Kato Homology of Arithmetic Schemes and Higher Class Field Theory over Local Fields

Dedicated to Kazuya Kato on the occasion of his 50th birthday

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Abstract. For arithmetical schemes X, K. Kato introduced certain complexes $C^{r,s}(X)$ of Gersten-Bloch-Ogus type whose components involve Galois cohomology groups of all the residue fields of X. For specific (r, s), he stated some conjectures on their homology generalizing the fundamental isomorphisms and exact sequences for Brauer groups of local and global fields. We prove some of these conjectures in small degrees and give applications to the class field theory of smooth projecive varieties over local fields, and finiteness questions for some motivic cohomology groups over local and global fields.

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1. Introduction

The following two facts are fundamental in the theory of global and local fields. Let k be a global field, namely either a finite extension of $\mathbb{Q}$ or a function field in one variable over a finite field. Let $\mathbb{P}$ be the set of all places of k, and denote by k_v the completion of k at $v \in \mathbb{P}$. For a field L let $Br(L)$ be its Brauer group, and identify the Galois cohomology group $H^1(L, \mathbb{Q}/\mathbb{Z})$ with the group of the continuous characters on the absolute Galois group of L with values in $\mathbb{Q}/\mathbb{Z}$.

(1-1) For a finite place v, with residue field F_v, there are natural isomorphisms

$$Br(k_v) \xrightarrow{\cong} H^1(F_v, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \mathbb{Q}/\mathbb{Z},$$

where the first map is the residue map and the second is the evaluation of characters at the Frobenius element. For an archimedean place v there is an injection

$$Br(k_v) \xrightarrow{\cong} H^1(k_v, \mathbb{Q}/\mathbb{Z}) \hookrightarrow \mathbb{Q}/\mathbb{Z}.$$

(1-2) There is an exact sequence

$$0 \longrightarrow Br(k) \xrightarrow{\alpha} \bigoplus_{v\in\mathbb{P}} Br(k_v) \xrightarrow{\beta} \mathbb{Q}/\mathbb{Z} \longrightarrow 0,$$

where α is induced by the restrictions and β is the sum of the maps in (1-1).

In [K1] Kazuya Kato proposed a fascinating framework of conjectures that generalizes the stated facts to higher dimensional arithmetic schemes. In order to review these conjectures, we introduce some notations. For a field L and an integer $n > 0$ define the following Galois cohomology groups: If n is invertible in L, let $H^i(L, \mathbb{Z}/n\mathbb{Z}(j)) = H^i(L, \mu_n^{\otimes j})$ where μ_n is the Galois module of n-th roots of unity. If n is not invertible in L and L is of characteristic $p > 0$, let

$$H^i(L, \mathbb{Z}/n\mathbb{Z}(j)) = H^i(L, \mathbb{Z}/m\mathbb{Z}(j)) \oplus H^{i-j}(L, W_r\Omega^i_{L,log})$$

where $n = mp^r$ with $(p, m) = 1$. Here $W_r\Omega^i_{L,log}$ is the logarithmic part of the de Rham-Witt sheaf $W_r\Omega^i_L$ [Il, I 5.7]. Then one has a canonical identification $H^2(L, \mathbb{Z}/n\mathbb{Z}(1)) = Br(L)[n]$ where $[n]$ denotes the n-torsion part.
For an excellent scheme X and integers $n, r, s > 0$, and under certain assumptions (which are always satisfied in the cases we consider), Kato defined a homological complex $C^{r,s}(X, \mathbb{Z}/n\mathbb{Z})$ of Bloch-Ogus type (cf. [K1], §1):

$$\cdots \bigoplus_{x\in X_i} H^{r+i}(k(x), \mathbb{Z}/n\mathbb{Z}(s+i)) \to \bigoplus_{x\in X_{i-1}} H^{r+i-1}(k(x), \mathbb{Z}/n\mathbb{Z}(s+i-1)) \to \cdots$$
$$\cdots \to \bigoplus_{x\in X_1} H^{r+1}(k(x), \mathbb{Z}/n\mathbb{Z}(s+1)) \to \bigoplus_{x\in X_0} H^r(k(x), \mathbb{Z}/n\mathbb{Z}(s)).$$

Here $X_i = \{x \in X | \dim\overline{\{x\}} = i\}$, $k(x)$ denotes the residue field of x, and the term $\bigoplus_{x\in X_i}$ is placed in degree i. The differentials are certain residue maps generalizing the maps

$$Br(k_v)[n] = H^2(k_v, \mathbb{Z}/n\mathbb{Z}(1)) \longrightarrow H^1(F_v, \mathbb{Z}/n\mathbb{Z})$$

alluded to in (1-1). More precisely, they rely on the fact that one has canonical residue maps $H^i(K, \mathbb{Z}/n(j)) \to H^{i-1}(F, \mathbb{Z}/n(j-1))$ for a discrete valuation ring with fraction field K and residue field F.

Definition 1.1 *We define the Kato homology of X with coefficient in $\mathbb{Z}/n\mathbb{Z}$ as*

$$H_i^{r,s}(X, \mathbb{Z}/n\mathbb{Z}) = H_i(C^{r,s}(X, \mathbb{Z}/n\mathbb{Z})).$$

Note that $H_i^{r,s}(X,\mathbb{Z}/n\mathbb{Z}) = 0$ for $i \notin [0,d],\ d = \dim X$. Kato's conjectures concern the following special values of (r,s).

Definition 1.2 *If X is of finite type over $\mathbb{Z}$, we put*

$$H_i^K(X,\mathbb{Z}/n\mathbb{Z}) = H_i^{1,0}(X,\mathbb{Z}/n\mathbb{Z}).$$

If X is of finite type over a global field or its completion at a place, we put

$$H_i^K(X,\mathbb{Z}/n\mathbb{Z}) = H_i^{2,1}(X,\mathbb{Z}/n\mathbb{Z}).$$

For a prime ℓ we define the Kato homology groups of X with coefficient in $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ as the direct limit of those with coefficient in $\mathbb{Z}/\ell^\nu\mathbb{Z}$ for $\nu > 0$.

The first conjecture of Kato is a generalization of (1-2) ([K1], 0.4).

Conjecture A *Let X be a smooth connected projective variety over a global field k. For $v \in \mathbb{P}$, let $X_v = X \times_k k_v$. Then the restriction maps induce isomorphisms*

$$H_i^K(X,\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} \bigoplus_{v\in\mathbb{P}} H_i^K(X_v,\mathbb{Z}/n\mathbb{Z}) \quad \text{for } i > 0,$$

and an exact sequence

$$0 \to H_0^K(X,\mathbb{Z}/n\mathbb{Z}) \to \bigoplus_{v\in\mathbb{P}} H_0^K(X_v,\mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

If $\dim(X) = 0$, we may assume $X = \mathrm{Spec}(k)$. Then $H_i(X,\mathbb{Z}/n\mathbb{Z}) = H_i(X_v,\mathbb{Z}/n\mathbb{Z}) = 0$ for $i > 0$, and $H_0(X,\mathbb{Z}/n\mathbb{Z}) = Br(k)[n]$ and $H_0(X_v,\mathbb{Z}/n\mathbb{Z}) = Br(k_v)[n]$. Thus, in this case conjecture A is equivalent to (1-2). In case $\dim(X) = 1$, conjecture A was proved by Kato [K1]. The following is shown in [J4].

Theorem 1.3 *Conjecture A holds if* $\mathrm{ch}(k) = 0$ *and if one replaces the coefficients $\mathbb{Z}/n\mathbb{Z}$ with $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ for any prime ℓ.*

The main objective of this paper is to study the generalization of (1-1) to the higher dimensional case. Let A be a henselian discrete valuation ring with finite residue field F of characteristic p. Let K be the quotient field of A. Let $S = \mathrm{Spec}(A)$ and assume given the diagram

$$\begin{array}{ccccc} X_\eta & \xrightarrow{j_X} & X & \xleftarrow{i_X} & X_s \\ \downarrow f_\eta & & \downarrow f & & \downarrow f_s \\ \eta & \xrightarrow{j} & S & \xleftarrow{i} & s \end{array} \tag{1-3}$$

in which s and η are the closed and generic point of S, respectively, the squares are cartesian, and f is flat of finite type. Then Kato defined a canonical *residue map*

$$\Delta^i_{X,n} \ :\ H_i^K(X_\eta,\mathbb{Z}/n\mathbb{Z}) \to H_i^K(X_s,\mathbb{Z}/n\mathbb{Z}),$$

and stated the following second conjecture ([K1], 5.1), which he proved for $\dim X_\eta = 1$.

Conjecture B *If f is proper and X is regular, $\Delta^i_{X,n}$ is an isomorphism for all $n > 0$ and all $i \geq 0$.*

If $X = S$, then $\Delta^0_{X,n}$ is just the map $H^2(K, \mathbb{Z}/n(1)) \to H^1(F, \mathbb{Z}/n)$ in (1-1). In general, conjecture B would allow to compute the Kato homology of X_η by that of the special fiber X_s. Our investigations are also strongly related to Kato's third conjecture ([K1], 0.3 and 0.5):

Conjecture C *Let $\mathcal{X}$ be a connected regular projective scheme of finite type over $\mathbb{Z}$. Then*

$$\widetilde{H}^K_i(\mathcal{X}, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} \begin{cases} 0 & \text{if } i \neq 0, \\ \mathbb{Z}/n\mathbb{Z} & \text{if } i = 0. \end{cases}$$

Here the modified Kato homology $\widetilde{H}^K_i(\mathcal{X}, \mathbb{Z}/n\mathbb{Z})$ is defined as the homology of the modified Kato complex

$$\widetilde{C}^{1,0}(\mathcal{X}, \mathbb{Z}/n\mathbb{Z}) := Cone(\ C^{1,0}(\mathcal{X}, \mathbb{Z}/n\mathbb{Z})[1] \to C^{2,1}(\mathcal{X} \times_{\mathbb{Z}} \mathbb{R}, \mathbb{Z}/n\mathbb{Z})\ ).$$

The map $\widetilde{H}^K_0(\mathcal{X}, \mathbb{Z}/n) \to \mathbb{Z}/n\mathbb{Z}$ is induced by the maps $H^1(k(x), \mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z}$ for $x \in \mathcal{X}_0$ given by the evaluation of characters at the Frobenius (note that $k(x)$ is a finite field for $x \in \mathcal{X}_0$), together with the maps $H^2(k(y), \mathbb{Z}/n\mathbb{Z}(1)) = Br(k(y))[n] \hookrightarrow \mathbb{Z}/n\mathbb{Z}$ for $y \in (\mathcal{X} \times_{\mathbb{Z}} \mathbb{R})_0$. The canonical map $\widetilde{H}^K_i(\mathcal{X}, \mathbb{Z}/n\mathbb{Z}) \to H^K_i(\mathcal{X}, \mathbb{Z}/n\mathbb{Z})$ is an isomorphism if $\mathcal{X}(\mathbb{R})$ is empty or if n is odd.

Conjecture C in case $\dim(\mathcal{X}) = 1$ is equivalent to (the n-torsion part of) the classical exact sequence (1-2) for $k = k(\mathcal{X})$, the function field of $\mathcal{X}$. In case $\dim(\mathcal{X}) = 2$ conjecture C is proved in [K1] and [CTSS], as a consequence of the class field theory of $\mathcal{X}$. The other known results concern the case that $\mathcal{X} = Y$ is a smooth projective variety over a finite field F: In [Sa4] it is shown that $H^K_3(Y, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$ if $\ell \neq \mathrm{ch}(F)$ and $\dim(Y) = 3$. This is generalized in [CT] and [Sw] where the $\mathbb{Q}_\ell/\mathbb{Z}_\ell$-coefficient version of Conjecture C in degrees $i \leq 3$ is proved for all primes ℓ and for Y of arbitrary dimension over F.

As we have seen, conjecture C can be regarded as another generalization of (1-2). In fact, conjectures A, B, and C are not unrelated: If $\mathcal{X}$ is flat over $\mathbb{Z}$, it is geometrically connected over $\mathcal{O}_k$, the ring of integers in some number field k. Then the generic fiber $X = \mathcal{X}_k$ is smooth, and we get a commutative diagram with exact rows

$$(1\text{-}4)\qquad \begin{array}{ccccccccc} 0 & \to & \oplus_v C^{1,0}(Y_v) & \to & \oplus_v C^{1,0}(\mathcal{X}_v) & \to & \oplus_v C^{2,1}(X_{k_v})[-1] & \to & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \to & \oplus_v C^{1,0}(Y_v) & \to & C^{1,0}(\mathcal{X}) & \to & C^{2,1}(X)[-1] & \to & 0. \end{array}$$

Here $\mathcal{X}_v = \mathcal{X} \times_{\mathcal{O}_k} \mathcal{O}_v$ for the ring of integers $\mathcal{O}_v$ in k_v, and $Y_v = \mathcal{X} \times_{\mathcal{O}_k} F_v$ is the fiber over v, if v is finite. If v is infinite, we let $Y_v = \emptyset$ and $C^{1,0}(\mathcal{X}_v, \mathbb{Z}/n\mathbb{Z}) = C^{2,1}(X_v, \mathbb{Z}/n\mathbb{Z})[-1]$. Thus conjecture B for $\mathcal{X}_v$ means that $C^{1,0}(\mathcal{X}_v)$ is acyclic for finite v, and any two of the conjectures imply the third one in this case.

On the other hand, conjecture C for a smooth projective variety over a finite field allows to compute the Kato homology of X_s in (1-3), at least in the case of semi-stable reduction: Assume that X is proper over S in (1-3), and that the reduced special fiber $Y = (X_s)_{red}$ is a strict normal crossings variety. In §3 we construct a *configuration map*

$$\gamma^i_{X_s,n} \ : \ H^K_i(X_s, \mathbb{Z}/n\mathbb{Z}) \to H_i(\Gamma_{X_s}, \mathbb{Z}/n\mathbb{Z}).$$

Here Γ_{X_s}, the *configuration (or dual) complex* of X_s, is a simplicial complex whose $(r-1)$-simplices ($r \geq 1$) are the connected components of

$$Y^{[r]} = \coprod_{1 \leq j_1 < \cdots < j_r \leq N} Y_{j_1} \cap \cdots \cap Y_{j_r},$$

where $Y_1, \ldots, Y_N$ are the irreducible components of Y. This complex has been studied very often in the literature for a curve X/S, in which case Γ_{X_s} is a graph. In case $X = \mathrm{Spec}(\mathcal{O}_K)$, $\gamma^0_{X_s,n}$ is nothing but the map $H^1(F, \mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z}$ in (1-1). For a prime ℓ, let

$$\gamma^i_{X_s,\ell^\infty} \ : \ H^K_i(X_s, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H_i(\Gamma_{X_s}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

be the inductive limit of $\gamma^i_{X_s,\ell^\nu}$ for $\nu > 0$. Then we show in 3.9:

Theorem 1.4 *The map $\gamma^j_{X_s,n}$ is an isomorphism if Conjecture C is true in degree i for all $i \leq j$ and for any connected component of $Y^{[r]}$, for all $r \geq 1$. The analogous fact holds with $\mathbb{Q}_\ell/\mathbb{Z}_\ell$-coefficients. In particular, $\gamma^j_{X_s,n}$ is an isomorphism for $j = 0, 1, 2$ and all $n > 0$, and $\gamma^3_{X_s,\ell^\infty}$ is an isomorphism for all primes ℓ.*

Our main results on Conjecture B now are as follows.

Theorem 1.5 *Let n be invertible in K, and assume that X is proper over S.*

(1) *If X_η is connected, one has isomorphisms*

$$H^K_0(X_\eta, \mathbb{Z}/n\mathbb{Z}) \xrightarrow[\sim]{\Delta^0_{X,n}} H^K_0(X_s, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}/n\mathbb{Z}.$$

(2) *If X is regular, $\Delta^1_{X,n}$ is an isomorphism.*

In the proof of Theorem 1.5, given in §5, an important role is played by the class field theory for varieties over local fields developed in [Bl], [Sa1] and [KS1].

In §6 we propose a strategy to show the $\mathbb{Q}_\ell/\mathbb{Z}_\ell$-coefficient version of Conjecture B in degrees ≥ 2 (cf. Proposition 6.4 and the remark at the end of §6) and then show the following result. Fix a prime ℓ different from $\mathrm{ch}(K)$. Passing to the inductive limit, the maps Δ^i_{X,ℓ^ν} induce

$$\Delta^i_{X,\ell^\infty} \ : \ H^K_i(X_\eta, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H^K_i(X_s, \mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

Theorem 1.6 *Let X be regular, projective over S, and with strict semistable reduction. Then Δ^2_{X,ℓ^∞} is an isomorphism and Δ^3_{X,ℓ^∞} is surjective.*

We note that the combination of Theorems 1.4, 1.5 and 1.6 gives a simple description of $H_i^K(X_\eta, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ for $i \leq 2$, in terms of the configuration complex of X_s.

The method of proof for 1.6 is as follows. In [K1] Kato defined the complexes $C^{r,s}(X, \mathbb{Z}/n)$ and the residue map $\Delta^i_{X,n}$ by using his computations with symbols in the Galois cohomology of discrete valuation fields of mixed characteristic [BK]. To handle these objects more globally and to obtain some compatibilities, we give an alternative definition in terms of a suitable *étale homology theory* , in particular for schemes over discrete valuation rings, in §2.

We will have to use the fact that the complexes defined here, following the method of Bloch and Ogus [BO], agree with the Kato complexes, as defined in [K1], because our constructions rely on the Bloch-Ogus method, while we have to use several results in the literature stated for Kato's definition (although even there the agreement is sometimes used implicitely). For the proof that the complexes agree (up to some signs) we refer the reader to [JSS].

Given this setting, the residue map $\Delta^i_{X,n}$ is then studied in §4 by a square

$$\begin{array}{ccc} H^{et}_{a-2}(X_\eta, \mathbb{Z}/n\mathbb{Z}(-1)) & \xrightarrow{\epsilon_{X_\eta}} & H^K_a(X_\eta, \mathbb{Z}/n\mathbb{Z}) \\ \downarrow \Delta^{et}_X & & \downarrow \Delta^K_X \\ H^{et}_{a-1}(X_s, \mathbb{Z}/n\mathbb{Z}(0)) & \xrightarrow{\epsilon_{X_s}} & H^K_a(X_s, \mathbb{Z}/n\mathbb{Z}), \end{array} \tag{1-5}$$

in which the groups on the left are étale homology groups, and the maps ϵ are constructed by the theory in §2. The shifts of degrees by -2 and -1 correspond to the fact that the cohomological dimensions of K and F are 2 and 1, respectively. If X_η is smooth of pure dimension d, then $H^{et}_{a-2}(X_\eta, \mathbb{Z}/n\mathbb{Z}(-1)) \cong H^{2d-a+2}_{et}(X_\eta, \mathbb{Z}/n\mathbb{Z}(d+1))$, similarly for X_s. But X_s will not in general be smooth, and then étale cohomology does not work. The strategy is to show that Δ^{et}_X and ϵ_{X_s} are bijective and that ϵ_{X_η} is surjective, at least if the coefficients are replaced by $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ (to use weight arguments), and if X is replaced by a suitable "good open" U (to get some vanishing in cohomology).

The proof of the p-part, i.e., for $\mathbb{Q}_p/\mathbb{Z}_p$ with $p = \mathrm{ch}(F)$, depends on two results not published yet. One is the purity for logarithmic de Rham-Witt sheaves stated in formula (4-2) (taken from [JSS]) and in Proposition 4.12 (due to K. Sato [Sat3]). The other is a calculation for p-adic vanishing cycles sheaves, or rather its consequence as stated in Lemma 4.22. It just needs the assumption $p \geq \dim(X_\eta)$ and will be contained in [JS]. If we only use the results from [BK], [H] and [Ts2], we need the condition $p \geq \dim(X_\eta) + 3$, and have to assume $p \geq 5$ in Theorem 1.6.

Combining Theorems 1.5 and 1.6 with Theorem 1.3 one obtains the following result concerning conjecture C (cf. (1-4)).

Theorem 1.7 *Let k be a number field with ring of integers $\mathcal{O}_k$. Let $f : \mathcal{X} \to S$ be a regular proper flat geometrically connected scheme over $S := \mathrm{Spec}(\mathcal{O}_k)$. Assume* *that $\mathcal{X}$ has strict semistable reduction around every closed fiber of f. Then we have*

$$\widetilde{H}_i^K(\mathcal{X}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \begin{cases} 0 & \text{if } 1 \le i \le 3, \\ \mathbb{Q}/\mathbb{Z} & \text{if } i = 0. \end{cases}$$

We give an application of the above results to the class field theory of surfaces over local fields. Let K be a non-archimedean local field as in (1-3), and let V be a proper variety over $\eta = \mathrm{Spec}(K)$. Then we have the *reciprocity map for V*

$$\rho_V \ : \ SK_1(V) \to \pi_1^{ab}(V)$$

introduced in the works [Bl], [Sa1] and [KS1]. Here $\pi_1^{ab}(V)$ is the abelian algebraic fundamental group of V and

$$SK_1(V) = \mathrm{Coker}(\bigoplus_{x \in V_1} K_2(y) \xrightarrow{\partial} \bigoplus_{x \in V_0} K_1(x))$$

where $K_q(x)$ denotes the q-th algebraic K-group of $k(x)$, and ∂ is induced by tame symbols. The definition of ρ_V will be recalled in §5. For an integer $n > 0$ prime to $\mathrm{ch}(K)$ let

$$\rho_{V,n} \ : \ SK_1(V)/n \to \pi_1^{ab}(V)/n$$

denote the induced map. There exists the fundamental exact sequence (cf. §5)

$$H_2^K(V, \mathbb{Z}/n\mathbb{Z}) \to SK_1(V)/n \xrightarrow{\rho_{V,n}} \pi_1^{ab}(V)/n \to H_1^K(V, \mathbb{Z}/n\mathbb{Z}) \to 0. \tag{1-6}$$

Combined with 1.5 (2) and 1.4 it describes the cokernel of $\rho_{V,n}$ - which is the quotient $\pi_1^{ab}(V)^{c.d.}$ of the abelianized fundamental group classifying the covers in which every point of V splits completely - in terms of the first configuration homology of the reduction in the case of semi-stable reduction. This generalizes the results for curves in [Sa1]. Moreover, (1-6) immediately implies that $\rho_{V,n}$ is injective if $\dim(V) = 1$, which was proved in [Sa1] assuming furthermore that V is smooth. In general $\mathrm{Ker}(\rho_{V,n})$ is controlled by the Kato homology $H_2^K(V, \mathbb{Z}/n\mathbb{Z})$. Sato [Sat2] constructed an example of a proper smooth surface V over K for which $\rho_{V,n}$ is not injective, which implies that the first map in the above sequence is not trivial in general. The following conjecture plays an important role in controlling $\mathrm{Ker}(\rho_{V,n})$. Let L be a field, and let ℓ be a prime different from $\mathrm{ch}(L)$.

Conjecture $BK_q(L, \ell)$: The group $H^q(L, \mathbb{Q}_\ell/\mathbb{Z}_\ell(q))$ is divisible.

This conjecture is a direct consequence of the Bloch-Kato conjecture asserting the surjectivity of the symbol map $K_q^M(L) \to H^q(L, \mathbb{Z}/\ell\mathbb{Z}(q))$ from Milnor K-theory to Galois cohomology. The above form is weaker if restricted to particular fields L, but known to be equivalent if stated for all fields. By Kummer theory, $BK_1(L, \ell)$ holds for any L and any ℓ. The celebrated work of [MS] shows that $BK_2(L, \ell)$ holds for any L and any ℓ. Voevodsky [V] proved $BK_q(L, 2)$ for any L and any q.

Quite generally, the validity of this conjecture would allow to extend results from $\mathbb{Q}_\ell/\mathbb{Z}_\ell$-coefficients to arbitrary coefficients, by the following result (cf. Lemma 7.3; for extending 1.3 and 1.7 one would need $BK_q(L, \ell)$ over number fields):

Lemma *Let V be of finite type over K, and let ℓ be a prime. Assume that either $\ell = \mathrm{ch}(K)$, or that $BK_{i+1}(K(x), \ell)$ holds for all $x \in V_i$ and $BK_i(K(x), \ell)$ holds for all $x \in V_{i-1}$. Then we have an exact sequence*

$$0 \to H_{i+1}^K(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell)/\ell^\nu \to H_i^K(V, \mathbb{Z}/\ell^\nu) \to H_i^K(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell)[\ell^\nu] \to 0.$$

In §7 we combine this observation with considerations about norm maps, to obtain the following results on surfaces. Let P be a set (either finite or infinite) of rational primes different from $\mathrm{ch}(K)$. Call an abelian group P-divisible if it is ℓ-divisible for all $\ell \in P$.

Theorem 1.8 *Let V be an irreducible, proper and smooth surface over K. Assume $BK_3(K(V), \ell)$ for all $\ell \in P$, where $K(V)$ is the function field of V.*

(1) *Then* $\mathrm{Ker}(\rho_V)$ *is the direct sum of a finite group and an P-divisible group.*

(2) *Assume further that there exists a finite extension K'/K and an alteration (=surjective, proper, generically finite morphism) $f : W \to V \times_K K'$ such that W has a semistable model X' over A', the ring of integers in K', with $H_2(\Gamma_{X'_{s'}}, \mathbb{Q}) = 0$ for its special fibre $X'_{s'}$. Then* $\mathrm{Ker}(\rho_V)$ *is P-divisible.*

(3) *If V has good reduction, then the reciprocity map induces isomorphisms*

$$\rho_{V,\ell^\nu} : SK_1(V)/\ell^\nu \xrightarrow{\cong} \pi_1^{ab}(V)/\ell^\nu$$

for all $\ell \in P$ and all $\nu > 0$. In particular, $\mathrm{Ker}(\rho_V)$ *is ℓ-divisible.*

Theorem 1.9 *Assume that V is an irreducible variety of dimension 2 over a local field K. Assume $BK_3(K(V), \ell)$ for all $\ell \in P$. If V is not proper (resp. proper), $SK_1(V)$ (resp.* $\mathrm{Ker}(N_{V/K})$*) is the direct sum of a finite group and a P-divisible group. Here $N_{V/K} : SK_1(V) \to K^*$ is the norm map introduced in §6.*

We remark that Theorem 1.8 generalizes [Sa1] where the kernel of the reciprocity map for curves over local fields is shown to be divisible under no assumption. Another remark is that Szamuely [Sz] has studied the reciprocity map for varieties over local fields and its kernel. His results require stronger assumptions than ours while it affirms that the kernel is uniquely divisible. We note however that Sato's example in [Sat2] also implies that the finite group in Theorem 1.8 is non-trivial in general.

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2. Kato complexes and Bloch-Ogus theory

It is well-known, although not made precise in the literature, that for a smooth variety over a field, one may construct the Kato complexes via the niveau spectral sequence for étale cohomology constructed by Bloch and Ogus [BO]. In this paper we will however need the Kato complexes for singular varieties and for schemes over discrete valuation rings, again not smooth. It was a crucial observation for us that for these one gets similar results by using étale *homology* (whose definition is somewhat subtle for p-coefficients with p not invertible on the scheme). This fits also well with the required functorial behavior of the Kato complexes, which is of 'homological' nature: covariant for proper morphisms, and contravariant for open immersions.
In several instances one could use still use étale *cohomology*, by embedding the schemes into a smooth ambient scheme and taking étale cohomology *with supports* (cf. 2.2 (b) and 2.3 (f)). But then the covariance for arbitrary proper morphisms became rather unnatural, and there were always annoying degree shifts in relation to the Kato homology. Therefore we invite the readers to follow our homological approach.

The following definition formalizes the properties of a homology of Borel-Moore type. It is useful for dealing with étale and Kato homology together, and for separating structural compatibilities from explicit calculations.

A. General results Let $\mathcal{C}$ be a category of noetherian schemes such that for any object X in $\mathcal{C}$, every closed immersion $i : Y \hookrightarrow X$ and every open immersion $j : V \hookrightarrow X$ is (a morphism) in $\mathcal{C}$.

Definition 2.1 (a) Let $\mathcal{C}_*$ be the category with the same objects as $\mathcal{C}$, but where morphisms are just the proper maps in $\mathcal{C}$. A homology theory on $\mathcal{C}$ is a sequence of covariant functors

$$H_a(-) :\ \mathcal{C}_* \to (\text{abelian groups}) \qquad (a \in \mathbb{Z})$$

satisfying the following conditions:

(i) For each open immersion $j : V \hookrightarrow X$ in $\mathcal{C}$, there is a map $j^* : H_a(X) \to H_a(V)$, associated to j in a functorial way.

(ii) If $i : Y \hookrightarrow X$ is a closed immersion in X, with open complement $j : V \hookrightarrow X$, there is a long exact sequence (called localization sequence)

$$\cdots \xrightarrow{\delta} H_a(Y) \xrightarrow{i_*} H_a(X) \xrightarrow{j^*} H_a(V) \xrightarrow{\delta} H_{a-1}(Y) \longrightarrow \cdots.$$

(The maps δ are called the connecting morphisms.) This sequence is functorial with respect to proper maps or open immersions, in an obvious way.

(b) A morphism between homology theories H and H' is a morphism $\phi : H \to H'$ of functors on $\mathcal{C}_*$, which is compatible with the long exact sequences from (ii).

Before we go on, we note the two examples we need.

Examples 2.2 (a) This is the basic example. Let S be a noetherian scheme, and let $\mathcal{C} = Sch_{sft}/S$ be the category of schemes which are separated and of finite type over S. Let $\Lambda = \Lambda_S \in D^b(S_{ét})$ be a bounded complex of étale sheaves on S. Then one gets a homology theory $H = H^\Lambda$ on $\mathcal{C}$ by defining

$$H_a^\Lambda(X) := H_a(X/S;\Lambda) := H^{-a}(X_{ét}, R\,f^!\Lambda)$$

for a scheme $f : X \to S$ in Sch_{sft}/S (which may be called the étale homology of X over (or relative) S with values in Λ). Here $Rf^!$ is the right adjoint of $Rf_!$ defined in [SGA 4.3, XVIII, 3.1.4]. For a proper morphism $g : Y \to X$ between schemes $f_Y : Y \to S$ and $f_X : X \to S$ in Sch_{sft}/S, the trace (=adjunction) morphism $tr : g_*Rg^! \to id$ induces a morphism

$$R(f_Y)_*Rf_Y^!\Lambda = R(f_X)_*Rg_*Rg^!Rf_X^!\Lambda \xrightarrow{tr} R(f_X)_*Rf_X^!\Lambda$$

which gives the covariant functoriality, and the contravariant functoriality for open immersions is given by restriction. The long exact localization sequence 1.1 (ii) comes from the exact triangle

$$i_*Rf_Y^!\Lambda = i_*Ri^!Rf_X^!\Lambda \to Rf_X^!\Lambda \to Rj_*j^*Rf_X^!\Lambda = Rj_*Rf_V^!\Lambda \to .$$

(b) Sometimes (but not always) it suffices to consider the following more down to earth version (avoiding the use of homology and Grothendieck-Verdier duality). Let X be a fixed noetherian scheme, and let $\mathcal{C} = Sub(X)$ be the category of subschemes of X, regarded as schemes over X (Note that this implies that there is at most one morphism between two objects). Let $\Lambda = \Lambda_X$ be an étale sheaf (resp. a bounded below complex of étale sheaves on X). Then one gets a homology theory $H = H^{\Lambda_X}$ on $Sub(X)$ by defining

$$H_a^\Lambda(Z) := H_a(Z/X;\Lambda) := H_Z^{-a}(U_{ét}, \Lambda|U),$$

as the étale cohomology (resp. hypercohomology) with supports in Z, where U is any open subscheme of X containing Z as a closed subscheme. For the proper morphisms in $Sub(X)$, which are the inclusions $Z' \hookrightarrow Z$, the covariantly associated maps are the canonical maps $H_{Z'}^{-a}(U_{ét}, \Lambda|U) \to H_Z^{-a}(U_{ét}, \Lambda|U)$. The contravariant functoriality for open subschemes is given by the obvious restriction maps. We may extend everything to the equivalent category $Im(X)$ of immersions $S \hookrightarrow X$, regarded as schemes over X, and we will identify $Sub(X)$ and $Im(X)$.

Remarks 2.3 (a) For any homology theory H and any integer N, we get a shifted homology theory $H[N]$ defined by setting $H[N]_a(Z) = H_{a+N}(Z)$ and multiplying the connecting morphisms by $(-1)^N$.
(b) If H is a homology theory on $\mathcal{C}$, then for any scheme X in $\mathcal{C}$ the restriction of H to the subcategory $\mathcal{C}/X$ of schemes over X is again a homology theory.

(c) Let H be a homology theory on $\mathcal{C}/X$), and let $Z \hookrightarrow X$ be an immersion. Then the groups

$$H_a^{(Z)}(T) := H_a(T \times_X Z)$$

again define a homology theory on $\mathcal{C}/X$. For an open immersion $j : U \hookrightarrow X$ (resp. closed immersion $i : Z \hookrightarrow X$) one has an obvious morphism of homology theories $j^* : H \to H^{(U)}$ (resp. $i_* : H^{(Y)} \to H$).
(d) In the situation of 2.2 (a), let $X \in Ob(Sch_{sft}/S)$. Then by functoriality of $f \rightsquigarrow Rf^!$ the restriction of H^{Λ_S} to $\mathcal{C}/X = Sch_{sft}/X$ can be identified with H^{Λ_X} for $\Lambda_X = Rf_X^!\Lambda_S$.

(e) Since for a subscheme $ji : Z \overset{i}{\hookrightarrow} U \overset{j}{\hookrightarrow} X$, with i closed and j open immersion, we have

$$H_Z^{-a}(U_{\acute{e}t}, \Lambda_X|U) = H^{-a}(Z_{\acute{e}t}, Ri^!(j^*\Lambda_X) = H^{-a}(Z_{\acute{e}t}, R(ji)^!\Lambda_X)$$

the notation $H_a(Z/X;\Lambda_X)$ has the same meaning in 2.2 (b) as in 2.2 (a), and the restriction of H^{Λ_S} to $Sub(X)$ coincides with H^{Λ_X} from 2.2 (b).
(f) If moreover $f_X : X \to S$ is smooth of pure dimension d and $\Lambda_S = \mathbb{Z}/n(b)$ for integers n and b with n invertible on S, then by purity we have $Rf_X^!\mathbb{Z}/n(b) = \mathbb{Z}/n(b+d)[2d]$, so that H^{Λ_S} restricted to $Sub(X)$ is $H^{\Lambda_X}[2d]$ for $\Lambda_X = \mathbb{Z}/n(b+d)$.
(g) In the situation of 2.2 (a), any morphism $\psi : \Lambda_S \to \Lambda'_S$ in $D^b(S_{\acute{e}t})$ induces a morphism between the associated homology theories. Similarly for 2.2 (b) and a morphism $\psi : \Lambda_X \to \Lambda'_X$ of (complexes of) sheaves on X.

The axioms in 2.1 already imply the following property, which is known for example 2.2.
Let $Y, Z \subset X$ be a closed subschemes with open complement $U, V \subset X$, respectively. Then we get an infinite diagram of localization sequences

$$\begin{array}{ccccccccccc}
\dots & H_{a-1}(Y\cap Z) & \to & H_{a-1}(Z) & \to & H_{a-1}(U\cap Z) & \xrightarrow{\delta} & H_{a-2}(Y\cap Z) & \dots \\
 & \uparrow\delta & & \uparrow\delta & & \uparrow\delta & (-) & \uparrow\delta & \\
\dots & H_a(Y\cap V) & \to & H_a(V) & \to & H_a(U\cap V) & \xrightarrow{\delta} & H_{a-1}(Y\cap V) & \dots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\dots & H_a(Y) & \to & H_a(X) & \to & H_a(U) & \xrightarrow{\delta} & H_{a-1}(Y) & \dots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\dots & H_a(Y\cap Z) & \to & H_a(Z) & \to & H_a(U\cap Z) & \xrightarrow{\delta} & H_{a-1}(Y\cap Z) & \dots
\end{array}$$

Lemma 2.4 *The above diagram is commutative, except for the squares marked* (−)*, which anticommute.*

Proof For all squares except for the one with the four δ's, the commutativity follows from the functoriality in 2.1 (ii), so it only remains to consider that square marked (-). Since $(Y \cup Z) \setminus (Y \cap Z)$ is the disjoint union of $Y \setminus Z = Y \cap V$ and $Z \setminus Y = U \cap Z$, from 2.1 (ii) we have an isomorphism

$$H_{a-1}((Y \cup Z) \setminus (Y \cap Z)) \cong H_{a-1}(U \cap Z) \oplus H_{a-1}(Y \cap V)$$

and a commutative diagram from the respective localization sequences

$$\begin{array}{ccccc}
H_{a-1}(X) & \to & H_{a-1}(X \setminus (Y \cap Z)) & & \\
\uparrow & & \uparrow & & \\
H_{a-1}(Y \cup Z) & \to & H_{a-1}(U \cap Z) \oplus H_{a-1}(Y \cap V) & \stackrel{\delta+\delta}{\to} & H_{a-2}(Y \cap Z) \\
\uparrow & & \uparrow (\delta, \delta) & & \\
H_a(X \setminus (Y \cup Z)) & = & H_a(U \cap V) & & .
\end{array}$$

As indicated, the connecting morphisms are given by the product $\alpha = (\delta, \delta)$ and the sum $\beta = \delta + \delta$, respectively, of the connecting morphisms from the square marked (-), as one can see by applying the functoriality 2.1 (ii). Now the diagram implies that the composition $\beta \circ \alpha$ is zero, hence the claim.

Corollary 2.5 *The maps* $\delta : H_a(T \times_X U) \to H_{a-1}(T \times_X Y)$, *for* $T \in \mathcal{C}/X$, *define a morphism of homology theories* $\delta : H^{(U)}[1] \to H^{(Y)}$.

We shall also need the following Mayer-Vietoris property.

Lemma 2.6 *Let* $X = X_1 \cup X_2$ *be the union of two closed subschemes* $i_\nu : X_\nu \hookrightarrow X$, *and let* $k_\nu : X_1 \cap X_2 \hookrightarrow X_\nu$ *be the closed immersions of the (scheme-theoretic) intersection. Then there is a long exact Mayer-Vietoris sequence*

$$\to H_a(X_1 \cap X_2) \stackrel{(k_{1*}, -k_{2*})}{\longrightarrow} H_a(X_1) \oplus H_a(X_2) \stackrel{i_{1*}+i_{2*}}{\longrightarrow} H_a(X) \stackrel{\delta}{\to} H_{a-1}(X_1 \cap X_2) \to .$$

This sequence is functorial with respect to proper maps, localization sequences and morphisms of homology theories, in the obvious way.

Proof The exact sequence is induced in a standard way (via the snake lemma) from the commutative ladder of localization sequences

$$\begin{array}{ccccccccc}
.. & H_a(X_2) & \stackrel{i_{2*}}{\to} & H_a(X) & \to & H_a(X \setminus X_2) & \to & H_{a-1}(X_2) & .. \\
 & \uparrow k_{2*} & & \uparrow i_{1*} & & \| & & \uparrow & \\
.. & H_a(X_1 \cap X_2) & \stackrel{k_{1*}}{\to} & H_a(X_1) & \to & H_a(X_1 \setminus X_1 \cap X_2) & \to & H_{a-1}(X_1 \cap X_2) & ..
\end{array}$$

The functorialities are clear from the functoriality of this diagram.

Now we come to the main object of this chapter. As in [BO] one proves the existence of the following *niveau spectral sequence*, by using the niveau filtration on the homology and the method of exact couples.

Proposition 2.7 *If H is a homology theory on $\mathcal{C}$, then, for every $X \in Ob(\mathcal{C})$, there is a spectral sequence of homological type*

$$E^1_{r,q}(X) = \bigoplus_{x \in X_r} H_{r+q}(x) \quad \Rightarrow \quad H_{r+q}(X).$$

Here $X_r = \{x \in X \mid \dim x = r\}$ and

$$H_a(x) = \lim_{\rightarrow} H_a(V)$$

for $x \in X$, where the limit is over all open non-empty subschemes $V \subseteq \overline{\{x\}}$. This spectral sequence is covariant with respect to proper morphisms in $\mathcal{C}$ and contravariant with respect to open immersions.

Remarks 2.8 (a) Since we shall partially need it, we briefly recall the construction of this spectral sequence. As in [BO], for any scheme $T \in \mathcal{C}$ let $\mathcal{Z}_r = \mathcal{Z}_r(T)$ be the set of closed subsets $Z \subset T$ of dimension $\leq r$, ordered by inclusion, and let $\mathcal{Z}_r/\mathcal{Z}_{r-1}(T)$ be the set of pairs $(Z, Z') \in \mathcal{Z}_r \times \mathcal{Z}_{r-1}$ with $Z' \subset Z$, again ordered by inclusion. For every $(Z, Z') \in \mathcal{Z}_r/\mathcal{Z}_{r-1}(X)$, one then has an exact localization sequence

$$\ldots \to H_a(Z') \to H_a(Z) \to H_a(Z \setminus Z') \xrightarrow{\delta} H_{a-1}(Z') \to \ldots,$$

and the limit of these, taken over $\mathcal{Z}_r/\mathcal{Z}_{r-1}(X)$, defines an exact sequence denoted

$$\ldots H_a(\mathcal{Z}_{r-1}(X)) \to H_a(\mathcal{Z}_r(X)) \to H_a(\mathcal{Z}_r/\mathcal{Z}_{r-1}(X)) \xrightarrow{\delta} H_{a-1}(\mathcal{Z}_{r-1}(X)) \ldots.$$

The collection of these sequences for all r, together with the fact that one has $H_*(\mathcal{Z}_r(X)) = 0$ for $r < 0$ and $H_*(\mathcal{Z}_r(X)) = H_*(X)$ for $r \geq \dim X$, gives the spectral sequence in a standard way, e.g., by exact couples. Here

$$E^1_{r,q}(X) = H_{r+q}(\mathcal{Z}_r/\mathcal{Z}_{r-1}(X)) = \bigoplus_{x \in X_r} H_{r+q}(x).$$

The differentials are easily described, e.g., in the same way as in [J3] for a filtered complex (by renumbering from cohomology to homology). In particular, the E^1-differentials are the compositions

$$H_{r+q}(\mathcal{Z}_r/\mathcal{Z}_{r-1}(X)) \xrightarrow{\delta} H_{r+q-1}(\mathcal{Z}_{r-1}(X)) \to H_{r+q-1}(\mathcal{Z}_{r-1}/\mathcal{Z}_{r-2}(X)).$$

Moreover the 'edge isomorphisms' $E^\infty_{r,q} \cong E^{r,q}_{r+q}$ are induced by

$$H_{r+q}(\mathcal{Z}_r/\mathcal{Z}_{r-1}(X) \leftarrow H_{r+q}(\mathcal{Z}_r(X)) \to H_{r+q}(\mathcal{Z}_\infty(X)) = H_{r+q}(X)$$

(b) This shows that the differential

$$d^1_{r,q} : \bigoplus_{x \in X_r} H_{r+q}(x) \ \to \ \bigoplus_{x \in X_{r-1}} H_{r+q-1}(x)$$

has the following description. For $x \in X_r$ and $y \in X_{r-1}$ define

$$\delta_X^{loc}\{x,y\} := \delta_{X,a}^{loc}\{x,y\} : H_a(x) \to H_{a+1}(y)$$

as the map induced by the connecting maps $H_a(V \setminus \overline{\{y\}}) \xrightarrow{\delta} H_{a-1}(V \cap \overline{\{y\}})$ from 2.1 (ii), for all open $V \subset \overline{\{x\}}$. Then the components of $d^1_{r,q}$ are the $\delta^{loc}_{X,r+q}\{x,y\}$. Note that $\delta_X^{loc}\{x,y\} = 0$ if y is not contained in $\overline{\{x\}}$.
(c) Every morphism $\phi : H \to H'$ between homology theories induces a morphism between the associated niveau spectral sequences.

We note some general results for fields and discrete valuation rings.

Proposition 2.9 *Let $S = Spec(F)$ for a field F, let X be separated and of finite type over F, and let H be a homology theory on $Sub(X)$. If $i : Y \hookrightarrow X$ is a closed subscheme and $j : U = X \setminus Y \hookrightarrow X$ is the open complement, then the following holds.*
(a) For all r,q the sequence

$$0 \to E^1_{r,q}(Y) \xrightarrow{i_*} E^1_{r,q}(X) \xrightarrow{j^*} E^1_{r,q}(U) \to 0$$

is exact.
(b) The connecting morphisms $\delta : H_a(Z \cap U) \to H_{a-1}(Z \cap Y)$, for $T \in Sub(X)$, induce a morphism of spectral sequences

$$\delta : E^1_{r,q}(U)^{(-)} \longrightarrow E^1_{r-1,q}(Y),$$

where the superscript $^{(-)}$ means that all differentials in the original spectral sequence (but not the edge isomorphisms $E^\infty_{r,q} \cong E^{r,q}_{r+q}$) are multiplied by -1.

Proof (a): One has always $X_r \cap Y = Y_r$, and since X is of finite type over a field, we also have $X_r \cap U = U_r$.
(b): This morphism is induced by the morphism of homology theories $\delta : H^{(U)}[1] \to H^{(Y)}$ and the construction of the spectral sequences, noting the following. For a closed subset $Z \subset U$ let $\overline{Z}$ be the closure in X and $\delta(Z) = \overline{Z} \cap Y$. For $(Z, Z') \in \mathcal{Z}_r/\mathcal{Z}_{r-1}(U)$ one then has $(\delta(Z), \delta(Z')) \in \mathcal{Z}_{r-1}/\mathcal{Z}_{r-2}(Y)$, and a commutative diagram via localization sequences

$$\begin{array}{ccccccccc}
.. & H_a(\overline{Z'}) & \to & H_a(\overline{Z}) & \to & H_a(\overline{Z} \setminus \overline{Z'}) & \xrightarrow{\delta} & H_{a-1}(\overline{Z'}) & .. \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
.. & H_a(\delta(Z')) & \to & H_a(\delta(Z)) & \to & H_a(\delta(Z) \setminus \delta(Z')) & \xrightarrow{\delta} & H_{a-1}(\delta(Z')) & .. \\
 & \uparrow \delta & & \uparrow \delta & & \uparrow \delta & & \uparrow \delta & \\
.. & H_{a+1}(Z') & \to & H_{a+1}(Z) & \to & H_{a+1}(Z \setminus Z') & \xrightarrow{-\delta} & H_a(Z') & ..
\end{array}$$

This shows that one gets a map of the exact couples defining the spectral sequences and hence of the spectral sequences themselves, with the claimed shift and change of signs. Note that every differential in the spectral sequence involves a connecting morphism once, whereas the edge isomorphisms do not involve any connecting morphism; this gives the signs in $E^{(-)}$.

Corollary 2.10 *Let $\mathcal{C}$ be a subcategory of $Sch_{sft}/Spec(F)$. For every fixed q, the family of functors $(E^2_{r,q})_{r\in\mathbb{Z}}$ defines a homology theory on $\mathcal{C}$.*

Proof The functoriality for proper morphisms and immersions comes from that of the spectral sequence noted in 2.8 (c). Moreover, in the situation of 2.9, we get an exact sequence of complexes

$$0 \to E^1_{\bullet,q}(Y) \to E^1_{\bullet,q}(X) \to E^1_{\bullet,q}(U) \to 0,$$

whose associated long exact cohomology sequence is the needed long exact sequence

$$\ldots \to E^2_{r,q}(Y) \to E^2_{r,q}(X) \to E^2_{r,q}(U) \xrightarrow{\delta} E^2_{r-1,q}(Y) \to \ldots .$$

Its functoriality for proper morphisms and open immersions comes from the functoriality of the mentioned exact sequence of complexes.

Remark 2.11 By the construction in 2.9 (b), the components of the maps δ on E^1-level,

$$\delta : E^1_{r,q}(U) = \bigoplus_{x\in U_r} H_{r+q}(x) \to \bigoplus_{y\in Y_{r-1}} H_{r+q}(y) = E^1_{r-1,q}(Y)$$

are the maps $\delta^{loc}_X\{x,y\}$. This also shows that the associated maps on the E^2-level coincide with the connecting morphisms in 2.10.

We now turn to discrete valuation rings.

Proposition 2.12 *Let $S = Spec(A)$ for a discrete valuation ring A, let X be separated of finite type over S, and let H be a homology theory on $Sub(X)$. Let η and s be the generic and closed point of S, respectively, and write $Z_\eta = Z\times_S \eta$ and $Z_s = Z\times_S s$ for any $Z\in Ob(Sub(X))$.*
(a) The connecting morphisms $\delta : H_a(Z_\eta) \to H_{a-1}(Z_s)$ induce a morphism of spectral sequences

$$\Delta_X : E^1_{r,q}(X_\eta)^{(-)} \;\to\; E^1_{r,q-1}(X_s),$$

where the superscript $^{(-)}$ has the same meaning as in 2.9. This morphism is functorial with respect to closed and open immersions, so that one gets commutative diagrams

$$\begin{array}{ccccccccc} 0 & \to & E^1_{r,q}(Y_\eta)^{(-)} & \to & E^1_{r,q}(X_\eta)^{(-)} & \to & E^1_{r,q}(U_\eta)^{(-)} & \to & 0 \\ & & \downarrow \Delta_Y & & \downarrow \Delta_X & & \downarrow \Delta_U & & \\ 0 & \to & E^1_{r,q}(Y_s) & \to & E^1_{r,q}(X_s) & \to & E^1_{r,q}(U_s) & \to & 0 \end{array}$$

for every closed subscheme Y in X, with open complement U.
(b) If X is proper over S, the open immersion $j : X_\eta \to X$ induces a morphism of spectral sequences

$$j^* : E^1_{r,q}(X) \;\to\; E^1_{r-1,q}(X_\eta)$$

such that

$$0 \to E^1_{r,q}(X_s) \xrightarrow{i_*} E^1_{r,q}(X) \xrightarrow{j^*} E^1_{r-1,q}(X_\eta) \to 0$$

is exact for all r and q, where $i : X_s \hookrightarrow X$ is the closed immersion of the special fiber X_s into X.

Proof. (a): As in 2.9 (b), this morphism is induced by the morphism of homology theories $\delta : H^{(X_\eta)}[1] \to H^{(X_s)}$ and the construction of the spectral sequences, noting the following in the present case: For $Z \in \mathcal{Z}_r(X_\eta)$ one now has $\delta(Z) = \overline{Z} \cap X_s \in \mathcal{Z}_r(X_s)$, where $\overline{Z}$ denotes the closure of Z in X.
(b): If $X \to S$ is proper, then $X_r \cap X_\eta = (X_\eta)_{r-1}$.

Remarks 2.13 (a) Proposition 2.12 (b) will in general be false if X is not proper over S, because $X_r \cap X_\eta$ will in general be different from $(X_\eta)_{r-1}$ (e.g., for $X = \mathrm{Spec}(K)$).
(b) By definition of Δ_X, the components of the map on E^1-level,

$$\Delta_X : \bigoplus_{x \in (X_\eta)_r} H_{r+q}(x) \longrightarrow \bigoplus_{x \in (X_s)_r} H_{r+q+1}(x)$$

are the maps $\delta^{loc}_X\{x, y\}$.

We study now two important special cases of Example 2.2 (a) (resp.(b)).

B. Étale homology over fields Let $S = \mathrm{Spec}(F)$ for a field F, and fix integers n and b. We consider two cases.

(i) n is invertible in F, and b is arbitrary.

(ii) F is a perfect field of characteristic $p > 0$, and $n = p^m$ for a positive integer m. Then we only consider the case $b = 0$.

We consider the homology theory

$$H_a(X/F, \mathbb{Z}/n(b)) := H_a(X/S; \mathbb{Z}/n(-b)) = H^{-a}(X_{\acute{e}t}, Rf^!\mathbb{Z}/n(-b))$$

(for $f : X \to S$) of 2.2 (a) on Sch_{sft}/S associated to the following complex of étale sheaves $\mathbb{Z}/n(-b)$ on S. In case (i) we take the usual $(-b)$-fold Tate twist of the constant sheaf $\mathbb{Z}/n$ and get the homology theory considered by Bloch and Ogus in [BO]. In case (ii) we define the complex of étale sheaves

$$\mathbb{Z}/p^m(i) := \mathbb{Z}/p^m(i)_T := W_m\Omega^i_{T,log}[-i],$$

for every T of finite type over F and every non-negative integer i, so that

$$H_a(X/F, \mathbb{Z}/n(b)) = H^{-a+b}(X, Rf^!W_m\Omega^{-b}_{F,log}).$$

Here $W_m\Omega^i_{T,log}$ is the logarithmic de Rham-Witt sheaf defined in [Il]. Note that $\mathbb{Z}/n(0)$ is just the constant sheaf $\mathbb{Z}/n$, and that $W_m\Omega^i_{F,log}$ is not defined for $i<0$ and 0 for $i>0$ (That is why we just consider $b=0$ in case (ii)).

The niveau spectral sequence 2.7 associated to our étale homology is

$$E^1_{r,q}(X/F,\mathbb{Z}/n(b)) = \bigoplus_{x\in X_r} H_{r+q}(x/F,\mathbb{Z}/n(b)) \Rightarrow H_{r+q}(X/F,\mathbb{Z}/n(b)).$$

Theorem 2.14 *Let X be separated and of finite type over F.*
(a) There are canonical isomorphisms

$$H_a(x/F,\mathbb{Z}/n(b)) \cong H^{2r-a}(k(x),\mathbb{Z}/n(r-b)) \quad \textit{for} \;\; x\in X_r.$$

(b) If the cohomological ℓ-dimension $cd_\ell(F)\le c$ for all primes ℓ dividing n, then one has $E^1_{r,q}(X/F,\mathbb{Z}/n(b))=0$ for all $q<-c$, and, in particular, canonical edge morphisms

$$\epsilon(X/F): H_{a-c}(X/F,\mathbb{Z}/n(b)) \longrightarrow E^2_{a,-c}(X/F,\mathbb{Z}/n(b)).$$

(c) If X is smooth of pure dimension d over F, then there are canonical isomorphisms

$$H_a(X/F,\mathbb{Z}/n(b)) \cong H^{2d-a}(X_{\acute{e}t},\mathbb{Z}/n(d-b)).$$

Proof (c): If $f: X\to \mathrm{Spec}(F)$ is smooth of pure dimension d, then one has a canonical isomorphism of sheaves

$$\text{(2-1)} \qquad \alpha_X : Rf^!\mathbb{Z}/n(-b)_S \cong \mathbb{Z}/n(d-b)[2d],$$

and (c) follows by taking the cohomology. In case (i) the isomorphism α_X is the Poincaré duality proved in [SGA 4.3, XVIII, 3.2.5]. In case (ii) it amounts to a purity isomorphism $Rf^!\mathbb{Z}/p^m \cong W_m\Omega^d_{X,log}[d]$ which is proved in [JSS].
Independently of [JSS] we note the following. In the case of a finite field F (which suffices for the later applications) we may deduce (c) in case (ii) from results of Moser [Mo] as follows. By [Mo] we have a canonical isomorphism of finite groups

$$Ext^i(\mathcal{F},W_m\Omega^d_{X,log}) \cong H^{d+1-i}_c(X,\mathcal{F})^\vee,$$

for any constructible $\mathbb{Z}/p^m$-sheaf $\mathcal{F}$ on X. Here $M^\vee = Hom(M,\mathbb{Z}/p^m)$ for a $\mathbb{Z}/p^m$-module M. Applying this to $\mathcal{F}=\mathbb{Z}/p^m$, we get an isomorphism

$$\text{(2-2)} \qquad H^i(X,W_m\Omega^d_{X,log}) \cong H^{d+1-i}_c(X,\mathbb{Z}/p^m)^\vee.$$

On the other hand, by combining Artin-Verdier duality [SGA 4.3, XVIII, 3.1.4]) and duality for Galois cohomology over F (cf. also 5.3 (2) below), one gets a canonical isomorphism of finite groups

$$\text{(2-3)} \qquad H_j(X,\mathbb{Z}/p^m(0))) = H^{-j}(X,Rf^!\mathbb{Z}/p^m) \cong H^{1+j}_c(X,\mathbb{Z}/p^m)^\vee.$$

Putting together (2-2) and (2-3) we obtain (c):

$$H^{2d-a}(X,\mathbb{Z}/p^m(d)) \stackrel{def}{=} H^{d-a}(X,W_m\Omega^d_{X,log}) \cong H_a(X,\mathbb{Z}/p^m(0))).$$

(a): By topological invariance of étale cohomology we may assume that F is perfect also in case (i). Then every point $x \in X_r$ has an open neighbourhood $V \subset \overline{\{x\}}$ which is smooth of dimension r over F. Thus (a) follows from (c) and the compatibility of étale cohomology with limits.
(b): If $x \in X_r$, then $k(x)$ is of transcendence degree r over F, and hence $cd_\ell(F) \leq c$ implies $cd_\ell(k(x)) \leq c+r$. Hence in case (i) $H_{r+q}(x/F,\mathbb{Z}/n(b)) = H^{r-q}(k(x),\mathbb{Z}/n(r-b)) = 0$ for $r-q > c+r$, i.e., $q < -c$. In case (ii), since $cd_p(L) \leq 1$ for every field of characteristic $p > 0$, we have $H_{r+q}(x/F,\mathbb{Z}/p^m(0)) = H^{-q}(k(x),W_m\Omega^r_{log}) = 0$ for $-q > 1$, which shows the claim unless $cd_p(F) = 0$. In this case we may assume that F is algebraically closed, by a usual norm argument, because every algebraic extension of F has degree prime to p [Se, I 3.3 Cor. 2]. Then $H^i(k(x),W_m\Omega^r_{log}) = 0$ for $i > 0$ by a result of Suwa ([Sw, Lem. 2.1], cf. the proof of Theorem 3.5 (a) below), because $k(x)$ is the limit of smooth affine F-algebras by perfectness of F.

We shall need the following result from [JSS].

Lemma 2.15 *Via the isomorphisms 2.14 (a), the homological complex*

$E^1_{\bullet,q}(X/F,\mathbb{Z}/n(b))$:

$$\dots \bigoplus_{x\in X_r} H_{r+q}(x/F,\mathbb{Z}/n(b)) \to \bigoplus_{x\in X_{r-1}} H_{r+q-1}(x/F,\mathbb{Z}/n(b))\dots$$

$$\dots \to \bigoplus_{x\in X_0} H_q(x/F,\mathbb{Z}/n(b))$$

(with the last term placed in degree zero) coincides with the Kato complex

$C_n^{-q,-b}(X)$:

$$\dots \bigoplus_{x\in X_r} H^{r-q}(k(x),\mathbb{Z}/n(r-b)) \to \bigoplus_{x\in X_{r-1}} H^{r-q-1}(k(x),\mathbb{Z}/n(r-b-1))\dots$$

$$\dots \to \bigoplus_{x\in X_0} H^{-q}(k(x),\mathbb{Z}/n(-b))$$

up to signs.

We also note the following functoriality.

Lemma 2.16 *The edge morphisms ϵ from 2.14 (b) define a morphism of homology theories on Sch_{sft}/F*

$$\epsilon : H_{\bullet-c}(-/F,\mathbb{Z}/n(b)) \longrightarrow E^2_{\bullet,-c}(-/F,\mathbb{Z}/n(b)) \stackrel{2.15}{=} H_\bullet(C_n^{-c,-b}(-))$$

Proof Note that the target is a homology theory by 2.10. The functoriality for proper morphisms and open immersions is clear from the functoriality of the niveau spectral sequence. The compatibility with the connecting morphisms of localization sequences follows from 2.8 (b) and remark 2.11.

C. Étale homology over discrete valuation rings Let $S = \operatorname{Spec} A$ for a discrete valuation ring A with residue field F and fraction field K. Let $j : \eta = \operatorname{Spec}(K) \hookrightarrow S$ be the open immersion of the generic point, and let $i : s = Spec(F) \hookrightarrow S$ be the closed immersion of the special point. Let n and b be integers. We consider two cases:

(i) n is invertible on S and b is arbitrary.

(ii) K is a field of characteristic 0, F is a perfect field of characteristic $p > 0$, $n = p^m$ for a positive integer m, and $b = -1$.

We consider the homology theory

$$H_a(X/S, \mathbb{Z}/n(b)) := H_a(X/S; \mathbb{Z}/n(-b)_S) = H^{-a}(X_{\acute{e}t}, Rf^!\mathbb{Z}/n(-b)_S)$$

(for $f : X \to S$) of 2.2 (a) on Sch_{sft}/S associated to the complex $\mathbb{Z}/n(-b)_S \in D^b(S_{\acute{e}t})$ defined below. The associated niveau spectral sequence is

$$E^1_{p,q}(X/S, \mathbb{Z}/n(b)) = \bigoplus_{x \in X_p} H_{p+q}(x/S, \mathbb{Z}/n(b)) \Rightarrow H_{p+q}(X/S, \mathbb{Z}/n(b)).$$

In case (i), $\mathbb{Z}/n(-b)_S$ is the usual Tate twist of the constant sheaf $\mathbb{Z}/n$ on S. In case (ii) it is the complex of étale sheaves on S

$$\mathbb{Z}/n(1)_S := Cone(Rj_*(\mathbb{Z}/n(1))_\eta \xrightarrow{\sigma} i_*(\mathbb{Z}/n)_s[-1])[-1]$$

considered in [JSS].

For the convenience of the reader, we add some explanation. By definition, $(\mathbb{Z}/n)_s$ is the constant sheaf with value $\mathbb{Z}/n$ on s, and $(\mathbb{Z}/n(1))_\eta$ is the locally constant sheaf $\mathbb{Z}/n(1) = \mu_n$ of n-th roots of unity on η. Note that n is invertible on η. The complex $Rj_*(\mathbb{Z}/n(1))_\eta$ is concentrated in degrees 0 and 1: Pulling back by j^* one gets $(\mathbb{Z}/n(1))_\eta$, concentrated in degree zero, and pulling back by i^* the stalk of the i-th cohomology sheaf is $H^i(K_{sh}, \mu_n)$, where K_{sh} is the strict Henselization of K. Since K_{sh} has cohomological dimension at most 1, the claim follows. Given this, and adjunction for i, the morphism σ is determined by a map $i^*R^1j_*(\mathbb{Z}/n(1))_\eta \to (\mathbb{Z}/n)_s$. Since sheaves on s are determined by their stalks as Galois modules, it suffices to describe the map on stalks

$$H^1(K_{sh}, \mu_n) = K_{sh}^\times/(K_{sh}^\times)^n \longrightarrow \mathbb{Z}/n$$

which we take to be the map induced by the normalized valuation.
We remark that $(\mathbb{Z}/n(1))_S$ is well-defined up to unique isomorphism, although forming a cone is not in general a well-defined operation in the derived category. But in our case, the source A of σ is concentrated in degrees 0 and 1, and the target B is concentrated in degree 1, so that $Hom(A[1], B) = 0$ in the derived category, and we can apply [BBD, 1.1.10].

Let X be separated of finite type over S, and use the notations s, η, X_s and X_η from Proposition 2.12.

Lemma 2.17 *There are isomorphisms of spectral sequences*

$$\begin{array}{rcl} E^1_{r,q}(X_\eta/S,\mathbb{Z}/n(b)) & \cong & E^1_{r,q}(X_\eta/\eta,\mathbb{Z}/n(b)) \\ E^1_{r,q}(X_s/S,\mathbb{Z}/n(b)) & \cong & E^1_{r,q+2}(X_s/s,\mathbb{Z}/n(b+1)). \end{array}$$

Proof. One has canonical isomorphisms

$$\begin{array}{rcl} j^*\mathbb{Z}/n(-b)_S & \cong & \mathbb{Z}/n(-b)_\eta \\ Ri^!\mathbb{Z}/n(-b)_S & \cong & \mathbb{Z}/n(-b-1)_s[-2]. \end{array}$$

This is clear for j^*. For $i^!$ it is the purity for discrete valuation rings [SGA 5, I,5.1] in case (i), and follows from the definition of $\mathbb{Z}/n(1)_S$ in case (ii). Thus the claim follows from remarks 2.3 (d) and (g), which imply isomorphisms of homology theories on Sch_{sft}/η and Sch_{sft}/s, respectively.

$$\begin{array}{rcl} H_a(X_\eta/S;\mathbb{Z}/n(-b)) & \cong & H_a(X_\eta/\eta;\mathbb{Z}/n(-b)) \\ H_a(X_s/S;\mathbb{Z}/n(-b)) & \cong & H_{a+2}(X_s/s;\mathbb{Z}/n(-b-1)). \end{array}$$

Definition 2.18 Define the *residue morphism*

$$\Delta_X : C_n^{-a,-b}(X_\eta)^{(-)} \to C_n^{-a-1,-b-1}(X_s)$$

between the Kato complexes by the commutative diagram

$$\begin{array}{ccc} C_n^{-a,-b}(X_\eta)^{(-)} & \xrightarrow{\Delta_X} & C_n^{-a-1,-b-1}(X_s) \\ \|\wr\ 2.15 & & \|\wr\ 2.15 \\ E^1_{\bullet,a}(X_\eta/\eta,\mathbb{Z}/n(b))^{(-)} & & E^1_{\bullet,a+1}(X_s/s,\mathbb{Z}/n(b+1)) \\ \|\wr\ 2.17 & & \|\wr\ 2.17 \\ E^1_{\bullet,a}(X_\eta/S,\mathbb{Z}/n(b))^{(-)} & \underset{2.12(a)}{\xrightarrow{\Delta_X}} & E^1_{\bullet,a-1}(X_s/S,\mathbb{Z}/n(b)) \end{array}$$

Remark 2.19 By 2.12 (a), the residue map is compatible with restrictions for open immersions and push-forwards for closed immersion. Thus, if Y is closed in X, with open complement $U = X \setminus Y$, we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & C_n^{-a,-b}(Y_\eta) & \to & C_n^{-a,-b}(X_\eta) & \to & C_n^{-a,-b}(U_\eta) & \to & 0 \\ & & \downarrow \Delta_Y & & \downarrow \Delta_X & & \downarrow \Delta_U & & \\ 0 & \to & C^{-a-1,-b-1}(Y_s) & \to & C_n^{-a-1,-b-1}(X_s) & \to & C^{-a-1,-b-1}(U_s) & \to & 0 \end{array}$$

In view of 2.13 (b), the following is proved in [JSS].

Lemma 2.20 *For $x \in (X_\eta)_r$ and $y \in (X_s)_r$ the component*

$$\Delta_X\{x,y\} : H^{r+a+1}(k(x), \mathbb{Z}/n(r-b+1)) \to H^{r+a}(k(y), \mathbb{Z}/n(r-b))$$

of Δ_X coincides with the residue map $\delta_X^{Kato}\{x,y\}$ used by Kato in the complex $C^{a,-b}(X)$.

This gives the relationship between étale homology and the Kato complexes also in the case of a discrete valuation ring:

Corollary 2.21 *If X is proper over S, then the following holds.*
(a) The residue map Δ_X from 2.18 coincides with the map considered by Kato in Conjecture B (cf. the introduction).
(b) The homological complex

$E^1_{\bullet,q}(X/S, \mathbb{Z}/n(b))$:

$$\ldots \to \bigoplus_{x\in X_r} H_{r+q}(x/S, \mathbb{Z}/n(b)) \to \bigoplus_{x\in X_{r-1}} H_{r+q-1}(x/S, \mathbb{Z}/n(b)) \to \ldots$$

$$\ldots \to \bigoplus_{x\in X_0} H_q(x/S, \mathbb{Z}/n(b))$$

(with the last term placed in degree zero) coincides with the Kato complex

$C_n^{-q-2,-b-1}$:

$$\ldots \bigoplus_{x\in X_r} H^{r-q-2}(k(x), \mathbb{Z}/n(r-b-1)) \to \bigoplus_{x\in X_{r-1}} H^{r-q-3}(k(x), \mathbb{Z}/n(r-b-2)) \ldots$$

$$\ldots \to \bigoplus_{x\in X_0} H^{-q-2}(k(x), \mathbb{Z}/n(-b-1)).$$

Proof. Since $X_r \cap X_s = (X_s)_r$ and $X_r \cap X_\eta = (X_\eta)_{r-1}$, we have

$$H_a(x/S, \mathbb{Z}/n(b)) = H^{2r-a-2}(k(x), \mathbb{Z}/n(r-b-1)) \quad \text{for all} \quad x \in X_r.$$

Hence the components agree in (b). It then follows from 2.13 (b), 2.15 and Kato's definitions that (a) and (b) are equivalent, and that (a) holds by lemma 2.20.

3. Kato complexes and étale homology over finite fields

3.1 In this section, F is a finite field of characteristic $p > 0$, and $n > 0$ is an integer. Let

$$Y \xrightarrow{f} \mathrm{Spec}(F) \quad \rightsquigarrow \quad H_a^{et}(Y/F, \mathbb{Z}/n(0)) := H^{-a}(Y_{\acute{e}t}, Rf^!\mathbb{Z}/n(0))$$

be the étale homology with coefficients $\mathbb{Z}/n(0)$ over F, and let

$$E^1_{r,q}(Y/F,\mathbb{Z}/n(0)) \quad \Rightarrow \quad H^{et}_{r+q}(Y/F,\mathbb{Z}/n(0)).$$

be the associated niveau spectral sequence (cf. 2.A). Since F has cohomological dimension 1, theorems 2.14 (b) and theorem 2.21 give a canonical edge morphism

$$\epsilon_Y : H^{et}_{a-1}(Y/F,\mathbb{Z}/n(0)) \to E^2_{a,-1}(Y/F,\mathbb{Z}/n(0)) \;=\; H_a(C^{1,0}(Y,\mathbb{Z}/n)) \;=\; H^K_a(Y,\mathbb{Z}/n)$$

from étale to Kato homology which we want to study more closely for certain varieties.

For the étale homology we use the Hochschild-Serre spectral sequence, which in our case just becomes the collection of short exact sequences

$$0 \to H^{et}_{a+1}(\overline{Y}/\overline{F},\mathbb{Z}/n(0))_\Gamma \xrightarrow{\alpha} H^{et}_a(Y/F,\mathbb{Z}/n(0)) \xrightarrow{\beta} H^{et}_a(\overline{Y}/\overline{F},\mathbb{Z}/n(0))^\Gamma \to 0,$$

where $\Gamma = Gal(\overline{F}/F)$, for an algebraic closure $\overline{F}$ of F, is the absolute Galois group of F, and $\overline{Y} = Y\times_F\overline{F}$. Here we have used that the cohomological dimension of Γ is 1 , and that one has a canonical isomorphisms $H^1(\Gamma,M)\cong M_\Gamma$ for any Γ-module M. If we pass to the inductive limit, we get versions with coefficients $\mathbb{Q}_\ell/\mathbb{Z}_\ell(0)$ or $\mathbb{Q}/\mathbb{Z}(0)$ which we will treat as well. In the following, we shall suppress F and $\overline{F}$ in the notations.

For the Kato homology the following conjecture by Kato, which is a special case of conjecture C in the introduction, will play an important role. Let ℓ be a prime and let ν be a natural number or ∞. If $\nu=\infty$, we define $\mathbb{Z}/\ell^\infty := \mathbb{Q}_\ell/\mathbb{Z}_\ell$.

Conjecture $K(F,\mathbb{Z}/\ell^\nu)$ If X is a connected smooth projective variety over F, then

$$H^K_i(X,\mathbb{Z}/\ell^\nu) \xrightarrow{\cong} \begin{cases} 0 & \text{if } i\neq 0,\\ \mathbb{Z}/\ell^\nu & \text{if } i=0.\end{cases}$$

3.2 More precisely, we want to study ϵ_Y for strict normal crossings varieties, i.e., reduced separated varieties Y with smooth irreducible components $Y_1,\dots,Y_N$ intersecting transversally. Let

$$Y_{i_1,\dots,i_s} := Y_{i_1}\times_Y \ldots \times_Y Y_{i_s}$$

be the scheme-theoretic intersection of $Y_{i_1},\dots,Y_{i_s}$, and write

$$Y^{[s]} := \coprod_{1\le i_1<\cdots<i_s\le N} Y_{i_1,\dots,i_s}$$

for the disjoint union of the s-fold intersections of the Y_i, for $s>0$. By assumption, all $Y^{[s]}$ are smooth. Denote by

$$i^{[s]} : Y^{[s]} \longrightarrow Y$$

the canonical morphism induced by the immersions $Y_{i_1,\ldots,i_s} \hookrightarrow Y$, and let

$$\delta_\nu : Y^{[s]} \longrightarrow Y^{[s-1]} \qquad (\nu = 1, \ldots, s)$$

be the morphism induced by the closed immersions

$$Y_{i_1,\ldots,i_s} \hookrightarrow Y_{i_1,\ldots,\hat{i_\nu},\ldots,i_s}.$$

Definition 3.3 A *good divisor* on Y is a reduced closed subscheme $Z \subset Y$ of pure codimension 1 such that

(a) Z intersects all subschemes $Y_{i_1,\ldots,i_s}$ transversally.

(b) $U = Y \setminus Z$ is affine.

Lemma 3.4 *Let X be a smooth proper variety over a field L, and let D be a smooth divisor on X such that $X \setminus D$ is affine. If X is connected of dimension > 1, then D is connected.*

Proof We may assume that X is geometrically connected over L, and, by base change, that L is algebraically closed. Let $d = \dim X$, and let ℓ be a prime invertible in L. Then we get an exact Gysin sequence

$$0 = H^{2d-1}_{et}(U, \mathbb{Z}/\ell\mathbb{Z}) \to H^{2d-2}_{et}(D, \mathbb{Z}/\ell\mathbb{Z}) \to H^{2d}_{et}(X, \mathbb{Z}/\ell\mathbb{Z}) \to H^{2d}_{et}(U, \mathbb{Z}/\ell\mathbb{Z}) = 0$$

where the vanishing comes from weak Lefschetz (note that $2d - 1 > d = \dim U$). Since $\dim_{\mathbb{Z}/\ell\mathbb{Z}} H^{2d}_{et}(X, \mathbb{Z}/\ell\mathbb{Z})$ is the number of proper connected components of a purely d-dimensional smooth variety X, the claim follows.

If Z is a good divisor, then it is again a strict normal crossing variety. By the lemma, the intersections $Y_i \cap Z$ are the connected components of Z, unless $\dim(Y_i) = 1$. If Y is projective, then a good divisor always exists over any infinite field extension of F by the Bertini theorem. The main result of this section is:

Theorem 3.5 *Let Y be a proper strict normal crossings variety of pure dimension d over F, and let ℓ be a prime number. Let $Z \subset Y$ be a good divisor, and let $U = Y \setminus Z$ be the open complement.*
(a) One has $H^{et}_{a-1}(U, \mathbb{Z}/\ell^\nu(0)) = 0 = H^{et}_a(\overline{U}, \mathbb{Z}/\ell^\nu)$ for $a < d$ and all $\nu \geq 1$.
(b) If the Kato conjecture $K(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ holds in degrees $\leq m$, then the map

$$\epsilon_U : H^{et}_{a-1}(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(0)) \longrightarrow H_a(C^{1,0}(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell)) = H^K_a(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

is an isomorphism for all $a \leq min(m, d)$.

Proof We prove the theorem by double induction on the dimension and the number N of components of Y. If dim $Y = 0$, then the claim is trivially true. Now let Y be of pure dimension d, and assume the statements are true in smaller dimension.

(1) If $N = 1$, i.e., Y has only one component, then Y is smooth and proper. Let $Z \subset Y$ be a good divisor. Then $U = Y \setminus Z$ is affine. For $\ell \neq p$ one then has $H_a^{et}(\overline{U}, \mathbb{Z}/\ell^\nu(0)) = H_{et}^{2d-a}(\overline{U}, \mathbb{Z}/\ell^\nu(d)) = 0$ for $a < d$ by weak Lefschetz. For $\ell = p$ one has $H_a^{et}(\overline{U}, \mathbb{Z}/p^r(0)) = H_{et}^{2d-a}(\overline{U}, \mathbb{Z}/p^r(d)) = H^{d-a}(\overline{U}, W_r\Omega_{U,log}^d) = 0$ for $d - a > 0$, i.e., $a < d$ as well, by 2.14 (c) and the weak Lefschetz theorem proved in [Sw, Lemma 2.1]. By the exact sequence

$$0 \to H_a^{et}(\overline{U}, \mathbb{Z}/\ell^\nu(0))_\Gamma \to H_{a-1}^{et}(U, \mathbb{Z}/\ell^\nu(0)) \to H_{a-1}^{et}(\overline{U}, \mathbb{Z}/\ell^\nu(0))^\Gamma \to 0$$

we see that $H_{a-1}^{et}(U, \mathbb{Z}/\ell^\nu(0)) = 0$ for $a < d$ as claimed in (a).

For (b) we may assume that Y is geometrically connected. First let dim $Y = 1$. In this case $H_1^{et}(\overline{Y}, \mathbb{Q}/\mathbb{Z}(0)) = H_{et}^1(\overline{Y}, \mathbb{Q}/\mathbb{Z}(1)) = \mathrm{Tor}(Pic(\overline{Y}))$ is the torsion of the Jacobian of $\overline{Y}$, hence $H_1^{et}(\overline{Y}, \mathbb{Q}/\mathbb{Z}(0))_\Gamma = 0$, by Weil's theorem. The map

$$H_0^{et}(Z, \mathbb{Q}/\mathbb{Z}(0)) \to H_0^{et}(Y, \mathbb{Q}/\mathbb{Z}(0))$$

is therefore identified with the Gysin map

$$H_{et}^0(\overline{Z}, \mathbb{Q}/\mathbb{Z}(0))^\Gamma \to H_{et}^2(\overline{Y}, \mathbb{Q}/\mathbb{Z}(1))^\Gamma \cong \mathbb{Q}/\mathbb{Z},$$

which is surjective: It has a left inverse up to isogeny, and the target is divisible. Hence the upper row in the following commutative diagram of localization sequences is exact

$$\begin{array}{ccccccccc}
0 & \to & H_0^{et}(U, \mathbb{Q}/\mathbb{Z}(0)) & \to & H_{-1}^{et}(Z, \mathbb{Q}/\mathbb{Z}(0)) & \to & H_{-1}^{et}(Y, \mathbb{Q}/\mathbb{Z}(0)) & \to & 0 \\
 & & \downarrow \epsilon_U & & \downarrow \epsilon_Z & & \downarrow \epsilon_Y & & \\
0 & \to & H_1^K(U, \mathbb{Q}/\mathbb{Z}) & \to & H_0^K(Z, \mathbb{Q}/\mathbb{Z}) & \to & H_0^K(Y, \mathbb{Q}/\mathbb{Z}) & \to & 0.
\end{array}$$

Note that $H_{-1}^{et}(U, \mathbb{Q}/\mathbb{Z}(0)) = 0$ by the first step. Since the Kato conjecture is known for Z and Y (cf. the introduction), one has $H_1^K(Y, \mathbb{Q}/\mathbb{Z}) = 0$, and an isomorphism $H_0^K(Y, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$ via the trace map to Spec(F). Therefore ϵ_Z and ϵ_Y are isomorphisms, and we conclude that the bottom sequence is exact (recall that Kato homology gives a homology theory in the sense of 2.1), ϵ_U is an isomorphism, and $H_0^K(U, \mathbb{Q}/\mathbb{Z}) = 0$. This settles the case dim $Y = 1$.

Now assume dim $Y > 1$. The long exact localization sequence for the Kato homology,

$$\dots H_a^K(Y, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H_a^K(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H_{a-1}^K(Z, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \dots$$

$$\to H_1^K(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H_0^K(Z, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow{\beta} H_0^K(Y, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H_0^K(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to 0$$

then shows that $H_a^K(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$ for all $0 \leq a \leq m$ if the Kato conjecture is true in dimensions $\leq m$ for Y and Z. Note that β is an isomorphism since Z is connected for dim $Y > 1$ by lemma 3.4. So it remains to show that

$H^{et}_{d-1}(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(0)) = 0$ in the considered case as well. The above weak Lefschetz results imply that

$$H^{et}_d(\overline{U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(0))_\Gamma \cong H^{et}_{d-1}(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(0))$$

is divisible, and a quotient of $H^{et}_d(\overline{U}, \mathbb{Q}_\ell(0))_\Gamma$. But the exact sequence

$$H^{et}_d(\overline{Y}, \mathbb{Q}_\ell(0)) \to H^{et}_d(\overline{U}, \mathbb{Q}_\ell(0)) \to H^{et}_{d-1}(\overline{Z}, \mathbb{Q}_\ell(0))$$

shows that the middle group is mixed of weights $-d$ and $-d+1$, since $H^{et}_a(\overline{X}, \mathbb{Q}_\ell(b)) \cong H^{2d-a}_{et}(\overline{X}, \mathbb{Q}_\ell(d-b))$ is pure of weight $b-a$ for a smooth proper variety X by Deligne's proof of the Weil conjecture (for the case $\ell = p$ one uses results of Katz-Messing cf. [G-S]). Hence $H^{et}_d(\overline{U}, \mathbb{Q}_\ell(0))_\Gamma = 0$ for $d > 1$.
(2) Finally we carry out the induction step for the induction on N, the number of components of Y. Let

$$Y' = \bigcup_{i=1}^{N-1} Y_i.$$

Then $Y' \cap Y_N$ is a strict normal crossings variety, and $Z \cap Y'$, $Z \cap Y_N$ and $Z \cap Y' \cap Y_N$ are good divisors on Y', Y_N and $Y' \cap Y_N$, respectively. Write $\overset{\circ}{Y} = U = Y \setminus Z$, $\overset{\circ}{Y'} = Y' \setminus Z$ and $\overset{\circ}{Y_N} = Y_N \setminus Z$, and note that $\overset{\circ}{Y'} \cap \overset{\circ}{Y_N} = (Y' \cap Y_N) \setminus Z$. By 2.6 and 2.16 we get a commutative diagram of Mayer-Vietoris sequences

$$\begin{array}{ccccccc}
..H^{et}_{a-1}(\overset{\circ}{Y'} \cap \overset{\circ}{Y_N}) & \to & H^{et}_{a-1}(\overset{\circ}{Y'}) \oplus H^{et}_{a-1}(\overset{\circ}{Y_N}) & \to & H^{et}_{a-1}(\overset{\circ}{Y}) & \to & H^{et}_{a-2}(\overset{\circ}{Y'} \cap \overset{\circ}{Y_N}).. \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
..\ H^K_a(\overset{\circ}{Y'} \cap \overset{\circ}{Y_N}) & \to & H^K_a(\overset{\circ}{Y'}) \oplus H^K_a(\overset{\circ}{Y_N}) & \to & H^K_a(\overset{\circ}{Y}) & \to & H^K_{a-1}(\overset{\circ}{Y'} \cap \overset{\circ}{Y_N})..
\end{array}$$

by taking the maps ϵ as vertical maps. Here we abbreviated $H^{et}_a(-)$ for $H^{et}_a(-, \ell^\nu(0))$ and $H^K_a(-)$ for $H^K_a(-, \ell^\nu)$, respectively, where $\nu \in \mathbb{N} \cup \{\infty\}$ is fixed. By induction on N, (a) holds for $\overset{\circ}{Y'}, \overset{\circ}{Y_N}$ and $\overset{\circ}{Y'} \cap \overset{\circ}{Y_N}$, hence also for $\overset{\circ}{Y} = U$ by the upper row. Now let $\nu = \infty$. By induction the vertical maps are then isomorphisms for $\overset{\circ}{Y'}$ and $\overset{\circ}{Y_N}$ if $a \le \min(d, m)$, and for $\overset{\circ}{Y'} \cap \overset{\circ}{Y_N}$ if $a \le \min(d-1, m)$, so we conclude by the 5-lemma. In fact, in case $a = d \le m$ note that $H^K_d(\overset{\circ}{Y'} \cap \overset{\circ}{Y_N}) = 0$. So even if $H^{ét}_{d-1}(\overset{\circ}{Y'} \cap \overset{\circ}{Y_N})$ may be non-zero (because $d > d-1 = \dim Y' \cap Y_N$), the 5-lemma in its stronger form applies.

We now come to the proof of theorem 1.4 in the introduction. We use the following spectral sequence.

Proposition 3.6 *Let Y be a noetherian scheme, let $\mathcal{Y} = (Y_1, \ldots, Y_N)$ be an ordered tuple of closed subschemes with $Y = \bigcup Y_i$, and let the notations be as in 3.2 (although we do not assume any normal crossing condition). Let n, r and b be integers such that the Kato complex $C^{r,b}(Y, \mathbb{Z}/n)$ is defined.*
(a) There is a spectral sequence of homological type

$$E^1_{s,t}(\mathcal{Y}, \mathbb{Z}/n) = H_t(C^{r,b}(Y^{[s+1]}, \mathbb{Z}/n)) \ \Rightarrow \ H_{s+t}(C^{r,b}(Y, \mathbb{Z}/n)),$$

in which the E^1-differential is $d^1_{s,t} = \sum_{\nu=1}^{s+1}(-1)^{\nu+1}(\delta_\nu)_*$, with $(\delta_\nu)_*$ being the homomorphism induced by the map δ_ν from 3.2.
(b) One has $E^1_{s,t}(\mathcal{Y}, \mathbb{Z}/n) = 0$ for all $t < 0$ and hence canonical edge morphisms

$$e_a^{\mathcal{Y},n}: \; H_a(C^{r,b}(Y, \mathbb{Z}/n)) \to E^2_{a,0}(\mathcal{Y}, \mathbb{Z}/n) = H_a(H_0(C^{r,b}(Y^{[\bullet+1]}, \mathbb{Z}/n))).$$

(c) Let Y be of finite type over the finite field F. Define the following complex

$$C(\mathcal{Y}, \mathbb{Z}/n): \qquad \ldots \to (\mathbb{Z}/n)^{\pi_0(Y^{[s+1]})} \xrightarrow{d_s} (\mathbb{Z}/n)^{\pi_0(Y^{[s]})} \to \; \ldots \; \to (\mathbb{Z}/n)^{\pi_0(Y^{[1]})}.$$

Here $\pi_0(Z)$ is the set of connected components of a scheme Z, the last term of the complex is placed in degree 0, and the differential d_s is $\sum_{\nu=1}^{s+1}(-1)^{\nu+1}(\delta_\nu)_*$, where $(\delta_\nu)_*$ is the obvious homomorphism induced by the map δ_ν from 3.2. If Y is proper, then there is a canonical homomorphism

$$tr: \; E^2_{a,0}(\mathcal{Y}) \longrightarrow H_a(\mathcal{Y}, \mathbb{Z}/n) := H_a(C(\mathcal{Y}, \mathbb{Z}/n)).$$

If Y is a proper strict normal crossing variety, and $\mathcal{Y} = (Y_1, \ldots, Y_N)$ consists of the irreducible components of Y, then this map is an isomorphism.

Proof (a): Write C for $C^{r,b}$. There is an exact sequence of complexes

$$\ldots \to C(Y^{[s+1]}, \mathbb{Z}/n) \xrightarrow{d_s} C(Y^{[s]}, \mathbb{Z}/n) \to \; \ldots \; \to C(Y^{[1]}, \mathbb{Z}/n) \xrightarrow{\pi_*} C(Y, \mathbb{Z}/n) \to 0,$$

where $d_s = \sum_{\nu=1}^{s+1}(-1)^{\nu+1}(\delta_\nu)_*$, and π_* is induced by the covering map $\pi : Y^{[1]} \to Y$. The exactness is standard. A simple proof can be given by using induction on the number of components and the exact sequence of complexes

$$0 \to C(Y', \mathbb{Z}/n\mathbb{Z}) \to C(Y, \mathbb{Z}/n\mathbb{Z}) \to C(Y_N \setminus Y', \mathbb{Z}/n\mathbb{Z}) \to 0,$$

for Y, Y' and Y_N as in the proof of theorem 3.5.
Then the spectral sequence is induced by the naive filtration of the above sequence of complexes, i.e.,by the second filtration of the double complex $C_\bullet(Y^{[\bullet]}, \mathbb{Z}/n\mathbb{Z})$, whose associated total complex is quasiisomorphic to $C(Y, \mathbb{Z}/n\mathbb{Z})$.
(b): The first claim is clear, since the Kato complex $C(-, \mathbb{Z}/n\mathbb{Z})$ is zero in negative degrees, and the second claim follows from this.
(c): If Y is proper, then the covariant functoriality gives a trace map

$$tr: C^{1,0}(Y, \mathbb{Z}/n\mathbb{Z}) \to C^{1,0}(\mathrm{Spec}(F), \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z},$$

inducing a map $tr : H_0(C^{1,0}(Y, \mathbb{Z}/n\mathbb{Z})) \to \mathbb{Z}/n\mathbb{Z}$. This is functorial in Y, and by restricting to the connected components we get a morphism of complexes

$$E^1_{\bullet,0}(\mathcal{Y}, \mathbb{Z}/n) = H_0(C^{1,0}(Y^{[\bullet+1]}, \mathbb{Z}/n)) \xrightarrow{tr} (\mathbb{Z}/n)^{\pi_0(Y^{[\bullet+1]})} = C(\mathcal{Y}, \mathbb{Z}/n)$$

giving the wanted map tr. It is an isomorphism for strict normal crossings varieties, since $tr : H_0(C^{1,0}(X, \mathbb{Z}/n\mathbb{Z})) \to \mathbb{Z}/n\mathbb{Z}$ is an isomorphism for smooth proper connected X (the known case $i = 0$ of Kato's conjecture $K(F, \mathbb{Z}/n)$, cf. [CT], [Sw], or 5.2 below).

As before, we also have versions with coefficients in $\mathbb{Z}/\ell^\infty := \mathbb{Q}_\ell/\mathbb{Z}_\ell$ for a prime number ℓ of the complexes and statements above.

Definition 3.7 Let Y be a proper strict normal crossings variety over F. If $Y_1, \ldots, Y_N$ are the irreducible components of Y, and $\mathcal{Y} = (Y_1, \ldots, Y_N)$, then we define $C(\Gamma_Y, \mathbb{Z}/n\mathbb{Z}) := C(\mathcal{Y}, \mathbb{Z}/n\mathbb{Z})$ and $H_a(\Gamma_Y, \mathbb{Z}/n\mathbb{Z}) := H_a(\mathcal{Y}, \mathbb{Z}/n\mathbb{Z})$ and call it the *configuration chain complex* and *configuration homology* of Y, respectively.

Remark 3.8 If Y is a (proper) strict normal crossings variety, let us define its *configuration (or dual) complex* Γ_Y as the simplicial complex, whose $s-$simplices correspond to the conncted components of $Y^{[s+1]}$, with the face maps given by the δ_ν. Then the group $H_a(\Gamma_Y, \mathbb{Z}/n\mathbb{Z})$, as defined above, does in fact compute the a-th homology of Γ_Y with coefficients in $\mathbb{Z}/n\mathbb{Z}$. If Y has dimension d, then Γ_Y has at most dimension d, and $C_a(\Gamma_Y, \mathbb{Z}/n\mathbb{Z}) = 0$ for $a > d$. If Y is a curve, then Γ_Y is a graph and is also called the intersection graph of Y.

Theorem 3.9 *Let Y be a proper strict normal crossings variety of pure dimension d over F, let ℓ be a prime number, and let ν be a natural number or ∞. Then there is a canonical map*

$$\gamma = \gamma_a^{Y,\ell^\nu} : H_a(Y, \mathbb{Z}/\ell^\nu) \longrightarrow H_a(\Gamma_Y, \mathbb{Z}/\ell^\nu),$$

where we define $\mathbb{Z}/\ell^\infty := \mathbb{Q}_\ell/\mathbb{Z}_\ell$. This map is an isomorphism for all $a \leq m$ if the Kato conjecture $K(F, \mathbb{Z}/\ell^\nu)$ is true in degrees $\leq m$.

Proof We define γ_a^{Y,ℓ^ν} as the composition

$$\gamma_a^{Y,\ell^\nu} : \; H_a(C^{1,0}(Y, \mathbb{Z}/\ell^\nu)) \xrightarrow{e_a^{\mathcal{Y},\ell^\nu}} E^2_{a,0}(\mathcal{Y}, \mathbb{Z}/n) \xrightarrow{tr} H_a(C(\Gamma_Y, \mathbb{Z}/\ell^\nu)),$$

where $\mathcal{Y} = (Y_1, \ldots, Y_N)$. By 3.6 (b) , tr is an isomorphism. If the Kato conjecture $K(F, \mathbb{Z}/\ell^\nu)$ holds in degrees $\leq m$, then one has $E^1_{p,q}(\mathcal{Y}, \mathbb{Z}/n) = 0$ for all $q = 1, \ldots, m$, so that $e_a^{\mathcal{Y},\ell^\nu}$ is an isomorphism in degrees $a \leq m$.

Remark 3.10 As the referee points out, we have morphisms of complexes

$$C^{1,0}(Y, \mathbb{Z}/\ell^\nu) \overset{quis}{\leftarrow} totC^{1,0}_\bullet(Y^{[\bullet]}, \mathbb{Z}/\ell^\nu) \; \rightarrow \; E^1_{\bullet,0}(\mathcal{Y}, \mathbb{Z}/\ell^\nu) \overset{tr}{\rightarrow} C(\Gamma_Y, \mathbb{Z}/\ell^\nu)$$

which induce γ and are quasi-isomorphisms if $K(F, \mathbb{Z}/\ell^\nu)$ holds.

4. The residue map in étale homology

We consider the same situation and notations as in 2.C: Hence we have $S = \mathrm{Spec}(A)$ for a discrete valuation ring A, with generic point $\eta = \mathrm{Spec}(K)$ and special point $s = \mathrm{Spec}(F)$. But in this section we assume that A is henselian.

4.1 Let $f : X \to S$ be a scheme which is separated and flat of finite type over S. We get a diagram with cartesian squares

$$\begin{array}{ccccc} X_\eta & \xrightarrow{j_X} & X & \xleftarrow{i_X} & X_s \\ \downarrow f_\eta & & \downarrow f & & \downarrow f_s \\ \eta & \xrightarrow{j} & S & \xleftarrow{i} & s \end{array}$$

The connecting morphisms $\delta : H_a(X_\eta/S, \mathbb{Z}/n(-1)) \to H_{a-1}(X_s/S, \mathbb{Z}/n(-1))$ in étale homology over S together with lemma 2.17 give residue maps in étale homology

$$\Delta_X : \ H_a(X_\eta, \mathbb{Z}/n(-1)) \longrightarrow H_{a+1}(X_s, \mathbb{Z}/n(0)),$$

which we want to study under suitable conditions. We shall also consider the versions with $\mathbb{Z}/n$ replaced by $\mathbb{Q}/\mathbb{Z}$ or $\mathbb{Q}_\ell/\mathbb{Z}_\ell$. Here we omit K and F in the notation for the homology of schemes over K and F, respectively, as in section 3. For a fixed algebraic closure $\overline{F}$ of F, let $\Gamma = \mathrm{Gal}(\overline{F}/F)$, $\overline{s} = \mathrm{Spec}(\overline{F})$, and $X_{\overline{s}} = X_s \times_s \overline{s}$. Similarly, let $\overline{K}$ be a fixed separable closure of K and write $\overline{\eta} = \mathrm{Spec}(\overline{K})$ and $X_{\overline{\eta}} = X_\eta \times_\eta \overline{\eta}$. We shall also consider the homology groups of $X_{\overline{s}}/\overline{F}$ and $X_{\overline{\eta}}/\overline{K}$, and omit the fields $\overline{F}$ and $\overline{K}$ in the notations as well.

Let $f : X \longrightarrow S$ be regular of pure relative dimension $d \geq 1$ and with strict semi-stable reduction. Hence X_s is a strict normal crossings variety over F.

Definition 4.2 By a *good divisor* on X we mean a divisor $Z \hookrightarrow X$ which is flat over S and for which Z_s is a good divisor in X_s in the sense of 3.3.

Proposition 4.3 *(a) If Z is a good divisor on X, then Z is regular and has strict semi-stable reduction, and is of pure relative dimension $d-1$.*

(b) If X is projective over S and F has infinitely many elements, then there is always a good divisor.

Proof (cf. also [JS]) Let $Y_1, \ldots, Y_M$ be the irreducible components of the special fibre $Y := X_s$. Then all r-fold intersections

$$Y_{i_1,\ldots,i_r} := Y_{i_1} \cap Y_{i_2} \cap \ldots Y_{i_r}$$

$(1 \leq i_1 < i_2 < \ldots < i_r \leq M)$ are smooth. If $X \hookrightarrow \mathbb{P}^N_S$ is a projective embedding and F is infinite, then by the Bertini theorem there is a hyperplane $H_0 \subset \mathbb{P}^N_F = \mathbb{P}^N_S \times_S s$, defined over F, intersecting all (irreducible components of all) varieties

$Y_{i_1,\dots,i_r}$ transversally. Let $H \subset \mathbb{P}^N_S$ be any hyperplane lifting H_0. Then the scheme-theoretic intersection $Z = H \cdot X = H \times_{\mathbb{P}^N_S} X$ is flat over S (the generic fibre is non-empty), and is a good divisor, by definition. It thus suffices to show (a).
We may assume that F is algebraically closed. Let x be a closed point of X_s which is contained in Z. The completion $\hat{\mathcal{O}}_{X,x}$ of the local ring $\mathcal{O}_{X,x}$ is isomorphic to

$$B = A[[x_1, \dots, x_{d+1}]]/\langle x_1 \dots x_r - \pi\rangle \qquad (1 \le r \le d+1)$$

where π is a prime element in A. Since X is regular, Z is defined by one local equation at x. Let f be the image of the local equation in B, and let $\mathfrak{n} \subseteq B$ be the maximal ideal. Then $f \in \mathfrak{n}$, and the elements $x_1, \dots, x_r$ are the images of the local equations for $Y_{i_1}, \dots, Y_{i_r}$ for suitable $1 \le i_1 < \dots < i_r \le M$. Thus the trace of $Y_{i_1,\dots,i_r}$ in $\hat{\mathcal{O}}_{X,x} \cong B$ corresponds to the quotient

$$B' = B/\langle x_1, \dots, x_r\rangle \cong F[[x_{r+1}, \dots, x_{d+1}]] \quad .$$

Since $x \in H$ and, by assumption, H does not intersect the zero-dimensional varieties $Y_{i_1,\dots,i_{d+1}}$, we may assume $r < d+1$. Then H intersects $Y_{i_1,\dots,i_r}$ transversally at x if and only if the image of f in B' lies in $\mathfrak{n}' - (\mathfrak{n}')^2$, where $\mathfrak{n}'$ is the maximal ideal of B'. Since $\mathfrak{n}'/(\mathfrak{n}')^2 \cong \mathfrak{n}/(\mathfrak{n}^2 + \langle x_1, \dots, x_r\rangle)$ we see that f has non-zero image in $\mathfrak{n}/(\mathfrak{n}^2 + \langle x_1, \dots, x_r\rangle)$.
Now the elements $x_i \bmod \mathfrak{n}^2$ $(i = 1, \dots, d+1)$ form an F-basis of $\mathfrak{n}/\mathfrak{n}^2$. Hence we have

$$f \equiv \sum_{i=1}^{d+1} a_i x_i \mod \mathfrak{n}^2$$

with elements $a_i \in A$ which are determined modulo $\langle \pi\rangle$. By our condition on f, a_i must be a unit for one i with $i > r$, and by possibly renumbering and multiplying f by a unit we may assume $i = d+1$, and $a_{d+1} = 1$. But then

$$B/\langle f\rangle \cong A[[x_1, \dots, x_d]]/\langle x_1 \dots x_r - \pi\rangle$$

which proves the claim. Note that the irreducible components of $(H\cdot X)_s = H_s \cdot X_s$ are the connected components of the smooth varieties $H_s \cdot Y_i$.

Good divisors Z and "good opens" $U = X \setminus Z$ are useful because of the following:

Theorem 4.4 *Assume that F is a finite field of characteristic p, and that X is proper of pure relative dimension d and has strict semi-stable reduction. Let $Z \hookrightarrow X$ be a good divisor, and let $U = X \setminus Z$ be the open complement.*
(a) One has $H_{a-2}(U_\eta, \mathbb{Z}/n(-1)) = H_{a-1}(U_s, \mathbb{Z}/n(0)) = 0$ if $a < d$.
(b) The map

$$\Delta_U : \ H_{a-2}\big(U_\eta, \mathbb{Z}/n(-1)\big) \to H_{a-1}\big(U_s, \mathbb{Z}/n(0)\big)$$

is an isomorphism for all n prime to p and all $a \le d$.
(c) Assume $\mathrm{ch}(K) = 0$. The map

$$\Delta_U : \ H_{d-2}(U_\eta, \mathbb{Q}_p/\mathbb{Z}_p(-1)) \to H_{d-1}(U_s, \mathbb{Q}_p/\mathbb{Z}_p(0))$$

is a surjective isogeny, and it is an isomorphism if $p \ge d$ or if X is smooth over S.

The rest of this section is devoted to the proof of this theorem. Part (a) for U_s was proved in 3.5 (a), and it follows directly from weak Lefschetz for U_η: Since U_η is smooth of dimension d, we have $H_{a-2}(U_\eta, \mathbb{Z}/n(o)) \cong H^{2d-a+2}(U_\eta, \mathbb{Z}/n(d))$, which is zero for $a < d$ by weak Lefschetz, since K has cohomological dimension at most 2. Now we turn to parts (b) and (c).

4.5 From the localization sequence together with Lemma 2.17 we get a long exact sequence

$$\cdots \to H_a(X/S, \mathbb{Z}/n(-1)) \to H_a(X_\eta, \mathbb{Z}/n(-1)) \stackrel{\Delta_X}{\to}$$

$$H_{a+1}\big(X_s, \mathbb{Z}/n(0)\big) \to H_{a-1}(X/S, \mathbb{Z}/n(-1)) \to \dots .$$

This sequence shows that kernel and cokernel of the residue map Δ_X are controlled by the homology groups $H_\bullet(X/S, \mathbb{Z}/n(-1)) = H^{-\bullet}(X,\, Rf^!\,\mathbb{Z}/n(1)_S\,)$. We study these cohomology groups in the following. First we consider the prime-to-p case, where $\mathbb{Z}/n(1)_S$ is the usual sheaf $\mathbb{Z}/n(1) = \mu_n$, the first Tate twist of the constant sheaf $\mathbb{Z}/n$.

Lemma 4.6 *Let n be prime to p, and let b be any integer. There are canonical isomorphisms*

(*a*) $$Rf^!\mathbb{Z}/n(b) \xrightarrow{\sim} \mathbb{Z}/n(b+d)[2d]$$

(*b*) $$H_a(X/S, \mathbb{Z}/n(b)) \xrightarrow{\sim} H^{2d-a}\big(X, \mathbb{Z}/n(d-b)\big)$$

Proof (b) follows from (a), and (a) is a special case of Grothendieck's purity conjecture for excellent schemes, a proof of which has been announced by Gabber and written down by Fujiwara [Fu].

Lemma 4.7 *For X and U as in 4.4, and n prime to p, one has*
(a) $H^i(X, \mathbb{Z}/n(j)) \cong H^i(X_s, \mathbb{Z}/n(j))$ for all $i, j \in \mathbb{Z}$,
(b) $H^i(U, \mathbb{Z}/n(j)) \cong H^i(U_s, \mathbb{Z}/n(j))$ for all $i, j \in \mathbb{Z}$.

Proof Part (a) follows from the proper base change theorem. For (b) let $g : U \to S$ be the structural morphism, and let $g_s : U_s \to s$ be the base change to $s = \mathrm{Spec}\ F$. Then it follows from [R-Z, 2.18 and 2.19] that the base change morphism

$$i^*Rg_*\mathbb{Z}/n(b) \to Rg_{s*}\mathbb{Z}/n(b),$$

is an isomorphism, too, for the complement $U = X \setminus Z$ of a good divisor Z, and we obtain (b).

With this, we can now prove Theorem 4.4 (b): Recall that, by assumption, U_s has dimension d and is affine. By weak Lefschetz and the Hochschild-Serre exact sequences

$$0 \to H^{i-1}(U_{\overline{s}}, \mathbb{Z}/n(j))_\Gamma \to H^i(U_s, \mathbb{Z}/n(j)) \to H^i(U_{\overline{s}}, \mathbb{Z}/n(j))^\Gamma \to 0$$

we conclude that $H^i(U_s, \mathbb{Z}/n(j)) = 0$ for $i > d+1$. On the other hand, by 4.5 and 4.6 (b) we have a long exact sequence

$$\begin{array}{llllll} \to & H^{2d-a+2}(U_s, \mathbb{Z}/n(-1)) & \to & H_{a-2}(U_\eta, \mathbb{Z}/n(-1)) & \stackrel{\Delta_U}{\to} & H_{a-1}(U_s, \mathbb{Z}/n(0)) \\ \to & H^{2d-a+3}(U_s, \mathbb{Z}/n(-1)) & \to & & & . \end{array}$$

This evidently shows the claimed bijectivity of Δ_U for all $a \leq d$.

4.8 Now we prove Theorem 4.4 (c). We first fix some notation. Let A denote a henselian discrete valuation ring with perfect (not necessarily finite) residue field F of characteristic $p > 0$ and with quotient field K of characteristic 0. Let $S = \mathrm{Spec}(A)$ and assume given a diagram as in 4.1:

$$\begin{array}{ccccc} X_\eta & \xrightarrow{j_X} & X & \xleftarrow{i_X} & X_s \\ \downarrow f_\eta & & \downarrow f & & \downarrow f_s \\ \eta & \xrightarrow{j} & S & \xleftarrow{i} & s \end{array}$$

Assume that X is proper and generically smooth with strict semistable reduction of relative dimension d over S. Let L (resp. M_X) be the log structure on S (resp. X) associated to $s \hookrightarrow S$ (resp. $X_s \hookrightarrow X$) and let L_s (resp. M_{X_s}) be its inverse image on s (resp. X_s). We have the Cartesian square of morphisms of log-schemes

$$\begin{array}{ccc} (X_s, M_{X_s}) & \to & (X, M_X) \\ \downarrow f_s & & \downarrow f \\ (s, L_s) & \to & (S, L) \end{array}$$

where f and f_s is log smooth. Assume now given $Z \subset X$, a good divisor in the sense of Definition 4.2 and let $U = X - Z$. We have the diagram of immersions

$$\begin{array}{ccccc} U_s & \xrightarrow{i_U} & U & \xleftarrow{j_U} & U_\eta \\ \downarrow \phi & & \downarrow \phi & & \downarrow \phi \\ X_s & \xrightarrow{i_X} & X & \xleftarrow{j_X} & X_\eta \\ \uparrow \tau & & \uparrow \tau & & \uparrow \tau \\ Z_s & \xrightarrow{i_Z} & Z & \xleftarrow{j_Z} & Z_\eta \end{array}$$

Let M_U (resp. M_Z) be the log structure on X (resp. Z) associated to $X_s \cup Z \subset X$ (resp. $Z_s \hookrightarrow Z$) and let M_{U_s} (resp. M_{Z_s}) be its inverse image on X_s (resp. Z_s). By definition we have $M_X = \mathcal{O}_X \cap (j_X)_* \mathcal{O}^*_{U_\eta}$, $M_U = \mathcal{O}_X \cap (\phi j_U)_* \mathcal{O}^*_{U_\eta}$, and $M_Z = \mathcal{O}_Z \cap (j_Z)_* \mathcal{O}^*_{Z_\eta}$. Let

$$W_n\omega^\cdot_{X_s}, \quad W_n\omega^\cdot_{X_s}(\log Z), \quad W_n\omega^\cdot_{Z_s}$$

denote the de Rham-Witt complexes associated to the log smooth schemes (X_s, M_{X_s}), (X_s, M_{U_s}), and (Z_s, M_{Z_s}) over (s, L_s) respectively (cf. [HK]). For $n = 1$ we use the simplified notation $\omega^{\cdot}_{X_s}$, $\omega^{\cdot}_{X_s}(\log Z)$, and $\omega^{\cdot}_{Z_s}$ for the de Rham-Witt complexes. They coincide with the complexes of logarithmic differentials defined in [K3], (1.9). We have the map of sheaves on $(X_s)_{et}$ (cf. [HK],(4.9)).

$$dlog\ :\ (M_{U_s}^{gp})^{\otimes q} \to W_n\omega^q_{X_s}(\log Z);\ a_1 \otimes \cdots \otimes a_q \to dlog(a_1) \wedge \cdots \wedge dlog(a_q).$$

We define the etale subsheaf $W_n\omega^q_{X_s}(\log Z)_{log} \subset W_n\omega^q_{X_s}(\log Z)$ to be the image of the above map. In a similar way we define the subsheaves

$$W_n\omega^q_{X_s,log} \subset W_n\omega^q_{X_s} \quad \text{and} \quad W_n\omega^q_{Z_s,log} \subset W_n\omega^q_{Z_s}.$$

Lemma 4.9 *The sheaf* $W_n\omega^q_{X_s}(\log Z)_{log}$ *is flat over* $\mathbb{Z}/p^n\mathbb{Z}$ *and we have an exact sequence*

$$0 \to W_n\omega^q_{X_s}(\log Z)_{log} \xrightarrow{p^m} W_{m+n}\omega^q_{X_s}(\log Z)_{log} \to W_m\omega^q_{X_s}(\log Z)_{log} \to 0.$$

The analogous facts hold for $W_n\omega^q_{X_s,log}$ *and* $W_n\omega^q_{Z_s,log}$.

Proof This follows from [H], (2.6) and [Lo] (1.5.4).

By [H], (1.6.1) and [Ts3], (3.1.5) the sheaf $R^m(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(m)$ is generated by symbols $\{x_1, \ldots, x_m\} = \{x_1\} \cup \ldots \cup \{x_m\}$ with local sections $x_1, \ldots, x_m$ of M_U^{gp}, i.e., by the cup products of the images $\{x_1\}, \ldots, \{x_m\}$ of these elements under the Kummer map $M_U^{gp} \to R^1(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(1)$. There exists a natural morphism

$$\text{(4-1)} \qquad \delta_m^{sym}\ :\ i_X^* R^m(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(m) \longrightarrow W_n\omega^m_{X_s}(\log Z)_{log}$$

sending $\{x_1 \ldots x_m\}$ to $dlog(x_1) \wedge \ldots \wedge dlog(x_m)$. In case $m = d+1$, $W_n\omega^{d+1}_{X_s,log} = 0$ and $R^{d+1}(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(d+1)$ is generated by symbols $\{\pi_K, x_1, \ldots, x_d\}$ with $x_1, \ldots, x_d$ as above and π_K, a prime element of K. Then there exists the natural morphism

$$\delta_{tame}\ :\ i_X^* R^{d+1}(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(d+1) \longrightarrow W_n\omega^d_{X_s}(\log Z)_{log}$$

which maps $\{\pi_K, x_1, \ldots, x_d\}$ to $dlog(x_1) \wedge \ldots \wedge dlog(x_d)$.

Lemma 4.10 $R^q(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(d+1) = 0$ *for* $q > d+1$.

Proof Since ϕj_U is affine, the stalk of $R^q(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(d+1)$ at a geometric point $\overline{x}$ over $x \in X_s$ is a limit of groups $H^q(V, \mathbb{Z}/p^n\mathbb{Z}(d+1))$ where V is an affine variety of dimension d over K^{ur}, the maximal unramified extension of K. The lemma now follows from the fact that $cd(K^{ur}) = 1$ and $cd(V \times \overline{K}) = d$, cf. [SGA4], XIV 3.1.

By the above lemma δ_{tame} induces a morphism in $D^b(X_{et})$

$$\delta_{tame}: \ R(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(d+1) \to (i_X)_*W_n\omega^d_{X_s}(\log Z)_{log}[-d-1]$$

Proposition 4.11 *Let $g: U \to S$, $g_\eta: U_\eta \to \eta$, and $g_s: U_s \to s$ be the structural morphisms. There is a canonical isomorphism in $D^b(X_{s,\acute{e}t})$,*

$$\alpha_{X,Z}: \ R\phi_* Rg^!_s\mathbb{Z}/p^n\mathbb{Z}(0)_s \cong W_n\omega^d_{X_s}(\log Z)_{log}[d]$$

which fits into the following commutative diagram

$$\begin{array}{ccc} H^{2d-a}(U_\eta, \mathbb{Z}/p^n\mathbb{Z}(d+1)) & \xrightarrow{\delta_{tame}} & H^{d-a-1}(X_s, W_n\omega^d_{X_s}(\log Z)_{log}) \\ \uparrow \cong \textit{2.14 (c)} & & \uparrow \cong \alpha_{X,Z} \\ H^{-a}(U_\eta, Rg^!_\eta\mathbb{Z}/p^n\mathbb{Z}(1)_\eta) & \xrightarrow{\delta} & H^{-a-1}(U_s, Rg^!_s\mathbb{Z}/p^n\mathbb{Z}(0)_s) \\ \| & & \| \\ H_a(U_\eta, \mathbb{Z}/p^n\mathbb{Z}(-1)) & \xrightarrow{\Delta_U} & H_{a+1}(U_s, \mathbb{Z}/p^n\mathbb{Z}(0)) \end{array}$$

where δ is induced by the map $\sigma: Rj_*\mathbb{Z}/p^n\mathbb{Z}(1)_\eta \to Ri_*\mathbb{Z}/p^n\mathbb{Z}(0)_s[-1]$ in §2 C by base change for $g^!$.

Proof The commutativity of the lower diagram is a direct consequence of the definition of Δ_U. In [JSS] it is proved that there is a canonical isomorphism,

$$\alpha_{U_s}: Rg^!_s\mathbb{Z}/p^n\mathbb{Z}(0)_s \cong W_n\omega^d_{U_s,log}[d] \tag{4-2}$$

which fits into the following commutative diagram, in which α_{U_η} is the purity isomorphism for the smooth morphism g_η (cf. (2-1))

$$\begin{array}{ccc} R(j_U)_*\mathbb{Z}/p^n\mathbb{Z}(d+1)[2d] & \xrightarrow{\delta_{tame}} & (i_U)_*W_n\omega^d_{U_s,log}[d-1] \\ \uparrow \cong \alpha_{U_\eta} & & \uparrow \cong \alpha_{U_s} \\ R(j_U)_*Rg^!_\eta\mathbb{Z}/p^n\mathbb{Z}(1)_\eta & \xrightarrow{\delta} & (i_U)_*Rg^!_s\mathbb{Z}/p^n\mathbb{Z}(0)_s[-1]. \end{array}$$

Applying $R\phi_*$ to it, and noting $\phi j_U = j_X\phi$, we get the commutative diagram

$$\begin{array}{ccc} R(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(d+1)[2d] & \xrightarrow{\delta_{tame}} & (i_X)_*W_n\omega^d_{X_s}(\log Z)_{log}[d-1] \\ \| & & \downarrow \gamma \\ R(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(d+1)[2d] & \xrightarrow{\delta_{tame}} & (i_X)_*R\phi_*W_n\omega^d_{U_s,log}[d-1] \\ \uparrow \cong & & \uparrow \cong \\ R(\phi j_U)_*Rg^!_\eta\mathbb{Z}/p^n\mathbb{Z}(1)_\eta & \xrightarrow{\delta} & (i_X)_*R\phi_*Rg^!_s\mathbb{Z}/p^n\mathbb{Z}(0)_s[-1] \end{array}$$

where $W_n\omega^d_{U_s,log} = \phi^*W_n\omega^d_{X_s}(\log Z)_{log}$ and γ is the adjunction map. Thus Proposition 4.11 follows from the following result shown in [Sat3].

Proposition 4.12 *The adjunction induces an isomorphism*

$$W_n\omega^d_{X_s}(\log Z)_{log} \xrightarrow{\cong} R\phi_*W_n\omega^d_{U_s,log}.$$

In view of Proposition 4.11, Theorem 4.4 (c) is now implied by the following result.

Theorem 4.13 *Assume that F is finite. Let*

$$\delta^{\infty}_{tame} \;:\; H^{d+2}(U_\eta, \mathbb{Q}_p/\mathbb{Z}_p(d+1)) \to H^1(X_s, W_\infty\omega^d_{X_s}(\log Z)_{log})$$

be obtained from the maps δ_{tame} in Proposition 4.11 by passing to the inductive limit over n, where $W_\infty\omega^q_{X_s}(\log Z)_{log} = \varinjlim_n W_n\omega^q_{X_s}(\log Z)_{log}$ (with transition maps denoted p^m in 4.9). Then δ^∞_{tame} is a surjective isogeny and it is an isomorphism if $p \geq d$ or if X is smooth over S.

In order to show this theorem, we want to compare the map δ_{tame} in Proposition 4.11 with another one, to be able to quote some results of Sato [Sat1] and Tsuji [Ts3]. Let $\overline{K}$ be a fixed separable closure of K and recall the notation in 4.1. Let $\overline{A}$ be the integral closure of A in $\overline{K}$ and put $\overline{S} = \mathrm{Spec}(\overline{A})$ and $\overline{\eta} = \mathrm{Spec}(\overline{K})$. By base change, we obtain the diagram

$$\begin{array}{ccccc}
U_{\overline{s}} & \xrightarrow{\bar{i}_U} & \overline{U} & \xleftarrow{\bar{j}_U} & U_{\overline{\eta}} \\
\downarrow\phi & & \downarrow\phi & & \downarrow\phi \\
X_{\overline{s}} & \xrightarrow{\bar{i}_X} & \overline{X} & \xleftarrow{\bar{j}_X} & X_{\overline{\eta}} \\
\uparrow\tau & & \uparrow\tau & & \uparrow\tau \\
Z_{\overline{s}} & \xrightarrow{\bar{i}_Z} & \overline{Z} & \xleftarrow{\bar{j}_Z} & Z_{\overline{\eta}}
\end{array}$$

where $\overline{X} = X \times_S \overline{S}$ and so on. Passing to the limit over the base changes by all finite extensions of K contained in $\overline{K}$, the maps δ^{sym}_m from (4-1) induce a morphism

$$\text{(4-3)} \qquad \overline{\delta}^{sym}_m : \bar{i}^*_X R^m(\phi\bar{j}_U)_*\mathbb{Z}/p^n\mathbb{Z}(m) \to W_n\omega^m_{X_{\overline{s}}}(\log Z)_{log}.$$

of etale sheaves on $X_{\overline{s}}$. It induces the map

$$\text{(4-4)} \qquad \overline{\delta}^{sym}_d : H^d_{et}(U_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z}(d)) \to H^0(X_{\overline{s}}, W_n\omega^d_{X_{\overline{s}}}(\log Z)_{log}).$$

Lemma 4.14 *Assume that F is finite. Let $G = \mathrm{Gal}(\overline{K}/K)$ (resp. $\Gamma = \mathrm{Gal}(\overline{F}/F)$) be the absolute Galois group of K (resp. F). Then the following diagram is commutative*

$$\begin{array}{ccc}
H^d(U_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z}(d))_G & \xrightarrow{\overline{\delta}^{sym}_d} & H^0(X_{\overline{s}}, W_n\omega^d_{X_{\overline{s}}}(\log Z)_{log})_\Gamma \\
\downarrow \phi_K & & \downarrow \phi_F \\
H^2(K, H^d(U_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z}(d+1))) & & H^1(F, H^0(X_{\overline{s}}, W_n\omega^d_{X_{\overline{s}}}(\log Z)_{log})) \\
\downarrow \cong (a) & & \downarrow \cong (b) \\
H^{d+2}(U_\eta, \mathbb{Z}/p^n\mathbb{Z}(d+1)) & \xrightarrow{\delta_{tame}} & H^1(X_s, W_n\omega^d_{X_s}(\log Z)_{log})
\end{array}$$

where ϕ_K (resp. ϕ_F) are the isomorphisms coming from the duality theorems for Galois cohomology of K (resp. F), and where the isomorphisms (a) and (b) come from the Hochschild-Serre spectral sequences.

Proof (We are indebted to every reader who finds a simpler proof.) The maps (a) and (b) are isomorphisms since $H^r(U_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z}(d+1)) = 0$ for $r > d$ by [SGA4], XIV 3.1 and $H^s(X_{\overline{s}}, W_n\omega^d_{X_{\overline{s}}}(\log Z)_{log}) = H_{d-s}(U_{\overline{s}}, \mathbb{Z}/p^n\mathbb{Z}(0)) = 0$ for $s > 0$ by Proposition 4.11 and Theorem 3.5. Recall that local duality (resp. duality for Γ) gives isomorphisms

$$\phi_K : M_G \xrightarrow{\sim} H^2(K, M(1)) \qquad (\text{resp.} \quad \phi_F : N_\Gamma \xrightarrow{\sim} H^1(F, N)\ )$$

for any discrete torsion G-module M (resp. Γ-module N). For the proof of the commutativity we first note that we may pass to any finite extension K'/K. In fact, the diagram obtained by base change to $S' = \mathrm{Spec}(\mathcal{O}_{K'}) = \{\eta', s'\}$ is related to the previous one by the corestrictions from K' to K (resp. $F' = k(s')$ to F) and the trace maps from $U'_{\eta'}$ to U_η (resp. $X'_{s'}$ to X_s). This is clear for the top and the vertical maps of the diagram. For the bottom, we may translate back to homology via 4.11, and then have to show that the following diagram commutes:

$$\begin{array}{ccc} H_a(U'_{\eta'}, \mathbb{Z}/p^n\mathbb{Z}(-1)) & \xrightarrow{\Delta'_U} & H_{a+1}(U'_{s'}, \mathbb{Z}/p^n\mathbb{Z}(0)) \\ \downarrow tr_{K'/K} & & \downarrow tr_{F'/F} \\ H_a(U_\eta, \mathbb{Z}/p^n\mathbb{Z}(-1)) & \xrightarrow{\Delta_U} & H_{a+1}(U_s, \mathbb{Z}/p^n\mathbb{Z}(0)) \end{array}.$$

This commutativity follows from the covariance of étale homology over S (and its compatibility with localization sequences) once we show that the étale homology of U' over S' (which is considered in the first line) coincides with the étale homology of U' over S. This holds, however, because $Rg^!(\mathbb{Z}/p^n(1))_S = (\mathbb{Z}/p^n(1))_{S'}$ for $g : S' \to S$ and the complexes defined in §2 C , as one easily sees (a detailed proof can be found in [JSS]).

Thus it suffices to consider elements which are in the image of the map $H^d(U_\eta, , \mathbb{Z}/p^n\mathbb{Z}(d)) \to H^d(U_{\overline{\eta}}, , \mathbb{Z}/p^n\mathbb{Z}(d))^G \to H^d(U_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z}(d))_G$. Let

$$tr : H^2(K, \mathbb{Z}/p^n\mathbb{Z}(1)) \underset{\sim}{\longrightarrow} H^1(F, \mathbb{Z}/p^n\mathbb{Z}) \underset{\sim}{\xrightarrow{tr}} \mathbb{Z}/p^n\mathbb{Z}$$

be the canonical (trace) isomorphisms of (1-1) in the introduction, and let

$$\chi_K \in H^2(K, \mathbb{Z}/p^n\mathbb{Z}(1)) \qquad \text{and} \qquad \chi_F \in H^1(F, \mathbb{Z}/p^n\mathbb{Z})$$

be the inverse images of $1 \in \mathbb{Z}/p^n\mathbb{Z}$ under the isomorphisms. Then the composition

$$H^d(U_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z}(d))^G \longrightarrow H^d(U_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z}(d))_G \xrightarrow{\phi_K} H^2(K, H^d(U_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z}(d+1)))$$

is the cup product with χ_K. Now, by the compatibility of cup product with Hochschild-Serre spectral sequences we have a commutative diagram of cup product pairings

$$\begin{array}{ccccccc} & H^2(U_\eta, 1) & \times & H^d(U_\eta, d) & \xrightarrow{\cup} & H^{d+2}(U_\eta, d+1) \\ & \uparrow & & \downarrow & & \uparrow \\ & H^1(\Gamma, H^1(U^{ur}_\eta, 1)) & \times & H^d(U^{ur}_\eta, d)^\Gamma & \xrightarrow{\cup} & H^1(\Gamma, H^{d+1}(U^{ur}_\eta, d+1)) \\ (4\text{-}5) & \uparrow & & \downarrow & & \uparrow \\ & H^1(\Gamma, H^1(K^{ur}, 1)) & \times & H^d(U_{\overline{\eta}}, d)^G & \xrightarrow{\cup} & H^1(\Gamma, H^1(K^{ur}, H^d(U_{\overline{\eta}}, d+1))) \\ & \|\wr & & \| & & \|\wr \\ & H^2(K, 1) & \times & H^d(U_{\overline{\eta}}, d)^G & \xrightarrow{\cup} & H^2(K, H^d(U_{\overline{\eta}}, d+1)). \end{array}$$

Here we have omitted the coefficients $\mathbb{Z}/p^n$ (but indicated the Tate twists), K^{ur} is the maximal unramified extension of K, $\Gamma = \mathrm{Gal}(K^{ur}/K)$, and the vertical maps are restrictions or come from the obvious Hochschild-Serre spectral sequences. The middle diagram comes from the commutative diagram

$$\begin{array}{ccccc} H^1(U_\eta^{ur},1) & \times & H^d(U_\eta^{ur},d) & \xrightarrow{\cup} & H^{d+1}(U_\eta^{ur},d+1) \\ \uparrow & & \downarrow & & \uparrow \\ H^1(K^{ur},1) & \times & H^0(K^{ur},H^d(U_{\overline{\eta}},d)) & \xrightarrow{\cup} & H^1(K^{ur},H^d(U_{\overline{\eta}},d+1), \end{array}$$

where the bottom cup product is induced by the pairing $H^0 \times H^d \to H^{d+1}$ for $U_{\overline{\eta}}$. The right vertical composition in (4-5) is the isomorphism (a) in Lemma 4.14. Putting things together, we get the following diagram
(4-6)

$$\begin{array}{ccccc} H^d(U_\eta,d) & \xrightarrow{\alpha} & H^{d+1}(U_\eta,d+1) & \xrightarrow{\delta_{tame}} & H^0(X_s,W_n\omega^d) \\ \downarrow & & \downarrow & & \downarrow \\ H^d(U_\eta^{ur},\mathbb{Z}/p^n\mathbb{Z}(d))^\Gamma & \xrightarrow{\beta} & H^{d+1}(U_\eta^{ur},d+1)_\Gamma & \xrightarrow{\delta_{tame}} & H^0(X_{\overline{s}},W_n\omega^d)_\Gamma \\ \| & & \downarrow \phi_F & & \downarrow \phi_F \\ H^d(U_\eta^{ur},\mathbb{Z}/p^n\mathbb{Z}(d))^\Gamma & \xrightarrow{\gamma} & H^1(\Gamma,H^{d+1}(U_\eta^{ur},d+1)) & \xrightarrow{\delta_{tame}} & H^1(\Gamma,H^0(X_{\overline{s}},W_n\omega^d)) \\ \downarrow res\phi_K & & \downarrow & & \downarrow \\ H^2(K,H^d(U_{\overline{\eta}},d+1)) & \xrightarrow{(a)} & H^{d+2}(U_\eta,d+1) & \xrightarrow{\delta_{tame}} & H^1(X_s,W_n\omega^d), \end{array}$$

where $W_n\omega^d$ stands for $W_n\omega^d_{X_s}(logZ)_{log}$. Here γ is the cup products with the image $\tilde{\chi}_K$ of χ_K in $H^1(\Gamma,H^1(U_\eta^{ur},1))$, and $res\phi_K$ is the composition of $res : H^d(U_\eta^{ur},d)^\Gamma \to H^d(U_{\overline{\eta}},d)^G$ and ϕ_K. Hence the bottom of the diagram is commutative by (4-5). The diagram involving β and γ is commutative if β is cup product with the element x corresponding to $\tilde{\chi}_K$ under the isomorphism $\phi_F : H^1(U_\eta^{ur},1)_\Gamma \to H^1(\Gamma,H^1(U_\eta^{ur},1))$. Consider the commutative diagram

$$\begin{array}{ccccc} H^2(K,\mathbb{Z}/p^n\mathbb{Z}(1)) & \to & H^1(\Gamma,H^1(K^{ur},\mathbb{Z}/p^n\mathbb{Z}(1))) & \to & H^1(\Gamma,\mathbb{Z}/p^n\mathbb{Z}(1)) \\ & & \uparrow \phi_F & & \uparrow \phi_F \\ & & H^1(K^{ur},\mathbb{Z}/p^n\mathbb{Z}(1))_\Gamma & \xrightarrow{v} & \mathbb{Z}/p^n\mathbb{Z}. \end{array}$$

Here the left horizonal map comes from the Hochschild-Serre spectral sequence, and the two right horizontal maps are induced by the Kummer isomorphism $H^1(K^{ur},\mathbb{Z}/p^n\mathbb{Z}(1)) \cong (K^{ur})^\times/p^n$ and the normalized valuation $v : (K^{ur})^\times \to \mathbb{Z}$. The composition in the top row is the map considered above and maps χ_K to χ_F, and the right vertical map sends 1 to χ_F. This shows that the element x is the image $\{\pi_K\}$ of a uniformizing element π_K of K under the Kummer isomorphism and the map $H^1(K^{ur},\mathbb{Z}/p^n\mathbb{Z}(1)) \to H^1(U_\eta^{ur},\mathbb{Z}/p^n\mathbb{Z}(1)_\Gamma$. Note that π_K is also a uniformizing element of K^{ur}. Now we see that the diagram involving α and β is commutative, if α is the cup product with $\{\pi_K\}$, which now denotes the image of π_K under $K^\times/p^n \to H^1(K,\mathbb{Z}/p^n\mathbb{Z}(1)) \to H^1(U_\eta,\mathbb{Z}/p^n\mathbb{Z}(1))$.
Thus, diagram (4-6) is commutative with $\alpha = \{\pi_K\}\cup$. Now, in view of the observation in the beginning of the proof, we obtain the commutativity in 4.14 by observing that the composition

$$H^d(U_\eta,d) \xrightarrow{\{\pi_K\}\cup} H^{d+1}(U_\eta,d+1) \xrightarrow{\delta_{tame}} H^0(X_s,W_n\omega^d)$$

in the top of (4-6) is just δ_d^{sym}, since on sheaf level, δ_d^{sym} is the composition

$$R^d(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(d) \overset{\{\pi_K\}\cup}{\rightarrow} R^{d+1}(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(d+1) \overset{\delta_{tame}}{\rightarrow} (i_X)_* W_n\omega^d_{X_s}(logZ)_{log},$$

where now $\{\pi_K\}$ denotes the global section of $R^1(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(1)$ corresponding to π_K.

We start the proof of Theorem 4.13. First we note that $H^1(X_s, W_\infty\omega^d_{X_s}(\log Z)_{log})$ is of cofinite type, by [GS, 4.18] and Lemma 4.16 below. We then claim that it is divisible. Indeed, considering the long exact sequence arising from the sequence (cf. Lemma 4.9)

$$0 \to W_n\omega^d_{X_s}(\log Z)_{log} \to W_\infty\omega^d_{X_s}(\log Z)_{log} \xrightarrow{p^n} W_\infty\omega^d_{X_s}(\log Z)_{log} \to 0,$$

the claim follows from the fact that $H^2(X_s, W_n\omega^d_{X_s}(\log Z)_{log}) = 0$ by Proposition 4.11 and Theorem 3.5 (a). Thus the first assertion of Theorem 4.13 follows from Lemmma 4.14 and the following result.

Proposition 4.15 *Let*

$$\overline{\delta}_d^{sym} : \ H^d(U_{\overline{\eta}}, \mathbb{Z}_p(d))_I \to \varprojlim_n H^0(X_{\overline{s}}, W_n\omega^d_{X_{\overline{s}}}(\log Z)_{log})$$

be the the map induced by the maps $\overline{\delta}_d^{sym}$ from (4-4), where $I \subset G$ is the inertia subgroup. Then $\overline{\delta}_d^{sym}$ has torsion kernel and cokernel.

We need the following two lemmas.

Lemma 4.16 *For every integers $n, t > 0$ the sequence*

$$0 \to W_n\omega^t_{X_s,log} \to W_n\omega^t_{X_s}(\log Z)_{log} \xrightarrow{Res_Z} \tau_* W_n\omega^{t-1}_{Z_s,log} \to 0.$$

is exact where Res_Z is the residue along $Z \subset X$.

Proof By Lemma 4.9 it suffices to show the exactness of the sequence

$$0 \to \omega^t_{X_s,log} \to \omega^t_{X_s}(\log Z)_{log} \to \tau_*\omega^{t-1}_{Z_s,log} \to 0. \tag{4-7}$$

It follows immediately from the definition of log-differentials that the sequences

$$0 \to \omega^t_{X_s} \to \omega^t_{X_s}(\log Z) \to \tau_*\omega^{t-1}_{Z_s} \to 0,$$

is exact. Except for the surjectivity of the last map, the exactness of (4-7) then follows from the commutative exact diagram

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \to & \omega^t_{X_s,log} & \to & \omega^t_{X_s}(\log Z)_{log} & \to & \tau_*\omega^{t-1}_{Z_s,log} \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \to & \omega^t_{X_s,d=0} & \to & \omega^t_{X_s}(\log Z)_{d=0} & \to & \tau_*\omega^{t-1}_{Z_s,d=0} \\
 & \downarrow C-1 & & \downarrow C-1 & & \downarrow C-1 \\
0 \to & \omega^t_{X_s} & \to & \omega^t_{X_s}(\log Z) & \to & \tau_*\omega^{t-1}_{Z_s}
\end{array}$$

where the vertical exact sequences follow from [Ts3], Th.A.3 and A.4. Finally the surjectivity onto $\tau_*\omega^{t-1}_{Z_s,log}$ is a consequence of the fact that $\omega^q_{Z_s,log}$ is generated by *dlog*-differentials. This completes the proof of Lemma 4.16.

Lemma 4.17 *For every integer $m > 0$ write*

$$\begin{aligned} M_n^m(X) &= (i_X)^* R^m (j_X)_* \mathbb{Z}/p^n\mathbb{Z}(m), \\ M_n^m(U) &= (i_X)^* R^m (\phi j_U)_* \mathbb{Z}/p^n\mathbb{Z}(m), \\ M_n^m(Z) &= (i_Z)^* R^m (j_Z)_* \mathbb{Z}/p^n\mathbb{Z}(m). \end{aligned}$$

Then we have a commutative diagram with exact horizontal sequences

$$\begin{array}{ccccccc} & M_n^m(X) & \to & M_n^m(U) & \xrightarrow{\beta} & M_n^{m-1}(Z) & \to 0 \\ & \downarrow & & \downarrow \delta_m^{sym} & & \downarrow & \\ 0 \to & W_n\omega^m_{X_s,log} & \to & W_n\omega^m_{X_s}(\log Z)_{log} & \xrightarrow{Res_Z} & \tau_* W_n\omega^{m-1}_{Z_s,log} & \to 0 \end{array}$$

where the vertical arrows are the morphism δ_m^{sym} defined for U in (4-1), and its variants for X and Z, respectively. The upper sequence is induced by the distinguished triangle

$$R(j_X)_*\mathbb{Z}/p^n\mathbb{Z}(m)_{X_\eta} \to R(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(m)_{U_\eta} \to R(\tau j_Z)_*\mathbb{Z}/p^n\mathbb{Z}(m-1)_{Z_\eta}[-1]$$

arising from the localization theory for the smooth pair $Z_\eta \hookrightarrow X_\eta$.

Proof We remark that the above sheaves the p-adic vanishing cycles are generated by local symbols. One can check $\beta(\{s, x_1, \ldots, x_{m-1}\}) = \{\overline{x}_1, \ldots, \overline{x}_{m-1}\}$, where s is a local equation of Z in X and x_i (resp. $\overline{x}_i$) with $1 \le i \le m-1$ are local sections of M_X^{gp} (resp. the image of x_i in M_Z^{gp}). This shows the surjectivity of β. The commutativity of the diagram is verified by using the description of the values of δ_m^{sym} on symbols. This completes the proof of Lemma 4.17.

We proceed with the proof of Proposition 4.15. Writing

$$\begin{aligned} \overline{M}_n^m(X) &= (\bar{i}_X)^* R^m (\bar{j}_X)_* \mathbb{Z}/p^n\mathbb{Z}(m), \\ \overline{M}_n^m(U) &= (\bar{i}_X)^* R^m (\phi \bar{j}_U)_* \mathbb{Z}/p^n\mathbb{Z}(m), \\ \overline{M}_n^m(Z) &= (\bar{i}_Z)^* R^m (\bar{j}_Z)_* \mathbb{Z}/p^n\mathbb{Z}(m), \end{aligned}$$

we have the exact sequence

$$\cdots \to \overline{M}_n^{m-1}(U)(1) \to \overline{M}_n^{m-2}(Z)(1) \to \overline{M}_n^m(X) \to \overline{M}_n^m(U) \to \overline{M}_n^{m-1}(Z) \to \cdots .$$

Hence the surjectivity of β in Lemma 4.17 implies the exactness of

$$0 \to \overline{M}_n^m(X) \to \overline{M}_n^m(U) \to \overline{M}_n^{m-1}(Z) \to 0.$$

Thus, taking the cohomology and passing to the limit over finite extensions of K, the diagram in Lemma 4.17, with $m = d$, gives rise to the commutative diagram

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ H^d(X_{\overline{\eta}}, \mathbb{Z}_p(d))_I & \rightarrow & \varprojlim_n H^0(X_{\overline{s}}, W_n\omega^d_{X_{\overline{s}},log}) \\ \downarrow & & \downarrow \\ H^d(U_{\overline{\eta}}, \mathbb{Z}_p(d))_I & \xrightarrow{\overline{\delta}^{sym}_d} & \varprojlim_n H^0(X_{\overline{s}}, W_n\omega^d_{X_{\overline{s}}}(\log Z)_{log}) \\ \downarrow & & \downarrow \\ H^d(Z_{\overline{\eta}}, \mathbb{Z}_p(d))_I & \rightarrow & \varprojlim_n H^0(Z_{\overline{s}}, W_n\omega^d_{Z_{\overline{s}},log}) \\ \downarrow & & \downarrow \\ H^{d+1}(X_{\overline{\eta}}, \mathbb{Z}_p(d))_I & \rightarrow & \varprojlim_n H^1(X_{\overline{s}}, W_n\omega^d_{X_{\overline{s}},log}) \end{array}$$

where the second horizontal arrow is the map $\overline{\delta}^{sym}_d$ for U from Proposition 4.14, and the first one is its analogue for X, induced by the analogue of (4-3)

$$\overline{\delta}^{sym}_{X,d} :\ \overline{i}^*_X R^d\overline{j}_{X*}\mathbb{Z}/p^n\mathbb{Z}(d) \rightarrow W_n\omega^d_{X_{\overline{s}},log}.$$

The fourth horizontal arrow is induced by $\overline{\delta}^{sym}_{X,d}$ as well, by noting the fact that $R^q\overline{j}_{X*}\mathbb{Z}/p^n\mathbb{Z}(d) = 0$ for $q > d$, which can be shown by the same argument as in Lemma 4.10. By similar reasoning, the third horizontal map is induced by the corresponding morphism $\overline{\delta}^{sym}_{Z,d-1}$ for Z. The left vertical sequence comes from the localization exact sequence

$$H^d(X_{\overline{\eta}}, \mathbb{Z}_p(d)) \rightarrow H^d(U_{\overline{\eta}}, \mathbb{Z}_p(d)) \rightarrow H^{d-1}(Z_{\overline{\eta}}, \mathbb{Z}_p(d-1)) \xrightarrow{\tau_*} H^{d+1}(X_{\overline{\eta}}, \mathbb{Z}_p(d))$$

via taking coinvariants under I, and it remains exact modulo torsion, since τ_* is split surjective modulo torsion by the hard Lefschetz theorem. Using the semistable comparison theorem on the comparison of p-adic étale cohomology and log-crystalline cohomology for proper semistable families, one can show (cf. [Sat1], Lemma 3.3 and [Ts3], (3.1.12) and (3.2.7)) that the first horizontal arrow and the last two ones are isomorphisms modulo torsion. Hence the second arrow is an isomorphism modulo torsion, too. This completes the proof of Proposition 4.15.

Next we show the second claim in Theorem 4.13, i.e., that δ^{∞}_{tame} is an isomorphism provided $p \geq d$. Denote by $\mathbb{Z}/p^n\mathbb{Z}(d+1)_{(X,Z)}$ the mapping fiber of

$$\delta_{tame} :\ R(\phi j_U)_*\mathbb{Z}/p^n\mathbb{Z}(d+1) \rightarrow (i_X)_*W_n\omega^d_{X_s}(\log Z)_{log}[-d-1]$$

and let $\mathbb{Q}_p/\mathbb{Z}_p(d+1)_{(X,Z)} = \varinjlim_n \mathbb{Z}/p^n\mathbb{Z}(d+1)_{(X,Z)}$. By definition we have the exact sequence

$$H^{d+2}(X, \mathbb{Q}_p/\mathbb{Z}_p(d+1)_{(X,Z)}) \rightarrow H^{d+2}(U_\eta, \mathbb{Q}_p/\mathbb{Z}_p(d+1)) \xrightarrow{\delta^{\infty}_{tame}} H^1(X_s, W_\infty\omega^d_{X_s}(\log Z)_{log}).$$

We know already that $\mathrm{Ker}(\delta^{\infty}_{tame})$ is finite, so its vanishing follows once we show that $H^{d+2}(X, \mathbb{Q}_p/\mathbb{Z}_p(d+1)_{(X,Z)})$ is divisible. In view of the distinguished triangle

$$\mathbb{Z}/p\mathbb{Z}(d+1)_{(X,Z)} \to \mathbb{Q}_p/\mathbb{Z}_p(d+1)_{(X,Z)} \xrightarrow{p} \mathbb{Q}_p/\mathbb{Z}_p(d+1)_{(X,Z)} \to,$$

which follows from Lemma 4.9, it suffices to show $H^{d+3}(X, \mathbb{Z}/p\mathbb{Z}(d+1)_{(X,Z)}) = 0$. Thus the claim follows from the following.

Theorem 4.18 *Assuming $p \geq d$, $H^q(X, \mathbb{Z}/p\mathbb{Z}(d+1)_{(X,Z)}) = 0$ for $q \geq d+3$.*

We remark that F is not assumed to be finite in Theorem 4.18. We need the following three lemmas.

Lemma 4.19 *Let the assumption be as above. Then there exists a trace map*

$$H^d(X_s, \omega^d_{X_s}) \xrightarrow{\cong} F.$$

For a locally free $\mathcal{O}_{X_s}$-module $\mathcal{M}$ the natural pairing

$$\begin{aligned} H^j(X_s, \omega^i_{X_s}(\log Z) \otimes \mathcal{M}) \times H^{d-j}(X_s, \mathcal{H}om_{\mathcal{O}_{X_s}}(\mathcal{M} \otimes \mathcal{O}_X(Z), \omega^{d-i}_{X_s}(\log Z))) \\ \to H^d(X_s, \omega^d_{X_s}) \xrightarrow{\cong} F \end{aligned}$$

is a perfect pairing of finite-dimensional F-vector spaces.

Proof By the isomorphism just after [Ts1], Th. 2.21, $\omega^d_{X_s}$ placed at degree d is the dualizing complex for X_s. The assertion follows from the isomorphisms

$$(\omega^i_{X_s}(\log Z))^{\vee} \otimes \omega^d_{X_s}(\log Z) \cong \omega^{d-i}_{X_s}(\log Z), \quad \omega^d_{X_s}(\log Z) = \omega^d_{X_s} \otimes \mathcal{O}(Z).$$

Lemma 4.20 *For any ample line bundle $\mathcal{L}$ on X_s, we have*

$$H^j(X_s, \omega^i_{X_s}(\log Z) \otimes \mathcal{L}^{-1}) = 0 \quad \text{for } i+j < \min\{d, p\}.$$

Proof The assertion follows from [K3], Th. 4.12 by the same argument as the proof of [DI], Cor. 2.8.

Lemma 4.21 $H^j(X_s, \omega^i_{X_s}(\log Z)) = 0$ *for* $i + j > \max\{d, 2d-p\}$.

Proof By Lemma 4.19 it suffices to show $H^j(X_s, \omega^i_{X_s}(\log Z) \otimes \mathcal{O}_X(Z)^{-1}) = 0$ for $i+j < d$, which follows from Lemma 4.20 since $\mathcal{O}_X(Z)$ is ample.

We start the proof of Theorem 4.18. Write ω^t for $\omega^t_{X_s}(\log Z)_{log}$, as well as

$$B^t = \mathrm{Im}(d : \omega^{t-1} \to \omega^t), \quad \text{and} \quad Z^t = \mathrm{Ker}(d : \omega^t \to \omega^{t+1}).$$

We may assume that K contains the p-th roots of unity, and can omit all Tate twists. Recall that $\mathbb{Z}/p\mathbb{Z}(d+1)_{(X,Z)}$ is concentrated in degree $[0, d+1]$.

Lemma 4.22 *Assume $p \geq d$. For every $t \leq d+1$, the homology sheaf $\mathcal{H}^t(i_X^*\mathbb{Z}/p\mathbb{Z}(d+1)_{(X,Z)})$ has a finite filtration whose subquotients are*

$$\omega_{log}^t, \quad \omega_{log}^{t-1}, \quad B^t, \quad B^{t-1} \quad \text{and} \quad \omega^{t-1}/B^{t-1}.$$

Proof By the Bloch-Kato-Hyodo theorems for the sheaf of the p-adic vanishing cycles $(i_X)^*R^t(\phi j_U)_*\mathbb{Z}/p\mathbb{Z}(t)$, as proved in [BK], [H] and, in particular, [Ts2 Proposition A.15], this holds under the condition $p \geq d+3$, which comes from the use of the syntomic complexes. For the result under the weaker assumption $p \geq d$, which uses the special structure of the good open U, we refer the reader to [JS].

By this result, it suffices to show the following.

Lemma 4.23 *Assume $p \geq d$. We have $H^s(X_s, Q) = 0$ for $s+t \geq d+3$ and for each of the above subquotients Q.*

Proof Write $Y = X_s$. By Lemma 4.21 and the assumption $p \geq d$ we have

(1) $H^s(Y, \omega^t) = 0$ for $s+t > d$ so that $H^s(Y, \omega^{t-1}) = 0$ for $s+t > d+1$.

Via the Cartier isomorphism (cf. [K3], Th. 4.12 and [Ts2], Th. A.3) we get

(2) $H^s(Y, Z^t/B^t) = 0$ for $s+t > d$.

Now we use descending induction on t to show that

(a_t) $H^s(Y, \omega^t/B^t) = 0$ for $s+t > d$,

(b_t) $H^s(Y, B^{t+1}) = 0$ for $s+t > d$,

(c_t) $H^s(Y, Z^t) = 0$ for $s+t > d+1$.

We start with the case $t = d$ where (a_d) (resp. (c_d)) follows from (2) (resp. (1)) by noting $Z^d = \omega^d$, and (b_d) follows from $B^{d+1} = 0$. Now assume that we have shown (a_t), (b_t), (c_t) for some $t \leq d$. For the induction step we use the exact sequence

$$H^{s-1}(Y, Z^t/B^t) \to H^s(Y, B^t) \to H^s(Y, Z^t).$$

We have $H^{s-1}(Y, Z^t/B^t) = 0$ if $s-1+t > d$ by (2) and $H^s(Y, Z^t) = 0$ if $s+t > d+1$ by (c_t) so that $H^s(Y, B^t) = 0$ if $s+t > d+1$, which implies (b_{t-1}). Next we look at the exact sequence

$$H^{s-1}(Y, B^t) \to H^s(Y, Z^{t-1}) \to H^s(Y, \omega^{t-1})$$

associated to the exact sequence $0 \to Z^{t-1} \to \omega^{t-1} \xrightarrow{d} B^t \to 0$. We have $H^{s-1}(Y, B^t) = 0$ if $s-1+t-1 > d$ by (b_{t-1}) and $H^s(Y, \omega^{t-1}) = 0$ if $s+t-1 > d$ by (1) so that $H^s(Y, Z^{t-1}) = 0$ if $s+t > d+2$, which implies (c_{t-1}). Now consider the exact sequence

$$H^s(Y, Z^{t-1}/B^{t-1}) \to H^s(Y, \omega^{t-1}/B^{t-1}) \to H^s(Y, \omega^{t-1}/Z^{t-1}).$$

We have $H^s(Y, Z^{t-1}/B^{t-1}) = 0$ if $s+t-1 > d$ by (2) and $H^s(Y, \omega^{t-1}/Z^{t-1}) \cong H^s(Y, B^t) = 0$ if $s+t-1 > d$ by (b_{t-1}) so that $H^s(Y, \omega^{t-1}/B^{t-1}) = 0$ if $s+t > d+1$, which implies (a_{t-1}). This completes the proof of (a_t), (b_t), (c_t) for $\forall t \leq d$. Note that (b_t) implies $H^s(Y, B^{t-1}) = 0$ for $s+t > d+2$ and (a_t) implies $H^s(Y, \omega^{t-1}/B^{t-1}) = 0$ for $s+t > d+1$.

Finally we look at the exact sequence

$$H^{s-1}(Y, \omega^t/B^t) \to H^s(Y, \omega^t_{log}) \to H^s(Y, \omega^t)$$

associated to the exact sequence

$$0 \to \omega^t_{log} \to \omega^t \xrightarrow{1-C^{-1}} \omega^t/B^t \to 0.$$

We have $H^{s-1}(Y, \omega^t/B^t) = 0$ if $s-1+t > d$ by (a_t) and $H^s(Y, \omega^t) = 0$ if $s+t > d$ by (1) so that $H^s(Y, \omega^t_{log}) = 0$ for $s+t > d+1$ and hence $H^s(Y, \omega^{t-1}_{log}) = 0$ for $s+t > d+2$ by the exact sequence. This completes the proof of Lemma 4.23 and hence that of Theorem 4.18.

It remains to show the last claim of Theorem 4.13, i.e., that δ^∞_{tame} is an isomorphism if X is smooth over S. It suffices to show $H^{d+2}(U_\eta, \mathbb{Q}_p/\mathbb{Z}_p(d+1)) = 0$ assuming $d \geq 2$. With $G = \mathrm{Gal}(\overline{K}/K)$, we have isomorphisms

$$H^{d+2}(U_\eta, \mathbb{Q}_p/\mathbb{Z}_p(d+1)) \simeq H^2(K, H^d(U_{\overline{\eta}}, \mathbb{Q}_p/\mathbb{Z}_p(d+1))) \simeq \mathrm{Hom}(H^d_c(U_{\overline{\eta}}, \mathbb{Z}_p)^G, \mathbb{Q}_p/\mathbb{Z}_p)$$

by local duality (cf. Lemma 4.14) and Poincaré duality. By the weak Lefschetz theorem $H^d_c(U_{\overline{\eta}}, \mathbb{Z}_p)$ is torsion free. Therefore it suffices to show $H^d_c(U_{\overline{\eta}}, \mathbb{Q}_p)^G = 0$. Noting the exact sequence

$$H^{d-1}(Z_{\overline{\eta}}, \mathbb{Q}_p) \to H^d_c(U_{\overline{\eta}}, \mathbb{Q}_p) \to H^d(X_{\overline{\eta}}, \mathbb{Q}_p)$$

and that Z is smooth over S by the assumption, this vanishing follows from:

Lemma 4.24 *If X is proper and smooth over S, then for any $i > 0$ and any G-subquotient V of $H^i(X_{\overline{\eta}}, \mathbb{Q}_p)$ one has $V^G = 0$.*

Proof By the B_{cris}-comparison isomorphism ([FM], [Fa])

$$H^i(X_{\overline{\eta}}, \mathbb{Q}_p) \otimes B_{cris} \simeq H^i_{cris}(X_s/W(F)) \otimes B_{cris},$$

the claim follows from the Weil conjecture for the crystalline cohomology ([KM]). In fact, $V^G \cong D_{cris}(V)^{F=1}$, where $D_{cris}(V)$ is a subquotient of $H^i_{cris}(X_s/W(F)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and F is induced by the crystalline Frobenius.

5. Δ^0_X , Δ^1_X, and class field theory

Let A be a henselian discrete valuation ring with finite residue field F of characteristic p. Let K be the quotient field of A. Let $S = \mathrm{Spec}(A)$ and consider a diagram like (1-3) in the introduction

$$\begin{array}{ccccc} X_\eta & \xrightarrow{j_X} & X & \xleftarrow{i_X} & X_s \\ \downarrow f_\eta & & \downarrow f & & \downarrow f_s \\ \eta & \xrightarrow{j} & S & \xleftarrow{i} & s \end{array}$$

In this section we prove Theorem 1.5, by using the relation between the Kato complexes of X_η and X_s and the class field theory developed in [Bl], [Sa1] and [KS1].

Definition 5.1 *For a scheme V of finite type over a field we put*

$$SK_1(V) = \mathrm{Coker}(\bigoplus_{x\in V_1} K_2(x) \xrightarrow{\partial} \bigoplus_{x\in V_0} K_1(x)),$$

$$CH_0(V) = \mathrm{Coker}(\bigoplus_{x\in V_1} K_1(x) \xrightarrow{\partial} \bigoplus_{x\in V_0} K_0(x)),$$

where K_* denotes algebraic K-groups and boundary maps are induced by localization theory for algebraic K-theory.

The components of the differential ∂ for $SK_1(V)$ (resp. $CH_0(V)$) are given by tame symbols (resp. valuations). By definition $CH_0(V)$ is the Chow group of zero-cycles on V. In case V is smooth of pure dimension d, $SK_1(V)$ coincides with Bloch's higher Chow group $CH^{d+1}(V,1)$ by [La], Lem. 2.8.

Note that $\mathrm{cd}(K) = 2$ and $\mathrm{cd}(F) = 1$ (cf. [Se], Ch.II, §2 and §4). By Proposition 2.12 (a) and Lemma 2.15 we have the commutative diagram

$$\text{(5-1)}\qquad \begin{array}{ccc} H^{et}_{i-2}(X_\eta,\mathbb{Z}/n\mathbb{Z}(-1)) & \xrightarrow{\epsilon^i_{X_\eta}} & H^K_i(X_\eta,\mathbb{Z}/n\mathbb{Z}) \\ \downarrow \Delta_X & & \downarrow \Delta_X \\ H^{et}_{i-1}(X_s,\mathbb{Z}/n\mathbb{Z}(0)) & \xrightarrow{\epsilon^i_{X_s}} & H^K_i(X_s,\mathbb{Z}/n\mathbb{Z}), \end{array}$$

where the vertical maps are the residue maps (2.12 (a), 2.18) and the horizontal maps are edge homomorphisms (2.14 (b)) of the following spectral sequences (note 2.14 (a))

$$\text{(5-2)}\qquad E^1_{p,q}(X_\eta) = \bigoplus_{x\in (X_\eta)_p} H^{p-q}(x,\mathbb{Z}/n\mathbb{Z}(p+1)) \Rightarrow H^{et}_{p+q}(X_\eta,\mathbb{Z}/n\mathbb{Z}(-1)),$$

$$\text{(5-3)}\qquad E^1_{p,q}(X_s) = \bigoplus_{x\in (X_s)_p} H^{p-q}(x,\mathbb{Z}/n\mathbb{Z}(p)) \Rightarrow H^{et}_{p+q}(X_s,\mathbb{Z}/n\mathbb{Z}(0)).$$

Lemma 5.2 *The maps $\epsilon^0_{X_\eta}$ and $\epsilon^0_{X_s}$ are isomorphims and we have the commutative diagram with exact horizontal sequences*

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Cok}(\epsilon^2_{X_\eta}) & \to & SK_1(X_\eta)/n & \xrightarrow{\alpha_{X_\eta}} & H^{et}_{-1}(X_\eta,\mathbb{Z}/n(-1)) & \xrightarrow{\epsilon^1_{X_\eta}} & H^K_1(X_\eta,\mathbb{Z}/n) & \to 0 \\ & \downarrow \Delta_X & & \downarrow \partial_X & & \downarrow \Delta_X & & \downarrow \Delta_X & \\ 0 \to & \mathrm{Cok}(\epsilon^2_{X_s}) & \to & CH_0(X_s)/n & \xrightarrow{\alpha_{X_s}} & H^{et}_0(X_s,\mathbb{Z}/n(0)) & \xrightarrow{\epsilon^1_{X_s}} & H^K_1(X_s,\mathbb{Z}/n) & \to 0 \end{array}$$

Here ∂_X comes from the localization theory for algebraic K-theory on X.

Proof Lemma 2.14 implies that $E^1_{p,q}(X_\eta) = 0$ unless $p \geq q$ and $q \geq -2$ and $E^1_{p,q}(X_s) = 0$ unless $p \geq q$ and $q \geq -1$ and that

$$E^2_{p,-2}(X_\eta) = H^K_p(X_\eta, \mathbb{Z}/n\mathbb{Z}) \quad \text{and} \quad E^1_{p,-1}(X_s) = H^K_p(X_s, \mathbb{Z}/n\mathbb{Z}),$$
$$E^2_{0,-1}(X_\eta) \cong SK_1(X_\eta)/n \quad \text{and} \quad E^2_{0,0}(X_s) \cong CH_0(X_s)/n.$$

Here the isomorphisms in the second row follows from the following commutative diagrams established in [K1], Lem. 1.4

$$\begin{array}{ccc} \bigoplus_{x\in(X_\eta)_1} K_2(x)/n & \to & \bigoplus_{x\in(X_\eta)_0} K_1(x)/n \\ \downarrow \wr & & \downarrow \wr \\ \bigoplus_{x\in(X_\eta)_1} H^2(x, \mathbb{Z}/n\mathbb{Z}(2)) & \to & \bigoplus_{x\in(X_\eta)_0} H^1(x, \mathbb{Z}/n\mathbb{Z}(1)) \end{array}$$

$$\begin{array}{ccc} \bigoplus_{x\in(X_s)_1} K_1(x)/n & \to & \bigoplus_{x\in(X_s)_0} K_0(x)/n \\ \downarrow \wr & & \downarrow \wr \\ \bigoplus_{x\in(X_s)_1} H^1(x, \mathbb{Z}/n\mathbb{Z}(1)) & \to & \bigoplus_{x\in(X_s)_0} H^1(x, \mathbb{Z}/n\mathbb{Z}) \end{array}$$

where the top side maps are the boundary maps of the Gersten complex for algebraic K-theory and the bottom side maps are the d_1-differentials of the spectral sequences (5-2) and (5-3). The vertical maps are the Galois symbol maps and they are isomorphisms by Kummer theory and [MS]. The proposition follows easily from these facts.

Let

$$tr_s : H^1(s, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}/n\mathbb{Z} \quad \text{and} \quad tr_\eta : H^2(\eta, \mathbb{Z}/n\mathbb{Z}(1)) \xrightarrow{\cong} \mathbb{Z}/n\mathbb{Z}$$

be the evaluation at the Frobenius substitution of the finite field F, and the composite of tr_s and the residue map $H^2(\eta, \mathbb{Z}/n\mathbb{Z}(1)) \to H^1(s, \mathbb{Z}/n\mathbb{Z})$, respectively. For a scheme Z denote by $D^b_c(Z, \mathbb{Z}/n\mathbb{Z})$ the derived category of complexes of étale sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules on Z with bounded constructible cohomology sheaves.

Lemma 5.3 *Assume that f is proper.*
(1) For any $K \in D^b_c(X_\eta, \mathbb{Z}/n\mathbb{Z})$ the pairing

$$H^i(X_\eta, D_{X_\eta}(K)) \times H^{2-i}(X_\eta, K) \to H^2(X_\eta, Rf^!_\eta \mathbb{Z}/n(1)) \xrightarrow{tr_{X_\eta}} H^2(\eta, \mathbb{Z}/n(1)) \xrightarrow{tr_\eta} \mathbb{Z}/n\mathbb{Z}$$

is a perfect pairing of finite groups. Here $D_{X_\eta}(K) = R\mathcal{H}om(K, Rf^!_\eta \mathbb{Z}/n\mathbb{Z}(1))$ and tr_{X_η} is induced by the trace morphisms $Rf_{\eta *} Rf^!_\eta \mathbb{Z}/n\mathbb{Z}(1) \to \mathbb{Z}/n\mathbb{Z}(1)$.
(2) For any $K \in D^b_c(X_s, \mathbb{Z}/n\mathbb{Z})$ the pairing

$$H^i(X_s, D_{X_s}(K)) \times H^{1-i}(X_s, K) \to H^1(X_s, Rf^!_s \mathbb{Z}/n\mathbb{Z}) \xrightarrow{tr_{X_s}} H^1(s, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{tr_s} \mathbb{Z}/n\mathbb{Z}$$

is a perfect pairing of finite groups. Here $D_{X_s}(K) = R\mathcal{H}om(K, Rf^!_s \mathbb{Z}/n\mathbb{Z})$ and tr_{X_s} is induced by the trace morphisms $Rf_{s *} Rf^!_s \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$.

Proof This follows immediately from the Artin-Verdier duality for f_η and f_s together with the duality theorems for Galois cohomology of K and F (cf. [Sa3] and [CTSS]).

Proof of Theorem 1.5 (1) Assume that f is proper and X_η is connected. Then the bijectivity of $\Delta_X^0 = \Delta_X^{K,0}$ immediately follows from the following commutative diagram deduced from Lemmas 5.2 and 5.3

$$\begin{array}{ccccc} H_0^K(X_\eta, \mathbb{Z}/n\mathbb{Z}) & \overset{\epsilon^0_{X_\eta}}{\underset{\sim}{\longleftarrow}} & H^2(X_\eta, Rf_\eta^!\mathbb{Z}/n\mathbb{Z}(1)) & \overset{tr_{X_\eta}}{\underset{\sim}{\longrightarrow}} & H^2(\eta, \mathbb{Z}/n\mathbb{Z}(1)) \\ \downarrow \Delta_X^{K,0} & & \downarrow \Delta_X^{et,-2} & & \downarrow\wr\ \Delta_S^{et,-2} \\ H_0^K(X_s, \mathbb{Z}/n\mathbb{Z}) & \overset{\epsilon^0_{X_s}}{\underset{\sim}{\longleftarrow}} & H^1(X_s, Rf_s^!\mathbb{Z}/n\mathbb{Z}) & \overset{tr_{X_s}}{\underset{\sim}{\longrightarrow}} & H^1(s, \mathbb{Z}/n\mathbb{Z}) \end{array}$$

Proof of Theorem 1.5 (2) We need to recall the class field theory of X_η and X_s developed in [Bl], [KS1] and [Sa1]. For a scheme Z we let

$$\pi_1^{ab}(Z) = \mathrm{Hom}(H^1(Z_{et}, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$$

be the abelian algebraic fundamental group of Z. Let V be a proper scheme over K. For each $x \in V_0$ we have the map

$$K(x)^* \to \mathrm{Gal}(K(x)^{ab}/K(x)) = \pi_1^{ab}(x) \to \pi_1^{ab}(V),$$

where the first map is the reciprocity map for $K(x)$ which is a henselian discrete valuation field with finite residue field. The second map comes from the covariant functoriality of π_1^{ab}. Taking the sum of β_x over $x \in V_0$, we get the map

$$\tilde{\rho}_V \ : \ \bigoplus_{x \in V_0} K(x)^* \to \pi_1^{ab}(V).$$

Now the reciprocity law proved in [Sa1] and [Sa2] implies that $\tilde{\rho}_V$ factors through

$$\rho_V \ : \ SK_1(V) \to \pi_1^{ab}(V).$$

If Y is a proper scheme over a finite field F, we have the reciprocity map

$$\rho_Y \ : \ CH_0(Y) \to \pi_1^{ab}(Y)$$

defined in a similar way by using the following map for each $x \in Y_0$

$$\mathbb{Z} \to \mathrm{Gal}(F(x)^{ab}/F(x)) = \pi_1^{ab}(x) \to \pi_1^{ab}(Y),$$

where the first map sends $1 \in \mathbb{Z}$ to the Frobenius substitution over $F(x)$.

Now we return to the situation in Lemma 5.2. By 5.3 we have the isomorphism

$$H_{-1}^{et}(X_\eta, \mathbb{Z}/n\mathbb{Z}(-1)) \xrightarrow{\cong} \pi_1^{ab}(X_\eta)/n \quad (\text{resp. } H_0^{et}(X_s, \mathbb{Z}/n\mathbb{Z}(0)) \xrightarrow{\cong} \pi_1^{ab}(X_s)/n),$$

and we claim that the composite map of the isomorphism and the map α_{X_η} (resp. α_{X_s}) coincides with ρ_{X_η} (resp. ρ_{X_s}). Indeed, for $x \in (X_\eta)_0$ let $i_x : x \to X_\eta$ be the inclusion and put $\pi_x = f_\eta i_x : x \to \eta$. Consider the following composite map β_x

$$H^1(x, \mathbb{Z}/n(1)) \cong H^1(x, Ri_x^! Rf_\eta^! \mathbb{Z}/n(1)) \to H^1(X_\eta, Rf_\eta^! \mathbb{Z}/n(1)) = H_{-1}^{et}(X_\eta, \mathbb{Z}/n{-1}),$$

where the isomorphism comes from $Ri_x^! Rf_\eta^! \mathbb{Z}/n\mathbb{Z}(1) = R\pi_x^! \mathbb{Z}/n\mathbb{Z}(1) = \pi_x^* \mathbb{Z}/n\mathbb{Z}(1)$. By definition α_{X_η} is induced by the sum over $x \in (X_\eta)_0$ of the composite of β_x and the map $K(x)^*/n \xrightarrow{\cong} H^1(x, \mathbb{Z}/n\mathbb{Z}(1))$. Via the duality in Lemma 5.3 (1) and the local duality for Galois cohomology of $K(x)$, β_x is identified with the dual of $H^1(X_\eta, \mathbb{Z}/n\mathbb{Z}) \to H^1(x, \mathbb{Z}/n\mathbb{Z})$, the restriction map via i_x. This proves the desired assertion for ρ_{X_η}. The assertion for ρ_{X_s} is shown by the same argument. Thus we get the commutative diagram with exact rows

$$\begin{array}{ccccc}
SK_1(X_\eta)/n & \xrightarrow{\rho_{X_\eta}/n} & \pi_1^{ab}(X_\eta)/n & \to & H_1^K(X_\eta, \mathbb{Z}/n\mathbb{Z}) \to 0 \\
\downarrow \partial_X & & \downarrow \delta_X & & \downarrow \Delta_X \\
CH_0(X_s)/n & \xrightarrow{\rho_{X_s}/n} & \pi_1^{ab}(X_s)/n & \to & H_1^K(X_s, \mathbb{Z}/n\mathbb{Z}) \to 0
\end{array}$$

where δ_X is the dual of $H^1(X_s, \mathbb{Z}/n\mathbb{Z}) \cong H^1(X, \mathbb{Z}/n\mathbb{Z}) \to H^1(X_\eta, \mathbb{Z}/n\mathbb{Z})$, and hence is the specialization map on fundamental groups. By definition of the reciprocity map, the cokernel of ρ_{X_η} is the quotient $\pi_1^{ab}(X_\eta)^{c.d.}$ classifying the abelian coverings in which every closed point of X_η is completely decomposed. Similarly $\mathrm{Coker}(\rho_{X_s}) = \pi_1^{ab}(X_s)^{c.d.}$, where the latter classifies the completely decomposed abelian coverings of X_s. Therefore Theorem 1.5 (2) follows from the next lemma.

Lemma 5.4 *If $f : X \to S$ is proper, with X regular, then the specialization map $\delta_X : \pi_1(X_\eta) \to \pi_1(X_s)$ (where we omitted suitable base points) is surjective and induces an isomorphism*

$$\pi_1^{ab}(X_\eta)^{c.d.} \xrightarrow{\sim} \pi_1^{ab}(X_s)^{c.d.}.$$

Proof The map δ_X factorizes as $\pi_1(X_\eta) \twoheadrightarrow \pi_1(X) \xleftarrow{\sim} \pi_1(X_s)$, in which the first map is surjective because X is normal [SGA 1] V 8.2, and the second map is an isomorphism because X is proper [Ar] (3.1), (3.4). The second claim of the lemma then follows from [Sa2], Proposition 3.12, which characterizes the image of $H^1(X_s, \mathbb{Q}/\mathbb{Z}) \to H^1(X_\eta, \mathbb{Q}/\mathbb{Z})$ as consisting of those characters on $\pi_1^{ab}(X_\eta)$ whose associated character on $SK_1(X_\eta)$ factors through $\partial_X : SK_1(X_\eta) \to CH_0(X_s)$.

6. Δ_X^2, Δ_X^3, and finiteness results for Kato homology

Let the notations be as in the beginning of the previous section. Recall that K is the quotient field of A, a henselian discrete valuation ring with finite residue field F of characteristic p. In this section we prove a crucial result that allows to control the second Kato homology of varieties over K by étale homology (Theorem 6.1. It enables us complete the proof of Theorem 1.6 and to deduce finiteness results for Kato homology over local and global fields. We also present a strategy to show Conjecture B in general (cf. Proposition 6.4). In the whole section, ℓ denotes a prime different from $\mathrm{ch}(K)$. Let V be a separated scheme of finite type over K, and let

$$\epsilon_V^i \ : \ H_{i-2}^{et}(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)) \to H_i^K(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

be the map considered in (5-1) (for $V = X_\eta$). Recall that for a proper scheme V over K, we have the norm map

$$N_{V/K} \ : \ SK_1(V) \to K^*$$

induced by the sum of the norm maps $K(x)^* \to K^*$ for $x \in V_0$ (cf. [Sa1]).

Theorem 6.1 *(1) If V is affine, then ϵ_V^2 is surjective.*
(2) If V is proper and geometrically irreducible over K, then $\mathrm{Coker}(\epsilon_V^2)$ *is finite, and vanishes if $\ell \not| \, |\mathrm{Coker}(N_{V/K})|$.*

For the proof of Theorem 6.1, we need some preliminaries, in particular the following generalization of $N_{V/K}$. If $f : V \to M$ is a proper morphism with V and M of finite type over L, there is a norm map

$$N_{V/M} \ : \ SK_1(V) \to SK_1(M)$$

induced by the norm maps $K(x)^* \to K(f(x))^*$ for $x \in V_0$. For $M = \mathrm{Spec}(K)$ we have $N_{V/M} = N_{V/K}$.

Lemma 6.2 *(1) For a proper non-empty scheme V over K,* $\mathrm{Coker}(N_{V/K})$ *is finite.*
(2) For a proper surjective morphism $f : V \to M$ with V and M of finite type over K, $\mathrm{Coker}(N_{V/M})$ *is of finite exponent.*

Proof The first assertion is clear, since $N_{L/K}(L^*) \subset K^*$ is of finite index for any finite extension L/K. As for the second we may clearly assume that V and M are irreducible. If f is finite and flat, there is a map $f^* : SK_1(M) \to SK_1(V)$ induced by the natural inclusions $K(y)^* \to \oplus K(x)^*$ for $y \in M_0$ where the sum ranges over all $x \in V_0$ such that $f(x) = y$. We have $N_{V/M} f^* = [K(V) : K(M)]$, from which the assertion follows. In general, we proceed by induction on $\dim(M)$. If $\dim(M) = 0$, the claim follows from the first assertion of the lemma. By induction on $\dim(M)$ we may then replace V/M by $f^{-1}(U)/U$ for any non-empty open subset $U \subset M$ in view of the commutative diagram

$$\begin{array}{ccccccc}
SK_1(f^{-1}(Z)) & \to & SK_1(V) & \to & SK_1(f^{-1}(U)) & \to 0 \\
\downarrow & & \downarrow & & \downarrow & \\
SK_1(Z) & \to & SK_1(M) & \to & SK_1(U) & \to 0
\end{array}$$

where $Z = M \setminus U$ and the vertical maps are the norm maps. However, replacing M with some non-empty open subset, we may assume that there exists a finite flat morphism $f' : N \to M$ and a proper morphism $g : N \to V$ such that $f \circ g = f'$. We have $N_{V/M} \cdot N_{N/V} = N_{N/M}$. Thus the desired assertion follows from the finite flat case. This completes the proof of Lemma 6.2.

Proposition 6.3 *(1) If V is affine, then $SK_1(V) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0$.*
(2) If V is irreducible and not proper over K, then $SK_1(V) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0$.
(3) If V is geometrically irreducible and proper over K, $\mathrm{Ker}(N_{V/K}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0$.

Proof If V is a proper smooth curve, the claim follows from the class field theory for curves over local fields [Sa1], Th. 4.1 and Th. 5.1. In what follows we reduce to this crucial case. First we note that claim (2) follows from claim (3). Indeed, for irreducible non-proper U there is an open immersion $U \subset V$ such that V is irreducible and proper over K with $Z := V \setminus U$ non-empty [N]. By possibly enlarging K, we may assume that V is geometrically irreducible over K. In view of the exact sequence

$$SK_1(Z) \to SK_1(V) \to SK_1(U) \to 0, \tag{6-1}$$

it suffices to show the surjectivity of $SK_1(Z) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to SK_1(V) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$. We have the commutative diagram

(6-2)

$$\begin{array}{ccccccc} 0 \to & \mathrm{Ker}(N_{Z/K}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell & \to & SK_1(Z) \otimes \mathbb{Q}/\mathbb{Z} & \stackrel{N_{Z/K}}{\to} & K^* \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell & \to 0 \\ & \downarrow \alpha & & \downarrow \beta & & \| & \\ 0 \to & \mathrm{Ker}(N_{V/K}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell & \to & SK_1(V) \otimes \mathbb{Q}/\mathbb{Z} & \stackrel{N_{V/K}}{\to} & K^* \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell & \to 0\ . \end{array}$$

Since $\mathrm{Coker}(N_{Z/K})$ and the torsion part of K^* are finite, the horizontal sequences are exact up to finite groups. By the assumption we have $\mathrm{Ker}(N_{V/K}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0$ so that β is surjective up to finite groups, hence surjective.

In particular, we see that claim (1) holds for smooth affine curves over K. By 6.2 (2) it also holds for an arbitrary irreducible affine curve V over K, since there is a finite surjective morphism $C \to V$ with C affine and smooth. Now let V be arbitrary affine. Since every closed point of V lies on some irreducible curve $Z \hookrightarrow V$, the natural maps $SK_1(Z) \to SK_1(V)$ induce a surjection $\bigoplus_{Z \subset V} SK_1(Z) \to SK_1(V)$, where Z ranges over all irreducible closed subschemes of V with $\dim(Z) = 1$. As these are necessarily affine, claim (1) follows for V.

As for claim (3), let V is proper and geometrically irreducible over K. By Chow's lemma there is a proper birational morphism $Z \to V$ with Z projective, and we have diagram (6-2), exact up to finite groups, also for this morphism. By Lemma 6.2 (2) the map β then is surjective, hence α is surjective as well. Thus we may assume that V is projective. We proceed by induction on $\dim(V)$. In case $\dim(V) = 1$ we may replace V with its normalization (by Lemma 6.2), and we may pass to any inseparable extension L of K, because the norm induces an isomorphism $L^* \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to K^* \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$. We thus reduce to the treated case of proper smooth curves. Assume $\dim(V) > 1$. Then, by Bertini's theorem, there is a hyperplane section $Z \subset V$ which is defined over K and geometrically irreducible over K. Then $U = V \setminus Z$ is affine, and we conclude from claim (1) that $SK_1(U) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0$. Now we consider (6-1) and (6-2) for this triple (V, Z, U). From (6-1) we get that the map β in (6-2) is surjective, and hence so is α. By induction on the dimension we may assume that $\mathrm{Ker}(N_{Z/K}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0$, and hence we also get $\mathrm{Ker}(N_{V/K}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0$ as wanted.

Now we show Theorem 6.1. In case V is affine, it follows immediately from 6.3 (1) and the exact sequence (cf. Lemma 5.2)

$$H_0^{et}(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)) \xrightarrow{\epsilon_V^2} H_2^K(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to SK_1(V) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \xrightarrow{\alpha_V} H_{-1}^{et}(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)).$$

In case $f : V \to \mathrm{Spec}(K)$ is proper, we use the commutative diagram

$$\begin{array}{ccccc} SK_1(V)\otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell & \xrightarrow{\alpha_V} & H^{et}_{-1}(V,\mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)) & = & H^1(V, Rf^!\mathbb{Q}_\ell/\mathbb{Z}_\ell(1))) \\ \downarrow \mu & & \downarrow f_* & & \downarrow tr_{V/K} \\ K^*\otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell & \xrightarrow{\cong} & H^{et}_{-1}(\mathrm{Spec}(K),\mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)) & = & H^1(K,\mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \end{array}$$

where μ is induced by $N_{V/K}$. By the above two diagrams we have $\mathrm{Coker}(\epsilon^2_V) = \mathrm{Ker}(\alpha_V) \subset \mathrm{Ker}(\mu)$. But by 6.2 (1) and 6.3 (3), $\mathrm{Ker}(\mu)$ is finite and vanishes if $\ell \not| \, |\mathrm{Coker}(N_{V/K})|$. This completes the proof of Theorem 6.1.

Now we turn to the proof of Theorem 1.6. In what follows we assume that X is projective and generically smooth with strict semistable reduction over $S = \mathrm{Spec}(A)$. Assume also that we are given $Z \subset X$, a good divisor in the sense of Definition 4.2. Write $U = X \setminus Z$. We fix a prime ℓ different from $\mathrm{ch}(K)$ and consider the residue maps

$$\Delta^i_X : H^K_i(X_\eta,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H^K_i(X_s,\mathbb{Q}_\ell/\mathbb{Z}_\ell)$$
$$\Delta^i_Z : H^K_i(Z_\eta,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H^K_i(Z_s,\mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

Our strategy is to use induction on $\dim(X)$. Let

$$\epsilon^i_{U_\eta} \;:\; H^{et}_{i-2}(U_\eta,\mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)) \to H^K_i(U_\eta,\mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

be the map considered in (5-1).

Proposition 6.4 *Fix an integer $q \geq 2$. Assume the following conditions hold.*

(1) The Kato conjecture $K(F,\mathbb{Q}_\ell/\mathbb{Z}_\ell)$ (cf. 3.1) is true in degree $\leq q+1$.

(2) $\epsilon^q_{U_\eta}$ is surjective.

(3) Δ^{q-1}_Z and Δ^q_Z are isomorphisms and Δ^{q+1}_Z is surjective.

(4) One of the following conditions is satisfied:

- *(a) X is smooth over S.*
- *(b) $\ell \neq p := \mathrm{ch}(F)$.*
- *(c) $\ell = p \geq q$.*
- *(d) $q < d$.*

Then Δ^q_X is an isomorphism and Δ^{q+1}_X is surjective.

Proof For $i = q, q+1$ we look at the following commutative diagram of Kato homology groups obtained from 2.17 and 2.18, with the coefficients $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ omitted in the notation.

$$\begin{array}{ccccccccc} H^K_{i+1}(U_\eta) & \to & H^K_i(Z_\eta) & \to & H^K_i(X_\eta) & \to & H^K_i(U_\eta) & \to & H^K_{i-1}(Z_\eta) \\ \downarrow \Delta^{i+1}_U & & \downarrow \Delta^i_Z & & \downarrow \Delta^i_X & & \downarrow \Delta^i_U & & \downarrow \Delta^{i-1}_Z \\ H^K_{i+1}(U_s) & \to & H^K_i(Z_s) & \to & H^K_i(X_s) & \to & H^K_i(U_s) & \to & H^K_{i-1}(Z_s) \end{array}$$

Since $H^K_i(X_\eta) = H^K_i(X_s) = 0$ for $i > d := \dim(X_\eta)$, we may assume $q \leq d$. The proposition follows from the diagram by using the following result.

Lemma 6.5 *Assume $q \leq d$ and the conditions 6.4 (1) and (2) for q.*

(1) Δ_U^i *is surjective for $i \leq q+1$.*

(2) $H_q^K(U_\eta, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = H_q^K(U_s, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$ *if $q < d$.*

(3) Δ_U^q *is an isomorphism if one of the conditions in 6.4 (4) is satisfied.*

Proof Consider the commutative diagram

$$\begin{array}{ccc} H_{i-2}^{et}(U_\eta, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)) & \xrightarrow{\epsilon_{U_\eta}^i} & H_i^K(U_\eta, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \\ \downarrow \Delta_U^{i,et} & & \downarrow \Delta_U^i \\ H_{i-1}^{et}(U_s, \mathbb{Q}_\ell/\mathbb{Z}_\ell(0)) & \xrightarrow{\epsilon_{U_s}^i} & H_i^K(U_s, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \end{array}$$

where the vertical maps are the respective residue maps. The left residue map $\Delta^{i,et}$ is surjective if $i \leq d$ by Theorem 4.4, and by Theorem 3.5, condition 6.4 (1) implies that $\epsilon_{U_s}^i$ is an isomorphism for $i \leq \min(q+1, d)$. This proves the first assertion. For the second assertion note that $H_{q-2}^{et}(U_\eta, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)) = H_{q-1}^{et}(U_s, \mathbb{Q}_\ell/\mathbb{Z}_\ell(0)) = 0$ if $q < d$ by Theorem 3.5 and Theorem 4.4. Thus the claim follows since $\epsilon_{U_s}^q$ and $\epsilon_{U_\eta}^q$ are surjective by the assumptions 6.4 (1) and (2). For (3) we may assume $q = d$ by (2). Then the assertion follows since, by Theorem 4.4, the residue map $\Delta_U^{i,et}$ is an isomorphism for $i = d(= q)$ if one of the conditions 6.4 (4)(a),(b),(c) is satisfied.

We can now prove Theorem 1.6. We proceed by induction on $\dim(X)$. First we claim that we may assume the existence of a good divisor $Z \subset X$. Indeed, by Proposition 4.3 such a divisor exists after replacing F with a finite extension of degree prime to ℓ and K with the corresponding unramified extension. Then the claim follows for the original F and K by a standard norm argument. Given a good divisor, Theorem 1.6 (1) follows from Proposition 6.4 with $q = 2$: Condition (1) is satisfied since the Kato conjecture $K(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is known in degrees ≤ 3, as recalled in the introduction. Condition (2) is satisfied by Lemma 6.1 (1). Condition (3) is satisfied by the induction hypothesis and Theorem 1.5. Condition (4) is satisfied since every prime is not less than 2. This completes the proof of Theorem 1.6.

Remark 6.6 Proposition 6.4 tells that, assuming that the Kato conjecture $K(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ holds, an essential obstacle against showing Conjecture B in degrees > 2 is the surjectivity of $\epsilon_{U_\eta}^i$ for $i > 2$. In case $i = 2$ the class field theory for curves over local fields [Sa1] plays a crucial role to prove it (cf. the proof of Lemma 6.1). In case $i > 2$ we do not have any effective means to approach this problem at present.

We close this section with the following applications of Theorems 6.1 and 1.6.

Corollary 6.7 *Let V be a scheme of finite type over K.*
(1) $H_i^K(V, \mathbb{Z}/n\mathbb{Z})$ is finite for $i = 0, 1$ and for all $n > 0$ invertible in K.
(2) $H_2^K(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is of cofinite type.

Proof There is an affine open subscheme $U \hookrightarrow V$ with complement $Z = V \setminus U$ such that $\dim(Z) < \dim(V)$. By induction on $\dim(V)$ and by the exact sequences

$$H_i^K(Z, \mathbb{Z}/n\mathbb{Z}) \to H_i^K(V, \mathbb{Z}/n\mathbb{Z}) \to H_i^K(U, \mathbb{Z}/n\mathbb{Z})$$

we may thus assume that V is affine. By the following Lemma, the claim then follows from Lemma 5.2 for H_0^K and H_1^K, and from Theorem 6.1 (1) for H_2^K.

Lemma 6.8 *If V separated, and n invertible in K, $H_i^{et}(V, \mathbb{Z}/n\mathbb{Z}(j))$ is finite for for all $i, j \in \mathbb{Z}$. In particular, $H_i^{et}(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ is of cofinite type for all $i, j \in \mathbb{Z}$.*

Proof It suffices to show the first claim; the second then follows via the Kummer sequence $0 \to \mathbb{Z}/\ell\mathbb{Z}(-j) \to \mathbb{Q}_\ell/\mathbb{Z}_\ell(-j) \xrightarrow{\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell(-j) \to 0$. For an open immersion $V \hookrightarrow V'$ with complement $Z = V' \setminus V$, we have the exact sequence

$$H_i^{et}(Z, \mathbb{Z}/n\mathbb{Z}(j)) \to H_i^{et}(V', \mathbb{Z}/n\mathbb{Z}(j)) \to H_i^{et}(V, \mathbb{Z}/n\mathbb{Z}(j)) \to H_{i-1}^{et}(Z, \mathbb{Z}/n\mathbb{Z}(j)).$$

By induction on $\dim(V)$ and embedding V into a proper K-scheme [N], we may thus assume that $g : V \to \mathrm{Spec}(K)$ is proper. By Lemma 5.3 we are reduced to show the finiteness of $H^a(V, \mathbb{Z}/n\mathbb{Z}(b)) = H^a(K, Rg_*\mathbb{Z}/n\mathbb{Z}(b))$ for proper g, which is a consequence of [SGA4], XIV Th.1.1 and [Se], Ch.II §5.2.

Corollary 6.9 *If X is smooth and projective over S, then $H_2^K(X_\eta, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$.*

Proof This is an immediate consequence of Theorems 1.6 and 1.4.

Corollary 6.10 *For a projective smooth variety Z over a number field, $H_2^K(Z, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is of cofinite type.*

Proof This follows from Theorem 1.3 and Corollaries 6.7 and 6.9 by noting the following: Let the notation be as in Theorem 1.3. For an imaginary place v, one has $H_i^K(Z_v, \mathbb{Q}/\mathbb{Z}) = 0$ because $cd(k(x)) = p$ for $x \in (Z_v)_p$. For a real place v, $H_i^K(Z_v, \mathbb{Q}/\mathbb{Z})$ is a 2-torsion finite group by results of [Sch] 19.5.1 and 17.7.

7. An application: The kernel of the reciprocity map

Let the notations be as in the beginning of §5. Recall that K is the quotient field of A, a henselian discrete valuation ring with finite residue field F of characteristic p. We assume that f_η is proper. Let $k(X_\eta)$ denote the function field of X_η. In this section we study the kernel of the reciprocity map

$$\rho_{X_\eta} \ : \ SK_1(X_\eta) \to \pi_1^{ab}(X_\eta)$$

and prove Theorems 1.8 and 1.9. For an integer $n > 0$ prime to $\mathrm{ch}(K)$ let

$$\rho_{X_\eta, n} \ : \ SK_1(X_\eta)/n \to \pi_1^{ab}(X_\eta)/n$$

be the induced map. By Lemmas 5.2 and 5.3 we have an exact sequence

$$\text{(7-1)} \qquad \mathrm{Coker}(\epsilon^2_{X_\eta,n}) \to SK_1(X_\eta)/n \xrightarrow{\rho_{X_\eta,n}} \pi_1^{ab}(X_\eta)/n \to H_1^K(X_\eta, \mathbb{Z}/n\mathbb{Z}) \to 0$$

where $\epsilon^i_{X_\eta,n} : H^{et}_{i-2}(X_\eta, \mathbb{Z}/n\mathbb{Z}(-1)) \to H_i^K(X_\eta, \mathbb{Z}/n\mathbb{Z})$ is as in (5-1). We need the following theorem.

Theorem 7.1 *Let V be a smooth proper geometrically irreducible variety over K. Let $\pi_1^{ab}(V)^{geo}$ be the kernel of the natural surjection $\pi_1^{ab}(V) \to \mathrm{Gal}(K^{ab}/K)$. There is an exact sequence*

$$0 \to T \to \pi_1^{ab}(V)^{geo} \to \hat{\mathbb{Z}}^r \to 0,$$

where T is finite and r is the F-rank of the special fiber of the Néron model of the Albanese variety of V.

For the pro-ℓ-part, with $\ell \neq p := \mathrm{ch}(F)$, this is essentially due to Grothendieck. The result for the pro-p-part is due to T. Yoshida [Y].

Theorem 7.2 *For V as in Theorem 7.1, the image of* $\mathrm{Ker}(N_{V/K})$ *under $\rho_V : SK_1(V) \to \pi_1^{ab}(V)$ is finite.*

Proof By Theorem 7.1 it suffices to show that the image is torsion. By a similar argument as in the proof of Proposition 6.3 (1) we may reduce to the case that V is a proper smooth curve. Then the assertion follows from [Sa1], Th. 4.1, together with [Y], Th. 5.1 (for the p-part if $\mathrm{ch}(K) = p$).

Lemma 7.3 *Let V be of finite type over K. For a prime ℓ we have an exact sequence*

$$0 \to H^K_{i+1}(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell)/\ell^\nu \to H_i^K(V, \mathbb{Z}/\ell^\nu\mathbb{Z}) \to H_i^K(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell)[\ell^\nu] \to 0$$

either if $\ell = \mathrm{ch}(K)$, or if $\ell \neq \mathrm{ch}(K)$, $BK_{i+1}(K(x), \ell)$ (cf. the introduction) holds for all $x \in V_i$ and $BK_i(K(x), \ell)$ holds for all $x \in V_{i-1}$.

Proof (cf. [CT], §2) We use the Kummer sequences of étale sheaves

$$\begin{aligned} 0 \to \mathbb{Z}/\ell^\nu\mathbb{Z}(r) \to \mathbb{Q}_\ell/\mathbb{Z}_\ell(r) \xrightarrow{\ell^\nu} \mathbb{Q}_\ell/\mathbb{Z}_\ell(r) \to 0 \quad &(\text{for } \ell \neq \mathrm{ch}(K)),\\ 0 \to W_\nu\Omega^r_{log} \to W_\infty\Omega^r_{log} \xrightarrow{\ell^\nu} W_\infty\Omega^r_{log} \to 0 \quad &(\text{for } \ell = \mathrm{ch}(K)). \end{aligned}$$

For $x \in V_j$ they give rise to the exact sequence

$$\begin{aligned} H^{j+1}(K(x), \mathbb{Q}_\ell/\mathbb{Z}_\ell(j+1)) \xrightarrow{\ell^\nu} H^{j+1}(K(x), \mathbb{Q}_\ell/\mathbb{Z}_\ell(j+1)) \to H^{j+2}(K(x), \mathbb{Z}/\ell^\nu\mathbb{Z}(j+1))\\ \xrightarrow{\iota} H^{j+2}(K(x), \mathbb{Q}_\ell/\mathbb{Z}_\ell(j+1)) \xrightarrow{\ell^\nu} H^{j+2}(K(x), \mathbb{Q}_\ell/\mathbb{Z}_\ell(j+1)) \to 0, \end{aligned}$$

(where we let $H^s(K(x),\mathbb{Q}_\ell/\mathbb{Z}_\ell(r)) := H^{s-r}(K(x),W_\infty\Omega^r_{log})$, cf. 2 B and the introduction). The surjectivity of the last map follows from $H^{j+3}(K(x),\mathbb{Z}/\ell^\nu\mathbb{Z}(j+1)) = 0$, which follows from [Se], Ch.II §4 and §6 for $\ell \neq \mathrm{ch}(K)$. For $\ell = p = \mathrm{ch}(K)$, the vanishing of $H^2(K(x),W_\nu\Omega^{j+1})$ follows from $cd_p(K(x)) \leq 1$, cf. the proof of 2.14 (c). In case $\ell \neq \mathrm{ch}(K)$ the assumption $BK_{j+1}(K(x),\ell)$ says that $H^{j+1}(K(x),\mathbb{Q}_\ell/\mathbb{Z}_\ell(j+1))$ is divisible so that ι is injective. In case $\ell = \mathrm{ch}(K)$ we get the same conclusion by [BK], Th. 2.1. The desired assertion now follows from the sequence of Kato complexes

$$0 \to C^{2,1}(V,\mathbb{Z}/\ell^\nu\mathbb{Z}) \to C^{2,1}(V,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow{\ell^\nu} C^{2,1}(V,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \to 0$$

and the exactness properties derived from the above exact sequences for $j = i-1, i, i+1$.

Lemma 7.4 *Let V and W be irreducible and smooth of dimension d over a field L and let $f : W \to V$ be proper and generically finite of degree N. For any integers $r, s,$, and any integer n invertible in L, the cokernel of the map*

$$f_* \ : \ H^{r,s}_d(W,\mathbb{Z}/n\mathbb{Z}) \to H^{r,s}_d(V,\mathbb{Z}/n\mathbb{Z})$$

induced by f in the Kato homology is annihilated by N.

Proof We have the commutative diagram

$$\begin{array}{ccc} H^{r,s}_d(W,\mathbb{Z}/n\mathbb{Z}) & \stackrel{\hookrightarrow}{\longrightarrow} & H^{d+r}(L(W),\mathbb{Z}/n\mathbb{Z}(d+s)) \\ \downarrow f_* & & \downarrow Cor_{L(W)/L(V)} \\ H^{r,s}_d(V,\mathbb{Z}/n\mathbb{Z}) & \stackrel{\hookrightarrow}{\longrightarrow} & H^{d+r}(L(V),\mathbb{Z}/n\mathbb{Z}(d+s)) \end{array}$$

where $Cor_{L(W)/L(V)}$ is the corestriction map for Galois cohomology. We claim that the restriction map

$$Res_{L(W)/L(V)} \ : \ H^{d+r}(L(V),\mathbb{Z}/n\mathbb{Z}(d+s)) \to H^{d+r}(L(W),\mathbb{Z}/n\mathbb{Z}(d+s))$$

induces $f^* : H^{r,s}_d(V,\mathbb{Z}/n\mathbb{Z}) \to H^{r,s}_d(W,\mathbb{Z}/n\mathbb{Z})$, which proves the lemma since the composite map $Cor_{L(W)/L(V)} \circ Res_{L(W)/L(V)}$ is the multiplication by N. If f is flat, the assertion follows from the contravariant functoriality of the Kato complexes for flat morphisms. In general it follows from the canonical isomorphism

$$H^{r,s}_d(W,\mathbb{Z}/n\mathbb{Z}) \cong H^0(W_{Zar},\mathcal{H}^{d+r}(\mathbb{Z}/n\mathbb{Z}(d+s)))$$

and the similar isomorphism for V following from [BO] by the assumption of smoothness. Here $\mathcal{H}^{d+r}(\mathbb{Z}/n\mathbb{Z}(d+s))$ is the Zariski sheaf associated to $U \to H^{d+r}(U_{et},\mathbb{Z}/n\mathbb{Z}(d+s))$.

Lemma 7.5 *Assume that X_η is irreducible and proper of dimension 2. If $\ell \neq \mathrm{ch}(K)$ is a prime such that $BK_3(K(X_\eta),\ell)$ holds, then $|\mathrm{Ker}(\rho_{X_\eta,\ell^\nu})|$ is bounded with respect to ν.*

Proof Lemma 5.2 induces the following commutative diagram
(7-2)

$$\begin{array}{ccccc}
H_0^{et}(X_\eta, \mathbb{Z}/\ell^\nu\mathbb{Z}(-1)) & \to & H_2^K(X_\eta, \mathbb{Z}/\ell^\nu\mathbb{Z}) & \to & \mathrm{Ker}(\rho_{X_\eta,\ell^\nu}) \to 0 \\
\downarrow \alpha & & \downarrow \gamma & & \\
H_0^{et}(X_\eta, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1))[\ell^\nu] & \xrightarrow{\epsilon[\ell^\nu]} & H_2^K(X_\eta, \mathbb{Q}_\ell/\mathbb{Z}_\ell)[\ell^\nu] & & \\
\downarrow \hookrightarrow & & \downarrow \hookrightarrow & & \\
H_0^{et}(X_\eta, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)) & \xrightarrow{\epsilon} & H_2^K(X_\eta, \mathbb{Q}_\ell/\mathbb{Z}_\ell) & &
\end{array}$$

where the upper horizontal sequence is exact, α is surjective, γ is an isomorphism by Lemma 7.3, and $\mathrm{Coker}(\epsilon)$ is finite by Theorem 6.1. Since $H_0^{et}(X_\eta, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1))$ is of cofinite type by Lemma 6.8, this implies that $\mathrm{Ker}(\rho_{X_\eta,\ell^\nu})$ is finite and that

$$\begin{aligned}
|\mathrm{Ker}(\rho_{X_\eta,\ell^\nu})| = |\mathrm{Coker}(\epsilon[\ell^\nu])| &\leq |\mathrm{Coker}(\epsilon)| \cdot |\mathrm{Ker}(\epsilon)/\ell^\nu| \\
&\leq |\mathrm{Coker}(\epsilon)| \cdot |\mathrm{Ker}(\epsilon)/\mathrm{Div}(\mathrm{Ker}(\epsilon))|,
\end{aligned}$$

where $\mathrm{Div}(A)$ denotes the maximal divisible subgroup of an abelian group A. This proves Lemma 7.5.

Lemma 7.6 *With assumption be as Theorem 1.8 (1), let I_P be the set of all integers whose prime divisors belong to P. Then $\mathrm{Ker}(\rho_{X_\eta,n})$ is bounded with respect to $n \in I_P$.*

Proof By Lemma 7.5 it suffices to show $\mathrm{Ker}(\rho_{X_\eta,\ell^\nu}) = 0$ for almost all $\ell \in P$, $\ell \neq \mathrm{ch}(F)$. We claim that we may assume that X is projective over S having semistable reduction over S. Indeed, by [dJ] there exists a finite morphism $S' \to S$ and an alteration $\widetilde{X} \to X$ with $\widetilde{X}/S'$ satisfying the condition. Noting that (7-1) is covariantly functorial for proper morphisms, we get the commutative diagram

$$\begin{array}{ccccc}
H_2^K(\widetilde{X}_\eta, \mathbb{Z}/\ell^\nu\mathbb{Z}) & \to & \mathrm{Ker}(\rho_{\widetilde{X}_\eta,\ell^\nu}) & \to 0 \\
\downarrow & & \downarrow & \\
H_2^K(X_\eta, \mathbb{Z}/\ell^\nu\mathbb{Z}) & \to & \mathrm{Ker}(\rho_{X_\eta,\ell^\nu}) & \to 0
\end{array}$$

Hence the claim follows from Lemma 7.4. Then by Proposition 4.3 and a finite étale base change we may furthermore assume that there exists a very good divisor $Z \subset X$ in the sense of Definition 3.3. Let $U = X \setminus Z$.

Claim 1 *If $\ell \in P$, then $\mathrm{Ker}(\rho_{X_\eta,\ell^\nu}) \subset \mathrm{Im}(SK_1(Z_\eta))$ for all ν.*

Proof We have the commutative diagram

$$\begin{array}{ccccccc}
SK_1(Z_\eta)/\ell^\nu & \to & SK_1(X_\eta)/\ell^\nu & \to & SK_1(U_\eta)/\ell^\nu & \to 0 \\
& & \downarrow \alpha_{X_\eta} & & \downarrow \alpha_{U_\eta} & \\
& & H_{-1}^{et}(X_\eta, \mathbb{Z}/\ell^\nu\mathbb{Z}(-1)) & \to & H_{-1}^{et}(U_\eta, \mathbb{Z}/\ell^\nu\mathbb{Z}(-1)) &
\end{array}$$

in which α_{X_η} can be identified with ρ_{X_η,ℓ^ν} (cf. 5.2 and 5.3). Thus it suffices to show that α_{U_η} is an injection. By Lemma 5.2 we have the commutative diagram

$$\begin{array}{ccccc} H_0^{et}(U_\eta,\mathbb{Z}/\ell^\nu\mathbb{Z}(-1)) & \xrightarrow{\epsilon_{U_\eta}} & H_2^K(U_\eta,\mathbb{Z}/\ell^\nu\mathbb{Z}) & \to & \mathrm{Ker}(\alpha_{U_\eta})\to 0 \\ \downarrow & & \downarrow & & \\ H_0^{et}(U_\eta,\mathbb{Q}_\ell/\mathbb{Z}_\ell(-1))[\ell^\nu] & \xrightarrow{\cong} & H_2^K(U_\eta,\mathbb{Q}_\ell/\mathbb{Z}_\ell)[\ell^\nu] & & \end{array}$$

where the upper horizontal sequence is exact and the left vertical map is surjective. The right vertical arrow is an isomorphism by Lemma 7.3 and the lower horizontal map is an isomorphism by the proof of Theorem 1.6 (note that $\ell\neq \mathrm{ch}(F)$). This proves the desired assertion.

Claim 2 *If* $\ell \not| |\mathrm{Coker}(N_{Z_\eta/K})|$, *then* $\mathrm{Ker}(\rho_{X_\eta,\ell^\nu})\subset \mathrm{Im}(\mathrm{Ker}(N_{Z_\eta/K}))$.

Proof We have the commutative diagram

$$\begin{array}{cccccccc} 0\to & \mathrm{Ker}(N_{Z_\eta/K})/\ell^\nu & \to & SK_1(Z_\eta)/\ell^\nu & \to & K^*/\ell^\nu & \to 0 \\ & \downarrow & & \downarrow & & \| & \\ 0\to & \mathrm{Ker}(N_{X_\eta/K})/\ell^\nu & \to & SK_1(X_\eta)/\ell^\nu & \to & K^*/\ell^\nu & \to 0 \\ & & & \downarrow \rho_{X_\eta,\ell^\nu} & & \downarrow \rho_{\eta,\ell^\nu} & \\ & & & \pi_1^{ab}(X_\eta)/\ell^\nu & \to & \pi_1^{ab}(\eta)/\ell^\nu & \end{array}$$

where ρ_{η,ℓ^ν} is an isomorphism by local class field theory. Under the assumption of the claim, the horizontal sequences are exact. Now the claim follows easily from Claim 1 by simple diagram chasing.

To finish the proof of Lemma 7.6 it suffices to show $\mathrm{Ker}(N_{Z_\eta/K})$ is ℓ-divisible for almost all ℓ. Since $\dim(Z_\eta)=1$, this follows from the class field theory for curves over local fields. Indeed, the kernel of

$$\mathrm{Ker}(N_{Z_\eta/K})\hookrightarrow SK_1(Z_\eta)\xrightarrow{\rho_V}\pi_1^{ab}(Z_\eta)$$

is ℓ-divisible by [Sa1], Th.5.1 so that the assertion follows from Theorem 7.2.

Proof of Theorem 1.8(1): Consider the commutative diagram

$$\begin{array}{cccccc} 0\to & \mathrm{Ker}(\rho_{X_\eta}) & \to & SK_1(X_\eta) & \to & \pi_1^{ab}(X_\eta) \\ & \downarrow & & \downarrow \pi_1 & & \downarrow \pi_2 \\ 0\to & \varprojlim_{n\in I_P}\mathrm{Ker}(\rho_{X_\eta,n}) & \to & \varprojlim_{n\in I_P} SK_1(X_\eta)/n & \to & \varprojlim_{n\in I_P}\pi_1^{ab}(X_\eta)/n \end{array}$$

Theorem 7.1 implies that

$$\varprojlim_{n\in I_P}\pi_1^{ab}(X_\eta)/n=\prod_{\ell\in P}\pi_1^{ab}(X_\eta)(\ell)\quad\text{and}\quad \mathrm{Ker}(\pi_2)=\prod_{\ell\notin P}\pi_1^{ab}(X_\eta)(\ell).$$

and that the torsion part of the former is finite and that the latter is P-torsion free, namely $nx=0$ with $n\in I_P$ and $x\in\mathrm{Ker}(\pi_2)$ implies $x=0$. Lemma 7.6 implies $\varprojlim_{n\in I_P}\mathrm{Ker}(\rho_{X_\eta,n})$ is finite so that $\mathrm{Ker}(\pi_1)$ is P-divisible by Lemma 7.7 below. Hence the diagram implies that $\mathrm{Ker}(\rho_{X_\eta})$ is an extension of a finite group by a P-divisible group. Now Theorem 1.8(1) follows from Lemma 7.8 below.

Lemma 7.7 *Assume given an abelian group A, a projective system of abelian groups $\{B_n\}_{n\in I_P}$ and a projective system $\{A/nA \xrightarrow{\varphi_n} B_n\}_{n\in I_P}$ of homomorphisms. Write*

$$\hat{\varphi} := \varprojlim_{n\in I_P} \varphi_n \ :\ \hat{A} \to \hat{B}, \quad (\hat{A} = \varprojlim_{n\in I_P} A/nA,\ \hat{B} := \varprojlim_{n\in I_P} B_n)$$

Assume that there exists $0 \neq N \in I_P$ such that $N\cdot\hat{B}_{tor} = 0$ and that $N\cdot\mathrm{Ker}(\hat{\varphi}) = 0$. Then $D := \cap_{n\in I_P} nA$ is the maximal P-divisible subgroup of A.

Proof It is evident that D contains the maximal P-divisible subgroup of A. Thus it suffices to show that D is P-divisible. Let $\pi : A \to \hat{A}$ be the natural map and put $D' = \mathrm{Ker}(\hat{\varphi}\circ\pi)$. By the assumption $D = \mathrm{Ker}(\pi) \subset D'$ and $N\cdot D' \subset D$. Take $x \in D$. For any $n \in I_P$ there exists $y \in A$ such that $x = nN^2y$. Then we have $0 = \hat{\varphi}(\pi(x)) = nN^2\hat{\varphi}(\pi(y))$ so that $\hat{\varphi}(\pi(y)) \in \hat{B}_{tor}$. By the assumption this implies $N\hat{\varphi}(\pi(y)) = \hat{\varphi}(\pi(Ny)) = 0$. Hence $z := Ny \in D'$ so that $x = n(Nz) \in n\cdot D$. This completes the proof of Lemma 7.7.

Lemma 7.8 *Assume given $0 \to D \to A \xrightarrow{\pi} T \to 0$, an exact sequence of abelian groups, where T is torsion and D is P-divisible. Write $T = T(P)\oplus T'$ where $T(P)$ is P-torsion and T' has no P-torsion element. Put $D' = \pi^{-1}(T')$. Then D' is P-divisible and we have $A \cong D' \oplus T(P)$.*

Proof We have the exact sequence $0 \to D \to D' \to T' \to 0$. By definition T' is P-divisible and hence D' is P-divisible. We show $A \cong D' \oplus T(P)$. It suffices to construct a map $s_\ell : T\{\ell\} \to A$ for each $\ell \in P$ such that $\pi\circ s_\ell$ is the identity, where $M\{\ell\}$ denotes the ℓ-primary torsion part of an abelian group M. We have an exact Tor-sequence

$$0 \to D'\{\ell\} \to A\{\ell\} \to T\{\ell\} \to 0 = D' \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$$

which can be viewed as an exact sequence of $\mathbb{Z}_\ell$-modules. Since $D'\{\ell\}$ is ℓ-divisible, it is an injective $\mathbb{Z}_\ell$-module (cf. [HS], Th. 7.1). Thus the above sequence splits and we get the desired map s_ℓ. This completes the proof of Lemma 7.8.

Proof of Theorem 1.9: If V is smooth and proper, the claim follows from Theorem 1.8 (1) and Theorem 7.2. If V is proper, there exists an alteration $\widetilde{V} \to V$ with $\widetilde{V}$ smooth and proper [dJ], and we get a commutative diagram

$$\begin{array}{ccc} SK_1(\widetilde{V}) & \stackrel{N_{\widetilde{V}/V}}{\to} & SK_1(V) \\ \downarrow N_{\widetilde{V}/K} & & \downarrow N_{V/K} \\ K^* & = & K^* \end{array}$$

in which $N_{\widetilde{V}/V}$ has finite cokernel by Lemma 6.2. Hence the claim for V follows from that for $\widetilde{V}$. If V is not proper, take an open immersion $V \hookrightarrow W$ with W

proper over K, and let $Z = W \setminus V$. Then we have a commutative diagram with exact top row

$$\begin{array}{ccccccc} SK_1(Z) & \to & SK_1(W) & \to & SK_1(V) & \to 0 \\ \downarrow N_{Z/K} & & \downarrow N_{W/K} & & & \\ K^* & = & K^* & & & \end{array}$$

It gives a map $\mathrm{Ker}(N_{W/K}) \to SK_1(V)$ with finite cokernel, which shows that it suffices to consider W, i.e. the case that V is proper. This completes the proof of Theorem 1.9.

Proof of Theorem 1.8 (2) and (3): Lemmas 5.2 and 5.3 induce the exact sequence for $n \in I_P$

$$H_2^K(X_\eta, \mathbb{Q}/\mathbb{Z})[n] \to SK_1(X_\eta)/n \xrightarrow{\rho_{X_\eta,n}} \pi_1^{ab}(X_\eta)/n \tag{7-3}$$

where we used that $H_2^K(X_\eta, \mathbb{Z}/n\mathbb{Z}) \cong H_2^K(X_\eta, \mathbb{Q}/\mathbb{Z})[n]$ by Lemma 7.3. The assumption $H_2(\Gamma_{\widetilde{X}_s}, \mathbb{Q}) = 0$ implies that $H_2(\Gamma_{\widetilde{X}_s}, \mathbb{Q}/\mathbb{Z})$ is finite. On the other hand, we have maps

$$H_2^K(\widetilde{X}_\eta, \mathbb{Q}/\mathbb{Z}') \xrightarrow{\Delta^2_{\widetilde{X}}} H_2^K(\widetilde{X}_s, \mathbb{Q}/\mathbb{Z}') \xrightarrow{\gamma_{\widetilde{X}_s}} H_2^K(\Gamma_{\widetilde{X}_s}, \mathbb{Q}/\mathbb{Z}')\ , \quad \mathbb{Q}/\mathbb{Z}' = \bigoplus_{\ell \neq \mathrm{ch}(K)} \mathbb{Q}_\ell/\mathbb{Z}_\ell,$$

where $\Delta^2_{\widetilde{X}}$ is an isomorphism by Theorem 1.6 and $\gamma_{\widetilde{X}_s}$ is an isomorphism by Theorem 1.4. Hence $H_2^K(\Gamma_{\widetilde{X}_s}, \mathbb{Q}/\mathbb{Z}')$ is finite, and by Lemma 7.4, $H_2^K(X_\eta, \mathbb{Q}/\mathbb{Z}')$ is finite as well. Thus, passing to the limit, the sequences (7-3) induces an injection

$$\varprojlim_{n \in I_P} SK_1(X_\eta)/n \hookrightarrow \varprojlim_{n \in I_P} \pi_1^{ab}(X_\eta)/n,$$

because $\varprojlim A[n] = 0 = \varprojlim^1 A[n]$ for any finite abelian group A. Now Theorem 1.8 (2) follows from Lemma 7.7 by the same argument as in the proof of Theorem 1.8(1). Finally Theorem 1.8 (3) follows from 1.8 (2) together with Theorems 1.4 and 1.5, because in the case of good reduction the complex Γ_{X_s} is contractible.

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