ then $\mu(O) = |a|^{-1/2}$. If $a\mathcal{O} = t_2^i\mathcal{O}$ then in the last formula of section 8 one should insert the coefficient $X^{-i/2}$ on the right hand side. If $f_0(\alpha) = f(a\alpha)$ belongs to Q_F or Q'_F then $\mathcal{F}(f) = |a|^{1/2}\,\mathcal{F}_0(f_0)$ where $\mathcal{F}_0$ is the transform with respect to character $\psi_0(\alpha) = \psi(a\alpha)$ of conductor O and measure μ_0 such that $\mu_0(O) = 1$. In particular, $\mathcal{F}(char_O) = |a|^{-1/2}char_{aO}$.

13. Generally, given an integral domain A with principal ideal $P = tA$ and projection $A \longrightarrow A/P = B$ and a shift invariant measure on B, similar to the previous one defines a measure and integration on A. For example, the analogue of the ring $\mathcal{A}$ is the minimal ring which contains sets $\alpha + t^ip^{-1}(S)$, where S is from a class of measurable subsets of B; the space of integrable functions is generated as a vector space by functions $\alpha \to g \circ p(t^{-i}\alpha)$ extended by zero outside t^iA, where g is an integrable function on B. Such a measure and integration is natural to call lifts of the corresponding measure and integration on the base B.

In particular, if $B = \mathbb{A}_k$ for a global field k of positive characteristic, then one has a lift μ_1 of the measure $\mu_{\mathbb{A}_k}$, a lift μ_2 of the counting measure of k (so $\mu_2(a + tA) = 1$), and a lift μ_3 of the measure on $\mathbb{A}_k/k$. If the measures are normalized such that $\mu_{\mathbb{A}_k} = \mu_k \otimes \mu_{\mathbb{A}_k/k}$ (of course, μ_k is a counting measure), then $\mu_1 = \mu_2 \otimes \mu_3$.

Remarks.

1. T. Satoh [S], A.N. Parshin [P6], and M. Kapranov [Kp] suggested two other very different approaches to define a measure on two dimensional fields (notice that Satoh's work was an attempt to use Wiener's measure). The work [P6] suggested to use an ind-pro description for a generalization of Haar measure to two dimensional local fields, see also [Kp]. The works [P6] and [Kp] don't introduce a measure, and deal with distributions and functions; local zeta integrals are not discussed.

After this part had been written, the author was informed by A.N. Parshin that the formula $\mu(t_2^it_1^jO) = q^{-j}X^i$ was briefly discussed in a short message [P5], however, it led to unresolved at that stage paradoxes.

2. In this work we do not need a general theory of analysis on non locally compact abelian groups, since the case of higher dimensional local fields and natural adelic objects composed of them is enough for applications in number theory. Undoubtedly, there is a more general theory of harmonic analysis on non locally compact groups than presented above, see also the next remark.

3. The existence of Haar measure on locally compact abelian groups can be proved in the shortest and most elegant conceptual way by viewing it as

induced from the counting measure on a covering hyperfinite abelian group (e.g. [Gor1]); the same is also true for the Fourier transform [Gor2].

Conceptually, the integration theory of this work is supposed to be induced by integration on hyper locally compact abelian groups (or hyper hyper finite groups). This would also give an extension of this theory to a more general class of non locally compact groups than those of this part.

2. Dimension two local zeta integral

Using the theory of the previous part we explain how to integrate over topological K-groups of higher local fields, define a local zeta integral and discuss its properties and new phenomena of the dimension two situation. Using a covering of a so called topological Milnor K_2^t-group of a two dimensional local field (which is the central object in explicit local class field theory) by the product of the group of units with itself we introduce integrals over the K_2^t-group. For this we use additional maps $\mathfrak{r}, \mathfrak{t}, \mathfrak{o}$ whose meaning can only be finally clarified in constructions of the global part [F4]. In section 17 we define the main object – a higher dimensional local zeta integral associated to a function on the field and a continuous character of the K_2^t-group, followed by concrete examples. Several first properties of the zeta integral, in analogy with the one dimensional case, are discussed and proved. In the case of formal power series over archimedean local fields zeta integrals introduced in section 23 are not really new in comparison to one dimensional integrals; this agrees with the fact that the class field theory for such fields does not really require K_2^t.

14. On $F^\times$ one has the induced from $F \oplus F$ topology with respect to

$$F^\times \longrightarrow F \oplus F, \quad \alpha \mapsto (\alpha, \alpha^{-1}),$$

and the shift-invariant measure

$$\mu_{F^\times} = (1 - q^{-1})^{-1}\, \mu / |\ |.$$

Explicit higher class field theory describes abelian extensions of F via closed subgroups in the topological Milnor K-group $K_2^t(F) = K_2(F)/\Lambda_2(F)$ where $\Lambda_2(F)$ is the intersection of all open neighborhoods of zero in the strongest topology on $K_2(F)$ in which the subtraction in $K_2(F)$ and the map

$$F^\times \times F^\times \longrightarrow K_2(F), \quad (a, b) \mapsto \{a, b\}$$

are sequentially continuous. For more details see [IHLF, sect. 6 part I] and [F3]. Denote by $UK_2^t(F)$ the subgroup generated by symbols $\{u, \alpha\}$, $u \in U, \alpha \in F^\times$.

15. Introduce a module on $K_2^t(F)$:

$$| \;|_2 \colon K_2^t(F) \longrightarrow \mathbb{R}_+^\times, \quad \alpha \mapsto q^{-v(\alpha)},$$

where $v\colon K_2^t(F) \longrightarrow K_0(\mathbb{F}_q) = \mathbb{Z}$ is the composite of two boundary homomorphisms.

Definition. Introduce a subgroup

$$T = \mathcal{O}^\times \times \mathcal{O}^\times = \{(t_1^j u_1, t_1^l u_2) : j, l \in \mathbb{Z}, u_i \in U\}$$

of $F^\times \times F^\times$. The closure of T in $F \times F$ equals $\mathcal{O} \times \mathcal{O}$.

A specific feature of the two dimensional theory is that one needs to use auxiliary maps $\mathfrak{o}, \mathfrak{o}'$ to modify the integral in such a way that later on, in the adelic work in [F4] one gets the right factors for transformed functions and their zeta integrals.

Definition. Introduce a map (morphism of multiplicative structures)

$$\mathfrak{o}' \colon T \longrightarrow F \times F, \quad (t_1^j u_1, t_1^l u_2) \mapsto (t_1^{2j} u_1, t_1^{2l} u_2)$$

and denote by $\mathfrak{o}$ the bijection $\mathfrak{o}'(T) \longrightarrow T$.

For a complex valued continuous function f whose domain includes T and is a subset of $F \times F$ form $f \circ \mathfrak{o} \colon \mathfrak{o}'(T) \longrightarrow \mathbb{C}$, then extend it by continuity to the closure of $\mathfrak{o}'(T)$ in $F \times F$ and by zero outside the closure, denote the result by $f_\mathfrak{o} \colon F \times F \to \mathbb{C}$.

To be able to apply the transform $\mathcal{F}$ introduce an extension $\mathrm{e}(g)$ of a function $g = f_\mathfrak{o} \colon F \times F \longrightarrow \mathbb{C}$ as the continuous extension on $F \times F$ of

$$\mathrm{e}(g)(\alpha_1, \alpha_2) = g(\alpha_1, \alpha_2) + \sum_{1 \leqslant i \leqslant 3} g((\alpha_1, \alpha_2)\, \nu_i), \quad (\alpha_1, \alpha_2) \in F^\times \times F^\times,$$

where $\nu_1 = (t_1^{-1}, t_1^{-1})$, $\nu_2 = (t_1^{-1}, 1)$, $\nu_3 = (1, t_1^{-1})$.

For example, if $f = char_{(t_1^j O, t_1^l O)}$ then $f_\mathfrak{o}(\alpha_1, \alpha_2) = 1$ for $(\alpha_1, \alpha_2) \in F^\times \times F^\times$ only if $\alpha_1 \in t_1^{2k} U, \alpha_2 \in t_1^{2m} U$ for $k \geqslant j, m \geqslant l$, hence $f_\mathfrak{o}$ is not a continuous function; but $\mathrm{e}(f_\mathfrak{o}) = char_{(t_1^{2j} O, t_1^{2l} O)}$ is.

Definition. For a function $f \colon T \longrightarrow \mathbb{C}$ such that $\mathrm{e}(f_\mathfrak{o}) \in R_{F \times F}$ define its transform

$$\widehat{f} = \mathcal{F}(\mathrm{e}(f_\mathfrak{o})) \circ \mathfrak{o}' \colon T \longrightarrow \mathbb{C}.$$

16. Due to the well known useful equality

$$\{1-\alpha, 1-\beta\} = \{\alpha, 1+\alpha\beta/(1-\alpha)\} + \{1-\beta, 1+\alpha\beta/(1-\alpha)\},$$

and a topological argument we have surjective maps ([IHLF, sect. 6 part I]):

$$\mathfrak{r}: T \longrightarrow K_2^t(F), \quad \mathfrak{r}\big((t_1^j, t_1^l)(u_1,u_2)\big) = \min(j,l)\,\{t_1,t_2\} + \{t_1,u_1\} + \{u_2,t_2\},$$
$$\mathfrak{t}: T \longrightarrow K_2^t(F), \quad \mathfrak{t}\big((t_1^j, t_1^l)(u_1,u_2)\big) = (j+l)\,\{t_1,t_2\} + \{t_1,u_1\} + \{u_2,t_2\}.$$

For $(\alpha_1,\alpha_2) \in T$ we have

$$|\mathfrak{t}(\alpha_1,\alpha_2)|_2 = |\alpha_1|\,|\alpha_2| = |(\alpha_1,\alpha_2)|.$$

The homomorphism $\mathfrak{t}$ plays in important role in the study of topological Milnor K-groups of higher local fields ([P3], [F3]). If we denote $H = \{(\alpha,\beta) : \alpha,\beta \in \langle t_1\rangle U, \alpha\beta^{-1} \in U\}$, then $\mathfrak{r}(H) = K_2^t(F)$. Note that if $K_2^t(F)$ is replaced by the usual $K_2(F)$ then none of the previous maps is surjective. The topology of $K_2^t(F)$ is the quotient topology of the sequential saturation of the multiplicative topology on H (cf. [F3], [IHLF, sect. 6 part I]).

The maps $\mathfrak{r},\mathfrak{t}$ depend on the choice of t_1,t_2 but the induced map to $K_2^t(F)/\ker(\partial)$ (see section 18 for the definition) does not depend on the choice of t_2. The homomorphism $\mathfrak{t}$ (which depends on the choice of t_1,t_2) is much more convenient to use for integration on $K_2^t(F)$ than the canonically defined map $F^\times \times F^\times \longrightarrow K_2^t(F)$.

For a function $h: K_2^t(F) \longrightarrow \mathbb{C}$ denote by $h_\mathfrak{r}: F\times F \longrightarrow \mathbb{C}$ ($h_\mathfrak{t}: F\times F \longrightarrow \mathbb{C}$) the composite $h\circ\mathfrak{r}$ (resp. the composite $h\circ\mathfrak{t}$) extended by zero outside T.

17. Let $\chi: K_2^t(F) \longrightarrow \mathbb{C}^\times$ be a continuous homomorphism. Similarly to the one dimensional case one easily proves that $\chi = \chi_0|\ |_2^s$ ($s\in\mathbb{C}$), where χ_0 is a lift of a character of $UK_2^t(F)$ such that $\chi_0(\{t_1,t_2\}) = 1$. The group $V = 1 + t_1 O$ of principal units has the property: every open subgroup contains V^{p^n} for sufficiently large n, c.f. [Z1, Lemma 1.6]. This property and the definitions imply that $|\chi_0(K_2^t(F))| = 1$. Put $s = s(\chi)$.

Definition. For a function f such that $char_{\mathfrak{o}'(T)}\, f_\mathfrak{o} \in R_{F\times F}$ introduce

$$\int_T f\, d\mu_T = (1-q^{-1})^{-2} \int_{\mathfrak{o}'(T)} f_\mathfrak{o}\, d\mu_{F\times F}.$$

Notice that $\int_T f\, d\mu_T = (1-q^{-1})^{-2} \int_{F\times F} char_T\, f\,|\ |\, d\mu_{F\times F}$.

For a function $f: F\times F \longrightarrow \mathbb{C}$ continuous on T and a continuous quasi-character $\chi: K_2^t(F) \longrightarrow \mathbb{C}^\times$ introduce a zeta function

$$\zeta(f,\chi) = \int_T f(\alpha)\,\chi_\mathfrak{t}(\alpha)\,|\mathfrak{t}(\alpha)|_2^{-2}\, d\mu_T(\alpha)$$

as an element of $\mathbb{C}((X))$ if the integral converges.

From the definitions we obtain $\zeta(f,\chi)$

$$= (1-q^{-1})^{-2} \sum_{j,l\in\mathbb{Z}} (q^{-s})^{j+l} \int_{U\times U} f(t_1^j u_1, t_1^l u_2)\, \chi_0\big(\mathfrak{t}(u_1,u_2)\big)\, d\mu(u_1)\, d\mu(u_2).$$

If for fixed j,l the value $f(t_1^j u_1, t_1^l u_2)$ is constant $= f_0(j,l)$, then we get

$$\zeta(f,\chi) = (1-q^{-1})^{-2} \sum_{j,l\in\mathbb{Z}} (q^{-s})^{j+l}\, f_0(j,l) \int_{U\times U} \chi_0\big(\mathfrak{t}(u_1,u_2)\big)\, d\mu(u_1)\, d\mu(u_2).$$

Examples.

1. Let $f = char_{OK_2^t(F)_{\mathfrak{r}}}$ where $OK_2^t(F) = v^{-1}(\{0,1,\dots\})$. So $f_0(j,l) = 1$ if $j,l \geqslant 0$ and $f_0(j,l) = 0$ otherwise. Then

$$\zeta(f, |\ |_2^s) = \left(\frac{1}{1-q^{-s}}\right)^2$$

which converges for $\mathrm{Re}(s) > 0$, and has a single valued meromorphic continuation to the whole complex place. Note that for this specific choice of f the local zeta integral *does not involve* X.

2. More generally, for a function $f \in Q_{F\times F}$ the local zeta integral $\zeta(f,\chi)$ converges for $\mathrm{Re}(s) > 0$ to a rational function in q^s and therefore has a meromorphic continuation to the whole complex plane.

To show this, one can assume that $f(\alpha_1,\alpha_2) = char_{A_1}(\alpha_1) char_{A_2}(\alpha_2)$ with distinguished sets A_1, A_2, each of which is either of type $t_1^l(a + t_1^r O)$, $a \in U$, $r > 0$, or of type $t_1^m O$. For example, if A_1 of the first type and A_2 of the second type, then

$$\zeta(f,\chi) = (1-q^{-1})^{-2} \sum_{j\geqslant m} (q^{-s})^{j+l} \int_{(a+t_1^r O)\,\times U} \chi_0\big(\mathfrak{t}(u_1,u_2)\big)\, d\mu(u_1)\, d\mu(u_2),$$

which for $\mathrm{Re}(s) > 0$ converges to $(1-q^{-1})^{-2}(q^{-s})^{m+l}(1-q^{-s})^{-1}\, c$ where $c = \int_{(a+t_1^r O)\times U} \chi_0\big(\mathfrak{t}(u_1,u_2)\big)\, d\mu(u_1)\, d\mu(u_2)$, and therefore equals to this rational function of q^s on the whole plane.

3. For $f \in Q_{F\times F}$ and $(\alpha_1,\alpha_2) \in T$ define, in analogy with A. Weil's definition in [W1]

$$W(f)(\alpha_1,\alpha_2) = f(\alpha_1,\alpha_2) - f(t_1^{-1}\alpha_1,\alpha_2) - f(\alpha_1, t_1^{-1}\alpha_2) + f(t_1^{-1}\alpha_1, t_1^{-1}\alpha_2).$$

Then from the definitions and the properties of μ we get

$$\zeta(W(f), |\ |_2^s) = (1-q^{-s})^2\, \zeta(f, |\ |_2^s).$$

In particular, let $g = W(f)$, $f|_T = \big(char_{OK_2^t(F)}\big)_{\mathfrak{r}}$. Then

$$\zeta(g, |\ |_2^s) = 1.$$

18. Let $\partial: K_2(F) \longrightarrow K_1(E)$ be the boundary homomorphism. Note that $\partial(\Lambda_2(F)) = 1$. Define

$$\lambda: K_1(E) \longrightarrow K_2^t(F)/\ker(\partial), \quad \alpha \mapsto \{\tilde{\alpha}, t_2\}$$

where $\tilde{\alpha} \in \mathcal{O}$ is a lifting of $\alpha \in E^\times$. Then $\partial(\lambda(\alpha)) = \alpha$.

Let $f: K_2^t(F) \longrightarrow \mathbb{C}$ be a continuous function which factorizes through the quotient $K_2^t(F)/\ker(\partial)$. Let $\chi: K_2^t(F) \longrightarrow \mathbb{C}^\times$ be a weakly ramified continuous quasi-character, i.e. $\chi_0(\ker(\partial)) = 1$, so χ is induced by a (not necessarily unramified) quasi-character of the first residue field E of F. Then using Lemma in section 8 one immediately deduces that

$$\zeta(f_{\mathfrak{r}}, \chi) = \zeta_E(f \circ \lambda, \chi \circ \lambda)^2$$

where ζ_E is the one dimensional zeta integral on E which corresponds to the normalized Haar measure on E.

Similarly, if F is a mixed characteristic field, and K is the algebraic closure of $\mathbb{Q}_p$ in F, then one has a residue map $\mathfrak{d}: K_2^t(F) \longrightarrow K_1(K)$, see e.g. [Ka2, sect. 2], the map $\mathfrak{d}$ is $-$res defined there.

Assume that a prime element of K is a t_2-local parameter of F, then we can identify $F = K\{\{t_1\}\}$. Define

$$\mathfrak{l}: K_1(K) \longrightarrow K_2^t(F), \quad \alpha \mapsto \{t_1, \alpha\},$$

then $\mathfrak{d}(\mathfrak{l}(\alpha) = \alpha$.

Let $f: K_2^t(F) \longrightarrow \mathbb{C}$ and a quasi-character χ have analogous to the above properties but with respect to $\mathfrak{d}$. Then similarly one has

$$\zeta(f_{\mathfrak{r}}, \chi) = \zeta_K(f \circ \mathfrak{l}, \chi \circ \mathfrak{l})^2.$$

Thus, the two dimensional zeta integral for F links zeta integrals for both E and K, local fields of characteristic p and 0.

19. Put $\widehat{\chi} = \chi^{-1}|\ |_2^2$.

PROPOSITION. *Let $g, h \in Q_{F\times F}$ be continuous functions, and let $\chi: K_2^t(F) \longrightarrow \mathbb{C}^\times$ be a continuous quasi-character. For $0 < \operatorname{Re} s(\chi) < 2$ (and therefore for all s) one has a local functional equation*

$$\zeta(g, \chi)\, \zeta(\widehat{h}, \widehat{\chi}) = \zeta(\widehat{g}, \widehat{\chi})\, \zeta(h, \chi).$$

Proof. One proof is similar to Example 2 in the previous section and reduces the verification to the case where both g and h are of the simple form $char_{A_1} char_{A_2}$. We indicate another, similar to dimension one, with one new feature – a factor $k(\beta)$ which corresponds to the transform $\widehat{f}$ involving the map $\mathfrak{o}$.

For $\alpha = (\alpha_1, \alpha_2)$ denote $\alpha^{-1} = (\alpha_1^{-1}, \alpha_2^{-1})$, $\alpha\beta = (\alpha_1\beta_1, \alpha_2\beta_2)$, $|\alpha| = |\alpha_1|\,|\alpha_2|$, $\mu(\alpha) = \mu_{F\times F}(\alpha_1, \alpha_2)$, $\psi(\alpha) = \psi(\alpha_1)\psi(\alpha_2)$. Put

$$k(\beta) = \psi(\beta) + q^{-2}\psi(\nu_1^{-1}\beta) + q^{-1}\psi(\nu_2^{-1}\beta) + q^{-1}\psi(\nu_3^{-1}\beta),$$

ν_i defined in section 15. We will use $|\mathfrak{to}(\alpha)|_2^2 = |\alpha|$ for $\alpha \in \mathfrak{o}'(T)$.

We have

$$\begin{aligned}
\zeta(g,\chi)\,\zeta(\widehat{h},\widehat{\chi}) &= \iint_{T\times T} g(\alpha)\,\widehat{h}(\beta)\,\chi_{\mathfrak{t}}(\alpha\beta^{-1})\,|\beta|^2\,|\alpha|^{-2}\,|\beta|^{-2}\,d\mu_T(\alpha)\,d\mu_T(\beta)\\
&= \int_{F\times F}\int_{F\times F} g_{\mathfrak{o}}(\alpha)\,\mathcal{F}(\mathrm{e}(h_{\mathfrak{o}}))(\beta)\,\chi_{\mathfrak{to}}(\alpha\beta^{-1})\,|\alpha|^{-1}\,d\mu(\alpha)\,d\mu(\beta)\\
&= \iiint_{F^6} g_{\mathfrak{o}}(\alpha)\;\mathrm{e}(h_{\mathfrak{o}})(\gamma)\,\psi(\beta\gamma)\,\chi_{\mathfrak{to}}(\alpha\beta^{-1})\,|\alpha|^{-1}\,d\mu(\alpha)\,d\mu(\beta)\,d\mu(\gamma)\\
&= \iiint_{F^6} g_{\mathfrak{o}}(\alpha)\,h_{\mathfrak{o}}(\gamma)\,\psi(\beta\gamma)\,\chi_{\mathfrak{to}}(\alpha\beta^{-1})\,|\alpha|^{-1}\,d\mu(\alpha)\,d\mu(\beta)\,d\mu(\gamma)\\
&+ \sum_{1\leqslant i\leqslant 3}\iiint_{F^6} g_{\mathfrak{o}}(\alpha)\,h_{\mathfrak{o}}(\gamma)\,\psi(\beta\nu_i^{-1}\gamma)\,\chi_{\mathfrak{to}}(\alpha\beta^{-1})\,|\alpha|^{-1}\,d\mu(\alpha)\,d\mu(\beta)\,d\mu(\nu_i^{-1}\gamma)\\
&= \iiint_{F^6} g_{\mathfrak{o}}(\alpha)\,h_{\mathfrak{o}}(\gamma)\,k(\beta\gamma)\,\chi_{\mathfrak{to}}(\alpha\beta^{-1})\,|\alpha|^{-1}\,d\mu(\alpha)\,d\mu(\beta)\,d\mu(\gamma)\\
&\quad(\beta\to\gamma^{-1}\beta)\\
&= \iiint_{F^6} g_{\mathfrak{o}}(\alpha)\,h_{\mathfrak{o}}(\gamma)\,\chi_{\mathfrak{to}}(\alpha\gamma)\,|\alpha\gamma|^{-1}\,d\mu(\alpha)\,d\mu(\gamma)\,k(\beta)\,\chi_{\mathfrak{to}}(\beta^{-1})\,d\mu(\beta)
\end{aligned}$$

(due to $\mathfrak{to}$ the integrals are actually taken over $\mathfrak{o}'(T)$ where one can apply the Fubini property). The symmetry in α, γ implies the local functional equation.

Remark. Undoubtedly, one can extend this proposition to a larger class of continuous functions $g, h\colon F\times F \longrightarrow \mathbb{C}$.

20. Similar to section 12, we need to take care of zeta integrals which involve normalized measures corresponding to shifted characters. The zeta integral should be more correctly written as $\zeta(f, \chi|\,|_2^s, \mu)$ since it depends on the normalization of the measure μ. Above we used the normalized measure μ such that $\mu(O) = 1$ (and the conductor of ψ is O). More generally, let the conductor of a character ψ' be $t_1^d O$, so $\psi'(\alpha) = \psi_0(t_1^{-d}u\alpha)$ for some unit u, ψ_0 was defined in section 3. Put $\psi(\alpha) = \psi_0(t_1^{-2d}u\alpha)$ and use ψ and $\psi(\alpha_1, \alpha_2)$ for the transform $\mathcal{F}$ of functions in Q_F and $Q_{F\times F}$ as in sections 9–10. Then self dual measures μ on F and $F\times F$ with respect to ψ satisfy $\mu(O) = q^d$, $\mu(O,O) = q^{2d}$.

Let $f = char_{c\{t_1,t_2\}+OK_2^t(F)_{\mathfrak{r}}}$, so $f(t_1^j u_1, t_1^l u_2) = 1$ if $j, l \geqslant c$ and $= 0$ otherwise. Now $\mathcal{F}(\mathrm{e}(f_{\mathfrak{o}}))(t_1^j u_1, t_1^l u_2) = q^{2d-4c}$ if $j, l \geqslant 2d - 2c$ and $= 0$ otherwise; and $\widehat{f}(t_1^j u_1, t_1^l u_2) = q^{2d-4c}$ if $j, l \geqslant d - c$ and $= 0$ otherwise. Notice that $\widehat{\widehat{f}} = f$

on T. We get

$$\zeta(f,|\ |_2^s,\mu) = q^{2d-2cs}\left(\frac{1}{1-q^{-s}}\right)^2,$$

$$\zeta(\widehat{f},|\ |_2^{2-s},\mu) = q^{2d}\,q^{2d-4c}\,q^{-2(d-c)(2-s)}\left(\frac{1}{1-q^{s-2}}\right)^2 = q^{2s(d-c)}\left(\frac{1}{1-q^{s-2}}\right)^2,$$

$$\zeta(f,|\ |_2^s,\mu)\,\zeta(\widehat{f},|\ |_2^{2-s},\mu)^{-1} = q^{-2d(s-1)}\left(\frac{1}{1-q^{-s}}\right)^2\left(\frac{1}{1-q^{s-2}}\right)^{-2}.$$

The local constant for $|\ |_2^s,\mu$ is $q^{-2d(s-1)}$. This calculation will be quite useful in [F4].

21. The previous constructions of the maps $\mathfrak{o}', \mathfrak{r}, \mathfrak{t}, \mathrm{e}$ are suitable for unramified characters, but are not good enough for ramified characters.

Let a weakly ramified character χ of $K_2^t(F)$ be induced by a character $\overline{\chi}$ of the field E with conductor $1+\overline{t_1}^{\,r}O_E$, $r>0$. By analogy with the one dimensional case it seems reasonable to calculate the zeta integral $\zeta(f,\chi)$ for $f\colon T\longrightarrow\mathbb{C}$ which is defined by $(t_1^j u_1, t_1^l u_2)\mapsto 1$ if and only if $j\geqslant 0, l=0, u_2\in 1+t_1^r O$. Using the dimension one calculation in [T] one easily obtains

$$\zeta(f,\chi) = \frac{1}{1-q^{-s}}\,(1-q^{-1})^{-1}\,q^{-r},$$

$$\zeta(\widehat{f},\widehat{\chi}) = \frac{1}{1-q^{s-2}}\,(1-q^{-1})^{-1}\,q^{-(s+1)r/2-\delta}\,\rho_0(\overline{\chi}),$$

where $\rho_0(\overline{\chi})$ is the root number as in [T, 2.5], $\delta=s/2$ if r is odd and $=0$ if r is even.

Notice that unlike the case of unramified characters in section 20, $\widehat{\widehat{f}}$ differs from f on T due to the difference between $\mathrm{e}(\widehat{f}_{\mathfrak{o}})$ and $\mathcal{F}(\mathrm{e}(f_{\mathfrak{o}}))$; so for the current choice of $\mathfrak{o}$ and e as above there is no canonical local constant associated to χ.

This may be related to the well known difficulty of constructing a general higher ramification theory which is still not available. It is expected that one can refine $\mathfrak{o}$, e, $\mathfrak{r},\mathfrak{t}$ to get canonical local constants for ramified characters as well.

22. One can slightly modify the definition of the zeta integral by extending T to the set

$$T' = \{(t_2^m t_1^j u_1, t_2^n t_1^l u_2)\}, \qquad \text{where } j,l\in\mathbb{Z},\ u_i\in U,\ m,n\geqslant 0,\ mn=0.$$

The set T' will be useful in the study of global zeta integrals in [F4].

Introduce a set $\Gamma = \{(t_2^m,1): m\geqslant 0\}\bigcup\{(1,t_2^m): m\geqslant 1\}$, then $T' = T\times\Gamma$. Extend $\mathfrak{r},\mathfrak{t}$ symmetrically to T' by $(m>0)$

$$\mathfrak{r}(t_2^m t_1^j u_1, t_1^l u_2) = l\,\{t_1,t_2\} + \{t_1,u_1\} + \{u_2,t_2\},$$

$$\mathfrak{t}(t_2^m t_1^j u_1, t_1^l u_2) = \{t_1,u_1\} + \{u_2,t_2\}.$$

Extend $\mathfrak{o}'$ to $\mathfrak{o}': T' \longrightarrow F^\times \times F^\times$ by $(t_2^m t_1^j u_1, t_2^n t_1^l u_2) \mapsto (t_2^m t_1^{2j} u_1, t_2^n t_1^{2l} u_2)$ and denote by $\mathfrak{o}$ the inverse bijection $\mathfrak{o}'(T') \longrightarrow T'$.

For a function $f: T' \longrightarrow \mathbb{C}$ define $f_{\mathfrak{o}}: F \times F \longrightarrow \mathbb{C}$ as the composite $f \circ \mathfrak{o}$ extended by zero outside $\mathfrak{o}'(T')$. Introduce

$$\int_{T'} f \, d\mu_{T'} = (1-q^{-1})^{-2} \int_{F\times F} f_{\mathfrak{o}} \, d\mu_{F\times F}.$$

Let a function $f: F \times F \longrightarrow \mathbb{C}$ be continuous on T' and let

$$f(t_2^m \alpha_1, \alpha_2) = f(\alpha_1, t_2^m \alpha_2) = f(\alpha_1, \alpha_2)$$

for every $(\alpha_1, \alpha_2) \in T$, $m > 0$. Then for a continuous quasi-character $\chi: K_2^t(F) \longrightarrow \mathbb{C}^\times$ such that $\chi(UK_2^t(F)) = 1$ we have

$$\int_{T'} f(\alpha) \, \chi_{\mathfrak{t}}(\alpha) \, |\mathfrak{t}(\alpha)|_2^{-2} \, d\mu_{T'}(\alpha) = \zeta(f, \chi).$$

This follows from the following observation: the function $f(\alpha)\, \chi_{\mathfrak{t}}(\alpha) |\mathfrak{t}(\alpha)|_2^{-2}$ on the set $(t_2^m t_1^j u_1, t_1^l u_2)$, $m > 0$, depends on $t_1^l u_2$ only, and therefore one can write the integral $\int_{T'} f(\alpha)\, \chi_{\mathfrak{t}}(\alpha) |\mathfrak{t}(\alpha)|_2^{-2} \, d\mu_{T'}(\alpha)$ as the sum of $\zeta(f,\chi)$ and the sum of the product of two integrals one of which is the integral of a constant function over $t_2^m \mathcal{O} \setminus t_2^{m+1}\mathcal{O}$, and hence is zero by Example 8.

23. We describe elements of the theory for $K_2^t(K((t)))$ where K is an archimedean local field. Class field theory for such fields is described in [P2] and in [KS]. In this case class field theory does not really match nicely the structure of the K_2^t-group of the field, and this is reflected in the constructions of this section.

Introduce the sequential topology on $L = K((t))$ as in section 1. Define

$$K_2^t(L) = K_2(L)/\Lambda_2(L), \qquad \Lambda_2(L) = \Lambda_2'(L) + \{K^\times, 1+tK[[t]]\} + \{t, 1+tK[[t]]\},$$

$\Lambda_2'(L)$ is the intersection of all neighborhoods of zero in the strongest topology on $K_2(L)$ in which the subtraction in $K_2(L)$ and the map

$$L^\times \times L^\times \longrightarrow K_2(L), \quad (a,b) \mapsto \{a,b\}$$

are sequentially continuous. Then $K_2^t(L)$ is generated by $\{a,t\}$, $a \in K^\times$ and $\{-1,-1\}$ (which is zero if $\sqrt{-1} \in K$). We have $\partial(\Lambda_2(L)) = 1$ where ∂ is the boundary map. ∂ induces $K_2^t(L) \longrightarrow K_1(K)$.

Introduce a module on $K_2^t(L)$: $|\alpha|_2 = |\partial(\alpha)|_K$.

A continuous homomorphism $\chi: K_2^t(L) \longrightarrow \mathbb{C}^\times$ can be written as $\chi = \chi_0 | \ |_2^s$ ($s \in \mathbb{C}$), where $\chi_0(\{K^\times, t_2\}) = 1$ and $|\chi_0(K_2^t(L))| = 1$ (so χ_0 is determined by its value on $\{-1,-1\}$; this symbol corresponds via class field theory to the totally ramified extension $K((t^{1/2}))/K((t)))$.

Let

$$\mathfrak{r}\colon K^\times \longrightarrow K_2^t(L), \quad \alpha \mapsto \{\alpha, t\}$$
$$\mathfrak{t}\colon T = K^\times \times K^\times \longrightarrow K_2^t(L), \quad (\alpha,\beta) \mapsto \{\alpha\beta, t\}$$

(totally ramified characters with respect to L/K are ignored).

For a function $f\colon L \times L \longrightarrow \mathbb{C}$ continuous on T and rapidly decaying at $(\pm\infty, \pm\infty)$ and a continuous quasi-character $\chi\colon K_2^t(L) \longrightarrow \mathbb{C}^\times$ introduce a zeta function

$$\zeta(f,\chi) = \int_{K\times K} f(\alpha,\beta)\,\chi_{\mathfrak{t}}(\alpha,\beta)\,\frac{d\mu(\alpha,\beta)}{|\alpha|_K\,|\beta|_K}.$$

Define $\lambda\colon K_1(K) \longrightarrow K_2^t(L)/\ker(\partial)$ by $\alpha \mapsto \{\tilde{\alpha}, t\}$. Let $f\colon K_2^t(L) \longrightarrow \mathbb{C}$ factorize through $K_2^t(L)/\ker(\partial)$. Let $\chi\colon K_2^t(L) \longrightarrow \mathbb{C}^\times$ be a continuous quasi-character such that $\chi_0(\ker(\partial)) = 1$ (i.e. χ is an unramified character with respect to L/K). Then

$$\zeta(g,\chi) = \zeta_1(f\circ\lambda, \chi\circ\lambda)^2,$$

where $g(\alpha,\beta) = f_{\mathfrak{r}}(\alpha)\,f_{\mathfrak{r}}(\beta)$, and ζ_1 is the one dimensional zeta integral.

Remark. The well known technique in dimension one which extends the local integrals associated to representations of the multiplicative group to representations of algebraic groups is likely to give a similar extension of this work.

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