Stably Cayley Semisimple Groups

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Abstract. A linear algebraic group G over a field k is called a Cayley group if it admits a Cayley map, i.e., a G-equivariant birational isomorphism over k between the group variety G and its Lie algebra $\mathrm{Lie}(G)$. A prototypical example is the classical "Cayley transform" for the special orthogonal group $\mathbf{SO}_n$ defined by Arthur Cayley in 1846. A linear algebraic group G is called stably Cayley if $G \times S$ is Cayley for some split k-torus S. We classify stably Cayley semisimple groups over an arbitrary field k of characteristic 0.

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To Alexander Merkurjev on the occasion of his 60th birthday

0 Introduction

Let k be a field of characteristic 0 and $\bar{k}$ a fixed algebraic closure of k. Let G be a connected linear algebraic k-group. A birational isomorphism $\phi\colon G \overset{\simeq}{\dashrightarrow} \mathrm{Lie}(G)$ is called a *Cayley map* if it is equivariant with respect to the conjugation action of G on itself and the adjoint action of G on its Lie algebra $\mathrm{Lie}(G)$, respectively. A linear algebraic k-group G is called *Cayley* if it admits a Cayley map, and *stably Cayley* if $G \times_k (\mathbb{G}_{\mathrm{m},k})^r$ is Cayley for some $r \geq 0$. Here $\mathbb{G}_{\mathrm{m},k}$ denotes the multiplicative group over k. These notions were introduced by Lemire, Popov and Reichstein [LPR]; for a more detailed discussion and numerous classical examples we refer the reader to [LPR, Introduction]. The main results of [LPR] are the classifications of Cayley and stably Cayley *simple* groups over an algebraically closed field k of characteristic 0. Over an arbitrary field k of characteristic 0 stably Cayley *simple* k-groups, stably Cayley *simply connected* semisimple k-groups and stably Cayley *adjoint* semisimple k-groups were classified in the paper [BKLR] of Borovoi, Kunyavskiĭ, Lemire and Reichstein. In the present paper, building on results of [LPR] and [BKLR], we classify all stably Cayley *semisimple* k-groups (not necessarily simple, or simply connected, or adjoint) over an arbitrary field k of characteristic 0.

By a semisimple (or reductive) k-group we always mean a *connected* semisimple (or reductive) k-group. We shall need the following result of [BKLR] extending [LPR, Theorem 1.28].

Theorem 0.1 ([BKLR, Theorem 1.4]). *Let k be a field of characteristic 0 and G an absolutely simple k-group. Then the following conditions are equivalent:*

(a) *G is stably Cayley over k;*

(b) *G is an arbitrary k-form of one of the following groups:*

$$\mathbf{SL}_3,\ \mathbf{PGL}_2,\ \mathbf{PGL}_{2n+1}\ (n \geq 1),\ \mathbf{SO}_n\ (n \geq 5),\ \mathbf{Sp}_{2n}\ (n \geq 1),\ \mathbf{G}_2,$$

or an inner *k-form of $\mathbf{PGL}_{2n}$ $(n \geq 2)$.*

In this paper we classify stably Cayley semisimple groups over an *algebraically closed* field k of characteristic 0 (Theorem 0.2) and, more generally, over an *arbitrary* field k of characteristic 0 (Theorem 0.3). Note that Theorem 0.2 was conjectured in [BKLR, Remark 9.3].

Theorem 0.2. *Let k be an algebraically closed field of characteristic 0 and G a semisimple k-group. Then G is stably Cayley if and only if G decomposes into a direct product $G_1 \times_k \cdots \times_k G_s$ of its normal subgroups, where each G_i $(i = 1, \dots, s)$ either is a stably Cayley simple k-group (i.e., isomorphic to one of the groups listed in Theorem 0.1) or is isomorphic to the stably Cayley semisimple k-group $\mathbf{SO}_4$.*

Theorem 0.3. *Let G be a semisimple k-group over a field k of characteristic 0 (not necessarily algebraically closed). Then G is stably Cayley over k if and only if G decomposes into a direct product $G_1 \times_k \cdots \times_k G_s$ of its normal k-subgroups, where each G_i $(i = 1, \dots, s)$ is isomorphic to the Weil restriction $R_{l_i/k} G_{i,l_i}$ for some finite field extension l_i/k, and each G_{i,l_i} is either a stably Cayley absolutely simple group over l_i (i.e., one of the groups listed in Theorem 0.1) or an l_i-form of the semisimple group $\mathbf{SO}_4$ (which is always stably Cayley, but is not absolutely simple and can be not l_i-simple).*

Note that the “if” assertions in Theorems 0.2 and 0.3 follow immediately from the definitions.

The rest of the paper is structured as follows. In Section 1 we recall the definition of a quasi-permutation lattice and state some known results, in particular, an assertion from [LPR, Theorem 1.27] that reduces Theorem 0.2 to an assertion on lattices. In Sections 2 and 3 we construct certain families of non-quasi-permutation lattices. In particular, we correct an inaccuracy in [BKLR]; see Remark 2.5. In Section 4 we prove (in the language of lattices) Theorem

0.2 in the special case when G is isogenous to a direct product of simple groups of type $\mathbf{A}_{n-1}$ with $n \geq 3$. In Section 5 we prove (again in the language of lattices) Theorem 0.2 in the general case. In Section 6 we deduce Theorem 0.3 from Theorem 0.2. In Appendix A we prove in terms of lattices only, that certain quasi-permutation lattices are indeed quasi-permutation.

1 Preliminaries on quasi-permutation groups and on character lattices

In this section we gather definitions and known results concerning quasi-permutation lattices, quasi-invertible lattices and character lattices that we need for the proofs of Theorems 0.2 and 0.3. For details see [BKLR, Sections 2 and 10] and [LPR, Introduction].

1.1. By a *lattice* we mean a pair (Γ, L) where Γ is a finite group acting on a finitely generated free abelian group L. We say also that L is a Γ-lattice. A Γ-lattice L is called a *permutation* lattice if it has a $\mathbb{Z}$-basis permuted by Γ. Following Colliot-Thélène and Sansuc [CTS], we say that two Γ-lattices L and L' are *equivalent*, and write $L \sim L'$, if there exist short exact sequences

$$0 \to L \to E \to P \to 0 \qquad \text{and} \qquad 0 \to L' \to E \to P' \to 0$$

with the same Γ-lattice E, where P and P' are permutation Γ-lattices. For a proof that this is indeed an equivalence relation see [CTS, Lemma 8, p. 182] or [Sw, Section 8]. Note that if there exists a short exact sequence of Γ-lattices

$$0 \to L \to L' \to Q \to 0$$

where Q is a permutation Γ-lattice, then, taking in account the trivial short exact sequence

$$0 \to L' \to L' \to 0 \to 0,$$

we obtain that $L \sim L'$. If Γ-lattices L, L', M, M' satisfy $L \sim L'$ and $M \sim M'$, then clearly $L \oplus M \sim L' \oplus M'$.

Definition 1.2. A Γ-lattice L is called a *quasi-permutation* lattice if there exists a short exact sequence

$$0 \to L \to P \to P' \to 0, \tag{1.1}$$

where both P and P' are permutation Γ-lattices.

Lemma 1.3 (well-known). *A Γ-lattice L is quasi-permutation if and only if $L \sim 0$.*

Proof. If L is quasi-permutation, then sequence (1.1) together with the trivial short exact sequence

$$0 \to 0 \to P \to P \to 0$$

shows that $L \sim 0$. Conversely, if $L \sim 0$, then there are short exact sequences

$$0 \to L \to E \to P \to 0 \qquad \text{and} \qquad 0 \to 0 \to E \to P' \to 0,$$

where P and P' are permutation lattices. From the second exact sequence we have $E \cong P'$, hence E is a permutation lattice, and then the first exact sequence shows that L is a quasi-permutation lattice. □

Definition 1.4. A Γ-lattice L is called *quasi-invertible* if it is a direct summand of a quasi-permutation Γ-lattice.

Note that if a Γ-lattice L is not quasi-invertible, then it is not quasi-permutation.

Lemma 1.5 (well-known). *If a Γ-lattice L is quasi-permutation (resp., quasi-invertible) and $L' \sim L$, then L' is quasi-permutation (resp., quasi-invertible) as well.*

Proof. If L is quasi-permutation, then using Lemma 1.3 we see that $L' \sim L \sim 0$, hence L' is quasi-permutation. If L is quasi-invertible, then $L \oplus M$ is quasi-permutation for some Γ-lattice M, and by Lemma 1.3 we have $L \oplus M \sim 0$. We see that $L' \oplus M \sim L \oplus M \sim 0$, and by Lemma 1.3 we obtain that $L' \oplus M$ is quasi-permutation, hence L' is quasi-invertible. □

Let $\mathbb{Z}[\Gamma]$ denote the group ring of a finite group Γ. We define the Γ-lattice J_Γ by the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{N} \mathbb{Z}[\Gamma] \to J_\Gamma \to 0,$$

where N is the norm map, see [BKLR, before Lemma 10.4]. We refer to [BKLR, Proposition 10.6] for a proof of the following result, due to Voskresenskiĭ [Vo1, Corollary of Theorem 7]:

Proposition 1.6. *Let $\Gamma = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, where p is a prime. Then the Γ-lattice J_Γ is not quasi-invertible.*

Note that if $\Gamma = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then $\operatorname{rank} J_\Gamma = 3$.

We shall use the following lemma from [BKLR]:

Lemma 1.7 ([BKLR, Lemma 2.8]). *Let $W_1, \dots, W_m$ be finite groups. For each $i = 1, \dots, m$, let V_i be a finite-dimensional $\mathbb{Q}$-representation of W_i. Set $V := V_1 \oplus \dots \oplus V_m$. Suppose $L \subset V$ is a free abelian subgroup, invariant under $W := W_1 \times \dots \times W_m$. If L is a quasi-permutation W-lattice, then for each $i = 1, \dots, m$ the intersection $L_i := L \cap V_i$ is a quasi-permutation W_i-lattice.*

We shall need the notion, due to [LPR] and [BKLR], of the character lattice of a reductive k-group G over a field k. Let $\bar{k}$ be a separable closure of k. Let $T \subset G$ be a maximal torus (defined over k). Set $\overline{T} = T \times_k \bar{k}$, $\overline{G} = G \times_k \bar{k}$. Let $\mathsf{X}(\overline{T})$

denote the character group of $\overline{T} := T \times_k \bar{k}$. Let $W = W(\overline{G}, \overline{T}) := \mathcal{N}_G(\overline{T})/\overline{T}$ denote the Weyl group, it acts on $\mathsf{X}(\overline{T})$. Consider the canonical Galois action on $\mathsf{X}(\overline{T})$, it defines a homomorphism $\mathrm{Gal}(\bar{k}/k) \to \mathrm{Aut}\,\mathsf{X}(\overline{T})$. The image $\mathrm{im}\,\rho \subset \mathrm{Aut}\,\mathsf{X}(\overline{T})$ normalizes W, hence $\mathrm{im}\,\rho \cdot W$ is a subgroup of $\mathrm{Aut}\,\mathsf{X}(\overline{T})$. By the character lattice of G we mean the pair $\mathcal{X}(G) := (\mathrm{im}\,\rho \cdot W, \mathsf{X}(\overline{T}))$ (up to an isomorphism it does not depend on the choice of T). In particular, if k is algebraically closed, then $\mathcal{X}(G) = (W, \mathsf{X}(T))$.

We shall reduce Theorem 0.2 to an assertion about quasi-permutation lattices using the following result due to [LPR]:

Proposition 1.8 ([LPR, Theorem 1.27], see also [BKLR, Theorem 1.3]). *A reductive group G over an algebraically closed field k of characteristic 0 is stably Cayley if and only if its character lattice $\mathcal{X}(G)$ is quasi-permutation, i.e., $\mathsf{X}(T)$ is a quasi-permutation $W(G,T)$-lattice.*

We shall use the following result due to Cortella and Kunyavskiĭ [CK] and to Lemire, Popov and Reichstein [LPR].

Proposition 1.9 ([CK], [LPR]). *Let D be a connected Dynkin diagram. Let $R = R(D)$ denote the corresponding root system, $W = W(D)$ denote the Weyl group, $Q = Q(D)$ denote the root lattice, and $P = P(D)$ denote the weight lattice. We say that L is an* intermediate lattice between Q and P *if $Q \subset L \subset P$ (note that $L = Q$ and $L = P$ are possible). Then the following list gives (up to an isomorphism) all the pairs (D, L) such that L is a quasi-permutation intermediate $W(D)$-lattice between $Q(D)$ and $P(D)$:*

$$Q(\mathbf{A}_n),\ Q(\mathbf{B}_n),\ P(\mathbf{C}_n),\ \mathcal{X}(\mathbf{SO}_{2n})\ \textit{(then } D = \mathbf{D}_n\textit{)},$$

or D is any connected Dynkin diagram of rank 1 or 2 (i.e. $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{B}_2$, or $\mathbf{G}_2$) and L is any lattice between $Q(D)$ and $P(D)$, (i.e., either $L = P(D)$ or $L = Q(D)$).

Proof. The positive result (the assertion that the lattices in the list are indeed quasi-permutation) follows from the assertion that the corresponding groups are stably Cayley (or that their generic tori are stably rational), see the references in [CK], Section 3. See Appendix A below for a proof of this positive result in terms of lattices only. The difficult part of Proposition 1.9 is the negative result (the assertion that all the other lattices are not quasi-permutation). This was proved in [CK, Theorem 0.1] in the cases when $L = Q$ or $L = P$, and in [LPR, Propositions 5.1 and 5.2] in the cases when $Q \subsetneq L \subsetneq P$ (this can happen only when $D = \mathbf{A}_n$ or $D = \mathbf{D}_n$). ☐

Remark 1.10. It follows from Proposition 1.9 that, in particular, the following lattices are quasi-permutation: $Q(\mathbf{A}_1)$, $P(\mathbf{A}_1)$, $P(\mathbf{A}_2)$, $P(\mathbf{B}_2)$, $Q(\mathbf{C}_2)$, $Q(\mathbf{G}_2) = P(\mathbf{G}_2)$, $Q(\mathbf{D}_3) = Q(\mathbf{A}_3)$, $\mathcal{X}(\mathbf{SL}_4/\mu_2) = \mathcal{X}(\mathbf{SO}_6)$.

2 A family of non-quasi-permutation lattices

In this section we construct a family of non-quasi-permutation (even non-quasi-invertible) lattices.

2.1. We consider a Dynkin diagram $D \sqcup \Delta$ (disjoint union). We assume that $D = \bigsqcup_{i \in I} D_i$ (a finite disjoint union), where each D_i is of type $\mathbf{B}_{l_i}$ ($l_i \geq 1$) or $\mathbf{D}_{l_i}$ ($l_i \geq 2$) (and where $\mathbf{B}_1 = \mathbf{A}_1$, $\mathbf{B}_2 = \mathbf{C}_2$, $\mathbf{D}_2 = \mathbf{A}_1 \sqcup \mathbf{A}_1$, and $\mathbf{D}_3 = \mathbf{A}_3$ are permitted). We denote by m the cardinality of the finite index set I. We assume that $\Delta = \bigsqcup_{\iota=1}^{\mu} \Delta_\iota$ (disjoint union), where Δ_ι is of type $\mathbf{A}_{2n_\iota - 1}$, $n_\iota \geq 2$ ($\mathbf{A}_3 = \mathbf{D}_3$ is permitted). We assume that $m \geq 1$ and $\mu \geq 0$ (in the case $\mu = 0$ the diagram Δ is empty).

For each $i \in I$ we realize the root system $R(D_i)$ of type $\mathbf{B}_{l_i}$ or $\mathbf{D}_{l_i}$ in the standard way in the space $V_i := \mathbb{R}^{l_i}$ with basis $(e_s)_{s \in S_i}$ where S_i is an index set consisting of l_i elements; cf. [Bou, Planche II] for $\mathbf{B}_l$ ($l \geq 2$) (the relevant formulas for $\mathbf{B}_1$ are similar) and [Bou, Planche IV] for $\mathbf{D}_l$ ($l \geq 3$) (again, the relevant formulas for $\mathbf{D}_2$ are similar). Let $M_i \subset V_i$ denote the lattice generated by the basis vectors $(e_s)_{s \in S_i}$. Let $P_i \supset M_i$ denote the weight lattice of the root system D_i. Set $S = \bigsqcup_i S_i$ (disjoint union). Consider the vector space $V = \bigoplus_i V_i$ with basis $(e_s)_{s \in S}$. Let $M_D \subset V$ denote the lattice generated by the basis vectors $(e_s)_{s \in S}$, then $M_D = \bigoplus_i M_i$. Set $P_D = \bigoplus_i P_i$.

For each $\iota = 1, \ldots, \mu$ we realize the root system $R(\Delta_\iota)$ of type $\mathbf{A}_{2n_\iota - 1}$ in the standard way in the subspace V_ι of vectors with zero sum of the coordinates in the space $\mathbb{R}^{2n_\iota}$ with basis $\varepsilon_{\iota,1}, \ldots, \varepsilon_{\iota,2n_\iota}$; cf. [Bou, Planche I]. Let Q_ι be the root lattice of $R(\Delta_\iota)$ with basis $\varepsilon_{\iota,1} - \varepsilon_{\iota,2},\ \varepsilon_{\iota,2} - \varepsilon_{\iota,3},\ \ldots,\ \varepsilon_{\iota,2n_\iota-1} - \varepsilon_{\iota,2n_\iota}$, and let $P_\iota \supset Q_\iota$ be the weight lattice of $R(\Delta_\iota)$. Set $Q_\Delta = \bigoplus_\iota Q_\iota$, $P_\Delta = \bigoplus_\iota P_\iota$.

Set

$$W := \prod_{i \in I} W(D_i) \times \prod_{\iota=1}^{\mu} W(\Delta_\iota), \quad L' = M_D \oplus Q_\Delta = \bigoplus_{i \in I} M_i \oplus \bigoplus_{\iota=1}^{\mu} Q_\iota,$$

then W acts on L' and on $L' \otimes_{\mathbb{Z}} \mathbb{R}$. For each i consider the vector

$$x_i = \sum_{s \in S_i} e_s \in M_i,$$

then $\frac{1}{2} x_i \in P_i$. For each ι consider the vector

$$\xi_\iota = \varepsilon_{\iota,1} - \varepsilon_{\iota,2} + \varepsilon_{\iota,3} - \varepsilon_{\iota,4} + \cdots + \varepsilon_{\iota,2n_\iota-1} - \varepsilon_{\iota,2n_\iota} \in Q_\iota,$$

then $\frac{1}{2}\xi_\iota \in P_\iota$; see [Bou, Planche I]. Write

$$\xi'_\iota = \varepsilon_{\iota,1} - \varepsilon_{\iota,2}, \quad \xi''_\iota = \varepsilon_{\iota,3} - \varepsilon_{\iota,4} + \cdots + \varepsilon_{\iota,2n_\iota-1} - \varepsilon_{\iota,2n_\iota},$$

then $\xi_\iota = \xi'_\iota + \xi''_\iota$. Consider the vector

$$v = \frac{1}{2} \sum_{i \in I} x_i + \frac{1}{2} \sum_{\iota=1}^{\mu} \xi_\iota = \frac{1}{2} \sum_{s \in S} e_s + \frac{1}{2} \sum_{\iota=1}^{\mu} \xi_\iota \in P_D \oplus P_\Delta.$$

Set

$$L = \langle L', v \rangle, \tag{2.1}$$

then $[L : L'] = 2$ because $v \in \frac{1}{2}L' \smallsetminus L'$. Note that the sublattice $L \subset P_D \oplus P_\Delta$ is W-invariant. Indeed, the group W acts on $(P_D \oplus P_\Delta)/(M_D \oplus Q_\Delta)$ trivially.

Proposition 2.2. *We assume that $m \geq 1$, $m + \mu \geq 2$. If $\mu = 0$, we assume that not all of D_i are of types $\mathbf{B}_1$ or $\mathbf{D}_2$. Then the W-lattice L as in* (2.1) *is not quasi-invertible, hence not quasi-permutation.*

Proof. We consider a group $\Gamma = \{e, \gamma_1, \gamma_2, \gamma_3\}$ of order 4, where $\gamma_1, \gamma_2, \gamma_3$ are of order 2. The idea of our proof is to construct an embedding

$$j \colon \Gamma \to W$$

in such a way that L, viewed as a Γ-lattice, is equivalent to its Γ-sublattice L_1, and L_1 is isomorphic to a direct sum of a Γ-sublattice $L_0 \simeq J_\Gamma$ of rank 3 and a number of Γ-lattices of rank 1. Since by Proposition 1.6 J_Γ is not quasi-invertible, this will imply that L_1 and L are not quasi-invertible Γ-lattices, and hence L is not quasi-invertible as a W-lattice. We shall now fill in the details of this argument in four steps.

Step 1. We begin by partitioning each S_i for $i \in I$ into three (non-overlapping) subsets $S_{i,1}$, $S_{i,2}$ and $S_{i,3}$, subject to the requirement that

$$|S_{i,1}| \equiv |S_{i,2}| \equiv |S_{i,3}| \equiv l_i \pmod 2 \text{ if } D_i \text{ is of type } \mathbf{D}_{l_i}. \tag{2.2}$$

We then set U_1 to be the union of the $S_{i,1}$, U_2 to be the union of the $S_{i,2}$, and U_3 to be the union of the $S_{i,3}$, as i runs over I.

Lemma 2.3. (i) *If $\mu \geq 1$, the subsets $S_{i,1}$, $S_{i,2}$ and $S_{i,3}$ of S_i can be chosen, subject to* (2.2)*, so that $U_1 \neq \emptyset$.*

(ii) *If $\mu = 0$ (and $m \geq 2$), the subsets $S_{i,1}$, $S_{i,2}$ and $S_{i,3}$ of S_i can be chosen, subject to* (2.2)*, so that $U_1, U_2, U_3 \neq \emptyset$.*

To prove the lemma, first consider case (i). For all i such that D_i is of type $\mathbf{D}_{l_i}$ with *odd* l_i, we partition S_i into three non-empty subsets of odd cardinalities. For all the other i we take $S_{i,1} = S_i$, $S_{i,2} = S_{i,3} = \emptyset$. Then $U_1 \neq \emptyset$ (note that $m \geq 1$) and (2.2) is satisfied.

In case (ii), if one of the D_i is of type $\mathbf{D}_{l_i}$ where $l_i \geq 3$ is *odd,* then we partition S_i for each such D_i into three non-empty subsets of odd cardinalities. We partition all the other S_i as follows:

$$S_{i,1} = S_{i,2} = \emptyset \text{ and } S_{i,3} = S_i. \tag{2.3}$$

Clearly $U_1, U_2, U_3 \neq \emptyset$ and (2.2) is satisfied.

If there is no D_i of type $\mathbf{D}_{l_i}$ with odd $l_i \geq 3$, but one of the D_i, say for $i = i_0$, is $\mathbf{D}_l$ with *even* $l \geq 4$, then we partition S_{i_0} into two non-empty subsets $S_{i_0,1}$ and

$S_{i_0,2}$ of even cardinalities, and set $S_{i_0,3} = \emptyset$. We partition the sets S_i for $i \neq i_0$ as in (2.3) (note that by our assumption $m \geq 2$). Once again, $U_1, U_2, U_3 \neq \emptyset$ and (2.2) is satisfied.

If there is no D_i of type $\mathbf{D}_{l_i}$ with $l_i \geq 3$ (odd or even), but one of the D_i, say for $i = i_0$, is of type $\mathbf{B}_l$ with $l \geq 2$, we partition S_{i_0} into two non-empty subsets $S_{i_0,1}$ and $S_{i_0,2}$, and set $S_{i_0,3} = \emptyset$. We partition the sets S_i for $i \neq i_0$ as in (2.3) (again, note that $m \geq 2$). Once again, $U_1, U_2, U_3 \neq \emptyset$ and (2.2) is satisfied.

Since by our assumption not all of D_i are of type $\mathbf{B}_1$ or $\mathbf{D}_2$, we have exhausted all the cases. This completes the proof of Lemma 2.3. $\square$

Step 2. We continue proving Proposition 2.2. We construct an embedding $\Gamma \hookrightarrow W$.

For $s \in S$ we denote by c_s the automorphism of L taking the basis vector e_s to $-e_s$ and fixing all the other basis vectors. For $\iota = 1, \dots, \mu$ we define $\tau_\iota^{(12)} = \mathrm{Transp}((\iota,1),(\iota,2)) \in W_\iota$ (the transposition of the basis vectors $\varepsilon_{\iota,1}$ and $\varepsilon_{\iota,2}$). Set

$$\tau_\iota^{>2} = \mathrm{Transp}((\iota,3),(\iota,4)) \cdot \ \cdots \ \cdot \mathrm{Transp}((\iota,2n_\iota - 1),(\iota,2n_\iota)) \in W_\iota.$$

Write $\Gamma = \{e, \gamma_1, \gamma_2, \gamma_3\}$ and define an embedding $j\colon \Gamma \hookrightarrow W$ as follows:

$$j(\gamma_1) = \prod_{s \in S \smallsetminus U_1} c_s \ \cdot \ \prod_{\iota=1}^{\mu} \tau_\iota^{(12)} \tau_\iota^{>2}\,;$$

$$j(\gamma_2) = \prod_{s \in S \smallsetminus U_2} c_s \ \cdot \ \prod_{\iota=1}^{\mu} \tau_\iota^{(12)}\,;$$

$$j(\gamma_3) = \prod_{s \in S \smallsetminus U_3} c_s \ \cdot \ \prod_{\iota=1}^{\mu} \tau_\iota^{>2}\,.$$

Note that if D_i is of type $\mathbf{D}_{l_i}$, then by (2.2) for $\varkappa = 1,2,3$ the cardinality $\#(S_i \smallsetminus S_{i,\varkappa})$ is even, hence the product of c_s over $s \in S_i \smallsetminus S_{i,\varkappa}$ is contained in $W(D_i)$ for all such i, and therefore, $j(\gamma_\varkappa) \in W$. Since $j(\gamma_1)$, $j(\gamma_2)$ and $j(\gamma_3)$ commute, are of order 2, and $j(\gamma_1) j(\gamma_2) = j(\gamma_3)$, we see that j is a homomorphism. If $\mu \geq 1$, then, since $2n_1 \geq 4$, clearly $j(\gamma_\varkappa) \neq 1$ for $\varkappa = 1,2,3$, hence j is an embedding. If $\mu = 0$, then the sets $S \smallsetminus U_1$, $S \smallsetminus U_2$ and $S \smallsetminus U_3$ are nonempty, and again $j(\gamma_\varkappa) \neq 1$ for $\varkappa = 1,2,3$, hence j is an embedding.

Step 3. We construct a Γ-sublattice L_0 of rank 3. Write a vector $\mathbf{x} \in L$ as

$$\mathbf{x} = \sum_{s \in S} b_s e_s + \sum_{\iota=1}^{\mu} \sum_{\nu=1}^{2n_\iota} \beta_{\iota,\nu} \varepsilon_{\iota,\nu},$$

where $b_s\,, \beta_{\iota,\nu} \in \frac{1}{2}\mathbb{Z}$. Set $n' = \sum_{\iota=1}^{\mu} (n_\iota - 1)$. Define a Γ-equivariant homomorphism

$$\phi\colon L \to \mathbb{Z}^{n'}, \quad \mathbf{x} \mapsto (\beta_{\iota,2\lambda-1} + \beta_{\iota,2\lambda})_{\iota=1,\dots,\mu,\ \lambda=2,\dots,n_\iota}$$

(we skip $\lambda = 1$). We obtain a short exact sequence of Γ-lattices

$$0 \to L_1 \to L \xrightarrow{\phi} \mathbb{Z}^{n'} \to 0,$$

where $L_1 := \ker \phi$. Since Γ acts trivially on $\mathbb{Z}^{n'}$, we have $L_1 \sim L$. Therefore, it suffices to show that L_1 is not quasi-invertible.

Recall that

$$v = \frac{1}{2}\sum_{s\in S} e_s + \frac{1}{2}\sum_{\iota=1}^{\mu} \xi_\iota.$$

Set $v_1 = \gamma_1 \cdot v,\ v_2 = \gamma_2 \cdot v,\ v_3 = \gamma_3 \cdot v$. Set

$$L_0 = \langle v, v_1, v_2, v_3\rangle.$$

We have

$$v_1 = \frac{1}{2}\sum_{s\in U_1} e_s - \frac{1}{2}\sum_{s\in U_2\cup U_3} e_s - \frac{1}{2}\sum_{\iota=1}^{\mu} \xi_\iota,$$

whence

$$v + v_1 = \sum_{s\in U_1} e_s. \tag{2.4}$$

We have

$$v_2 = \frac{1}{2}\sum_{s\in U_2} e_s - \frac{1}{2}\sum_{s\in U_1\cup U_3} e_s + \frac{1}{2}\sum_{\iota=1}^{\mu} (-\xi'_\iota + \xi''_\iota),$$

whence

$$v + v_2 = \sum_{s\in U_2} e_s + \sum_{\iota=1}^{\mu} \xi''_\iota. \tag{2.5}$$

We have

$$v_3 = \frac{1}{2}\sum_{s\in U_3} e_s - \frac{1}{2}\sum_{s\in U_1\cup U_2} e_s + \frac{1}{2}\sum_{\iota=1}^{\mu} (\xi'_\iota - \xi''_\iota),$$

whence

$$v + v_3 = \sum_{s\in U_3} e_s + \sum_{\iota=1}^{\mu} \xi'_\iota. \tag{2.6}$$

Clearly, we have

$$v + v_1 + v_2 + v_3 = 0.$$

Since the set $\{v, v_1, v_2, v_3\}$ is the orbit of v under Γ, the sublattice $L_0 = \langle v, v_1, v_2, v_3\rangle \subset L$ is Γ-invariant. If $\mu \geq 1$, then $U_1 \neq \emptyset$, and we see from (2.4), (2.5) and (2.6) that $\operatorname{rank} L_0 \geq 3$. If $\mu = 0$, then $U_1, U_2, U_3 \neq \emptyset$, and again we see from (2.4), (2.5) and (2.6) that $\operatorname{rank} L_0 \geq 3$. Thus $\operatorname{rank} L_0 = 3$ and $L_0 \simeq J_\Gamma$, whence by Proposition 1.6 L_0 is not quasi-invertible.

Step 4. We show that L_0 is a direct summand of L_1. Set $m' = |S|$.

First assume that $\mu \geq 1$. Choose $u_1 \in U_1 \subset S$. Set $S' = S \smallsetminus \{u_1\}$. For each $s \in S'$ (i.e., $s \neq u_1$) consider the one-dimensional (i.e., of rank 1) lattice $X_s = \langle e_s \rangle$. We obtain $m' - 1$ Γ-invariant one-dimensional sublattices of L_1.

Denote by Υ the set of pairs (ι, λ) such that $1 \leq \iota \leq \mu$, $1 \leq \lambda \leq n_\iota$, and if $\iota = 1$, then $\lambda \neq 1, 2$. For each $(\iota, \lambda) \in \Upsilon$ consider the one-dimensional lattice

$$\Xi_{\iota,\lambda} = \langle \varepsilon_{\iota,2\lambda-1} - \varepsilon_{\iota,2\lambda} \rangle.$$

We obtain $-2 + \sum_{\iota=1}^{\mu} n_\iota$ one-dimensional Γ-invariant sublattices of L_1.

We show that

$$L_1 = L_0 \oplus \bigoplus_{s \in S'} X_s \oplus \bigoplus_{(\iota,\lambda) \in \Upsilon} \Xi_{\iota,\lambda}. \tag{2.7}$$

Set $L_1' = \left\langle\, L_0,\, (X_s)_{s \neq u_1},\, (\Xi_{\iota,\lambda})_{(\iota,\lambda)\in\Upsilon} \,\right\rangle$, then

$$\operatorname{rank} L_1' \leq 3 + (m'-1) - 2 + \sum_{\iota=1}^{\mu} n_\iota = m' + \sum_{\iota=1}^{\mu} (2n_\iota - 1) - \sum_{\iota=1}^{\mu} (n_\iota - 1) = \operatorname{rank} L_1. \tag{2.8}$$

Therefore, it suffices to check that $L_1' \supset L_1$. The set

$$\{v\} \cup \{e_s \mid s \in S\} \cup \{\varepsilon_{\iota,2\lambda-1} - \varepsilon_{\iota,2\lambda} \mid 1 \leq \iota \leq \mu, 1 \leq \lambda \leq n_\iota\}$$

is a set of generators of L_1. By construction $v, v_1, v_2, v_3 \in L_0 \subset L_1'$. We have $e_s \in X_s \subset L_1'$ for $s \neq u_1$. By (2.4) $\sum_{s \in U_1} e_s \in L_1'$, hence $e_{u_1} \in L_1'$. By construction

$$\varepsilon_{\iota,2\lambda-1} - \varepsilon_{\iota,2\lambda} \in L_1', \quad \text{for all } (\iota, \lambda) \neq (1,1), (1,2)\,.$$

From (2.6) and (2.5) we see that

$$\sum_{\iota=1}^{\mu} (\varepsilon_{\iota,1} - \varepsilon_{\iota,2}) \in L_1', \quad \sum_{\iota=1} \xi_\iota'' \in L_1'\,.$$

Thus

$$\varepsilon_{1,1} - \varepsilon_{1,2} \in L_1', \quad \varepsilon_{1,3} - \varepsilon_{1,4} \in L_1'\,.$$

We conclude that $L_1' \supset L_1$, hence $L_1 = L_1'$. From dimension count (2.8) we see that (2.7) holds.

Now assume that $\mu = 0$. Then for each $\varkappa = 1, 2, 3$ we choose an element $u_\varkappa \in U_\varkappa$ and set $U_\varkappa' = U_\varkappa \smallsetminus \{u_\varkappa\}$. We set $S' = U_1' \cup U_2' \cup U_3' = S \smallsetminus \{u_1, u_2, u_3\}$. Again for $s \in S'$ (i.e., $s \neq u_1, u_2, u_3$) consider the one-dimensional lattice $X_s = \langle e_s \rangle$. We obtain $m' - 3$ one-dimensional Γ-invariant sublattices of $L_1 = L$. We show that

$$L_1 = L_0 \oplus \bigoplus_{s \in S'} X_s\,. \tag{2.9}$$

Set $L_1' = \langle L_0, (X_s)_{s\in S'}\rangle$, then

$$\operatorname{rank} L_1' \leq 3 + m' - 3 = m' = \operatorname{rank} L_1. \tag{2.10}$$

Therefore, it suffices to check that $L_1' \supset L_1$. The set $\{v\} \cup \{e_s \mid s \in S\}$ is a set of generators of $L_1 = L$. By construction $v, v_1, v_2, v_3 \in L_1'$ and $e_s \in L_1'$ for $s \neq u_1, u_2, u_3$. We see from (2.4), (2.5), (2.6) that $e_s \in L_1'$ also for $s = u_1, u_2, u_3$. Thus $L_1' \supset L_1$, hence $L_1' = L_1$. From dimension count (2.10) we see that (2.9) holds.

We see that in both cases $\mu \geq 1$ and $\mu = 0$, the sublattice L_0 is a direct summand of L_1. Since by Proposition 1.6 L_0 is not quasi-invertible as a Γ-lattice, it follows that L_1 and L are not quasi-invertible as Γ-lattices. Thus L is not quasi-invertible as a W-lattice. This completes the proof of Proposition 2.2. □

Remark 2.4. Since $\text{Ш}^2(\Gamma, J_\Gamma) \cong \mathbb{Z}/2\mathbb{Z}$ (Voskresenskiĭ, see [BKLR, Section 10] for the notation and the result), our argument shows that $\text{Ш}^2(\Gamma, L) \cong \mathbb{Z}/2\mathbb{Z}$.

Remark 2.5. The proof of [BKLR, Lemma 12.3] (which is a version with $\mu = 0$ of Lemma 2.3 above) contains an inaccuracy, though the lemma as stated is correct. Namely, in [BKLR] we write that if there exists i such that Δ_i is of type $\mathbf{D}_{l_i}$ where $l_i \geq 3$ is odd, then we partition S_i for *one* such i into three non-empty subsets $S_{i,1}$, $S_{i,2}$ and $S_{i,3}$ of odd cardinalities, and we partition all the other S_i as in [BKLR, (12.4)]. However, this partitioning of the sets S_i into three subsets does not satisfy [BKLR, (12.3)] for *other* i such that Δ_i is of type $\mathbf{D}_{l_i}$ with odd l_i. This inaccuracy can be easily corrected: we should partition S_i for *each* i such that Δ_i is of type $\mathbf{D}_{l_i}$ with odd l_i into three non-empty subsets of odd cardinalities.

3 More non-quasi-permutation lattices

In this section we construct another family of non-quasi-permutation lattices.

3.1. For $i = 1, \dots, r$ let $Q_i = \mathbb{Z}\mathbf{A}_{n_i-1}$ and $P_i = \Lambda_{n_i}$ denote the root lattice and the weight lattice of $\mathbf{SL}_{n_i}$, respectively, and let $W_i = \mathfrak{S}_{n_i}$ denote the corresponding Weyl group (the symmetric group on n_i letters) acting on P_i and Q_i. Set $F_i = P_i/Q_i$, then W_i acts trivially on F_i. Set

$$Q = \bigoplus_{i=1}^r Q_i, \quad P = \bigoplus_{i=1}^r P_i, \quad W = \prod_{i=1}^r W_i,$$

then $Q \subset P$ and the Weyl group W acts on Q and P. Set

$$F = P/Q = \bigoplus_{i=1}^r F_i,$$

then W acts trivially on F.

We regard $Q_i = \mathbb{Z}\mathbf{A}_{n_i-1}$ and $P_i = \Lambda_{n_i}$ as the lattices described in Bourbaki [Bou, Planche I]. Then we have an isomorphism $F_i \cong \mathbb{Z}/n_i\mathbb{Z}$. Note that for each $1 \le i \le r$, the set $\{\alpha_{\varkappa,i} \mid 1 \le \varkappa \le n_i - 1\}$ is a $\mathbb{Z}$-basis of Q_i.

Set $c = \gcd(n_1, \dots, n_r)$; we assume that $c > 1$. Let $d > 1$ be a divisor of c. For each $i = 1, \dots, r$, let $\nu_i \in \mathbb{Z}$ be such that $1 \le \nu_i < d$, $\gcd(\nu_i, d) = 1$, and assume that $\nu_1 = 1$. We write $\boldsymbol{\nu} = (\nu_i)_{i=1}^r \in \mathbb{Z}^r$. Let $\overline{\boldsymbol{\nu}}$ denote the image of $\boldsymbol{\nu}$ in $(\mathbb{Z}/d\mathbb{Z})^r$. Let $S_{\boldsymbol{\nu}} \subset (\mathbb{Z}/d\mathbb{Z})^r \subset \bigoplus_{i=1}^r \mathbb{Z}/n_i\mathbb{Z} = F$ denote the cyclic subgroup of order d generated by $\overline{\boldsymbol{\nu}}$. Let $L_{\boldsymbol{\nu}}$ denote the preimage of $S_{\boldsymbol{\nu}} \subset F$ in P under the canonical epimorphism $P \twoheadrightarrow F$, then $Q \subset L_{\boldsymbol{\nu}} \subset P$.

Proposition 3.2. *Let W and the W-lattice $L_{\boldsymbol{\nu}}$ be as in Subsection 3.1. In the case $d = 2^s$ we assume that $\sum n_i > 4$. Then $L_{\boldsymbol{\nu}}$ is not quasi-permutation.*

This proposition follows from Lemmas 3.3 and 3.8 below.

Lemma 3.3. *Let $p|d$ be a prime. Then for any subgroup $\Gamma \subset W$ isomorphic to $(\mathbb{Z}/p\mathbb{Z})^m$ for some natural m, the Γ-lattices $L_{\boldsymbol{\nu}}$ and $L_{\mathbf{1}} := L_{(1,\dots,1)}$ are equivalent for any $\boldsymbol{\nu} = (\nu_1, \dots, \nu_r)$ as above (in particular, we assume that $\nu_1 = 1$).*

Note that this lemma is trivial when $d = 2$.

3.4. We compute the lattice $L_{\boldsymbol{\nu}}$ explicitly. First let $r = 1$. We have $Q = Q_1$, $P = P_1$. Then P_1 is generated by Q_1 and an element $\omega \in P_1$ whose image in P_1/Q_1 is of order n_1. We may take

$$\omega = \frac{1}{n_1}[(n_1-1)\alpha_1 + (n_1-2)\alpha_2 + \dots + 2\alpha_{n_1-2} + \alpha_{n_1-1}],$$

where $\alpha_1, \dots, \alpha_{n_1-1}$ are the simple roots, see [Bou, Planche I]. There exists exactly one intermediate lattice L between Q_1 and P_1 such that $[L : Q_1] = d$, and it is generated by Q_1 and the element

$$w = \frac{n_1}{d}\omega = \frac{1}{d}[(n_1-1)\alpha_1 + (n_1-2)\alpha_2 + \dots + 2\alpha_{n_1-2} + \alpha_{n_1-1}].$$

Now for any natural r, the lattice $L_{\boldsymbol{\nu}}$ is generated by Q and the element

$$w_{\boldsymbol{\nu}} = \frac{1}{d}\sum_{i=1}^r \nu_i[(n_i-1)\alpha_{1,i} + (n_i-2)\alpha_{2,i} + \dots + 2\alpha_{n_i-2,i} + \alpha_{n_i-1,i}].$$

In particular, $L_{\mathbf{1}}$ is generated by Q and

$$w_{\mathbf{1}} = \frac{1}{d}\sum_{i=1}^r [(n_i-1)\alpha_{1,i} + (n_i-2)\alpha_{2,i} + \dots + 2\alpha_{n_i-2,i} + \alpha_{n_i-1,i}].$$

3.5. *Proof of Lemma 3.3.* Recall that $L_{\boldsymbol\nu} = \langle Q, w_{\boldsymbol\nu}\rangle$ with

$$Q = \langle \alpha_{\varkappa,i}\rangle\,, \quad \text{where} \quad i = 1,\dots,r,\ \varkappa = 1,\dots,n_i - 1.$$

Set $Q_{\boldsymbol\nu} = \langle \nu_i \alpha_{\varkappa,i}\rangle$. Denote by $\mathfrak{T}_{\boldsymbol\nu}$ the endomorphism of Q that acts on Q_i by multiplication by ν_i. We have $Q_{\mathbf{1}} = Q,\ Q_{\boldsymbol\nu} = \mathfrak{T}_{\boldsymbol\nu} Q_{\mathbf{1}},\ w_{\boldsymbol\nu} = \mathfrak{T}_{\boldsymbol\nu} w_{\mathbf{1}}$. Consider

$$\mathfrak{T}_{\boldsymbol\nu} L_{\mathbf{1}} = \langle Q_{\boldsymbol\nu}, w_{\boldsymbol\nu}\rangle.$$

Clearly the W-lattices $L_{\mathbf{1}}$ and $\mathfrak{T}_{\boldsymbol\nu} L_{\mathbf{1}}$ are isomorphic. We have an embedding of W-lattices $Q \hookrightarrow L_{\boldsymbol\nu}$, in particular, an embedding $Q \hookrightarrow L_{\mathbf{1}}$, which induces an embedding $\mathfrak{T}_{\boldsymbol\nu} Q \hookrightarrow \mathfrak{T}_{\boldsymbol\nu} L_{\mathbf{1}}$. Set $M_{\boldsymbol\nu} = L_{\boldsymbol\nu}/\mathfrak{T}_{\boldsymbol\nu} L_{\mathbf{1}}$, then we obtain a homomorphism of W-modules $Q/\mathfrak{T}_{\boldsymbol\nu} Q \to M_{\boldsymbol\nu}$, which is an isomorphism by Lemma 3.6 below.

Now let $p|d$ be a prime. Let $\Gamma \subset W$ be a subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^m$ for some natural m. As in [LPR, Proof of Proposition 2.10], we use Roiter's version [Ro, Proposition 2] of Schanuel's lemma. We have exact sequences of Γ-modules

$$0 \to \mathfrak{T}_{\boldsymbol\nu} L_{\mathbf{1}} \to L_{\boldsymbol\nu} \to M_{\boldsymbol\nu} \to 0,$$
$$0 \to Q \xrightarrow{\mathfrak{T}_{\boldsymbol\nu}} Q \to M_{\boldsymbol\nu} \to 0.$$

Since all ν_i are prime to p, we have $|\Gamma| \cdot M_{\boldsymbol\nu} = p^m M_{\boldsymbol\nu} = M_{\boldsymbol\nu}$, and by [Ro, Corollary of Proposition 3] the morphisms of $\mathbb{Z}[\Gamma]$-modules $L_{\boldsymbol\nu} \to M_{\boldsymbol\nu}$ and $Q \to M_{\boldsymbol\nu}$ are projective in the sense of [Ro, § 1]. Now by [Ro, Proposition 2] there exists an isomorphism of Γ-lattices $L_{\boldsymbol\nu} \oplus Q \simeq \mathfrak{T}_{\boldsymbol\nu} L_{\mathbf{1}} \oplus Q$. Since Q is a quasi-permutation W-lattice, it is a quasi-permutation Γ-lattice, and by Lemma 3.7 below, $L_{\boldsymbol\nu} \sim \mathfrak{T}_{\boldsymbol\nu} L_{\mathbf{1}}$ as Γ-lattices. Since $\mathfrak{T}_{\boldsymbol\nu} L_{\mathbf{1}} \simeq L_{\mathbf{1}}$, we conclude that $L_{\boldsymbol\nu} \sim L_{\mathbf{1}}$. □

Lemma 3.6. *With the above notation* $L_{\boldsymbol\nu}/\mathfrak{T}_{\boldsymbol\nu} L_{\mathbf{1}} \simeq Q/\mathfrak{T}_{\boldsymbol\nu} Q = \bigoplus_{i=2}^r Q_i/\nu_i Q_i$.

Proof. We have $\mathfrak{T}_{\boldsymbol\nu} L_{\mathbf{1}} = \langle S_{\boldsymbol\nu}\rangle$, where $S_{\boldsymbol\nu} = \{\nu_i \alpha_{\varkappa,i}\}_{i,\varkappa} \cup \{w_{\boldsymbol\nu}\}$. Note that

$$d w_{\boldsymbol\nu} = \sum_{i=1}^r \nu_i[(n_i-1)\alpha_{1,i} + (n_i-2)\alpha_{2,i} + \dots + 2\alpha_{n_i-2,i} + \alpha_{n_i-1,i}].$$

We see that $d w_{\boldsymbol\nu}$ is a linear combination with integer coefficients of $\nu_i \alpha_{\varkappa,i}$ and that $\alpha_{n_1-1,1}$ appears in this linear combination with coefficient 1 (because $\nu_1 = 1$). Set $B'_{\boldsymbol\nu} = S_{\boldsymbol\nu} \smallsetminus \{\alpha_{n_1-1,1}\}$, then $\langle B'_{\boldsymbol\nu}\rangle \ni \alpha_{n_1-1,1}$, hence $\langle B'_{\boldsymbol\nu}\rangle = \langle S_{\boldsymbol\nu}\rangle = \mathfrak{T}_{\boldsymbol\nu} L_{\mathbf{1}}$, thus $B'_{\boldsymbol\nu}$ is a basis of $\mathfrak{T}_{\boldsymbol\nu} L_{\mathbf{1}}$. Similarly, the set $B_{\boldsymbol\nu} := \{\alpha_{\varkappa,i}\}_{i,\varkappa} \cup \{w_{\boldsymbol\nu}\} \smallsetminus \{\alpha_{n_1-1,1}\}$ is a basis of $L_{\boldsymbol\nu}$. Both bases $B_{\boldsymbol\nu}$ and $B'_{\boldsymbol\nu}$ contain $\alpha_{1,1},\dots,\alpha_{n_1-2,1}$ and $w_{\boldsymbol\nu}$. For all $i = 2,\dots,r$ and all $\varkappa = 1,\dots,n_i - 1$, the basis $B_{\boldsymbol\nu}$ contains $\alpha_{\varkappa,i}$, while $B'_{\boldsymbol\nu}$ contains $\nu_i \alpha_{\varkappa,i}$. We see that the homomorphism of W-modules $Q/\mathfrak{T}_{\boldsymbol\nu} Q = \bigoplus_{i=2}^r Q_i/\nu_i Q_i \to L_{\boldsymbol\nu}/\mathfrak{T}_{\boldsymbol\nu} L_{\mathbf{1}}$ is an isomorphism. □

Lemma 3.7. *Let Γ be a finite group, A and A' be Γ-lattices. If $A\oplus B \sim A'\oplus B'$, where B and B' are quasi-permutation Γ-lattices, then $A \sim A'$.*

Proof. Since B and B' are quasi-permutation, by Lemma 1.3 they are equivalent to 0, and we have

$$A = A \oplus 0 \sim A \oplus B \sim A' \oplus B' \sim A' \oplus 0 = A'.$$

This completes the proof of Lemma 3.7 and hence of Lemma 3.3. □

To complete the proof of Proposition 3.2 it suffices to prove the next lemma.

Lemma 3.8. *Let $p|d$ be a prime. Then there exists a subgroup $\Gamma \subset W$ isomorphic to $(\mathbb{Z}/p\mathbb{Z})^m$ for some natural m such that the Γ-lattice $L_{\mathbf{1}} := L_{(1,\dots,1)}$ is not quasi-permutation.*

3.9. Denote by U_i the space $\mathbb{R}^{n_i}$ with canonical basis $\varepsilon_{1,i}\,,\ \varepsilon_{2,i}\,,\ \dots,\ \varepsilon_{n_i,i}$. Denote by V_i the subspace of codimension 1 in U_i consisting of vectors with zero sum of the coordinates. The group $W_i = \mathfrak{S}_{n_i}$ (the symmetric group) permutes the basis vectors $\varepsilon_{1,i}\,,\ \varepsilon_{2,i}\,,\ \dots,\ \varepsilon_{n_i,i}$ and thus acts on U_i and V_i. Consider the homomorphism of vector spaces

$$\chi_i\colon U_i \to \mathbb{R}, \quad \sum_{\lambda=1}^{n_i} \beta_{\lambda,i}\,\varepsilon_{\lambda,i} \ \mapsto \ \sum_{\lambda=1}^{n_i} \beta_{\lambda,i}$$

taking a vector to the sum of its coordinates. Clearly this homomorphism is W_i-equivariant, where W_i acts trivially on $\mathbb{R}$. We have short exact sequences

$$0 \to V_i \to U_i \xrightarrow{\chi_i} \mathbb{R} \to 0.$$

Set $U = \bigoplus_{i=1}^r U_i,\ V = \bigoplus_{i=1}^r V_i$. The group $W = \prod_{i=1}^r W_i$ naturally acts on U and V, and we have an exact sequence of W-spaces

$$0 \to V \to U \xrightarrow{\chi} \mathbb{R}^r \to 0, \tag{3.1}$$

where $\chi = (\chi_i)_{i=1,\dots,r}$ and W acts trivially on $\mathbb{R}^r$.

Set $n = \sum_{i=1}^r n_i$. Consider the vector space $\overline{U} := \mathbb{R}^n$ with canonical basis $\overline{\varepsilon}_1, \overline{\varepsilon}_2, \dots, \overline{\varepsilon}_n$. Consider the natural isomorphism

$$\varphi\colon U = \bigoplus_i U_i \xrightarrow{\sim} \overline{U}$$

that takes $\varepsilon_{1,1}, \varepsilon_{2,1}, \dots, \varepsilon_{n_1,1}$ to $\overline{\varepsilon}_1, \overline{\varepsilon}_2, \dots, \overline{\varepsilon}_{n_1}$, takes $\varepsilon_{1,2}, \varepsilon_{2,2}, \dots, \varepsilon_{n_2,2}$ to $\overline{\varepsilon}_{n_1+1}, \overline{\varepsilon}_{n_1+2}, \dots, \overline{\varepsilon}_{n_1+n_2}$, and so on. Let $\overline{V}$ denote the subspace of codimension 1 in $\overline{U}$ consisting of vectors with zero sum of the coordinates. Sequence (3.1) induces an exact sequence of W-spaces

$$0 \to \varphi(V) \to \overline{V} \xrightarrow{\psi} \mathbb{R}^r \xrightarrow{\Sigma} \mathbb{R} \to 0. \tag{3.2}$$

Here $\psi = (\psi_i)_{i=1,\dots,r}$, where ψ_i takes a vector $\sum_{j=1}^{n} \beta_j \overline{\varepsilon}_j \in \overline{V}$ to $\sum_{\lambda=1}^{n_i} \beta_{n_1+\dots+n_{i-1}+\lambda}$, and the map Σ takes a vector in $\mathbb{R}^r$ to the sum of its coordinates. Note that W acts trivially on $\mathbb{R}^r$ and $\mathbb{R}$.

We have a lattice $Q_i \subset V_i$ for each $i = 1,\dots,r$, a lattice $Q = \bigoplus_i Q_i \subset \bigoplus_i V_i$, and a lattice $\overline{Q} := \mathbb{Z}\mathbf{A}_{n-1}$ in $\overline{V}$ with basis $\overline{\varepsilon}_1 - \overline{\varepsilon}_2, \dots, \overline{\varepsilon}_{n-1} - \overline{\varepsilon}_n$. The isomorphism φ induces an embedding of $Q = \bigoplus_i Q_i$ into $\overline{Q}$. Under this embedding

$$\begin{aligned}
&\alpha_{1,1} \mapsto \overline{\alpha}_1,\ \alpha_{2,1} \mapsto \overline{\alpha}_2,\ \dots,\ \alpha_{n_1-1,1} \mapsto \overline{\alpha}_{n_1-1},\\
&\alpha_{1,2} \mapsto \overline{\alpha}_{n_1+1},\ \alpha_{2,2} \mapsto \overline{\alpha}_{n_1+2},\ \dots,\ \alpha_{n_2-1,2} \mapsto \overline{\alpha}_{n_1+n_2-1},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\alpha_{1,r} \mapsto \overline{\alpha}_{n_1+n_2+\dots+n_{r-1}+1},\ \dots,\ \alpha_{n_r-1,r} \mapsto \overline{\alpha}_{n-1},
\end{aligned}$$

while $\overline{\alpha}_{n_1}, \overline{\alpha}_{n_1+n_2},\ \dots,\ \overline{\alpha}_{n_1+n_2+\dots+n_{r-1}}$ are skipped.

3.10. We write L for $L_{\mathbf{1}}$ and w for $w_{\mathbf{1}} \in \frac{1}{d}Q$, where $Q = \bigoplus_i Q_i$. Then

$$w = \sum_{i=1}^{r} w_i, \quad w_i = \frac{1}{d}[(n_i - 1)\alpha_{1,i} + \dots + \alpha_{n_i-1,i}].$$

Recall that

$$Q_i = \mathbb{Z}\mathbf{A}_{n_i-1} = \{(a_j) \in \mathbb{Z}^{n_i} \mid \sum_{j=1}^{n_i} a_j = 0\}.$$

Set

$$\overline{w} = \frac{1}{d}\sum_{j=1}^{n-1}(n-j)\overline{\alpha}_j.$$

Set $\Lambda_n(d) = \langle \overline{Q}, \overline{w} \rangle$. Note that $\Lambda_n(d) = Q_n(n/d)$ with the notation of [LPR, Subsection 6.1]. Set

$$N = \varphi(Q \otimes_{\mathbb{Z}} \mathbb{R}) \cap \Lambda_n(d) = \varphi(V) \cap \Lambda_n(d).$$

Lemma 3.11. $\varphi(L) = N$.

Proof. Write $j_1 = n_1,\ j_2 = n_1 + n_2,\ \dots,\ j_{r-1} = n_1 + \dots + n_{r-1}$. Set $J = \{1, 2, \dots, n-1\} \smallsetminus \{j_1, j_2, \dots, j_{r-1}\}$. Set

$$\mu = \frac{1}{d}\sum_{j \in J}(n-j)\overline{\alpha}_j = \overline{w} - \sum_{i=1}^{r-1}\frac{n - j_i}{d}\overline{\alpha}_{j_i}.$$

Note that $d|n$ and $d|j_i$ for all i, hence the coefficients $(n - j_i)/d$ are integral, and therefore $\mu \in \Lambda_n(d)$. Since also $\mu \in \varphi(Q \otimes_{\mathbb{Z}} \mathbb{R})$, we see that $\mu \in N$.

Let $y \in N$. Then

$$y = b\overline{w} + \sum_{j=1}^{n-1} a_j \overline{\alpha}_j$$

where $b, a_j \in \mathbb{Z}$, because $y \in \Lambda_n(d)$. We see that in the basis $\overline{\alpha}_1, \dots, \overline{\alpha}_{n-1}$ of $\Lambda_n(d) \otimes_{\mathbb{Z}} \mathbb{R}$, the element y contains $\overline{\alpha}_{j_i}$ with coefficient

$$b\frac{n - j_i}{d} + a_{j_i}.$$

Since $y \in \varphi(Q \otimes_{\mathbb{Z}} \mathbb{R})$, this coefficient must be 0:

$$b\frac{n - j_i}{d} + a_{j_i} = 0.$$

Consider

$$\begin{aligned} y - b\mu = y - b\left(\overline{w} - \sum_{i=1}^{r-1} \frac{n - j_i}{d}\overline{\alpha}_{j_i}\right) &= y - b\overline{w} + \sum_{i=1}^{r-1} \frac{b(n - j_i)}{d}\overline{\alpha}_{j_i} \\ &= \sum_{j=1}^{n-1} a_j\overline{\alpha}_j + \sum_{i=1}^{r-1} \frac{b(n - j_i)}{d}\overline{\alpha}_{j_i} = \sum_{j\in J} a_j\overline{\alpha}_j\,, \end{aligned}$$

where $a_j \in \mathbb{Z}$. We see that $y \in \langle\overline{\alpha}_j \ (j \in J), \mu\rangle$ for any $y \in N$, hence $N \subset \langle\overline{\alpha}_j \ (j \in J), \mu\rangle$. Conversely, $\mu \in N$ and $\overline{\alpha}_j \in N$ for $j \in J$, hence $\langle\overline{\alpha}_j \ (j \in J), \mu\rangle \subset N$, thus

$$N = \langle\overline{\alpha}_j \ (j \in J), \mu\rangle. \tag{3.3}$$

Now

$$\varphi(w) = \frac{1}{d}\left[\sum_{j=1}^{n_1-1}(n_1 - j)\overline{\alpha}_j + \sum_{j=1}^{n_2-1}(n_2 - j)\overline{\alpha}_{n_1+j} + \dots + \sum_{j=1}^{n_r-1}(n_r - j)\overline{\alpha}_{j_{r-1}+j}\right]$$

while

$$\mu = \frac{1}{d}\left[\sum_{j=1}^{n_1-1}(n - j)\overline{\alpha}_j + \sum_{j=1}^{n_2-1}(n - n_1 - j)\overline{\alpha}_{n_1+j} + \dots + \sum_{j=1}^{n_r-1}(n_r - j)\overline{\alpha}_{j_{r-1}+j}\right].$$

Thus

$$\mu = \varphi(w) + \frac{n - n_1}{d}\sum_{j=1}^{n_1-1}\overline{\alpha}_j + \frac{n - n_1 - n_2}{d}\sum_{j=1}^{n_2-1}\overline{\alpha}_{n_1+j} + \dots + \frac{n_r}{d}\sum_{j=1}^{n_r-1}\overline{\alpha}_{j_{r-1}+j}\,,$$

where the coefficients

$$\frac{n - n_1}{d}, \quad \frac{n - n_1 - n_2}{d}, \quad \dots, \quad \frac{n_r}{d}$$

are integral. We see that

$$\langle\overline{\alpha}_j \ (j \in J),\ \mu\rangle = \langle\overline{\alpha}_j \ (j \in J),\ \varphi(w)\rangle. \tag{3.4}$$

From (3.3) and (3.4) we obtain that

$$N = \langle\overline{\alpha}_j\,(j \in J),\,\mu\rangle = \langle\overline{\alpha}_j\,(j \in J),\,\varphi(w)\rangle = \varphi(L). \qquad \square$$

3.12. Now let $p|\gcd(n_1,\dots,n_r)$. Recall that $W=\prod_{i=1}^r \mathfrak{S}_{n_i}$. Since $p|n_i$ for all i, we can naturally embed $(\mathfrak{S}_p)^{n_i/p}$ into $\mathfrak{S}_{n_i}$. We obtain a natural embedding

$$\Gamma := (\mathbb{Z}/p\mathbb{Z})^{n/p} \hookrightarrow (\mathfrak{S}_p)^{n/p} \hookrightarrow W.$$

In order to prove Lemma 3.8, it suffices to prove the next Lemma 3.13. Indeed, if n has an odd prime factor p, then by Lemma 3.13 L is not quasi-permutation. If $n=2^s$, then we take $p=2$. By the assumptions of Proposition 3.2, $n>4=2^2$, and again by Lemma 3.13 L is not quasi-permutation. This proves Lemma 3.8.

Lemma 3.13. *If either p odd or $n>p^2$, then L is not quasi-permutation as a Γ-lattice.*

Proof. By Lemma 3.11 it suffices to show that N is not quasi-permutation. Since $N=\Lambda_n(d)\cap\varphi(V)$, we have an embedding

$$\Lambda_n(d)/N \hookrightarrow \overline{V}/\varphi(V).$$

By (3.2) $\overline{V}/\varphi(V)\simeq\mathbb{R}^{r-1}$ and W acts on $\overline{V}/\varphi(V)$ trivially. Thus $\Lambda_n(d)/N\simeq\mathbb{Z}^{r-1}$ and W acts on $\mathbb{Z}^{r-1}$ trivially. We have an exact sequence of W-lattices

$$0\to N\to\Lambda_n(d)\to\mathbb{Z}^{r-1}\to 0,$$

with trivial action of W on $\mathbb{Z}^{r-1}$. We obtain that $N\sim\Lambda_n(d)$ as a W-lattice, and hence, as a Γ-lattice. Therefore, it suffices to show that $\Lambda_n(d)=Q_n(n/d)$ is not quasi-permutation as a Γ-lattice if either p is odd or $n>p^2$. This is done in [LPR] in the proofs of Propositions 7.4 and 7.8. This completes the proof of Lemma 3.13 and hence those of Lemma 3.8 and Proposition 3.2. ☐

4 Quasi-permutation lattices – case $\mathbf{A}_{n-1}$

In this section we prove Theorem 0.2 in the special case when G is isogenous to a direct product of groups of type $\mathbf{A}_{n-1}$ for $n\geq 3$.

We maintain the notation of Subsection 3.1. Let L be an intermediate lattice between Q and P, i.e., $Q\subset L\subset P$ ($L=Q$ are $L=P$ are possible). Let S denote the image of L in F, then L is the preimage of $S\subset F$ in P. Since W acts trivially on F, the subgroup $S\subset F$ is W-invariant, and therefore, the sublattice $L\subset P$ is W-invariant.

Theorem 4.1. *With the notation of Subsection 3.1 assume that $n_i\geq 3$ for all $i=1,2,\dots,r$. Let L between Q and P be an intermediate lattice, and assume that $L\cap P_i=Q_i$ for all i such that $n_i=3$ or $n_i=4$. If L is a quasi-permutation W-lattice, then $L=Q$.*

Proof. We prove the theorem by induction on r. The case $r=1$ follows from our assumptions if $n_1=3$ or $n_1=4$, and from Proposition 1.9 if $n_1>4$.

We assume that $r > 1$ and that the assertion is true for $r-1$. We prove it for r.

For i between 1 and r we set

$$Q_i' = \bigoplus_{j\neq i} Q_j, \quad P_i' = \bigoplus_{j\neq i} P_j, \quad F_i' = \bigoplus_{j\neq i} F_j, \quad W_i' = \prod_{j\neq i} W_j,$$

then $Q_i' \subset Q$, $P_i' \subset P$, $F_i' \subset F$ and $W_i' \subset W$. If L is a quasi-permutation W-lattice, then by Lemma 1.7 $L \cap P_i'$ is a quasi-permutation W_i'-lattice, and by the induction hypothesis $L \cap P_i' = Q_i'$.

Now let $Q \subset L \subset P$, and assume that $L \cap P_i' = Q_i'$ for all $i = 1, \dots, r$. We shall show that if $L \neq Q$ then L is not a quasi-permutation W-lattice. This will prove Theorem 4.1.

Assume that $L \neq Q$. Set $S = L/Q \subset F$, then $S \neq 0$. We first show that $(L \cap P_i')/Q_i' = S \cap F_i'$. Indeed, clearly $(L \cap P_i')/Q_i' \subset L/Q \cap P_i'/Q_i' = S \cap F_i'$. Conversely, let $f \in S \cap F_i'$, then f can be represented by some $l \in L$ and by some $p \in P_i'$, and $q := l - p \in Q$. Since $L \supset Q$, we see that $p = l - q \in L \cap P_i'$, hence $f \in (L \cap P_i')/Q_i'$, and therefore $S \cap F_i' \subset (L \cap P_i')/Q_i'$. Thus $(L \cap P_i')/Q_i' = S \cap F_i'$.

By assumption we have $L \cap P_i' = Q_i'$, and we obtain that $S \cap F_i' = 0$ for all $i = 1, \dots, r$. Let $S_{(i)}$ denote the image of S under the projection $F \to F_i$. We have a canonical epimorphism $p_i \colon S \to S_{(i)}$ with kernel $S \cap F_i'$. Since $S \cap F_i' = 0$, we see that $p_i \colon S \to S_{(i)}$ is an isomorphism. Set $q_i = p_i \circ p_1^{-1} \colon S_{(1)} \to S_{(i)}$, it is an isomorphism.

We regard $Q_i = \mathbb{Z}\mathbf{A}_{n_i-1}$ and $P_i = \Lambda_{n_i}$ as the lattices described in [Bou, Planche I]. Then we have an isomorphism $F_i \cong \mathbb{Z}/n_i\mathbb{Z}$. Since $S_{(i)}$ is a subgroup of the cyclic group $F_i \cong \mathbb{Z}/n_i\mathbb{Z}$ and $S \cong S_{(i)}$, we see that S is a cyclic group, and we see also that $|S|$ divides n_i for all i, hence $d := |S|$ divides $c := \gcd(n_1, \dots, n_r)$.

We describe all subgroups S of order d in $\bigoplus_{i=1}^r \mathbb{Z}/n_i\mathbb{Z}$ such that $S \cap (\bigoplus_{j\neq i} \mathbb{Z}/n_j\mathbb{Z}) = 0$ for all i. The element $a_i := n_i/d + n_i\mathbb{Z}$ is a generator of $S_{(i)} \subset F_i = \mathbb{Z}/n_i\mathbb{Z}$. Set $b_i = q_i(a_1)$. Since b_i is a generator of $S_{(i)}$, we have $b_i = \overline{\nu}_i a_i$ for some $\overline{\nu}_i \in (\mathbb{Z}/d\mathbb{Z})^\times$. Let $\nu_i \in \mathbb{Z}$ be a representative of $\overline{\nu}_i$ such that $1 \le \nu_i < d$, then $\gcd(\nu_i, d) = 1$. Moreover, since $q_1 = \mathrm{id}$, we have $b_1 = a_1$, hence $\overline{\nu}_1 = 1$ and $\nu_1 = 1$. We obtain an element $\boldsymbol{\nu} = (\nu_1, \dots, \nu_r)$. With the notation of Subsection 3.1 we have $S = S_{\boldsymbol{\nu}}$ and $L = L_{\boldsymbol{\nu}}$.

By Proposition 3.2 $L_{\boldsymbol{\nu}}$ is not a quasi-permutation W-lattice. Thus L is not quasi-permutation, which completes the proof of Theorem 4.1. $\square$

5 Proof of Theorem 0.2

Lemma 5.1 (well-known). *Let P_1 and P_2 be abelian groups. Set $P = P_1 \oplus P_2 = P_1 \times P_2$, and let $\pi_1 \colon P \to P_1$ denote the canonical projection. Let $L \subset P$ be a* *subgroup. If $\pi_1(L) = L \cap P_1$, then*

$$L = (L \cap P_1) \oplus (L \cap P_2).$$

Proof. Let $x \in L$. Set $x_1 = \pi_1(x) \in \pi_1(L)$. Since $\pi_1(L) = L \cap P_1$, we have $x_1 \in L \cap P_1$. Set $x_2 = x - x_1$, then $x_2 \in L \cap P_2$. We have $x = x_1 + x_2$. This completes the proof of Lemma 5.1. $\square$

5.2. Let I be a finite set. For any $i \in I$ let D_i be a connected Dynkin diagram. Let $D = \bigsqcup_i D_i$ (disjoint union). Let Q_i and P_i be the root and weight lattices of D_i, respectively, and W_i be the Weyl group of D_i. Set

$$Q = \bigoplus_{i \in I} Q_i, \quad P = \bigoplus_{i \in I} P_i, \quad W = \prod_{i \in I} W_i.$$

5.3. We construct certain quasi-permutation lattices L such that $Q \subset L \subset P$.

Let $\{\, \{i_1, j_1\}, \dots, \{i_s, j_s\} \,\}$ be a set of non-ordered pairs in I such that D_{i_l} and D_{j_l} for all $l = 1, \dots, s$ are of type $\mathbf{B}_1 = \mathbf{A}_1$ and all the indices $i_1, j_1, \dots, i_s, j_s$ are distinct. Fix such an l. We write $\{i, j\}$ for $\{i_l, j_l\}$ and we set $D_{i,j} := D_i \sqcup D_j$, $Q_{i,j} := Q_i \oplus Q_j$, $P_{i,j} := P_i \oplus P_j$. We regard $D_{i,j}$ as a Dynkin diagram of type $\mathbf{D}_2$, and we denote by $M_{i,j}$ the intermediate lattice between $Q_{i,j}$ and $P_{i,j}$ isomorphic to $\mathcal{X}(\mathbf{SO}_4)$, the character lattice of the group $\mathbf{SO}_4$; see Section 1, after Lemma 1.7. Let f_i be a generator of the lattice Q_i of rank 1, and let f_j be a generator of Q_j, then $P_i = \langle \frac{1}{2} f_i \rangle$ and $P_j = \langle \frac{1}{2} f_j \rangle$. Set $e_1^{(l)} = \frac{1}{2}(f_i + f_j)$, $e_2^{(l)} = \frac{1}{2}(f_i - f_j)$, then $\{e_1^{(l)}, e_2^{(l)}\}$ is a basis of $M_{i,j}$, and

$$M_{i,j} = \left\langle Q_{i,j}, e_1^{(l)} \right\rangle, \qquad P_{i,j} = \left\langle M_{i,j}, \frac{1}{2}(e_1^{(l)} + e_2^{(l)}) \right\rangle. \tag{5.1}$$

We have $M_{i,j} \cap P_i = Q_i$, $M_{i,j} \cap P_j = Q_j$, and $[M_{i,j} : Q_{i,j}] = 2$. Concerning the Weyl group, we have

$$W(D_{i,j}) = W(D_i) \times W(D_j) = W(\mathbf{D}_2) = \mathfrak{S}_2 \times \{\pm 1\},$$

where the symmetric group $\mathfrak{S}_2$ permutes the basis vectors $e_1^{(l)}$ and $e_2^{(l)}$ of $M_{i,j}$, while the group $\{\pm 1\}$ acts on $M_{i,j}$ by multiplication by scalars. We say that $M_{i,j}$ is an *indecomposable quasi-permutation lattice* (it corresponds to the semisimple Cayley group $\mathbf{SO}_4$ which does not decompose into a direct product of its normal subgroups).

Set $I' = I \smallsetminus \bigcup_{l=1}^{s} \{i_l, j_l\}$. For $i \in I'$ let M_i be any quasi-permutation intermediate lattice between Q_i and P_i (such an intermediate lattice exists if and only if D_i is of one of the types $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$, $\mathbf{G}_2$, see Proposition 1.9). We say that M_i is a *simple quasi-permutation lattice* (it corresponds to a stably Cayley simple group). We set

$$L = \bigoplus_{l=1}^{s} M_{i_l, j_l} \oplus \bigoplus_{i \in I'} M_i. \tag{5.2}$$

We say that a lattice L as in (5.2) is a *direct sum of indecomposable quasi-permutation lattices and simple quasi-permutation lattices.* Clearly L is a quasi-permutation W-lattice.

Theorem 5.4. *Let D, Q, P, W be as in Subsection 5.2. Let L be an intermediate lattice between Q and P, i.e., $Q \subset L \subset P$ (where $L = Q$ and $L = P$ are possible). If L is a quasi-permutation W-lattice, then L is as in* (5.2). *Namely, then L is a direct sum of indecomposable quasi-permutation lattices $M_{i,j}$ for some set of pairs $\{\{i_1, j_1\}, \dots, \{i_s, j_s\}\}$ and some family of simple quasi-permutation intermediate lattices M_i between Q_i and P_i for $i \in I'$.*

Remark 5.5. The set of pairs $\{\{i_1, j_1\}, \dots, \{i_s, j_s\}\}$ in Theorem 5.4 is uniquely determined by L. Namely, a pair $\{i, j\}$ belongs to this set if and only if the Dynkin diagrams D_i and D_j are of type $\mathbf{B}_1 = \mathbf{A}_1$ and

$$L \cap P_i = Q_i\,, \quad L \cap P_j = Q_j\,, \quad \text{while } L \cap (P_i \oplus P_j) \neq Q_i \oplus Q_j\,.$$

Proof of Theorem 5.4. We prove the theorem by induction on $m = |I|$, where I is as in Subsection 5.2. The case $m = 1$ is trivial.

We assume that $m \geq 2$ and that the theorem is proved for all $m' < m$. We prove it for m. First we consider three special cases.

Split case. Assume that for some subset $A \subset I$, $A \neq I$, $A \neq \emptyset$, we have $\pi_A(L) = L \cap P_A$, where $P_A = \bigoplus_{i \in A} P_i$ and $\pi_A \colon P \to P_A$ is the canonical projection. Then by Lemma 5.1 we have $L = (L \cap P_A) \oplus (L \cap P_{A'})$, where $A' = I \smallsetminus A$. By Lemma 1.7 $L \cap P_A$ is a quasi-permutation W_A-lattice, where $W_A = \prod_{i \in A} W_i$. By the induction hypothesis the lattice $L \cap P_A$ is a direct sum of indecomposable quasi-permutation lattices and simple quasi-permutation lattices. Similarly, $L \cap P_{A'}$ is such a direct sum. We conclude that $L = (L \cap P_A) \oplus (L \cap P_{A'})$ is such a direct sum, and we are done.

$\mathbf{A}_{n-1}$-*case.* Assume that all D_i are of type $\mathbf{A}_{n_i - 1}$, where $n_i \geq 3$ (so $\mathbf{A}_1$ is not permitted). We assume also that when $n_i = 3$ and when $n_i = 4$ (that is, for $\mathbf{A}_2$ and for $\mathbf{A}_3 = \mathbf{D}_3$) we have $L \cap P_i = Q_i$ (for $n_i > 4$ this is automatic because $L \cap P_i$ is a quasi-permutation W_i-lattice, see Proposition 1.9). In this case by Theorem 4.1 we have $L = Q = \bigoplus Q_i$, hence L is a direct sum of simple quasi-permutation lattices, and we are done.

$\mathbf{A}_1$-*case.* Assume that all D_i are of type $\mathbf{A}_1$. Then by [BKLR, Theorem 18.1] the lattice L is a direct sum of indecomposable quasi-permutation lattices and simple quasi-permutation lattices, and we are done.

Now we shall show that these three special cases exhaust all the quasi-permutation lattices. In other words, we shall show that if $Q \subset L \subset P$ and L is not as in one of these three cases, then L is not quasi-permutation. This will complete the proof of the theorem.

For the sake of contradiction, let us assume that $Q \subset L \subset P$, that L is not in one of the three special cases above, and that L is a quasi-permutation W-lattice.

We shall show in three steps that L is as in Proposition 2.2. By Proposition 2.2, L is not quasi–permutation, which contradicts our assumptions. This contradiction will prove the theorem.

Step 1. For $i \in I$ consider the intersection $L \cap P_i$, it is a quasi-permutation W_i-lattice (by Lemma 1.7), hence D_i is of one of the types $\mathbf{A}_{n-1}$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$, $\mathbf{G}_2$ (by Proposition 1.9). Note that $\pi_i(L) \neq L \cap P_i$ (otherwise we are in the split case).

Now assume that for some $i \in I$, the Dynkin diagram D_i is of type $\mathbf{G}_2$ or $\mathbf{C}_n$ for some $n \geq 3$, or D_i is of type $\mathbf{A}_2$ and $L \cap P_i \neq Q_i$. Then $L \cap P_i$ is a quasi-permutation W_i-lattice (by Lemma 1.7), hence $L \cap P_i = P_i$ (by Proposition 1.9). Since $P_i \supset \pi_i(L) \supset L \cap P_i$, we obtain that $\pi_i(L) = L \cap P_i$, which is impossible. Thus no D_i can be of type $\mathbf{G}_2$ or $\mathbf{C}_n$, $n \geq 3$, and if D_i is of type $\mathbf{A}_2$ for some i, then $L \cap P_i = Q_i$.

Thus all D_i are of types $\mathbf{A}_{n-1}$, $\mathbf{B}_n$ or $\mathbf{D}_n$, and if D_i is of type $\mathbf{A}_2$ for some $i \in I$, then $L \cap P_i = Q_i$. Since L is not as in the $\mathbf{A}_{n-1}$-case, we may assume that one of the D_i, say D_1, is of type $\mathbf{B}_n$ for some $n \geq 1$ ($\mathbf{B}_1 = \mathbf{A}_1$ is permitted), or of type $\mathbf{D}_n$ for some $n \geq 4$, or of type $\mathbf{D}_3$ with $L \cap P_1 \neq Q_1$. Indeed, otherwise *all* D_i are of type $\mathbf{A}_{n_i-1}$ for $n_i \geq 3$, and in the cases $\mathbf{A}_2$ ($n_i = 3$) and $\mathbf{A}_3$ ($n_i = 4$) we have $L \cap P_i = Q_i$, i.e., we are in the $\mathbf{A}_{n-1}$-case, which contradicts our assumptions.

Step 2. In this step, using the Dynkin diagram D_1 of type $\mathbf{B}_n$ or $\mathbf{D}_n$ from the previous step, we construct a quasi-permutation sublattice $L' \subset L$ of index 2 such that L' is as in (5.2). First we consider the cases $\mathbf{B}_n$ and $\mathbf{D}_n$ separately.

Assume that D_1 is of type $\mathbf{B}_n$ for some $n \geq 1$ ($\mathbf{B}_1 = \mathbf{A}_1$ is permitted). We have $[P_1 : Q_1] = 2$. Since $P_1 \supset \pi_1(L) \supsetneq L \cap P_1 \supset Q_1$, we see that $\pi_1(L) = P_1$ and $L \cap P_1 = Q_1$. Set $M_1 = Q_1$. We have $\pi_1(L) = P_1$, $L \cap P_1 = M_1$, and $[P_1 : M_1] = 2$.

Now assume that D_1 is of type $\mathbf{D}_n$ for some $n \geq 4$, or of type $\mathbf{D}_3$ with $L \cap P_1 \neq Q_1$. Set $M_1 = L \cap P_1$, then M_1 is a quasi-permutation W_1-lattice by Lemma 1.7, and it follows from Proposition 1.9 that $(W_1, M_1) \simeq \mathcal{X}(\mathbf{SO}_{2n})$, where $\mathcal{X}(\mathbf{SO}_{2n})$ denotes the character lattice of $\mathbf{SO}_{2n}$; see Section 1, after Lemma 1.7. It follows that $[M_1 : Q_1] = 2$ and $[P_1 : M_1] = 2$. Since $P_1 \supset \pi_1(L) \supsetneq L \cap P_1 = M_1$, we see that $\pi_1(L) = P_1$. Again we have $\pi_1(L) = P_1$, $L \cap P_1 = M_1$, and $[P_1 : M_1] = 2$.

Now we consider together the cases when D_1 is of type $\mathbf{B}_n$ for some $n \geq 1$ and when D_1 is of type $\mathbf{D}_n$ for some $n \geq 3$, where in the case $\mathbf{D}_3$ we have $L \cap P_1 \neq Q_1$. Set

$$L' := \ker[L \xrightarrow{\pi_1} P_1 \to P_1/M_1].$$

Since $\pi_1(L) = P_1$, and $[P_1 : M_1] = 2$, we have $[L : L'] = 2$. Clearly we have $\pi_1(L') = M_1$. Set

$$L_1^\dagger := \ker[\pi_1 \colon L \to P_1] = L \cap P_1',$$

where $P_1' = \bigoplus_{i\neq 1} P_i$. Since L is a quasi-permutation W-lattice, by Lemma 1.7 the lattice $L_1^\dagger$ is a quasi-permutation W_1'-lattice, where $W_1' = \prod_{i\neq 1} W_i$. By the induction hypothesis, $L_1^\dagger$ is a direct sum of indecomposable quasi-permutation lattices and simple quasi-permutation lattices as in (5.2). Since $M_1 = L \cap P_1$, we have $M_1 \subset L' \cap P_1$, and $L' \cap P_1 \subset L \cap P_1 = M_1$, hence $L' \cap P_1 = M_1 = \pi_1(L')$, and by Lemma 5.1 we have $L' = M_1 \oplus L_1^\dagger$. Since M_1 is a simple quasi-permutation lattice, we conclude that L' is a direct sum of indecomposable quasi-permutation lattices and simple quasi-permutation lattices as in (5.2), and $[L : L'] = 2$.

Step 3. In this step we show that L is as in Proposition 2.2. We write

$$L' = \bigoplus_{l=1}^{s}(L' \cap P_{i_l,j_l}) \oplus \bigoplus_{i\in I'}(L' \cap P_i),$$

where $P_{i_l,j_l} = P_{i_l} \oplus P_{j_l}$, the Dynkin diagrams D_{i_l} and D_{j_l} are of type $\mathbf{A}_1 = \mathbf{B}_1$, and $L' \cap P_{i_l,j_l} = M_{i_l,j_l}$ as in (5.1). For any $i \in I'$, we have $[\pi_i(L) : \pi_i(L')] \leq 2$, because $[L : L'] = 2$. Furthermore, for $i \in I'$ we have

$$\pi_i(L') = L' \cap P_i \subset L \cap P_i \subsetneq \pi_i(L),$$

hence $[\pi_i(L) : (L\cap P_i)] = 2$ and $L'\cap P_i = L\cap P_i$. Similarly, for any $l = 1, \dots, s$, if we write $i = i_l,\ j = j_l$, then we have

$$M_{i,j} = L' \cap P_{i,j} \subset L \cap P_{i,j} \subsetneq \pi_{i,j}(L) \subset P_{i,j}, \qquad [P_{i,j} : M_{i,j}] = 2,$$

whence $\pi_{i,j}(L) = P_{i,j}$, $L \cap P_{i,j} = M_{i,j}$, and therefore $[\pi_{i,j}(L) : (L \cap P_{i,j})] = [P_{i,j} : M_{i,j}] = 2$ and $L' \cap P_{i,j} = M_{i,j} = L \cap P_{i,j}$.

We view the Dynkin diagram $D_{i_l} \sqcup D_{j_l}$ of type $\mathbf{A}_1 \sqcup \mathbf{A}_1$ corresponding to the pair $\{i_l, j_l\}$ $(l = 1, \dots, s)$ as a Dynkin diagram of type $\mathbf{D}_2$. Thus we view L' as a direct sum of indecomposable quasi-permutation lattices and simple quasi-permutation lattices corresponding to Dynkin diagrams of type $\mathbf{B}_n$, $\mathbf{D}_n$ and $\mathbf{A}_n$.

We wish to show that L is as in Proposition 2.2. We change our notation in order to make it closer to that of Proposition 2.2.

As in Subsection 2.1, we now write D_i for Dynkin diagrams of types $\mathbf{B}_{l_i}$ and $\mathbf{D}_{l_i}$ only, appearing in L', where $\mathbf{B}_1 = \mathbf{A}_1$, $\mathbf{B}_2 = \mathbf{C}_2$, $\mathbf{D}_2 = \mathbf{A}_1 \sqcup \mathbf{A}_1$ and $\mathbf{D}_3 = \mathbf{A}_3$ are permitted, but for $\mathbf{D}_{l_i}$ with $l_i = 2, 3$ we require that

$$L \cap P_i = M_i := \mathcal{X}(\mathbf{SO}_{2l_i}).$$

We write $L_i' := L\cap P_i = L'\cap P_i$. We have $[\pi_i(L) : L_i'] = 2$, hence $[P_i : L_i'] \geq 2$. If D_i is of type $\mathbf{B}_{l_i}$, then $[P_i : L_i'] = 2$. If D_i is of type $\mathbf{D}_{l_i}$, then $L_i' = L\cap P_i \neq Q_i$, for $\mathbf{D}_2$ and $\mathbf{D}_3$ by our assumption and for $\mathbf{D}_{l_i}$ with $l_i \geq 4$ because $L \cap P_i$ is a quasi-permutation W_i-lattice (see Proposition 1.9); again we have $[P_i : L_i'] = 2$.

We see that for all i we have $[P_i : L'_i] = 2$, $\pi_i(L) = P_i$, and the lattice $L'_i = M_i$ is as in Subsection 2.1. We realize the root system $R(D_i)$ of type $\mathbf{B}_{l_i}$ or $\mathbf{D}_{l_i}$ in the standard way (cf. [Bou, Planches II, IV]) in the space $V_i := \mathbb{R}^{l_i}$ with basis $(e_s)_{s\in S_i}$, then L'_i is the lattice generated by the basis vectors $(e_s)_{s\in S_i}$ of V_i, and we have $P_i = \langle L'_i, \frac{1}{2}x_i\rangle$, where

$$x_i = \sum_{s\in S_i} e_s \in L'_i\,.$$

In particular, when D_i is of type $\mathbf{D}_2$ we have $x_i = e_1^{(l)} + e_2^{(l)}$ with the notation of formula (5.1).

As in Subsection 2.1, we write Δ_ι for Dynkin diagrams of type $\mathbf{A}_{n'_\iota-1}$ appearing in L', where $n'_\iota \geq 3$ and for $\mathbf{A}_3 = \mathbf{D}_3$ we require that $L \cap P_\iota = Q_\iota$. We write $L'_\iota := L \cap P_\iota = L' \cap P_\iota$. Then $L'_\iota = Q_\iota$ for all ι, for $\mathbf{A}_2$ by Step 1, for $\mathbf{A}_3$ by our assumption, and for other $\mathbf{A}_{n'_\iota-1}$ because L'_ι is a quasi-permutation W_ι-lattice; see Proposition 1.9. We have $\pi_\iota(L) \supsetneq L \cap P_\iota = L'_\iota$ and $[\pi_\iota(L) : L'_\iota] = [\pi_\iota(L) : \pi_\iota(L')] \leq 2$ (because $[L : L'] = 2$). It follows that $[\pi_\iota(L) : L'_\iota] = 2$, i.e., $[\pi_\iota(L) : Q_\iota] = 2$. We know that P_ι/Q_ι is a cyclic group of order n'_ι. Since it has a subgroup $\pi_\iota(L)/Q_\iota$ of order 2, we conclude that n'_ι is even, $n'_\iota = 2n_\iota$ (where $2n_\iota \geq 4$), and $\pi_\iota(L)/Q_\iota$ is the unique subgroup of order 2 of the cyclic group P_ι/Q_ι of order $2n_\iota$. As in Subsection 2.1, we realize the root system Δ_ι of type $\mathbf{A}_{2n_\iota-1}$ in the standard way (cf. [Bou, Planche I]) in the subspace V_ι of vectors with zero sum of the coordinates in the space $\mathbb{R}^{2n_\iota}$ with basis $\varepsilon_{\iota,1}, \dots, \varepsilon_{\iota,2n_\iota}$. We set

$$\xi_\iota = \varepsilon_{\iota,1} - \varepsilon_{\iota,2} + \varepsilon_{\iota,3} - \varepsilon_{\iota,4} + \dots + \varepsilon_{\iota,2n_\iota-1} - \varepsilon_{\iota,2n_\iota},$$

then $\xi_\iota \in L'_\iota$ and $\frac{1}{2}\xi_\iota \in \pi_\iota(L) \smallsetminus L'_\iota$ (cf. [Bou, Planche I, formula (VI)]), hence $\pi_\iota(L) = \langle L'_\iota, \frac{1}{2}\xi_\iota\rangle$.

Now we set

$$v = \frac{1}{2}\sum_{i\in I} x_i + \frac{1}{2}\sum_{\iota=1}^{\mu} \xi_\iota\,.$$

We claim that

$$L = \langle L', v\rangle.$$

Proof of the claim. Let $w \in L \smallsetminus L'$, then $L = \langle L', w\rangle$, because $[L : L'] = 2$. Set $z_i = \frac{1}{2}x_i - \pi_i(w)$, then $z_i \in L'_i \subset L'$, because $\frac{1}{2}x_i, \pi_i(w) \in \pi_i(L) \smallsetminus L'_i$. Similarly, we set $\zeta_\iota = \frac{1}{2}\xi_\iota - \pi_\iota(w)$, then $\zeta_\iota \in L'_\iota \subset L'$. We see that

$$v = w + \sum_i z_i + \sum_\iota \zeta_\iota,$$

where $\sum_i z_i + \sum_\iota \zeta_\iota \in L'$, and the claim follows. □

It follows from the claim that L is as in Proposition 2.2 (we use the assumption that we are not in the $\mathbf{A}_1$-case). Now by Proposition 2.2 L is not quasi-invertible, hence not quasi-permutation, which contradicts our assumptions. This contradiction proves Theorem 5.4. $\square$

Proof of Theorem 0.2. Theorem 0.2 follows immediately from Theorem 5.4 by virtue of Proposition 1.8. $\square$

6 Proof of Theorem 0.3

In this section we deduce Theorem 0.3 from Theorem 0.2.

Let G be a stably Cayley semisimple k-group. Then $\overline{G} := G \times_k \bar{k}$ is stably Cayley over an algebraic closure $\bar{k}$ of k. By Theorem 0.2, $G_{\bar{k}} = \prod_{j\in J} G_{j,\bar{k}}$ for some finite index set J, where each $G_{j,\bar{k}}$ is either a stably Cayley simple group or is isomorphic to $\mathbf{SO}_{4,\bar{k}}$. (Recall that $\mathbf{SO}_{4,\bar{k}}$ is stably Cayley and semisimple, but is not simple.) Here we write $G_{j,\bar{k}}$ for the factors in order to emphasize that they are defined over $\bar{k}$. By Remark 5.5 the collection of direct factors $G_{j,\bar{k}}$ is determined uniquely by $\overline{G}$. The Galois group $\mathrm{Gal}(\bar{k}/k)$ acts on $G_{\bar{k}}$, hence on J. Let Ω denote the set of orbits of $\mathrm{Gal}(\bar{k}/k)$ in J. For $\omega \in \Omega$ set $G^\omega_{\bar{k}} = \prod_{j\in\omega} G_{j,\bar{k}}$, then $\overline{G} = \prod_{\omega\in\Omega} G^\omega_{\bar{k}}$. Each $G^\omega_{\bar{k}}$ is $\mathrm{Gal}(\bar{k}/k)$-invariant, hence it defines a k-form G^ω_k of $G^\omega_{\bar{k}}$. We have $G = \prod_{\omega\in\Omega} G^\omega_k$.

For each $\omega \in \Omega$ choose $j = j_\omega \in \omega$. Let l_j/k denote the Galois extension in $\bar{k}$ corresponding to the stabilizer of j in $\mathrm{Gal}(\bar{k}/k)$. The subgroup $G_{j,\bar{k}}$ is $\mathrm{Gal}(\bar{k}/l_j)$-invariant, hence it comes from an l_j-form G_{j,l_j}. By the definition of Weil's restriction of scalars (see e.g. [Vo2, Subsection 3.12]) $G^\omega_k \cong R_{l_j/k} G_{j,l_j}$, hence $G \cong \prod_{\omega\in\Omega} R_{l_j/k} G_{j,l_j}$. Each G_{j,l_j} is either absolutely simple or an l_j-form of $\mathbf{SO}_4$.

We complete the proof using an argument from [BKLR, Proof of Lemma 11.1]. We show that G_{j,l_j} is a direct factor of $G_{l_j} := G\times_k l_j$. It is clear from the definition that $G_{j,\bar{k}}$ is a direct factor of $G_{\bar{k}}$ with complement $G'_{\bar{k}} = \prod_{i\in J\smallsetminus\{j\}} G_{i,\bar{k}}$. Then $G'_{\bar{k}}$ is $\mathrm{Gal}(\bar{k}/l_j)$-invariant, hence it comes from some l_j-group G'_{l_j}. We have $G_{l_j} = G_{j,l_j} \times_{l_j} G'_{l_j}$, hence G_{j,l_j} is a direct factor of G_{l_j}.

Recall that G_{j,l_j} is either a form of $\mathbf{SO}_4$ or absolutely simple. If it is a form of $\mathbf{SO}_4$, then clearly it is stably Cayley over l_j. It remains to show that if G_{j,l_j} is absolutely simple, then G_{j,l_j} is stably Cayley over l_j. The group $G_{\bar{k}}$ is stably Cayley over $\bar{k}$. Since $G_{j,\bar{k}}$ is a direct factor of the stably Cayley $\bar{k}$-group $G_{\bar{k}}$ over the algebraically closed field $\bar{k}$, by [LPR, Lemma 4.7] $G_{j,\bar{k}}$ is stably Cayley over $\bar{k}$. Comparing [LPR, Theorem 1.28] and [BKLR, Theorem 1.4], we see that G_{j,l_j} is either stably Cayley over l_j (in which case we are done) or an outer form of $\mathbf{PGL}_{2n}$ for some $n \geq 2$. Thus assume by the way of contradiction that G_{j,l_j} is an outer form of $\mathbf{PGL}_{2n}$ for some $n \geq 2$. Then by [BKLR, Example 10.7] the character lattice of G_{j,l_j} is not quasi-invertible, and by [BKLR, Proposition 10.8] the group G_{j,l_j} cannot be a direct factor of a stably Cayley l_j-group. This contradicts the fact that G_{j,l_j} is a direct factor of the stably Cayley l_j-group G_{l_j}. We conclude that G_{j,l_j} cannot be an outer form of $\mathbf{PGL}_{2n}$ for any $n \geq 2$. Thus G_{j,l_j} is stably Cayley over l_j, as desired. $\square$

A Appendix: Some quasi-permutation character lattices

The positive assertion of Proposition 1.9 above is well known. It is contained in [CK, Theorem 0.1] and in [BKLR, Theorem 1.4]. However, [BKLR] refers to [CK, Theorem 0.1], and [CK] refers to a series of results on rationality (rather than only stable rationality) of corresponding generic tori. In this appendix for the reader's convenience we provide a proof of the following positive result in terms of lattices only.

Proposition A.1. *Let G be any form of one of the following groups*

$$\mathbf{SL}_3,\ \mathbf{PGL}_n\ (n \text{ odd}),\ \mathbf{SO}_n\ (n \geq 3),\ \mathbf{Sp}_{2n},\ \mathbf{G}_2$$

or an inner *form of $\mathbf{PGL}_n$ (n even). Then the character lattice of G is quasi-permutation.*

Proof. $\mathbf{SO}_{2n+1}$. Let L be the character lattice of $\mathbf{SO}_{2n+1}$ (including $\mathbf{SO}_3$). Then the Dynkin diagram is $D = \mathbf{B}_n$. The Weyl group is $W = \mathfrak{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$. Then $L = \mathbb{Z}^n$ with the standard basis $e_1, \dots, e_n$. The group $\mathfrak{S}_n$ naturally permutes $e_1, \dots, e_n$, while $(\mathbb{Z}/2\mathbb{Z})^n$ acts by sign changes. Since W permutes the basis up to $\pm$ sign, the W-lattice L is quasi-permutation, see [Lo, § 2.8].

$\mathbf{SO}_{2n}$, *any form, inner or outer.* Let L be the character lattice of $\mathbf{SO}_{2n}$ (including $\mathbf{SO}_4$). Then the Dynkin diagram is $D = \mathbf{D}_n$, with root system $R = R(D)$. We consider the pair (A, L) where $A = \mathrm{Aut}(R, L)$, then (A, L) is isomorphic to the character lattice of $\mathbf{SO}_{2n+1}$, hence is quasi-permutation.

$\mathbf{Sp}_{2n}$. The character lattice of $\mathbf{Sp}_{2n}$ is isomorphic to the character lattice of $\mathbf{SO}_{2n+1}$, hence is quasi-permutation.

$\mathbf{PGL}_n$, *inner form.* The character lattice of $\mathbf{PGL}_n$ is the root lattice $L = Q$ of $\mathbf{A}_{n-1}$. It is a quasi-permutation $\mathfrak{S}_n$-lattice, cf. [Lo, Example 2.8.1].

$\mathbf{PGL}_n$, *outer form, n odd.* Let P be the *weight* lattice of $\mathbf{A}_{n-1}$, where $n \geq 3$ is odd. Then P is generated by elements $e_1, \dots, e_n$ subject to the relation

$$e_1 + \cdots + e_n = 0.$$

The automorphism group $A = \mathrm{Aut}(\mathbf{A}_{n-1})$ is the product of $\mathfrak{S}_n$ and $\mathfrak{S}_2$. The group A acts on P as follows: $\mathfrak{S}_n$ permutes $e_1, \dots, e_n$, and the nontrivial element of $\mathfrak{S}_2$ takes each e_i to $-e_i$.

We denote by M the A-lattice of rank $2n+1$ with basis $s_1, \dots, s_n, t_1, \dots, t_n, u$. The group $\mathfrak{S}_n$ permutes s_i and permutes t_i ($i = 1, \dots, n$), and the nontrivial element of $\mathfrak{S}_2$ permutes s_i and t_i for each i. The group A acts trivially on u. Clearly M is a permutation lattice.

We define an A-epimorphism $\pi\colon M \to P$ as follows:

$$\pi\colon \quad s_i \mapsto e_i, \quad t_i \mapsto -e_i, \quad u \mapsto 0.$$

Set $M' = \ker \pi$, it is an A-lattice of rank $n+2$. We show that it is a permutation lattice. We write down a set of $n+3$ generators of M':

$$\rho_i = s_i + t_i, \quad \sigma = s_1 + \cdots + s_n, \quad \tau = t_1 + \cdots + t_n, \quad u.$$

There is a relation

$$\rho_1 + \cdots + \rho_n = \sigma + \tau.$$

We define a new set of $n+2$ generators:

$$\tilde{\rho}_i = \rho_i + u, \quad \tilde{\sigma} = \sigma + \tfrac{n-1}{2}u, \quad \tilde{\tau} = \tau + \tfrac{n-1}{2}u,$$

where $\frac{n-1}{2}$ is integral because n is odd. We have

$$\tilde{\rho}_1 + \cdots + \tilde{\rho}_n - \tilde{\sigma} - \tilde{\tau} = u,$$

hence this new set indeed generates M', hence it is a basis. The group $\mathfrak{S}_n$ permutes $\tilde{\rho}_1, \ldots, \tilde{\rho}_n$, while $\mathfrak{S}_2$ permutes $\tilde{\sigma}$ and $\tilde{\tau}$. Thus A permutes our basis, and therefore M' is a permutation lattice. We have constructed a left resolution of P:

$$0 \to M' \to M \to P \to 0,$$

(with permutation lattices M and M'), which by duality gives a right resolution of the *root* lattice $Q \cong P^\vee$ of $\mathbf{A}_{n-1}$:

$$0 \to Q \to M^\vee \to (M')^\vee \to 0$$

with permutation lattices $M^\vee$ and $(M')^\vee$. Thus the character lattice Q of $\mathbf{PGL}_n$ is a quasi-permutation A-lattice for odd n.

The assertion that the character lattice of G is quasi-permutation in the remaining cases $\mathbf{SL}_3$ and $\mathbf{G}_2$ follows from the next Lemma A.2.

Lemma A.2 ([BKLR, Lemma 2.5]). *Let Γ be a finite group and L be any Γ-lattice of rank $r = 1$ or 2. Then L is quasi-permutation.*

This lemma, which is a version of [Vo2, § 4.9, Examples 6 and 7], was stated in [BKLR] without proof. For the sake of completeness we supply a short proof here.

We may assume that Γ is a maximal finite subgroup of $\mathbf{GL}_r(\mathbb{Z})$. If $r = 1$, then $\mathbf{GL}_1(\mathbb{Z}) = \{\pm 1\}$, and the lemma reduces to the case of the character lattice of $\mathbf{SO}_3$ treated above.

Now let $r = 2$. Up to conjugation there are two maximal finite subgroups of $\mathbf{GL}_2(\mathbb{Z})$, they are isomorphic to the dihedral groups D_8 (of order 8) and to D_{12} (of order 12), resp., see e.g. [Lo, § 1.10.1, Table 1.2]. The group D_8 is the group of symmetries of a square, and in this case it suffices to show that the character lattice of $\mathbf{SO}_5$ is quasi-permutation, which we have done above. The group D_{12} is the group of symmetries of a regular hexagon, and in this case it suffices to show that the character lattice of $\mathbf{PGL}_3$ (outer form) is quasi-permutation, which we have done above as well. This completes the proofs of Lemma A.2 and Proposition A.1. □

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