Coleman Power Series for K_2 and p-Adic Zeta Functions of Modular Forms

dedicated to Professor Kazuya Kato

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Abstract. For a usual local field of mixed characteristic $(0, p)$, we have the theory of Coleman power series [Co]. By applying this theory to the norm compatible system of cyclotomic elements, we obtain the p-adic Riemann zeta function of Kubota-Leopoldt [KL]. This application is very important in cyclotomic Iwasawa theory.

In [Fu1], the author defined and studied Coleman power series for K_2 for certain class of local fields. The aim of this paper is following the analogy with the above classical case, to obtain p-adic zeta functions of various cusp forms (both in one variable attached to cusp forms, and in two variables attached to ordinary families of cusp forms) by Amice-Vélu, Vishik, Greenberg-Stevens, and Kitagawa,... by applying the K_2 Coleman power series to the norm compatible system of Beilinson elements defined by Kato [Ka2] in the projective limit of K_2 of modular curves.

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Contents

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1. Introduction

1.1. Let p be any prime number. For a complete discrete valuation field H of mixed characteristic $(0,p)$, with perfect residue field, we have the theory of Coleman power series, as reviewed in section 2. One of the important applications of this theory is the construction of the p-adic Riemann zeta function of Kubota and Leopoldt [KL], by applying the theory to the norm compatible system of cyclotomic elements.
In the paper [Fu1], we have obtained "K_2-version of Coleman power series" for a certain class of local fields. Following the analogy with the case of the usual Coleman power series above, the aim of the present paper is to show that by applying the theory of K_2 Coleman power series to the norm compatible system of Beilinson elements in the projective limit of K_2 of modular curves defined by Kato [Ka2], we obtain p-adic zeta functions of various cusp forms, both in one variable attached to cusp forms (cf.Amice-Vélu [AV], Vishik [Vi]), and two variables attached to universal family of ordinary cusp forms (cf. Greenberg-Stevens [GS], Kitagawa [Ki]).

1.2. We describe our result briefly reviewing the classical result on the p-adic Riemann zeta function.
Let denote by $\zeta_{p^n} \in \overline{\mathbb{Q}_p}$ a primitive p^n-th root of unity and assume $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for all $n \geq 1$. We write $Q(\mathbb{Z}_p[[G_\infty]])$ for the total quotient ring of the completed group ring $\mathbb{Z}_p[[G_\infty]] = \varprojlim_n \mathbb{Z}_p[(\mathbb{Z}/p^n\mathbb{Z})^\times]$ and $G_\infty = \mathbb{Z}_p^\times$ is regarded as the Galois group $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p)$ associated to the cyclotomic p extension of $\mathbb{Q}_p$ via the cyclotomic character.
Iwasawa [Iw] discovered a relationship between the norm compatibles system of cyclotomic elements $(1-\zeta_{p^n})_n \in \varprojlim_n \mathbb{Q}_p(\zeta_{p^n})^\times$ and the p-adic Riemann zeta function $\zeta_{p\text{-adic}} \in Q(\mathbb{Z}_p[[G_\infty]])$ of Kubota and Leopoldt [KL]. The relation of these two appears in the theory of the usual Coleman power series as follows. The theory of Coleman power series for the multiplicative group induces a map $\mathcal{C}$ and $\mathcal{C}$ sends $(1-\zeta_{p^n})_n \in \varprojlim_n \mathbb{Q}_p(\zeta_{p^n})^\times$ to $\zeta_{p\text{-adic}} \in Q(\mathbb{Z}_p[[G_\infty]])$:

$$\mathcal{C} : \varprojlim_n \mathbb{Q}_p(\zeta_{p^n})^\times \xrightarrow{\text{via Coleman power series}} Q(\mathbb{Z}_p[[G_\infty]]),$$

$$\mathcal{C}((1-\zeta_{p^n})_n) = \zeta_{p\text{-adic}}.$$

The purpose of this paper is, by pursuing the analogy with this work, to obtain p-adic zeta functions in one variable attached to cusp forms, and in two variables attached to ordinary families of modular forms, whose existences are already known (for one-variable zeta functions cf. Amice-Vélu [AV], Vishik [Vi],..., and

two-variable zeta functions associated to ordinary families of cusp forms cf. Greenberg-Stevens [GS], Kitagawa [Ki],...).
Let

$$\mathsf{H} = (\varprojlim_n (\mathbb{Z}/p^n\mathbb{Z}[[q]][\frac{1}{q}]))[\frac{1}{p}],$$

where q is an indeterminate. This is a complete discrete valuation field of mixed characteristic $(0,p)$ whose residue field k is an imperfect field satisfying $[\mathsf{k} : \mathsf{k}^p] = p$. As reviewed in section 2, for H, we have a theory of K_2 Coleman power series [Fu1].
Let N be a positive integer which is prime to p. Let us denote by $Y(Np^n, p^n)$ the modular curve corresponding to the subgroup $\Gamma(Np^n, p^n) = \{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z})\ ;\ \alpha \equiv 1(Np^n), \beta \equiv 0(Np^n), \gamma \equiv 0(p^n), \delta \equiv 1(p^n)\}$ whose total constant field is $\mathbb{Q}(\zeta_{p^n})$.
In his paper [Ka2], Kato discovered a norm compatible system of Beilinson elements belonging to $\varprojlim_n K_2(Y(Np^n, p^n))$. We study the image of this Beilinson-Kato system under a map $\mathcal{C}_N$ below which is defined by using our K_2 Coleman power series following the analogy with the classical map $\mathcal{C}$. We call this image a universal zeta modular form (see section 4).

$$\mathcal{C}_N : \varprojlim_n K_2(Y(Np^n, p^n)) \xrightarrow{\text{via } K_2 \text{ Coleman power series}} Q(O_{\mathsf{H}}[[G_\infty^{(1)} \times G_\infty^{(2)}]])$$

$$\mathcal{C}_N(\text{Beilinson-Kato system}) = \text{the universal zeta modular form.}$$

(Precisely we will define the universal zeta modular form as an element obtained from this image with a suitable modification, cf. section 4.) Here $G_\infty^{(1)} \cong G_\infty^{(2)} \cong G_\infty$, $G_\infty^{(1)}$ is a group of diamond operators acting on the space of p-adic modular forms (refer to sections 3 and 4), and $G_\infty^{(2)} = \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p)$. Further $Q(O_{\mathsf{H}}[[G_\infty^{(1)} \times G_\infty^{(2)}]])$ denotes the total quotient ring of the completed group ring $O_{\mathsf{H}}[[G_\infty^{(1)} \times G_\infty^{(2)}]]$.
Theorem 6.2 which is one of our two main results will state that the above universal zeta modular form produces p-adic zeta functions (in one variable) of eigen cusp forms which are not necessarily ordinary.
Theorem 7.3 which is the other main theorem asserts roughly the following.

Theorem 1.3 (cf. Theorem 7.3). *We assume $p \geq 5$. Let $\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}$ be the ordinary part of the ring of Hecke operators of level Np^∞ acting on the space of the p-adic cusp forms of level Np^∞ (cf. section 3). The universal zeta modular form above produces, by the method in section 7, a p-adic zeta function in two variables*

$$L_{p\text{-}adic}^{\mathrm{ord,univ}} \in (\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}/\mathcal{I}_{Np^\infty}^{\mathrm{ord}})[[G_\infty^{(2)}]][\frac{1}{a}]$$

which displays property (1.1) below. Here $\mathcal{I}_{Np^\infty}^{\mathrm{ord}} \subset \mathfrak{h}_{Np^\infty}^{\mathrm{ord}}$ is a certain ideal (see3.7 in section 3), and $a \in \mathfrak{h}_{Np^\infty}^{\mathrm{ord}}[[G_\infty^{(2)}]]$ is a certain non-zerodivisor.

Let f be an ordinary p-stabilized newform of tame conductor N (for the definition of an ordinary p-stabilized newform, see 7.2.1 in section 7) of weight $k \geq 2$. Attached to f, we have a ring homomorphism κ_f :

$$\kappa_f : \mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty} \longrightarrow \overline{\mathbb{Z}_p} \ ; \ T(n) \mapsto a_n(f) \quad (n \geq 1),$$

with $a_n(f)$ such that $T(n)f = a_n(f)f$. Suppose κ_f satisfies some "suitable" condition. Then κ_f induces a homomorphism which is also denoted by κ_f :

$$\kappa_f : (\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty})[[G_\infty]][\frac{1}{a}] \longrightarrow Q(\overline{\mathbb{Z}_p}[[G^{(2)}_\infty]]),$$

and concerning the image $L^{\mathrm{ord,univ}}_{p\text{-}adic}(\kappa_f)$ of $L^{\mathrm{ord,univ}}_{p\text{-}adic}$ under this homomorphism κ_f, we have

$$L^{\mathrm{ord,univ}}_{p\text{-}adic}(\kappa_f) = p\text{-}adic\ zeta\ function\ of\ f \in (O_M[[G^{(2)}_\infty]]) \otimes_{O_M} M. \tag{1.1}$$

Here M is the finite extension $\mathbb{Q}_p(a_n(f); n \geq 1)$ of $\mathbb{Q}_p$.

For the precise statement, see Theorem 7.3 in section 7.

The above $L^{\mathrm{ord,univ}}_{p\text{-adic}}$ is essentially the two-variable p-adic zeta function associated to ordinary families of cusp forms which has been already given by Greenberg-Stevens [GS], Kitagawa [Ki],..., by another method. The significance of our p-adic zeta function is that the coefficients in the p-adic zeta function belong to the ring of Hecke operators as above. Hence our $L^{\mathrm{ord,univ}}_{p\text{-adic}}$ is a p-adic zeta function associated with the universal family of ordinary cusp forms. By another method, Ochiai ([Oc]) has also constructed this kind of two-variable p-adic zeta function.
The author found that Panchishkin [Pa1], [Pa2] gave a new way of the construction of p-adic zeta functions of modular forms by using something similar to our universal zeta modular form at almost the same time as the author gave talks in the conferences in the autumn of 2000 as described in the proceedings [Fu2], [Fu3] in Japanese. Our aim is to obtain p-adic zeta functions of modular forms by applying K_2 Coleman power series to the norm compatible system of Beilinson elements in K_2 of modular curves.

1.4. The organization of this paper is as follows.
In section 2, we review the theory of Coleman power series both in the classical case and in the case for K_2 [Fu1].
In section 3, we review the theory of p-adic modular forms (cf. [Hi1]).
In section 4, we define and study a "universal zeta modular form" which is obtained from the image of Beilinson-Kato system under $\mathcal{C}_N$ appearing in 1.2. In our construction, p-adic properties of p-adic zeta functions are deduced from the p-adic properties of the universal zeta modular form and the relation between the universal zeta modular form and special values of zeta functions of modular forms.
In section 5, we review the theory of p-adic zeta functions of modular forms.

In section 6, we prove our theorem (Theorem 6.2) on the construction of one-variable p-adic zeta functions of eigen cusp forms which are not necessarily ordinary.
In section 7, we prove our theorem (Theorem 7.3) on the construction of p-adic zeta functions in two variables, which are attached to universal families of ordinary cusp forms.

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In this paper, for a complete discrete valuation field L, O_L denotes the ring of integers of L.
For a ring R, $Q(R)$ denotes the total quotient ring of R.
We also fix once and for all an embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}_p}$.

2. Review of Coleman power series for K_2

In this section we give a brief review of the theory of Coleman power series both in the usual case (in 2.1 – 2.2) and our K_2-version case (in 2.3 – 2.5).

2.1. We review the classical case of the usual Coleman power series. The existence of Coleman power series were discovered by Coates and Wiles [CW] and almost immediately, Coleman [Co] generalized their approach by an alternative method. The theory of Coleman power series has been obtained for general Lubin-Tate groups, but here we review the theory only for the formal multiplicative group.
Let H be a complete discrete valuation field of mixed characteristic $(0,p)$ with perfect residue field k. We assume that H is absolutely unramified, i.e. p is a prime element of O_H. We denote by $O_H[[\varepsilon-1]] = \varprojlim_n O_H[\varepsilon^{\pm 1}]/(\varepsilon-1)^n$ the coordinate ring of the formal completion of the multiplicative group over O_H, and by $O_H((\varepsilon-1))$ the Laurent series ring $O_H[[\varepsilon-1]][1/(\varepsilon-1)]$. Let σ denote the Frobenius automorphism of O_H. We extend σ to an endomorphism of $O_H((\varepsilon-1))$ by putting $\sigma(\varepsilon)=\varepsilon$. We define a ring homomorphism

$$\varphi : O_H((\varepsilon-1)) \longrightarrow O_H[[\varepsilon-1]][\frac{1}{\varepsilon^p-1}]$$

by $\varphi(f)(\varepsilon) = (\sigma f)(\varepsilon^p)$, and

$$\mathrm{N} : O_H((\varepsilon-1))^\times \longrightarrow O_H((\varepsilon-1))^\times$$

to be the norm operator induced by the homomorphism φ. We write $(O_H((\varepsilon-1))^\times)^{\mathrm{N}=1}$ for the group of all units f in $O_H((\varepsilon-1))$ which satisfy $\mathrm{N}(f)=f$. Now let ζ_{p^n} denote a primitive p^n-th root of unity, and assume $\zeta_{p^{n+1}}^p = \zeta_{p^n}$

for all $n \geq 1$. Put $H_n = H(\zeta_{p^n})$. The aim in this case is to study $\varprojlim_n H_n^\times$, where the projective limit is taken with respect to the norm maps in the tower of fields H_n $(n = 1, 2, 3, \dots)$.

THEOREM 2.2 (Coleman [Co]). *We have an isomorphism*

$$\Psi : (O_H((\varepsilon - 1))^\times)^{\mathrm{N}=1} \xrightarrow{\cong} \varprojlim_n H_n^\times$$

given by $\Psi(f(\varepsilon)) = ((\sigma^{-n} f)(\zeta_{p^n}))_{n=1,2,3,\dots}$.

2.3. Now we review our case of K_2-version of Theorem 2.2. (See [Fu1] for more details.)
Let H be a complete discrete valuation field of characteristic 0, whose residue field k is an imperfect field of characteristic p satisfying $[\mathsf{k} : \mathsf{k}^p] = p$. We assume that H is absolutely unramified. We fix once and for all a p-base b of k, and a lifting q of b to H (all of our subsequent constructions depend on these choices). We define $\sigma : O_{\mathsf{H}} \to O_{\mathsf{H}}$ to be the unique ring homomorphism satisfying $\sigma(q) = q^p$, and the action of σ on k is given by raising to the p-th power. For simplicity, let us write

$$S = O_{\mathsf{H}}[[\varepsilon - 1]], \quad S' = O_{\mathsf{H}}((\varepsilon - 1)) = S[\frac{1}{\varepsilon - 1}].$$

We extend σ to an endomorphism of S' by putting $\sigma(\varepsilon) = \varepsilon$. We then define a ring homomorphism

$$\varphi : S' \longrightarrow S[\frac{1}{\varepsilon^p - 1}]$$

by $\varphi(f)(\varepsilon) = (\sigma f)(\varepsilon^p)$. For any ring A, let $K_2(A)$ denote Quillen's K_2 group of A ([Qu]). Since $S[1/(\varepsilon^p - 1)]$ is a free S'-module of rank p^2 via φ, we have the K_2 norm map (see [Qu], §4, Transfer maps)

$$K_2(S[\frac{1}{\varepsilon^p - 1}]) \longrightarrow K_2(S').$$

The composition of this with

$$K_2(S') \longrightarrow K_2(S[\frac{1}{\varepsilon^p - 1}])$$

induced by the inclusion map $S' \hookrightarrow S[1/(\varepsilon^p - 1)]$, gives rise to a K_2 norm map

$$\mathrm{N} : K_2(S') \longrightarrow K_2(S').$$

We consider the following tower of fields above H. We take a p^n-th root q^{1/p^n} of q in a fixed algebraic closure $\overline{\mathsf{H}}$ of H and assume that $(q^{1/p^{n+1}})^p = q^{1/p^n}$ for all $n \geq 1$. We define

$$\mathsf{H}_n = \mathsf{H}(\zeta_{p^n}, q^{1/p^n}).$$

Moreover, we define a ring homomorphism

$$\theta_n : O_{\mathsf{H}} \longrightarrow O_{\mathsf{H}(q^{1/p^n})}$$

by specifying that $\theta_n(q) = q^{1/p^n}$, and that the induced map $\mathsf{k} \longrightarrow \mathsf{k}(b^{1/p^n})$ on the residue fields is the isomorphism $x \mapsto x^{1/p^n}$. For $n \geq 1$, θ_n induces a ring homomorphism

$$h_n : S' \longrightarrow \mathsf{H}_n$$

given by $h_n(\sum_{m=-r}^{\infty} a_m(\varepsilon - 1)^m) = \sum_{m=-r}^{\infty} \theta_n(a_m)(\zeta_{p^n} - 1)^m$. Thus we obtain a map

$$\Psi_n : K_2(S') \longrightarrow K_2(\mathsf{H}_n) \quad (n \geq 1)$$

which is induced by h_n. In order to state our theorem, we need to introduce certain completions $\hat{K}_2$ of our K_2-groups (see 2.5 below for the definition). All of the above homomorphisms give rise to corresponding maps between the completed K_2-groups, which we can check easily from the definition of the completions in 2.5 below and which we denote by the same symbol. We also write $\hat{K}_2(S')^{\mathrm{N}=1}$ for the subgroup of elements f in $\hat{K}_2(S')$ which satisfy $\mathrm{N}(f) = f$.

Instead of $\varprojlim_n H_n^\times$, we study the projective limit $\varprojlim_n \hat{K}_2(\mathsf{H}_n)$ with respect to the norm maps in the tower of fields H_n $(n = 1, 2, 3, \dots)$.

Theorem 2.4 ([Fu1], Theorem 1.5). *We have an isomorphism*

$$\Psi : \hat{K}_2(S')^{\mathrm{N}=1} \xrightarrow[\cong]{} \varprojlim_n \hat{K}_2(\mathsf{H}_n)$$

given by $\Psi(f) = (\Psi_n(f) : n = 1, 2, 3, \dots)$.

2.5. We describe the completions of the K_2 groups appearing in Theorem 2.4. We introduce the completions $\hat{K}_2(A)$ in the following two cases by which the completions in Theorem 2.4 follows.

(i) $A = S'$.

(ii)A is a complete discrete valuation field L.

Let $r \geq 1$.

In the case of (i), let $U^{(r)} = 1 + (p, (\varepsilon - 1))^r S$, a subgroup of $S^\times$, where $(p, (\varepsilon - 1))$ is the ideal of S generated by the elements in ( ).

In the case of (ii), let $U^{(r)} = U_L^{(r)}$, where $U_L^{(r)}$ is the r-th unit group of L, i.e. $1 + m_L^r \subset O_L^\times$ for the maximal ideal m_L of L.

We define a subgroup $\mathcal{U}^{(r)} K_2(A)$ of $K_2(A)$ for a ring A of the type (i) or (ii), as the one which is generated by $\{a, A^\times\}$ for all $a \in U^{(r)} \subset A^\times$, and define

$$\hat{K}_2(A) = \varprojlim_r K_2(A)/\mathcal{U}^{(r)} K_2(A).$$

3. Review of p-adic modular forms

In this section, we briefly review the necessary facts for us on the theory of p-adic modular forms. We follow Hida [Hi1] to which we refer for more details. (See also Katz [Katz1], [Katz2], etc.)

3.1. Let L be a finite extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}_p}$, and we take a finite extension L_0 of $\mathbb{Q}$ which is dense in L under the p-adic topology. For $k \geq 0$ and for $M \geq 1$, let $M_k(X_1(M); L_0)$ be the space of modular forms for $\Gamma_1(M)$ of weight k with Fourier coefficients in L_0. We define

$$M_k(X_1(M); L) = M_k(X_1(M); L_0) \otimes_{L_0} L.$$

Similarly we define the space of cusp forms $S_k(X_1(M); L_0)$ and $S_k(X_1(M); L)$. As in [Hi1], §1, it has been known that the spaces $M_k(X_1(M); L)$, $S_k(X_1(M); L)$ are independent of the choice of a subfield L_0 in the evident sense.

Now for $j \geq 0$ we put

$$M^j(X_1(M); L) = \bigoplus_{k=0}^{j} M_k(X_1(M); L),$$

$$S^j(X_1(M); L) = \bigoplus_{k=1}^{j} S_k(X_1(M); L),$$

which are embedded in $L[[q]]$ via the summation of q-expansions, and

$$M^j(X_1(M); O_L) = M^j(X_1(M); L) \cap O_L[[q]],$$

$$S^j(X_1(M); O_L) = S^j(X_1(M); L) \cap O_L[[q]].$$

We define $\overline{M}(X_1(M); O_L)$ as the closure of $\bigcup_{j \geq 1} M^j(X_1(M); O_L)$ in $O_L[[q]]$ for the p-adic topology, and $\overline{S}(X_1(M); O_L)$ to be the closure of $\bigcup_{j \geq 1} S^j(X_1(M); O_L)$ in $O_L[[q]]$ for the p-adic topology.

For an integer $N \geq 1$ which is prime to p, it has been proven that $\overline{M}(X_1(Np^t); O_L)$ and $\overline{S}(X_1(Np^t); O_L)$ are independent of the choice of $t \geq 1$, as in Hida [Hi1], §1, Cor. 1.2 (i), and (1.19a), respectively. For simplicity we put

$$\overline{M}_{Np^\infty} = \overline{M}(X_1(Np^t); \mathbb{Z}_p), \quad \overline{S}_{Np^\infty} = \overline{S}(X_1(Np^t); \mathbb{Z}_p)$$

for any $t \geq 1$.

We introduced the above notation in a general situation for our later use, however in the rest of this section, we always take $L = \mathbb{Q}_p$.

3.2. We review the definition of the rings of Hecke operators $\mathcal{H}_{Np^\infty}$ and $\mathfrak{h}_{Np^\infty}$ acting on $\overline{M}_{Np^\infty}$ and $\overline{S}_{Np^\infty}$, respectively.

For $t \geq 1$ and $j \geq 1$, let $\mathcal{H}^j(X_1(Np^t); \mathbb{Z}_p)$ (resp. $\mathfrak{h}^j(X_1(Np^t); \mathbb{Z}_p)$) be the $\mathbb{Z}_p$-subalgebra of $\mathbb{Z}_p$-endomorphism ring of $M^j(X_1(Np^t); \mathbb{Z}_p)$ (resp. $S^j(X_1(Np^t); \mathbb{Z}_p)$) generated over $\mathbb{Z}_p$ by $T(n)$ ($n \geq 1$).

We put

$$\mathcal{H}(X_1(Np^t); \mathbb{Z}_p) = \varprojlim_j \mathcal{H}^j(X_1(Np^t); \mathbb{Z}_p),$$

$$\mathfrak{h}(X_1(Np^t); \mathbb{Z}_p) = \varprojlim_j \mathfrak{h}^j(X_1(Np^t); \mathbb{Z}_p),$$

where the inverse limits are taken by natural homomorphisms given by the restriction of operators.
As in Hida [Hi1], §1, (1.15a) and (1.19a), respectively, $\mathcal{H}(X_1(Np^t);\mathbb{Z}_p)$ and $\mathfrak{h}(X_1(Np^t);\mathbb{Z}_p)$ do not depend on $t \geq 1$. For simplicity we put

$$\mathcal{H}_{Np^\infty} = \mathcal{H}(X_1(Np^t);\mathbb{Z}_p), \quad \mathfrak{h}_{Np^\infty} = \mathfrak{h}(X_1(Np^t);\mathbb{Z}_p)$$

for any $t \geq 1$. The rings $\mathcal{H}_{Np^\infty}$ and $\mathfrak{h}_{Np^\infty}$ act on $\overline{M}_{Np^\infty}$ and $\overline{S}_{Np^\infty}$, respectively. The ring $\mathfrak{h}_{Np^\infty}$ is, in fact, a quotient of $\mathcal{H}_{Np^\infty}$ by the annihilator in $\mathcal{H}_{Np^\infty}$ of $\overline{S}_{Np^\infty}$.

3.3. By the action of $\mathcal{H}_{Np^\infty}$ on $\overline{M}_{Np^\infty}$ in 3.2, we have a canonical map

$$\mathfrak{i} : \overline{M}_{Np^\infty} \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{H}_{Np^\infty}, \mathbb{Z}_p) \ ; \ f \mapsto (T(n) \mapsto a_1(T(n)f))$$

$(f \in \overline{M}_{Np^\infty})$ where $a_1(T(n)f)$ is the coefficient of q in the q-expansion of $T(n)f \in \overline{M}_{Np^\infty}$.
This map will play an important role later in the construction of our two-variable p-adic zeta function $L^{\mathrm{ord,univ}}_{p\text{-adic}}$.

3.4. For $j \geq 1$ let $e^j = \lim_{n \longrightarrow \infty} T(p)^{n!}$ in $\mathcal{H}^j(X_1(Np^t);\mathbb{Z}_p)$ or $\mathfrak{h}^j(X_1(Np^t);\mathbb{Z}_p)$, and $e = \varprojlim_j e^j$. Then $e^2 = e$.
We denote by $\mathcal{H}^{\mathrm{ord}}_{Np^\infty}$ the ordinary part $e \cdot \mathcal{H}_{Np^\infty}$ of $\mathcal{H}_{Np^\infty}$ and by $\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}$ the ordinary part $e \cdot \mathfrak{h}_{Np^\infty}$ of $\mathfrak{h}_{Np^\infty}$.
Let $P^{\mathrm{ord}}_{Np^\infty} \subset \mathcal{H}^{\mathrm{ord}}_{Np^\infty}$ (resp. $p^{\mathrm{ord}}_{Np^\infty} \subset \mathfrak{h}^{\mathrm{ord}}_{Np^\infty}$) be the annihilator of the old forms (resp. old cusp forms) of level $N'p^t$ for all N' such that $N'|N$ and $N' < N$. For the precise definition of $P^{\mathrm{ord}}_{Np^\infty}$ (resp. $p^{\mathrm{ord}}_{Np^\infty}$), see [Hi1], §3. In the case $N = 1$, we have

$$P^{\mathrm{ord}}_{p^\infty} = \mathcal{H}^{\mathrm{ord}}_{p^\infty}, \quad p^{\mathrm{ord}}_{p^\infty} = \mathfrak{h}^{\mathrm{ord}}_{p^\infty}.$$

On $P^{\mathrm{ord}}_{Np^\infty}$ and $p^{\mathrm{ord}}_{Np^\infty}$, Hida showed Proposition 3.6 below which is important for us. Preceding it, we set up notation.

3.5. We define a group $G^{(1)}_\infty$ which is endowed with an isomorphism to $\mathbb{Z}_p^\times$ and which acts on the space $\overline{M}_{Np^\infty}$ in the following way.
Firstly for $x \in (\mathbb{Z}/Np^t\mathbb{Z})^\times$ we denote by $\langle x \rangle$ the endomorphism $\begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}^*$ on $M_k(X_1(Np^t);\mathbb{Q})$ induced by the action of $\begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix} \in GL_2(\mathbb{Z}/Np^t\mathbb{Z})$ on $X(Np^t, Np^t)$. (The action of $GL_2(\mathbb{Z}/Np^t\mathbb{Z})$ on $X(Np^t, Np^t)$ induces an endomorphism on $M_k(X_1(Np^t);\mathbb{Q})$ by the fact that $M_k(X_1(Np^t);\mathbb{Q})$ may be regarded as the fixed part of $M_k(X(Np^t, Np^t))$ by the group $\{\begin{pmatrix} u & v \\ w & x \end{pmatrix} \in GL_2(\mathbb{Z}/Np^t\mathbb{Z}) \ ; \ u \equiv 1(Np^t), w \equiv 0(Np^t)\}$, where $M_k(X(Np^t, Np^t))$ is the space of modular forms on $X(Np^t, Np^t)$ of weight k as, for example, in [Ka2], §3 (3.3.1), and §4.)

We also use the same notation $\langle x\rangle$ for the endomorphism on $M_k(X_1(Np^t);\mathbb{Q}_p)$ induced by the above.
For $a \in \mathbb{Z}_p^\times$, let $g_a^{(1)} \in G_\infty^{(1)}$ denote the element corresponding to a under the given isomorphism. For $f = \sum_k f_k \in \bigcup_j M^j(X_1(Np^t);\mathbb{Q}_p)$ with $f_k \in M_k(X_1(Np^t);\mathbb{Q}_p)$ and $t \geq 1$, we define the action of $g_a^{(1)} \in G_\infty^{(1)}$ as

$$g_a^{(1)} \cdot f = \sum_k a^{k-2}\langle a'\rangle f_k,$$

where $a' \in (\mathbb{Z}/Np^t\mathbb{Z})^\times$ is the element such that $a' \equiv a(p^t)$ and $a' \equiv 1(N)$. This action of $G_\infty^{(1)}$ may be extended to $\overline{M}_{Np^\infty}$.
We remark that the relation between this action of $G_\infty^{(1)}$ and the action of $\mathbb{Z}_p^\times$ in Hida [Hi1], §3, (3.1) is

$$f \mid a = a^2 g_a^{(1)} \cdot f,$$

where $f \mid a$ denotes the image of f under the action of a in the meaning of Hida.
We put $\Lambda = \mathbb{Z}_p[[G_\infty^{(1)}]]$. By the above action of $G_\infty^{(1)}$ on $\overline{M}_{Np^\infty}$, we have a ring homomorphism

$$\Lambda \longrightarrow \mathcal{H}_{Np^\infty}.$$

We see that via this homomorphism $\mathfrak{h}_{Np^\infty}$, $\mathcal{H}_{Np^\infty}^{\mathrm{ord}}$, and $\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}$ become also Λ-algebras, and $P_{Np^\infty}^{\mathrm{ord}}$, and $p_{Np^\infty}^{\mathrm{ord}}$ are Λ-modules.

PROPOSITION 3.6 (Hida [Hi1] Corollary 3.3). *We assume $p \geq 5$.*
(1) *The rings $\mathcal{H}_{Np^\infty}^{\mathrm{ord}}$ and $\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}$ are finitely generated projective modules over Λ.*
(2) *The ideal $P_{Np^\infty}^{\mathrm{ord}}$ (resp. $p_{Np^\infty}^{\mathrm{ord}}$) is a finitely generated projective module over Λ. Moreover the intersection of $P_{Np^\infty}^{\mathrm{ord}}$ (resp. $p_{Np^\infty}^{\mathrm{ord}}$) and the nilradical of $\mathcal{H}_{Np^\infty}^{\mathrm{ord}}$ (resp. $\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}$) is null.*

Proof. For the proof, see [Hi1]. □

3.7. We define an ideal

$$\mathcal{I}_{Np^\infty}^{\mathrm{ord}} \subset \mathfrak{h}_{Np^\infty}^{\mathrm{ord}}$$

to be the annihilator of $p_{Np^\infty}^{\mathrm{ord}} \subset \mathfrak{h}_{Np^\infty}^{\mathrm{ord}}$. Then the natural map

$$p_{Np^\infty}^{\mathrm{ord}} \otimes_\Lambda Q(\Lambda) \longrightarrow (\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}/\mathcal{I}_{Np^\infty}^{\mathrm{ord}}) \otimes_\Lambda Q(\Lambda)$$

is an isomorphism, where the both hands sides are semisimple algebras over $Q(\Lambda)$ (cf. [Hi1], §3).

4. UNIVERSAL ZETA MODULAR FORM

In this section, we define and study a"universal zeta modular form" which is obtained from the norm compatible system of Beilinson elements defined by Kato [Ka2], via K_2 Coleman power series. The p-adic properties of p-adic zeta functions of modular forms are deduced from the p-adic property of the universal zeta modular form and the relations between the universal zeta modular form and zeta values. In 4.1, we define a map

$$\mathcal{C}_N : \varprojlim_n K_2(Y(Np^n, p^n)) \to Q(O_{\mathsf{H}}[[G_\infty^{(1)} \times G_\infty^{(2)}]])$$

(see 4.1 for the details) by using K_2 Coleman power series, and Proposition 4.4 shows that

$$\mathcal{C}_N : \underset{\text{(reviewed in 4.2)}}{\text{Beilinson-Kato system}} \mapsto \underset{\text{(defined in 4.3).}}{\text{the universal zeta modular form}}$$

In 4.5, we explain the properties of the universal zeta modular form concerning the relation with special values of zeta functions of cusp forms.
In what follows,

$$\mathsf{H} = (\varprojlim_n (\mathbb{Z}/p^n\mathbb{Z}[[q]][\frac{1}{q}]))[\frac{1}{p}].$$

We fix a system $(\zeta_{p^n})_{n\geq 1}$ of primitive p^n-th roots of unity which satisfy $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for all $n \geq 1$. For $q \in \mathsf{H}$ and $n \geq 1$, we fix p^n-th roots q^{1/p^n} of q in $\overline{\mathsf{H}}$ which satisfy $(q^{1/p^{n+1}})^p = q^{1/p^n}$ for all $n \geq 1$. Let N denote a positive integer such that $(N,p) = 1$.

4.1. By using K_2 Coleman power series, we define a map

$$\mathcal{C}_N : \varprojlim_n K_2(Y(Np^n, p^n)) \longrightarrow Q(O_{\mathsf{H}}[[G_\infty^{(1)} \times G_\infty^{(2)}]]),$$

where the left hand side is the inverse limit of K_2 of modular curves (cf. 4.1.1) taken with respect to the norm maps, and on the right hand side, the group $G_\infty^{(1)}$ is as in section 2 (we will review this in 4.3) and the group $G_\infty^{(2)}$ is the Galois group $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p)$.
We put $S = O_{\mathsf{H}}[[\varepsilon - 1]]$ and $S' = O_{\mathsf{H}}[[\varepsilon - 1]][1/(\varepsilon - 1)]$. Let $G_\infty = \mathbb{Z}_p^\times$. The map $\mathcal{C}_N$ is defined as the following composition:

$$\begin{aligned}\mathcal{C}_N :&\varprojlim_n K_2(Y(Np^n, p^n)) \\ &\xrightarrow{\mathcal{E}_N} \varprojlim_n \hat{K}_2(\mathsf{H}_n)[[G_\infty]] \\ &\xrightarrow[\cong]{\mathrm{Col}} \hat{K}_2(S')^{\mathrm{N}=1}[[G_\infty]] \\ &\xrightarrow{d\log} \Omega_S^2(\log)[[G_\infty]] = \frac{S}{\varepsilon - 1} \cdot d\log(q) \wedge d\log(\varepsilon)[[G_\infty]] \\ &\xrightarrow[\cong]{} \frac{S}{\varepsilon - 1}[[G_\infty]] \to Q(O_{\mathsf{H}}[[G_\infty^{(1)} \times G_\infty^{(2)}]]).\end{aligned} \tag{4.1}$$

We explain each term and each arrow in the composition (4.1).

4.1.1. For $M_1, M_2 \geq 1$ such that $M_1 + M_2 \geq 5$, let $Y(M_1, M_2)$ be the modular curve over $\mathbb{Q}$, which represents the functor

$$\begin{aligned} S \mapsto &(\text{the set of isomorphism classes of pairs } (E,\ \iota) \\ &\quad \text{where } E \text{ is an elliptic curve over } S \text{ and } \iota \text{ is} \\ &\quad \text{an injective homomorphism} \\ &\quad \mathbb{Z}/M_1\mathbb{Z} \times \mathbb{Z}/M_2\mathbb{Z} \to E \text{ of group schemes over } S). \end{aligned}$$

For $M \geq 3$ such that $M_1|M$ and $M_2|M$, we have

$$Y(M_1, M_2) = G\backslash Y(M, M), \tag{4.2}$$

where G is the group $\{\begin{pmatrix} u & v \\ w & x \end{pmatrix} \in GL_2(\mathbb{Z}/M\mathbb{Z})\ ;\ u \equiv 1(M_1), v \equiv 0(M_1), w \equiv 0(M_2), x \equiv 1(M_2)\}$.
We define $Y(M_1, M_2)$ for $M_1, M_2 \geq 1$, $M_1 + M_2 < 5$, by (4.2).

4.1.2. We explain $A[[G_\infty]]$ for an abelian group A. For a set J, $\mathbb{Z}[J]$ denotes a free $\mathbb{Z}$-module on the set J. We define $G_n = (\mathbb{Z}/p^n\mathbb{Z})^\times$, $A[G_n] = A \otimes_{\mathbb{Z}} \mathbb{Z}[G_n]$, and $A[[G_\infty]] = \varprojlim_n A[G_n]$.

4.1.3. Let

$$\Omega^1_{S/\mathbb{Z}}(\log) = (\Omega^1_{S/\mathbb{Z}} \oplus S \otimes_{\mathbb{Z}} S'^{\times})/\mathcal{N},$$

where $\Omega^1_{S/\mathbb{Z}}$ is the module of the absolute differential forms and $\mathcal{N}$ is the S-submodule of the direct sum which is generated by elements $(-da, a \otimes a)$ for $a \in S \cap S'^{\times}$. In $\Omega^1_{S/\mathbb{Z}}(\log)$, we denote the class $(0, 1 \otimes a)$ for $a \in S'^{\times}$ by $d\log(a)$. For $r \geq 1$, let $\Omega^r_{S/\mathbb{Z}}(\log) = \bigwedge^r_S \Omega^1_{S/\mathbb{Z}}(\log)$, and define

$$\Omega^r_S(\log) = \varprojlim_n \Omega^r_{S/\mathbb{Z}}(\log)/p^n\Omega^r_{S/\mathbb{Z}}(\log).$$

Then we have $\Omega^1_S(\log)$ is a free S-module and

$$\Omega^1_S(\log) = S \cdot d\log(q) \oplus S \cdot d\log(\varepsilon - 1), \quad \Omega^2_S(\log) = S \cdot d\log(q) \wedge d\log(\varepsilon - 1),$$

$$\Omega^r_S(\log) = 0 \text{ for } \ r \geq 3.$$

4.1.4. We explain the definition of the map $\mathcal{E}_N$ in (4.1) (cf. [Fu1], §6). This map is induced by the following map for $n \geq 1$ satisfying $Np^n + p^n \geq 5$

$$K_2(Y(Np^n, p^n)) \longrightarrow K_2(\mathsf{H}_n)[G_n] \quad ; \quad x \mapsto \sum_{u \in G_n} (\sum_{w \in \mathbb{Z}/p^n\mathbb{Z}} x_{(u,w)}) g_u.$$

Here $x_{(u,w)} \in K_2(\mathsf{H}_n)$ is the pull-back of x under the following composition:

$$\mathrm{Spec}(\mathsf{H}_n) \to \mathrm{Spec}(\mathsf{H}_n(q^{1/N})) \to Y(Np^n, p^n), \tag{4.3}$$

where $q^{1/N} \in \overline{\mathsf{H}}$ is a N-th root of q.

The first map in (4.3) is given by the homomorphism

$$\mathsf{H}_n(q^{1/N}) \to \mathsf{H}_n \ \ ; \quad \sum_{i=-\infty}^{\infty} a_i q^{i/Np^n} \mapsto \sum_{i=-\infty}^{\infty} a_i q^{i/p^n}.$$

We define the second map of (4.3). Let $\mathfrak{E}_q$ be the elliptic curve over O_{H} which is obtained from the Tate curve over $\mathbb{Z}[[q]][1/q]$ with q-invariant q. For each $m \geq 1$, we have ${}_m\mathfrak{E}_q(O_{\overline{\mathsf{H}}}) = \{q^{a/m}\zeta_m^b \bmod q^{\mathbb{Z}} \ ; \ a, b \in \mathbb{Z}\}$, where ${}_m\mathfrak{E}_q = \mathrm{Ker}(m : \mathfrak{E}_q \to \mathfrak{E}_q)$. Now we define the second map of (4.3) by the open immersion corresponding to

$$(\mathfrak{E}_q \otimes_{O_{\mathsf{H}}} \mathsf{H}_n(q^{1/N}),\ q^{u'/Np^n} \bmod q^{\mathbb{Z}}, q^{w/p^n}\zeta_{p^n} \bmod q^{\mathbb{Z}}) \quad \text{over } \mathsf{H}_n(q^{1/N}).$$

Here $u' \in (\mathbb{Z}/Np^n\mathbb{Z})^\times$ is the element such that $u' \equiv u(p^n)$ and $u' \equiv 1(N)$.

4.1.5. The second arrow of (4.1), which is an isomorphism, is by Theorem 2.4 in section 2 on K_2 Coleman power series.

4.1.6. We explain the map $d\log$ in (4.1). It is the map induced by the map $d\log : \hat{K}_2(S') \longrightarrow \Omega^2_S(\log)$ characterized by $\{\alpha_1, \alpha_2\} \mapsto d\log(\alpha_1) \wedge d\log(\alpha_2)$, where $\alpha_1, \alpha_2 \in S'^\times$ and $\{\alpha_1, \alpha_2\} \in \hat{K}_2(S')$ is the symbol. (The group $\hat{K}_2(S')$ is topologically generated by the symbols. Refer to [Fu1], §4, 4.11.)

4.1.7. We explain the last arrow in (4.1). We firstly define a map

$$S[[G_\infty]] \longrightarrow O_{\mathsf{H}}[[G_\infty^{(1)} \times G_\infty^{(2)}]] \tag{4.4}$$

to be the O_{H}-homomorphism associated to

$$\varepsilon^a g_u \mapsto \begin{cases} u^{-1} g_u^{(1)} g_{u^{-1}a}^{(2)} & \text{if } (a,p) = 1 \\ 0 & \text{if } (a,p) \neq 1, \end{cases}$$

for $a \in \mathbb{Z}$ and $u \in \mathbb{Z}_p^\times$. Here for $u \in \mathbb{Z}_p^\times$, $g_u^{(1)} \in G_\infty^{(1)}$ denotes the corresponding element, and $g_u^{(2)} \in G_\infty^{(2)}$ denotes the corresponding element to u via the cyclotomic character $\chi_{\text{cyclo}} : G_\infty^{(2)} \xrightarrow{\cong} \mathbb{Z}_p^\times$. Next for an integer d' which is prime to p, let $\nu_{d'} : \frac{S}{\varepsilon-1} \to \frac{S}{\varepsilon-1}$ be the O_{H}-homomorphism given by sending $f(\varepsilon)/(\varepsilon-1)$ ($f(\varepsilon) \in S$) to $f(\varepsilon^{d'})/(\varepsilon^{d'}-1)$. It follows from the definition that the image $(1 - d'\nu_{d'})(\frac{S}{\varepsilon-1})$ is contained in S. Now the last map in (4.1) is defined as the composition

$$\begin{aligned} \frac{S}{\varepsilon-1}[[G_\infty]] &\xrightarrow{1-d'\nu_{d'}} S[[G_\infty]] \xrightarrow{(4.4)} O_{\mathsf{H}}[[G_\infty^{(1)} \times G_\infty^{(2)}]] \\ &\xrightarrow{\cdot(1-d' g_{d'}^{(2)})^{-1}} Q(O_{\mathsf{H}}[[G_\infty^{(1)} \times G_\infty^{(2)}]]), \end{aligned}$$

where $1 - d'\nu_{d'}$ is applied only for the coefficient $\frac{S}{\varepsilon-1}$ of G_∞. It is easily seen that the map $\mathcal{C}_N$ is independent of the choice of d'.

4.2. Let c and d be integers satisfying $(c, 6Np) = 1$ and $(d, 6p) = 1$. We review the norm compatible system of Beilinson elements defined by Kato in [Ka2]

$$({}_{c,d}z_{Np^n,p^n})_n \in \varprojlim_n K_2(Y(Np^n, p^n)), \tag{4.5}$$

where the projective limit is taken with respect to the norm maps, in the form which is enough for us here (the details are found in [Ka2]).

For $n \geq 1$ satisfying $Np^n + p^n \geq 5$, the element (4.5) is given as

$${}_{c,d}z_{Np^n,p^n} = \{{}_cg_{Np^n,0}, {}_dg_{0,p^n}\},$$

where ${}_cg_{Np^n,0} \in \mathcal{O}(Y(Np^n,1))^\times$ and ${}_dg_{0,p^n} \in \mathcal{O}(Y(1,p^n))^\times$ are Siegel units, and we introduce their properties which are necessary for us here (see, for example, [Ka2] for the details).

For integers M and c such that $M \geq 3$ and $(c, 6M) = 1$, we have an element ${}_c\theta_E$ of $\mathcal{O}(E \setminus {}_cE)^\times$ which has the following property. Here E is the universal elliptic curve over $Y(M,M)$, ${}_cE = \mathrm{Ker}(c : E \to E)$, and $\mathcal{O}(E \setminus {}_cE)^\times$ is the affine ring. For $\tau \in \mathfrak{H}$ and $z \in \mathbb{C} \setminus c^{-1}(\mathbb{Z}\tau + \mathbb{Z})$, the value at z of ${}_c\theta_E$, on the elliptic curve $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$, is

$$q^{(1/12)(c^2-1)}(-t)^{(1/2)(c-c^2)} \cdot \gamma_q(t)^{c^2}\gamma_q(t^c)^{-1},$$

where $q = \exp(2\pi i\tau)$, $t = \exp(2\pi iz)$, and

$$\gamma_q(t) = \prod_{j\geq 0}(1 - q^jt)\prod_{j\geq 1}(1 - q^jt^{-1}).$$

Now the Siegel unit ${}_cg_{\alpha,\beta}$ for $(\alpha, \beta) = (a/M, b/M) \in (((1/M)\mathbb{Z})/\mathbb{Z})^2 \setminus \{(0,0)\}$ $(a, b \in \mathbb{Z})$ may be defined by

$${}_cg_{\alpha,\beta} = \iota^*_{\alpha,\beta}({}_c\theta_E) \in \mathcal{O}(Y(M,M))^\times.$$

Here

$$\iota_{\alpha,\beta} = ae_1 + be_2 : Y(M,M) \to E \setminus {}_cE$$

with the canonical basis (e_1, e_2) of E over $Y(M,M)$. In the case $\alpha = 0$, ${}_cg_{0,\beta} \in \mathcal{O}(Y(1,M))^\times$ and in the case $\beta = 0$, ${}_cg_{\alpha,0} \in \mathcal{O}(Y(M,1))^\times$.)

In [Ka2] (cf. [Sc]), it was shown that ${}_{c,d}z_{Np^n,p^n}$ $(n \geq 1)$ form a projective system with respect to the norm maps.

In the paper [Ka2], Kato always used norm compatible systems $({}_{c,d}z_{Mp^n,M'p^n})_n \in \varprojlim_n K_2(Y(Mp^n, M'p^n))$ $(M, M' \geq 1,\ (M + M')p^n \geq 5)$ satisfying the condition that $M|M'$ in application. However clearly the system $({}_{c,d}z_{Np^n,p^n})_n \in \varprojlim_n K_2(Y(Np^n,p^n))$ which we use does not satisfy this condition. When Kato used a system, for example, to construct a p-adic zeta function of an eigen cusp form f, he considered the "f^*-component" of the system, where f^* is the dual cusp form of f (see 6.5.1 in section 6 for the definition of the dual cusp form, and for the meaning of "component", refer to section 6). But our method needs to study the "f-component" of the system. So we must slightly modify his system in application.

4.3. We define a"universal zeta modular form"

$$z^{\mathrm{univ}}_{Np^\infty} \in \overline{M}_{Np^\infty}[[G^{(1)}_\infty \times G^{(2)}_\infty]][\frac{1}{g}](\subset O_{\mathsf{H}}[[G^{(1)}_\infty \times G^{(2)}_\infty]][\frac{1}{g}])$$

which yields special values at $s = r$ ($r \in \mathbb{Z}, 1 \leq r \leq k-1$) of the zeta functions of modular forms of weight $k \geq 2$ and level Np^t for $t \geq 0$. Here $\overline{M}_{Np^\infty}$ is as in section 3. Moreover $G^{(1)}_\infty \cong G^{(2)}_\infty \cong G_\infty = \mathbb{Z}_p^\times$, and $G^{(1)}_\infty$ is , as before, the group acting on the space of p-adic modular forms $\overline{M}_{Np^\infty}$ whose action is characterized by $g^{(1)}_a f = a^{k-2}\langle a'\rangle f$ for $f \in M_k(X_1(Np^t);\mathbb{Q}_p)$, $a \in \mathbb{Z}_p^\times$, and $a' \in (\mathbb{Z}/Np^t\mathbb{Z})^\times$ such that $a' \equiv a(p^t)$ and $a' \equiv 1(N)$. Here $\langle a'\rangle$ is as in 3.5. The group $G^{(2)}_\infty$ is the Galois group $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p)$.

We define $z^{\mathrm{univ}}_{Np^\infty}$ as an element of $O_{\mathsf{H}}[[G^{(1)}_\infty \times G^{(2)}_\infty]][1/g]$ and in 4.5.5, we will prove that it belongs to the subspace $\overline{M}_{Np^\infty}[[G^{(1)}_\infty \times G^{(2)}_\infty]][1/g]$.

Firstly we define $F_{N,1}, F_{N,2} \in \mathsf{H}[[G_\infty]] = \varprojlim_n \mathsf{H}[G_n]$ to be

$$F_{N,1} = (\sum_{\substack{i\geq 1\\(i,p)=1}} \sum_{j\geq 1} q^{Nij})(g_i - g_{-i}) + \varprojlim_n (\sum_{a\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \zeta_{\equiv a(p^n)}(0)\cdot g_a),$$

$$\begin{aligned} F_{N,2} = (&\sum_{\substack{i\geq 1, i\equiv 1(N)\\(i,p)=1}} \sum_{j\geq 1} q^{ij}\cdot g_i - \sum_{\substack{i\geq 1, i\equiv -1(N)\\(i,p)=1}} \sum_{j\geq 1} q^{ij}\cdot g_{-i}) \\ &+ \varprojlim_n (\sum_{\substack{a\in(\mathbb{Z}/Np^n\mathbb{Z})^\times\\ a\equiv 1(N)}} \zeta_{\equiv a(Np^n)}(0)\cdot g_a). \end{aligned}$$

Here for $a \in \mathbb{Z}_p^\times$ or $a \in (\mathbb{Z}/p^n\mathbb{Z})^\times$, g_a represents the corresponding element of G_∞ or G_n, respectively. For $M, m \in \mathbb{Z}$, $M \geq 1$, and $a \in \mathbb{Z}/M\mathbb{Z}$, $\zeta_{\equiv a(M)}(m)$ is the evaluation at $s = m$ of the partial Riemann zeta function

$$\zeta_{\equiv a(M)}(s) = \sum_{\substack{j\geq 1\\ j\equiv a \bmod M}} j^{-s},$$

and $\sum_{\substack{a\in(\mathbb{Z}/Np^n\mathbb{Z})^\times\\ a\equiv 1(N)}} \zeta_{\equiv a(Np^n)}(0)\cdot g_a$ belongs to $H[G_n]$.

We define the product $F_{N,1}\cdot F_{N,2} \in \mathsf{H}[[G_\infty \times G_\infty]]$ naturally (by the rule $xg_a \cdot yg_b \mapsto xyg_{a,1}g_{b,2}$ with $x, y \in \mathsf{H}$, $a, b \in \mathbb{Z}_p^\times$, where $g_{a,1}$ (resp, $g_{b,2}$) means the corresponding element of the first (resp. the second) G_∞).

Now we define the universal zeta modular form $z^{\mathrm{univ}}_{Np^\infty}$ to be the image of $F_{N,1}\cdot F_{N,2}$ under the isomorphism of rings over H

$$\mathsf{H}[[G_\infty \times G_\infty]] \longrightarrow \mathsf{H}[[G^{(1)}_\infty \times G^{(2)}_\infty]] \;; \tag{4.6}$$

$$xg_{a,1}g_{b,2} \mapsto xg^{(1)}_b g^{(2)}_{ab^{-1}} \quad (x \in \mathsf{H}, a, b \in \mathbb{Z}_p^\times)$$

$$(F_{N,1}\cdot F_{N,2} \mapsto z^{\mathrm{univ}}_{Np^\infty}).$$

For integers c, d' such that $(c,p) = 1$ and $(d',p) = 1$, we have

$$(1 - c^{-1} g_{c^{-1}}^{(1)} g_c^{(2)})(1 - d' g_{d'}^{(2)}) z_{Np^\infty}^{\mathrm{univ}} \in O_{\mathsf{H}}[[G_\infty^{(1)} \times G_\infty^{(2)}]]. \tag{4.7}$$

This follows from the fact that

$$(1 - c^{-1} g_{c^{-1}}) \cdot \varprojlim_n \sum_{\substack{a \in (\mathbb{Z}/Np^n\mathbb{Z})^\times \\ a \equiv 1 (N)}} \zeta_{\equiv a(Np^n)}(0) \cdot g_a \in O_{\mathsf{H}}[[G_\infty]].$$

By (4.7), we obtain that $z_{Np^\infty}^{\mathrm{univ}} \in O_{\mathsf{H}}[[G_\infty^{(1)} \times G_\infty^{(2)}]][1/g]$ for a non-zero divisor $g \in \mathbb{Z}[[G_\infty^{(1)} \times G_\infty^{(2)}]]$.

Thus the universal zeta modular form is something like a product of two "Λ-adic Eisenstein series" in the sense of Hida in [Hi3], Chapter 7, §7.1.

The following proposition describes the relation between the norm compatible system of Beilinson elements and the universal zeta modular form.

PROPOSITION 4.4 . *Let c and d be as in* 4.2. *We further assume that $c \equiv 1$ mod N. Then we have*

$$\mathcal{C}_N(({}_{c,d}z_{Np^n,p^n})_n) = (c^2 - c g_{c^{-1}}^{(1)} g_c^{(2)})(d^2 - d g_d^{(2)}) \cdot z_{Np^\infty}^{\mathrm{univ}}.$$

Proof. Firstly we consider a composition $\mathcal{C}'_{N,d'}$ determined by the relation that $(1 - d' g_{d'}^{(2)}) \cdot \mathcal{C}_N = (4.6) \circ \mathcal{C}'_{N,d'}$, where d' is, as in 4.1.7, an integer which is prime to p. Namely, $\mathcal{C}'_{N,d'}$ is as follows:

$$\begin{aligned} \mathcal{C}'_{N,d'} : \varprojlim_n K_2(Y(Np^n, p^n)) &\to \frac{S}{\varepsilon - 1}[[G_\infty]] \\ &\xrightarrow{s} O_{\mathsf{H}}[[G_\infty]][[G_\infty]], \end{aligned}$$

where the first arrow is the composition of the first four maps in (4.1), and the map s is defined by the composition

$$s : \frac{S}{\varepsilon - 1}[[G_\infty]] \xrightarrow{1 - d'\nu_{d'}} S[[G_\infty]] \to O_{\mathsf{H}}[[G_\infty]][[G_\infty]].$$

Here the second map is the O_{H}-homomorphism associated to

$$\varepsilon^a g_u \mapsto \begin{cases} u^{-1} g_{a,1} g_{u,2} & \text{if } (a,p) = 1 \\ 0 & \text{if } (a,p) \neq 1, \end{cases}$$

for $a \in \mathbb{Z}$ and $u \in \mathbb{Z}_p^\times$.

For the proof of Proposition 4.4, by the definition of $z_{Np^\infty}^{\mathrm{univ}}$, our task is to show that under the map $\mathcal{C}'_{N,d'}$, the norm compatible system of Beilinson elements $({}_{c,d}z_{Np^n,p^n})_n$ is sent to $(c^2 - c g_{c^{-1},2})(d^2 - d g_{d,1})(1 - d' g_{d',1}) \cdot F_{N,1} \cdot F_{N,2}$, where $F_{N,1}, F_{N,2}$ are as in the definition of $z_{Np^\infty}^{\mathrm{univ}}$.

We prove the above assertion by showing the result of the computation of the image of $({}_{c,d}z_{Np^n,p^n})_n$ under each step in the composition defining $\mathcal{C}'_{N,d'}$.

Step 1. Firstly we compute the image of ${}_{c,d}z_{Np^n,p^n} = \{{}_c g_{Np^n,0}, {}_d g_{0,p^n}\}$ under $K_2(Y(Np^n,p^n)) \to K_2(\mathsf{H}_n(q^{1/N}))$ which is given by the pull-back by the latter map of (4.3). Directly from the definition, we have that the image is $\{A_1, B_1\} \in K_2(\mathsf{H}_n(q^{1/N}))$. Here the element $A_1 \in \mathsf{H}_n(q^{1/N})^\times$ is obtained from ${}_c\theta_E$, where E is the universal elliptic curve over $Y(Np^n, Np^n)$, by putting $t = q^{u'/Np^n}$ with $u' \in (\mathbb{Z}/Np^n\mathbb{Z})^\times$ such that $u' \equiv u(p^n)$ and $u' \equiv 1(N)$. The element $B_1 \in \mathsf{H}_n(q^{1/N})^\times$ is obtained from ${}_d\theta_E$, where E is the universal elliptic curve over $Y(p^n, p^n)$ by putting $t = q^{w/p^n}\zeta_{p^n}$. From this, it is easy to have the following result:

$$
\begin{aligned}
&\mathcal{E}_N(({}_{c,d}z_{Np^n,p^n})_n)\\
&= (\{\prod_{u_n \in (\mathbb{Z}/p^n\mathbb{Z})^\times} q^{(N/12)(c^2-1)}(-q^{u'_n/p^n})^{(1/2)(c-c^2)}\gamma_{q^N}(q^{u'_n/p^n})^{c^2}\gamma_{q^N}(q^{cu'_n/p^n})^{-1},\\
&\quad \prod_{w_n \in \mathbb{Z}/p^n\mathbb{Z}} q^{(N/12)(d^2-1)}(-q^{Nw'_n/p^n}\zeta_{p^n})^{(1/2)(d-d^2)}\\
&\quad \gamma_{q^N}(q^{Nw'_n/p^n}\zeta_{p^n})^{d^2}\gamma_{q^N}(q^{dNw'_n/p^n}\zeta_{p^n}^d)^{-1}\}g_{u_n})_n\\
&\qquad \in \varprojlim_n \hat{K}_2(\mathsf{H}_n)[G_n] = (\varprojlim_n \hat{K}_2(\mathsf{H}_n))[[G_\infty]],
\end{aligned}
$$

where for $u_n \in (\mathbb{Z}/p^n\mathbb{Z})^\times$, u'_n is an integer such that $u'_n \equiv 1(N)$ and $u'_n \equiv u_n(p^n)$, and for $w_n \in \mathbb{Z}/p^n\mathbb{Z}$, w'_n is an integer such that $w'_n \equiv w_n(p^n)$.
Step 2. From the definition, we obtain that under the notation in the composition (4.1)

$$
\mathrm{Col} \circ \mathcal{E}_N(({}_{c,d}z_{Np^n,p^n})_n) \in \hat{K}_2(S')^{\mathrm{N}=1}[[G_\infty]]
$$

coincides with the image of (A_2, B_2) with $A_2 \in O_{\mathsf{H}}^\times[[G_\infty]]$ and $B_2 \in S'^\times$ given below under the natural map

$$
O_{\mathsf{H}}^\times[[G_\infty]] \times S'^\times \longrightarrow \hat{K}_2(S')[[G_\infty]]\ ;\ (x_u g_u,\ y) \mapsto \{x_u, y\}g_u.
$$

The elements A_2, B_2 are as follows:

$$
\begin{aligned}
A_2 = \varprojlim_n(&\prod_{u_n \in (\mathbb{Z}/p^n\mathbb{Z})^\times} (-q^{(-c^2\zeta_{\equiv u'_n(Np^n)}(-1)+\zeta_{\equiv cu'_n(Np^n)}(-1))})g_{u_n})\\
&\cdot \prod_{\substack{i\geq 1, i\equiv 1(N)\\(i,p)=1}} (1-q^i)^{c^2}g_i \cdot \prod_{\substack{i\geq 1, i\equiv -1(N)\\(i,p)=1}} (1-q^i)^{c^2}g_{-i}\\
&\cdot \prod_{\substack{i\geq 1, i\equiv 1(N)\\(i,p)=1}} (1-q^i)^{-1}g_{c^{-1}i} \cdot \prod_{\substack{i\geq 1, i\equiv -1(N)\\(i,p)=1}} (1-q^i)^{-1}g_{-c^{-1}i},
\end{aligned}
$$

where u'_n is as before.

$$
\begin{aligned}
B_2 = \varprojlim_n \Big(\prod_{w_n \in \mathbb{Z}/p^n\mathbb{Z}} & (-q^{N(-d^2\zeta_{\equiv w_n(p^n)}(-1)+\zeta_{\equiv dw_n(p^n)(-1)})}\Big) \\
& \cdot \prod_{i\geq 1}(1-q^{Ni}\varepsilon)^{d^2} \cdot \prod_{i\geq 1}(1-q^{Ni}\varepsilon^{-1})^{d^2} \\
& \cdot \prod_{i\geq 1}(1-q^{Ni}\varepsilon^{d})^{-1} \cdot \prod_{i\geq 1}(1-q^{Ni}\varepsilon^{-d})^{-1} \\
& \cdot (1-\varepsilon)^{d^2}(1-\varepsilon^d)^{-1}\varepsilon^{(1/2)(d-d^2)}.
\end{aligned}
$$

Step 3. By the definition of the map $d\log$, we find easily that

$$d\log \circ \mathrm{Col} \circ \mathcal{E}_N(({}_{c,d}z_{Np^n,p^n})_n) \in \Omega^2_S(\log)[[G_\infty]]$$

coincides with the image of (A_3, B_3) with $A_3 \in O_{\mathsf{H}}[[G_\infty]]$ and $B_3 \in \frac{S}{\varepsilon-1}$ given below under the map

$$O_{\mathsf{H}}[[G_\infty]] \times \frac{S}{\varepsilon-1} \longrightarrow \Omega^2_S(\log)[[G_\infty]] \ ;$$

$$(xg_u, y) \mapsto xy \cdot d\log(q) \wedge d\log(\varepsilon) \cdot g_u$$

for $x \in O_{\mathsf{H}}$, $u \in \mathbb{Z}_p^\times$, $y \in \frac{S}{\varepsilon-1}$.
The elements A_3, B_3 are as follows:

$$
\begin{aligned}
A_3 = & \Big(\sum_{\substack{i\geq 1, i\equiv 1(N)\\ (i,p)=1}} \sum_{j\geq 1} iq^{ij}\Big)(c^2\cdot g_i - g_{c^{-1}i}) \\
& + \Big(\sum_{\substack{i\geq 1, i\equiv -1(N)\\ (i,p)=1}} \sum_{j\geq 1} iq^{ij}\Big)(c^2\cdot g_{-i} - g_{-c^{-1}i}) \\
& + \varprojlim_n \Big(\sum_{\substack{a\in(\mathbb{Z}/Np^n\mathbb{Z})^\times \\ a\equiv 1(N)}} \zeta_{\equiv a(Np^n)}(-1)(c^2\cdot g_a - g_{c^{-1}a})\Big).
\end{aligned}
$$

$$
\begin{aligned}
B_3 = & \, d^2\Big(\sum_{i\geq 1}\sum_{j\geq 1} q^{Nij}\varepsilon^j - \sum_{i\geq 1}\sum_{j\geq 1} q^{Nij}\varepsilon^{-j}\Big) \\
& - d\Big(\sum_{i\geq 1}\sum_{j\geq 1} q^{Nij}\varepsilon^{dj} - \sum_{i\geq 1}\sum_{j\geq 1} q^{Nij}\varepsilon^{-dj}\Big) \\
& + d^2\frac{\varepsilon}{1-\varepsilon} - d\frac{\varepsilon^d}{1-\varepsilon^d} + \frac{1}{2}(d-d^2).
\end{aligned}
$$

Step 4. By the definition of the map s, we see that

$$s \circ d\log \circ \mathrm{Col} \circ \mathcal{E}_N(({}_{c,d}z_{Np^n,p^n})_n) \in O_{\mathsf{H}}[[G_\infty]][[G_\infty]]$$

coincides with the image of $((1-d'g_{d'})B_4, A_4)$ with $A_4 \in O_{\mathsf{H}}[[G_\infty]]$ and $B_4 \in Q(O_{\mathsf{H}}[[G_\infty]])$ $((1-d'g_{d'})B_4 \in O_{\mathsf{H}}[[G_\infty]])$ given below under the natural O_{H}-homomorphism

$$O_{\mathsf{H}}[[G_\infty]] \times O_{\mathsf{H}}[[G_\infty]] \longrightarrow O_{\mathsf{H}}[[G_\infty]][[G_\infty]] \ ;$$
$$(xg_a, yg_b) \mapsto xyg_{a,1}g_{b,2} \quad (a, b \in \mathbb{Z}_p^\times).$$

The elements A_4, B_4 are as follows:

$$\begin{aligned} A_4 = &\Big(\sum_{\substack{i \geq 1, i \equiv 1 (N) \\ (i,p)=1}} \sum_{j \geq 1} q^{ij}\Big)(c^2 \cdot g_i - c \cdot g_{c^{-1}i}) \\ &- \Big(\sum_{\substack{i \geq 1, i \equiv -1 (N) \\ (i,p)=1}} \sum_{j \geq 1} q^{ij}\Big)(c^2 \cdot g_{-i} - c \cdot g_{-c^{-1}i}) \\ &+ \varprojlim_n \Big(\sum_{\substack{a \in (\mathbb{Z}/Np^n\mathbb{Z})^\times \\ a \equiv 1 (N)}} \zeta_{\equiv a (Np^n)}(0)(c^2 \cdot g_a - c \cdot g_{c^{-1}a})\Big). \end{aligned}$$

$$\begin{aligned} B_4 = &\, d^2\Big(\sum_{i \geq 1} \sum_{\substack{j \geq 1 \\ (j,p)=1}} q^{Nij} g_j - \sum_{i \geq 1} \sum_{\substack{j \geq 1 \\ (j,p)=1}} q^{Nij} g_{-j}\Big) \\ &- d\Big(\sum_{i \geq 1} \sum_{\substack{j \geq 1 \\ (j,p)=1}} q^{Nij} g_{dj} - \sum_{i \geq 1} \sum_{\substack{j \geq 1 \\ (j,p)=1}} q^{Nij} g_{-dj}\Big) \\ &+ d^2 \varprojlim_n \Big(\sum_{a_n \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \zeta_{\equiv a_n (p^n)}(0) \cdot g_{a_n}\Big) \\ &- d \varprojlim_n \Big(\sum_{a_n \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \zeta_{\equiv a_n (p^n)}(0) \cdot g_{da_n}\Big). \end{aligned}$$

By comparing A_4 with F_1 and B_4 with F_2, we obtain the assertion of Proposition 4.4. $\square$

4.5. We prove that $z_{Np^\infty}^{\mathrm{univ}}$ which has been defined as an element of $O_{\mathsf{H}}[[G_\infty^{(1)} \times G_\infty^{(2)}]][[1/g]$, in fact, belongs to the subspace $\overline{M}_{Np^\infty}[[G_\infty^{(1)} \times G_\infty^{(2)}]][1/g]$. We further show the relation between $z_{Np^\infty}^{\mathrm{univ}}$ and special values of zeta functions of cusp forms. Preceding this, in 4.5.1 – 4.5.4, we review the zeta modular forms in [Ka2], which were defined basing on the works of Shimura [Sh], and whose period integrals yield special values of the zeta functions of cusp forms. In 4.5.5, we show that $z_{Np^\infty}^{\mathrm{univ}}$ is contained in the subspace $\overline{M}_{Np^\infty}[[G_\infty^{(1)} \times G_\infty^{(2)}]][1/g]$, and then in 4.5.6, we describe the relation between $z_{Np^\infty}^{\mathrm{univ}}$ and the zeta modular forms reviewed in 4.5.1 – 4.5.4.

4.5.1. We review some Eisenstein series appearing, for example, in [Ka2], §3. For $M_1, M_2 \geq 1$ such that $M_1 + M_2 \geq 5$, as before, let $M_j(X(M_1, M_2))$ be the space of modular forms on $X(M_1, M_2)$ of weight $j \geq 1$.

Let $M \geq 3$, and $x, y \in ((1/M)\mathbb{Z})/\mathbb{Z}$. We review the q-expansions of Eisenstein series

$$F_{x,y}^{(j)}\ ((j,x,y) \neq (2,0,0)),\ E_{x,y}^{(j)}\ (j \neq 2),\ \tilde{E}_{x,y}^{(2)} \in M_j(X(M,M)),$$

following Kato [Ka2], §4. (In the case $x = 0$, these modular forms are, in fact, elements of $M_j(X(1,M))$ and in the case $y = 0$, they are, in fact, elements of $M_j(X(M,1))$.) For $\gamma \in \mathbb{Q}/\mathbb{Z}$, we define

$$\zeta(\gamma, s) = \sum_{\substack{m \in \mathbb{Q}, m>0 \\ m \bmod \mathbb{Z} \equiv \gamma}} m^{-s}, \qquad \zeta^*(\gamma, s) = \sum_{m=1}^{\infty} \exp(2\pi i \gamma m) \cdot m^{-s}.$$

For (i) $F = F_{x,y}^{(j)}$ $((j,x,y) \neq (2,0,0))$, (ii) $F = E_{x,y}^{(j)}$ $(j \neq 2)$, or (iii) $F = \tilde{E}_{x,y}^{(2)}$, we write $F = \sum_{\substack{m \in \mathbb{Q} \\ m \geq 0}} a_m q^m$ $(q = \exp(2\pi i \tau))$.

In the case of (i), we assume that $(j,x,y) \neq (2,0,0)$. Then a_m for $m > 0$ can be obtained from the equation

$$\sum_{\substack{m \in \mathbb{Q} \\ m>0}} a_m m^{-s} = \zeta(x, s-j+1)\zeta^*(y,s) + (-1)^j \zeta(-x, s-j+1)\zeta^*(-y,s).$$

In the case $j \neq 1$, $a_0 = \zeta(x, 1-j)$.
In the case $j = 1$, $a_0 = \zeta(x,0)$ if $x \neq 0$, and $a_0 = (1/2)(\zeta^*(y,0) - \zeta^*(-y,0))$ if $x = 0$.

In the case of (ii), we assume that $j \neq 2$. Then a_m for $m > 0$ can be obtained from the equation

$$\sum_{\substack{m \in \mathbb{Q} \\ m>0}} a_m m^{-s} = \zeta(x,s)\zeta^*(y, s-j+1) + (-1)^j \zeta(-x,s)\zeta^*(-y, s-j+1).$$

In the case $j \neq 1$, $a_0 = 0$ if $x \neq 0$, and $a_0 = \zeta^*(y, 1-j)$ if $x = 0$.
In the case $j = 1$, $a_0 = \zeta(x,0)$ if $x \neq 0$, and $a_0 = (1/2)(\zeta^*(y,0) - \zeta^*(-y,0))$ if $x = 0$.

In the case of (iii), the a_m for $m > 0$ can be obtained from the equation

$$\sum_{\substack{m \in \mathbb{Q} \\ m>0}} a_m m^{-s} = \zeta(x,s)\zeta^*(y,s-1) + \zeta(-x,s)\zeta^*(-y,s-1) - 2\zeta(s)\zeta(s-1).$$

If $x \neq 0$, $a_0 = 0$, and if $x = 0$, $a_0 = \zeta^*(y,-1) - \zeta(-1)$.

4.5.2. We review the zeta modular forms in [Ka2], §§4 and 5, which were defined basing on the works of Shimura [Sh]. These zeta modular forms yield special values of zeta functions of modular forms by period integrals (concerning this, refer to [Ka2], §5).
Let k, r, m, n be integers such that $k \geq 2$, $1 \leq r \leq k-1$, $1 \leq m \leq n$, and $N(p^n + p^m) \geq 5$. Further for an integer M, let prime(M) denote the set of all of the prime divisors of M.
In the case $r \neq 2$, the zeta modular forms are as follows:

$$z^{(k,r)}_{Np^n,Np^n}(k,r,k-1)$$
$$= (r-1)!^{-1}\cdot (Np^n)^{k-r-2}(Np^n)^{-r}\cdot F^{(k-r)}_{1/Np^n,0}\cdot E^{(r)}_{0,1/Np^n}\in M_k(X(Np^n,Np^n)),$$

$$z^{(k,r)}_{1,Np^m,Np^n}(k,r,k-1,0(1),\mathrm{prim}(Np))$$
$$= \mathrm{Tr}_{Np^m}(z^{(k,r)}_{Np^n,Np^n}(k,r,k-1,0(1),\mathrm{prim}(Np)))\in M_k(X_1(Np^m);\mathbb{Q}(\zeta_{Np^n})).$$

We remark that $M_k(X_1(Np^m);\mathbb{Q}(\zeta_{Np^n}))$ may be regarded as the fixed part of $M_k(X(Np^n,Np^n))$ by the group $\{\begin{pmatrix} u & v \\ w & x\end{pmatrix}\in GL_2(\mathbb{Z}/Np^n\mathbb{Z})\ ;\ u\equiv 1(Np^m), w\equiv 0(Np^m), ux-vw\equiv 1(Np^n)\}$. In the above

$$\mathrm{Tr}_{Np^m}: M_k(X(Np^n,Np^n))\longrightarrow M_k(X_1(Np^m);\mathbb{Q}(\zeta_{Np^n}))$$

denotes the trace map.
Let c and d be integers such that $(c,Np)=1$ and $(d,p)=1$. In the case $r=2$, the zeta modular forms are as follows:

$$\begin{aligned} {}_{c,d}z^{(k,2)}_{Np^n,Np^n}&(k,2,k-1)\\ =&(Np^n)^{k-4}(Np^n)^{-2}c^2d^2\\ &\cdot(F^{(k-2)}_{1/Np^n,0}-c^{2-k}\cdot F^{(k-2)}_{c/Np^n,0})\cdot(\tilde{E}^{(2)}_{0,1/Np^n}-\tilde{E}^{(2)}_{0,d/Np^n})\\ &\qquad\in M_k(X(Np^n,Np^n))),\end{aligned}$$

$$\begin{aligned} {}_{c,d}z^{(k,2)}_{1,Np^m,Np^n}&(k,2,k-1,0(1),\mathrm{prim}(Np))\\ =&\ \mathrm{Tr}_{Np^m}({}_{c,d}z^{(k,2)}_{Np^n,Np^n}(k,2,k-1,0(1),\mathrm{prim}(Np)))\\ &\quad\in M_k(X_1(Np^m);\mathbb{Q}(\zeta_{Np^n})).\end{aligned}$$

The above zeta modular forms provide the value at $s=r$ of the zeta functions of modular forms of weight k by period integrals.
In our method, modular forms whose q-expansions belong to $\mathbb{Z}_{(p)}[[q]]$ or $\mathbb{Q}[[q]]$ are important. So we analyze zeta modular forms from this viewpoint. Firstly for $j\in\mathbb{Z}$, $j\geq 1$, and $a\in((1/M)\mathbb{Z})/\mathbb{Z}$ satisfying $(j,a)\neq(2,0)$, directly from the definition we have

$$\sum_{x\in((1/M)\mathbb{Z})/\mathbb{Z}} F^{(j)}_{a,x}\in M_j(X_1(M);\mathbb{Q}).$$

LEMMA 4.5.3 . *We assume that $r\neq 2$.*

(1) *In* $M_k(X_1(Np^n);\mathbb{Q}(\zeta_{Np^n}))$, *we have*

$$\begin{aligned}
&z^{(k,r)}_{1,Np^n,Np^n}(k,r,k-1,0(1),\mathrm{prim}(Np))\\
&\quad = (r-1)!^{-1}\cdot(Np^n)^{k-r-2}(Np^n)^{-r}\cdot \sum_{x\in((1/Np^n)\mathbb{Z})/\mathbb{Z}} F^{(k-r)}_{1/Np^n,x}\cdot E^{(r)}_{0,1/Np^n}\\
&\quad = (r-1)!^{-1}\cdot(Np^n)^{k-r-2}(Np^n)^{-2}\\
&\qquad \cdot(\sum_{a\in\mathbb{Z}/Np^n\mathbb{Z}}(\sum_{x,y\in((1/Np^n)\mathbb{Z})/\mathbb{Z}} F^{(k-r)}_{1/Np^n,x}\cdot F^{(r)}_{a/Np^n,y})\cdot\zeta^a_{Np^n}.
\end{aligned}$$

(2) *Let*

$$\mathrm{Tr}_{Np^n,Np^m}: M_k(X_1(Np^n);\mathbb{Q}(\zeta_{Np^n}))\longrightarrow M_k(X_1(Np^m);\mathbb{Q}(\zeta_{Np^n}))$$

be the trace map. In $M_k(X_1(Np^m);\mathbb{Q}(\zeta_{Np^n}))$, *we have*

$$\begin{aligned}
&z^{(k,r)}_{1,Np^m,Np^n}(k,r,k-1,0(1),\mathrm{prim}(Np))\\
&\quad = \mathrm{Tr}_{Np^n,Np^m}(z^{(k,r)}_{1,Np^n,Np^n}(k,r,k-1,0(1),\mathrm{prim}(Np)))\\
&\quad = (r-1)!^{-1}\cdot(Np^m)^{k-r-2}(Np^n)^{-2}\\
&\qquad \cdot(\sum_{a\in\mathbb{Z}/Np^n\mathbb{Z}} T(p)^{n-m}\\
&\qquad\quad (\sum_{x\in((1/Np^m)\mathbb{Z})/\mathbb{Z}}\sum_{y\in((1/Np^n)\mathbb{Z})/\mathbb{Z}}(F^{(k-r)}_{1/Np^m,x}\cdot F^{(r)}_{a/Np^n,y}))\cdot\zeta^a_{Np^n}).
\end{aligned}$$

Here $T(p)=U(p)$ *is the Hecke operator on the space* $M_k(X_1(Np^n);\mathbb{Q})$.

(3) *Let*

$$\mathrm{tr}_{Np^m,p^n}: M_k(X_1(Np^m);\mathbb{Q}(\zeta_{Np^n}))\longrightarrow M_k(X_1(Np^m);\mathbb{Q}(\zeta_{p^n}))$$

be the trace map. In $M_k(X_1(Np^m);\mathbb{Q}(\zeta_{p^n}))$, *we have*

$$\begin{aligned}
&\mathrm{tr}_{Np^n,p^n}(z^{(k,r)}_{1,Np^m,Np^n}(k,r,k-1,0(1),\mathrm{prim}(Np)))\\
&\quad = (r-1)!^{-1}\cdot(Np^m)^{k-r-2}(Np^n)^{-2}\cdot N\\
&\qquad \cdot\prod_{\substack{l:\mathrm{prime}\\ l|N}}(1-l^{-r}T(l)\nu_{l^{-1}})\\
&\qquad (\sum_{a\in\mathbb{Z}/p^n\mathbb{Z}} T(p)^{n-m}\\
&\qquad\quad (\sum_{x\in((1/Np^m)\mathbb{Z})/\mathbb{Z}}\sum_{y\in((1/Np^n)\mathbb{Z})/\mathbb{Z}}(F^{(k-r)}_{1/Np^m,x}\cdot F^{(r)}_{Na/Np^n,y}))\cdot\zeta^a_{p^n}),
\end{aligned}\tag{4.8}$$

where for $x\in(\mathbb{Z}/p^n\mathbb{Z})^\times$, ν_x *is the corresponding element of* $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})$ *via the cyclotomic character.*

Proof. (1) The first equality is direct from the definition. The equality ($r \neq 2$)

$$E^{(r)}_{0,1/Np^n} = (Np^n)^{r-2} \cdot \sum_{a \in \mathbb{Z}/Np^n\mathbb{Z}} \sum_{y \in ((1/Np^n)\mathbb{Z})/\mathbb{Z}} F^{(r)}_{a/Np^n, y} \cdot \zeta^a_{Np^n}$$

which can be obtained by computation, shows the second equality.

(2) (3) The results are immediate from the definitions.

□

The following Lemma 4.5.4 describes the case $r = 2$.

Lemma 4.5.4 . *We use the same notation as in Lemma* 4.5.3.

(1) *In* $M_k(X_1(Np^n); \mathbb{Q}(\zeta_{Np^n}))$, *we have*

$$\begin{aligned}
&{}_{c,d}z^{(k,2)}_{1,Np^n,Np^n}(k, 2, k-1, 0(1), \mathrm{prim}(Np)) \\
&= (Np^n)^{k-4}(Np^n)^{-2}c^2d^2 \\
&\cdot \sum_{\substack{a \in \mathbb{Z}/Np^n\mathbb{Z} \\ a \neq 0}} \sum_{x,y \in ((1/Np^n)\mathbb{Z})/\mathbb{Z}} (F^{(k-2)}_{1/Np^n, x} - c^{2-k} \cdot F^{(k-2)}_{c/Np^n, x}) \\
&\qquad \cdot (F^{(2)}_{a/Np^n, y} \cdot \zeta^a_{Np^n} - F^{(2)}_{a/Np^n, y} \cdot \zeta^{da}_{Np^n}).
\end{aligned}$$

(2) *In* $M_k(X_1(Np^m); \mathbb{Q}(\zeta_{Np^n}))$, *we have*

$$\begin{aligned}
&{}_{c,d}z^{(k,2)}_{1,Np^m,Np^n}(k, 2, k-1, 0(1), \mathrm{prim}(Np)) \\
&= (Np^m)^{k-4}(Np^n)^{-2}c^2d^2 \\
&\cdot T(p)^{n-m}(\sum_{\substack{a \in \mathbb{Z}/Np^n\mathbb{Z} \\ a \neq 0}} \sum_{x \in ((1/Np^m)\mathbb{Z})/\mathbb{Z}} \sum_{y \in ((1/Np^n)\mathbb{Z})/\mathbb{Z}} \\
&((F^{(k-2)}_{1/Np^m, x} - c^{2-k} \cdot F^{(k-2)}_{c/Np^m, x}) \cdot (F^{(2)}_{a/Np^n, y} \cdot \zeta^a_{Np^n} - F^{(2)}_{a/Np^n, y} \cdot \zeta^{da}_{Np^n}))).
\end{aligned}$$

(3) *In* $M_k(X_1(Np^m); \mathbb{Q}(\zeta_{p^n}))$, *we have*

$$\begin{aligned}
&\mathrm{tr}_{Np^n,p^n}({}_{c,d}z^{(k,2)}_{1,Np^m,Np^n}(k, 2, k-1, 0(1), \mathrm{prim}(Np))) \\
&= (Np^m)^{k-4}(Np^n)^{-2}c^2d^2 \cdot N \\
&\cdot \prod_{\substack{l:\text{prime} \\ l \mid N}} (1 - l^{-r}T(l)\nu_{l^{-1}}) \\
&(T(p)^{n-m}(\sum_{\substack{a \in \mathbb{Z}/p^n\mathbb{Z} \\ a \neq 0}} \sum_{x \in ((1/Np^m)\mathbb{Z})/\mathbb{Z}} \sum_{y \in ((1/Np^n)\mathbb{Z})/\mathbb{Z}} \\
&((F^{(k-2)}_{1/Np^m, x} - c^{2-k} \cdot F^{(k-2)}_{c/Np^m, x}) \cdot (F^{(2)}_{Na/Np^n, y} \cdot \zeta^a_{p^n} - F^{(2)}_{Na/Np^n, y} \cdot \zeta^{da}_{p^n})))).
\end{aligned} \tag{4.9}$$

Proof. (1) The equality follows from the equality

$$\tilde{E}^{(2)}_{0,1/Np^n} - \tilde{E}^{(2)}_{0,d/Np^n} = \sum_{\substack{a\in\mathbb{Z}/Np^n\mathbb{Z}\\ a\neq 0}} \sum_{y\in((1/Np^n)\mathbb{Z})/\mathbb{Z}} (F^{(2)}_{a/Np^n,y}\cdot\zeta^a_{Np^n} - F^{(2)}_{a/Np^n,y}\cdot\zeta^{da}_{Np^n})$$

which can be obtained by computation.

(2) (3) The results are immediate from the definitions.

□

4.5.5. We prove that $z^{\mathrm{univ}}_{Np^\infty} \in \overline{M}_{Np^\infty}[[G^{(1)}_\infty \times G^{(2)}_\infty]][1/g]$.
For any $j \in \mathbb{Z}$, we define an isomorphism of rings

$$\chi^j : \mathbb{Z}_p[[G_\infty]] \underset{\cong}{\longrightarrow} \mathbb{Z}_p[[G_\infty]]\ ;\ g_a \mapsto a^j g_a \qquad (a \in \mathbb{Z}_p^\times),$$

and for $a_1, a_2 \in \mathbb{Z}$, we define an isomorphism of rings over O_{H}

$$O_{\mathsf{H}}[[G^{(1)}_\infty \times G^{(2)}_\infty]] \xrightarrow{(\chi^{a_1},\chi^{a_2})} O_{\mathsf{H}}[[G^{(1)}_\infty \times G^{(2)}_\infty]]\ ;$$

$$x \cdot g^{(1)}_{b_1} \cdot g^{(2)}_{b_2} \mapsto x b_1^{a_1} b_2^{a_2} \cdot g^{(1)}_{b_1} \cdot g^{(2)}_{b_2}$$

for $x \in O_{\mathsf{H}}$, $b_1, b_2 \in \mathbb{Z}_p^\times$.
Let c and d' be integers which are prime to p. We put

$${}_{c,d'}z^{\mathrm{univ}}_{Np^\infty} = (1 - c^{-1} g^{(1)}_{c^{-1}} g^{(2)}_c)(1 - d' g^{(2)}_{d'}) z^{\mathrm{univ}}_{Np^\infty} \in O_{\mathsf{H}}[[G^{(1)}_\infty \times G^{(2)}_\infty]].$$

Let a_1 and a_2 be integers such that $0 \le a_2 \le a_1$. We regard $((1 - c^{a_2-a_1-1} g^{(1)}_{c^{-1}} g^{(2)}_c)(1 - d'^{a_2+1} g^{(2)}_{d'}))^{-1} \cdot {}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\chi^{a_1},\chi^{a_2})$ as an element of $\mathsf{H}[[G^{(1)}_\infty \times G^{(2)}_\infty]]$ and write $z^{\mathrm{univ}}_{Np^\infty}(\chi^{a_1},\chi^{a_2})$ for it. Then directly from the definitions, $z^{\mathrm{univ}}_{Np^\infty}(\chi^{a_1},\chi^{a_2})$ coincides with the image of the product

$$\begin{aligned}
&\Big(\Big(\sum_{\substack{i\geq 1\\ (i,p)=1}} \sum_{j\geq 1} i^{a_2} q^{Nij}\Big)(g_{i,1} - (-1)^{a_2} g_{-i,1}) + \varprojlim_n \Big(\sum_{a\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \zeta_{\equiv a(p^n)}(-a_2)\cdot g_{a,1}\Big)\Big)\\
&\quad \cdot \Big(\Big(\sum_{\substack{i\geq 1, i\equiv 1(N)\\ (i,p)=1}} \sum_{j\geq 1} i^{a_1-a_2} q^{ij}\cdot g_{i,2} - (-1)^{a_1-a_2} \sum_{\substack{i\geq 1, i\equiv -1(N)\\ (i,p)=1}} \sum_{j\geq 1} i^{a_1-a_2} q^{ij}\cdot g_{-i,2}\Big)\\
&\quad + \varprojlim_n \Big(\sum_{\substack{a\in(\mathbb{Z}/Np^n\mathbb{Z})^\times\\ a\equiv 1(N)}} \zeta_{\equiv a(Np^n)}(-a_1+a_2)\cdot g_{a,2}\Big)\Big)
\end{aligned}$$

under the map (4.6). Hence concerning the image $z^{\mathrm{univ}}_{Np^\infty}(\chi^{a_1},\chi^{a_2})|_{(n,n)}$ of $z^{\mathrm{univ}}_{Np^\infty}(\chi^{a_1},\chi^{a_2})$ under the projection $\mathsf{H}[[G^{(1)}_\infty \times G^{(2)}_\infty]] \to \mathsf{H}[G^{(1)}_n \times G^{(2)}_n]$, we find that

$$z^{\mathrm{univ}}_{Np^\infty}(\chi^{a_1},\chi^{a_2})|_{(n,n)} = (Np^n)^{a_1-a_2-1}N^{-1}(p^n)^{a_2-1}$$
$$\sum_{\substack{b_1\in(\mathbb{Z}/Np^n\mathbb{Z})^\times \\ b_1\equiv 1(N)}}\ \sum_{b_2\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\Big(\sum_{x,y((1/Np^n)\mathbb{Z})/\mathbb{Z}} F^{(a_1-a_2+1)}_{b_1/Np^n,x}\cdot F^{(a_2+1)}_{Nb_2/Np^n,y}\Big)\cdot g^{(1)}_{b_1}\cdot g^{(2)}_{b_1^{-1}b_2} \tag{4.10}$$

with $F^{(a_1-a_2+1)}_{b_1/Np^n,x}$ and $F^{(a_2+1)}_{Nb_2/Np^n,y}$ in 4.5.1. From this, we obtain that

$$z^{\mathrm{univ}}_{Np^\infty}(\chi^{a_1},\chi^{a_2})|_{(n,n)} \in M_{a_1+2}(X_1(Np^n);\mathbb{Q})[G^{(1)}_n\times G^{(2)}_n](\subset \mathsf{H}[G^{(1)}_n\times G^{(2)}_n])$$

and hence

$$_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\chi^{a_1},\chi^{a_2})|_{(n,n)}$$
$$\in M_{a_1+2}(X_1(Np^n);\mathbb{Z}_{(p)})[G^{(1)}_n\times G^{(2)}_n](\subset O_{\mathsf{H}}[G^{(1)}_n\times G^{(2)}_n]).$$

Here $M_{a_1+2}(X_1(Np^n);\mathbb{Z}_{(p)}) = M_{a_1+2}(X_1(Np^n);\mathbb{Q})\cap\mathbb{Z}_{(p)}[[q]]$. The latter fact implies that $z^{\mathrm{univ}}_{Np^\infty}\in \overline{M}_{Np^\infty}[[G^{(1)}_\infty\times G^{(2)}_\infty]][1/g]$.

4.5.6. We see the relation between $z^{\mathrm{univ}}_{Np^\infty}$ and special values of zeta functions of cusp forms. This relation is described by the relation between the universal zeta modular form $z^{\mathrm{univ}}_{Np^\infty}$ and the zeta modular forms reviewed in 4.5.2 – 4.5.4. This relation will play an important role in sections 6 and 7.
Let

$$\cdot\zeta_{p^n}:\mathbb{Q}[G^{(2)}_n]\longrightarrow\mathbb{Q}(\zeta_{p^n}) \tag{4.11}$$

be the $\mathbb{Q}$-linear map given by the action of $G^{(2)}_n$ on ζ_{p^n} such that $g^{(2)}_a\mapsto\zeta^a_{p^n}$ $(a\in(\mathbb{Z}/p^n\mathbb{Z})^\times)$. We consider the image of $z^{\mathrm{univ}}_{Np^\infty}(\chi^{a_1},\chi^{a_2})|_{(n,n)}$ under the map

$$\cdot\zeta_{p^n}: M_{a_1+2}(X_1(Np^n);\mathbb{Q})[G^{(1)}_n\times G^{(2)}_n]\to M_{a_1+2}(X_1(Np^n);\mathbb{Q}(\zeta_{p^n}))[G^{(1)}_n] \tag{4.12}$$

induced by the map (4.11). By the caluculation until now, we see that the above image is

$$(Np^n)^{a_1-a_2-1}N^{-1}(p^n)^{a_2-1}\sum_{\substack{b_1\in(\mathbb{Z}/Np^n\mathbb{Z})^\times \\ b_1\equiv 1(N)}}\ \sum_{b_2\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\Big(\sum_{x,y\in((1/Np^n)\mathbb{Z})/\mathbb{Z}} F^{(a_1-a_2+1)}_{b_1/Np^n,x}\cdot F^{(a_2+1)}_{Nb_2/Np^n,y}\Big)\cdot\zeta^{b_1^{-1}b_2}_{p^n}\cdot g^{(1)}_{b_1}. \tag{4.13}$$

We assume $a_2\neq 1$. Putting $a_1=k-2$, $a_2=r-1$ in (4.13), $m=n$ in (4.8) in Lemma 4.5.3 (3), and comparing (4.13) with (4.8), we see that the element (4.13) is closely related to the element (4.8) in Lemma 4.5.3 (3). Roughly speaking, for an eigen cusp form f, "f-component" of $\sum_{x\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\psi(x)\nu_x\cdot(4.8)$, with a character $\psi:(\mathbb{Z}/p^n\mathbb{Z})^\times\to\overline{\mathbb{Q}}^\times$ $(n\geq 0)$, yields $L_{(Np)}(f,\psi,r)$ by period integrals. Here $L_{(Np)}(f,\psi,s)$ denotes the function obtained from

$L(f,\psi,s) = \sum_{i=1}^{\infty} a_i(f)\psi(i)i^{-s}$, where $a_i(f)$ is given by $T(i)f = a_i(f)f$, by removing prime(Np) factors of $L(f,\psi,s)$. We will see in section 6 that $\sum_{x\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \psi(x)\nu_x \cdot (4.13)$ yields $L_{(p)}(f,\psi,r)$. For the details, see section 6.
In the case $a_2 = 1$, the above statements must be modified as follows.
The image of $(1-c^{-a_1}g^{(1)}_{c^{-1}}g^{(2)}_c)(1-g^{(2)}_d)z^{\text{univ}}_{Np^\infty}(\chi^{a_1},\chi)|_{(n,n)}$ under the map (4.12) is closely related to the element in Lemma 4.5.4 (3).

5. Review of p-adic zeta functions of modular forms

In this section, we review the result of Amice-Vélu [AV] and Vishik [Vi] (Theorem 5.5) which concerns the existence and the characterizing properties of p-adic zeta functions of modular forms.

As referred above, in the rest of this paper, N denotes a positive integer which is prime to p.

5.1. Let

$$f = \sum_{n\geq 1} a_n(f)q^n \in M_k(X(1,Np^t)) \otimes \mathbb{C}$$

be a normalized eigen cusp form of weight $k \geq 2$ of level Np^t for some $t \geq 0$. We assume that the conductor of f is divisible by N. We further assume that t is the smallest integer ≥ 0 such that $f \in M_k(X(1,Np^t)) \otimes \mathbb{C}$. Set $K = \mathbb{Q}(a_n(f); n \geq 1)$. We take a prime λ of K which is above p, and let K_λ be the completion of K by λ.
Suppose that there exists an element $\alpha \in \overline{K_\lambda}^\times$ satisfying $v_p(\alpha) < k-1$ for the additive valuation v_p of $\overline{K_\lambda}$ normalized by $v_p(p) = 1$, and

$$1-\alpha p^{-s} \quad | \quad (p\text{-factor of } L(f,s))^{-1} \qquad \text{in } \overline{\mathbb{Q}_p}[p^{-s}],$$

where $L(f,s) = \sum_{n\geq 1} a_n(f)n^{-s}$ is the complex zeta function of f. Then the p-adic zeta function of f may be defined for each α satisfying the above conditions. We fix such α and suppress α in the notation of p-adic zeta functions. We will review the characterizing properties of a p-adic zeta function in 5.5.

5.2. We give a review of the space $\mathrm{H}_{K_\lambda,k-1}$ to which the p-adic zeta function of f belongs. We first set up the notation. For the natural decomposition $\mathbb{Z}_p^\times = \mathbb{F}_p^\times \times (1+p\mathbb{Z}_p)$ in the case $p \neq 2$ (resp. $\mathbb{Z}_2^\times = \{\pm 1\} \times (1+4\mathbb{Z}_2)$), let u be a topological generator of the second component $1+p\mathbb{Z}_p$ (resp. $1+4\mathbb{Z}_2$). We denote $\mathbb{F}_p^\times$ (resp. $\{\pm\}$) by Δ. As before $G^{(2)}_\infty$ is the Galois group $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p)$ which is endowed with an isomorphism to $\mathbb{Z}_p^\times$ via the cyclotomic character. Now for a finite extension L of $\mathbb{Q}_p$ and for a positive integer d, we define

$$\mathrm{H}_{L,d} = \{\sum_{\substack{n\geq 0\\ a\in\Delta}} c_{n,a}\cdot g^{(2)}_a \cdot (g^{(2)}_u - 1)^n \in L[\Delta][[g^{(2)}_u - 1]] \ ;$$
$$\lim_{n\longrightarrow\infty} |c_{n,a}|_p n^{-d} = 0 \text{ for all } a \in \Delta\}.$$

Here $|\ |_p$ is the multiplicative valuation of L normalized by $|p|_p = 1/p$. The space $\mathrm{H}_{L,d}$ is independent of the choice of u in the evident sense.
We have

$$O_L[[G^{(2)}_\infty]] \otimes_{O_L} L \subset \mathrm{H}_{L,1}, \quad \text{and} \quad \mathrm{H}_{L,i} \subset \mathrm{H}_{L,j} \ \text{ for } 1 \leq i \leq j.$$

Here the first inclusion is given by the natural map.
We put

$$\mathrm{H}_{L,\infty} = \bigcup_{d \geq 1} \mathrm{H}_{L,d},$$

then $\mathrm{H}_{L,\infty}$ is a ring.
For any positive integer d and for any subset U of $\mathbb{Z}$, we define a map

$$i_U : \mathrm{H}_{L,d} \longrightarrow \prod_{j \in U} L[[G^{(2)}_\infty]] = \prod_{j \in U} \varprojlim_n L[G^{(2)}_n]$$

$$\sum_{\substack{n \geq 0 \\ a \in \Delta}} c_{n,a} \cdot g^{(2)}_a \cdot (g^{(2)}_u - 1)^n \mapsto (\sum_{\substack{n \geq 0 \\ a \in \Delta}} c_{n,a} \cdot a^j g^{(2)}_a \cdot (u^j g^{(2)}_u - 1)^n)_j.$$

It is known that for any $d \geq 1$, the map $i_{\mathbb{Z}}$ is injective. Moreover for any different d integers $r_1, \ldots, r_d$, the map $i_{\{r_1,\ldots,r_d\}}$ is already injective:

$$i_{\{r_1,\ldots,r_d\}} : \mathrm{H}_{L,d} \hookrightarrow \prod_{j \in \{r_1,\ldots,r_d\}} L[[G^{(2)}_\infty]] (\subset \prod_{j \in \mathbb{Z}} L[[G^{(2)}_\infty]]).$$

Concerning the above injection, we have a proposition (cf. [AV]).

Proposition 5.3 . *Let*

$$\mathbb{H}_{L,d} \subset \prod_{j \in \{1,\ldots,d\}} \varprojlim_n L[G^{(2)}_n] = \prod_{j \in \{1,\ldots,d\}} L[[G^{(2)}_\infty]]$$

be the subspace consisting of elements

$$\mu = (\mu_j)_j = ((\mu_{j,n})_n)_j \in \prod_{j \in \{1,\ldots,d\}} \varprojlim_n L[G^{(2)}_n]$$

satisfying conditions (i) *and* (ii) *below. Then the map*

$$i_{\{1,\ldots,d\}} : \mathrm{H}_{L,d} \hookrightarrow \prod_{j \in \{1,\ldots,d\}} L[[G^{(2)}_\infty]]$$

induces a bijection from $\mathrm{H}_{L,d}$ *onto* $\mathbb{H}_{L,d}$*.*
(i) *For any* $j = 1, \ldots, d$*,*

$$\lim_{n \longrightarrow \infty} p^{dn} \mu_{j,n} = 0.$$

(ii) *For* $n \geq 1$*, let* $\phi_n : (\mathbb{Z}/p^n\mathbb{Z})^\times \longrightarrow \mathbb{Z}_p^\times$ *be a lifting, namely it is a map such that the composition* $(\mathbb{Z}/p^n\mathbb{Z})^\times \xrightarrow{\phi_n} \mathbb{Z}_p^\times \xrightarrow{\text{proj.}} (\mathbb{Z}/p^n\mathbb{Z})^\times$ *coincides with the identity map. For* $j \in \mathbb{Z}$*, let* $X^j_{\phi_n} : L[G^{(2)}_n] \to L[G^{(2)}_n]$ *be the* L*-homomorphism* *induced by* $g_a^{(2)} \mapsto \phi_n(a)^j g_a^{(2)}$ *for any* $a \in (\mathbb{Z}/p^n\mathbb{Z})^\times$. *Now for* $1 \le i \le d$ *and for any* ϕ_n *satisfying the above condition,*

$$\lim_{n \longrightarrow \infty} p^{(d-i+1)n} \sum_{j=0}^{i-1} (-1)^{i-j-1} \binom{i-1}{j} \mu_{j+1,n}(X_{\phi_n}^{-j}) = 0.$$

Here $\mu_{j,n}(X_{\phi_n}^{-j}) \in L[G_n^{(2)}]$ *is the image of* $\mu_{j,n}$ *under* $X_{\phi_n}^{-j}$.

Proof. For the proof, see [AV]. □

In section 6, we will construct the p-adic zeta function of f as an element of $\prod_{1\le j\le k-1} K_\lambda(\alpha)[[G_\infty^{(2)}]]$, and we will prove that it is contained in $\mathbb{H}_{K_\lambda(\alpha),k-1}$ by using Proposition 5.3.

5.4. In this subsection we give a preliminary discussion to introduce Theorem 5.5 concerning the existence and the characterizing properties of p-adic zeta functions.
In the rest of this section, we assume that $a_p(f) \neq 0$.

5.4.1. As in [Ka2], §6, we define $S(f)$ to be

$$S(f) = (M_k(X_1(Np^t);\mathbb{Q}) \otimes_{\mathbb{Q}} K)/(T(n) \otimes 1 - 1 \otimes a_n(f)\ ;\ n \ge 1),$$

which is the quotient of $M_k(X_1(Np^t);\mathbb{Q}) \otimes_{\mathbb{Q}} K$ by the K-subspace generated by $T(n) \otimes 1 - 1 \otimes a_n(f)$ for $n \ge 1$. This $S(f)$ is a one dimensional K-vector space.

5.4.2. We define $V_K(f)$ to be the quotient of $H^1(Y(1,Np^t)(\mathbb{C}),\ \mathrm{Sym}_{\mathbb{Z}}^{k-2}(R^1\lambda_*(\mathbb{Z})) \otimes_{\mathbb{Z}} K)$ by the K-subspace generated by the images of $T(n) \otimes 1 - 1 \otimes a_n(f)$ for $n \ge 1$. Here $\lambda : E \to Y(1,Np^t)$ is the universal elliptic curve. This $V_K(f)$ is a two dimensional K-vector space.

5.4.3. We put $V_{\mathbb{C}}(f) = V_K(f) \otimes_K \mathbb{C}$, and let

$$\mathrm{per}_f : S(f) \longrightarrow V_{\mathbb{C}}(f)$$

be the one induced by the period map (cf. for example, [Ka2], §5, 5.4)

$$\mathrm{per}_{1,Np^t} : M_k(X(1,Np^t)) \otimes \mathbb{C} \longrightarrow H^1(Y(1,Np^t)(\mathbb{C}), \mathrm{Sym}_{\mathbb{Z}}^{k-2}(R^1\lambda_*(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{C}).$$

5.4.4. For the $\mathbb{C}$-linear map

$$\iota : V_{\mathbb{C}}(f) \longrightarrow V_{\mathbb{C}}(f)$$

induced by the complex conjugation on $Y(1,Np^t)(\mathbb{C})$ and $E(\mathbb{C})$, and for $x \in V_{\mathbb{C}}(f)$, we define

$$x^+ = \frac{1}{2}(1+\iota)(x), \quad x^- = \frac{1}{2}(1-\iota)(x).$$

Now we take an element $\gamma \in V_K(f)$ such that $\gamma^+ \neq 0$, $\gamma^- \neq 0$. For $\omega \in S(f)$ and for the above $\gamma \in V_K(f)$, we define $\Omega(\omega,\gamma)_+, \Omega(\omega,\gamma)_- \in \mathbb{C}$ as

$$\mathrm{per}_f(\omega) = \Omega(\omega,\gamma)_+ \cdot \gamma^+ + \Omega(\omega,\gamma)_- \cdot \gamma^-.$$

5.4.5. For $x \in (\mathbb{Z}/Np^t\mathbb{Z})^\times$, let $\langle x\rangle$ be as in 3.5 in section 3.
Let

$$\epsilon_f : (\mathbb{Z}/Np^t\mathbb{Z})^\times \longrightarrow \overline{\mathbb{Q}}^\times$$

be the character defined by $\langle x\rangle f = \epsilon_f(x)\cdot f$ for $x \in (\mathbb{Z}/Np^t\mathbb{Z})^\times$.

5.4.6. As before, let $\chi_{\text{cyclo}} : G_\infty^{(2)} \xrightarrow{\cong} \mathbb{Z}_p^\times$ be the cyclotomic character. For $j \in \mathbb{Z}$, we regard

$$\chi_{\text{cyclo}}^j : G_\infty^{(2)} \longrightarrow \mathbb{Z}_p^\times \quad ; \quad g_a^{(2)} \mapsto a^j \quad (a \in \mathbb{Z}_p^\times)$$

also as a character $\mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$; $a \mapsto a^j$.
For $\mu = \sum_{\substack{n\geq 0\\ a\in\Delta}} c_{n,a}\cdot g_a^{(2)}\cdot(g_u-1)^n \in \mathrm{H}_{L,d}$ and a continuous character $\psi : \mathbb{Z}_p^\times \to \overline{\mathbb{Q}_p}^\times$, we define

$$\mu(\psi) = \sum_{\substack{n\geq 0\\ a\in\Delta}} c_{n,a}\cdot\psi(a)\cdot(\psi(u)-1)^n.$$

Let α be as in 5.1.

THEOREM 5.5 ([AV],[Vi]). *Let $h = \min\{n \in \mathbb{Z} \ ; \ n \geq 1, v_p(\alpha) < n\}(\leq k-1)$. For $\gamma \in V_K(f)$ such that $\gamma^+ \neq 0$, $\gamma^- \neq 0$, and for a non-zero $\omega \in S(f)$, we have a function*

$$L_{p\text{-}adic}(f)_{\omega,\gamma} \in \mathrm{H}_{K_\lambda,h} \subset \mathrm{H}_{K_\lambda,k-1},$$

characterized by the properties (i) *and* (ii) *below. In particular, if $v_p(\alpha) = 0$, $L_{p\text{-}adic}(f)$ belongs to the subspace*

$$L_{p\text{-}adic}(f)_{\omega,\gamma} \in O_{K_\lambda}[[G_\infty^{(2)}]] \otimes_{O_{K_\lambda}} K_\lambda \subset \mathrm{H}_{K_\lambda,1}.$$

(i) *Let $\psi : (\mathbb{Z}/p^n\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ be a character with conductor p^n $(n \geq 1)$. We put $\pm = (-1)^{k-r-1}\psi(-1)\epsilon_f(-1)$. Then for any integer r such that $1 \leq r \leq k-1$, we have*

$$\begin{aligned} &L_{p\text{-}adic}(f)_{\omega,\gamma}(\chi_{\text{cyclo}}^r\psi^{-1}) \\ &\quad = (r-1)!\cdot p^{nr}\alpha^{-n}\cdot G(\psi,\zeta_{p^n})^{-1}\cdot(2\pi i)^{k-r-1}\cdot\frac{1}{\Omega(\omega,\gamma)_\pm}\cdot L(f,\psi,r), \end{aligned}$$

where $G(\psi,\zeta_{p^n})$ denotes the Gauss sum $\sum_{x\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\psi(x)\zeta_{p^n}^x$ and $L(f,\psi,r)$ is the evaluation at $s = r$ of the function $L(f,\psi,s) = \sum_{i=1}^\infty a_i(f)\psi(i)i^{-s}$.

(ii) *We put $\pm = (-1)^{k-r-1}\epsilon_f(-1)$. For any integer r such that $1 \leq r \leq k-1$, we have*

$$\begin{aligned} &L_{p\text{-}adic}(f)_{\omega,\gamma}(\chi_{\text{cyclo}}^r) \\ &\quad = (r-1)!\cdot(2\pi i)^{k-r-1}\cdot\frac{1}{\Omega(\omega,\gamma)_\pm} \\ &\qquad \cdot(1-p^{r-1}\alpha^{-1})(1-\epsilon_f(p)p^{k-r-1}\alpha^{-1})\cdot L(f,r). \end{aligned}$$

REMARK 5.5.1 . (1) *In fact, both hands sides of the above equality belong to* $\overline{\mathbb{Q}}$.

(2) *The function in Theorem* 5.5 *can be characterized by only property* (i).

The function $L_{p\text{-adic}}(f)_{\omega,\gamma}$ in Theorem 5.5 is called the p-adic zeta function of f.

5.6. We have a canonical isomorphism

$$\mathrm{H}_{K_\lambda,d} \cong \mathrm{H}_{K_\lambda,d}/(g_{-1}^{(2)}-1) \times \mathrm{H}_{K_\lambda,d}/(g_{-1}^{(2)}+1).$$

For $x \in \mathrm{H}_{K_\lambda,d}$, we call its image in $\mathrm{H}_{K_\lambda,d}/(g_{-1}^{(2)}-1)$ (resp. $\mathrm{H}_{K_\lambda,d}/(g_{-1}^{(2)}+1)$) under the above isomorphism the $+$-part (resp. $-$-part) of x. Moreover we call $\mathrm{H}_{K_\lambda,d}/(g_{-1}^{(2)}-1)$ (resp. $\mathrm{H}_{K_\lambda,d}/(g_{-1}^{(2)}+1)$) the $+$-part (resp. $-$-part) of $\mathrm{H}_{K_\lambda,d}$.

5.7. As in [Ka2], §6, we define $\delta(f,k-1,0(1)) \in V_K(f)$ to be the image of $\delta(k,k-1,0(1)) \in H^1(Y(1,Np^t)(\mathbb{C}),\mathrm{Sym}_{\mathbb{Z}}^{k-2}(R^1\lambda_*(\mathbb{Z}))\otimes_{\mathbb{Z}}\mathbb{Q})$ in [Ka2], §5, 5.4. It is known that $\delta(f,k-1,0(1))^+ = 0$, and if $L(f,k-1)\neq 0$, $\delta(f,k-1,0(1)) = \delta(f,k-1,0(1))^- \neq 0$. In what follows, in the case $L(f,k-1)\neq 0$, we take $\delta(f,k-1,0(1)) \in V_K(f)$ as $\gamma \in V_K(f)$, we consider $(-1)^k\cdot\epsilon_f(-1)$-part of the p-adic zeta function of f, and we suppress γ in the notation of p-adic zeta functions.

6. THE RESULT ON ONE-VARIABLE p-ADIC ZETA FUNCTION

Let the notation and the setting be as in section 5. Suppose f is an eigen cusp form of weight $k\geq 2$ and of level Np^t with $t\geq 1$ which satisfies the condition in 5.1. In 6.1, for a certain subspace A of $\overline{M}_{Np^\infty}[[G_\infty^{(1)}\times G_\infty^{(2)}]]\otimes_{\mathbb{Z}_p[[G_\infty^{(1)}\times G_\infty^{(2)}]]} Q(\mathbb{Z}_p[[G_\infty^{(1)}\times G_\infty^{(2)}]])$ to which the universal zeta modular form $z_{Np^\infty}^{\mathrm{univ}}$ belongs, we define a map "to take f-component"

$$\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},t} : A \longrightarrow \prod_{r\in\{1,\dots,k-1\}} K_\lambda[[G_\infty^{(2)}]][G_t^{(1)}].$$

The main theorem (Theorem 6.2) of this section is, roughly speaking, that if $L(f,k-1)\neq 0$,

$$\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},t} \;:\; z_{Np^\infty}^{\mathrm{univ}} \;\mapsto\; p\text{-adic zeta function of } f$$

(see Theorem 6.2 for the precise statement).

6.1. We define the subspace A and the map $\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},t}$ in a more general forms $M[[G_\infty^{(1)}\times G_\infty^{(2)}]]_{I_1,I_2}$ (see 6.1.4) and $\mathfrak{L}_{f,I_1,I_2,i}$ (see 6.1.6), respectively. (For simplicity we assumed that $t\geq 1$ in the above, but in fact, we can treat also the case $t=0$, as seen below.) We begin with some preliminaries. Let f, K, and the other setting be as in 5.1 in section 5.

6.1.1. As in 5.1, we fix α under the notation there. We first consider the case that $t=0$. We denote the p-factor of $L(f,s)$ by $(1-\alpha p^{-s})^{-1}(1-\beta p^{-s})^{-1}$ with $\beta\in\overline{K_\lambda}^\times$.

Let

$$f_\alpha = f - \beta \cdot \varphi_q(f) \in M_k(X(1, Np)) \otimes \mathbb{C},$$

where $\varphi_q(f) = \sum_{n \geq 1} a_n(f) q^{pn}$ with $f = \sum_{n \geq 1} a_n(f) q^n$. We have

$$T(p) f_\alpha = \alpha \cdot f_\alpha,$$

$$L(f_\alpha, s) = (1 - \beta p^{-s}) L(f, s),$$

and $\epsilon_{f_\alpha} : (\mathbb{Z}/Np\mathbb{Z})^\times \longrightarrow \overline{\mathbb{Q}}^\times$ is the one induced by $(\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\epsilon_f} \overline{\mathbb{Q}}^\times$. This f_α is an eigen cusp form but is not a newform.

6.1.2. For f in 5.1, we put $\mathfrak{f} = f$ in the case that $t \geq 1$ with t in 5.1, and $\mathfrak{f} = f_\alpha$ in the case $t = 0$, where f_α is as in 6.1.1. Moreover let Np^m denote the level of $\mathfrak{f}$. (Namely $m = t$ in the case that $\mathfrak{f} = f$ and $m = 1$ in the case that $\mathfrak{f} = f_\alpha$ as above.) Further put

$$L = \begin{cases} K_\lambda & t \geq 1 \\ K_\lambda(\alpha) & t = 0. \end{cases}$$

6.1.3. Let $j, s \in \mathbb{Z}$, and let $M^j(X_1(Np^s); \mathbb{Q})$ be as in section 3. The space $(\cup_{j \geq 2, s \geq 1} M^j(X_1(Np^s); \mathbb{Q}) \otimes_{\mathbb{Q}} L)/(T(n) \otimes 1 - 1 \otimes a_n(\mathfrak{f})\ ;\ n \geq 1)$ is a one dimensional L-vector space in which the class of f is a base. We define a map $\mathrm{pr}_{\mathfrak{f}}$ by the following composition

$$\begin{aligned} \mathrm{pr}_{\mathfrak{f}} : & \bigcup_{j \geq 2\ s \geq 1} M^j(X_1(Np^s); \mathbb{Q}) \\ & \to \Big(\bigcup_{j \geq 2, s \geq 1} M^j(X_1(Np^s); \mathbb{Q}) \otimes_{\mathbb{Q}} L \Big)/(T(n) \otimes 1 - 1 \otimes a_n(\mathfrak{f})\ ;\ n \geq 1) \\ & \to L, \end{aligned} \tag{6.1}$$

where the first map is the natural projection and the second map is by sending the class of f to 1.

6.1.4. Let $I_1, I_2 \subset \mathbb{Z}$ be subsets. We denote by $M[[G_\infty^{(1)} \times G_\infty^{(2)}]]_{I_1, I_2}$ the $\mathbb{Z}_p[[G_\infty^{(1)} \times G_\infty^{(2)}]]$-submodule of $\overline{M}_{Np^\infty}[[G_\infty^{(1)} \times G_\infty^{(2)}]]$ defined in the following way:

$$\begin{aligned} & M[[G_\infty^{(1)} \times G_\infty^{(2)}]]_{I_1, I_2} \\ & \quad := \{ x \in \overline{M}_{Np^\infty}[[G_\infty^{(1)} \times G_\infty^{(2)}]]\ ; \\ & \qquad x(\chi^{a_1}, \chi^{a_2})|_{(n,n)} \in \Big(\bigcup_{j \geq 2,\ s \geq 1} M^j(X_1(Np^s); \mathbb{Z}_p) \Big)[G_n^{(1)} \times G_n^{(2)}] \\ & \qquad \text{for any } (a_1, a_2) \in I_1 \times I_2 \text{ and } n \geq 1 \}. \end{aligned}$$

6.1.5. Let c, d' be integers which are prime to p. By the result in 4.5.5, we find that for any $k \geq 2$,

$$\begin{aligned} {}_{c,d'}z^{\mathrm{univ}}_{Np^\infty} &= (1 - c^{-1} g^{(1)}_{c^{-1}} g^{(2)}_{c})(1 - d' g^{(2)}_{d'}) z^{\mathrm{univ}}_{Np^\infty} \\ &\in M[[G^{(1)}_\infty \times G^{(2)}_\infty]]_{\{k-2\},\{0,\dots,k-2\}}. \end{aligned} \tag{6.2}$$

6.1.6. Let i be a positive integer. We define a map "to take $\mathfrak{f}$-component"

$$\mathfrak{L}_{f,I_1,I_2,i} : M[[G^{(1)}_\infty \times G^{(2)}_\infty]]_{I_1,I_2} \longrightarrow \prod_{(a_1,a_2)\in I_1\times I_2} L[[G^{(2)}_\infty]][G^{(1)}_i]$$

as follows:

$$\mathfrak{L}_{f,I_1,I_2,i} = \prod_{(a_1,a_2)\in I_1\times I_2} \mathfrak{L}_{f,a_1,a_2,i}$$

for

$$\mathfrak{L}_{f,a_1,a_2,i} : M[[G^{(1)}_\infty \times G^{(2)}_\infty]]_{I_1,I_2} \longrightarrow L[[G^{(2)}_\infty]][G^{(1)}_i].$$

The map $\mathfrak{L}_{f,a_1,a_2,i}$ is defined as the composition

$$\begin{aligned} \mathfrak{L}_{f,a_1,a_2,i} : M[[G^{(1)}_\infty \times G^{(2)}_\infty]]_{I_1,I_2} &\xrightarrow{(\chi^{a_1},\chi^{a_2})} M[[G^{(1)}_\infty \times G^{(2)}_\infty]]_{\{0\},\{0\}} \\ &\xrightarrow{\text{proj.}} M[[G^{(2)}_\infty]][G^{(1)}_i]_{\{0\},\{0\}} \\ &\xrightarrow{\mathrm{pr}_{\mathfrak{f}}} L[[G^{(2)}_\infty]][G^{(1)}_i], \end{aligned}$$

where $M[[G^{(2)}_\infty]][G^{(1)}_i]_{\{0\},\{0\}}$ denotes the image of $M[[G^{(1)}_\infty \times G^{(2)}_\infty]]_{\{0\},\{0\}}$ under the natural projection $\overline{M}_{Np^\infty}[[G^{(1)}_\infty \times G^{(2)}_\infty]] \to \overline{M}_{Np^\infty}[[G^{(2)}_\infty]][G^{(1)}_i]$, and the last map is given by $\mathrm{pr}_{\mathfrak{f}}$ (6.1) for each coefficients of $G^{(2)}_n$ and by taking $\varprojlim_n$.

For $x \in M[[G^{(1)}_\infty \times G^{(2)}_\infty]]_{I_1,I_2} \otimes_{\mathbb{Z}_p[[G^{(1)}_\infty \times G^{(2)}_\infty]]} Q(\mathbb{Z}_p[[G^{(1)}_\infty \times G^{(2)}_\infty]])$, if there is a non-zerodivisor $g \in \mathbb{Z}_p[[G^{(1)}_\infty \times G^{(2)}_\infty]]$ such that $g(\chi^{a_1}, \chi^{a_2})$ is invertible in $\mathbb{Q}_p[[G^{(1)}_\infty \times G^{(2)}_\infty]]$ for all $(a_1, a_2) \in I_1 \times I_2$ and $gx \in M[[G^{(1)}_\infty \times G^{(2)}_\infty]]_{I_1,I_2}$, then we define

$$\begin{aligned} \mathfrak{L}_{f,I_1,I_2,i}(x) = \mathfrak{L}_{f,I_1,I_2,i}(gx) \cdot &\prod_{(a_1,a_2)\in I_1\times I_2} g(a_1,a_2)^{-1} \\ \in &\prod_{(a_1,a_2)\in I_1\times I_2} L[[G^{(2)}_\infty]][G^{(1)}_i]. \end{aligned}$$

6.1.7. By (6.2), we find that $\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},m}(z^{\mathrm{univ}}_{Np^\infty}) \in \prod_{a_2\in\{0,\dots,k-2\}} L[[G^{(2)}_\infty]][G^{(1)}_m]$ can be defined in the sense at the end of 6.1.6. In what follows, by putting $a_2 = r-1$, we write $\prod_{r\in\{1,\dots,k-1\}} L[[G^{(2)}_\infty]][G^{(1)}_m]$ instead of $\prod_{a_2\in\{0,\dots,k-2\}} L[[G^{(2)}_\infty]][G^{(1)}_m]$. Furthermore for $L[[G^{(2)}_\infty]]$, we define $+$ and $-$ parts in the same way as in 5.6, and we define the $*$-part of $\prod_{r\in\{1,\dots,k-1\}} L[[G^{(2)}_\infty]][G^{(1)}_m]$ with $* = +$ or $* = -$ by

$\prod_{r\in\{1,\dots,k-1\}} L[[G_\infty^{(2)}]]^{*\cdot(-1)^r}[G_m^{(1)}]$, where $L[[G_\infty^{(2)}]]^{*\cdot(-1)^r}$ is the $*\cdot(-1)^r$-part of $L[[G_\infty^{(2)}]]$.

We state our main theorem in this section.
In the situation of Theorem 5.5, we take the class of f as $\omega \in S(\mathfrak{f})$, and suppress the ω in the notation of p-adic zeta functions appearing below. Since we will assume $L(\mathfrak{f}, k-1) \neq 0$, we take $\delta(\mathfrak{f}, k-1, 0(1)) \in V_K(\mathfrak{f}) \setminus \{0\}$ as γ, and suppress γ in the notation of p-adic zeta functions, as referred in 5.7.

Theorem 6.2 . *Put $\pm = (-1)^k\epsilon_f(-1)$. Let $h = \min\{n \in \mathbb{Z}\ ;\ n \geq 1, v_p(\alpha) < n\}(\leq k-1)$, as in Theorem 5.5. Then we have*

$$\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},m}(z_{Np^\infty}^{\mathrm{univ}})^{\pm} \in \prod_{r\in\{1,\dots,k-1\}} L[[G_\infty^{(2)}]][G_m^{(1)}] \tag{6.3}$$

is contained in the subspace

$$i_{\{1,\dots,k-1\}}(\mathrm{H}_{L,h})[G_m^{(1)}].$$

Here $\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},m}(z_{Np^\infty}^{\mathrm{univ}})^{\pm}$ represents the $\pm$-part. Moreover if $v_p(\alpha) = 0$, (6.3) belongs to $i_{\{1,\dots,k-1\}}(O_L[[G_\infty^{(2)}]] \otimes_{O_L} L)[G_m^{(1)}]$.
Concerning the relation with p-adic zeta function, we have the following result. Suppose $L(\mathfrak{f}, k-1) \neq 0$. In the rest of this theorem, we identify an element of $\mathrm{H}_{L,h}$ with its image under $i_{\{1,\dots,k-1\}} : \mathrm{H}_{L,h} \hookrightarrow \prod_{r\in\{1,\dots,k-1\}} L[[G_\infty^{(2)}]]$.

(1) *In the case $\mathfrak{f} = f$, we have*

$$\begin{aligned}&\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},m}(z_{Np^\infty}^{\mathrm{univ}})^{\pm}\\ &\qquad = \alpha^m \sum_{a\in(\mathbb{Z}/p^m\mathbb{Z})^\times} L_{p\text{-}adic}(f)^{\pm}\epsilon_f(a'^{-1}) \cdot g_a^{(1)} \in \prod_{r\in\{1,\dots,k-1\}} L[[G_\infty^{(2)}]][G_m^{(1)}].\end{aligned}$$

Here $a' \in (\mathbb{Z}/Np^m\mathbb{Z})^\times$ is the element such that $a' \equiv 1(N)$ and $a' \equiv a(p^m)$.

(2) *In the case $\mathfrak{f} = f_\alpha$, we have*

$$\begin{aligned}&\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},1}(z_{Np^\infty}^{\mathrm{univ}})^{\pm}\\ &\qquad = \alpha \sum_{a\in(\mathbb{Z}/p\mathbb{Z})^\times} L_{p\text{-}adic}(f_\alpha)^{\pm} \cdot g_a^{(1)} \in \prod_{r\in\{1,\dots,k-1\}} L[[G_\infty^{(2)}]][G_1^{(1)}].\end{aligned}$$

(3) *In the above, we only considered the $(-1)^k\epsilon_f(-1)$-parts of $L_{p\text{-}adic}(\mathfrak{f})$, and we put the assumption that $L(\mathfrak{f}, k-1) \neq 0$ which always holds in the case $k \geq 3$. However we can obtain by the method in 6.7 below, the whole $L_{p\text{-}adic}(\mathfrak{f})$, including the $(-1)^{k-1}\epsilon_f(-1)$-part, without the assumption that $L(\mathfrak{f}, k-1) \neq 0$.*

Remark 6.2.1 . *In Theorem 6.2 (2), we present the p-adic zeta function of f_α instead of the p-adic zeta function of f. By the characterizing property of p-adic zeta functions in Theorem 5.5, their p-adic zeta functions are a multiple of the other by a non-zero constant.*

The rest of this section is devoted to the proof of Theorem 6.2. In 6.3, we prove that $\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},m}(z^{\mathrm{univ}}_{Np^\infty})^{\pm}$ is contained in $i_{\{1,\dots,k-1\}}(\mathrm{H}_{L,h})[G^{(1)}_m]$. Then in 6.4 – 6.7, we prove that $\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},m}(z^{\mathrm{univ}}_{Np^\infty})$ displays the characterizing property (i) of the p-adic zeta function in Theorem 5.5. (Cf. Remark 5.5.1 in section 5.)

6.3. We show that $\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},m}(z^{\mathrm{univ}}_{Np^\infty})^{\pm}$ belongs to $i_{\{1,\dots,k-1\}}(\mathrm{H}_{L,h})$ $[G^{(1)}_m]$.
First, we give a preliminary discussion which will be important also for the proof on the characterizing properties of p-adic zeta functions.

6.3.1. We define a homomorphism

$$\varphi_q : \mathsf{H} \longrightarrow \mathsf{H}$$

by $\varphi_q(\sum_{i=-\infty}^{\infty} a_i q^i) = \sum_{i=-\infty}^{\infty} a_i q^{pi}$ ($a_i \in \mathbb{Q}_p$, the valuation of a_i is bounded below, and $a_i \to 0$ when $i \to -\infty$). Let

$$\mathrm{Tr}_q : \mathsf{H} \longrightarrow \mathsf{H}$$

be the trace map associated to φ_q.
We use the same symbol Tr_q

$$\mathsf{H}[G^{(1)}_{a_1} \times G^{(2)}_{a_2}] \longrightarrow \mathsf{H}[G^{(1)}_{a_1} \times G^{(2)}_{a_2}] \qquad (a_1, a_2 \geq 1)$$

for the map induced by Tr_q on the coefficients.
For an element x of $L[[G^{(1)}_\infty \times G^{(2)}_\infty]]$ and for positive integers a_1, a_2, we denote by $x|_{(a_1,a_2)}$ the image of x under the natural projection

$$L[[G^{(1)}_\infty \times G^{(2)}_\infty]] \to L[G^{(1)}_{a_1} \times G^{(2)}_{a_2}].$$

Proposition 6.3.2 . *Let a_1, a_2 be integers such that $0 \leq a_2 \leq a_1$. Then for all positive integers n and m such that $n \geq m$, we have*

$$\mathrm{Tr}_q^{n-m}(z^{\mathrm{univ}}_{Np^\infty}(\chi^{a_1}, \chi^{a_2})|_{(m,n)}) \in M_{a_1+2}(X_1(Np^m); \mathbb{Q})[G^{(1)}_m \times G^{(2)}_n],$$

$$\mathrm{Tr}_q^{n-m}({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\chi^{a_1}, \chi^{a_2})|_{(m,n)}) \in M_{a_1+2}(X_1(Np^m); \mathbb{Z}_{(p)})[G^{(1)}_m \times G^{(2)}_n],$$

where $z^{\mathrm{univ}}_{Np^\infty}(\chi^{a_1}, \chi^{a_2})$ is as in 4.5.5.

Proof. Clearly $*|_{(m,n)}$ is the image of $*|_{(n,n)}$ under the projection $\mathsf{H}[G^{(1)}_n \times G^{(2)}_n] \to \mathsf{H}[G^{(1)}_m \times G^{(2)}_n]$. By (4.10), we know $z^{\mathrm{univ}}_{Np^\infty}(\chi^{a_1}, \chi^{a_2})|_{(n,n)}$ which is an element of $M_{a_1+2}(X_1(Np^n); \mathbb{Q})[G^{(1)}_n \times G^{(2)}_n]$ precisely, and hence we can calculate $\mathrm{Tr}_q^{n-m}(z^{\mathrm{univ}}_{Np^\infty}(\chi^{a_1}, \chi^{a_2}))|_{(m,n)}$. By this calculation, we find that the element $\mathrm{Tr}_q^{n-m}(z^{\mathrm{univ}}_{Np^\infty}(\chi^{a_1}, \chi^{a_2}))|_{(m,n)}$ belongs to $M_{a_1+2}(X_1(Np^m); \mathbb{Q})[G^{(1)}_m \times G^{(2)}_n]$. The assertion for ${}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}$ follows from this. $\square$

In 6.3.3 – 6.3.6, we show that the element $\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},m}(z^{\mathrm{univ}}_{Np^\infty})$ belongs to $i_{\{1,\dots,k-1\}}(\mathrm{H}_{L,h} \otimes_{\mathbb{Z}_p[[G^{(2)}_\infty]]} \mathbb{Z}_p[[G^{(2)}_\infty]][1/a])[G^{(1)}_m] = i_{\{1,\dots,k-1\}}(\mathrm{H}_{L,h}[1/a])[G^{(1)}_m]$

($\subset$ $(i_{\{1,\dots,k-1\}}(\mathrm{H}_{L,h}) \otimes_{\mathbb{Z}_p[[G_\infty^{(2)}]]} Q(\mathbb{Z}_p[[G_\infty^{(2)}]]))[G_m^{(1)}]$) with a certain non-zerodivisor $a \in \mathbb{Z}_p[[G_\infty^{(2)}]]$ which satisfies that $a(\chi^r)$ $(r = 1,\dots,k-1)$ are invertible in $\mathbb{Q}_p[[G_\infty^{(2)}]]$. In 6.3.7 – 6.3.12, we prove that $\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},m}(z_{Np^\infty}^{\mathrm{univ}})^{\pm}$ is, in fact, contained in $i_{\{1,\dots,k-1\}}(\mathrm{H}_{L,h})[G_m^{(1)}]$.

6.3.3. We prove that

$$\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},m}({}_{c,d'}z_{Np^\infty}^{\mathrm{univ}}) \in i_{\{1,\dots,k-1\}}(\mathrm{H}_{L,h})[G_m^{(1)}], \tag{6.4}$$

which means that $\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},m}(z_{Np^\infty}^{\mathrm{univ}}) \in i_{\{1,\dots,k-1\}}(\mathrm{H}_{L,h}[1/a])[G_m^{(1)}]$ with a non-zerodivisor $a \in \mathbb{Z}_p[[G_\infty^{(2)}]]$ (which satisfies that $a(\chi^r)$ $(r = 1,\dots,k-1)$ are invertible in $\mathbb{Q}_p[[G_\infty^{(2)}]]$). It follows directly from the definition that the projection $M_k(X_1(Np^n);\mathbb{Q}) \longrightarrow S(\mathfrak{f})$ $(n \geq m)$ commutes with the Hecke operator $T(p) = U(p) = \mathrm{Tr}_q$. By this and by the fact that the action of $T(p)$ on $S(\mathfrak{f})$ coincides with the multiplication by α, we obtain

$$\begin{aligned}&\mathfrak{L}_{f,\{k-2\},\{r-1\},m}({}_{c,d'}z_{Np^\infty}^{\mathrm{univ}})|_{(m,n)}\\ &\qquad = \alpha^{m-n}\cdot \mathrm{pr}_{\mathfrak{f}}(\mathrm{Tr}_q^{n-m}({}_{c,d'}z_{Np^\infty}^{\mathrm{univ}}(\chi^{k-2},\chi^{r-1})|_{(m,n)})).\end{aligned} \tag{6.5}$$

Therefore for the proof of (6.4) it is enough to show that

$$\begin{aligned}\prod_{r\in\{1,\dots,k-1\}} &\varprojlim_n (\alpha^{-n}\cdot \mathrm{pr}_{\mathfrak{f}}(\mathrm{Tr}_q^{n-m}({}_{c,d'}z_{Np^\infty}^{\mathrm{univ}}(\chi^{k-2},\chi^{r-1})|_{(m,n)})))\\ &\qquad\qquad \in i_{\{1,\dots,k-1\}}(\mathrm{H}_{L,h})[G_m^{(1)}].\end{aligned} \tag{6.6}$$

We denote the r-component $(r \in \{1,\dots,k-1\})$ of the left hand side of (6.6) by

$$\mu_r(\mathfrak{f}) = (\mu_{r,n}(\mathfrak{f}))_n \in \varprojlim_n L[G_n^{(2)}][G_m^{(1)}] = L[[G_\infty^{(1)}]][G_m^{(1)}].$$

In order to prove the assertion (6.6), by Proposition 5.3 in section 5, it is sufficient to show the following two assertions:
One is to show that

$$\lim_{n\longrightarrow\infty} p^{hn}\mu_{r,n}(\mathfrak{f}) = 0 \quad \text{for all } r \in \{1,\dots,k-1\}. \tag{6.7}$$

The other is to prove that for any $d \in \mathbb{Z}$ such that $h \leq d \leq k-1$, $(\mu_r(\mathfrak{f}))_{r=1,\dots,d}$ satisfy the condition (ii) in Proposition 5.3.

6.3.4. We check that $(\mu_r(\mathfrak{f}))_{r=1,\dots,k-1}$ satisfy (6.7) above.
By Proposition 6.3.2 which shows

$$\mathrm{Tr}_q^{n-m}({}_{c,d'}z_{Np^\infty}^{\mathrm{univ}}(\chi^{k-2},\chi^{r-1})|_{(m,n)}) \in M_k(X_1(Np^m);\mathbb{Z}_{(p)})[G_m^{(1)}][G_n^{(2)}]$$

for all $n \geq m$ and by the fact that $M_k(X_1(Np^m);\mathbb{Z}_{(p)})$ is a finitely generated $\mathbb{Z}_{(p)}$-module, we have that the image of $M_k(X_1(Np^m);\mathbb{Z}_{(p)})$ under $\mathrm{pr}_{\mathfrak{f}} : M_k(X_1(Np^m);\mathbb{Q}) \longrightarrow L$ is contained in $a\cdot O_L$ for some $a \in L^\times$.

From this, p^{nh} times $\mu_{r,n}(\mathfrak{f})$ is contained in $p^{nh}\cdot\alpha^{-n}a\cdot O_L[G_m^{(1)}][G_n^{(2)}]$. As $v_p(\alpha)<h$, we obtain $\lim_{n\to\infty}p^{nh}\cdot\alpha^{-n}a\cdot O_L=0$ which implies $\lim_{n\to\infty}p^{nh}\mu_{r,n}(\mathfrak{f})=0$ as desired.

6.3.5. We show that $(\mu_r(\mathfrak{f}))_{r=1,\dots,d}$ satisfy the condition (ii) of Proposition 5.3 for any $d\in\mathbb{Z}$ such that $h\le d\le k-1$.
We use the notation in Proposition 5.3 (ii). We write
${}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\chi^{k-2},\chi^j X_{\phi_n}^{-j})|_{(m,n)}$ for $({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\chi^{k-2},\chi^j)|_{(m,n)})(\mathrm{id},X_{\phi_n}^{-j})\in M_k(X_1(Np^n);\mathbb{Z}_{(p)})[G_m^{(1)}][G_n^{(2)}]$.
We can prove that

$$\begin{aligned}&\sum_{j=0}^{i-1}(-1)^{i-j-1}\binom{i-1}{j}\mu(\mathfrak{f})_{j+1,n}(X_{\phi_n}^{-j})\\ &\quad=\sum_{j=0}^{i-1}(-1)^{i-j-1}\binom{i-1}{j}\cdot\alpha^{-n}\cdot\mathrm{pr}_{\mathfrak{f}}(\mathrm{Tr}_q^{n-m}({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\chi^{k-2},\chi^jX_{\phi_n}^{-j})|_{(m,n)}))\end{aligned}$$

in Proposition 5.3 coincides with

$$\alpha^{-n}\cdot\mathrm{pr}_{\mathfrak{f}}(\mathrm{Tr}_q^{n-m}(\sum_{j=0}^{i-1}(-1)^{i-j-1}\binom{i-1}{j}\cdot({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\chi^{k-2},\chi^jX_{\phi_n}^{-j})|_{(m,n)}))).\tag{6.8}$$

Moreover we have

$$\begin{aligned}&\sum_{j=0}^{i-1}(-1)^{i-j-1}\binom{i-1}{j}\cdot({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\chi^{k-2},\chi^jX_{\phi_n}^{-j})|_{(m,n)})\\ &\quad\in M_k(X_1(Np^n);p^{n(i-1)}\mathbb{Z}_{(p)})[G_m^{(1)}][G_n^{(2)}]\end{aligned}$$

which follows from the general argument that

$$\sum_{j=0}^{i-1}(-1)^{i-j-1}\binom{i-1}{j}\cdot x(\chi^jX_{\phi_n}^{-j})|_n\in p^{n(i-1)}\mathbb{Z}_{(p)}[G_n]$$

for any $x\in\mathbb{Z}_{(p)}[[G_\infty]]$. Here $x(\chi^jX_{\phi_n}^{-j})|_n$ represents $x(\chi^j)|_n(X_{\phi_n}^{-j})$ for the image $x(\chi^j)|_n\in\mathbb{Z}_{(p)}[G_n]$ of $x(\chi^j)\in\mathbb{Z}_{(p)}[[G_\infty]]$ under the projection. By this, we find (6.8) is contained in $\alpha^{-n}\cdot a\cdot p^{n(i-1)}O_L[G_m^{(1)}][G_n^{(2)}]$ with $a\in L^\times$ in 6.3.4. As $v_p(\alpha)<h\le d$, we obtain

$$\lim_{n\longrightarrow\infty}p^{(d-i+1)n}\sum_{j=0}^{i-1}(-1)^{i-j-1}\binom{i-1}{j}\mu(\mathfrak{f})_{j+1,n}(X_{\phi_n}^{-j})=0,$$

which says that $(\mu_r(\mathfrak{f}))_{r=1,\dots,d}$ $(h\le d\le k-1)$ satisfy the condition (ii) of Proposition 5.3, as desired.

The above arguments conclude our claim (6.4) in 6.3.3.

6.3.6. The argument in 6.3.4 shows that if $v_p(\alpha)=0$,

$$\mu_r(\mathfrak{f})\in(O_L[[G_\infty^{(2)}]]\otimes_{O_L}L)[G_m^{(1)}].$$

This and the argument in 6.3.5 show that in this case, $\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},m}(z^{\mathrm{univ}}_{Np^\infty})$ is contained in $i_{\{1,\dots,k-1\}}((O_L[[G^{(2)}_\infty]] \otimes_{O_L} L)[1/a])[G^{(1)}_m]$.

Now we show that $\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},m}(z^{\mathrm{univ}}_{Np^\infty})^\pm \in i_{\{1,\dots,k-1\}}(\mathrm{H}_{L,h})[G^{(1)}_m]$. The main line of the proof is roughly as follows. Let M be an integer ≥ 1 which is prime to Np. By using ${}_{c,d'}z_{M,Np^\infty}$ which is similar with the element ${}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}$ in 4.5.5 and which is defined in 6.3.7 and 6.3.8, we will construct a function $\mathcal{L}_{\phi,f,\{k-2\},\{0,\dots,k-2\},m}({}_{c,d'}z_{M,Np^\infty}) \in i_{\{1,\dots,k-1\}}(\mathrm{H}_{L(\phi),h})[G^{(1)}_m]$ in 6.3.9. Here ϕ denotes a character $(\mathbb{Z}/M\mathbb{Z})^\times \longrightarrow \overline{\mathbb{Q}}^\times$. Assertions in 6.3.11 and 6.3.12 say that $\mathcal{L}_{\phi,f,\{k-2\},\{0,\dots,k-2\},m}({}_{c,d'}z_{M,Np^\infty})$ is, in fact, a function which is a multiple of our $\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},m}(z^{\mathrm{univ}}_{Np^\infty})$ by a non-zerodivisor in $\mathbb{Z}_p[[G^{(2)}_\infty]]$ which is prime to $a \in \mathbb{Z}_p[[G_\infty]]$ in 6.3.3.

6.3.7. Let M be a positive integer such that $(M, Np) = 1$. We define an element

$$z_{M,Np^\infty} \in \overline{M}_{MNp^\infty}[[G_{Mp^\infty}{}^{(1)} \times G_{Mp^\infty}{}^{(2)}]][\frac{1}{g'}]$$

with a certain non-zerodivisor

$$g' \in \mathbb{Z}[[G_{Mp^\infty}{}^{(1)} \times G_{Mp^\infty}{}^{(2)}]] \subset \overline{M}_{MNp^\infty}[[G_{Mp^\infty}{}^{(1)} \times G_{Mp^\infty}{}^{(2)}]],$$

which is similar to the universal zeta modular form $z^{\mathrm{univ}}_{Np^\infty}$. In fact, in the case $M = 1$, we have $z_{1,Np^\infty} = z^{\mathrm{univ}}_{Np^\infty}$. Here

$$G_{Mp^\infty}{}^{(1)} \cong G_{Mp^\infty}{}^{(2)} \cong G_{Mp^\infty} = (\mathbb{Z}/M\mathbb{Z})^\times \times \mathbb{Z}_p^\times,$$

the group $G_{Mp^\infty}{}^{(1)}$ is the one acting on the space $\overline{M}_{MNp^\infty}$ in the following way. For $a = (a_1, a_2) \in (\mathbb{Z}/M\mathbb{Z})^\times \times \mathbb{Z}_p^\times$, the action of the corresponding element $\mathfrak{g}^{(1)}_a \in G_{Mp^\infty}{}^{(1)}$ on $f = \sum_k f_k \in \bigcup_j M^j(X_1(MNp^t); \mathbb{Q}_p)$ with $f_k \in M_k(X_1(MNp^t); \mathbb{Q}_p)$ and $t \geq 1$ is given as

$$\mathfrak{g}^{(1)}_a \cdot f = \sum_k a_2^{k-2} \langle a' \rangle f_k,$$

where $a' \in (\mathbb{Z}/MNp^t\mathbb{Z})^\times$ is the element such that $a' \equiv a(Mp^t)$ and $a' \equiv 1(N)$. The group $G_{Mp^\infty}{}^{(2)}$ is the Galois group $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{Mp^\infty})/\mathbb{Q}_p)$ which is endowed with an isomorphism to $(\mathbb{Z}/M\mathbb{Z})^\times \times \mathbb{Z}_p^\times$ via the cyclotomic character.
The element z_{M,Np^∞} is the image of the product $F'_{N,1} \cdot F'_{N,2} \in \mathsf{H}[[G_{Mp^\infty}]][[G_{Mp^\infty}]]$ with $F'_{N,1}, F'_{N,2} \in \mathsf{H}[[G_{Mp^\infty}]]$ below under the isomorphism of rings over H

$$\mathsf{H}[[G_{Mp^\infty}]][[G_{Mp^\infty}]] \to \mathsf{H}[[G_{Mp^\infty}{}^{(1)}]][[G_{Mp^\infty}{}^{(2)}]]\ ;\ x\mathfrak{g}_{a,1}\mathfrak{g}_{b,2} \mapsto x\mathfrak{g}^{(1)}_b \mathfrak{g}^{(2)}_{ab^{-1}}$$

$(x \in \mathsf{H}, a, b \in \mathbb{Z}_p^\times \times (\mathbb{Z}/M\mathbb{Z})^\times)$, where $\mathfrak{g}_a \in G_{Mp^\infty}$ is the corresponding element to a. Here

$$F'_{N,1} = (\sum_{\substack{i\geq 1\\(i,Mp)=1}} \sum_{j\geq 1} q^{Nij})(\mathfrak{g}_i - \mathfrak{g}_{-i}) + \varprojlim_n (\sum_{a\in(\mathbb{Z}/Mp^n\mathbb{Z})^\times} \zeta_{\equiv a(Mp^n)}(0)\cdot \mathfrak{g}_a),$$

$$F'_{N,2} = (\sum_{\substack{i\geq 1, i\equiv 1(N)\\(i,Mp)=1}} \sum_{j\geq 1} q^{ij}\cdot \mathfrak{g}_i - \sum_{\substack{i\geq 1, i\equiv -1(N)\\(i,Mp)=1}} \sum_{j\geq 1} q^{ij}\cdot \mathfrak{g}_{-i})$$
$$+ \varprojlim_n (\sum_{\substack{a\in(\mathbb{Z}/NMp^n\mathbb{Z})^\times\\ a\equiv 1(N)}} \zeta_{\equiv a(NMp^n)}(0)\cdot \mathfrak{g}_a).$$

Originally z_{M,Np^∞} belongs to $\mathsf{H}[[{G_{Mp^\infty}}^{(1)} \times {G_{Mp^\infty}}^{(2)}]]$, but we can prove that $z_{M,Np^\infty} \in \overline{M}_{MNp^\infty}[[{G_{Mp^\infty}}^{(1)} \times {G_{Mp^\infty}}^{(2)}]][1/g']$. This follows from the following lemma which can be proven in the same way as for $z^{\mathrm{univ}}_{Np^\infty}$.
For $n \geq 1$, we write G_{Mp^n} for the group $(\mathbb{Z}/M\mathbb{Z})^\times \times G_n$.

Lemma 6.3.8 . (1) *Let c, d' be integers which are prime to p. Then*

$${}_{c,d'}z_{M,Np^\infty} := (1-c^{-1}\mathfrak{g}^{(1)}_{c^{-1}}\mathfrak{g}^{(2)}_c)(1-d'\mathfrak{g}^{(2)}_{d'})z_{M,Np^\infty} \in O_{\mathsf{H}}[[{G_{Mp^\infty}}^{(1)} \times {G_{Mp^\infty}}^{(2)}]].$$

(2) *For $i \in \mathbb{Z}$, let us denote by $\chi^i : G_{Mp^\infty} \to G_{Mp^\infty}$ the map induced by χ^i on the component $\mathbb{Z}_p^\times$. Let a_1, a_2 be integers such that $0 \leq a_2 \leq a_1$. We define $z_{M,Np^\infty}(\chi^{a_1}, \chi^{a_2})$ in the same way as for $z^{\mathrm{univ}}_{Np^\infty}$ in section 4. Concerning the image of the natural projection, we have*

$$z_{M,Np^\infty}(\chi^{a_1},\chi^{a_2})|_{(n,n)} \in M_{a_1+2}(X_1(MNp^n);\mathbb{Q})[G^{(1)}_{Mp^n}\times G^{(2)}_{Mp^n}]$$
$$(\subset \mathsf{H}[G^{(1)}_{Mp^n}\times G^{(2)}_{Mp^n}]),$$

$${}_{c,d'}z_{M,Np^\infty}(\chi^{a_1},\chi^{a_2})|_{(n,n)} \in M_{a_1+2}(X_1(MNp^n);\mathbb{Z}_{(p)})[G^{(1)}_{Mp^n}\times G^{(2)}_{Mp^n}]$$
$$(\subset O_{\mathsf{H}}[G^{(1)}_{Mp^n}\times G^{(2)}_{Mp^n}]).$$

(3) *Let a_1 and a_2 be as in (2). For any integers n, m such that $n \geq m \geq 1$, we have*

$$\mathrm{Tr}^{n-m}_q(z_{M,Np^\infty}(\chi^{a_1},\chi^{a_2})|_{(m,n)}) \in M_{a_1+2}(X_1(MNp^m);\mathbb{Q})[G^{(1)}_{Mp^m}][G^{(2)}_{Mp^n}].$$

$$\mathrm{Tr}^{n-m}_q({}_{c,d'}z_{M,Np^\infty}(\chi^{a_1},\chi^{a_2})|_{(m,n)}) \in M_{a_1+2}(X_1(MNp^m);\mathbb{Z}_{(p)})[G^{(1)}_{Mp^m}][G^{(2)}_{Mp^n}].$$

6.3.9. Let $\phi : (\mathbb{Z}/M\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ be a character whose conductor is M. Let $\mathfrak{f}_\phi$ be the modular form given by

$$\mathfrak{f}_\phi = \sum_{n\geq 1} a_n(\mathfrak{f})\phi(n)q^n.$$

It is an eigen cusp form of level MNp^m with zeta function $L(\mathfrak{f}_\phi, s) = L(\mathfrak{f}, \phi, s)$.

In the same manner as in 6.3.3 (6.5), we can show that

$$(\alpha^{-n}\phi(p)^{-n}\cdot \mathrm{pr}_{\mathfrak{f}_\phi}(\mathrm{Tr}_q^{n-m}({}_{c,d'}z_{M,Np^\infty}(\chi^{k-2},\chi^{r-1})|_{(m,n)})))_n \tag{6.9}$$

belongs to $\varprojlim_n L(\phi)[G^{(2)}_{Mp^n}][G^{(1)}_{Mp^m}]$, where $L(\phi)$ is the field generated over L by the values of ϕ. Here remark that $a_p(\mathfrak{f}_\phi)=a_p(\mathfrak{f})\phi(p)=\alpha\phi(p)$. We consider the image of the element (6.9) under the $L(\phi)$-homomorphism given by

$$L(\phi)[[G^{(2)}_{Mp^\infty}]][G^{(1)}_{Mp^m}]\to L(\phi)[[G^{(2)}_\infty]][G^{(1)}_m]\ ;\ x\mathfrak{g}^{(2)}_a\mathfrak{g}^{(1)}_b\mapsto x\phi(ab^2)g^{(2)}_a g^{(1)}_b \tag{6.10}$$

($x\in L(\phi),\mathfrak{g}_*\in G_{Mp^\infty},g_*\in G_\infty$). We write $\mathcal{L}_{\phi,f,\{k-2\},\{r-1\},m}({}_{c,d'}z_{M,Np^\infty})\in L(\phi)[[G^{(2)}_\infty]][G^{(1)}_m]$ for this image, and we put

$$\mathcal{L}_{\phi,f,\{k-2\},\{0,\dots,k-2\},m}({}_{c,d'}z_{M,Np^\infty})= \prod_{r\in\{1,\dots,k-1\}}\mathcal{L}_{\phi,f,\{k-2\},\{r-1\},m}({}_{c,d'}z_{M,Np^\infty})\in\prod_{r\in\{1,\dots,k-1\}}L(\phi)[[G^{(2)}_\infty]][G^{(1)}_m].$$

Concerning $\mathcal{L}_{\phi,f,\{k-2\},\{0,\dots,k-2\},m}({}_{c,d'}z_{M,Np^\infty})$, we have the following propositions, which are crucial to our purpose.

Proposition 6.3.10 . *The element $\mathcal{L}_{\phi,f,\{k-2\},\{0,\dots,k-2\},m}({}_{c,d'}z_{M,Np^\infty})$ is contained in the subspace $i_{\{1,\dots,k-1\}}(\mathrm{H}_{L(\phi),h})[G^{(1)}_m]$.*

Proof. We can prove Proposition 6.3.10 in the same manner as in 6.3.1 – 6.3.6. □

In the rest of 6.3, we identify an element of $\mathrm{H}_{L(\phi),h}$ and its image under $i_{\{1,\dots,k-1\}}:\mathrm{H}_{L(\phi),h}\hookrightarrow\prod_{r\in\{1,\dots,k-1\}}L(\phi)[[G^{(2)}_\infty]]$.

Proposition 6.3.11 . *Assume $L(\mathfrak{f}_\phi,k-1)=L(\mathfrak{f},\phi,k-1)\neq 0$ and $\phi(-1)=1$. Then we have*

$$\begin{aligned}&\mathcal{L}_{\phi,f,\{k-2\},\{0,\dots,k-2\},m}({}_{c,d'}z_{M,Np^\infty})^{\pm}\\ &\quad= x\cdot(\prod_{r\in\{1,\dots,k-1\}}(\prod_{\substack{l:\text{prime}\\ l|M}}(1-a_l(\mathfrak{f})l^{-r}g^{(2)}_{l^{-1}}+\epsilon_{\mathfrak{f}}(l)l^{k-1-2r}g^{(2)}_{l^{-2}})\\ &\qquad\cdot(1-c^{r-k}\epsilon_{\mathfrak{f}}(c^{-1})\phi(c^{-1})g^{(2)}_c)(1-{d'}^r\phi(d')g^{(2)}_{d'}))\\ &\qquad\cdot\alpha^m\cdot\mathfrak{L}_{f,\{k-2\},\{0,\dots,k-2\},m}(z^{\mathrm{univ}}_{Np^\infty}))^{\pm}\\ &\qquad\quad\in\mathrm{H}_{L(\phi),h}[G^{(1)}_m]\end{aligned}$$

for some $x\in L(\phi)^\times$, where $\pm=(-1)^k\epsilon_{\mathfrak{f}}(-1)$.

Proof. An element of $\mathrm{H}_{L(\phi),h}[1/e](\subset\mathrm{H}_{L(\phi),h}\otimes_{\mathbb{Z}_p[[G^{(2)}_\infty]]}Q(\mathbb{Z}_p[[G^{(2)}_\infty]]))$ can be characterized by specializations $\mathrm{H}_{L(\phi),h}[1/e]\to\overline{L}$ induced by $\psi\circ\chi^{r_i}:G^{(2)}_\infty\to\overline{\mathbb{Q}}^\times$ for different h integers r_i ($i=1,\dots,h$) and all but finitely many Dirichlet characters ψ. So we can prove Proposition 6.3.11 by comparing the images of both hands sides under such specializations. The image of the right hand side under specializations will be studied in 6.4 – 6.6 below. The image of the left hand side may be obtained in the same way as for the right hand side, but we omit the details. $\square$

6.3.12. We finish proving that the elements in Theorem 6.2 are contained in $i_{\{1,\ldots,k-1\}}(\mathrm{H}_{L,h})[G_m^{(1)}]$.
We have

$$\begin{aligned}
&\mathfrak{L}_{f,\{k-2\},\{0,\ldots,k-2\},m}({}_{c,d'}z_{Np^\infty}^{\mathrm{univ}})\\
&\quad=\Big(\prod_{r\in\{1,\ldots,k-1\}}(1-c^{r-k}\epsilon_{\mathfrak{f}}(c^{-1})g_c^{(2)})(1-d'^r g_{d'}^{(2)})\Big)\\
&\quad\quad\cdot\mathfrak{L}_{f,\{k-2\},\{0,\ldots,k-2\},m}(z_{Np^\infty}^{\mathrm{univ}})\\
&\quad\quad\quad\in \mathrm{H}_{L,h}[G_m^{(1)}](\subset \mathrm{H}_{L,h}\otimes_{\mathbb{Z}_p[[G_\infty^{(2)}]]}Q(\mathbb{Z}_p[[G_\infty^{(2)}]])[G_m^{(1)}]),
\end{aligned}$$

which follows directly from the definition.
Since $(1-a_l(\mathfrak{f})g_{l^{-1}}^{(2)}+\epsilon_{\mathfrak{f}}(l)l^{k-1}g_{l^{-2}}^{(2)})$ for all of the prime numbers l which are prime to Np do not have a common divisor, by Propositions 6.3.10 and 6.3.11, it is sufficient to show the following assertion.
There exist c, d', M, and ϕ which satisfy the above given conditions and the following condition. For some characters $\psi_1,\psi_2:(\mathbb{Z}/p^n\mathbb{Z})^\times\longrightarrow\overline{\mathbb{Q}}^\times$ with conductor divisible by p, $\phi(c)=\psi_1(c)\neq 1$ and $\phi(d')=\psi_2(d')\neq 1$, and $\phi(-1)=1$ hold. We can take such elements, therefore we obtain the desired result.

6.4. We prove that our elements in Theorem 6.2 (1) and (2) satisfy the characterizing properties of $L_{p\text{-adic}}(\mathfrak{f})$ in Theorem 5.5. In fact, the difference between Theorem 6.2 (1) and (2) comes from the fact that we take $\mathrm{pr}_{\mathfrak{f}}$ instead of "pr_f". We treat the cases Theorem 6.2 (1) and (2) together.
We use the notation in Theorem 5.5.

6.4.1. As referred earlier, for $L[[G_\infty^{(2)}]]$, we define $+$ and $-$ parts in the same way as in 5.6, and for $*=+$ or $-$, we define the $*$-part of $L[[G_\infty^{(2)}]][G_m^{(1)}]$ by $L[[G_\infty^{(2)}]]^*[G_m^{(1)}]$.
In both cases of Theorem 6.2 (1) and (2), we have

$$\begin{aligned}
&\mathfrak{L}_{f,\{k-2\},\{0,\ldots,k-2\},m}(z_{Np^\infty}^{\mathrm{univ}})\\
&\quad=\prod_{r\in\{1,\ldots,k-1\}}\varprojlim_n(\alpha^{m-n}\cdot\mathrm{pr}_{\mathfrak{f}}(\mathrm{Tr}_q^{n-m}(z_{Np^\infty}^{\mathrm{univ}}(\chi^{k-2},\chi^{r-1})|_{(m,n)})))\\
&\quad\quad\in\prod_{r\in\{1,\ldots,k-1\}}\varprojlim_n L[G_n^{(2)}][G_m^{(1)}]=\prod_{r\in\{1,\ldots,k-1\}}L[[G_\infty^{(2)}]][G_m^{(1)}].
\end{aligned}\tag{6.11}$$

This follows from the equality (6.5). We will prove the assertion that the image of the coefficient belonging to $\varprojlim_n L[G_n^{(2)}] = L[[G_\infty^{(2)}]]$ of $g_1^{(1)}$ in

$$(\prod_{\substack{l:\text{prime}\\ l|N}} (1 - a_l(\mathfrak{f})l^{-r}g_{l^{-1}}^{(2)}) \cdot \varprojlim_n (\alpha^{-n} \cdot \mathrm{pr}_{\mathfrak{f}}(\mathrm{Tr}_q^{n-m}(z_{Np^\infty}^{\mathrm{univ}}(\chi^{k-2}, \chi^{r-1})|_{(m,n)}))))^{\pm\cdot(-1)^r}$$

$$\in \varprojlim_n L[G_n^{(2)}][G_m^{(1)}] = L[[G_\infty^{(2)}]][G_m^{(1)}], \tag{6.12}$$

where $\pm = (-1)^k \epsilon_{\mathfrak{f}}(-1)$ and $(\ )^{\pm\cdot(-1)^r}$ represents the $\pm \cdot (-1)^r$-part of the element in ( ), under the map $\psi^{-1} : L[[G_\infty^{(2)}]] \to \overline{L}$ coincides with the image of

$$(\prod_{\substack{l:\text{prime}\\ l|N}} (1 - a_l(\mathfrak{f})g_{l^{-1}}^{(2)}) \cdot L_{p\text{-adic}}(\mathfrak{f}))^{\pm}$$

under $\chi_{\mathrm{cyclo}}^r \psi^{-1} : \mathrm{H}_{L,h} \to \overline{L}$.
By the following facts, this assertion deduces Theorem 6.2.
Firstly we have the equality (6.11).
Secondly $\prod_{\substack{l:\text{prime}\\ l|N}}(1 - a_l(\mathfrak{f})g_{l^{-1}}^{(2)}) \in O_L[[G_\infty^{(2)}]] \otimes_{O_L} L$ is a non-zerodivisor of $\mathrm{H}_{L,h}$.
Finally by the results in 4.5.5, we have

$$(\mathrm{Tr}_q^{n-m}(z_{Np^\infty}^{\mathrm{univ}}(\chi^{k-2}, \chi^{r-1})|_{(m,n)}))_n = \sum_{a \in (\mathbb{Z}/p^m\mathbb{Z})^\times} \langle a'^{-1} \rangle x \cdot g_a^{(1)}$$

$$\in M_k(X_1(Np^m); \mathbb{Q})[G_m^{(1)}][[G_\infty^{(2)}]], \tag{6.13}$$

where x is the coefficient of $g_1^{(1)}$ in the left hand side, and $a' \in (\mathbb{Z}/Np^m\mathbb{Z})^\times$ is the element such that $a' \equiv a(p^m)$ and $a' \equiv 1(N)$. Hence the coefficient in $\varprojlim_n L[G_n^{(2)}] = L[[G_\infty^{(2)}]]$ of $g_a^{(1)}$ in (6.12) is $\epsilon_{\mathfrak{f}}(a'^{-1})$ times the coefficient of $g_1^{(1)}$ in (6.12). (Remark that in the case of Theorem 6.2 (2), $\epsilon_{\mathfrak{f}}(a') = 1$ holds.)
Thus we prove the assertion above.

6.4.2. We use the same notation as in Theorem 5.5. We consider the following composition

$$\psi_{\zeta_{p^n}} : L[[G_\infty^{(2)}]] \xrightarrow{\text{proj.}} L[G_n^{(2)}] \to \overline{L}, \tag{6.14}$$

where the second map is defined by

$$\sum_{u \in (\mathbb{Z}/p^n\mathbb{Z})^\times} a_u g_u^{(2)} \mapsto \sum_{x \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \psi(x) \sum_{u \in (\mathbb{Z}/p^n\mathbb{Z})^\times} a_u \zeta_{p^n}^{ux} \qquad (a_u \in L).$$

Lemma 6.4.3 . *Let μ be an element of $L[[G_\infty^{(2)}]]$. Then we have*

$$\mu(\psi^{-1}) = \psi_{\zeta_{p^n}}(\mu) \cdot G(\psi, \zeta_{p^n})^{-1} \in \overline{L}.$$

Proof. We can prove the lemma by direct computation. □

6.4.4. We assume $\psi(-1)=(-1)^{k-r}\epsilon_{\mathfrak{f}}(-1)$. By Lemma 6.4.3, our task is showing that the image of the coefficient of $g_1^{(1)}$ in (6.12) under the map (6.14) is

$$(r-1)!\cdot p^{nr}\cdot\alpha^{-n}\cdot(2\pi i)^{k-r-1}\cdot\frac{1}{\Omega(f,\delta(\mathfrak{f},k-1,1(0)))_{-}}\cdot L_{(Np)}(\mathfrak{f},\psi,r).$$

6.5. In order to prove our claim in 6.4.4, we need to review the result in [Ka2].

6.5.1. As in [Ka2], let

$$z_{Np^n}(\mathfrak{f},r,k-1,0(1),\mathrm{prim}(Np))\in S(\mathfrak{f})\otimes\mathbb{Q}(\zeta_{Np^n})\quad(r\neq 2),$$

$$_{c,d}z_{Np^n}(\mathfrak{f},2,k-1,0(1),\mathrm{prim}(Np))\in S(\mathfrak{f})\otimes\mathbb{Q}(\zeta_{Np^n})$$

be the images of $z^{(k,r)}_{1,Np^m,Np^n}(k,r,k-1,0(1),\mathrm{prim}(Np))$ in the case $r\neq 2$, and $_{c,d}z^{(k,2)}_{1,Np^m,Np^n}(k,2,k-1,0(1),\mathrm{prim}(Np))$ in the case $r=2$, both of which are elements of $M_k(X_1(Np^m);\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{Q}(\zeta_{Np^n})$, respectively, in 4.5.2 under the projection

$$M_k(X_1(Np^m);\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{Q}(\zeta_{Np^n})\longrightarrow S(\mathfrak{f})\otimes_{\mathbb{Q}}\mathbb{Q}(\zeta_{Np^n}).\tag{6.15}$$

Let $\mathfrak{f}^*=\sum_{n\geq 1}\overline{a_n}(\mathfrak{f})q^n$ denote the dual cusp form of $\mathfrak{f}$. Here $\overline{a_n}(\mathfrak{f})$ are the complex conjugates of $a_n(\mathfrak{f})$. This is also a normalized eigen cusp form. For $n\geq 1$ such that $(n,Np)=1$, it is known that

$$(T(n)\langle n^{-1}\rangle)(\mathfrak{f}^*)=a_n(\mathfrak{f})\mathfrak{f}^*.\tag{6.16}$$

For $x\in(\mathbb{Z}/Np^n\mathbb{Z})^\times$, let us denote by ν_x the corresponding element to x of $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{Np^n})/\mathbb{Q}_p)$ via the cyclotomic character.

Proposition 6.5.2 . *Assume $r\neq 2$. Let $\psi:(\mathbb{Z}/Np^n\mathbb{Z})^\times\to\overline{\mathbb{Q}}^\times$ be a character. We put $\pm=(-1)^{k-r-1}\psi(-1)$. Then we have*

$$\sum_{x\in(\mathbb{Z}/Np^n\mathbb{Z})^\times}\psi(x)\cdot\mathrm{per}_{\mathfrak{f}}((\langle x^{-1}\rangle\otimes\nu_x)(z_{Np^n}(\mathfrak{f},r,k-1,0(1),\mathrm{prim}(Np))))^{\pm}$$
$$=L_{(Np)}(\mathfrak{f}^*,\psi,r)\cdot(2\pi i)^{k-r-1}\cdot\delta(\mathfrak{f},k-1,0(1))^{\pm},$$

where $L_{(Np)}(\mathfrak{f}^*,\psi,s)$ denotes the function obtained from $L(\mathfrak{f}^*,\psi,s)$ by removing prime(Np) *factors.*

Proof. Remark that the definitions of the actions of Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_{Np^n})/\mathbb{Q})$ are different between in [Ka2] and here: The action of σ_x in [Ka2], §6, Theorem 6.6 on $S(\mathfrak{f})\otimes\mathbb{Q}(\zeta_{Np^n})$ is equal to the action of $\langle x^{-1}\rangle\otimes\nu_x$ in our notation. By this relation we see that the above equation is equivalent to 6.6 in [Ka2] which can be deduced from the work of Shimura [Sh]. □

In the case $r = 2$, Proposition 6.5.2 must be modified as

$$\sum_{x \in (\mathbb{Z}/Np^n\mathbb{Z})^\times} \psi(x) \cdot \mathrm{per}_{\mathfrak{f}}((\langle x^{-1} \rangle \otimes \nu_x)({}_{c,d}z_{Np^n}(\mathfrak{f}, 2, k-1, 0(1), \mathrm{prim}(Np))))^{\pm}$$
$$= c^2 d^2 \cdot (1 - c^{2-k}\psi(c)^{-1})(1 - \psi(d)^{-1}\epsilon_{\mathfrak{f}}(d))$$
$$\cdot L_{(Np)}(\mathfrak{f}^*, \psi, 2) \cdot (2\pi i)^{k-3} \cdot \delta(\mathfrak{f}, k-1, 0(1))^{\pm}.$$

The above proposition induces the following corollary.

COROLLARY 6.5.3 . *Assume $r \neq 2$. Let ψ be as in Proposition* 6.5.2. *We put $\pm = (-1)^{k-r-1}\psi(-1)\epsilon_{\mathfrak{f}}(-1)$. Then we have*

$$\sum_{x \in (\mathbb{Z}/Np^n\mathbb{Z})^\times} \psi(x) \cdot \mathrm{per}_{\mathfrak{f}}((1 \otimes \nu_x)(z_{Np^n}(\mathfrak{f}, r, k-1, 0(1), \mathrm{prim}(Np))))^{\pm}$$
$$= L_{(Np)}(\mathfrak{f}, \psi, r) \cdot (2\pi i)^{k-r-1} \cdot \delta(\mathfrak{f}, k-1, 0(1))^{\pm}.$$

Proof. By the definition of $\epsilon_{\mathfrak{f}}$, we have the equality

$$\sum_{x \in (\mathbb{Z}/Np^n\mathbb{Z})^\times} \psi(x)$$
$$\cdot \mathrm{per}_{\mathfrak{f}}((\langle x^{-1} \rangle \otimes \nu_x)(z_{Np^n}(\mathfrak{f}, r, k-1, 0(1), \mathrm{prim}(Np))))^{(-1)^{k-r-1}\psi(-1)}$$
$$= \sum_{x \in (\mathbb{Z}/Np^n\mathbb{Z})^\times} \psi(x)\epsilon_{\mathfrak{f}}(x^{-1})$$
$$\cdot \mathrm{per}_{\mathfrak{f}}((1 \otimes \nu_x)(z_{Np^n}(\mathfrak{f}, r, k-1, 0(1), \mathrm{prim}(Np))))^{(-1)^{k-r-1}\psi(-1)}.$$

By Proposition 6.5.2 and by this, we see that the left hand side of the equation in Corollary 6.5.3 is equal to

$$L_{(Np)}(\mathfrak{f}^*, \psi \cdot \epsilon_{\mathfrak{f}}, r) \cdot (2\pi i)^{k-r-1} \cdot \delta(\mathfrak{f}, k-1, 0(1))^{\pm}.$$

By (6.16) and by the fact that $\epsilon_{\mathfrak{f}}(n) = \epsilon_{\mathfrak{f}^*}(n^{-1})$ for $n \in \mathbb{Z}$ such that $(n, Np) = 1$, we have

$$L_{(Np)}(\mathfrak{f}^*, \psi \cdot \epsilon_{\mathfrak{f}}, s) = L_{(Np)}(\mathfrak{f}, \psi, s).$$

This shows the result. □

With a suitable modification, we have a similar result with Corollary 6.5.3, in the case $r = 2$.

As a corollary to Corollary 6.5.3, we obtain the following Corollary 6.5.4 which is important for the proof of our Theorem 6.2.
For $r \neq 2$, let $A(k, r)$ be the element of $M_k(X_1(Np^m); \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_{p^n})$ obtained from the right hand side of (4.8) in Lemma 4.5.3 (3) in section 4 by replacing "$a \in \mathbb{Z}/p^n\mathbb{Z}$" in the right hand side of (4.8) by "$a \in (\mathbb{Z}/p^n\mathbb{Z})^\times$". Namely,

$$\begin{aligned}A(k,r) =&(r-1)!^{-1}\cdot (Np^m)^{k-r-2}(Np^n)^{-2}\cdot N\\ &\cdot \prod_{\substack{l:\text{prime}\\ l\mid N}}(1-l^{-r}T(l)\nu_{l^{-1}})\\ &(\sum_{a\in(\mathbb{Z}/p^n\mathbb{Z})^\times} T(p)^{n-m}\\ &\quad(\sum_{x\in((1/Np^m)\mathbb{Z})/\mathbb{Z}}\sum_{y\in((1/Np^n)\mathbb{Z})/\mathbb{Z}}(F^{(k-r)}_{1/Np^m,x}\cdot F^{(r)}_{Na/Np^n,y}))\cdot\zeta^a_{p^n}).\end{aligned}$$

We define $A(\mathfrak{f},r)$ to be the image of $A(k,r)$ under the projection (6.15).

COROLLARY 6.5.4 . *Assume $r\neq 2$. Let $\psi:(\mathbb{Z}/p^n\mathbb{Z})^\times\to\overline{\mathbb{Q}}^\times$ be a character which does not factor through $(\mathbb{Z}/p^{n-1}\mathbb{Z})^\times$ and which satisfies $\psi(-1)=(-1)^{k-r}\epsilon_{\mathfrak{f}}(-1)$. Then we have*

$$\begin{aligned}&\sum_{x\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\psi(x)\cdot\mathrm{per}_{\mathfrak{f}}((1\otimes\nu_x)(A(\mathfrak{f},r)))^-\\ &\qquad\qquad = L_{(Np)}(\mathfrak{f},\psi,r)\cdot(2\pi i)^{k-r-1}\cdot\delta(\mathfrak{f},k-1,0(1))^-.\end{aligned}$$

Proof. We take ψ as a character which factors through $(\mathbb{Z}/p^n\mathbb{Z})^\times\to\overline{\mathbb{Q}}^\times$ and does not factor through $(\mathbb{Z}/p^{n-1}\mathbb{Z})^\times$, and apply Corollary 6.5.3. Then the part of the sum $\sum_{x\in(\mathbb{Z}/N\mathbb{Z})^\times}$ is equivalent to take the trace map tr_{Np^n,p^n} in Lemma 4.5.3 (3). Concerning the problem of changing a, since ψ in Corollary 6.5.4 is primitive, the iterated sum over $\sum_{x\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\psi(x)\zeta^{ax}_{p^n}$ for $(a,p)>1$ become zero. □

In the case $r=2$, Corollary 6.5.4 must be modified as follows. We take ${}_{c,d}A(k,2)$ as the element which is obtained from (4.9) by replacing "$a\in\mathbb{Z}/p^n\mathbb{Z}, a\neq 0$" in the right hand side of (4.9) by "$a\in(\mathbb{Z}/p^n\mathbb{Z})^\times$". We also define ${}_{c,d}A(\mathfrak{f},2)$ to be the image of ${}_{c,d}A(k,2)$ under the projection (6.15). Then we have

$$\begin{aligned}&\sum_{x\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\psi(x)\cdot\mathrm{per}_{\mathfrak{f}}((1\otimes\nu_x)({}_{c,d}A(\mathfrak{f},2)))^-\\ &\quad= c^2d^2\cdot((1-c^{2-k}\epsilon_{\mathfrak{f}}(c)^{-1}\psi(c)^{-1})(1-\psi(d)^{-1}))\\ &\qquad L_{(Np)}(\mathfrak{f},\psi,2)\cdot(2\pi i)^{k-3}\cdot\delta(\mathfrak{f},k-1,0(1))^-.\end{aligned}$$

6.6. Now we prove our claim in 6.4.4.
Since $T(l)\mathfrak{f}=a_l(\mathfrak{f})\mathfrak{f}$, (6.12) coincides with

$$\begin{aligned}&(\varprojlim_n(\alpha^{-n}\\ &\quad\cdot\mathrm{pr}_{\mathfrak{f}}(\prod_{\substack{l:\text{prime}\\ l\mid N}}(1-T(l)l^{-r}g^{(2)}_{l^{-1}})(\mathrm{Tr}^{n-m}_q(z^{\mathrm{univ}}_{Np^\infty}(\chi^{k-2},\chi^{r-1})|_{(m,n)})))))^{\pm\cdot(-1)^r}.\end{aligned}$$

So to analyze (6.12), we consider the image of

$$\prod_{\substack{l:\text{prime}\\ l|N}} (1 - T(l)l^{-r}g^{(2)}_{l^{-1}})(\mathrm{Tr}_q^{n-m}(z^{\mathrm{univ}}_{Np^\infty}(\chi^{k-2}, \chi^{r-1})|_{(m,n)})) \in M_k(X_1(Np^m); \mathbb{Q})[G^{(1)}_m \times G^{(2)}_n] \tag{6.17}$$

under the map

$$\cdot\zeta_{p^n} : M_k(X_1(Np^m); \mathbb{Q})[G^{(1)}_m \times G^{(2)}_n] \to M_k(X_1(Np^m); \mathbb{Q}(\zeta_{p^n}))[G^{(1)}_m] \tag{6.18}$$

defined in the similar way as (4.12).
In the case $r \neq 2$, by comparing (4.8) in Lemma 4.5.3 (3) and equation (4.13) in 4.5.6 with $a_1 = k-2$ and $a_2 = r-1$, we see that the coefficient of $g^{(1)}_1 \in G^{(1)}_m$ in the image of (6.17) under the map (6.18) is

$$(r-1)! \cdot p^{nr} \cdot A(k, r)$$

with $A(k, r)$ in Corollary 6.5.4. Thus under the composition

$$L[[G^{(2)}_\infty]][G^{(1)}_m] \xrightarrow{\text{proj.}} L[G^{(2)}_n][G^{(1)}_m] \xrightarrow{\cdot\zeta_{p^n}} L(\zeta_{p^n})[G^{(1)}_m] \xrightarrow{\cdot f} (L(\zeta_{p^n}) \cdot f)[G^{(1)}_m], \tag{6.19}$$

the coefficient of $g^{(1)}_1$ in the element (6.12) is sent to

$$\alpha^{-n} \cdot (r-1)! \cdot p^{nr} \cdot A(\mathfrak{f}, r).$$

Hence by Corollary 6.5.4 and by comparing the definition of the map (6.14) and the map in Corollary 6.5.4, we obtain that under the assumption in 6.4.4, $\mathrm{per}_{\mathfrak{f}}(f)^-$ times the image in question in 6.4.4 equals

$$\alpha^{-n} \cdot (r-1)! \cdot p^{nr} \cdot L_{(Np)}(\mathfrak{f}, \psi, r) \cdot (2\pi i)^{k-r-1} \cdot \delta(\mathfrak{f}, k-1, 0(1))^-.$$

Since $L(\mathfrak{f}, k-1) \neq 0$, we have $\delta(\mathfrak{f}, k-1, 0(1)) = \delta(\mathfrak{f}, k-1, 0(1))^- \neq 0$. As $\mathrm{per}_{\mathfrak{f}}(f)^- = \Omega(f, \delta(\mathfrak{f}, k-1, 1(0)))_- \cdot \delta(\mathfrak{f}, k-1, 0(1))^-$, we find that the claim in 6.4.4 is true in the case $r \neq 2$.
In the case $r = 2$, in the same way as for $r \neq 2$, we can show that the coefficient of $g^{(1)}_1$ in the image of $c^2d^2 \cdot (1 - c^{2-k}\epsilon_{\mathfrak{f}}(c^{-1})g^{(2)}_c)(1 - g^{(2)}_d)$ times element (6.17) under the map (6.18) is $p^{2n} \cdot {}_{c,d}A(k, 2)$ with ${}_{c,d}A(k, 2)$ just after Corollary 6.5.4. Hence the image of the coefficient of $g^{(1)}_1$ in $c^2d^2 \cdot (1 - c^{2-k}\epsilon_{\mathfrak{f}}(c^{-1})g^{(2)}_c)(1 - g^{(2)}_d)$ times (6.12) under the composition (6.19) is $\alpha^{-n} \cdot p^{2n} \cdot {}_{c,d}A(\mathfrak{f}, 2)$. From this, in the same way as for $r \neq 2$, we can show that the claim in 6.4.4 is true for $r = 2$.
The above arguments conclude our desired result that the left hands sides of the equations in Theorem 6.2 (1) and (2) satisfy the characterizing property of the right hands sides of them.

6.7. We explain that by our method, we can obtain the whole p-adic zeta function $L_{p\text{-adic}}(\mathfrak{f})$ without assuming that $L(\mathfrak{f}, k-1) \neq 0$.

Similar with $z^{\mathrm{univ}}_{Np^\infty}$, elements z_{M,Np^∞} in 6.3.7 may be obtained via K_2 Coleman power series from the system of Beilinson elements $({}_{c,d}z_{MNp^n,Mp^n})_n \in \varprojlim_n K_2(Y(MNp^n, Mp^n))$, with $c, d \in \mathbb{Z}$ such that $(c, 6MNp) = (d, 6Mp) = 1$ and $c \equiv 1(N)$. In this case, we take the field $\mathsf{H}(\zeta_M)$ instead of H.
We define $\mathcal{L}_{\phi,f,\{k-2\},\{0,\ldots,k-2\},m}(z_{M,Np^\infty}) \in \prod_{r\in\{1,\ldots,k-1\}} L(\phi)[[G^{(2)}_\infty]][G^{(1)}_m]$ by replacing ${}_{c,d'}z_{M,Np^\infty}$ by z_{M,Np^∞} in (6.9) in the construction of the function $\mathcal{L}_{\phi,f,\{k-2\},\{0,\ldots,k-2\},m}({}_{c,d'}z_{M,Np^\infty})$. Then in the same way as for Theorem 6.2, one can show that if $L(\mathfrak{f}_\phi, k-1) = L(\mathfrak{f}, \phi, k-1) \neq 0$,

$$\begin{aligned}
&\mathcal{L}_{\phi,f,\{k-2\},\{0,\ldots,k-2\},m}(z_{M,Np^\infty})^{\pm\cdot\phi(-1)} \\
&\quad = x \cdot \sum_{a\in(\mathbb{Z}/p^m\mathbb{Z})^\times} ((\prod_{r\in\{1,\ldots,k-1\}} (\prod_{\substack{l:\text{prime} \\ l|M}} (1 - a_l(\mathfrak{f}) l^{-r} g^{(2)}_{l^{-1}} + \epsilon_{\mathfrak{f}}(l) l^{k-1-2r} g^{(2)}_{l^{-2}}))) \\
&\qquad\quad \cdot L_{p\text{-adic}}(\mathfrak{f}))^{\pm\cdot\phi(-1)} \epsilon_{\mathfrak{f}}(a'^{-1}) g^{(1)}_a \in \prod_{r\in\{1,\ldots,k-1\}} L(\phi)[[G^{(2)}_\infty]][G^{(1)}_m]
\end{aligned}$$

with some $x \in L(\phi)^\times$, and $\pm = (-1)^k \epsilon_{\mathfrak{f}}(-1)$.
Moreover by replacing the pair (M, Np) by (N, p), we define a function. Let $\phi : (\mathbb{Z}/N\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ be a character whose conductor is N. Now we define $\mathcal{L}_{\phi,f,\{k-2\},\{0,\ldots,k-2\},m}(z_{N,p^\infty}) \in \prod_{r\in\{1,\ldots,k-1\}} L(\phi)[[G^{(2)}_\infty]][G^{(1)}_m]$ to be the image of

$$\begin{aligned}
&\prod_{r\in\{1,\ldots,k-1\}} (\alpha^{-n}\phi(p)^{-n} \cdot \mathrm{pr}_{\mathfrak{f}_\phi}(\mathrm{Tr}^{n-m}_q({}_{c,d'}z_{N,p^\infty}(\chi^{k-2}, \chi^{r-1})|_{(m,n)})))_n \\
&\qquad \in \prod_{r\in\{1,\ldots,k-1\}} L(\phi)[[G_{Np^\infty}]][G^{(1)}_{Np^m}]
\end{aligned}$$

under an $L(\phi)$-homomorphism given by

$$L(\phi)[[G^{(2)}_{Np^\infty}]][G^{(1)}_{Np^m}] \to L(\phi)[[G^{(2)}_\infty]][G^{(1)}_m] \ ; \ x\mathfrak{g}^{(2)}_a \mathfrak{g}^{(1)}_b \mapsto x\phi(ab^2) g^{(2)}_a g^{(1)}_b$$

$(x \in L(\phi), \mathfrak{g}_* \in G_{Np^\infty}, g_* \in G_\infty)$.
One can show that if $L(\mathfrak{f}_\phi, k-1) = L(\mathfrak{f}, \phi, k-1) \neq 0$,

$$\begin{aligned}
&\mathcal{L}_{\phi,f,\{k-2\},\{0,\ldots,k-2\},m}(z_{N,p^\infty})^{\pm\cdot\phi(-1)} \\
&\quad = x \cdot \sum_{a\in(\mathbb{Z}/p^m\mathbb{Z})^\times} (((\prod_{r\in\{1,\ldots,k-1\}} (\prod_{\substack{l:\text{prime} \\ l|N}} (1 - a_l(\mathfrak{f}) l^{-r} g^{(2)}_{l^{-1}})) \\
&\qquad\quad \cdot L_{p\text{-adic}}(\mathfrak{f}))^{\pm\cdot\phi(-1)}) \epsilon_{\mathfrak{f}}(a'^{-1}) g^{(1)}_a \in \prod_{r\in\{1,\ldots,k-1\}} L(\phi)[[G^{(2)}_\infty]][G^{(1)}_m]
\end{aligned}$$

with some $x \in L(\phi)^\times$, and $\pm = (-1)^k \epsilon_{\mathfrak{f}}(-1)$.

From the above equalities, we find that by taking various ϕ which satisfy $L(\mathfrak{f},\phi,k-1)\neq 0$, the whole (i.e. including the $(-1)^{k-1}\epsilon_{\mathfrak{f}}(-1)$-part of) $L_{p\text{-adic}}(\mathfrak{f})$ can be obtained by our method without assuming $L(\mathfrak{f},k-1)\neq 0$.

7. The result on two-variable p-adic zeta function

For a module A over $\mathbb{Z}_p[[G^{(1)}_\infty \times G^{(2)}_\infty]]$, we put $A_Q = A\otimes_{\mathbb{Z}_p[[G^{(1)}_\infty \times G^{(2)}_\infty]]} Q(\mathbb{Z}_p[[G^{(1)}_\infty \times G^{(2)}_\infty]])$. In 7.1, for a certain subspace B of $\overline{M}_{Np^\infty}[[G^{(1)}_\infty \times G^{(2)}_\infty]]_Q$ to which the universal zeta modular form $z^{\mathrm{univ}}_{Np^\infty}$ belongs, we define a map (in 7.1.3)

$$\mathrm{L}_N : B \longrightarrow (((\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty})\otimes_\Lambda Q(\Lambda))[[G^{(2)}_\infty]])_Q.$$

The main theorem (Theorem 7.3) of this section is that

$$\mathrm{L}_N \ : \ z^{\mathrm{univ}}_{Np^\infty} \ \mapsto \text{a “universal ordinary } p\text{-adic zeta function”},$$

where the universal ordinary p-adic zeta function is a p-adic zeta function in two variables associated to the universal family of ordinary cusp forms (see Theorem 7.3 for the details).

7.1. We define the subspace B and the map L_N.

7.1.1. We put $\Lambda = \mathbb{Z}_p[[G^{(1)}_\infty]]$ as in 3.5. Let us define a subspace $\overline{\mathfrak{m}}_\Lambda$ of $\overline{M}_{Np^\infty}[[G^{(1)}_\infty]]$ in the following way:

$$\overline{\mathfrak{m}}_\Lambda = \{x\in \overline{M}_{Np^\infty}[[G^{(1)}_\infty]] \ \ ; \ \ g^{(1)}_{a^{-1}}\cdot x_{n,b} = x_{n,ab} \quad \text{for all } a,b\in(\mathbb{Z}/p^n\mathbb{Z})^\times\}.$$

Here $x_{n,a}\in \overline{M}_{Np^\infty}$ are defined by $x|_n = \sum_{a\in(\mathbb{Z}/p^n\mathbb{Z})^\times} x_{n,a}g^{(1)}_a \in \overline{M}_{Np^\infty}[G^{(1)}_n]$ with the image $x|_n$ of x under the projection $\overline{M}_{Np^\infty}[[G^{(1)}_\infty]]\longrightarrow \overline{M}_{Np^\infty}[G^{(1)}_n]$.

Proposition 7.1.2 . (1) *We have*

$$z^{\mathrm{univ}}_{Np^\infty}\in \overline{\mathfrak{m}}_\Lambda[[G^{(2)}_\infty]][\frac{1}{g}](\subset \overline{M}_{Np^\infty}[[G^{(1)}_\infty\times G^{(2)}_\infty]][\frac{1}{g}]),$$

where g is as before.
(2) *Let $\mathfrak{i}:\overline{M}_{Np^\infty}\longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{H}_{Np^\infty},\mathbb{Z}_p)$ be as in* 3.3*. We denote by the same symbol $\mathfrak{i}$ the map*

$$\begin{aligned}\mathfrak{i}:\overline{M}_{Np^\infty}[[G^{(1)}_\infty]] &\longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{H}_{Np^\infty},\mathbb{Z}_p)[[G^{(1)}_\infty]]\\ &= \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{H}_{Np^\infty},\Lambda)\end{aligned}$$

induced by $\mathfrak{i}$ in 3.3*. Then*

$$\mathfrak{i}(\overline{\mathfrak{m}}_\Lambda)\subset \mathrm{Hom}_\Lambda(\mathcal{H}_{Np^\infty},\Lambda).$$

Proof. (1)The results in 4.5.5 deduce the assertion.
(2) The result follows directly from the definition. □

7.1.3. We define a map

$$\overline{\mathfrak{m}}_\Lambda \longrightarrow (\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty}) \otimes_\Lambda Q(\Lambda) \tag{7.1}$$

which induces

$$\mathrm{L}_N : \overline{\mathfrak{m}}_\Lambda[[G^{(2)}_\infty]][\frac{1}{g}] \longrightarrow ((\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty}) \otimes_\Lambda Q(\Lambda))[[G^{(2)}_\infty]][\frac{1}{g}]$$

for a non-zerodivisor $g \in \mathbb{Z}_p[[G^{(1)}_\infty \times G^{(2)}_\infty]]$.
The map (7.1) is defined as the following composition:

$$\begin{aligned}
\overline{\mathfrak{m}}_\Lambda &\xrightarrow{\mathrm{i}} \mathrm{Hom}_\Lambda(\mathcal{H}_{Np^\infty}, \Lambda) \xrightarrow{\mathrm{proj.}} \mathrm{Hom}_\Lambda(\mathcal{H}^{\mathrm{ord}}_{Np^\infty}, \Lambda) \\
&\to \mathrm{Hom}_{Q(\Lambda)}(P^{\mathrm{ord}}_{Np^\infty} \otimes_\Lambda Q(\Lambda), Q(\Lambda)) \\
&\to \mathrm{Hom}_{Q(\Lambda)}(p^{\mathrm{ord}}_{Np^\infty} \otimes_\Lambda Q(\Lambda), Q(\Lambda)) \\
&\to (\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty}) \otimes_\Lambda Q(\Lambda).
\end{aligned} \tag{7.2}$$

The second arrow is by the natural projection. The third arrow is the evident one. The fourth arrow is given as follows. By the definition of $P^{\mathrm{ord}}_{Np^\infty}$ and $p^{\mathrm{ord}}_{Np^\infty}$, it follows that the natural map $P^{\mathrm{ord}}_{Np^\infty} \to p^{\mathrm{ord}}_{Np^\infty}$ is surjective. The fourth arrow is given as the unique section $p^{\mathrm{ord}}_{Np^\infty} \otimes_\Lambda Q(\Lambda) \to P^{\mathrm{ord}}_{Np^\infty} \otimes_\Lambda Q(\Lambda)$ as algebras over $Q(\Lambda)$ of the surjective homomorphism between semisimple algebras $P^{\mathrm{ord}}_{Np^\infty} \otimes_\Lambda Q(\Lambda) \to p^{\mathrm{ord}}_{Np^\infty} \otimes_\Lambda Q(\Lambda)$ which is induced by the above surjective map. Finally the last arrow in (7.2) is defined in the following way. By Proposition 3.6, $p^{\mathrm{ord}}_{Np^\infty} \otimes_\Lambda Q(\Lambda) \cong (\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty}) \otimes_\Lambda Q(\Lambda)$ are finitely generated semisimple algebras over $Q(\Lambda)$. Hence we have an isomorphism

$$(\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty}) \otimes_\Lambda Q(\Lambda) \cong \mathrm{Hom}_{Q(\Lambda)}((\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty}) \otimes_\Lambda Q(\Lambda), Q(\Lambda)) \quad ; \tag{7.3}$$

$$a \mapsto (x \mapsto \mathrm{Tr}(a \cdot x)),$$

where Tr is the trace map

$$\mathrm{Tr} : \mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty} \otimes_\Lambda Q(\Lambda) \longrightarrow Q(\Lambda).$$

This map gives the last map of (7.2).

7.1.4. We define a universal ordinary p-adic zeta function

$$L^{\mathrm{ord,univ}}_{p\text{-adic}} \in (\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty})[[G^{(2)}_\infty]][\frac{1}{hg}],$$

where $h \in \mathfrak{h}^{\mathrm{ord}}_{Np^\infty}$ is a certain non-zerodivisor and g is as before, as

$$L^{\mathrm{ord,univ}}_{p\text{-adic}} = \mathrm{L}_N(z^{\mathrm{univ}}_{Np^\infty}).$$

From the definition, one can see that universal ordinary p-adic zeta function $L^{\mathrm{ord,univ}}_{p\text{-adic}}$ is an element of $(1/h)(\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty})[[G^{(2)}_\infty]][1/g]$, which is contained in $(\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty})[[G^{(2)}_\infty]][1/hg]$.

7.2. We review basic facts and results of Hida about ordinary eigen cusp forms.

7.2.1. Let $k \geq 2$ and $m \geq 1$. Let $f \in M_k(X(1, Np^m)) \otimes \mathbb{C}$ be a normalized eigen cusp form. We recall that f is called ordinary if the eigenvalue of $T(p)$ on f is a p-adic unit. Further f is called an ordinary p-stabilized newform of tame conductor N when f is ordinary, the conductor of f is divisible by N, and $m \geq 1$ (cf. [GS]). We call Np^m the level of an ordinary p-stabilized newform f of tame conductor N if m is the smallest (positive) integer such that $f \in M_k(X(1, Np^m)) \otimes \mathbb{C}$.
An ordinary p-stabilized newform of tame conductor N of level Np^m is either it is already a newform of level Np^m or it is an ordinary eigen cusp form f_α obtained from a newform f of level N when α is a p-adic unit by the method in 6.1.1 (cf.[GS]).

7.2.2. We call a ring homomorphism $\kappa : \mathfrak{h}^{\mathrm{ord}}_{Np^\infty} \longrightarrow \overline{\mathbb{Z}_p}$ satisfying the following conditions an N-primitive arithmetic point (cf. [GS]). For an integer i such that $i \geq 0$ and a Dirichlet character ψ of conductor p^n $(n \geq 0)$, let $\mathfrak{P}_{i,\psi}$ be the kernel of the $\mathbb{Z}_p$-homomorphism $\Lambda \to \overline{\mathbb{Z}_p}$ induced by $\psi \circ \chi^i : G^{(1)}_\infty \to \overline{\mathbb{Z}_p}^\times$. The condition for an N-primitive arithmetic point is that it factors through

$$\kappa : \mathfrak{h}^{\mathrm{ord}}_{Np^\infty} \to \mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/(\mathfrak{P}_{i,\psi} + \mathcal{I}^{\mathrm{ord}}_{Np^\infty}) \to \overline{\mathbb{Z}_p} \tag{7.4}$$

for some $i \geq 0$ and some ψ. Here $\mathcal{I}^{\mathrm{ord}}_{Np^\infty}$ is the ideal defined in 3.7 in section 3.

7.2.3. In the case $p \geq 5$, Hida [Hi2], §1, Corollary 1.3 proved that for an N-primitive arithmetic point κ which factors as (7.4), we have a unique ordinary p-stabilized newform $f = \sum_{n \geq 0} a_n(f) q^n$ of weight $i+2$ of tame conductor N such that $\kappa(T(n)) = a_n(f)$.

Theorem 7.3 . *We assume $p \geq 5$. The universal ordinary p-adic zeta function*

$$L^{\mathrm{ord,univ}}_{p\text{-}adic} \in (\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty})[[G^{(2)}_\infty]][\frac{1}{hg}]$$

defined in 7.1.4 *displays property* (7.6) *below. Let*

$$\kappa : \mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty} \longrightarrow \overline{\mathbb{Z}_p}$$

be an N-primitive arithmetic point. We write $f_\kappa = \sum_{n \geq 1} a_n(f_\kappa) q^n$ for the ordinary p-stabilized newform of tame conductor N attached in the sense of 7.2.3 *with κ. We denote the weight and level of f_κ by k and Np^m $(m \geq 1)$, respectively. We always have that $\kappa(g) \in \overline{\mathbb{Z}_p}[[G^{(2)}_\infty]]$ is a non-zero divisor. We assume that $\kappa(h) \neq 0$. Then κ induces the following homomorphism which is also denoted by κ :*

$$\kappa : (\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty})[[G^{(2)}_\infty]][\frac{1}{hg}] \longrightarrow Q(\overline{\mathbb{Z}_p}[[G^{(2)}_\infty]]). \tag{7.5}$$

Now if $L(f_\kappa, k-1) \neq 0$, then concerning the image $L^{\mathrm{ord,univ}}_{p\text{-}adic}(\kappa)$ of $L^{\mathrm{ord,univ}}_{p\text{-}adic}$ under (7.5)*, we have*

$$L^{\mathrm{ord,univ}}_{p\text{-}adic}(\kappa)^{\pm} = p^{m-1}(p-1) \cdot \kappa(T(p))^m \cdot L_{p\text{-}adic}(f_\kappa)^{\pm}(\chi) \in O_M[[G^{(2)}_\infty]] \otimes_{O_M} M. \tag{7.6}$$

Here concerning (one-variable) p-adic zeta function $L_{p\text{-}adic}(f_\kappa)$, we take the class of f_κ as $\omega \in S(f_\kappa)$ in the situation of Theorem 5.5. Moreover $\pm = (-1)^k\kappa(\langle -1\rangle)$, $L^{\mathrm{ord,univ}}_{p\text{-}adic}(\kappa)^\pm$ and $L_{p\text{-}adic}(f_\kappa)^\pm$ are the $\pm$-parts of $L^{\mathrm{ord,univ}}_{p\text{-}adic}(\kappa)$ and $L_{p\text{-}adic}(f_\kappa)$, respectively, and M is the finite extension $\mathbb{Q}_p(a_n(f_\kappa); n \geq 1)$ of $\mathbb{Q}_p$.

Remark 7.3.1 . (1) *Recall that as in Theorem 6.2, even for $p = 2, 3$, the p-adic zeta function $L_{p\text{-}adic}(f)$ of each ordinary p-stabilized newform f was constructed from $z^{\mathrm{univ}}_{Np^\infty}$.*

(2) *By the argument in 4.5.5, we obtain ${}_{c,d'}z^{\mathrm{univ}}_{Np^\infty} \in \overline{M}_{Np^\infty}[[G^{(1)}_\infty \times G^{(2)}_\infty]]$. From this and from the fact that $\kappa((1 - c^{-1}g^{(1)}_{c^{-1}}g^{(2)}_c)(1 - d'g^{(2)}_{d'})) \in \overline{\mathbb{Z}_p}[[G^{(2)}_\infty]]$ is a non-zerodivisor for any N-primitive arithmetic point κ, we find that $\kappa(g) \in \overline{\mathbb{Z}_p}[[G^{(2)}_\infty]]$ is a non-zerodivisor for any N-primitive arithmetic point κ.*

(3) *In the above, we put the assumption that $L(f_\kappa, k-1) \neq 0$. (As referred before, $L(f_\kappa, k-1) = 0$ occurs only in the case $k = 2$.) Moreover we only considered the $(-1)^k\kappa(\langle -1\rangle)$-parts of $L_{p\text{-}adic}(f_\kappa)$. However as explained briefly in 7.7 later, by using z_{M,Np^∞}, z_{N,p^∞}, and ϕ in 6.7, with some device, we can construct a two-variable p-adic zeta function which can provide the $(-1)^{k+1}\kappa(\langle -1\rangle)$-part of $L_{p\text{-}adic}(f_\kappa)$ for κ satisfying some conditions (see 7.7 for this condition) even though $L(f_\kappa, k-1) = 0$.*

We prove Theorem 7.3 by using Theorem 6.2 and the argument in the proof of it.
We use the notation in Theorem 7.3 and we assume $\kappa(h) \neq 0$.

7.4. For integers c, d' which are prime to p, we put

$$ {}_{c,d'}L^{\mathrm{ord,univ}}_{p\text{-adic}} = (1 - c^{-1}g^{(1)}_{c^{-1}}g^{(2)}_c)(1 - d'g^{(2)}_{d'})L^{\mathrm{ord,univ}}_{p\text{-adic}} \in (\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty})[\frac{1}{h}][[G_\infty]]. $$

One can see that ${}_{c,d'}L^{\mathrm{ord,univ}}_{p\text{-adic}} = \mathrm{L}_N({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty})$. We also put

$$ {}_{c,d'}L_{p\text{-adic}}(f_\kappa) = ((1 - c^{-k}\epsilon_{f_\kappa}(c^{-1})g^{(2)}_c)(1 - g^{(2)}_{d'})) \cdot L_{p\text{-adic}}(f_\kappa). $$

In 7.5 – 7.6, we will prove that under an N-primitive arithmetic point κ satisfying the condition in Theorem 7.3,

$$ {}_{c,d'}L^{\mathrm{ord,univ}}_{p\text{-adic}}(\kappa) = p^{m-1}(p-1) \cdot \kappa(T(p))^m \cdot {}_{c,d'}L_{p\text{-adic}}(f_\kappa)(\chi). \tag{7.7} $$

The result in Theorem 7.3 will follow from this.

7.5. We consider some consequences of Theorem 6.2 for the proof of (7.7).

7.5.1. Since f_κ is ordinary, $L_{p\text{-adic}}(f_\kappa)$ may be characterized by the specialization property in Theorem 5.5 (i) for only one r among $1, \ldots, k-1$ under the notation there.
We have that $S(f_\kappa)_M := M_k(X_1(Np^n); M)/(T(n) - \kappa(T(n))\ ;\ n \geq 1)$ is a one dimensional M-vector space with a base f_κ. We define

$\mathrm{pr}_{f_\kappa} : M_k(X_1(Np^n); M) \to M$ as the composition $M_k(X_1(Np^n); M) \xrightarrow{\text{proj.}} S(f_\kappa)_M \to M$, where the second map is the one given by sending the class of f_κ to 1. (In the case that f_κ is a newform, the map pr_{f_κ} coincides with the map $\mathrm{pr}_{\mathfrak{f}}$ in section 6. In the case $f_\kappa = f_\alpha$ for some newform f of conductor N and α as before, in section 6, we took f as the base of $S(f_\kappa)_M$. So if we use pr_{f_κ} in Theorem 6.2 instead of $\mathrm{pr}_{\mathfrak{f}}$, we obtain the p-adic zeta function of f_κ which takes the class of f_κ as ω in the notation in Theorem 5.5.) By Theorem 6.2, we obtain

$$\begin{aligned}(\varprojlim_n \mathrm{pr}_{f_\kappa}({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\chi^{k-2}, \chi^{r-1})|_{(m,n)}))^{\pm\cdot(-1)^r}\\ = \alpha^m \cdot \sum_{a\in(\mathbb{Z}/p^m\mathbb{Z})^\times} i_{\{r\}}({}_{c,d'}L_{p\text{-adic}}(f_\kappa)^\pm)\epsilon_{f_\kappa}(a'^{-1})g_a^{(1)} \in \varprojlim_n M[G_n^{(2)}][G_m^{(1)}],\end{aligned} \tag{7.8}$$

with $\pm = (-1)^k\kappa(\langle -1\rangle)$, where $a' \in (\mathbb{Z}/Np^m\mathbb{Z})^\times$ is the element such that $a' \equiv 1(N)$ and $a' \equiv a(p^m)$ for each a.

7.5.2. Let $\epsilon_p : G_\infty^{(1)} \to \mathbb{Z}_p^\times$ denote the character such that the restriction of κ to $G_\infty^{(1)}$ coincides with $\epsilon_p \circ \chi^{k-2}$. Then $\epsilon_p(g_a^{(1)}) = \epsilon_{f_\kappa}(a')$ for any $a \in (\mathbb{Z}/p^m\mathbb{Z})^\times$, and $a' \in (\mathbb{Z}/Np^m\mathbb{Z})^\times$ is the element such that $a' \equiv a(p^m)$ and $a' \equiv 1(N)$. Clearly the image of (7.8) under $\epsilon_p : M[[G_\infty^{(2)}]][G_m^{(1)}] \to M[[G_\infty^{(2)}]]$ is

$$p^{m-1}(p-1)\cdot\alpha^m\cdot i_{\{r\}}({}_{c,d'}L_{p\text{-adic}}(f_\kappa)^\pm).$$

As f_κ is ordinary, by Theorem 6.2, we know $L_{p\text{-adic}}(f_\kappa) \in O_M[[G_\infty^{(2)}]] \otimes_{O_M} M$, and hence we have

$$i_{\{r\}}({}_{c,d'}L_{p\text{-adic}}(f_\kappa)^\pm) = {}_{c,d'}L_{p\text{-adic}}(f_\kappa)^\pm(\chi^r) \in O_M[[G_\infty^{(2)}]] \otimes_{O_M} M.$$

7.5.3. By the commutative diagram

$$\begin{array}{ccc} M_k(X_1(Np^n);\mathbb{Z}_{(p)})[G_m^{(1)}] & \xrightarrow{\epsilon_p} & M_k(X_1(Np^n);\mathbb{Z}_{(p)}[\epsilon_p]) \\ \big\downarrow{\scriptstyle \mathrm{pr}_{f_\kappa}} & & \big\downarrow{\scriptstyle \mathrm{pr}_{f_\kappa}} \\ M[G_m^{(1)}] & \xrightarrow{\epsilon_p} & M, \end{array}$$

and (7.8), it follows that

$$\begin{aligned}\mathrm{pr}_{f_\kappa}({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\epsilon_p\circ\chi^{k-2}, \chi^{r-1}))^{\pm\cdot(-1)^r} = p^{m-1}(p-1)\cdot\alpha^m\cdot{}_{c,d'}L_{p\text{-adic}}(f_\kappa)^\pm(\chi^r)\\ \in O_M[[G_\infty^{(2)}]] \otimes_{O_M} M.\end{aligned} \tag{7.9}$$

Here we denote by $\mathrm{pr}_{f_\kappa}({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\epsilon_p\circ\chi^{k-2}, \chi^{r-1}))^{\pm\cdot(-1)^r}$ the inverse limit $(\varprojlim_n \mathrm{pr}_{f_\kappa}({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\epsilon_p\circ\chi^{k-2}, \chi^{r-1})|_n))^{\pm\cdot(-1)^r}$ with ${}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\epsilon_p\circ\chi^{k-2}, \chi^{r-1})|_n \in M_k(X_1(Np^n);\mathbb{Z}_{(p)}[\epsilon_p])[G_n^{(2)}]$.
Furthermore by (6.13), we have

$${}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\epsilon_p\circ\chi^{k-2}, \chi^{r-1})|_n \in M_k(X_1(Np^n), \epsilon_p; \mathbb{Z}_{(p)}[\epsilon_p])[G_n^{(2)}],$$

where $M_k(X_1(Np^n), \epsilon_p; \mathbb{Z}_{(p)}[\epsilon_p])$ is the sub $\mathbb{Z}_{(p)}[\epsilon_p]$-space of $M_k(X_1(Np^n); \mathbb{Z}_{(p)}[\epsilon_p])$ on which $G_\infty^{(1)}$ acts via $\epsilon_p \circ \chi^{k-2}$.

7.6. Now we study ${}_{c,d'}L_{p\text{-adic}}^{\mathrm{ord,univ}}$.

7.6.1. The following composition of Λ-homomorphisms

$$\begin{aligned}\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}/\mathcal{I}_{Np^\infty}^{\mathrm{ord}} &\to p_{Np^\infty}^{\mathrm{ord}} \otimes_\Lambda Q(\Lambda)\\ &\to P_{Np^\infty}^{\mathrm{ord}} \otimes_\Lambda Q(\Lambda) \to \mathcal{H}_{Np^\infty}^{\mathrm{ord}} \otimes_\Lambda Q(\Lambda)\end{aligned}$$

induces the Λ-homomorphism

$$\mathrm{Hom}_\Lambda(\mathcal{H}_{Np^\infty}^{\mathrm{ord}}, \Lambda) \longrightarrow \mathrm{Hom}_\Lambda(\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}/\mathcal{I}_{Np^\infty}^{\mathrm{ord}}, \frac{1}{h'}\Lambda) \tag{7.10}$$

with some non-zerodivisor $h' \in \Lambda$. It is easy to see that the composition of (7.10) and

$$\mathrm{Hom}_\Lambda(\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}/\mathcal{I}_{Np^\infty}^{\mathrm{ord}}, \frac{1}{h'}\Lambda) \to (\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}/\mathcal{I}_{Np^\infty}^{\mathrm{ord}}) \otimes_\Lambda Q(\Lambda),$$

which is given by (7.3), coincides with the composition in (7.2). From this, we see $(h')^{-1} \subset (h)^{-1}$.

The Λ-homomorphism $\overline{\mathsf{m}}_\Lambda[[G_\infty^{(2)}]] \xrightarrow{\mathrm{i}} \mathrm{Hom}_\Lambda(\mathcal{H}_{Np^\infty}, \Lambda)[[G_\infty^{(2)}]]$ and (7.10) give a Λ-homomorphism

$$\overline{\mathsf{m}}_\Lambda[[G_\infty^{(2)}]] \longrightarrow \mathrm{Hom}_\Lambda(\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}/\mathcal{I}_{Np^\infty}^{\mathrm{ord}}, \Lambda[\frac{1}{h'}])[[G_\infty^{(2)}]],$$

and this Λ-homomorphism induces

$$\begin{aligned}&\overline{\mathsf{m}}_{\Lambda/\mathfrak{P}_{k-2,\epsilon_p}}[[G_\infty^{(2)}]]\\ &\quad\to \mathrm{Hom}_{\Lambda/\mathfrak{P}_{k-2,\epsilon_p}}(\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}/(\mathfrak{P}_{k-2,\epsilon_p} + \mathcal{I}_{Np^\infty}^{\mathrm{ord}}),\ \Lambda[\frac{1}{h'}]/\mathfrak{P}_{k-2,\epsilon_p})[[G_\infty^{(2)}]]\end{aligned} \tag{7.11}$$

where $\overline{\mathsf{m}}_{\Lambda/\mathfrak{P}_{k-2,\epsilon_p}}$ denotes the image of $\overline{\mathsf{m}}_\Lambda$ under the map $\overline{M}_{Np^\infty}[[G_\infty^{(1)}]] \xrightarrow{\epsilon_p \circ \chi^{k-2}} \overline{M}_{Np^\infty} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\epsilon_p]$. The map (7.11) is well-defined as $\kappa(h) \neq 0$.

We put $L = \Lambda[1/h']/\mathfrak{P}_{k-2,\epsilon_p}$. Now the result of Hida [Hi2], Corollary 1.3 affirms that $\mathrm{Hom}_{\Lambda/\mathfrak{P}_{k-2,\epsilon_p}}(\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}/(\mathfrak{P}_{k-2,\epsilon_p} + \mathcal{I}_{Np^\infty}^{\mathrm{ord}}),\ \Lambda[1/h']/\mathfrak{P}_{k-2,\epsilon_p})$ is contained in $S_k^{\mathrm{ord},N}(X_1(Np^m), \epsilon_p; L)$, where $S_k^{\mathrm{ord},N}(X_1(Np^m), \epsilon_p; L)$ is the sub L-space of $e \cdot M_k(X_1(Np^m), \epsilon_p; L)$ with $e = \varprojlim_n T(p)^{n!}$, consisting of cusp forms with conductor divisible by N. Hence (7.11) induces a homomorphism

$$\overline{\mathsf{m}}_{\Lambda/\mathfrak{P}_{k-2,\epsilon_p}}[[G_\infty^{(2)}]] \to S_k^{\mathrm{ord},N}(X_1(Np^m), \epsilon_p; L)[[G_\infty^{(2)}]]. \tag{7.12}$$

7.6.2. We define $\mathsf{m}_\Lambda[[G_\infty^{(2)}]] = \overline{\mathsf{m}}_\Lambda[[G_\infty^{(2)}]] \cap M[[G_\infty^{(1)} \times G_\infty^{(2)}]]_{\{k-2\},\{0,\dots,k-2\}} (\subset \overline{M}_{Np^\infty}[[G_\infty^{(1)} \times G_\infty^{(2)}]])$ with $\overline{\mathsf{m}}_\Lambda$ in 7.1.1 and $M[[G_\infty^{(1)} \times G_\infty^{(2)}]]_{\{k-2\},\{0,\dots,k-2\}}$ in 6.1.4. We see that ${}_{c,d'}z_{Np^\infty}^{\mathrm{univ}} \in \mathsf{m}_\Lambda[[G_\infty^{(2)}]]$.

We define $\mathsf{m}_{\Lambda/\mathfrak{P}_{k-2,\epsilon_p}}[[G^{(2)}_\infty]]$ to be the image of of $\mathsf{m}_\Lambda[[G^{(2)}_\infty]]$ under the map $\overline{M}_{Np^\infty}[[G^{(1)}_\infty \times G^{(2)}_\infty]] \xrightarrow{(\epsilon_p\circ\chi^{k-2},\mathrm{id})} \overline{M}_{Np^\infty} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\epsilon_p][[G^{(2)}_\infty]]$. Then the map (7.12) induces a map

$$\mathfrak{j} : \mathsf{m}_{\Lambda/\mathfrak{P}_{k-2,\epsilon_p}}[[G^{(2)}_\infty]] \longrightarrow S^{\mathrm{ord},N}_k(X_1(Np^m), \epsilon_p; L)[[G^{(2)}_\infty]]. \tag{7.13}$$

By definition, the following diagram is commutative for each $n \geq 1$:

$$\begin{array}{ccc} \mathsf{m}_{\Lambda/\mathfrak{P}_{k-2,\epsilon_p}}[G^{(2)}_n] & \xrightarrow{\ \mathfrak{j}\ } & S^{\mathrm{ord},N}_k(X_1(Np^m), \epsilon_p; L)[G^{(2)}_n] \\ \big\downarrow{\scriptstyle \mathrm{pr}_{f_\kappa}} & & \big\downarrow{\scriptstyle \mathrm{pr}_{f_\kappa}} \\ M & \xrightarrow{\ \mathrm{id}\ } & M. \end{array} \tag{7.14}$$

We write $\mathrm{pr}_{f_\kappa}(\mathfrak{j}({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\epsilon_p \circ \chi^{k-2}, \mathrm{id})))$ for the inverse limit $\varprojlim_n(\mathrm{pr}_{f_\kappa}(\mathfrak{j}({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\epsilon_p \circ \chi^{k-2}, \chi^{r-1}))|_n))$. By the commutative diagram (7.14), we obtain

$$\mathrm{pr}_{f_\kappa}({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\epsilon_p \circ \chi^{k-2}, \mathrm{id})) = \mathrm{pr}_{f_\kappa}(\mathfrak{j}({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\epsilon_p \circ \chi^{k-2}, \mathrm{id}))).$$

Hence by this and by (7.9), if we prove the assertion in 7.6.3 below, Theorem 7.3 follows.

7.6.3. The assertion in question is as follows.
We have

$$\mathrm{pr}_{f_\kappa}(\mathfrak{j}({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\epsilon_p \circ \chi^{k-2}, \mathrm{id}))) = {}_{c,d'}L^{\mathrm{ord,univ}}_{p\text{-adic}}(\kappa).$$

We prove this assertion. As an element of the right hand side of (7.11), $\mathfrak{j}({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\epsilon_p \circ \chi^{k-2}, \mathrm{id}))$ is the element

$$x \mapsto \mathrm{Tr}_{k,\epsilon_p}({}_{c,d'}L^{\mathrm{ord,univ}}_{p\text{-adic}} \cdot x) \qquad (x \in \mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/(\mathfrak{P}_{k-2,\epsilon_p} + \mathcal{I}^{\mathrm{ord}}_{Np^\infty})),$$

where $\mathrm{Tr}_{k,\epsilon_p} : \mathrm{Hom}_{\Lambda/\mathfrak{P}_{k-2,\epsilon_p}}(\mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/(\mathfrak{P}_{k-2,\epsilon_p} + \mathcal{I}^{\mathrm{ord}}_{Np^\infty}), \Lambda/\mathfrak{P}_{k-2,\epsilon_p})$ is the trace map as $\Lambda/\mathfrak{P}_{k-2,\epsilon_p}$-modules. By the result of Hida [Hi2], Corollary 1.3, we find that

$$\mathrm{Tr}_{k,\epsilon_p} = \sum_{\kappa'} f_{\kappa'},$$

where $f_{\kappa'}$ runs over all ordinary p-stabilized newform $\in S^{\mathrm{ord},N}_k(X_1(Np^m), \epsilon_p; L)$ of tame conductor N attached to $\kappa' : \mathfrak{h}^{\mathrm{ord}}_{Np^\infty}/\mathcal{I}^{\mathrm{ord}}_{Np^\infty} \longrightarrow \overline{\mathbb{Z}_p}$ satisfying $\kappa'|_{G^{(1)}_\infty} = \epsilon_p \circ \chi^{k-2}$.

By the definition of ${}_{c,d'}L^{\mathrm{ord,univ}}_{p\text{-adic}}$, we obtain

$$\mathfrak{j}({}_{c,d'}z^{\mathrm{univ}}_{Np^\infty}(\epsilon_p \circ \chi^{k-2}, \mathrm{id})) = {}_{c,d'}L^{\mathrm{ord,univ}}_{p\text{-adic}}(\sum_{\kappa'} f_{\kappa'}) = \sum_{\kappa'} {}_{c,d'}L^{\mathrm{ord,univ}}_{p\text{-adic}}(\kappa') f_{\kappa'},$$

where in $\sum_{\kappa'}$, κ' runs as above. This concludes the assertion.

Therefore Theorem 7.3 is proven.

7.7. We briefly explain how to obtain a two-variable p-adic zeta function attached to the universal family of cusp forms which can provide $(-1)^{k+1}\kappa(\langle -1\rangle)$-part of $L_{p\text{-adic}}(f_\kappa)$ for κ satisfying a certain condition, even though $L(f_\kappa, k-1) = 0$.
We use the notation in 6.3.7 – 6.3.12 and in 6.7. As before, let M be a positive integer which is prime to Np.
We consider the image of z_{M,Np^∞} under the $\overline{M}_{MNp^\infty}$-homomorphism which is similar to (6.10) and which is given as follows

$$\overline{M}_{MNp^\infty}[[{G_{Mp^\infty}}^{(1)} \times {G_{Mp^\infty}}^{(2)}]] \longrightarrow \overline{M}_{MNp^\infty}[[G_\infty^{(1)} \times G_\infty^{(2)}]] \quad ; \tag{7.15}$$

$$x\mathfrak{g}_a^{(2)}\mathfrak{g}_b^{(1)} \mapsto x\phi(ab^2)g_a^{(2)}g_b^{(1)} \qquad (x \in \overline{M}_{MNp^\infty}, \mathfrak{g}_* \in G_{Mp^\infty}, g_* \in G_\infty).$$

We write $z_{M,Np^\infty,\phi}$ for the image of z_{M,Np^∞} under the above map.
Now by replacing N by NM in the definition of $\overline{\mathfrak{m}}_\Lambda$, we define $\overline{\mathfrak{m}}_\Lambda$. Then for this $\overline{\mathfrak{m}}_\Lambda$, the following which is similar to the result in Proposition 7.1.2 (1) holds. Namely we have

$$z_{M,Np^\infty,\phi} \in \overline{\mathfrak{m}}_\Lambda[[G_\infty^{(2)}]][\frac{1}{g''}](\subset \overline{M}_{NMp^\infty}[[G_\infty^{(1)} \times G_\infty^{(2)}]][\frac{1}{g''}]),$$

where g'' is the image of g' in 6.3.7 under the homomorphism (7.15). We study the image of $z_{M,Np^\infty,\phi}$ under the map

$$\overline{\mathfrak{m}}_\Lambda[[G_\infty^{(2)}]][\frac{1}{g''}] \longrightarrow ((\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}/\mathcal{I}_{Np^\infty}^{\mathrm{ord}}) \otimes_\Lambda Q(\Lambda))[[G_\infty^{(2)}]][\frac{1}{g''}] \tag{7.16}$$

which is defined as L_N. We denote this image by

$$L_{p\text{-adic},M,\phi}^{\mathrm{ord}} \in (\mathfrak{h}_{NMp^\infty}^{\mathrm{ord}}/\mathcal{I}_{NMp^\infty}^{\mathrm{ord}})[[G_\infty^{(2)}]][\frac{1}{h'g''}],$$

where $h' \in \mathfrak{h}_{NMp^\infty}^{\mathrm{ord}}$ is a certain non-zerodivisor. This function $L_{p\text{-adic},M,\phi}^{\mathrm{ord}}$ displays the property (7.17) below. For an N-primitive arithmetic point $\kappa : \mathfrak{h}_{Np^\infty}^{\mathrm{ord}}/\mathcal{I}_{Np^\infty}^{\mathrm{ord}} \to \overline{\mathbb{Z}_p}$, let $\kappa_\phi : \mathfrak{h}_{NMp^\infty}^{\mathrm{ord}}/\mathcal{I}_{NMp^\infty}^{\mathrm{ord}} \to \overline{\mathbb{Z}_p}$ be the NM-primitive arithmetic point characterized by $\kappa_\phi(T(n)) = \kappa(T(n))\phi(n)$. We always have that $\kappa_\phi(g'') \in \overline{\mathbb{Z}_p}[[G_\infty^{(2)}]]$ is a non-zero divisor. We assume that $\kappa_\phi(h') \neq 0$. Then κ_ϕ induces the following homomorphism which is also denoted by κ_ϕ :

$$\kappa_\phi : (\mathfrak{h}_{Np^\infty}^{\mathrm{ord}}/\mathcal{I}_{Np^\infty}^{\mathrm{ord}})[[G_\infty^{(2)}]][\frac{1}{h'g''}] \longrightarrow Q(\overline{\mathbb{Z}_p}[[G_\infty^{(2)}]]).$$

Now if $L(f_\kappa, \phi, k-1) \neq 0$, we have

$$
\begin{aligned}
&L^{\mathrm{ord}}_{p\text{-adic},M,\phi}(\kappa_\phi)^{\pm\cdot\phi(-1)} \\
&\quad = x \cdot \Big(\Big(\prod_{\substack{l:\mathrm{prime}\\ l|M}} (1 - a_l(\mathfrak{f}) g^{(2)}_{l^{-1}} + \epsilon_{\mathfrak{f}}(l) l^{k-1} g^{(2)}_{l^{-2}})\Big) \cdot L_{p\text{-adic}}(f_\kappa)\Big)^{\pm\cdot\phi(-1)}(\chi) \qquad (7.17)\\
&\qquad\qquad \in O_{M(\phi)}[[G^{(2)}_\infty]] \otimes_{O_{M(\phi)}} M(\phi)
\end{aligned}
$$

with some $x \in M(\phi)^\times$, and $\pm = (-1)^k \epsilon_{\mathfrak{f}}(-1)$.
By using the arguments in 6.3.7 – 6.3.12 and 6.7, we can prove the above assertion in the same way as for Theorem 6.2. Furthermore, by using z_{N,p^∞}, we can produce another two-variable p-adic zeta function attached to universal family of ordinary cusp forms in the same manner.
In this way, by taking various ϕ, we can construct two-variable p-adic zeta functions which can provide not only $(-1)^k\kappa(\langle -1\rangle)$-part but also $(-1)^{k+1}\kappa(\langle -1\rangle)$-part of $L_{p\text{-adic}}(f_\kappa)$ for κ satisfying the condition that $\kappa_\phi(h') \neq 0$, without assuming $L(f_\kappa, k-1) \neq 0$.

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