Quadric Surface Bundles over Surfaces

Dedicated to Sasha Merkurjev on his 60th birthday

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Abstract. Let $f : T \to S$ be a finite flat morphism of degree 2 between regular integral schemes of dimension ≤ 2 with 2 invertible, having regular branch divisor $D \subset S$. We establish a bijection between Azumaya quaternion algebras on T and quadric surface bundles with simple degeneration along D. This is a manifestation of the exceptional isomorphism ${}^2\mathsf{A}_1 = \mathsf{D}_2$ degenerating to the exceptional isomorphism $\mathsf{A}_1 = \mathsf{B}_1$. In one direction, the even Clifford algebra yields the map. In the other direction, we show that the classical algebra norm functor can be uniquely extended over the discriminant divisor. Along the way, we study the orthogonal group schemes, which are smooth yet nonreductive, of quadratic forms with simple degeneration. Finally, we provide two applications: constructing counter-examples to the local-global principle for isotropy, with respect to discrete valuations, of quadratic forms over surfaces; and a new proof of the global Torelli theorem for very general cubic fourfolds containing a plane.

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Introduction

A quadric surface bundle $\pi : Q \to S$ over a scheme S is the flat fibration in quadrics associated to a line bundle-valued quadratic form $q : \mathscr{E} \to \mathscr{L}$ of rank 4 over S. A natural class of quadric surface bundles over $\mathbb{P}^2$ appearing in algebraic geometry arise from cubic fourfolds $Y \subset \mathbb{P}^5$ containing a plane. Projection from the plane $\pi : \widetilde{Y} \to \mathbb{P}^2$, where $\widetilde{Y}$ is the blow-up of Y along the plane, yields a quadric surface bundle with degeneration along a sextic curve $D \subset \mathbb{P}^2$. If Y is sufficiently general then D is smooth and the double cover $T \to \mathbb{P}^2$ branched along D is a smooth K3 surface of degree 2. Over the surface T, the even Clifford algebra $\mathscr{C}_0$ associated to π becomes an Azumaya quaternion algebra representing a Brauer class $\beta \in {}_2\mathrm{Br}(T)$. For Y even more sufficiently general, the association $Y \mapsto (T, \beta)$ is injective: smooth cubic fourfolds Y and Y' giving rise to isomorphic data $(T, \beta) \cong (T', \beta')$ are linearly isomorphic. This result was originally obtained via Hodge theory by Voisin [60] in the course of her proof of the global Torelli theorem for cubic fourfolds.

In this work, we provide a vast algebraic generalization of this result to any regular integral scheme T of dimension ≤ 2, which is a finite flat double cover $T \to S$ of a regular scheme S, such that the branch divisor $D \subset S$ is regular. We establish a bijection between the isomorphism classes of quadric surface bundles on S having simple degeneration (see §1) with discriminant $T \to S$ and the isomorphism classes of Azumaya quaternion algebras on T whose norm to S is split (see §5). In one direction, the even Clifford algebra $\mathscr{C}_0$, associated to a quadric surface bundle on S with simple degeneration along D, gives rise to an Azumaya quaternion algebra on T. In the other, a generalization of the classical algebra norm functor $N_{T/S}$, applied to an Azumaya quaternion algebra on T with split norm to S, gives rise to a quadric surface bundle on S. Our main result is the following.

Theorem 1. *Let S be a regular integral scheme of dimension ≤ 2 with 2 invertible and $T \to S$ a finite flat morphism of degree 2 with regular branch divisor $D \subset S$. Then the even Clifford algebra and norm functors*

$$\left\{ \begin{array}{c} \textit{quadric surface bundles with} \\ \textit{simple degeneration along } D \\ \textit{and discriminant } T \to S \end{array} \right\} \underset{N_{T/S}}{\overset{\mathscr{C}_0}{\rightleftarrows}} \left\{ \begin{array}{c} \textit{Azumaya quaternion} \\ \textit{algebras over } T \textit{ with} \\ \textit{split norm to } S \end{array} \right\}$$

give rise to mutually inverse bijections.

This result can be viewed as a significant generalization of the exceptional isomorphism ${}^2\mathsf{A}_1 = \mathsf{D}_2$ correspondence over fields and rings (cf. [41, IV.15.B] and [43, §10]) to the setting of line bundle-valued quadratic forms with simple degeneration over schemes. Most of our work goes toward establishing fundamental local results concerning quadratic forms with simple degeneration (see §3) and the structure of their orthogonal group schemes, which are nonreductive (see §2). In particular, we prove that these group schemes are smooth (see Proposition 2.3) and realize a degeneration of exceptional isomorphisms ${}^2\mathsf{A}_1 = \mathsf{D}_2$ to $\mathsf{A}_1 = \mathsf{B}_1$. We also establish fundamental structural results concerning quadric surface bundles over schemes (see §1) and the formalism of gluing tensors over surfaces (see §4).

We also present two surprisingly different applications of our results. First, in §6, we provide a class of geometrically interesting quadratic forms that are counter-examples to the local-global principle to isotropy, with respect to discrete valuations, over the function field of any surface over an algebraically closed field of characteristic zero. This is made possible by the tight control we have over the degeneration divisors of norm forms of unramified quaternion algebras over function fields of ramified double covers of surfaces. Moreover, our class of counter-examples exists even over rational function fields, where the existence of such counterexamples was an open question.

Second, in §7, combining our main result with tools from the theory of moduli of twisted sheaves, we are able to provide a new proof of the result of Voisin mentioned above, concerning general complex cubic fourfolds containing a plane. Our method is algebraic in nature and could lead to similar results for other classes of complex fourfolds birational to quadric surface bundles over surfaces.

Our perspective comes from the algebraic theory of quadratic forms. We employ the even Clifford algebra of a line bundle-valued quadratic form constructed by Bichsel [14]. Bichsel–Knus [15], Caenepeel–van Oystaeyen [16] and Parimala–Sridharan [51, §4] give alternate constructions, which are all detailed in [3, §1.8]. In a similar vein, Kapranov [39, §4.1] (with further developments by Kuznetsov [44, §3]) considered the *homogeneous* Clifford algebra of a quadratic form—this is related to the *generalized* Clifford algebra of [15] and the *graded* Clifford algebra of [16]—to study the derived category of projective quadrics and quadric bundles. We focus on the even Clifford algebra as a sheaf of algebras, rather than its geometric manifestation as a relative Hilbert scheme of lines in the quadric bundle, as in [60, §1] and [37, §5]. In this context, we refer to Hassett–Tschinkel [36, §3] for a version of our result over smooth projective curves over an algebraically closed field.

Finally, our work on degenerate quadratic forms may also be of independent interest. There has been a recent focus on classification of degenerate (quadratic) forms from various number theoretic directions. An approach to Bhargava's [13] seminal construction of moduli spaces of "rings of low rank" over arbitrary base schemes is developed by Wood [62] where line bundle-valued degenerate forms (of higher degree) are crucial ingredients. In this context, a correspondence such as ours, established over $\mathbb{Z}$, could facilitate density results for discriminants of quaternion orders over quadratic extensions of number fields. In related developments, building on the work of Delone–Faddeev [25] over $\mathbb{Z}$ and Gross–Lucianovic [31] over local rings, Venkata Balaji [9], and independently Voight [59], used Clifford algebras of degenerate ternary quadratic forms to classify degenerations of quaternion algebras over arbitrary bases. In this context, our main result can be viewed as a classification of quaternary quadratic forms with squarefree discriminant in terms of their even Clifford algebras.

1 Reflections on simple degeneration

Let S be a noetherian separated integral scheme. A *(line bundle-valued) quadratic form* on S is a triple $(\mathscr{E}, q, \mathscr{L})$, where $\mathscr{E}$ is a locally free $\mathscr{O}_S$-module of finite rank and $q : \mathscr{E} \to \mathscr{L}$ is a morphism of sheaves, homogeneous of degree 2 for the action of $\mathscr{O}_S$, and such that the associated morphism of sheaves $b_q : \mathscr{E} \times \mathscr{E} \to \mathscr{L}$, defined on sections by $b_q(v, w) = q(v+w) - q(v) - q(w)$, is $\mathscr{O}_S$-bilinear. Equivalently, a quadratic form is an $\mathscr{O}_S$-module morphism $q : S_2\mathscr{E} \to \mathscr{L}$, see [57, Lemma 2.1] or [3, Lemma 1.1]. Here, $S^2\mathscr{E}$ and $S_2\mathscr{E}$ denote the second symmetric power and the submodule of symmetric second tensors of $\mathscr{E}$, respectively. There is a canonical isomorphism $S^2(\mathscr{E}^\vee) \otimes \mathscr{L} \cong \mathscr{H}om(S_2\mathscr{E}, \mathscr{L})$. A line bundle-valued quadratic form then corresponds to a global section

$$q \in \Gamma(S, \mathscr{H}om(S_2\mathscr{E}, \mathscr{L})) \cong \Gamma(S, S^2(\mathscr{E}^\vee) \otimes \mathscr{L}) \cong \Gamma(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})/S}(2) \otimes p^*\mathscr{L}),$$

where $p : \mathbb{P}(\mathscr{E}) = \mathbf{Proj}\, S^\bullet(\mathscr{E}^\vee) \to S$. There is a canonical $\mathscr{O}_S$-module *polar* morphism $\psi_q : \mathscr{E} \to \mathscr{H}om(\mathscr{E}, \mathscr{L})$ associated to b_q. A line bundle-valued quadratic form $(\mathscr{E}, q, \mathscr{L})$ is *regular* if ψ_q is an $\mathscr{O}_S$-module isomorphism. Otherwise, the *radical* $\mathrm{rad}(\mathscr{E}, q, \mathscr{L})$ is the sheaf kernel of ψ_q, which is a torsion-free subsheaf of $\mathscr{E}$. We will mostly dispense with the adjective "line bundle-valued." We define the *rank* of a quadratic form to be the rank of the underlying module. A *similarity* $(\varphi, \lambda_\varphi) : (\mathscr{E}, q, \mathscr{L}) \to (\mathscr{E}', q', \mathscr{L}')$ consists of $\mathscr{O}_S$-module isomorphisms $\varphi : \mathscr{E} \to \mathscr{E}'$ and $\lambda_\varphi : \mathscr{L} \to \mathscr{L}'$ such that $q'(\varphi(v)) = \lambda_\varphi \circ q(v)$ on sections. A similarity $(\varphi, \lambda_\varphi)$ is an *isometry* if $\mathscr{L} = \mathscr{L}'$ and λ_φ is the identity map. We write $\simeq$ for similarities and $\cong$ for isometries. Denote by $\mathbf{GO}(\mathscr{E}, q, \mathscr{L})$ and $\mathbf{O}(\mathscr{E}, q, \mathscr{L})$ the presheaves, on the flat (fppf) site on S, of similitudes and isometries of a quadratic form $(\mathscr{E}, q, \mathscr{L})$, respectively. These are sheaves and are representable by affine group schemes of finite presentation over S, indeed closed subgroupschemes of $\mathbf{GL}(\mathscr{E})$. The similarity factor defines a homomorphism $\lambda : \mathbf{GO}(\mathscr{E}, q, \mathscr{L}) \to \mathbf{G}_{\mathrm{m}}$ with kernel $\mathbf{O}(\mathscr{E}, q, \mathscr{L})$. If $(\mathscr{E}, q, \mathscr{L})$ has even rank $n = 2m$, then there is a homomorphism $\det/\lambda^m : \mathbf{GO}(\mathscr{E}, b, \mathscr{L}) \to \boldsymbol{\mu}_2$, whose kernel is denoted by $\mathbf{GO}^+(\mathscr{E}, q, \mathscr{L})$ (this definition of $\mathbf{GO}^+$ assumes 2 is invertible on S; in general it is defined as the kernel of the Dickson invariant). The similarity factor $\lambda : \mathbf{GO}^+(\mathscr{E}, q, \mathscr{L}) \to \mathbf{G}_{\mathrm{m}}$ has kernel denoted by $\mathbf{O}^+(\mathscr{E}, q, \mathscr{L})$. Denote by $\mathbf{PGO}(\mathscr{E}, q, \mathscr{L})$ the sheaf cokernel of the central subgroup scheme $\mathbf{G}_{\mathrm{m}} \to \mathbf{GO}(\mathscr{E}, q, \mathscr{L})$ of homotheties; similarly define $\mathbf{PGO}^+(\mathscr{E}, q, \mathscr{L})$. At every point where $(\mathscr{E}, q, \mathscr{L})$ is regular, these group schemes are smooth and reductive (see [26, II.1.2.6, III.5.2.3]) though not necessarily connected. In §2, we will study their structure over points where the form is not regular.

The *quadric bundle* $\pi : Q \to S$ associated to a nonzero quadratic form $(\mathscr{E}, q, \mathscr{L})$ of rank $n \geq 2$ is the restriction of $p : \mathbb{P}(\mathscr{E}) \to S$ via the closed embedding $j : Q \to \mathbb{P}(\mathscr{E})$ defined by the vanishing of the global section $q \in \Gamma_S(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})/S}(2) \otimes p^*\mathscr{L})$. Write $\mathscr{O}_{Q/S}(1) = j^*\mathscr{O}_{\mathbb{P}(\mathscr{E})/S}(1)$. We say that $(\mathscr{E}, q, \mathscr{L})$ is *primitive* if $q_x \neq 0$ at every point x of S, i.e., if $q : \mathscr{E} \to \mathscr{L}$ is an epimorphism. If q is primitive then $Q \to \mathbb{P}(\mathscr{E})$ has relative codimension 1 over S and $\pi : Q \to S$ is flat of relative dimension $n-2$, cf. [46, 8 Thm. 22.6].

We say that $(\mathscr{E}, q, \mathscr{L})$ is *generically regular* if q is regular over the generic point of S.
Define the *projective similarity* class of a quadratic form $(\mathscr{E}, q, \mathscr{L})$ to be the set of similarity classes of quadratic forms $(\mathscr{N} \otimes \mathscr{E}, \mathrm{id}_{\mathscr{N}^{\otimes 2}} \otimes q, \mathscr{N}^{\otimes 2} \otimes \mathscr{L})$ ranging over all line bundles $\mathscr{N}$ on S. Equivalently, this is the set of isometry classes $(\mathscr{N} \otimes \mathscr{E}, \phi \circ (\mathrm{id}_{\mathscr{N}^{\otimes 2}} \otimes q), \mathscr{L}')$ ranging over all isomorphisms $\phi : \mathscr{N}^{\otimes 2} \otimes \mathscr{L} \to \mathscr{L}'$ of line bundles on S. This is referred to as a *lax-similarity* class in [10]. The main result of this section shows that projectively similar quadratic forms yield isomorphic quadric bundles, while the converse holds under further hypotheses.
Let η be the generic point of S and $\pi : Q \to S$ a quadric bundle. Restriction to the generic fiber of π gives rise to a complex

$$0 \to \mathrm{Pic}(S) \xrightarrow{\pi^*} \mathrm{Pic}(Q) \to \mathrm{Pic}(Q_\eta) \to 0 \tag{1}$$

whose exactness we will study in Proposition 1.6 below.

Proposition 1.1. *Let $\pi : Q \to S$ and $\pi' : Q' \to S$ be quadric bundles associated to quadratic forms $(\mathscr{E}, q, \mathscr{L})$ and $(\mathscr{E}', q', \mathscr{L}')$. If $(\mathscr{E}, q, \mathscr{L})$ and $(\mathscr{E}', q', \mathscr{L}')$ are in the same projective similarity class then Q and Q' are S-isomorphic. The converse holds if q is assumed to be generically regular and (1) is assumed to be exact in the middle.*

Proof. Assume that $(\mathscr{E}, q, \mathscr{L})$ and $(\mathscr{E}', q', \mathscr{L}')$ are projectively similar with respect to an invertible $\mathscr{O}_S$-module $\mathscr{N}$ and $\mathscr{O}_S$-module isomorphisms $\varphi : \mathscr{E}' \to \mathscr{N} \otimes \mathscr{E}$ and $\lambda : \mathscr{L}' \to \mathscr{N}^{\otimes 2} \otimes \mathscr{L}$ preserving the quadratic forms. Let $p : \mathbb{P}(\mathscr{E}) \to S$ and $p' : \mathbb{P}(\mathscr{E}') \to S$ be the associated projective bundles and $h : \mathbb{P}(\mathscr{E}') \to \mathbb{P}(\mathscr{N} \otimes \mathscr{E})$ the S-isomorphism associated to $\varphi^\vee$. There is a natural S-isomorphism $g : \mathbb{P}(\mathscr{N} \otimes \mathscr{E}) \to \mathbb{P}(\mathscr{E})$ satisfying $g^* \mathscr{O}_{\mathbb{P}(\mathscr{E})/S}(1) \cong \mathscr{O}_{\mathbb{P}(\mathscr{E} \otimes \mathscr{N})/S}(1) \otimes p'^* \mathscr{N}$, see [34, II Lemma 7.9]. Denote by $f = g \circ h : \mathbb{P}(\mathscr{E}') \to \mathbb{P}(\mathscr{E})$ the composition. Then via the isomorphism

$$\Gamma\big(\mathbb{P}(\mathscr{E}'), f^*(\mathscr{O}_{\mathbb{P}(\mathscr{E})/S}(2) \otimes p^* \mathscr{L})\big) \to \Gamma\big(\mathbb{P}(\mathscr{E}'), \mathscr{O}_{\mathbb{P}(\mathscr{E}')/S}(2) \otimes p'^* \mathscr{L}'\big)$$

induced by $f^* \mathscr{O}_{\mathbb{P}(\mathscr{E})/S}(2) \cong \mathscr{O}_{\mathbb{P}(\mathscr{E}')}(2) \otimes (p'^* \mathscr{N})^{\otimes 2}$ and $p'^* \lambda^{-1} : (p'^* \mathscr{N})^{\otimes 2} \otimes p'^* \mathscr{L} \to p'^* \mathscr{L}'$, the global section $f^* s_q$ is taken to the global section $s_{q'}$, hence f restricts to a S-isomorphism $Q' \to Q$. The proof of the first claim is complete.
Now assume that $(\mathscr{E}, q, \mathscr{L})$ is generically regular and that $f : Q' \to Q$ is an S-isomorphism. First, we will prove that f can be extended to a S-isomorphism $\tilde{f} : \mathbb{P}(\mathscr{E}') \to \mathbb{P}(\mathscr{E})$ satisfying $\tilde{f} \circ j' = j \circ f$. To this end, considering the long exact sequence associated to applying p_* to the short exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}(\mathscr{E})/S}(-1) \otimes p^* \mathscr{L}^\vee \xrightarrow{s_q} \mathscr{O}_{\mathbb{P}(\mathscr{E})/S}(1) \to j_* \mathscr{O}_{Q/S}(1) \to 0. \tag{2}$$

and keeping in mind that $R^i p_* \mathscr{O}_{\mathbb{P}(\mathscr{E})/S}(-1) = 0$ for $i = 0, 1$, we arrive at an isomorphism $p_* \mathscr{O}_{\mathbb{P}(\mathscr{E})/S}(1) \cong \pi_* \mathscr{O}_{Q/S}(1)$. In particular, we have a canonical identification $\mathscr{E}^\vee = \pi_* \mathscr{O}_{Q/S}(1)$. We have a similar identification $\mathscr{E}'^\vee = \pi'_* \mathscr{O}_{Q'/S}(1)$.

We claim that $f^*\mathscr{O}_{Q/S}(1) \cong \mathscr{O}_{Q'/S}(1) \otimes \pi'^*\mathscr{N}$ for some line bundle $\mathscr{N}$ on S. Indeed, over the generic fiber, we have $f^*\mathscr{O}_{Q/S}(1)_\eta = f_\eta^*\mathscr{O}_{Q_\eta}(1) \cong \mathscr{O}_{Q'_\eta}(1)$ by the case of smooth quadrics (as q is generically regular) over a field, cf. [28, Lemma 69.2]. Then the exactness of (1) in the middle finishes the proof of the present claim.

Finally, by the projection formula and our assumption that $\pi' : Q' \to S$ is of positive relative dimension, we have that f induces an $\mathscr{O}_S$-module isomorphism

$$\begin{aligned} \mathscr{E}^\vee \otimes \mathscr{N}^\vee &\cong \pi_*\mathscr{O}_{Q/S}(1) \otimes \pi'_*\pi'^*\mathscr{N}^\vee \\ &\cong \pi_* f_*(f^*\mathscr{O}_{Q/S}(1) \otimes \pi'^*\mathscr{N}^\vee) \cong \pi'_*\mathscr{O}_{Q'/S}(1) = \mathscr{E}'^\vee \end{aligned}$$

with induced dual isomorphism $\varphi : \mathscr{E}' \to \mathscr{N} \otimes \mathscr{E}$. Now define $\tilde{f} : \mathbb{P}(\mathscr{E}') \to \mathbb{P}(\mathscr{E})$ to be the composition of the morphism $\mathbb{P}(\mathscr{E}') \to \mathbb{P}(\mathscr{N} \otimes \mathscr{E})$ defined by $\varphi^\vee$ with the natural S-isomorphism $\mathbb{P}(\mathscr{N} \otimes \mathscr{E}) \to \mathbb{P}(\mathscr{E})$, as earlier in this proof. Then by the construction of $\tilde{f}$, we have that $\tilde{f}^*\mathscr{O}_{\mathbb{P}(\mathscr{E})/S}(1) \cong \mathscr{O}_{\mathbb{P}(\mathscr{E}')/S}(1) \otimes p'^*\mathscr{N}$ and that $j \circ f = \tilde{f} \circ j'$ (an equality that is checked on fibers using [28, Thm. 69.3]). Equivalently, there exists an isomorphism $\tilde{f}^*(\mathscr{O}_{\mathbb{P}(\mathscr{E})(2)/S} \otimes p^*\mathscr{L}) \cong \mathscr{O}_{\mathbb{P}(\mathscr{E}')/S}(2) \otimes p'^*\mathscr{L}'$ taking f^*s_q to $s_{q'}$. However, as $\tilde{f}^*(\mathscr{O}_{\mathbb{P}(\mathscr{E})(2)/S} \otimes p^*\mathscr{L}) \cong \mathscr{O}_{\mathbb{P}(\mathscr{E}')/S}(2) \otimes p'^*(\mathscr{N}^{\otimes 2} \otimes \mathscr{L}')$, we have an isomorphism $p'^*\mathscr{L}' \cong p'^*(\mathscr{N}^{\otimes 2} \otimes \mathscr{L})$. Upon taking pushforward, we arrive at an isomorphism $\lambda : \mathscr{L}' \to \mathscr{N}^{\otimes 2} \otimes \mathscr{L}$. By the construction of φ and λ, it follows that (φ, λ) is a similarity $(\mathscr{E}, q, \mathscr{L}) \to (\mathscr{E}', q', \mathscr{L}')$, proving the converse. □

Definition 1.2. The determinant $\det \psi_q : \det \mathscr{E} \to \det \mathscr{E}^\vee \otimes \mathscr{L}^{\otimes n}$ gives rise to a global section of $(\det \mathscr{E}^\vee)^{\otimes 2} \otimes \mathscr{L}^{\otimes n}$, whose divisor of zeros is called the *discriminant divisor* D. The reduced subscheme associated to D is precisely the locus of points where the radical of q is nontrivial. If q is generically regular, then $D \subset S$ is closed of codimension one.

Definition 1.3. We say that a quadratic form $(\mathscr{E}, q, \mathscr{L})$ has *simple degeneration* if

$$\mathrm{rk}_{\kappa(x)} \operatorname{rad}(\mathscr{E}_x, q_x, \mathscr{L}_x) \leq 1$$

for every point x of S, where $\kappa(x)$ is the residue field of $\mathscr{O}_{S,x}$.

Our first lemma concerns the local structure of simple degeneration.

Lemma 1.4. *Let $(\mathscr{E}, q)$ be a quadratic form with simple degeneration over the spectrum of a local ring R with 2 invertible. Then $(\mathscr{E}, q) \cong (\mathscr{E}_1, q_1) \perp (R, <\pi>)$ where $(\mathscr{E}_1, q_1)$ is regular and $\pi \in R$.*

Proof. Over the residue field k, the form $(\mathscr{E}, q)$ has a regular subform $(\overline{\mathscr{E}}_1, \overline{q}_1)$ of corank one, which can be lifted to a regular orthogonal direct summand $(\mathscr{E}_1, q_1)$ of corank 1 of $(\mathscr{E}, q)$, cf. [8, Cor. 3.4]. This gives the required decomposition. Moreover, we can lift a diagonalization $\overline{q}_1 \cong <\overline{u}_1, \ldots, \overline{u}_{n-1}>$ with $\overline{u}_i \in k^\times$, to a diagonalization

$$q \cong <u_1, \ldots, u_{n-1}, \pi>,$$

with $u_i \in R^\times$ and $\pi \in R$. □

Let $D \subset S$ be a regular divisor. Assuming that S is normal, the local ring $\mathscr{O}_{S,D'}$ at the generic point of a component D' of D is a discrete valuation ring. When 2 is invertible on S, Lemma 1.4 shows that a quadratic form $(\mathscr{E}, q, \mathscr{L})$ with simple degeneration along D can be diagonalized over $\mathscr{O}_{S,D'}$ as

$$q \cong < u_1, \dots, u_{r-1}, u_r \pi^e >$$

where u_i are units and π is a parameter of $\mathscr{O}_{S,D'}$. We call $e \geq 1$ the *multiplicity* of the simple degeneration along D'. If e is even for every component of D, then there is a birational morphism $g : S' \to S$ such that the pullback of $(\mathscr{E}, q, \mathscr{L})$ to S' is regular. We will focus on quadratic forms with simple degeneration of multiplicity one along (all components of) D.
We can give a geometric interpretation of simple degeneration.

Proposition 1.5. *Let $\pi : Q \to S$ be the quadric bundle associated to a generically regular quadratic form $(\mathscr{E}, q, \mathscr{L})$ over S and $D \subset S$ its discriminant divisor. Then:*

a) q has simple degeneration if and only if the fiber Q_x of its associated quadric bundle has at worst isolated singularities for each closed point x of S;

b) if 2 is invertible on S and D is reduced, then any simple degeneration along D has multiplicity one;

c) if 2 is invertible on S and D is regular, then any degeneration along D is simple of multiplicity one;

d) if S is regular and q has simple degeneration, then D is regular if and only if Q is regular.

Proof. The first claim follows from the classical geometry of quadrics over a field: the quadric of a nondegenerate form is smooth while the quadric of a form with nontrivial radical has isolated singularity if and only if the radical has rank one. As for the second claim, the multiplicity of the simple degeneration is exactly the scheme-theoretic multiplicity of the divisor D. For the third claim, see [20, §3], [37, Rem. 7.1], or [4, Rem. 2.6]. The final claim is standard, cf. [11, I Prop. 1.2(iii)], [37, Lemma 5.2], or [4, Prop. 1.2.5]. □

We do not need the full flexibility of the following general result, but we include it for completeness.

Proposition 1.6. *Let $\pi : Q \to S$ be a flat proper separated morphism with geometrically integral fibers between noetherian integral separated locally factorial schemes and let η be the generic point of S. Then the complex of Picard groups (1) is exact.*

Proof. First, we argue that flat pullback and restriction to the generic fiber give rise to an exact sequence of Weil divisor groups

$$0 \to \mathrm{Div}(S) \xrightarrow{\pi^*} \mathrm{Div}(Q) \to \mathrm{Div}(Q_\eta) \to 0. \tag{3}$$

Indeed, as $\mathrm{Div}(Q_\eta) = \varinjlim \mathrm{Div}(Q_U)$, where the limit is taken over all dense open sets $U \subset S$ and we write $Q_U = Q \times_S U$, the exactness at right of sequence (3) then follows from the exactness of the excision sequence

$$Z^0(\pi^{-1}(S \smallsetminus U)) \to \mathrm{Div}(Q) \to \mathrm{Div}(Q_U) \to 0$$

cf. [30, 1 Prop. 1.8]. The sequence (3) is exact at left since π is surjective on codimension 1 points, providing a retraction of π^*. As for exactness in the middle, if a prime Weil divisor T on Q has trivial generic fiber then it is supported on the fibers over a closed subscheme of S not containing η. Since the fibers of π are irreducible, T must coincide with $\pi^{-1}(Z)$ for some prime Weil divisor Z of S. Thus T is in the image of π^*.
Second, we argue that there is an analogous exact sequence of principal Weil divisor groups

$$0 \to \mathrm{PDiv}(S) \xrightarrow{\pi^*} \mathrm{PDiv}(Q) \to \mathrm{PDiv}(Q_\eta) \to 0. \tag{4}$$

Indeed, since π is dominant, it induces an extension of function fields K_Q over K_S, and hence a well defined π^* on principal divisors, which is injective. Since $K_Q = K_{Q_\eta}$, restriction to the generic point is surjective on principal divisors. For the exactness in the middle, if $\mathrm{div}_Q(f)_\eta = 0$ then $f \in \Gamma(Q_\eta, \mathscr{O}_{Q_\eta}^\times)$, i.e., f has neither zeros nor poles on Q_η. Since Q_η is a proper geometrically integral K_S-scheme, $\Gamma(Q_\eta, \mathscr{O}_{Q_\eta}^\times) = K_S^\times$, and hence $f \in K_S^\times$. Thus $\mathrm{div}_Q(f)$ is in the image of π^*.
The snake lemma then induces an exact sequence of Weil divisor class groups

$$0 \to \mathrm{Cl}(S) \xrightarrow{\pi^*} \mathrm{Cl}(Q) \to \mathrm{Cl}(Q_\eta) \to 0.$$

As π is separated with geometrically integral fibers, Q_η is separated and integral. As Q is a noetherian locally factorial scheme, Q_η is as well. Hence all Weil divisor class groups coincide with Picard groups by [32, Cor. 21.6.10], immediately implying that the complex (1) is exact. ∎

Corollary 1.7. *Let S be a regular integral scheme with 2 invertible and $(\mathscr{E}, q, \mathscr{L})$ a quadratic form on S of rank ≥ 4 having at most simple degeneration along a regular divisor $D \subset S$. Let $\pi : Q \to S$ be the associated quadric bundle. Then the complex* (1) *is exact.*

Proof. First, recall that a quadratic form over a field contains a nondegenerate subform of rank ≥ 3 if and only if its associated quadric is irreducible, cf. [34, I Ex. 5.12]. Hence the fibers of π are geometrically irreducible. By Proposition 1.5, Q is regular. Quadratic forms with simple degeneration are primitive, hence π is flat. Thus we can apply all the parts of Proposition 1.6. ∎

We will define $\mathrm{Quad}_n^D(S)$ to be the set of projective similarity classes of line bundle-valued quadratic forms of rank n on S with simple degeneration of multiplicity one along an effective Cartier divisor D. An immediate consequence of Propositions 1.1 and 1.5 and Corollary 1.7 is the following.

Corollary 1.8. *For $n \geq 4$ and D reduced, the set $\mathrm{Quad}_n^D(S)$ is in bijection with the set of S-isomorphism classes of quadric bundles of relative dimension $n-2$ with isolated singularities in the fibers above D.*

Definition 1.9. Let $(\mathscr{E}, q, \mathscr{L})$ be a quadratic form of rank n on a scheme S, and $\mathscr{C}_0 = \mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$ be its even Clifford algebra (see [15] or [3, §1.8]), and $\mathscr{Z} = \mathscr{Z}(\mathscr{E}, q, \mathscr{L})$ be its center. Then $\mathscr{C}_0$ is a locally free $\mathscr{O}_S$-algebra of rank 2^{n-1}, cf. [40, IV.1.6]. The associated finite morphism $f : T \to S$ is called the *discriminant cover*. We remark that if S is locally factorial and q is generically regular of even rank then $\mathscr{Z}$ is a locally free $\mathscr{O}_S$-algebra of rank two, by (the remarks preceding) [40, IV Prop. 4.8.3], hence the discriminant cover $f : T \to S$ is finite flat of degree two. Below, we will arrive at the same conclusion under weaker hypotheses on S but assuming that q has simple degeneration.

Lemma 1.10 ([4, App. B]). *Let $(\mathscr{E}, q, \mathscr{L})$ be a quadratic form of even rank with simple degeneration of multiplicity one along $D \subset S$ and $f : T \to S$ its discriminant cover. Then $f^*\mathscr{O}(D)$ is a square in $\mathrm{Pic}(T)$ and the branch divisor of f is precisely D.*

By abuse of notation, we also denote by $\mathscr{C}_0 = \mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$ the $\mathscr{O}_T$-algebra associated to the $\mathscr{Z}$-algebra $\mathscr{C}_0 = \mathscr{C}(\mathscr{E}, q, \mathscr{L})$. The center $\mathscr{Z}$ is an étale algebra over every point of S where $(\mathscr{E}, q, \mathscr{L})$ is regular and $\mathscr{C}_0$ is an Azumaya algebra over every point of T lying over a point of S where $(\mathscr{E}, q, \mathscr{L})$ is regular. Now we prove [44, Prop. 3.13] over any integral scheme.

Proposition 1.11. *Let $(\mathscr{E}, q, \mathscr{L})$ be a quadratic form of even rank with simple degeneration over a scheme S with 2 invertible. Then the discriminant cover $T \to S$ is finite flat of degree two and $\mathscr{C}_0$ is an Azumaya $\mathscr{O}_T$-algebra.*

Proof. The desired properties are local for the étale topology, so we can assume that $S = \mathrm{Spec}\, R$ for a local ring R with 2 invertible, we can fix a trivialization of $\mathscr{L}$, and by Lemma 1.4 we can write $(\mathscr{E}, q) \cong (\mathscr{E}_1, q_1) \perp <\pi>$ with $\pi \in R$ (not necessarily nonzero) and $(\mathscr{E}_1, q_1) \cong <1, -1, \ldots, 1, -1, 1>$ a standard split quadratic form of odd rank. We have that $\mathscr{C}_0(\mathscr{E}_1, q_1)$ is (split) Azumaya over $\mathscr{O}_S$ and that $\mathscr{C}(<-\pi>)$ is $\mathscr{O}_S$-isomorphic to $\mathscr{Z}(\mathscr{E}, q)$.
Since $\mathscr{C}(<-\pi>) \cong R[\sqrt{-\pi}]$ is finite flat of degree two over S, the first claim is verified. For the second claim, by [40, IV Prop. 7.3.1], there are then $\mathscr{O}_S$-algebra isomorphisms

$$\mathscr{C}_0(\mathscr{E}, q) \cong \mathscr{C}_0(\mathscr{E}_1, q_1) \otimes_{\mathscr{O}_S} \mathscr{C}(<-\pi>) \cong \mathscr{C}_0(\mathscr{E}_1, q_1) \otimes_{\mathscr{O}_S} \mathscr{Z}(\mathscr{E}, q). \tag{5}$$

Thus étale locally, $\mathscr{C}_0(\mathscr{E}, q)$ is the base extension to $\mathscr{Z}(\mathscr{E}, q)$ of an Azumaya algebra over $\mathscr{O}_S$, hence can be regarded as an Azumaya $\mathscr{O}_T$-algebra. □

Over a field, we can now provide a strengthened version of [28, Prop. 11.6].

Proposition 1.12. *Let (V,q) be a quadratic form of even rank $n=2m$ over a field k of characteristic $\neq 2$. If $n \geq 4$ then the following are equivalent:*

a) The radical of q has rank at most 1.

b) The center $Z(q) \subset C_0(q)$ is a k-algebra of rank 2.

c) The algebra $C_0(q)$ is $Z(q)$-Azumaya of degree 2^{m-1}.

If $n=2$, then $C_0(q)$ is always commutative.

Proof. If q is nondegenerate (i.e., has trivial radical), then it is classical that $Z(q)$ is an étale quadratic algebra and $C_0(q)$ is an Azumaya $Z(q)$-algebra. If $\mathrm{rad}(q)$ has rank 1, generated by $v \in V$, then a straightforward computation shows that $Z(q) \cong k[\varepsilon]/(\epsilon^2)$, where $\epsilon \in vC_1(q) \cap Z(q) \smallsetminus k$. Furthermore, we have that $C_0(q) \otimes_{k[\varepsilon]/(\varepsilon^2)} k \cong C_0(q)/vC_1(q) \cong C_0(q/\mathrm{rad}(q))$ where $q/\mathrm{rad}(q)$ is nondegenerate of rank $n-1$, cf. [28, II §11, p. 58]. Proposition 1.11 implies that $C_0(q)$ is $Z(q)$-Azumaya of degree 2^{m-1}, proving *a)* $\Rightarrow$ *c)*
The fact that *c)* $\Rightarrow$ *b)* is clear from a dimension count. To prove *b)* $\Rightarrow$ *a)*, suppose that $\mathrm{rk}_k \,\mathrm{rad}(q) \geq 2$. In this case, the embedding $\bigwedge^2 \mathrm{rad}(q) \subset C_0(q)$ is central (and does not contain the central subalgebra generated by $V^{\otimes n}$, as q has rank > 2). More explicitly, if $e_1, e_2, \dots, e_n$ is an orthogonal basis of (V,q), then $k \oplus ke_1 \cdots e_n \oplus \bigwedge^2 \mathrm{rad}(q) \subset Z(q)$. Thus $Z(q)$ has k-rank at least $2 + \mathrm{rk}_k \bigwedge^2 \mathrm{rad}(q) \geq 3$. □

Finally, as a corollary of Proposition 1.12, we can deduce a converse to Proposition 1.11.

Proposition 1.13. *Let $(\mathscr{E}, q, \mathscr{L})$ be a quadratic form of even rank on an integral scheme S with discriminant cover $f : T \to S$. Then $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$ is an Azumaya $\mathscr{O}_T$-algebra if and only if $(\mathscr{E}, q, \mathscr{L})$ has simple degeneration or has rank 2 (and any degeneration).*

2 Orthogonal groups with simple degeneration

The main results of this section concern the special (projective) orthogonal group schemes of quadratic forms with simple degeneration over semilocal principal ideal domains.
Let S be a regular integral scheme. Recall, from Proposition 1.11, that if $(\mathscr{E}, q, \mathscr{L})$ is a line bundle-valued quadratic form on S with simple degeneration along a closed subscheme D of codimension 1, then the even Clifford algebra $\mathscr{C}_0(q)$ is an Azumaya algebra over the discriminant cover $T \to S$. The main result of this section is the following.

Theorem 2.1. *Let S be a regular scheme with 2 invertible, D a regular divisor, $(\mathscr{E}, q, \mathscr{L})$ a quadratic form of rank 4 on S with simple degeneration along D, $T \to S$ its discriminant cover, and $\mathscr{C}_0(q)$ its even Clifford algebra over T. The canonical homomorphism*

$$c : \mathbf{PGO}^+(q) \to R_{T/S}\mathbf{PGL}(\mathscr{C}_0(q)),$$

induced from the functor $\mathscr{C}_0$, is an isomorphism of S-group schemes.

The proof involves several preliminary general results concerning orthogonal groups of quadratic forms with simple degeneration and will occupy the remainder of this section.

Let $S = \operatorname{Spec} R$ be an affine scheme with 2 invertible, $D \subset S$ be the closed scheme defined by an element π in the Jacobson radical of R, and let $(V, q) = (V_1, q_1) \perp (R, <\pi>)$ be a quadratic form of rank n over S with q_1 regular and V_1 free. Let Q_1 be a Gram matrix of q_1. Then as an S-group scheme, $\mathbf{O}(q)$ is the subvariety of the affine space of block matrices

$$\begin{pmatrix} A & v \\ w & u \end{pmatrix} \quad \text{satisfying} \quad \begin{array}{l} A^t Q_1 A + \pi\, w^t w = Q_1 \\ A^t Q_1 v + u\pi\, w^t = 0 \\ v^t Q_1 v = (1-u^2)\pi \end{array} \tag{6}$$

where A is an invertible $(n-1)\times(n-1)$ matrix, v is an $n \times 1$ column vector, w is a $1 \times n$ row vector, and u a unit. Note that since A and Q_1 are invertible, the second relation in (6) implies that v is determined by w and u and that $\bar{v} = 0$ over R/π. In particular, if $\pi \neq 0$ and R is a domain then the third relation implies that $\bar{u}^2 = 1$ in R/π. Define $\mathbf{O}^+(q) = \ker(\det : \mathbf{O}(q) \to \mathbf{G}_{\mathrm{m}})$. If R is an integral domain then det factors through $\boldsymbol{\mu}_2$ and $\mathbf{O}^+(q)$ is the irreducible component of the identity.

Proposition 2.2. *Let R be a regular local ring with 2 invertible, $\pi \in \mathfrak{m}$ a nonzero element in the maximal ideal, and $(V, q) = (V_1, q_1) \perp (R, <\pi>)$ a quadratic form with q_1 regular of rank $n-1$ of R. Then $\mathbf{O}(q)$ and $\mathbf{O}^+(q)$ are smooth R-group schemes.*

Proof. Let K be the fraction field of R and k its residue field. First, we'll show that the equations in (6) define a local complete intersection morphism in the affine space $\mathbb{A}_R^{n^2}$ of $n \times n$ matrices over R. Indeed, the condition that the generic $n \times n$ matrix M over $R[x_1, \dots, x_{n^2}]$ is orthogonal with respect to a given symmetric $n \times n$ matrix Q over R can be written as the equality of symmetric matrices $M^t Q M = Q$ over $R[x_1, \dots, x_{n^2}][(\det M)^{-1}]$, hence giving $n(n+1)/2$ equations. Hence, the orthogonal group is the scheme defined by these $n(n+1)/2$ equations in the Zariski open of $\mathbb{A}_R^{n^2}$ defined by $\det M$.
Since q is generically regular of rank n, the generic fiber of $\mathbf{O}(q)$ has dimension $n(n-1)/2$. By (6), the special fiber of $\mathbf{O}^+(q)$ is isomorphic to the group scheme of rigid motions of the regular quadratic space (V_1, q_1), which is the semidirect product

$$\mathbf{O}^+(q) \times_R k \cong \mathbf{G}_{\mathrm{a}}^{n-1} \rtimes \mathbf{O}(q_{1,k}) \tag{7}$$

where $\mathbf{G}_{\mathrm{a}}^{n-1}$ acts in V_1 by translation and $\mathbf{O}(q_{1,k})$ acts on $\mathbf{G}_{\mathrm{a}}^{n-1}$ by conjugation. In particular, the special fiber of $\mathbf{O}^+(q)$ has dimension $(n-1)(n-2)/2+(n-1) = n(n-1)/2$, and similarly with $\mathbf{O}(q)$.

In particular, $\mathbf{O}(q)$ is a local complete intersection morphism. Since R is Cohen–Macaulay (being regular local) then $R[x_1,\dots,x_{n^2}][(\det M)^{-1}]$ is Cohen–Macaulay, and thus $\mathbf{O}(q)$ is Cohen–Macaulay. By the "miracle flatness" theorem, equidimensional and Cohen–Macaulay over a regular base implies that $\mathbf{O}(q) \to \operatorname{Spec} R$ is flat, cf. [32, Prop. 15.4.2] or [46, 8 Thm. 23.1]. Also $\mathbf{O}^+(q) \to \operatorname{Spec} R$ is flat. The generic fiber of $\mathbf{O}^+(q)$ is smooth since q is generically regular while the special fiber is smooth since it is a (semi)direct product of smooth schemes (recall that $\mathbf{O}(q_1)$ is smooth since 2 is invertible). Hence $\mathbf{O}^+(q) \to \operatorname{Spec} R$ is flat and has geometrically smooth fibers, hence is smooth. □

Proposition 2.3. *Let S be a regular scheme with 2 invertible and $(\mathscr{E}, q, \mathscr{L})$ a quadratic form of even rank on S with simple degeneration. Then the group schemes $\mathbf{O}(q)$, $\mathbf{O}^+(q)$, $\mathbf{GO}(q)$, $\mathbf{GO}^+(q)$, $\mathbf{PGO}(q)$, and $\mathbf{PGO}^+(q)$ are S-smooth. If $T \to S$ is the discriminant cover and $\mathscr{C}_0(q)$ is the even Clifford algebra of $(\mathscr{E}, q, \mathscr{L})$ over T, then $R_{T/S}\mathbf{GL}_1(\mathscr{C}_0(q))$, $R_{T/S}\mathbf{SL}_1(\mathscr{C}_0(q))$, and $R_{T/S}\mathbf{PGL}_1(\mathscr{C}_0(q))$ are smooth S-schemes.*

Proof. The S-smoothness of $\mathbf{O}(q)$ and $\mathbf{O}^+(q)$ follows from the fibral criterion for smoothness, with Proposition 2.2 handling points of S contained in the discriminant divisor. As $\mathbf{GO} \cong (\mathbf{O}(q)\times\mathbf{G}_{\mathrm{m}})/\boldsymbol{\mu}_2$, $\mathbf{GO}^+(q) \cong (\mathbf{O}^+(q)\times\mathbf{G}_{\mathrm{m}})/\boldsymbol{\mu}_2$, $\mathbf{PGO}(q) \cong \mathbf{GO}(q)/\mathbf{G}_{\mathrm{m}}$, $\mathbf{PGO}^+(q) \cong \mathbf{GO}^+(q)/\mathbf{G}_{\mathrm{m}}$ are quotients of S-smooth group schemes by flat closed subgroups, they are S-smooth. Finally, $\mathscr{C}_0(q)$ is an Azumaya $\mathscr{O}_T$-algebra by Proposition 1.11, hence $\mathbf{GL}_1(\mathscr{C}_0(q))$, $\mathbf{SL}_1(\mathscr{C}_0(q))$, and $\mathbf{PGL}_1(\mathscr{C}_0(q))$ are smooth T-schemes, hence their Weil restrictions via the finite flat map $T \to S$ are S-smooth by [21, App. A.5, Prop. A.5.2]. □

Remark 2.4. If the radical of q_s has rank ≥ 2 at a point s of S, a calculation shows that the fiber of $\mathbf{O}(q) \to S$ over s has dimension $> n(n-1)/2$. In particular, if q is generically regular over S then $\mathbf{O}(q) \to S$ is not flat. The smoothness of $\mathbf{O}(q)$ is a special feature of quadratic forms q with simple degeneration. Over a complete discretely valued ring, such $\mathbf{O}^+(q)$ can be viewed as an explicit model for one of the quasisplit Bruhat–Tits groups of type ${}^2\mathsf{D}_m$.

We will also make frequent reference to the classical version of Theorem 2.1 in the regular case, when the discriminant cover is étale.

Theorem 2.5. *Let S be a scheme and $(\mathscr{E}, q, \mathscr{L})$ a regular quadratic form of rank 4 with discriminant cover $T \to S$ and even Clifford algebra $\mathscr{C}_0(q)$ over T. The canonical homomorphism*

$$c : \mathbf{PGO}^+(q) \to R_{T/S}\mathbf{PGL}_1(\mathscr{C}_0(q)),$$

induced from the functor $\mathscr{C}_0$, is an isomorphism of S-group schemes.

Proof. The proof over affine schemes S in [43, §10] carries over immediately. See [41, IV.15.B] for the particular case of S the spectrum of a field. Also see [3, §5.3]. $\square$

Finally, we come to the proof of the main result of this section.

Proof of Theorem 2.1. We will use the following fibral criteria for relative isomorphisms (cf. [32, IV.4 Cor. 17.9.5]): let $g : X \to Y$ be a morphism of S-schemes locally of finite presentation over a scheme S and assume X is S-flat, then g is an S-isomorphism if and only if its fiber $g_s : X_s \to Y_s$ is an isomorphism over each geometric point s of S.
For each s in $S \smallsetminus D$, the fiber q_s is a regular quadratic form over $\kappa(s)$, hence the fiber $c_s : \mathbf{PGO}^+(q_s) \to R_{T/S}\mathbf{PGL}(\mathscr{C}_0(q_s))$ is an isomorphism by Theorem 2.5. We are thus reduced to considering the geometric fibers over points in D. Let $s = \operatorname{Spec} k$ be a geometric point of D. By Proposition 1.12, there is a natural identification of the fiber $T_s = \operatorname{Spec} k_\epsilon$, where $k_\epsilon = k[\epsilon]/(\epsilon^2)$.
We use the following criteria for isomorphisms of group schemes (cf. [41, VI Prop. 22.5]): let $g : X \to Y$ be a homomorphism of affine k-group schemes of finite type over an algebraically closed field k and assume that Y is smooth, then g is a k-isomorphism if and only if $g : X(k) \to Y(k)$ is an isomorphism on k-points and the Lie algebra map $\mathrm{d}g : \mathrm{Lie}(X) \to \mathrm{Lie}(Y)$ is an injective map of k-vector spaces.
First, we shall prove that c is an isomorphism on k points. Applying cohomology to the exact sequence

$$1 \to \boldsymbol{\mu}_2 \to \mathbf{O}^+(q) \to \mathbf{PGO}^+(q) \to 1,$$

we see that the corresponding sequence of k-points is exact since k is algebraically closed. Hence it suffices to show that $\mathbf{O}^+(q)(k) \to \mathbf{PGL}_1(\mathscr{C}_0(q))(k)$ is surjective with kernel $\boldsymbol{\mu}_2(k)$.
Write $q = q_1 \perp < 0 >$, where q_1 is regular over k. Denote by $\mathbf{E}$ the unipotent radical of $\mathbf{O}^+(q)$. We will now proceed to define the following diagram

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
 & & \boldsymbol{\mu}_2 \arrow[r, equal] \arrow[d] & \boldsymbol{\mu}_2 \arrow[d] & \\
1 \arrow[r] & \mathbf{E} \arrow[r] \arrow[d, dotted] & \mathbf{O}^+(q) \arrow[r] \arrow[d] & \mathbf{O}(q_1) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & I + \epsilon\, \mathfrak{c}_0(q) \arrow[r] & \mathbf{PGL}_1(\mathscr{C}_0(q)) \arrow[r] \arrow[d] & \mathbf{PGL}_1(\mathscr{C}_0(q_1)) \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 &
\end{tikzcd}$$

of groups schemes over k, and verify that it is commutative with exact rows and columns. This will finish the proof of the statement concerning c being an isomorphism on k-points. We have $H^1_{\text{ét}}(k,\mathbf{E})=0$ and also $H^1_{\text{ét}}(k,\boldsymbol{\mu}_2)=0$, as k is algebraically closed. Hence it suffices to argue after taking k-points in the diagram.

The central and right most vertical columns are induced by the standard action of the (special) orthogonal group on the even Clifford algebra. The right most column is an exact sequence

$$1 \to \boldsymbol{\mu}_2 \to \mathbf{O}(q_1) \cong \boldsymbol{\mu}_2 \times \mathbf{O}^+(q_1) \to \mathbf{PGL}_1(\mathscr{C}_0(q_1)) \to 1$$

arising from the split isogeny of type $\mathsf{A}_1 = \mathsf{B}_1$, cf. [41, IV.15.A]. The central row is defined by the map $\mathbf{O}^+(q)(k) \to \mathbf{O}(q_1)(k)$ defined by

$$\begin{pmatrix} A & v \\ w & u \end{pmatrix} \mapsto A$$

in the notation of (6). In particular, the group $\mathbf{E}(k)$ consists of block matrices of the form

$$\begin{pmatrix} I & 0 \\ w & 1 \end{pmatrix}$$

for $w \in \mathbb{A}^3(k)$. Since $\mathbf{O}(q_1)$ is semisimple, the kernel contains the unipotent radical $\mathbf{E}$, so coincides with it by a dimension count. The bottom row is defined as follows. By (5), we have $\mathscr{C}_0(q) \cong \mathscr{C}_0(q_1) \otimes_k \mathscr{Z}(q) \cong \mathscr{C}_0(q_1) \otimes_k k_\epsilon$. The map $\mathbf{PGL}_1(\mathscr{C}_0(q)) \to \mathbf{PGL}_1(\mathscr{C}_0(q_1)$ is thus defined by the reduction $k_\epsilon \to k$. This also identifies the kernel as $I + \epsilon\, \mathfrak{c}_0(q)$, where $\mathfrak{c}_0(q)$ is the affine scheme of reduced trace zero elements of $\mathscr{C}_0(q)$, which is identified with the Lie algebra of $\mathbf{PGL}_1(\mathscr{C}_0(q))$ in the usual way. The only thing to check is that the bottom left square commutes (since by (7), the central row is split). By the five lemma, it will then suffice to show that $\mathbf{E}(k) \to 1 + \epsilon\, \mathfrak{c}_0(q)(k)$ is an isomorphism.

To this end, we can diagonalize $q =< 1, -1, 1, 0 >$, since k is algebraically closed of characteristic $\neq 2$. Let $e_1, \dots, e_4$ be the corresponding orthogonal basis. Then $\mathscr{C}_0(q_1)(k)$ is generated over k by 1, e_1e_2, e_2e_3, and e_1e_3 and we have an identification $\varphi : \mathscr{C}_0(q_1)(k) \to M_2(k)$ given by

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1e_2 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2e_3 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_1e_3 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Similarly, $\mathscr{C}_0(q)$ is generated over $\mathscr{Z}(q) = k_\epsilon$ by 1, e_1e_2, e_2e_3, and e_1e_3, since we have

$$e_1e_4 = \epsilon\, e_2e_3, \quad e_2e_4 = \epsilon\, e_1e_3, \quad e_3e_4 = \epsilon\, e_1e_2, \quad e_1e_2e_3e_4 = \epsilon.$$

and we have an identification $\psi : \mathscr{C}_0(q) \to M_2(k_\epsilon)$ extending φ. With respect to this k_ϵ-algebra isomorphism, we have a group isomorphism $\mathbf{PGL}_1(\mathscr{C}_0(q))(k) = \mathbf{PGL}_2(k_\epsilon)$ and a Lie algebra isomorphism $\mathfrak{c}_0(q)(k) \cong \mathfrak{sl}_2(k)$, where $\mathfrak{sl}_2$ is the scheme of traceless 2×2 matrices. We claim that the map $\mathbf{E}(k) \to I + \epsilon\, \mathfrak{sl}_2(k)$

is explicitly given by

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ a & b & c & 1\end{pmatrix} \mapsto I - \frac{1}{2}\epsilon\begin{pmatrix} a & -b+c\\ b+c & -a\end{pmatrix}. \tag{8}$$

Indeed, let $\phi_{a,b,c} \in \mathbf{E}(S_0)$ be the orthogonal transformation whose matrix is displayed in (8), and $\sigma_{a,b,c}$ its image in $I + \epsilon\,\mathfrak{sl}_2(k)$, thought of as an automorphism of $\mathscr{C}_0(q)(k_\epsilon)$. Then we have

$$\begin{aligned}\sigma_{a,b,c}(e_1e_2) &= e_1e_2 + b\epsilon\, e_2e_3 - a\epsilon\, e_1e_3\\ \sigma_{a,b,c}(e_2e_3) &= e_2e_3 + c\epsilon\, e_1e_3 - b\epsilon\, e_1e_2\\ \sigma_{a,b,c}(e_1e_3) &= e_1e_3 + c\epsilon\, e_2e_3 - a\epsilon\, e_1e_2\end{aligned}$$

and $\sigma_{a,b,c}(\epsilon) = \epsilon$. It is then a straightforward calculation to see that

$$\sigma_{a,b,c} = \mathrm{ad}\big(1 - \frac{1}{2}\epsilon(c\,e_1e_2 + a\,e_2e_3 - b\,e_1e_3)\big),$$

where ad is conjugation in the Clifford algebra, and furthermore, that ψ takes $c\,e_1e_2 + a\,e_2e_3 - b\,e_1e_3$ to the 2×2 matrix displayed in (8). Thus the map $\mathbf{E}(k) \to I + \epsilon\,\mathfrak{sl}_2(k)$ is as stated, and in particular, is an isomorphism. Thus the diagram is commutative with exact rows and columns, and in particular, $c : \mathbf{PGO}^+(q) \to \mathbf{PGL}_1(\mathscr{C}_0(q))$ is an isomorphism on k-points.
Now we prove that the Lie algebra map dc is injective. Consider the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & I + x\,\mathfrak{so}(q)(k) \arrow[r] \arrow[d, "1+x\,\mathrm{d}c"] & \mathbf{O}^+(q)(k[x]/(x^2)) \arrow[r] \arrow[d, "c(k[x]/(x^2))"] & \mathbf{O}^+(q)(k) \arrow[r] \arrow[d] & 1\\
1 \arrow[r] & I + x\,\mathfrak{g}(k) \arrow[r] & \mathbf{PGL}_1(\mathscr{C}_0(q))(k[\epsilon,x]/(\epsilon^2,x^2)) \arrow[r] & \mathbf{PGL}_1(\mathscr{C}_0(q))(k_\epsilon) \arrow[r] & 1
\end{tikzcd}$$

where $\mathfrak{so}(q)$ and $\mathfrak{g}$ are the Lie algebras of $\mathbf{O}^+(q)$ and $R_{k_\epsilon/k}\mathbf{PGL}_1(\mathscr{C}_0(q))$, respectively.
The Lie algebra $\mathfrak{so}(q_1)$ of $\mathbf{O}(q_1)$ is identified with the scheme of 3×3 matrices A such that AQ_1 is skew-symmetric, where $Q_1 = \mathrm{diag}(1,-1,1)$. It is then a consequence of (6) that $I + x\,\mathfrak{so}(q)(k)$ consists of block matrices of the form

$$\begin{pmatrix} I + xA & 0\\ xw & 1\end{pmatrix}$$

for $w \in \mathbb{A}^3(k)$ and $A \in \mathfrak{so}(q_1)(k)$. Since

$$\begin{pmatrix} I + xA & 0\\ xw & 1\end{pmatrix} = \begin{pmatrix} I + xA & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} I & 0\\ xw & 1\end{pmatrix} = \begin{pmatrix} I & 0\\ xw & 1\end{pmatrix}\begin{pmatrix} I + xA & 0\\ 0 & 1\end{pmatrix},$$

we see that $I + x\,\mathfrak{so}(q)$ has a direct product decomposition $\mathbf{E} \times \big(I + x\,\mathfrak{so}(q_1)\big)$. We claim that the map $\mathfrak{h} \to \mathfrak{g}$ is explicitly given by the product map

$$\begin{pmatrix} I + xA & 0 \\ xw & 1 \end{pmatrix} \mapsto \big(I - \epsilon\beta(xw)\big)\big(I - \alpha(xA)\big) = I - x(\alpha(A) + \epsilon\beta(w))$$

where $\alpha : \mathfrak{so}(q_1) \to \mathfrak{sl}_2$ is the Lie algebra isomorphism

$$\begin{pmatrix} 0 & a & -b \\ a & 0 & c \\ b & c & 0 \end{pmatrix} \mapsto \frac{1}{2}\begin{pmatrix} a & -b+c \\ b+c & -a \end{pmatrix}$$

induced from the isomorphism $\mathbf{PSO}(q_1) \cong \mathbf{PGL}_2$ and $\beta : \mathbb{A}^3 \to \mathfrak{sl}_2$ is the Lie algebra isomorphism

$$(a\ \ b\ \ c) \mapsto \frac{1}{2}\begin{pmatrix} a & -b+c \\ b+c & -a \end{pmatrix}$$

as above. Thus $\mathrm{d}c : \mathfrak{so}(q) \to \mathfrak{g}$ is an isomorphism. □

Remark 2.6. The isomorphism of algebraic groups in the proof of Theorem 2.1 can be viewed as a degeneration of an isomorphism of semisimple groups of type ${}^2\mathsf{A}_1 = \mathsf{D}_2$ (on the generic fiber) to an isomorphism of nonreductive groups whose semisimplification has type $\mathsf{A}_1 = \mathsf{B}_1 = \mathsf{C}_1$ (on the special fiber).

3 Simple degeneration over semi-local rings

The semilocal ring R of a normal scheme at a finite set of points of codimension 1 is a semilocal Dedekind domain, hence a principal ideal domain. Let R_i denote the (finitely many) discrete valuation overrings of R contained in the fraction field K (the localizations at the height one prime ideals), $\widehat{R}_i$ their completions, and $\widehat{K}_i$ their fraction fields. If $\widehat{R}$ is the completion of R at its Jacobson radical $\mathrm{rad}(R)$ and $\widehat{K}$ the total ring of fractions, then $\widehat{R} \cong \prod_i \widehat{R}_i$ and $\widehat{K} \cong \prod_i \widehat{K}_i$. We call an element $\pi \in R$ a *parameter* if $\pi = \prod_i \pi_i$ is a product of parameters π_i of R_i.
We first recall a well-known result, cf. [17, §2.3.1].

Lemma 3.1. *Let R be a semilocal principal ideal domain and K its field of fractions. Let q be a regular quadratic form over R and $u \in R^\times$ a unit. If u is represented by q over K then it is represented by q over R.*

We now provide a generalization of Lemma 3.1 to the case of simple degeneration.

Proposition 3.2. *Let R be a semilocal principal ideal domain with 2 invertible and K its field of fractions. Let q be a quadratic form over R with simple degeneration of multiplicity one and let $u \in R^\times$ be a unit. If u is represented by q over K then it is represented by q over R.*

For the proof, we'll first need to generalize, to the degenerate case, some standard results concerning regular forms. If (V,q) is a quadratic form over a ring R and $v \in V$ is such that $q(v) = u \in R^\times$, then the *reflection* $r_v : V \to V$ through v given by

$$r_v(w) = w - u^{-1} b_q(v,w)\, v$$

is an isometry over R satisfying $r_v(v) = -v$ and $r_v(w) = w$ if $w \in v^\perp$.

Lemma 3.3. *Let R be a semilocal ring with 2 invertible. Let (V,q) be a quadratic form over R and $u \in R^\times$. Then $\mathbf{O}(V,q)(R)$ acts transitively on the set of vectors $v \in V$ such that $q(v) = u$.*

Proof. Let $v, w \in V$ be such that $q(v) = q(w) = u$. We first prove the lemma over any local ring with 2 invertible. Without loss of generality, we can assume that $q(v-w) \in R^\times$. Indeed, $q(v+w)+q(v-w) = 4u \in R^\times$ so that, since R is local, either $q(v+w)$ or $q(v-w)$ is a unit. If $q(v-w)$ is not a unit, then $q(v+w)$ is and we can replace w by $-w$ using the reflection r_w. Finally, by a standard computation, we have $r_{v-w}(v) = w$. Thus any two vectors representing u are related by a product of at most two reflection.
For a general semilocal ring, the quotient $R/\mathrm{rad}(R)$ is a product of fields. By the above argument, $\overline{v}$ can be transported to $-\overline{w}$ in each component by a product $\overline{\tau}$ of at most two reflections. By the Chinese remainder theorem, we can lift $\overline{\tau}$ to a product of at most two reflections τ of (V,q) transporting v to $-w+z$ for some $z \in \mathrm{rad}(R) \otimes_R V$. Replacing v by $-w+z$, we can assume that $v + w = z \in \mathrm{rad}(R) \otimes_R V$. Finally, $q(v+w) + q(v-w) = 4u$ and $q(v+w) \in \mathrm{rad}(R)$, thus $q(v-w)$ is a unit. As before, $r_{v-w}(v) = w$. ☐

Corollary 3.4. *Let R be a semilocal ring with 2 invertible. Then regular forms can be canceled, i.e., if q_1 and q_2 are quadratic forms and q a regular quadratic form over R with $q_1 \perp q \cong q_2 \perp q$, then $q_1 \cong q_2$.*

Proof. Regular quadratic forms over a semilocal ring with 2 invertible are diagonalizable. Hence we can reduce to the case of rank one form $q = (R, {<}u{>})$ for $u \in R^\times$. Let $\varphi : q_1 \perp (R\,w_1, {<}u{>}) \cong q_2 \perp (R\,w_2, {<}u{>})$ be an isometry. By Lemma 3.3, there is an isometry ψ of $q_2 \perp (R\,w_2, {<}u{>})$ taking $\varphi(w_1)$ to w_2, so that $\psi \circ \varphi$ takes w_1 to w_2. By taking orthogonal complements, φ thus induces an isometry $q_1 \cong q_2$. ☐

Lemma 3.5. *Let R be a complete discrete valuation ring with 2 invertible and K its fraction field. Let q be a quadratic form with simple degeneration of multiplicity one and let $u \in R^\times$. If u is represented by q over K then it is represented by q over R.*

Proof. For a choice of parameter π of R, write $(V,q) = (V_1,q_1) \perp (R\,e, {<}\pi{>})$ with q_1 regular. There are two cases, depending on whether $\overline{q}_1$ is isotropic over the residue field. First, if $\overline{q}_1$ is anisotropic, then q_1 only takes values with even valuation. Let $v \in V_K$ satisfy $q|_K(v) = u$ and write $v = \pi^n v_1 + a\pi^m e$ with $v_1 \in V_1$ such that $\overline{v}_1 \neq 0$ and $a \in R^\times$. Then we have $\pi^{2n} q_1(v_1) + a\pi^{2m+1} = u$.

By parity considerations, we see that $n = 0$ and $m \geq 0$ are forced, thus $v \in V$ and u is represented by q. Second, R being complete, if $\overline{q}_1$ is isotropic, then it splits off a hyperbolic plane, so represents u. $\square$

We now recall the theory of elementary hyperbolic isometries initiated by Eichler [27, Ch. 1] and developed in the setting of regular quadratic forms over rings by Wall [61, §5] and Roy [54, §5]. See also [48], [50], [56], and [8, III §2]. We will need to develop the theory for quadratic forms that are not necessarily regular.

Let R be a ring with 2 invertible, (V, q) a quadratic form over R, and (R^2, h) the hyperbolic plane with basis e, f. For $v \in V$, define E_v and E_v^* in $\mathbf{O}(q \perp h)(R)$ by

$$\begin{array}{lll} E_v(w) = w + b(v,w)e & E_v^*(w) = w + b(v,w)f, & \text{for } w \in V \\ E_v(e) \ = e & E_v^*(e) \ = -v - 2^{-1}q(v)f + e & \\ E_v(f) \ = -v - 2^{-1}q(v)e + f & E_v^*(f) \ = f. & \end{array}$$

Define the group of *elementary hyperbolic isometries* $\mathbf{EO}(q,h)(R)$ to be the subgroup of $\mathbf{O}(q \perp h)(R)$ generated by E_v and E_v^* for $v \in V$.

For $u \in R^\times$, define $\alpha_u \in \mathbf{O}(h)(R)$ by

$$\alpha_u(e) = ue, \qquad \alpha_u(f) = u^{-1}f$$

and $\beta_u \in \mathbf{O}(h)(R)$ by

$$\beta_u(e) = u^{-1}f, \qquad \beta_u(f) = ue.$$

Then $\mathbf{O}(h)(R) = \{\alpha_u : u \in R^\times\} \cup \{\beta_u : u \in R^\times\}$. One can verify the following identities:

$$\begin{array}{ll} \alpha_u^{-1} E_v \alpha_u = E_{u^{-1}v}, & \beta_u^{-1} E_v \beta_u = E_v^*, \\ \alpha_u^{-1} E_v^* \alpha_u = E_{u^{-1}v}^*, & \alpha_u^{-1} E_v^* \alpha_u = E_v. \end{array}$$

Thus $\mathbf{O}(h)(R)$ normalizes $\mathbf{EO}(q,h)(R)$.

If $R = K$ is a field and q is nondegenerate, then $\mathbf{EO}(q,h)(K)$ and $\mathbf{O}(h)(K)$ generate $\mathbf{O}(q \perp h)(K)$ (see [27, ch. 1]) so that

$$\mathbf{O}(q \perp h)(K) = \mathbf{EO}(q,h)(K) \rtimes \mathbf{O}(h)(K). \tag{9}$$

Proposition 3.6. *Let R be a semilocal principal ideal domain with 2 invertible and K its fraction field. Let $\widehat{R}$ be the completion of R at the radical and $\widehat{K}$ its fraction field. Let (V, q) be a quadratic form over R that is nondegenerate over K. Then every element $\varphi \in \mathbf{O}(q \perp h)(\widehat{K})$ is a product $\varphi_1\varphi_2$, where $\varphi_1 \in \mathbf{O}(q \perp h)(K)$ and $\mathbf{O}(q \perp h)(\widehat{R})$.*

Proof. We follow portions of the proof in [50, Prop. 3.1]. As topological rings, $\widehat{R}$ is open in $\widehat{K}$, and hence as topological groups, $\mathbf{O}(q \perp h)(\widehat{R})$ is open inside $\mathbf{O}(q \perp h)(\widehat{K})$. In particular, $\mathbf{O}(q \perp h)(\widehat{R}) \cap \mathbf{EO}(q,h)(\widehat{K})$ is open in $\mathbf{EO}(q,h)(\widehat{K})$. Since R is dense in $\widehat{R}$, K is dense in $\widehat{K}$, $V \otimes_R K$ is dense in $V \otimes_R \widehat{K}$, and hence $\mathbf{EO}(q,h)(K)$ is dense in $\mathbf{EO}(q,h)(\widehat{K})$.
Thus, by topological considerations, every element φ' of $\mathbf{EO}(q \perp h)(\widehat{K})$ is a product $\varphi'_1\varphi'_2$, where $\varphi'_1 \in \mathbf{EO}(q,h)(K)$ and $\varphi'_2 \in \mathbf{EO}(q,h)(\widehat{K}) \cap \mathbf{O}(q \perp h)(\widehat{R})$. Clearly, every element γ of $\mathbf{O}(h)(\widehat{K})$ is a product $\gamma_1\gamma_2$, where $\gamma_1 \in \mathbf{O}(h)(K)$ and $\gamma_2 \in \mathbf{O}(h)(\widehat{R})$.
The form $q \perp h$ is nondegenerate over $\widehat{K}$, so by (9), every $\varphi \in \mathbf{O}(q \perp h)(\widehat{K})$ is a product $\varphi'\gamma$, where $\varphi' \in \mathbf{EO}(q,h)(\widehat{K})$ and $\gamma \in \mathbf{O}(h)(\widehat{K})$. As above, we can write

$$\varphi = \varphi'\gamma = \varphi'_1\varphi'_2\gamma_1\gamma_2 = \varphi'_1\gamma_1(\gamma_1^{-1}\varphi'_2\gamma_1)\gamma_2.$$

Since $\mathbf{EO}(q,h)(\widehat{K})$ is a normal subgroup, $\gamma_1^{-1}\varphi'_2\gamma_1 \in \mathbf{EO}(q,h)(\widehat{K})$ and is thus a product $\psi_1\psi_2$, where $\psi_1 \in \mathbf{EO}(q,h)(K)$ and $\psi_2 \in \mathbf{EO}(q,h)(\widehat{K}) \cap \mathbf{O}(q \perp h)(\widehat{R})$. Finally, φ is a product $(\varphi'_1\gamma_1\psi_1)(\psi_2\gamma_2)$ of the desired form. □

Proof of Proposition 3.2. Let $\widehat{R}$ be the completion of R at the radical and $\widehat{K}$ the total ring of fractions. As $q|_K$ represents u, we have a splitting $q|_K \cong q_1 \perp <u>$. We have that $q|_{\widehat{R}} = \prod_i q|_{\widehat{R}_i}$ represents u over $\widehat{R} = \prod_i \widehat{R}_i$, by Lemma 3.5, since u is represented over $\widehat{K} = \prod_i \widehat{K}_i$. We thus have a splitting $q|_{\widehat{R}} \cong q_2 \perp <u>$. By Witt cancellation over $\widehat{K}$, we have an isometry $\varphi : q_1|_{\widehat{K}} \cong q_2|_{\widehat{K}}$, which by patching defines a quadratic form $\tilde{q}$ over R such that $\tilde{q}_K \cong q_1$ and $\tilde{q}|_{\widehat{R}} \cong q_2$. We claim that $q \perp <-u> \cong \tilde{q} \perp h$. Indeed, as $h \cong <u,-u>$, we have isometries

$$\psi^K : (q \perp <-u>)_K \cong (\tilde{q} \perp h)_K, \qquad \psi^{\widehat{R}} : (q \perp <-u>)_{\widehat{R}} \cong (\tilde{q} \perp h)_{\widehat{R}}.$$

By Proposition 3.6, there exists $\theta_1 \in \mathbf{O}(\tilde{q} \perp h)(\widehat{R})$ and $\theta_2 \in \mathbf{O}(\tilde{q} \perp h)(K)$ such that $\psi^{\widehat{R}}(\psi^K)^{-1} = \theta_1^{-1}\theta_2$. The isometries $\theta_1\psi^{\widehat{R}}$ and $\theta_2\psi^K$ then agree over $\widehat{K}$ and so patch to yield an isometry $\psi : q \perp <-u> \cong \tilde{q} \perp h$.
As $h \cong <u,-u>$, we have $q \perp <-u> \cong \tilde{q} \perp <u,-u>$. By Corollary 3.4, we can cancel the regular form $<-u>$, so that $q \cong \tilde{q} \perp <u>$. Thus q represents u over R. □

Lemma 3.7. *Let R be a discrete valuation ring and (E,q) a quadratic form of rank n over R with simple degeneration. If q represents $u \in R^\times$ then it can be diagonalized as $q \cong <u, u_2, \ldots, u_{n-1}, \pi>$ for $u_i \in R^\times$ and some parameter π.*

Proof. If $q(v) = u$ for some $v \in E$, then q restricted to the submodule $Rv \subset E$ is regular, hence (E,q) splits as $(R, <u>) \perp (Rv^\perp, q|_{Rv^\perp})$. Since $(Rv^\perp, q|_{Rv^\perp})$ has simple degeneration, we are done by induction. □

Corollary 3.8. *Let R be a semilocal principal ideal domain with 2 invertible and fraction field K. If quadratic forms q and q' with simple degeneration and multiplicity one over R are isometric over K, then they are isometric over R.*

Proof. Any quadratic form q with simple degeneration and multiplicity one has discriminant $\pi \in R/R^{\times 2}$ given by a parameter. Since $R^\times/R^{\times 2} \to K^\times/K^{\times 2}$ is injective, if q' is another quadratic form with simple degeneration and multiplicity one, such that $q|_K$ is isomorphic to $q|'_K$, then q and q' have the same discriminant.
Over each discrete valuation overring R_i of R, we thus have diagonalizations,

$$q|_{R_i} \cong <u_1, \dots, u_{r-1}, u_1 \cdots u_{r-1}\pi_i>, \quad q'|_{R'_i} \cong <u'_1, \dots, u'_{r-1}, u'_1 \cdots u'_{r-1}\pi_i>,$$

for a suitable parameter π_i of R_i, where $u_j, u'_j \in R_i^\times$. Now, since $q|_K$ and $q'|_K$ are isometric, $q'|_K$ represents u_1 over K, hence by Proposition 3.2, $q'|_{R_i}$ represents u_1 over R_i. Hence by Lemma 3.7, we have a further diagonalization

$$q'|_{R_i} \cong <u_1, u'_2, \dots, u'_{r-1}, u_1 u'_2 \cdots u'_{r-1}\pi_i>$$

with possibly different units u'_i. By cancellation over K, we have

$$<u_2, \dots, u_{r-1}, u_1 \cdots u_{r-1}\pi_i> |_K \cong <u'_2, \dots, u'_{r-1}, u'_1 \cdots u'_{r-1}\pi_i> |_K.$$

By an induction hypothesis over the rank of q, we have that

$$<u_2, \dots, u_{r-1}, u_1 \cdots u_{r-1}\pi_i> \cong <u'_2, \dots, u'_{r-1}, u'_1 \cdots u'_{r-1}\pi_i>$$

over R. By induction, we have the result over each R_i.
Thus $q|_{\widehat{R}} \cong q'|_{\widehat{R}}$ over $\widehat{R} = \prod_i \widehat{R}_i$. Consider the induced isometry $\psi^{\widehat{R}} : (q \perp h)|_{\widehat{R}} \cong (q' \perp h)|_{\widehat{R}}$ as well as the isometry $\psi^K : (q \perp h)|_K \cong (q' \perp h)|_K$ induced from the given one. By Proposition 3.6, there exists $\theta^{\widehat{R}} \in \mathbf{O}(q \perp h)(\widehat{R})$ and $\theta^K \in \mathbf{O}(q \perp h)(K)$ such that $\psi^{\widehat{R}-1}\psi^K = \theta^{\widehat{R}}\theta^{K-1}$. The isometries $\psi^{\widehat{R}}\theta^{\widehat{R}}$ and $\psi^K\theta^K$ then agree over $\widehat{K}$ and so patch to yield an isometry $\psi : q \perp h \cong q' \perp h$ over R. By Corollary 3.4, we then have an isometry $q \cong q'$. ∎

Remark 3.9. Let R be a semilocal principal ideal domain with 2 invertible, closed fiber D, and fraction field K. Let $\mathrm{QF}^D(R)$ be the set of isometry classes of quadratic forms on R with simple degeneration of multiplicity one along D. Corollary 3.8 says that $\mathrm{QF}^D(R) \to \mathrm{QF}(K)$ is injective, which can be viewed as an analogue of the Grothendieck–Serre conjecture for the (nonreductive) orthogonal group of a quadratic form with simple degeneration of multiplicity one over a discrete valuation ring. One might wonder if such a statement is true for more general nonreductive smooth R-group schemes.

Corollary 3.10. *Let R be a complete discrete valuation ring with 2 invertible and K its fraction field. If quadratic forms q and q' of even rank $n = 2m \geq 4$ with simple degeneration and multiplicity one over R are similar over K, then they are similar.*

Proof. Let $\psi : q|_K \simeq q'|_K$ be a similarity with factor $\lambda = u\pi^e$ where $u \in R^\times$ and π is a parameter whose square class we can assume is the discriminant of q and q'. If e is even, then $\pi^{e/2}\psi : q|_K \simeq q'|_K$ has factor u, so defines an isometry $q|_K \cong uq'|_K$. Hence by Corollary 3.8, there is an isometry $q \cong uq'$, hence a similarity $q \simeq q'$. If e is odd, then $\pi^{(e-1)/2}\psi^K$ defines an isometry $q|_K \cong u\pi q'|_K$. Writing $q \cong q_1 \perp <a\pi>$ and $q' = q_1' \perp <b\pi>$ for regular quadratic forms q_1 and q_1' over R and $a, b \in R^\times$, then $\pi q|_K \cong u\pi q'|_K \cong u\pi q_1' \perp <bu>$. Comparing first residues, we have that $\overline{q}_1$ and $<\overline{bu}>$ are equal in $W(k)$, where k is the residue field of R. Since R is complete, q_1 splits off the requisite number of hyperbolic planes, and so $q_1 \cong h^{m-1} \perp <(-1)^{m-1}a>$. Now note that $(-1)^{m-1}\pi$ is a similarity factor of the form $q|_K$. Finally, we have $(-1)^{m-1}\pi q|_K \cong q|_K \cong u\pi q'|_K$, so that $q|_K \cong (-1)^{m-1}uq'|_K$. Thus by Corollary 3.8, $q \cong (-1)^m uq'$ over R, so there is a similarity $q \simeq q'$ over R. □

We need the following relative version of Theorem 2.1.

Proposition 3.11. *Let R be a semilocal principal ideal domain with 2 invertible and K its fraction field. Let q and q' be quadratic forms of rank 4 over R with simple degeneration and multiplicity one. Given any R-algebra isomorphism $\varphi : \mathscr{C}_0(q) \cong \mathscr{C}_0(q')$ there exists a similarity $\psi : q \simeq q'$ such that $\mathscr{C}_0(\psi) = \varphi$.*

Proof. By Theorem 2.5, there exists a similarity $\psi^K : q \simeq q'$ such that $\mathscr{C}_0(\psi^K) = \varphi|_K$. Thus over $\widehat{R} = \prod_i \widehat{R}_i$, Corollary 3.10 applied to each component provides a similarity $\rho : q|_{\widehat{R}} \simeq q'|_{\widehat{R}}$. Now $\mathscr{C}_0(\rho)^{-1} \circ \varphi : \mathscr{C}_0(q)|_{\widehat{R}} \cong \mathscr{C}_0(q)|_{\widehat{R}}$ is a $\widehat{R}$-algebra isomorphism, hence by Theorem 2.1, is equal to $\mathscr{C}_0(\sigma)$ for some similarity $\sigma : q|_{\widehat{R}} \simeq q|_{\widehat{R}}$. Then $\psi^{\widehat{R}} = \rho \circ \sigma : q|_{\widehat{R}} \simeq q'|_{\widehat{R}}$ satisfies $\mathscr{C}_0(\psi^{\widehat{R}}) = \varphi|_{\widehat{R}}$. Let $\lambda \in K^\times$ and $u \in \widehat{R}^\times$ be the factor of ψ^K and $\psi^{\widehat{R}}$, respectively. Then $\psi^K|_{\widehat{K}}^{-1} \circ \psi^{\widehat{R}}|_{\widehat{K}} : q|_{\widehat{K}} \simeq q|_{\widehat{K}}$ has factor $u^{-1}\lambda \in \widehat{K}^\times$. But since $\mathscr{C}_0(\psi^K|_{\widehat{K}}^{-1} \circ \psi^{\widehat{R}}|_{\widehat{K}}) = \mathrm{id}$, we have that $\psi^K|_{\widehat{K}}^{-1} \circ \psi^{\widehat{R}}|_{\widehat{K}}$ is given by multiplication by $\mu \in \widehat{K}^\times$. In particular, $u^{-1}\lambda = \mu^2$ and thus the valuation of $\lambda \in K^\times$ is even in every R_i. Thus $\lambda = v\varpi^2$ with $v \in R^\times$ and so $\varpi\psi$ defines an isometry $q|_K \cong vq'|_K$. By Corollary 3.8, there's an isometry $\alpha : q \cong vq'$, i.e., a similarity $\alpha : q \simeq q'$. As before, $\mathscr{C}_0(\alpha)^{-1} \circ \varphi : \mathscr{C}_0(q) \cong \mathscr{C}_0(q)$ is a R-algebra isomorphism, hence by Theorem 2.1, is equal to $\mathscr{C}_0(\beta)$ for some similarity $\beta : q \simeq q$. Then we can define a similarity $\psi = \alpha \circ \beta : q \simeq q'$ over R, which satisfies $\mathscr{C}_0(\psi) = \varphi$. □

Finally, we need the following generalization of [19, Prop. 2.3] to the setting of quadratic forms with simple degeneration.

Proposition 3.12. *Let S be the spectrum of a regular local ring $(R, \mathfrak{m})$ of dimension ≥ 2 with 2 invertible and $D \subset S$ a regular divisor. Let (V, q) be a quadratic form over S such that $(V, q)|_{S \smallsetminus \{\mathfrak{m}\}}$ has simple degeneration of multiplicity one along $D \smallsetminus \{\mathfrak{m}\}$. Then (V, q) has simple degeneration along D of multiplicity one.*

Proof. First note that the discriminant of (V, q) (hence the subscheme D) is represented by a regular element $\pi \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$. Now assume, to get a contradiction, that the radical of $(V, q)_{\kappa(\mathfrak{m})}$, where $\kappa(\mathfrak{m})$ is the residue field at $\mathfrak{m}$, has dimension $r > 1$ and let $\overline{e}_1, \dots, \overline{e}_r$ be a $\kappa(\mathfrak{m})$-basis of the radical. Lifting to unimodular elements $e_1, \dots, e_r$ of V, we can complete to a basis $e_1, \dots, e_n$. Since $b_q(e_i, e_j) \in \mathfrak{m}$ for all $1 \leq i \leq r$ and $1 \leq j \leq n$, inspecting the Gram matrix M_q of b_q with respect to this basis, we find that $\det M_q \in \mathfrak{m}^r$, contradicting the description of the discriminant above. Thus the radical of (V, q) has rank 1 at $\mathfrak{m}$ and (V, q) has simple degeneration along D. Similarly, (V, q) also has multiplicity one at $\mathfrak{m}$, hence on S by hypothesis. $\square$

Corollary 3.13. *Let S be a regular integral scheme of dimension ≤ 2 with 2 invertible and D a regular divisor. Let $(\mathscr{E}, q, \mathscr{L})$ be a quadratic form over S that is regular over every codimension 1 point of $S \smallsetminus D$ and has simple degeneration of multiplicity one over every codimension one point of D. Then over S, the quadratic form q has simple degeneration along D of multiplicity one.*

Proof. Let $U = S \smallsetminus D$. The quadratic form $q|_U$ is regular except possibly at finitely many closed points. But regular quadratic forms over the complement of finitely many closed points of a regular surface extend uniquely by [19, Prop. 2.3]. Hence $q|_U$ is regular. The restriction $q|_D$ has simple degeneration at the generic point of D, hence along the complement of finitely many closed points of D. At each of these closed points, q has simple degeneration by Proposition 3.12. Thus q has simple degeneration along D. $\square$

4 Gluing tensors

In this section, we reproduce some results on gluing (or patching) tensor structures on vector bundles communicated to us by M. Ojanguren and inspired by Colliot-Thélène–Sansuc [19, §2, §6]. As usual, any scheme S is assumed to be noetherian.

Lemma 4.1. *Let S be a scheme of dimension n, $U \subset S$ a dense open subset, $x \in S \smallsetminus U$ a point of codimension 1 of S, $V \subset S$ a dense open neighborhood of x, and $W \subset U \cap V$ a dense open subset of S. Then there exists a dense open neighborhood V' of x such that $V' \cap U \subset W$.*

Proof. The closed set $Z = S \smallsetminus W$ is of dimension $n-1$, contains x, and has a decomposition into closed sets $Z = Z_1 \cup Z_2$, where $Z_1 = Z \cap (S \smallsetminus U)$ contains x and $Z_2 = \overline{Z \cap U}$. No irreducible component of Z_2 can contain x, otherwise it would contain (hence coincide with) the dimension $n-1$ set $\overline{\{x\}}$. Setting $V' = S \smallsetminus Z_2$, then $V' \subset S$ is a dense open neighborhood of x and satisfies $V' \cap U \subset W$. $\square$

Let $\mathscr{V}$ be a locally free $\mathscr{O}_S$-module (of finite rank). A *tensorial construction* $t(\mathscr{V})$ in $\mathscr{V}$ is any locally free $\mathscr{O}_S$-module that is a tensor product of modules $\bigwedge^j(\mathscr{V})$, $\bigwedge^j(\mathscr{V}^\vee)$, $S^j(\mathscr{V})$, or $S^j(\mathscr{V}^\vee)$. Let $\mathscr{L}$ be a line bundle on S. An *$\mathscr{L}$-valued tensor* $(\mathscr{V}, q, \mathscr{L})$ *of type* $t(\mathscr{V})$ on S is a global section $q \in \Gamma(S, t(\mathscr{V}) \otimes \mathscr{L})$ for some tensorial construction $t(\mathscr{V})$ in $\mathscr{V}$. For example, an $\mathscr{L}$-valued quadratic form is an $\mathscr{L}$-valued tensor of type $t(\mathscr{V}) = S^2(\mathscr{V}^\vee)$; an $\mathscr{O}_S$-algebra structure

on $\mathscr{V}$ is an $\mathscr{O}_S$-valued tensor of type $t(\mathscr{V}) = \mathscr{V}^\vee \otimes \mathscr{V}^\vee \otimes \mathscr{V}$. If $U \subset S$ is an open set, denote by $(\mathscr{V}, q, \mathscr{L})|_U = (\mathscr{V}|_U, q|_U, \mathscr{L}|_U)$ the restricted tensor over U. If $D \subset S$ is a closed subscheme, let $\mathscr{O}_{S,D}$ denote the semilocal ring at the generic points of D and $(\mathscr{V}, q, \mathscr{L})|_D = (\mathscr{V}, q, \mathscr{L}) \otimes_{\mathscr{O}_S} \mathscr{O}_{S,D}$ the associated tensor over $\mathscr{O}_{S,D}$. If S is integral and K its function field, we write $(\mathscr{V}, q, \mathscr{L})|_K$ for the stalk at the generic point.
A *similarity* between line bundle-valued tensors $(\mathscr{V}, q, \mathscr{L})$ and $(\mathscr{V}', q', \mathscr{L}')$ consists of a pair (φ, λ) where $\varphi : \mathscr{V} \cong \mathscr{V}'$ and $\lambda : \mathscr{L} \cong \mathscr{L}'$ are $\mathscr{O}_S$-module isomorphisms such that $t(\varphi) \otimes \lambda : t(\mathscr{V}) \otimes \mathscr{L} \cong t(\mathscr{V}') \otimes \mathscr{L}'$ takes q to q'. A similarity is an *isomorphism* if $\mathscr{L} = \mathscr{L}'$ and $\lambda = \mathrm{id}$.

Proposition 4.2. *Let S be an integral scheme, K its function field, $U \subset S$ a dense open subscheme, and $D \subset S \smallsetminus U$ a closed subscheme of codimension 1. Let $(\mathscr{V}^U, q^U, \mathscr{L}^U)$ be a tensor over U, $(\mathscr{V}^D, q^D, \mathscr{L}^D)$ a tensor over $\mathscr{O}_{S,D}$, and $(\varphi, \lambda) : (\mathscr{V}^U, q^U, \mathscr{L}^U)|_K \simeq (\mathscr{V}^D, q^D, \mathscr{L}^D)|_K$ a similarity of tensors over K. Then there exists a dense open set $U' \subset S$ containing U and the generic points of D and a tensor $(\mathscr{V}^{U'}, q^{U'}, \mathscr{L}^{U'})$ over U' together with similarities $(\mathscr{V}^U, q^U, \mathscr{L}^U) \cong (\mathscr{V}^{U'}, q^{U'}, \mathscr{L}^{U'})|_U$ and $(\mathscr{V}^D, q^D, \mathscr{L}^D) \cong (\mathscr{V}^{U'}, q^{U'}, \mathscr{L}^{U'})|_D$. A corresponding statement holds for isomorphisms of tensors.*

Proof. By induction on the number of irreducible components of D, gluing over one at a time, we can assume that D is irreducible. Choose an extension $(\mathscr{V}^V, q^V, \mathscr{L}^V)$ of $(\mathscr{V}^D, q^D, \mathscr{L}^D)$ to some open neighborhood V of D in S. Since $(\mathscr{V}^V, q^V, \mathscr{L}^V)|_K \simeq (\mathscr{V}^U, q^U, \mathscr{L}^U)|_K$, there exists an open subscheme $W \subset U \cap V$ over which $(\mathscr{V}^V, q^V, \mathscr{L}^V)|_W \simeq (\mathscr{V}^U, q^U, \mathscr{L}^U)|_W$. By Lemma 4.1, there exists an open neighborhood $V' \subset S$ of D such that $V' \cap U \subset W$. We can glue $(\mathscr{V}^U, q^U, \mathscr{L}^U)$ and $(\mathscr{V}^{V'}, q^{V'}, \mathscr{L}^{V'})$ over $U \cap V'$ to get a tensor $(\mathscr{V}^{U'}, q^{U'}, \mathscr{L}^{U'})$ over U' extending $(\mathscr{V}^U, q^U, \mathscr{L}^U)$, where $U' = U \cup V'$. But U' contains the generic points of D and we are done. □

For an open subscheme $U \subset S$, a closed subscheme $D \subset S \smallsetminus U$ of codimension 1, a *similarity gluing datum* (resp. *gluing datum*) is a triple $((\mathscr{V}^U, q^U, \mathscr{L}^U), (\mathscr{V}^D, q^D, \mathscr{L}^D), \varphi)$ consisting of a tensor over U, a tensor over $\mathscr{O}_{S,D}$, and a similarity (resp. an isomorphism) of tensors (φ, λ) : $(\mathscr{V}^U, q^U, \mathscr{L}^U)|_K \simeq (\mathscr{V}^D, q^D, \mathscr{L}^D)|_K$ over K. There is an evident notion of isomorphism between two (similarity) gluing data. Two isomorphic gluing data yield, by Proposition 4.2, tensors $(\mathscr{V}^{U'}, q^{U'}, \mathscr{L}^{U'})$ and $(\mathscr{V}^{U''}, q^{U''}, \mathscr{L}^{U''})$ over open dense subsets $U', U'' \subset S$ containing U and the generic points of D such that there is an open dense refinement $U''' \subset U' \cap U''$ over which we have $(\mathscr{V}^{U'}, q^{U'}, \mathscr{L}^{U'})|_{U'''} \simeq (\mathscr{V}^{U''}, q^{U''}, \mathscr{L}^{U''})|_{U'''}$.
Together with results of [19], we get a well-known result—*purity for division algebras* over surfaces—which we state in a precise way, due to Ojanguren, that is conducive to our usage. If K is the function field of a regular scheme S, we say that $\beta \in \mathrm{Br}(K)$ is *unramified* (along S) if it is contained in the image of the injection $\mathrm{Br}(\mathscr{O}_{S,x}) \to \mathrm{Br}(K)$ for all codimension 1 points x of S.

Theorem 4.3. *Let S be a regular integral scheme of dimension ≤ 2, K its function field, $D \subset S$ a closed subscheme of codimension 1, and $U = S \smallsetminus D$.*

a) If $\mathscr{A}^U$ is an Azumaya $\mathscr{O}_U$-algebra such that $\mathscr{A}^U|_K$ is unramified along D then there exists an Azumaya $\mathscr{O}_S$-algebra $\mathscr{A}$ such that $\mathscr{A}|_U \cong \mathscr{A}^U$.

b) If a central simple K-algebra A has Brauer class unramified over S, then there exists an Azumaya $\mathscr{O}_S$-algebra $\mathscr{A}$ such that $\mathscr{A}|_K \cong A$.

Proof. For *a)*, since $\mathscr{A}^U|_K$ is unramified along D, there exists an Azumaya $\mathscr{O}_{S,D}$-algebra $\mathscr{B}^D$ with $\mathscr{B}^D|_K$ Brauer equivalent to A.
We argue that we can choose $\mathscr{B}^D$ such that $\mathscr{B}^D|_K = A$. Indeed, writing $\mathscr{B}^D|_K = M_m(\Delta)$ for a division K-algebra Δ and choosing a maximal $\mathscr{O}_{S,D}$-order $\mathscr{D}^D$ of Δ, then $M_m(\mathscr{D}^D)$ is a maximal order of $\mathscr{B}^D|_K$. Any two maximal orders are isomorphic by [7, Prop. 3.5], hence $M_m(\mathscr{D}^D) \cong \mathscr{B}^D$. In particular, $\mathscr{D}^D$ is an Azumaya $\mathscr{O}_{S,D}$-algebra. Finally writing $A = M_n(\Delta)$, then $M_n(\mathscr{D}^D)$ is an Azumaya $\mathscr{O}_{S,D}$-algebra and is our new choice for $\mathscr{B}^D$.
Applying Proposition 4.2 to $\mathscr{A}^U$ and $\mathscr{B}^D$, we get an Azumaya $\mathscr{O}_{U'}$-algebra $\mathscr{A}^{U'}$ extending $\mathscr{A}^U$, where U' contains all points of S of codimension 1. Finally, by [19, Thm. 6.13] applied to the group $\mathbf{PGL}_n$ (where n is the degree of A), $\mathscr{A}^{U'}$ extends to an Azumaya $\mathscr{O}_S$-algebra $\mathscr{A}$ such that $\mathscr{A}|_U = \mathscr{A}^U$.
For *b)*, the K-algebra A extends, over some open subscheme $U \subset S$, to an Azumaya $\mathscr{O}_U$-algebra $\mathscr{A}^U$. If U contains all codimension 1 points, then we apply [19, Thm. 6.13] as above. Otherwise, $D = S \smallsetminus U$ has codimension 1 and we apply part (1). $\square$

Finally, we note that isomorphic Azumaya algebra gluing data on a regular integral scheme S of dimension ≤ 2 yield, by [19, Thm. 6.13], isomorphic Azumaya algebras on S.

5 The norm form $N_{T/S}$ for ramified covers

Let S be a regular integral scheme, $D \subset S$ a regular divisor, and $f : T \to S$ a ramified cover of degree 2 branched along D. Then T is a regular integral scheme. Let L/K be the corresponding quadratic extension of function fields. Let $U = S \smallsetminus D$, and for $E = f^{-1}(D)$, let $V = T \smallsetminus E$. Then $f|_V : V \to U$ is étale of degree 2. Let ι be the nontrivial Galois automorphism of T/S.
The following lemma is not strictly used in our construction but we need it for the applications in §6.

Lemma 5.1. *Let S be a regular integral scheme and $f : T \to S$ a finite flat cover of prime degree ℓ with regular branch divisor $D \subset S$ on which ℓ is invertible. Let L/K be the corresponding extension of function fields. Let d be a positive integer invertible on D.*

a) The corestriction map $N_{L/K} : \mathrm{Br}(L) \to \mathrm{Br}(K)$ restricts to a well-defined map $N_{T/S} : {}_d\mathrm{Br}(T) \to {}_d\mathrm{Br}(S)$.

b) If S has dimension ≤ 2 and $\mathscr{B}$ is an Azumaya $\mathscr{O}_T$-algebra of degree d representing $\beta \in \mathrm{Br}(T)$ then there exists an Azumaya $\mathscr{O}_S$-algebra of degree d^ℓ representing $N_{T/S}(\beta)$ whose restriction to U coincides with the classical étale norm algebra $N_{V/U}\mathscr{B}|_V$.

Proof. The hypotheses imply that T is regular integral and so by [6], we can consider $\mathrm{Br}(S) \subset \mathrm{Br}(K)$ and $\mathrm{Br}(T) \subset \mathrm{Br}(L)$. Let $\mathscr{B}$ be an Azumaya $\mathscr{O}_T$-algebra of degree d representing $\beta \in \mathrm{Br}(T)$. As V/U is étale of degree ℓ, the classical norm algebra $N_{V/U}(\mathscr{B}|_V)$ is an Azumaya $\mathscr{O}_U$-algebra of degree d^ℓ representing the class of $N_{L/K}(\beta) \in \mathrm{Br}(K)$. In particular, $N_{L/K}(\beta)$ is unramified at every point (of codimension 1) in U. As D is regular, it is a disjoint union of irreducible divisors and let D' be one such irreducible component. If $E' = f^*D'$, then $\mathscr{O}_{T,E'}$ is totally ramified over $\mathscr{O}_{S,D'}$ (since it is ramified of prime degree). In particular, $E' \subset T$ is an irreducible component of $E = f^*D$. The commutative diagram

$$\begin{tikzcd}
{}_d\mathrm{Br}(\mathscr{O}_{T,E'}) \arrow[r] \arrow[d, dotted] & {}_d\mathrm{Br}(L) \arrow[r, "\partial"] \arrow[d, "N_{L/K}"] & H^1(\kappa(E'), \mathbb{Z}/d\mathbb{Z}) \arrow[d, equal] \\
{}_d\mathrm{Br}(\mathscr{O}_{S,D'}) \arrow[r] & {}_d\mathrm{Br}(K) \arrow[r, "\partial"] & H^1(\kappa(D'), \mathbb{Z}/d\mathbb{Z})
\end{tikzcd}$$

of residue homomorphisms implies, since β is unramified along E', that $N_{L/K}(\beta)$ is unramified along D'. Thus $N_{L/K}(\beta)$ is an unramified class in $\mathrm{Br}(K)$, hence is contained in $\mathrm{Br}(S)$ by purity for the Brauer group (cf. [33, Cor. 1.10]). This proves part *a*.
By Theorem 4.3, $N_{V/U}(\mathscr{B}|_V)$ extends (since by part *a*, it is unramified along D) to an Azumaya $\mathscr{O}_S$-algebra of degree d^ℓ, whose generic fiber is $N_{L/K}(\beta)$. This proves part *b*. □

Remark 5.2. Following Deligne [2, Exp. XVII 6.3.13], for any finite flat morphism $f : T \to S$, there exists a natural trace morphism $\mathrm{Tr}_f : f_*\mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}}$ of sheaves of abelian groups on X. Taking flat (fppf) cohomology, we arrive at a homomorphism $H^2(\mathrm{Tr}_f) : H^2(T, \mathbf{G}_{\mathrm{m}}) \to H^2(S, \mathbf{G}_{\mathrm{m}})$. If we assume that the Brauer group and cohomological Brauer group of S coincide (e.g., S has an ample invertible sheaf [22] or is regular of dimension ≤ 2 [33, Cor. 2.2]), then we can refine this to a map $\mathrm{Br}(T) \to \mathrm{Br}(S)$. This map can be seen to coincide with the one constructed in Lemma 5.1, under the different set of hypotheses imposed there. We do not know how these norm constructions coincide with that defined by Ferrand [29].

Suppose that S has dimension ≤ 2. We are interested in finding a good extension of $N_{V/U}(\mathscr{B}|_V)$ to S. We note that if $\mathscr{B}$ has an involution of the first kind τ, then the corestriction involution $N_{V/U}(\tau|_V)$, given by the restriction of $\iota_*\tau|_V \otimes \tau|_V$ to $N_{V/U}(\mathscr{A}|_V)$, is of orthogonal type. If $N_{V/U}(\mathscr{B}|_V) \cong \mathscr{E}nd(\mathscr{E}^U)$ is split, then $N_{V/U}(\tau|_V)$ is adjoint to a regular line bundle-valued quadratic form $(\mathscr{E}^U, q^U, \mathscr{L}^U)$ on U unique up to projective similarity. The main result of this section is that this extends to a line bundle-valued quadratic form $(\mathscr{E}, q, \mathscr{L})$ on S with simple degeneration along a regular divisor D satisfying $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L}) \cong \mathscr{B}$.

Theorem 5.3. *Let S be a regular integral scheme of dimension ≤ 2 with 2 invertible and $f : T \to S$ a finite flat cover of degree 2 with regular branch divisor D. Let $\mathscr{B}$ be an Azumaya quaternion $\mathscr{O}_T$-algebra with standard involution τ. Suppose that $N_{V/U}(\mathscr{B}|_V)$ is split and $N_{V/U}(\tau|_V)$ is adjoint to a regular line bundle-valued quadratic form $(\mathscr{E}^U, q^U, \mathscr{L}^U)$ on U. There exists a line bundle-valued quadratic form $(\mathscr{E}, q, \mathscr{L})$ on S with simple degeneration along D with multiplicity one, which restricts to $(\mathscr{E}^U, q^U, \mathscr{L}^U)$ on U and such that $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L}) \cong \mathscr{B}$.*

First we need the following lemma. Let S be a normal integral scheme, K its function field, $D \subset S$ a regular divisor, and $\mathscr{O}_{S,D}$ the the semilocal ring at the generic points of D.

Lemma 5.4. *Let S be a normal integral scheme with 2 invertible, $T \to S$ a finite flat cover of degree 2 with regular branch divisor $D \subset S$, and L/K the corresponding extension of function fields. Under the restriction map $H^1_{\text{ét}}(U, \mathbb{Z}/2\mathbb{Z}) \to H^1_{\text{ét}}(K, \mathbb{Z}/2\mathbb{Z}) = K^\times/K^{\times 2}$, the class of the étale quadratic extension $[V/U]$ maps to a square class represented by a parameter $\pi \in K^\times$ of the semilocal ring $\mathscr{O}_{S,D}$.*

Proof. Consider any $\pi \in K^\times$ with $L = K(\sqrt{\pi})$. For any irreducible component D' of D, if $v_{D'}(\pi)$ is even, then we can modify π up to squares in K so that $v_{D'}(\pi) = 0$. But then T/S would be étale at the generic point of D', which is impossible. Hence, $v_{D'}(\pi)$ is odd for every irreducible component D' of D. Since $\mathscr{O}_{S,D}$ is a principal ideal domain, we can modify π up to squares in K so that $v_{D'}(\pi) = 1$ for every component D' of D. Under the restriction map $H^1_{\text{ét}}(U, \mathbb{Z}/2\mathbb{Z}) \to H^1_{\text{ét}}(K, \mathbb{Z}/2\mathbb{Z}) = K^\times/K^{\times 2}$, the class $[V/U]$ is mapped to the class $[L/K]$, which corresponds via Kummer theory to the square class (π). ☐

Proof of Theorem 5.3. If $D = \cup D_i$ is the irreducible decomposition of D and π_i is a parameter of $\mathscr{O}_{S,D_i}$, then $\pi = \prod_i \pi_i$ is a parameter of $\mathscr{O}_{S,D}$. Choose a regular quadratic form $(\mathscr{E}^U, q^U, \mathscr{L}^U)$ on U adjoint to $N_{V/S}(\sigma|_V)$. Since $\mathscr{O}_{S,D}$ is a principal ideal domain, modifying by squares over K, the form $q^U|_K$ has a diagonalization $< a_1, a_2, a_3, a_4 >$, where $a_i \in \mathscr{O}_{S,D}$ are squarefree. By Lemma 5.4, we can choose $\pi \in K^\times$ so that $[V/U] \in H^1_{\text{ét}}(U, \mathbb{Z}/2\mathbb{Z})$ maps to the square class (π). By Theorem 2.5, the class $[V/U]$ maps to the discriminant of $q^U|_K$. Since $\mathscr{O}_{S,D}$ is a principal ideal domain, $a_1 \cdots a_4 = \mu^2 \pi$, for some $\mu \in \mathscr{O}_{S,D}$. If π_i divides μ, then π_i divides exactly 3 of a_1, a_2, a_3, a_4, so that clearing squares from the entries of $\mu < a_1, a_2, a_3, a_4 >$ yields a form $< a_1', a_2', a_3', a_4' >$ over $\mathscr{O}_{S,D}$ with simple degeneration along D, which over K, is isometric to $\mu q^U|_K$. Define

$$(\mathscr{E}^D, q^D, \mathscr{L}^D) = (\mathscr{O}_{S,D}^4, < a_1', a_2', a_3', a_4' >, \mathscr{O}_{S,D}).$$

By definition, the identity map is a similarity $q^U|_K \simeq q^D|_K$ with similarity factor μ (up to $K^{\times 2}$). Our aim is to find a good similarity enabling a gluing to a quadratic form over S with simple degeneration along D and the correct even Clifford algebra.
First note that by the classical theory of ${}^2\mathsf{A}_1 = \mathsf{D}_2$ over V/U (cf. Theorem 2.5), we can choose an $\mathscr{O}_V$-algebra isomorphism $\varphi^U : \mathscr{C}_0(\mathscr{E}^U, q^U, \mathscr{L}^U) \to \mathscr{B}|_V$. Second, we can pick an $\mathscr{O}_{T,E}$-algebra isomorphism $\varphi^D : \mathscr{C}_0(q^D) \to \mathscr{B}|_E$, where $E = f^{-1}D$. Indeed, by the classical theory of ${}^2\mathsf{A}_1 = \mathsf{D}_2$ over L/K (cf. Theorem 2.5), the central simple algebras $\mathscr{C}_0(q^D)|_L$ and $\mathscr{B}|_L$ are isomorphic over L, hence they are isomorphic over the semilocal principal ideal domain $\mathscr{O}_{T,E}$. Now consider the L-isomorphism $\varphi^L = (\varphi^U|_L)^{-1} \circ \varphi^D|_L : \mathscr{C}_0(q^D)|_L \to \mathscr{C}_0(q^U)|_L$. Again by the classical theory of ${}^2\mathsf{A}_1 = \mathsf{D}_2$ over L/K (cf. Theorem 2.5), this is induced by a similarity $\psi^K : q^D|_K \to q^U|_K$, unique up to multiplication by scalars. By Proposition 4.2, the quadratic forms $(\mathscr{E}^U, q^U, \mathscr{L}^U)$ and $(\mathscr{E}^D, q^D, \mathscr{L}^D)$ glue, via the similarity ψ^K, to a quadratic form $(\mathscr{E}^{U'}, q^{U'}, \mathscr{L}^{U'})$ on a dense open subscheme $U' \subset S$ containing U and the generic points of D, hence all points of codimension 1. By [19, Prop. 2.3], the quadratic form $(\mathscr{E}^{U'}, q^{U'}, \mathscr{L}^{U'})$ extends uniquely to a quadratic form $(\mathscr{E}, q, \mathscr{L})$ on S since the underlying vector bundle $\mathscr{E}^{U'}$ extends to a vector bundle $\mathscr{E}$ on S (because S is a regular integral schemes of dimension ≤ 2). By Corollary 3.13, this extension has simple degeneration along D.
Finally, we argue that $\mathscr{C}_0(q) \cong \mathscr{B}$. We know that $q|_U = q^U$ and $q|_D = q^D$ and we have algebra isomorphisms $\varphi^U : \mathscr{C}_0(q)|_U \cong \mathscr{B}|_U$ and $\varphi^D : \mathscr{C}_0(q)|_D \cong \mathscr{B}|_D$ such that $\varphi^L = (\varphi^U|_L)^{-1} \circ \varphi^D|_L$. Hence the gluing data $(\mathscr{C}_0(q)|_U, \mathscr{C}_0(q)|_D, \varphi^L)$ is isomorphic to the gluing data $(\mathscr{B}|_U, \mathscr{B}|_D, \mathrm{id})$. Thus $\mathscr{C}_0(q)$ and $\mathscr{B}$ are isomorphic over an open subset $U' \subset S$ containing all codimension 1 points of S. Hence by [19, Thm. 6.13], these Azumaya algebras are isomorphic over S. ☐

Finally, we can prove our main result.

Proof of Theorem 1. Theorem 5.3 implies that $\mathscr{C}_0 : \mathrm{Quad}_2(T/S) \to \mathrm{Az}_2(T/S)$ is surjective. To prove the injectivity, let $(\mathscr{E}_1, q_1, \mathscr{L}_1)$ and $(\mathscr{E}_2, q_2, \mathscr{L}_2)$ be line bundle-valued quadratic forms of rank 4 on S with simple degeneration along D of multiplicity one such that there is an $\mathscr{O}_T$-algebra isomorphism $\varphi : \mathscr{C}_0(q_1) \cong \mathscr{C}_0(q_2)$. By the classical theory of ${}^2\mathsf{A}_1 = \mathsf{D}_2$ over V/U (cf. Theorem 2.5), we know that $\varphi|_U : \mathscr{C}_0(q_1)|_U \cong \mathscr{C}_0(q_2)|_U$ is induced by a similarity transformation $\psi^U : q_1|_U \simeq q_2|_U \otimes \mathscr{N}^U$, for some line bundle $\mathscr{N}^U$ on U, which we can assume is the restriction of a line bundle $\mathscr{N}$ on S. Replacing $(\mathscr{E}_2, q_2, \mathscr{L}_2)$ by $(\mathscr{E}_2 \otimes \mathscr{N}, q_2 \otimes {<}1{>}, \mathscr{L} \otimes \mathscr{N}^{\otimes 2})$, which is in the same projective similarity class, we can assume that $\psi^U : q_1|_U \simeq q_2|_U$. In particular, $\mathscr{L}_1|_U \cong \mathscr{L}_2|_U$ so that we have $\mathscr{L}_1 \cong \mathscr{L}_2 \otimes \mathscr{M}$ for some $\mathscr{M}|_U \cong \mathscr{O}_U$ by the exact excision sequence

$$A^0(D) \to \mathrm{Pic}(S) \to \mathrm{Pic}(U) \to 0$$

of Picard groups (really Weil divisor class groups), cf. [32, Cor. 21.6.10], [30, 1 Prop. 1.8].

By Theorem 3.11, we know that $\varphi|_D : \mathscr{C}_0(q_1)|_E \cong \mathscr{C}_0(q_2)|_E$ is induced by some similarity transformation $\psi^D : q_1|_D \simeq q_2|_D$. Thus $\psi^K = (\psi^D|_K)^{-1} \circ \psi^U|_K \in \mathbf{GO}(q_1|_K)$. Since $\mathscr{C}_0(\psi^U|_K) = \mathscr{C}_0(\psi^D|_K) = \varphi|_K$, we have that $\psi^K \in \mathbf{GO}(q_1|_K)$ is a homothety, multiplication by $\lambda \in K^\times$. As in §4, define a line bundle $\mathscr{P}$ on S by the gluing datum $(\mathscr{O}_U, \mathscr{O}_D, \lambda^{-1} : \mathscr{O}_U|_K \cong \mathscr{O}_D|_K)$. Then $\mathscr{P}$ comes equipped with isomorphisms $\rho^U : \mathscr{O}_U \cong \mathscr{P}|_U$ and $\rho^D : \mathscr{O}_D \cong \mathscr{P}|_D$ with $(\rho^D|_K)^{-1} \circ \rho^U|_K = \lambda^{-1}$. Then we have similarities $\psi^U \otimes \rho^U : q_1|_U \simeq q_2|_D \otimes \mathscr{P}|_U$ and $\psi^D \otimes \rho^D : q_1|_D \simeq q_2|_D \otimes \mathscr{P}|_D$ such that

$$(\psi^D \otimes \rho^D)|_K^{-1} \circ (\psi^U \otimes \rho^U)|_K = (\psi^D|_K^{-1} \psi^U|_K)(\rho^D|_K^{-1} \rho^U|_K) = \psi^K \lambda^{-1} = \mathrm{id}$$

in $\mathbf{GO}(q_1|_K)$. Hence, as in §4, $\psi^U \otimes \rho^U$ and $\psi^D \otimes \rho^D$ glue to a similarity $(\mathscr{E}_1, q_1, \mathscr{L}_1) \simeq (\mathscr{E}_2 \otimes \mathscr{P}, q_2 \otimes <1>, \mathscr{L}_2 \otimes \mathscr{P}^{\otimes 2})$. Thus $(\mathscr{E}_1, q_1, \mathscr{L}_1)$ and $(\mathscr{E}_2, q_2, \mathscr{L}_2)$ define the same element of $\mathrm{Quad}_2(T/S)$. □

6 Failure of the local-global principle for isotropy of quadratic forms over surfaces

In this section, we mention one application of the theory of quadratic forms with simple degeneration over surfaces. Let S be a regular proper integral scheme of dimension d over an algebraically closed field k of characteristic $\neq 2$. For a point x of X, denote by K_x the fraction field of the completion $\widehat{\mathscr{O}}_{S,x}$ of $\mathscr{O}_{S,x}$ at its maximal ideal.

Lemma 6.1. *Let S be a regular integral scheme of dimension d over an algebraically closed field k of characteristic $\neq 2$ and let $D \subset S$ be a divisor. Fix $i > 0$. If $(\mathscr{E}, q, \mathscr{L})$ is a quadratic form of rank $> 2^{d-i} + 1$ over S with simple degeneration along D then q is isotropic over K_x for all points x of S of codimension $\geq i$.*

Proof. The residue field $\kappa(x)$ of K_x has transcendence degree $\leq d - i$ over k and is hence a C_{d-i}-field. By hypothesis, q has, over K_x, a subform q_1 of rank $> 2^{d-i}$ that is regular over $\widehat{\mathscr{O}}_{S,x}$. Hence q_1 is isotropic over $\kappa(x)$, thus q is isotropic over the complete field K_x. □

As usual, denote by $K = k(S)$ the function field. We say that a quadratic form q over K is *locally isotropic* if q is isotropic over K_x for all points x of codimension one.

Corollary 6.2. *Let S be a proper regular integral surface over an algebraically closed field k of characteristic $\neq 2$ and let $D \subset S$ be a regular divisor. If $(\mathscr{E}, q, \mathscr{L})$ is a quadratic form of rank ≥ 4 over S with simple degeneration along D then q over K is locally isotropic.*

For a different proof of this corollary, see [49, §3]. However, quadratic forms with simple degeneration are mostly anisotropic.

Theorem 6.3. *Let S be a proper regular integral surface over an algebraically closed field k of characteristic $\neq 2$. Assume that ${}_2\mathrm{Br}(S)$ trivial. Let $T \to S$ be a finite flat morphism of degree 2 with regular branch divisor $D \subset S$. Then each nontrivial class in ${}_2\mathrm{Br}(T)$ gives rise to a locally isotropic, yet anisotropic, quadratic form over $k(S)$, unique up to similarity.*

Proof. Let $L = k(T)$ and $K = k(S)$. Let $\beta \in {}_2\mathrm{Br}(T)$ be nontrivial. By a result of Artin [1], the class $\beta|_L \in {}_2\mathrm{Br}(L)$ has index 2. Thus by purity for division algebras over regular surfaces (Theorem 4.3), there exists an Azumaya quaternion algebra $\mathscr{B}$ over T whose Brauer class is β. Since $N_{L/K}(\beta|_L)$ is unramified on S, by Lemma 5.1, it extends to an element of ${}_2\mathrm{Br}(S)$, which is assumed to be trivial. Hence $\mathscr{B} \in \mathrm{Az}_2(T/S)$.
By the classical theory of ${}^2\mathsf{A}_1 = \mathsf{D}_2$ over L/K (cf. Theorem 2.5), the quaternion algebra $\mathscr{B}|_L$ corresponds to a unique similarity class of quadratic form q^K of rank 4 on K. The crucial contribution of our work is that we can control the degeneration divisor of an extension of q^K to a quadratic form on S. Indeed, by Theorem 1, $\mathscr{B}$ corresponds to a unique projective similarity class of quadratic form $(\mathscr{E}, q, \mathscr{L})$ of rank 4 with simple degeneration along D that is generically similar, by the compatibility of the norm constructions in Theorems 2.5 and 5.3, to q^K. Thus by Corollary 6.2, q^K is locally isotropic.
A classical result in the theory of quadratic forms of rank 4 is that q^K is isotropic over K if and only if $\mathscr{C}_0(q^K)$ splits over L (since L/K is the discriminant extension of q^K), see [42, Thm. 6.3], [55, 2 Thm. 14.1, Lemma 14.2], or [8, II Prop. 5.3]. Hence q^K is anisotropic since $\mathscr{C}_0(q^K) = \mathscr{B}_L$ has nontrivial Brauer class β by construction. ☐

We can make Theorem 6.3 explicit as follows. Write $L = K(\sqrt{d})$. Let $\mathscr{B}$ be an Azumaya quaternion algebra over T, with $\mathscr{B}_L$ given by the quaternion symbol (a, b) over L. Since $N_{L/K}(\mathscr{B}_L)$ is trivial, the restriction-corestriction sequence shows that $\mathscr{B}|_L$ is the restriction of a class from ${}_2\mathrm{Br}(K)$, so we can choose $a, b \in K^\times$. The corresponding quadratic form over K (from Theorem 1) is then given, up to similarity, by $< 1, a, b, abd >$. Indeed, its similarity class is uniquely characterized by having discriminant d and even Clifford invariant (a, b) over L, see [42].
In order to produce counterexamples to the local-global principle for isotropy of quadratic forms over a given surface, we need branched double covers with nontrivial 2-torsion in their Brauer group. This always exists, at least assuming characteristic zero.

Proposition 6.4. *Let S be a smooth projective surface over an algebraically closed field k of characteristic zero. Then there exists a finite flat double cover $T \to S$ with smooth branch divisor $D \subset S$ such that ${}_2\mathrm{Br}(T) \neq 0$.*

Proof. Choose a very ample line bundle $\mathscr{N}$ on S. By Serre's theorem [34, II Thm. 5.17], there exists n_0 such that $\omega_S \otimes \mathscr{N}^{\otimes n}$ is generated by global sections for all $n \geq n_0$. We are free to enlarge n_0 as we wish. Write $\mathscr{M} = \mathscr{N}^{\otimes n_0}$.

Let $\varphi : S \to \mathbb{P}^N$ be the projective embedding associated to the very ample line bundle $\mathscr{M}^{\otimes 2}$. Then by Bertini's theorem, there exists a hyperplane $H \subset \mathbb{P}^N$ such that $D = H \cap S$ is a smooth divisor of S. As $\mathscr{M}^{\otimes 2} \cong \mathscr{O}_S(D)$, there exists a nonzero section $s \in \Gamma(S, \mathscr{M}^{\otimes 2})$ with D as divisor of zeros. Then s defines an $\mathscr{O}_S$-algebra structure on $\mathscr{O}_S \oplus \mathscr{M}^\vee$ and let $f : T \to S$ be the finite flat double cover associated to its relative spectrum, i.e., the cyclic double cover taking a square root of D. As D is smooth, T is a smooth projective surface. We will argue that taking the degree of the embedding φ large enough (i.e., taking n_0 large enough) will suffice.
The double cover is tame, so we have $\omega_T \cong f^*(\omega_S \otimes \mathscr{M})$. Then

$$H^0(T, \omega_T) \cong H^0(S, f_*\omega_T) \cong H^0(S, \omega_S) \oplus H^0(S, \omega_S \otimes \mathscr{M})$$

is a k-vector space of positive dimension, since $\omega_S \otimes \mathscr{M}$ is generated by global sections. Hence $h^{2,0}(T) = \dim_k H^0(T, \omega_{T/k}) > 0$. In general, the Hodge numbers are defined as $h^{p,q}(T) = \dim_k H^q(T, \Omega^p_{T/k})$. From the Kummer exact sequence, we derive a short exact sequence

$$0 \to \operatorname{Pic}(T) \otimes_{\mathbb{Z}} \mathbb{Z}_2 \to H^2_{\text{ét}}(T, \mathbb{Z}_2(1)) \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}_2/\mathbb{Z}_2, H^2_{\text{ét}}(T, \mathbf{G}_{\mathrm{m}})) \to 0.$$

As T is smooth and proper over a field, the 2-adic cohomology groups are of cofinite type, thus we get isomorphisms of 2-primary torsion subgroups

$$\operatorname{Br}(T)[2^\infty] \cong H^2(T, \mathbf{G}_{\mathrm{m}})[2^\infty] \cong (\mathbb{Q}_2/\mathbb{Z}_2)^{b_2(T)-\rho(T)} \times G,$$

for some finite group G, where $b_2(T) = \dim_{\mathbb{Z}_2} H^2_{\text{ét}}(T, \mathbb{Z}_2)$ is the 2nd 2-adic Betti number, and $\rho(T)$ is the rank of the Néron–Severi group of T. By the degeneration of the Hodge–de Rham spectral sequence for smooth projective varieties in characteristic zero, the Hodge decomposition yields $b_2(T) = h^{2,0}(T) + h^{1,1}(T) + h^{0,2}(T)$ and we note that $\rho(T) \leq h^{1,1}(T)$. Hence by construction, $b_2(T) - \rho(T) \geq 2h^{2,0}(T) > 0$. In particular, we have ${}_2\operatorname{Br}(T) \neq 0$. □

We remark that the characteristic zero hypothesis can be relaxed to the condition that the Hodge–de Rham spectral sequence degenerates at the first page, since all we used in the proof of Lemma 6.4 was the Hodge decomposition. By [24], for a smooth surface S over a perfect field of characteristic $\neq 2$, it is sufficient to assume that S admits a smooth lift to the Witt vectors $W_2(k)$ of length 2. In any case, we wonder whether it is possible to remove the characteristic zero hypothesis in general.

Corollary 6.5. *Let K be a field finitely generated of transcendence degree 2 over an algebraically closed field k of characteristic zero. Then there exist anisotropic quadratic forms q of rank 4 over K such that q_v is isotropic for every rank 1 discrete valuation v on K.*

Proof. By resolution of singularities and Chow's lemma, we can find a smooth projective connected surface S over k with function field K. If ${}_2\operatorname{Br}(S) \neq 0$, then as before, by purity for division algebras (Theorem 4.3) and Artin's result [1], any nontrivial β in ${}_2\mathrm{Br}(S)$ is represented by an Azumaya quaternion algebra $\mathscr{B}$ over S. Then the norm form $\mathrm{Nrd} : \mathscr{B} \to \mathscr{O}_S$ is locally isotropic by Tsen's theorem (cf. Lemma 6.1) yet is globally anisotropic. Hence we can assume that ${}_2\mathrm{Br}(S) = 0$. Appealing to Proposition 6.4, we have a finite flat morphism $T \to S$ of degree 2 with regular branch divisor such that ${}_2\mathrm{Br}(T) \neq 0$. We then apply Theorem 6.3 to provide the counterexamples. $\square$

Example 6.6. Let $T \to \mathbb{P}^2$ be a double cover branched over a smooth sextic curve over an algebraically closed field of characteristic $\neq 2$. Then T is a smooth projective K3 surface of degree 2. We remark that $b_2(T) = 22$ and that $\rho(T) \leq 20$. In fact, S admits a smooth lift to the Witt vectors by [23]. In particular, ${}_2\mathrm{Br}(T) \cong (\mathbb{Z}/2\mathbb{Z})^{22-\rho} \neq 0$, so that T gives rise to $2^{22-\rho} - 1$ similarity classes of locally isotropic yet anisotropic quadratic forms of rank 4 over $K = k(\mathbb{P}^2)$. This proves that the explicit Brauer classes constructed in [37] and [5] give rise to explicit quadratic forms that are counterexamples to the local-global principle.

We remark that while counter-examples to the local-global principle for isotropy of quadratic forms over the function field of a surfaces S could have previously been constructed from unramified quaternion algebras on S (cf. [18, Prop. 11]), such an approach cannot be used, for example, over rational surfaces.

7 A Torelli theorem for general cubic fourfolds containing a plane

Let Y be a *cubic fourfold*, i.e., a smooth cubic hypersurface of $\mathbb{P}^5 = \mathbb{P}(V)$ over $\mathbb{C}$. Let $W \subset V$ be a vector subspace of dimension three, $P = \mathbb{P}(W) \subset \mathbb{P}(V)$ the associated plane, and $P' = \mathbb{P}(V/W)$. If Y contains P, let $\widetilde{Y}$ be the blow-up of Y along P and $\pi : \widetilde{Y} \to P'$ the projection from P. The blow-up of $\mathbb{P}^5$ along P is isomorphic to the total space of the projective bundle $p : \mathbb{P}(\mathscr{E}) \to P'$, where $\mathscr{E} = W \otimes \mathscr{O}_{P'} \oplus \mathscr{O}_{P'}(-1)$, and in which $\pi : \widetilde{Y} \to P'$ embeds as a quadric surface bundle. The degeneration divisor of π is a sextic curve $D \subset P'$. It is known that D is smooth and π has simple degeneration along D if and only if Y does not contain any other plane meeting P, cf. [60, §1, Lemme 2]. In this case, the discriminant cover $T \to P'$ is a K3 surface of degree 2. All K3 surfaces considered will be smooth and projective.
We choose an identification $P' = \mathbb{P}^2$ and suppose, for the rest of this section, that $\pi : \widetilde{Y} \to P' = \mathbb{P}^2$ has simple degeneration. If Y contains another plane R disjoint from P, then $R \subset \widetilde{Y}$ is the image of a section of π, hence $\mathscr{C}_0(\pi)$ has trivial Brauer class over T by a classical result concerning quadratic forms of rank 4, cf. proof of Theorem 6.3. Thus if $\mathscr{C}_0(\pi)$ has nontrivial Brauer class $\beta \in {}_2\mathrm{Br}(T)$, then P is the unique plane contained in Y.
Given a scheme T with 2 invertible and an Azumaya quaternion algebra $\mathscr{B}$ on T, there is a standard choice of lift $[\mathscr{B}] \in H^2_{\text{ét}}(T, \boldsymbol{\mu}_2)$ of the Brauer class of $\mathscr{B}$,

defined in [52] by taking into account the standard symplectic involution on $\mathscr{B}$. Denote by $c_1 : \mathrm{Pic}(T) \to H^2_{\text{ét}}(T, \boldsymbol{\mu}_2)$ the mod 2 cycle class map arising from the Kummer sequence.

Definition 7.1. Let T be a K3 surface of degree 2 over k together with a *polarization* $\mathscr{F}$, i.e., an ample line bundle of self-intersection 2. For $\beta \in H^2_{\text{ét}}(T, \boldsymbol{\mu}_2)/\langle c_1(\mathscr{F})\rangle$, we say that a cubic fourfold Y *represents* β if Y contains a plane whose associated quadric bundle $\pi : \widetilde{Y} \to \mathbb{P}^2$ has simple degeneration and discriminant cover $f : T \to \mathbb{P}^2$ satisfying $f^*\mathscr{O}_{\mathbb{P}^2}(1) \cong \mathscr{F}$ and $[\mathscr{C}_0(\pi)] = \beta$.

Remark 7.2. For a K3 surface T of degree 2 with a polarization $\mathscr{F}$, not every class in $H^2_{\text{ét}}(T, \boldsymbol{\mu}_2)/\langle c_1(\mathscr{F})\rangle$ is represented by a cubic fourfold, though one can characterize such classes. Consider the cup product pairing in étale cohomology $H^2_{\text{ét}}(T, \boldsymbol{\mu}_2) \times H^2_{\text{ét}}(T, \boldsymbol{\mu}_2) \to H^4_{\text{ét}}(T, \boldsymbol{\mu}_2^{\otimes 2}) \cong \mathbb{Z}/2\mathbb{Z}$. Define

$$B(T, \mathscr{F}) = \{\overline{x} \in H^2_{\text{ét}}(T, \boldsymbol{\mu}_2)/\langle c_1(\mathscr{F})\rangle \;\big|\; x \cup c_1(\mathscr{F}) \neq 0\}.$$

Note that the natural map $B(T, \mathscr{F}) \to {}_2\mathrm{Br}(T)$ is injective if and only if $\mathrm{Pic}(T)$ is generated by $\mathscr{F}$. A consequence of the global description of the period domain for cubic fourfolds containing a plane is that for a K3 surface T of degree 2 with polarization $\mathscr{F}$, the subset of $H^2_{\text{ét}}(T, \boldsymbol{\mu}_2)/\langle c_1(\mathscr{F})\rangle$ represented by a cubic fourfolds containing a plane coincides with $B(T, f) \cup \{0\}$, cf. [58, §9.7] and [37, Prop. 2.1].

We can now state the main result of this section. Using Theorem 1 and results on twisted sheaves described below, we provide an algebraic proof of the following result, which is due to Voisin [60] (cf. [58, §9.7] and [37, Prop. 2.1]). See [12, Prop. 6.3] for a related result.

Theorem 7.3. *Let T be a general K3 surface of degree 2 with a polarization $\mathscr{F}$. Then each element of $B(T, \mathscr{F})$ is represented by a single cubic fourfold containing a plane up to isomorphism.*

We now explain the interest in this statement. The global Torelli theorem for cubic fourfolds states that a cubic fourfold Y is determined up to isomorphic by the polarized Hodge structure on $H^4(Y, \mathbb{Z})$. Here *polarization* means a class $h^2 \in H^4(Y, \mathbb{Z})$ of self-intersection 3. Voisin's approach [60] is to deal first with cubic fourfolds containing a plane, then apply a deformation argument to handle the general case. For cubic fourfolds containing a plane, we can give an alternate argument in the general case, assuming the global Torelli theorem for K3 surfaces of degree 2, which is a celebrated result of Piatetski-Shapiro and Shafarevich [53].

Proposition 7.4. *Assuming the global Torelli theorem holds for K3 surfaces of degree 2, the global Torelli theorem holds for general cubic fourfolds.*

Proof. Let Y be a cubic fourfold containing a plane P with discriminant cover $f : T \to \mathbb{P}^2$ and even Clifford algebra $\mathscr{C}_0$. Consider the cycle class of P

in $H^4(Y,\mathbb{Z})$. Then $\mathscr{F} = f^*\mathscr{O}_{\mathbb{P}^2}(1)$ is a polarization on T, which together with $[\mathscr{C}_0] \in H^2_{\text{ét}}(T,\boldsymbol{\mu}_2)$, determines the sublattice $\langle h^2, P\rangle^\perp \subset H^4(Y,\mathbb{Z})$. The key lattice-theoretic result we use is [60, §1, Prop. 3], which can be stated as follows: the polarized Hodge structure $H^2(T,\mathbb{Z})$ and the class $[\mathscr{C}_0] \in H^2_{\text{ét}}(T,\boldsymbol{\mu}_2)$ determines the Hodge structure of Y; conversely, the polarized Hodge structure $H^4(Y,\mathbb{Z})$ and the sublattice $\langle h^2, P\rangle$ determines the primitive Hodge structure of T, hence T itself by the global Torelli theorem for K3 surfaces of degree 2. Furthermore, if Y (and hence T) is general, then $H^4(Y,\mathbb{Z})$ and $\langle h^2, P\rangle$ determines the Brauer class $[\mathscr{C}_0]$ of the even Clifford algebra.
Now let Y and Y' be cubic fourfolds containing a plane P with associated discriminant covers T and T' and even Clifford algebras $\mathscr{C}_0$ and $\mathscr{C}_0'$. Assume that $\Psi : H^4(Y,\mathbb{Z}) \cong H^4(Y,\mathbb{Z})$ is an isomorphism of Hodge structures preserving the polarization h^2. By [35, Prop. 3.2.4], we can assume (by composing Ψ with a Hodge automorphism fixing h^2) that Ψ preserves the sublattice $\langle h^2, P\rangle$. By [60, §1, Prop. 3], Ψ induces an isomorphism $T \cong T'$, with respect to which $[\mathscr{C}_0] = [\mathscr{C}_0'] = \beta \in H^2_{\text{ét}}(T,\boldsymbol{\mu}_2) \cong H^2_{\text{ét}}(T',\boldsymbol{\mu}_2)$, for T general. Hence if there is at most a single cubic fourfold representing β up to isomorphism then $Y \cong Y'$. □

The following lemma, whose proof we could not find in the literature, holds for smooth cubic hypersurfaces $Y \subset \mathbb{P}^{2r+1}_k$ containing a linear subspace of dimension r over any field k. Since $\mathrm{Aut}(\mathbb{P}^{2r+1}_k) \cong \mathbf{PGL}_{2r+2}(k)$ acts transitively on the set of linear subspaces in $\mathbb{P}^{2r+1}_k$ of dimension r, any two cubic hypersurfaces containing linear subspaces of dimension r have isomorphic representatives containing a common such linear subspace.

Lemma 7.5. *Let Y_1 and Y_2 be smooth cubic hypersurfaces in $\mathbb{P}^{2r+1}_k$ containing a linear space P of dimension r. The associated quadric bundles $\pi_1 : \widetilde{Y}_1 \to \mathbb{P}^r_k$ and $\pi_2 : \widetilde{Y}_2 \to \mathbb{P}^r_k$ are $\mathbb{P}^r_k$-isomorphic if and only if the there is a linear isomorphism $Y_1 \cong Y_2$ fixing P.*

Proof. Any linear isomorphism $Y_1 \cong Y_2$ fixing P will induce an isomorphism of blow-ups $\widetilde{Y}_1 \cong \widetilde{Y}_2$ commuting with the projections from P. Conversely, assume that $\widetilde{Y}_1$ and $\widetilde{Y}_2$ are $\mathbb{P}^r_k$-isomorphic. Since $\mathbf{PGL}_{2r+2}(k)$ acts transitively on the set of linear subspaces of dimension r, without loss of generality, we can assume that $P = \{x_0 = \cdots = x_r = 0\}$ where $(x_0 : \cdots : x_r : y_0 : \cdots : y_r)$ are homogeneous coordinates on $\mathbb{P}^{2r+1}_k$. For $l = 1, 2$, write Y_l as

$$\sum_{0\le m\le n\le r} a^l_{mn} y_m y_n + \sum_{0\le p\le r} b^l_p y_p + c^l = 0$$

for homogeneous linear forms a^l_{mn}, quadratic forms b^l_p, and cubic forms c^l in $k[x_0,\dots,x_r]$. The blow-up of $\mathbb{P}^{2r+1}_k$ along P is identified with the total space of the projective bundle $\pi : \mathbb{P}(\mathscr{E}) \to \mathbb{P}^r_k$, where $\mathscr{E} = \mathscr{O}^{r+1}_{\mathbb{P}^r_k} \oplus \mathscr{O}_{\mathbb{P}^r_k}(-1)$. The homogeneous coordinates $y_0,\dots,y_r$ correspond, in the blow-up, to a basis of global sections of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$. Let z be a nonzero global section of of line bundle

$\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) \otimes \pi^* \mathscr{O}_{\mathbb{P}^r_k}(-1)$. Then z is unique up to scaling, as we have

$$\begin{aligned}\Gamma(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) \otimes \pi^* \mathscr{O}_{\mathbb{P}^r_k}(-1)) &\cong \Gamma(\mathbb{P}^r_k, \pi_* \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) \otimes \mathscr{O}_{\mathbb{P}^r_k}(-1)) \\ &= \Gamma(\mathbb{P}^r_k, \mathscr{E}^\vee \otimes \mathscr{O}_{\mathbb{P}^r_k}(-1)) = k\end{aligned}$$

by the projection formula. Thus $(y_0 : \cdots : y_r : z)$ forms a relative system of homogeneous coordinates on $\mathbb{P}(\mathscr{E})$ over $\mathbb{P}^r_k$. Then $\widetilde{Y}_l$ can be identified with the subscheme of $\mathbb{P}(\mathscr{E})$ defined by the global section

$$q_l(y_0, \ldots, y_r, z) = \sum_{0 \le m \le n \le r} a^l_{mn} y_m y_n + \sum_{0 \le p \le r} b^l_p y_p z + c^l z^2 = 0$$

of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(2) \otimes \pi^* \mathscr{O}_{\mathbb{P}^r_k}(1)$. Under these identifications, $\pi_l : \widetilde{Y}_l \to \mathbb{P}^r_k$ can be identified with the restriction of π to $\widetilde{Y}_l$, hence with the quadric bundle associated to the line bundle-valued quadratic form $(\mathscr{E}, q_l, \mathscr{O}_{\mathbb{P}^r_k}(1))$. Since Y_l and P are smooth, so is $\widetilde{Y}_l$. Thus $\pi_l : \widetilde{Y}_l \to \mathbb{P}^r_k$ is flat, being a morphism from a Cohen–Macaulay scheme to a regular scheme. Thus by Propositions 1.1 and 1.6, the $\mathbb{P}^r_k$-isomorphism $\widetilde{Y}_1 \cong \widetilde{Y}_2$ induces a projective similarity ψ between q_1 and q_2. But as $\mathscr{E} \otimes \mathscr{N} \cong \mathscr{E}$ implies $\mathscr{N}$ is trivial in $\mathrm{Pic}(\mathbb{P}^r_k)$, we have that $\psi : q_1 \simeq q_2$ is, in fact, a similarity. In particular, $\psi \in \mathbf{GL}(\mathscr{E})(\mathbb{P}^r_k)$, hence consists of a block matrix of the form

$$\begin{pmatrix} H & v \\ 0 & u \end{pmatrix}$$

where $H \in \mathbf{GL}(\mathscr{O}^{r+1}_{\mathbb{P}^r_k})(\mathbb{P}^r_k) = \mathbf{GL}_{r+1}(k)$ is a constant invertible matrix and $u \in \mathbf{GL}(\mathscr{O}_{\mathbb{P}^r_k}(-1))(\mathbb{P}^r_k) = \mathbf{G}_{\mathrm{m}}(\mathbb{P}^r_k) = k^\times$ is an invertible constant element, while $v \in \mathrm{Hom}_{\mathscr{O}_{\mathbb{P}^r_k}}(\mathscr{O}_{\mathbb{P}^r_k}(-1), \mathscr{O}^{r+1}_{\mathbb{P}^r_k}) = \Gamma(\mathbb{P}^r_k, \mathscr{O}_{\mathbb{P}^r_k}(1))^{\oplus(r+1)}$ consists of a vector of linear forms in $k[x_0, \ldots, x_r]$. Let $v = G \cdot (x_0, \ldots, x_r)^t$ for a matrix $G \in M_{r+1}(k)$. Then writing $H = (h_{ij})$ and $G = (g_{ij})$, we have that ψ acts as

$$x_i \mapsto x_i, \qquad y_i \mapsto \sum_{0 \le j \le r} (h_{ij} y_j + g_{ij} x_j z), \qquad z \mapsto uz$$

and satisfies $q_2(\psi(y_0), \ldots, \psi(y_r), \psi(z)) = \lambda q_1(y_0, \ldots, y_r, z)$ for some $\lambda \in k^\times$. Considering the matrix $J \in M_{2r+2}(k)$ with $(r+1) \times (r+1)$ blocks

$$J = \begin{pmatrix} uI & 0 \\ G & H \end{pmatrix}$$

as a linear automorphism of $\mathbb{P}^{2r+1}_k$, then J acts on $(x_0 : \cdots : x_r : y_0 : \cdots : y_r)$ as

$$x_i \mapsto ux_i, \qquad y_i \mapsto \sum_{0 \le j \le r} (h_{ij} y_j + g_{ij} x_j),$$

and hence satisfies $q_2(J(y_0), \ldots, J(y_r), 1) = u\lambda q_1(y_0, \ldots, y_r, 1)$ due to the homogeneity properties of x_i and z. Thus J is a linear automorphism taking Y_1 to Y_2 and fixes P. $\square$

Let T be a K3 surface. We shall freely use the notions of β-twisted sheaves, B-fields associated to β, the β-twisted Chern character, and β-twisted Mukai vectors from [38]. For a Brauer class $\beta \in {}_2\mathrm{Br}(T)$ we choose the rational B-field $\beta/2 \in H^2(T, \mathbb{Q})$. The β-twisted Mukai vector of a β-twisted sheaf $\mathscr{V}$ is

$$v^B(\mathscr{V}) = \mathrm{ch}^B(\mathscr{V})\sqrt{\mathrm{Td}_T} = \big(\mathrm{rk}\,\mathscr{V}, c_1^B(\mathscr{V}), \mathrm{rk}\,\mathscr{V} + \frac{1}{2}c_1^B(\mathscr{V}) - c_2^B(\mathscr{V})\big) \in H^*(T, \mathbb{Q})$$

where $H^*(T,\mathbb{Q}) = \bigoplus_{i=0}^2 H^{2i}(T,\mathbb{Q})$. As in [47], one introduces the Mukai pairing

$$(v, w) = v_2 \cup w_2 - v_0 \cup w_4 - v_4 \cup w_0 \in H^4(T, \mathbb{Q}) \cong \mathbb{Q}$$

for Mukai vectors $v = (v_0, v_2, v_4)$ and $w = (w_0, w_2, w_4)$.
By [63, Thm. 3.16], the moduli space of stable β-twisted sheaves $\mathscr{V}$ with Mukai vector $v = v^B(\mathscr{V})$ satisfying $(v, v) = 2n$ is isomorphic to the Hilbert scheme Hilb_T^{n+1}. In particular, when $(v, v) = -2$, this moduli space consists of one point; we give a direct proof of this fact inspired by [47, Cor. 3.6].

Lemma 7.6. *Let T be a K3 surface and $\beta \in {}_2\mathrm{Br}(T)$ with chosen B-field. Let $v \in H^*(T, \mathbb{Q})$ with $(v, v) = -2$. If $\mathscr{V}$ and $\mathscr{V}'$ are stable β-twisted sheaves with $v^B(\mathscr{V}) = v^B(\mathscr{V}') = v$ then $\mathscr{V} \cong \mathscr{V}'$.*

Proof. Assume that β-twisted sheaves $\mathscr{V}$ and $\mathscr{V}'$ have the same Mukai vector $v \in H^2(T, \mathbb{Q})$. Since $-2 = (v, v) = \chi(\mathscr{V}, \mathscr{V}) = \chi(\mathscr{V}, \mathscr{V}')$, a Riemann–Roch calculation shows that either $\mathrm{Hom}(\mathscr{V}, \mathscr{V}') \neq 0$ or $\mathrm{Hom}(\mathscr{V}, \mathscr{V}') \neq 0$. Without loss of generality, assume $\mathrm{Hom}(\mathscr{V}, \mathscr{V}') \neq 0$. Since $\mathscr{V}$ is stable, a nonzero map $\mathscr{V} \to \mathscr{V}'$ must be injective. Since $\mathscr{V}'$ is stable, the map is an isomorphism. □

Lemma 7.7. *Let T be a K3 surface of degree 2 and $\beta \in {}_2\mathrm{Br}(T)$ with chosen B-field. Let Y be a smooth cubic fourfold containing a plane whose even Clifford algebra $\mathscr{C}_0$ represents $\beta \in {}_2\mathrm{Br}(T)$. If $\mathscr{V}_0$ is a β-twisted sheaf associated to $\mathscr{C}_0$ then $(v^B(\mathscr{V}_0), v^B(\mathscr{V}_0)) = -2$. Furthermore, if T is general then $\mathscr{V}_0$ is stable.*

Proof. By the β-twisted Riemann–Roch theorem, we have

$$-(v^B(\mathscr{V}_0), v^B(\mathscr{V}_0)) = \chi(\mathscr{V}_0, \mathscr{V}_0) = \sum_{i=0}^{2} \mathrm{Ext}_T^i(\mathscr{V}_0, \mathscr{V}_0).$$

Then $v^B(\mathscr{V}_0) = 2$ results from the fact that $\mathscr{V}_0$ is a *spherical* object, i.e., $\mathrm{Ext}_T^i(\mathscr{V}_0, \mathscr{V}_0) = \mathbb{C}$ for $i = 0, 2$ and $\mathrm{Ext}^1(\mathscr{V}_0, \mathscr{V}_0) = 0$. Indeed, as in [45, Rem. 2.1], we have $\mathrm{Ext}_T^i(\mathscr{V}_0, \mathscr{V}_0) = H^i(\mathbb{P}^2, \mathscr{C}_0)$, which can be calculated directly using the fact that, as $\mathscr{O}_{\mathbb{P}^2}$-algebras,

$$\mathscr{C}_0 \cong \mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(-3) \oplus \mathscr{O}_{\mathbb{P}^2}(-1)^3 \oplus \mathscr{O}_{\mathbb{P}^2}(-2)^3 \tag{10}$$

If T is general, stability follows from [47, Prop. 3.14], cf. [63, Prop. 3.12]. □

Lemma 7.8. *Let T be a K3 surface of degree 2. Let Y and Y' be smooth cubic fourfolds containing a plane whose respective even Clifford algebras $\mathscr{C}_0$ and $\mathscr{C}_0'$ represent the same $\beta \in {}_2\mathrm{Br}(T)$. If T is general then $\mathscr{C}_0 \cong \mathscr{C}_0'$.*

Proof. Let $\mathscr{V}_0$ and $\mathscr{V}_0'$ be β-twisted sheaves associated to $\mathscr{C}_0$ and $\mathscr{C}_0$, respectively. A consequence of [45, Lemma 3.1] and (10) is that $v = v^B(\mathscr{V}_0) = v^B(\mathscr{V}_0' \otimes \mathscr{N})$ for some line bundle $\mathscr{N}$ on T. Replacing $\mathscr{V}_0'$ by $\mathscr{V}_0' \otimes \mathscr{N}^\vee$, we can assume that $v^B(\mathscr{V}_0) = v^B(\mathscr{V}_0')$. By Lemma 7.7, we have $(v, v) = -2$ and that $\mathscr{V}_0$ and $\mathscr{V}_0'$ are stable. Hence by Lemma 7.6, we have $\mathscr{V}_0 \cong \mathscr{V}_0'$ as β-twisted sheaves, hence $\mathscr{C}_0 \cong \mathscr{E}nd(\mathscr{V}_0) \cong \mathscr{E}nd(\mathscr{V}_0') \cong \mathscr{C}_0'$. □

Proof of Theorem 7.3. Suppose that Y and Y' are smooth cubic fourfolds containing a plane whose associated even Clifford algebras $\mathscr{C}_0$ and $\mathscr{C}_0'$ represent the same class $\beta \in B(T, \mathscr{F}) \subset H^2_{\text{ét}}(T, \boldsymbol{\mu}_2)/\langle c_1(\mathscr{F})\rangle \cong {}_2\mathrm{Br}(T)$. By Lemma 7.8, we have $\mathscr{C}_0 \cong \mathscr{C}_0'$. By Theorem 1, the quadric surface bundles $\pi : \widetilde{Y} \to \mathbb{P}^2$ and $\pi' : \widetilde{Y}' \to \mathbb{P}^2$ are $\mathbb{P}^2$-isomorphic. Finally, by Lemma 7.5, we have $Y \cong Y'$. □

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