Integral Mixed Motives in Equal Characteristic

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Abstract. For noetherian schemes of finite dimension over a field of characteristic exponent p, we study the triangulated categories of $\mathbf{Z}[1/p]$-linear mixed motives obtained from cdh-sheaves with transfers. We prove that these have many of the expected properties. In particular, the formalism of the six operations holds in this context. When we restrict ourselves to regular schemes, we also prove that these categories of motives are equivalent to the more classical triangulated categories of mixed motives constructed in terms of Nisnevich sheaves with transfers. Such a program is achieved by comparing these various triangulated categories of motives with modules over motivic Eilenberg-MacLane spectra.

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*This paper is dedicated to Alexander Merkurjev,
with many thanks and deep admiration.*

Contents

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The main advances of the actual theory of mixed motivic complexes over a field come from the fact they are defined integrally. Indeed, this divides the theory in two variants, the Nisnevich one and the étale one. With rational coefficients, the two theories agree and share their good properties. But with integral coefficients, their main success comes from the comparison of these two variants, the so-called Beilinson-Lichtenbaum conjecture which was proved by Voevodsky and gave the solution of the Bloch-Kato conjecture.

One of the most recent works in the theory has been devoted to extend the definitions in order to get the 6 operations of Grothendieck and to check they satisfy the required formalism; in chronological order: an unpublished work of Voevodsky, [Ayo07a], [CDa]. While the project has been finally completely realized with rational coefficients in [CDa], the case of integral coefficients remains unsolved. In fact, this is half true: the étale variant is now completely settled: see [Ayo14], [CDb].

But the Nisnevich variant is less mature. Markus Spitzweck [Spi] has constructed a motivic ring spectrum over any Dedekind domain, which allows to define motivic cohomology of arbitrary schemes, and even triangulated categories of motives on a general base (under the form of modules over the pullbacks of the motivic ring spectrum over $\mathrm{Spec}(\mathbf{Z})$). However, at this moment, there is no proof that Spitzweck's motivic cohomology satisfies the absolute purity theorem, and we do not know how to compare Spitzweck's construction with triangulated categories of motives constructed in the language of algebraic correspondences (except for fields). What is concretely at stake is the theory of algebraic cycles: we expect that motivic cohomology of a regular scheme in degree $2n$ and twist n agrees with the Chow group of n-codimensional cycles of X. Let us recall for example that the localization long exact sequence for higher Chow groups and the existence of a product of Chow groups of regular schemes are still open questions in the arithmetic case (*i.e.* for schemes of unequal residual characteristics). For sake of completeness, let us recall that the localization long exact sequence in equal characteristic already is the fruit of non trivial contributions of Spencer Bloch [Blo86, Blo94] and Marc Levine [Lev01]. Their work involves moving lemmas which are generalizations of the classical moving lemma used to understand the intersection product of cycles [Ful98].

Actually, Suslin and Voevodsky have already provided an intersection theoretic basis for the integral definition of Nisnevich motivic complexes: the theory of relative cycles of [VSF00, chap. 2]. Then, along the lines drawn by Voevodsky, and especially the homotopy theoretic setting realized by Morel and Voevodsky, it was at least possible to give a reasonable definition of such a theory over an arbitrary base, using Nisnevich sheaves with transfers over this base, and the methods of $\mathbf{A}^1$-homotopy and $\mathbf{P}^1$-stabilization: this was done in [CDa, Sec. 7]. Interestingly enough, the main technical issue of this construction is to prove that these motivic complexes satisfy the existence of the localization triangle:

$$j_!\, j^*(M) \to M \to i_*\, i^*(M) \to j_!\, j^*(M)[1]$$

for any closed immersion i with open complement j. This echoes much with the question of localization sequence for higher Chow groups.
In our unsuccessful efforts to prove this property with integral coefficients, we noticed two things: the issue of dealing with singular schemes (the property is true for smooth schemes over any base, and, with rational coefficients, for any closed immersion between excellent geometrically unibranch scheme); the fact this property implies cdh-descent (i.e. Nisnevich descent together with descent by blow ups). Moreover, in [CDb], we show that, at least for torsion coefficients, the localization property for étale motivic complexes is true without any restriction, but this is due to rigidity properties (*à la* Suslin) which only hold étale locally, and for torsion coefficients.
Therefore, the idea of replacing Nisnevich topology by a finer one, which allows to deal with singularities, but remains compatible with algebraic cycles, becomes obvious. The natural choice goes to the cdh-topology: in Voevodsky's work [VSF00], motivic (co)homology of smooth schemes over a field is naturally extended to schemes of finite type by cdh-descent in characteristic zero (or, more generally, if we admit resolution of singularities), and S. Kelly's thesis [Kel12] generalizes this result to arbitrary perfect fields of characteristic $p > 0$, at least with $\mathbf{Z}[1/p]$-linear coefficients.

In this work, we prove that if one restricts to noetherian schemes of finite dimension over a prime field (in fact, an arbitrary perfect field) k, and if we invert solely the characteristic exponent of k, then mixed motives built out of cdh-sheaves with transfers (Definition 1.5) do satisfy the localization property: Theorem 5.11. Using the work of Ayoub, it is then possible to get the complete 6 functors formalism for these cdh-motives. Note that we also prove that these cdh-motives agree with the Nisnevich ones for regular k-schemes – hence proving that the original construction done in [CDa, Def. 11.1.1] is meaningful if one restricts to regular schemes of equal characteristic and invert the residue characteristic (see Corollary 3.2 for a precise account).
The idea is to extend a result of Röndigs and Østvær, which identifies motivic complexes with modules over the motivic Eilenberg-MacLane spectrum over a field of characteristic 0. This was recently generalized to perfect fields of characteristic $p > 0$, up to inverting p, by Hoyois, Kelly and Østvær [HKØ]. Our

main result, proved in Theorem 5.1, is that this property holds for arbitrary noetherian k-schemes of finite dimension provided we use cdh-motives and invert the exponent characteristic p of k in their coefficients. For any noetherian k-scheme of finite dimension X with structural map $f : X \to \mathrm{Spec}(k)$, let us put $H\mathbf{Z}_{X/k} = \mathbf{L}f^*(H\mathbf{Z}_k)$. Then there is a canonical equivalence of triangulated categories

$$H\mathbf{Z}_{X/k}[1/p]\text{-Mod} \simeq \mathrm{DM}_{\mathrm{cdh}}(X, \mathbf{Z}[1/p])\,.$$

One of the ingredients is to prove this result for Nisnevich motivic complexes with $\mathbf{Z}[1/p]$-coefficients if one restricts to noetherian regular k-schemes of finite dimension: see Theorem 3.1. The other ingredient is to use Gabber's refinement of de Jong resolution of singularities by alteration via results and methods from Kelly's thesis.

We finally prove the stability of the notion of constructibility for cdh-motives up to inverting the characteristic exponent in Theorem 6.4. While the characteristic 0 case can be obtained using results of [Ayo07a], the positive characteristic case follows from a geometrical argument of Gabber (used in his proof of the analogous fact for torsion étale sheaves). We also prove a duality theorem for schemes of finite type over a field (7.3), and describe cycle cohomology of Friedlander and Voevodsky using the language of the six functors (8.11). In particular, Bloch's higher Chow groups and usual Chow groups of schemes of finite type over a field are are obtained via the expected formulas (see 8.12 and 8.13).

We would like to thank Offer Gabber for pointing out Bourbaki's notion of n-gonflement, $0 \leq n \leq \infty$, and Bradley Drew for having noticed a mistake in an earlier version of the proof of Theorem 7.3. We also want to warmly thank the referee for many precise and constructive comments and questions, which helped us to greatly improve the readability of this article.

Conventions

We will fix a perfect base field k of characteristic exponent p – the case where k is a prime field is enough. All the schemes appearing in the paper are assumed to be noetherian of finite dimension.

We will fix a commutative ring R which will serve as our coefficient ring.

1. Motivic complexes and spectra

In [VSF00, chap. 5], Voevodsky introduced the category of motivic complexes $\mathrm{DM}^{\mathit{eff}}_{-}(S)$ over a perfect field with integral coefficients, a candidate for a conjectural theory described by Beilinson. Since then, several generalizations to more general bases have been proposed.

In [CDa], we have introduced the following ones over a general base noetherian scheme S:

1.1. *The Nisnevich variant.*– Let Λ be the localization of $\mathbf{Z}$ by the prime numbers which are invertible in R. The first step is to consider the category $Sm^{cor}_{\Lambda,S}$

whose ojects are smooth separated S-schemes of finite type and morphisms between X and Y are *finite S-correspondences from X to Y with coefficients in* Λ (see [CDa, Def. 9.1.8] with $\mathscr{P}$ the category of smooth separated morphisms of finite type).[2] Taking the graph of a morphism between smooth S-schemes, one gets a faithful functor γ from the usual category of smooth S-schemes to the category $Sm^{cor}_{\Lambda,S}$.

Then one defines the category $\mathrm{Sh}^{tr}_{\mathrm{Nis}}(S,R)$ of *sheaves with transfers over* S as the category of presheaves F of R-modules over $Sm^{cor}_{\Lambda,S}$ whose restriction to the category of smooth S-schemes $F\circ\gamma$ is a sheaf for the Nisnevich topology. Essentially according to the original proof of Voevodsky over a field (see [CDa, 10.3.3 and 10.3.17] for details), this is a symmetric monoidal Grothendieck abelian category.

The category $\mathrm{DM}(S,R)$ of Nisnevich motivic spectra over S is defined by applying the process of $\mathbf{A}^1$-localization, and then of $\mathbf{P}^1$-stabilization, to the (adequate model category structure corresponding to the) derived category of $\mathrm{Sh}^{tr}_{\mathrm{Nis}}(S,R)$; see [CDa, Def. 11.1.1]. By construction, any smooth S-scheme X defines a (homological) motive $M_S(X)$ in $\mathrm{DM}(S,R)$ which is a compact object. Moreover, the triangulated category $\mathrm{DM}(S,R)$ is generated by Tate twists of such homological motives, i.e. by objects of the form $M_S(X)(n)$ for a smooth S-scheme X, and an integer $n\in\mathbf{Z}$.

Remark 1.2. When $S=\mathrm{Spec}(K)$ is the spectrum of a perfect field, the triangulated category $\mathrm{DM}(S,\mathbf{Z})$ contains as a full and faithful subcategory the category $\mathrm{DM}^{\mathit{eff}}_{-}(K)$ defined in [VSF00, chap. 5]. This follows from the description of $\mathbf{A}^1$-local objects in this case and from the cancellation theorem of Voevodsky [Voe10] (see for example [Dég11, Sec. 4] for more details).

1.3. *The generalized variants.*– This variant is an *enlargement*[3] of the previous context. However, at the same time, one can consider several possible Grothendieck topologies t: the Nisnevich topology $t=\mathrm{Nis}$, the cdh-topology $t=\mathrm{cdh}$, the étale topology $t=\text{ét}$, or the h-topology $t=\mathrm{h}$.

Instead of using the category $Sm^{cor}_{\Lambda,S}$, we consider the larger category $\mathscr{S}^{ft,cor}_{\Lambda,S}$ made by all separated S-schemes of finite type whose morphisms are made by the finite S-correspondences with coefficients in Λ as in the previous paragraph

[2]Recall: a finite S-correspondence from X to Y with coefficients in Λ is an algebraic cycle in $X\times_S Y$ with Λ-coefficients such that:

(1) its support is finite equidimensional over X,
(2) it is a relative cycles over X in the sense of Suslin and Voevodsky (cf. [VSF00, chap. 2]) - equivalently it is a special cycle over X (cf. [CDa, def. 8.1.28]),
(3) it is Λ-universal (cf. [CDa, def. 8.1.48]).

When X is geometrically unibranch, condition (2) is always fulfilled (cf. [CDa, 8.3.26]). When X is regular of the characteristic exponent of any residue field of X is invertible in Λ, condition (3) is always fulfilled (cf. [CDa, 8.3.29] in the first case). Everything gets much simpler when we work locally for the cdh-topology; see [VSF00, Chap. 2, 4.2].

Recall also for future reference this definition makes sense even if X and Y are singular of finite type over S.

[3]See [CDa, 1.4.13] for a general definition of this term.

(see again [CDa, 9.1.8] with $\mathscr{P}$ the class of all separated morphisms of finite type).

Then we can still define the category $\underline{\mathrm{Sh}}_t^{tr}(S,R)$ of generalized t-sheaves with transfers over S as the category of additive presheaves of R-modules over $\mathscr{S}^{ft,cor}_{\Lambda,S}$ whose restriction to $\mathscr{S}^{ft}_S$ is a sheaf for the cdh topology. This is again a well suited Grothendieck abelian category (by which we mean that, using the terminology of [CDa], when we let S vary, we get an abelian premotivic category which is compatible with the topology t; see [CDa, Sec. 10.4]). Moreover we have natural adjunctions:

$$\mathrm{Sh}^{tr}_{\mathrm{Nis}}(S,R) \underset{\rho^*}{\overset{\rho_!}{\rightleftarrows}} \underline{\mathrm{Sh}}^{tr}_{\mathrm{Nis}}(S,R) \overset{a^*_{\mathrm{cdh}}}{\rightleftarrows} \underline{\mathrm{Sh}}^{tr}_{\mathrm{cdh}}(S,R) \tag{1.3.1}$$

where ρ^* is the natural restriction functor and a^*_{cdh} is the associated cdh-sheaf with transfers functor (see *loc. cit.*)

Finally, one defines the category $\underline{\mathrm{DM}}_t(S,R)$ of generalized motivic t-spectra over S and coefficients in R as the triangulated category obtained by $\mathbf{P}^1$-stabilization and $\mathbf{A}^1$-localization of the (adequate model category structure corresponding to the) derived category of $\underline{\mathrm{Sh}}_t^{tr}(S,R)$.

Note that in the generalized context, any S-scheme X defines a (homological) t-motive $M_S(X)$ in $\underline{\mathrm{DM}}_t(S,R)$ which is a compact object and depends covariantly on X. This can even be extended to simplicial S-schemes (although we might then obtain non compact objects). Again, the triangulated category $\underline{\mathrm{DM}}_t(S,R)$ is generated by objects of the form $M_S(X)(n)$ for a smooth S-scheme X and an integer $n \in \mathbf{Z}$.

Thus, we have three variants of motivic spectra. Using the adjunctions (1.3.1) (which are Quillen adjunctions for suitable underlying model categories), one deduces adjunctions made by exact functors as follows:

$$\mathrm{DM}(S,R) \underset{\mathbf{R}\rho^*}{\overset{\mathbf{L}\rho_!}{\rightleftarrows}} \underline{\mathrm{DM}}(S,R) \overset{\mathbf{L}a^*_{\mathrm{cdh}}}{\rightleftarrows} \underline{\mathrm{DM}}_{\mathrm{cdh}}(S,R) \tag{1.3.2}$$

The following assertions are consequences of the model category structures used to get these derived functors:

(1) for any smooth S-scheme X and any integer $n \in \mathbf{Z}$, $\mathbf{L}\rho_!\big(M_S(X)(n)\big) = \underline{M}_S(X)(n)$.
(2) for any S-scheme X and any integer $n \in \mathbf{Z}$, $\mathbf{L}a^*_{\mathrm{cdh}}\big(\underline{M}_S(X)(n)\big) = \underline{M}_S(X)(n)$.

The following proposition is a formal consequence of these definitions:

PROPOSITION 1.4. *The category $\underline{\mathrm{DM}}_{\mathrm{cdh}}(S,R)$ is the localization of $\underline{\mathrm{DM}}(S,R)$ obtained by inverting the class of morphisms of the form:*

$$\underline{M}_S(X_\bullet) \xrightarrow{p_*} \underline{M}_S(X)$$

for any cdh*-hypercover p of any S-scheme X. Moreover, the functor a_{cdh} is the canonical projection functor.*

The definition that will prove most useful is the following one.

Definition 1.5. Let S be any noetherian scheme.
One defines the triangulated category $\mathrm{DM}_{\mathrm{cdh}}(S,R)$ of cdh-motivic spectra, as the full localizing triangulated subcategory of $\underline{\mathrm{DM}}_{\mathrm{cdh}}(S,R)$ generated by motives of the form $\underline{M}_S(X)(n)$ for a smooth S-scheme X and an integer $n \in \mathbf{Z}$.

1.6. These categories for various base schemes S are equipped with a basic functoriality (f^*, f_*, $f_\sharp$ for f smooth, $\otimes$ and $\underline{\mathrm{Hom}}$) satisfying basic properties. In [CDa], we have summarized these properties saying that $\mathrm{DM}(-,R)$ is a *premotivic triangulated category* – see 1.4.2 for the definition and 11.1.1 for the construction.

2. Modules over motivic Eilenberg-MacLane spectra

2.a. Symmetric Tate spectra and continuity.

2.1. Given a scheme X we write Sp_X for the category of symmetric T-spectra, where T denotes a cofibrant resolution of the projective line $\mathbf{P}^1$ over X (with the point at infinity as a base point, say) in the projective model structure of pointed Nisnevich simplicial sheaves of sets. We will consider Sp_X as combinatorial stable symmetric monoidal model category, obtained as the T-stabilization of the $\mathbf{A}^1$-localization of the projective model category structure on the category of pointed Nisnevich simplicial sheaves of sets on the site Sm_X of smooth separated X-schemes of finite type. The corresponding homotopy category

$$\mathrm{Ho}(\mathrm{Sp}_X) = \mathrm{SH}(X)$$

is thus the stable homotopy category of schemes over X, as considered by Morel, Voevodsky and various other authors. This defines a motivic triangulated category in the sense of [CDa]: in other words, thanks to Ayoub's thesis [Ayo07a, Ayo07b], we have the whole formalism of the six operations in SH. We note that the categories $\mathrm{SH}(X)$ can be defined as the homotopy categories of their $(\infty,1)$-categorical counterparts; see [Rob15, 2.3] and [Hoy14, Appendix C].

2.2. In [CDa], we have introduced the notion of continuity for a premotivic category $\mathcal{T}$ which comes from the a premotivic model category. In the sequel, we will need to work in a more slightly general context, in which we do not consider a monoidal structure. Therefore, we will recast the definition of continuity for complete triangulated Sm-fibred categories over Sch (see [CDa, 1.1.12, 1.3.13] for the definitions; in particular, the adjective 'complete' refers to the existence of right adjoints for the pullback functors).
Here Sch will be a full subcategory of the category of schemes stable by smooth base change and $\mathcal{F}$ will be a class of affine morphisms in Sch.[4]

[4]The examples we will use here are: Sch is the category of regular (excellent) k-schemes or the category of all noetherian finite dimensional (excellent) k-schemes; $\mathcal{F}$ is the category of dominant affine morphisms or the category of all affine morphisms.

Definition 2.3. Let $\mathcal{T}$ be a complete triangulated *Sm*-fibred category over *Sch* and c be a small family of cartesian sections $(c_i)_{i\in I}$ of $\mathcal{T}$.
We will say that $\mathcal{T}$ is *c-generated* if, for any scheme X in *Sch*, the family of objects $c_{i,X}$, $i \in I$, form a generating family of the triangulated category. We will then define $\mathcal{T}_c(X)$ as the smallest thick subcategory of $\mathcal{T}(X)$ which contains the elements of of the form $f_\sharp f^*(c_{i,X}) = f_\sharp(c_{i,Y})$, for any separated smooth morphism $f : Y \to X$ and any $i \in I$. The objects of $\mathcal{T}_c(X)$ will be called *c-constructible* (or simply *constructible*, when c is clearly determined by the context).

Remark 2.4. If for any $i \in I$, the objects $c_{i,X}$ are compact, then $\mathcal{T}_c(X)$ is the category of compact objects of $\mathcal{T}(X)$ and so does not depend on c.
When $\mathcal{T}$ has a symmetric monoidal structure, or in other words, is a premotivic category, and if we ask that c is stable by tensor product, then c is what we call a *set of twists* in [CDa, 1.1.d]. This is what happens in practice (e.g. for $\mathcal{T} = \mathrm{SH}$, DM or $\mathrm{DM}_{\mathrm{cdh}}$), and the family c consists of the Tate twist $\mathbb{1}_X(n)$ of the unit object for $n \in \mathbf{Z}$. Moreover, constructible objects coincide with compact objects for SH, DM and $\mathrm{DM}_{\mathrm{cdh}}$.

For short, a $(Sch, \mathcal{F})$-pro-scheme will be a pro-scheme $(S_\alpha)_{\alpha\in A}$ with values in *Sch*, whose transition morphisms are in $\mathcal{F}$, which admits a projective limit S in the category of schemes such that S belongs to *Sch*. The following definition is a slightly more general version of [CDa, 4.3.2].

Definition 2.5. Let $\mathcal{T}$ be a c-generated complete triangulated *Sm*-fibred category over *Sch*.
We say that $\mathcal{T}$ is *continuous with respect to* $\mathcal{F}$, if given any $(Sch, \mathcal{F})$-pro-scheme (X_α) with limit S, for any index α_0, any object E_{α_0} in $\mathcal{T}(X_{\alpha_0})$, and any $i \in I$, the canonical map

$$\varinjlim_{\alpha\geq\alpha_0} \mathrm{Hom}_{\mathcal{T}(X_\alpha)}(c_{i,X_\alpha}, E_\alpha) \to \mathrm{Hom}_{\mathcal{T}(X)}(c_{i,S}, E),$$

is bijective, where E_α is the pullback of E_{α_0} along the transition morphism $X_\alpha \to X_{\alpha_0}$, while E is the pullback of E_{α_0} along the projection $X \to X_{\alpha_0}$

Example 2.6.
(1) The premotivic category SH on the category of noetherian finite dimensional schemes satisfies continuity without restriction (*i.e.* $\mathcal{F}$ is the category of all affine morphisms). This is a formal consequence of [Hoy14, Proposition C.12] and of [Lur09, Lemma 6.3.3.6], for instance.
(2) According to [CDa, 11.1.4], the premotivic triangulated categories DM and $\mathrm{DM}_{\mathrm{cdh}}$, defined over the category of all schemes, are continuous with respect to dominant affine morphisms. (Actually, this example is the only reason why we introduce a restriction on the transition morphisms in the previous continuity property.)

The following proposition is a little variation on [CDa, 4.3.4], in the present slightly generalized context:

Proposition 2.7. *Let $\mathcal{T}$ be a c-generated complete triangulated Sm-fibred category over Sch which is continuous with respect to $\mathcal{F}$. Let (X_α) be a $(Sch, \mathcal{F})$-pro-scheme with projective limit X and let $f_\alpha : X \to X_\alpha$ be the canonical projection.*
For any index α_0 and any objects M_{α_0} and E_{α_0} in $\mathcal{T}(S_{\alpha_0})$, if M_{α_0} is c-constructible, then the canonical map

$$\varinjlim_{\alpha \geq \alpha_0} \mathrm{Hom}_{\mathcal{T}(S_\alpha)}(M_\alpha, E_\alpha) \to \mathrm{Hom}_{\mathcal{T}(S)}(M, E),$$

is bijective, where M_α and E_α are the respective pullbacks of M_{α_0} and E_{α_0} along the transition morphisms $S_\alpha \to S_{\alpha_0}$, while $M = f^*_{\alpha_0}(M_{\alpha_0})$ and $E = f^*_{\alpha_0}(E_{\alpha_0})$. Moreover, the canonical functor:

$$2\text{-}\varinjlim_{\alpha} \mathcal{T}_c(X_\alpha) \xrightarrow{2\text{-}\varinjlim_\alpha (f^*_\alpha)} \mathcal{T}_c(X)$$

is an equivalence of triangulated categories.

The proof is identical to that of *loc. cit.*

Proposition 2.8. *Let $f : X \to Y$ be a regular morphism of schemes. Then the pullback functor*

$$f^* : \mathrm{Sp}_Y \to \mathrm{Sp}_X$$

of the premotivic model category of Tate spectra (relative to simplicial sheaves) preserves stable weak $\mathbf{A}^1$-equivalences as well as $\mathbf{A}^1$-local fibrant objects.

Proof. This property is local in X so that replacing X (resp. Y) by a suitable affine open neighbourhood of any point $x \in X$ (resp. $f(x)$), we can assume that X and Y are affine.
Then, according to Popescu's theorem (as stated in Spivakovsky's article [Spi99, Th. 1.1]), the morphism f can be written as a projective limit of smooth morphisms $f_\alpha : X_\alpha \to Y$. By a continuity argument (in the context of sheaves of sets!), as each functor f^*_α commutes with small limits and colimits, we see that the functor f^* commutes with small colimits as well as with finite limits. These exactness properties imply that the functor f^* preserves stalkwise simplicial weak equivalences. One can also check that, for any Nisnevich sheaves E and F on Sm_Y, the canonical map

$$f^* \underline{\mathrm{Hom}}(E, F) \to \underline{\mathrm{Hom}}(f^*(E), f^*(F)) \tag{2.8.1}$$

is an isomorphism (where $\underline{\mathrm{Hom}}$ denotes the internal Hom of the category of sheaves), at least when E is a finite colimit of representable sheaves. Since the functor f^* preserves projections of the form $\mathbf{A}^1 \times U \to U$, this readily implies that, if L denotes the explicit $\mathbf{A}^1$-local fibrant replacement functor defined in [MV99, Lemma 3.21, page 93], then, for any simplicial sheaf E on Sm_Y, the map $f^*(E) \to f^*(L(E))$ is an $\mathbf{A}^1$-equivalence with fibrant $\mathbf{A}^1$-local codomain. Therefore, the functor f^* preserves both $\mathbf{A}^1$-equivalences and $\mathbf{A}^1$-local fibrant objects at the level of simplicial sheaves. Using the isomorphism (2.8.1), it is easy to see that f^* preserves $\mathbf{A}^1$-local motivic Ω-spectra. Given that one can turn a levelwise $\mathbf{A}^1$-local fibrant Tate spectrum into a motivic Ω-spectrum by a suitable filtered colimit of iterated T-loop space functors, we see that there exists a fibrant replacement functor R in Sp_Y such that, for any Tate spectrum E over Y, the map $f^*(E) \to f^*(R(E))$ is a stable $\mathbf{A}^1$-equivalence with fibrant codomain. This implies that f^* preserves stable $\mathbf{A}^1$-equivalences. $\square$

Corollary 2.9. *Let A be a commutative monoid in Sp_k. Given a regular k-scheme X with structural map $f : X \to \mathrm{Spec}(k)$, let us put $A_X = f^*(R)$. Then, for any k-morphism between regular k-schemes $\varphi : X \to Y$, the induced map $\mathbf{L}\varphi^*(A_Y) \to A_X$ is an isomorphism in $\mathrm{SH}(X)$.*

Proof. It is clearly sufficient to prove this property when $Y = \mathrm{Spec}(k)$, in which case this is a direct consequence of the preceding proposition. $\square$

We will use repeatedly the following easy fact to get the continuity property.

Lemma 2.10. *Let*

$$\varphi^* : \mathcal{T} \rightleftarrows \mathcal{T}' : \varphi_*$$

be an adjunction of complete triangulated Sm-fibred categories. We make the following assumptions:

(i) *There is a small family c of cartesian sections of $\mathcal{T}$ such that $\mathcal{T}$ is c-generated.*
(ii) *The functor φ_* is conservative (or equivalently, $\mathcal{T}'$ is $\varphi^*(c)$-generated; by abuse, we will then write $\varphi^*(c) = c$ and will say that $\mathcal{T}'$ is c-generated).*
(iii) *The functor φ_* commutes with the operation f^* for any morphism $f \in \mathcal{F}$.*

Then, if $\mathcal{T}$ is continuous with respect to $\mathcal{F}$, the same is true for $\mathcal{T}'$.

Proof. Let $c = (c_{i,?})_{i\in I}$. For any morphism $f : Y \to X$ in $\mathcal{F}$, any object $E \in \mathcal{T}'(X)$ and any $i \in I$, one has a canonical isomorphism:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{T}'(Y)}(c_{i,Y}, f^*(E)) &= \mathrm{Hom}_{\mathcal{T}'(Y)}(\varphi^*(c_{i,Y}), f^*(E))\\ &\simeq \mathrm{Hom}_{\mathcal{T}(Y)}(c_{i,Y}, \varphi_* f^*(E))\\ &\simeq \mathrm{Hom}_{\mathcal{T}(Y)}(c_{i,Y}, f^*\varphi_*(E))\,.\end{aligned}$$

This readily implies the lemma. $\square$

Example 2.11. Let Reg_k be the category of regular k-schemes with morphisms all morphisms of k-schemes.
Let $(A_X)_{X\in \mathit{Reg}_k}$ be a cartesian section of the category of commutative monoids in the category of Tate spectra (*i.e.* a strict commutative ring spectrum stable by pullbacks with respect to morphisms in Reg_k). In this case, we have defined in [CDa, 7.2.11] a premotivic model category over Reg_k whose fiber A_X-Mod over a scheme X in Reg_k is the homotopy category of the symmetric monoidal

stable model category of A_X-modules[5] (i.e. of Tate spectra over S, equiped with an action of the commutative monoid A_X). Since Corollary 2.9 ensures that $(A_X)_{X \in Reg_k}$ is a homotopy cartesian section in the sense of [CDa, 7.2.12], according to [CDa, 7.2.13], there exists a premotivic adjunction:

$$L_A : \mathrm{SH} \rightleftarrows A\text{-Mod} : \mathcal{O}_A$$

of triangulated premotivic categories over Reg_k, such that $L_A(E) = A_S \wedge E$ for any spectrum E over a scheme S in Reg_k. Lemma 2.10 ensures that A-Mod is continuous with respect to affine morphisms in Reg_k.

2.b. Motivic Eilenberg-MacLane spectra over regular k-schemes.

2.12. There is a canonical premotivic adjunction:

$$\varphi^* : \mathrm{SH} \rightleftarrows \mathrm{DM} : \varphi_* \tag{2.12.1}$$

(see [CDa, 11.2.16]). It comes from an adjunction of the premotivic model categories of Tate spectra built out of simplicial sheaves of sets and of complexes of sheaves with transfers respectively (see 1.1):

$$\tilde{\varphi}^* : \mathrm{Sp} \rightleftarrows \mathrm{Sp}^{tr} : \tilde{\varphi}_*. \tag{2.12.2}$$

In other words, we have $\varphi^* = \mathbf{L}\tilde{\varphi}^*$ and $\varphi_* = \mathbf{R}\tilde{\varphi}_*$ (strictly speaking, we can construct the functors $\mathbf{L}\tilde{\varphi}^*$ and $\mathbf{R}\tilde{\varphi}_*$ so that these equalities are true at the level of objects). Recall in particular from [CDa, 10.2.16] that the functor $\tilde{\varphi}_*$ is composed by the functor $\tilde{\gamma}_*$ with values in Tate spectra of Nisnevich sheaves of R-modules (without transfers), which *forgets transfers* and by the functor induced by the right adjoint of the Dold-Kan equivalence. We define, for any scheme X:

$$HR_X = \tilde{\varphi}_*(R_X)\,. \tag{2.12.3}$$

This is Voevodsky's motivic Eilenberg-MacLane spectrum over X, originally defined in [Voe98, 6.1]. In the case where $X = \mathrm{Spec}(K)$ for some commutative ring K, we sometimes write

$$HR_K = HR_{\mathrm{Spec}\,K}\,. \tag{2.12.4}$$

According to [CDa, 6.3.9], the functor $\tilde{\gamma}_*$ preserves (and detects) stable $\mathbf{A}^1$-equivalences. We deduce that the same fact is true for $\tilde{\varphi}_*$. Therefore, we have a canonical isomorphism

$$HR_X \simeq \varphi_*(R_X) \simeq \mathbf{R}\tilde{\varphi}_*(R_X)\,.$$

The Tate spectrum HR_X is a commutative motivic ring spectrum in the strict sense (*i.e.* a commutative monoid in the category Sp_X). We denote by HR_X-Mod the homotopy category of HR_X-modules. This defines a fibred triangulated category over the category of schemes; see [CDa, Prop. 7.2.11]. The functor $\tilde{\varphi}_*$ being weakly monoidal, we get a natural structure of a commutative monoid on $\tilde{\varphi}_*(M)$ for any symmetric Tate spectrum with transfers M.

[5] In order to apply this kind of construction, we need to know that the model category of symmetric Tate spectra in simplicial sheaves satisfies the monoid axiom of Schwede and Shipley [SS00]. This is proved explicitely in [Hoy15, Lemma 4.2], for instance.

This means that the Quillen adjunction (2.12.2) induces a Quillen adjunction from the fibred model category of HR-modules to the premotivic model category of symmetric Tate spectra with transfers[6], and thus defines an adjunction

$$t^* : HR_X\text{-Mod} \rightleftarrows \mathrm{DM}(X, R) : t_* \tag{2.12.5}$$

for any scheme X. For any object E of $\mathrm{SH}(X)$, there is a canonical isomorphism $t^*(HR_X \otimes^{\mathbf{L}} E) = \varphi^*(E)$. For any object M of $\mathrm{DM}(X, R)$, when we forget the HR_X-module structure on $t_*(M)$, we simply obtain $\varphi_*(M)$.

Let $f : X \to S$ be a regular morphism of schemes. Then according to Proposition 2.8, $f^* = \mathbf{L}f^*$. In particular, the isomorphism τ_f of $\mathrm{SH}(X)$ can be lifted as a morphism of strict ring spectra:

$$\tilde{\tau}_f : f^*(HR_S) \to HR_X. \tag{2.12.6}$$

Let Reg_k be the category of regular k-schemes as in Example 2.11.

Proposition 2.13. *The adjunctions* (2.12.5) *define a premotivic adjunction*

$$t^* : HR\text{-Mod} \rightleftarrows \mathrm{DM}(-, R) : t_*$$

over the category Reg_k of regular k-schemes.

Proof. We already know that this is a an adjunction of fibred categories over Reg_k and that t^* is (strongly) symmetric monoidal. Therefore, it is sufficient to check that t^* commutes with the operations $f_\sharp$ for any smooth morphism between regular k-scheme $f : X \to S$ (via the canonical exchange map). For this, it is sufficient to check what happens on free HR_X-modules (because we are comparing exact functors which preserve small sums, and because the smallest localizing subcategory of HR_X-Mod containing free HR_X-modules is HR_X-Mod). For any object E of $\mathrm{SH}(X)$, we have, by the projection formula in SH, a canonical isomorphism in HZ_S-Mod:

$$\mathbf{L}f_\sharp(HR_X \otimes^{\mathbf{L}} E) \simeq HR_S \otimes^{\mathbf{L}} \mathbf{L}f_\sharp(E)\,.$$

Therefore, formula $t^*(HR_X \otimes^{\mathbf{L}} E) = \varphi^*(E)$ tells us that t^* commutes with $f_\sharp$ when restricted to free HR_X-modules, as required. □

3. Comparison theorem: regular case

The aim of this section is to prove the following result:

Theorem 3.1. *Let R be a ring in which the characteristic exponent of k is invertible. Then the premotivic adjunction of Proposition 2.13 is an equivalence of premotivic categories over Reg_k. In particular, for any regular noetherian scheme of finite dimension X over k, we have a canonical equivalence of symmetric monoidal triangulated categories*

$$HR_X\text{-Mod} \simeq \mathrm{DM}(X, R)\,.$$

[6]The fact that the induced adjunction is a Quillen adjunction is obvious: this readily comes from the fact that the forgetful functor from HR-modules to symmetric Tate spectra preserves and detects weak equivalences as well as fibrations (by definition).

The preceding theorem tells us that the 6 operations constructed on $\mathrm{DM}(-, R)$ in [CDa, 11.4.5], behave appropriately if one restricts to regular noetherian schemes of finite dimension over k:

Corollary 3.2. *Consider the notations of paragraph 2.12.*

(1) *The functors φ^* and φ_* commute with the operations f^*, f_* (resp. $p_!$, $p^!$) for any morphism f (resp. separated morphism of finite type p) between regular k-schemes.*

(2) *The premotivic category $\mathrm{DM}(-, R)$ over Reg_k satisfies:*
   - *the localization property;*
   - *the base change formula ($g^* f_! \simeq f'_! g'^*$, with notations of* [CDa, 11.4.5, (4)]*);*
   - *the projection formula ($f_!(M \otimes f^*(N)) \simeq f_!(M) \otimes N$, with notations of* [CDa, 11.4.5, (5)]*).*

Proof. Point (1) follows from the fact the premotivic adjunction $(L_{HR}, \mathcal{O}_{HR})$ satisfies the properties stated for (φ^*, φ_*) and that they are true for (t^*, t_*) because it is an equivalence of premotivic categories, due to Theorem 3.1. The first statement of Point (2) follows from the fact that the localization property over Reg_k holds in HR-Mod, and from the equivalence $HR\text{-Mod} \simeq \mathrm{DM}(-, R)$ over Reg_k; the remaining two statements follow from Point (2) and the fact they are true for SH (see [Ayo07a] in the quasi-projective case and [CDa, 2.4.50] in the general case). □

The proof of Theorem 3.1 will be given in Section 3.c (page 165), after a few preparations. But before that, we will explain some of its consequences.

3.3. Let $f : X \to S$ be a morphism of schemes. Since (2.12.1) is an adjunction of fibred categories over the category of schemes, we have a canonical exchange transformation (see [CDa, 1.2.5]):

$$Ex(f^*, \varphi_*) : \mathbf{L}f^* \varphi_* \to \varphi_* \mathbf{L}f^*. \tag{3.3.1}$$

Evaluating this natural transformation on the object $\mathbb{1}_S$ gives us a map:

$$\tau_f : \mathbf{L}f^*(HR_S) \to HR_X.$$

Voevodsky conjectured in [Voe02] the following property:

Conjecture (Voevodsky). The map τ_f is an isomorphism.

When f is smooth, the conjecture is obviously true as $Ex(f^*, \varphi_*)$ is an isomorphism.

Remark 3.4. The preceding conjecture of Voevodsky is closely related to the localization property for DM. In fact, let us also mention the following result which was implicit in [CDa] – as it will not be used in the sequel we leave the proof as an exercise for the reader.[7]

[7] Hint: use the fact that φ_* commutes with $j_\sharp$ ([CDa, 6.3.11] and [CDa, 11.4.1]).

Proposition 3.5. *We use the notations of Par. 3.3. Let $i : Z \to S$ be a closed immersion. Then the following properties are equivalent:*

(i) *The premotivic triangulated category* DM *satisfies the localization property with respect to i (see* [CDa, 2.3.2]*).*
(ii) *The natural transformation $Ex(i^*, \varphi_*)$ is an isomorphism.*

From the case of smooth morphisms, we get the following particular case of the preceding conjecture.

Corollary 3.6. *The conjecture of Voevodsky holds for any morphism $f : X \to S$ of regular k-schemes.*

Proof. By transitivity of pullbacks, it is sufficient to consider the case where $f = p$ is the structural morphism of the k-scheme S, with k a prime field (in particular, with k perfect). Since DM is continuous with respect to projective systems of regular k-schemes with affine transition maps (because this is the case for HR-modules, using Theorem 3.1), we are reduced to the case where S is smooth over k, which is trivial. □

Remark 3.7. The previous result is known to have interesting consequences for the motivic Eilenberg-MacLane spectrum HR_X where X is an arbitrary noetherian regular k-scheme of finite dimension.
For example, we get the following extension of a result of Hoyois on a theorem first stated by Hopkins and Morel (for $p = 1$). Given a scheme X as above, the canonical map

$$MGL_X/(a_1, a_2, \ldots)[1/p] \to H\mathbf{Z}_X[1/p]$$

from the algebraic cobordism ring spectrum modulo generators of the Lazard ring is an isomorphism up to inverting the characteristic exponent of k. This was proved in [Hoy15], for the base field k, or, more generally, for any essentially smooth k-scheme X.
This shows in particular that $H\mathbf{Z}_X[1/p]$ is the universal oriented ring $\mathbf{Z}[1/p]$-linear spectrum over X with additive formal group law.
All this story remains true for arbitrary noetherian k-schemes of finite dimension if we are eager to replace $H\mathbf{Z}_X$ by its cdh-local version: this is one of the meanings of Theorem 5.1 below. Note that, since Spitweck's version of the motivic spectrum has the same relation with algebraic cobordism (see [Spi, Theorem 11.3]), it coincides with the cdh-local version of $H\mathbf{Z}_X$ as well, at least up to p-torsion.

Definition 3.8. Let X be a regular k-scheme with structural map $f : X \to \mathrm{Spec}(k)$. We define the *relative motivic Eilenberg-MacLane spectrum* of X/k by the formula

$$HR_{X/k} = f^*(HR_{\mathrm{Spec}(k)})$$

(where $f^* : \mathrm{Sp}_k \to \mathrm{Sp}_X$ is the pullback functor at the level of the model categories).

Remark 3.9. By virtue of Propositions 2.8 and Corollary 3.6, we have canonical isomorphisms

$$\mathbf{L}f^*(HR_{\mathrm{Spec}\,(k)}) \simeq HR_{X/k} \simeq HR_X\,.$$

Note that, the functor f^* being symmetric monoidal, each relative motivic Eilenberg-MacLane spectrum $HR_{X/k}$ is a commutative monoid in Sp_X. This has the following consequences.

Proposition 3.10. *For any regular k-scheme X, there is a canonical equivalence of symmetric monoidal triangulated categories*

$$HR_{X/k}\text{-Mod} \simeq HR_X\text{-Mod}\,.$$

In particular, the assignment $X \mapsto HR_X$-Mod defines a premotivic symmetric monoidal triangulated category HR-Mod over Reg_k, which is continuous with respect to any projective system of regular k-schemes with affine transition maps.
Moreover the forgetful functor

$$HR\text{-Mod} \to \mathrm{SH}$$

commutes with $\mathbf{L}f^*$ for any k-morphism $f : X \to Y$ between regular schemes, and with $\mathbf{L}f_\sharp$ for any smooth morphism of finite type between regular schemes.

Proof. Since the canonical morphism of commutative monoids $HR_{X/k} \to HR_X$ is a stable $\mathbf{A}^1$-equivalence the first assertion is a direct consequence of [CDa, Prop. 7.2.13]. The property of continuity is a particular case of Example 2.11, with $R_X = HR_{X/k}$. For the last part of the proposition, by virtue of the last assertion of [CDa, Prop. 7.1.11 and 7.2.12] we may replace (coherently) HR_X by a cofibrant monoid R_X (in the model category of monoids in Sp_X), in order to apply [CDa, Prop. 7.2.14]: The forgetful functor from R_X-modules to Sp_X is a left Quillen functor which preserves weak equivalences and commutes with f^* for any map f in Reg_k: therefore, this relation remains true after we pass to the total left derived functors. The case of $\mathbf{L}f_\sharp$ is similar. □

We now come back to the aim of proving Theorem 3.1.

3.a. Some consequences of continuity.

Lemma 3.11. *Consider the cartesian square of schemes below.*

$$\begin{tikzcd} X' \arrow[r, "q"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "p"] & Y \end{tikzcd}$$

We assume that Y' is the projective limit of a projective system of Y-schemes (Y_α) with affine flat transition maps, and make the following assumption. For any index α, if $p_\alpha : Y_\alpha \to Y$ denotes the structural morphism, the base change *morphism associated to the pullback square*

$$\begin{tikzcd} X_\alpha \arrow[r, "q_\alpha"] \arrow[d, "g_\alpha"'] & X \arrow[d, "f"] \\ Y_\alpha \arrow[r, "p_\alpha"] & Y \end{tikzcd}$$

in $\mathrm{DM}(Y_\alpha, R)$ *is an isomorphism:* $\mathbf{R}p_\alpha^* \mathbf{R}f_* \simeq \mathbf{R}g_{\alpha,*} \mathbf{L}q_\alpha^*$.
Then the base change morphism $\mathbf{L}p^* \mathbf{R}f_* \to \mathbf{R}g_* \mathbf{L}q^*$ *is invertible in* $\mathrm{DM}(Y', R)$.

Proof. We want to prove that, for any object E of $\mathrm{DM}(X, R)$, the map

$$\mathbf{L}p^* \mathbf{R}f_*(E) \to \mathbf{R}g_* \mathbf{L}q^*(E)$$

is invertible. For this, it is sufficient to prove that, for any constructible object M of $\mathrm{DM}(Y', R)$, the map

$$\mathrm{Hom}(M, \mathbf{L}p^* \mathbf{R}f_*(E)) \to \mathrm{Hom}(M, \mathbf{R}g_* \mathbf{L}q^*(E))$$

is bijective. Since $\mathrm{DM}(-, R)$ is continuous with respect to dominant affine morphisms, we may assume that there exists an index α_0 and a constructible object M_{α_0}, such that $M \simeq \mathbf{L}p_{\alpha_0}^*(M_{\alpha_0})$. For $\alpha > \alpha_0$, we will write M_α for the pullback of M_{α_0} along the transition map $Y_\alpha \to Y_{\alpha_0}$. By continuity, we have a canonical identification

$$\varinjlim_\alpha \mathrm{Hom}(M_\alpha, \mathbf{L}p_\alpha^* \mathbf{R}f_*(E)) \simeq \mathrm{Hom}(M, \mathbf{L}p^* \mathbf{R}f_*(E)) .$$

On the other hand, by assumption, we also have:

$$\begin{aligned} \varinjlim_\alpha \mathrm{Hom}(M_\alpha, \mathbf{L}p_\alpha^* \mathbf{R}f_*(E)) &\simeq \varinjlim_\alpha \mathrm{Hom}(M_\alpha, \mathbf{R}g_{\alpha,*} \mathbf{L}q_\alpha^*(E)) \\ &\simeq \varinjlim_\alpha \mathrm{Hom}(\mathbf{L}g_\alpha^*(M_\alpha), \mathbf{L}q_\alpha^*(E)) . \end{aligned}$$

The flatness of the maps $p_{\beta\alpha}$ ensures that the transition maps of the projective system (X_α) are also affine and dominant, so that, by continuity, we get the isomorphisms

$$\begin{aligned} \varinjlim_\alpha \mathrm{Hom}(\mathbf{L}g_\alpha^*(M_\alpha), \mathbf{L}q_\alpha^*(E)) &\simeq \mathrm{Hom}(\mathbf{L}g^*(M), \mathbf{L}q^*(E)) \\ &\simeq \mathrm{Hom}(M, \mathbf{R}g_* \mathbf{L}q^*(E)) , \end{aligned}$$

and this achieves the proof. □

Proposition 3.12. *Let* $i : Z \to S$ *be a closed immersion between regular* k*-schemes. Assume that* S *is the limit of a projective system of smooth separated* k*-schemes of finite type, with affine flat transition maps. Then* $\mathrm{DM}(-, R)$ *satisfies the localization property with respect to* i *(cf.* [CDa, Def. 2.3.2]*).*

Proof. According to [CDa, 11.4.2], the proposition holds when S is smooth of finite type over k – the assumption then implies that Z is smooth of finite type over k.

According to [CDa, 2.3.18], we have only to prove that for any smooth S-scheme X, putting $X_Z \times_S Z$, the canonical map in $\mathrm{DM}(S,R)$

$$M_S(X/X - X_Z) \to i_*\big(M_Z(X_Z)\big) \tag{3.12.1}$$

is an isomorphism. This property is clearly local for the Zariski topology, so that we can even assume that both X and S are affine.

Lifting the ideal of definition of Z, one can assume that Z lifts to a closed subscheme $i_\alpha : Z_\alpha \hookrightarrow S_\alpha$. We can also assume that i_α is regular (apply [EGA4, 9.4.7] to the normal cone of the i_α). Thus Z_α is smooth over k. Because X/S is affine of finite presentation, it can be lifted to a smooth scheme X_α/S_α and because X/S is smooth we can assume X_α/S_α is smooth.
Put $X_{Z,\alpha} = X_\alpha \times_{S_\alpha} Z_\alpha$. Then, applying localization with respect to i_α, we obtain that the canonical map:

$$M_{S_\alpha}(X_\alpha/X_\alpha - X_{Z,\alpha}) \to i_{\alpha *}(M_{Z_\alpha}\big(X_{Z,\alpha})\big) \tag{3.12.2}$$

is an isomorphism in $\mathrm{DM}(S_\alpha,R)$. Of course the analogue of (3.12.2) remains an isomorphism for any $\alpha' > \alpha$. Given $\alpha' > \alpha$, let us consider the cartesian square

$$\begin{tikzcd} Z_{\alpha'} \arrow[r, "i_{\alpha'}"] \arrow[d, "g"'] & S_{\alpha'} \arrow[d, "f"] \\ Z_\alpha \arrow[r, "i_\alpha"] & S_\alpha \end{tikzcd}$$

in which $f : X_{\alpha'} \to X_\alpha$ denotes the transition map. Then according to [CDa, Prop. 2.3.11(1)], the localization property with respect to i_α and $i_{\alpha'}$ implies that the canonical base change map $f^* i_{\alpha,*} \to i_{\alpha',*} g^*$ is an isomorphism. By virtue of Lemma 3.11, if $\varphi : S \to S_\alpha$ denote the canonical projection, the pullback square

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "\psi"'] & S \arrow[d, "\varphi"] \\ Z_\alpha \arrow[r, "i_\alpha"] & S_\alpha \end{tikzcd}$$

induces a base change isomorphism $\mathbf{L}\varphi^* i_{\alpha,*} \to i_* \mathbf{L}\psi^*$. Therefore, the image of the map (3.12.2) by $\mathbf{L}\varphi^*$ is isomorphic to the map (3.12.1), and this ends the proof. □

3.b. Motives over fields. This section is devoted to prove Theorem 3.1 when one restricts to field extensions of k:

Proposition 3.13. *Consider the assumptions of 3.1 and let K be an extension field of k. Then the functor*

$$t^* : HR_K\text{-Mod} \to \mathrm{DM}(K,R)$$

is an equivalence of symmetric monoidal triangulated categories.

In the case where K is a perfect field, this result is proved in [HKØ, 5.8] in a slightly different theoretical setting. The proof will be given below (page 164), after a few steps of preparation.

3.14. In the end, the main theorem will prove the existence of very general trace maps, but the proof of this intermediate result requires that we give a preliminary construction of traces in the following case.
Let K be an extension field of k, and $f : Y \to X$ be a flat finite surjective morphism of degree d between integral K-schemes. There is a natural morphism ${}^t f : R_X \to f_\sharp(R_Y)$ in $\underline{\mathrm{DM}}(X, R)$, defined by the transposition of the graph of f. The composition

$$f_\sharp(R_Y) \to R_X \xrightarrow{{}^t f} f_\sharp(R_Y)$$

is d times the identity of $f_\sharp(R_Y)$; see [CDa, Prop. 9.1.13]. Moreover, if f is radicial (i.e. if the field of functions on Y is a purely inseparable extension of the field of functions of X), then the composition

$$R_X \xrightarrow{{}^t f} f_\sharp(R_Y) \xrightarrow{f} R_X$$

is d times the identity of R_X; see [CDa, Prop. 9.1.14]. In other words, in the latter case, since p is invertible, the co-unit map $f_\sharp(R_Y) \to R_X$ is an isomorphism in $\underline{\mathrm{DM}}(X, R)$.

Lemma 3.15. *Under the assumptions of the previous paragraph, if f is radicial, then the pullback functor*

$$\mathbf{L}f^* : \mathrm{DM}(X, R) \to \mathrm{DM}(Y, R)$$

is fully faithful.

Proof. As the inclusion $\mathrm{DM}(-, R) \subset \underline{\mathrm{DM}}(-, R)$ is fully faithful and commutes with $\mathbf{L}f^*$, it is sufficient to prove that the functor

$$f^* : \underline{\mathrm{DM}}(X, R) \to \underline{\mathrm{DM}}(Y, R)$$

is fully faithful. In other words, we must see that the composition of f^* with its left adjoint $f_\sharp$ is isomorphic to the identity functor (through the co-unit of the adjunction). For any object M of $\underline{\mathrm{DM}}(X, R)$, we have a projection formula:

$$f_\sharp f^*(M) \simeq f_\sharp(R_Y) \otimes^{\mathbf{L}}_R M\,.$$

Therefore, it is sufficient to check that the co-unit

$$f_\sharp(R_Y) \simeq R_X$$

is an isomorphism. Since f is radicial, its degree must be a power of p, hence must be invertible in R. An inverse is provided by the map ${}^t f : R_X \to f_\sharp(R_Y)$. $\square$

3.16. These computations can be interpreted in terms of *HR*-modules as follows (we keep the assumptions of 3.14).
Using the internal Hom of $\underline{\mathrm{DM}}(X, R)$, one gets a morphism

$$Tr_f : \mathbf{R}f_*(R_Y) \to R_X$$

Since the right adjoint of the inclusion $\mathrm{DM}(-, R) \subset \underline{\mathrm{DM}}(-, R)$ commutes with $\mathbf{R}f_*$, the map Tr_f above can be seen as a map in $\mathrm{DM}(X, R)$. Similarly, since the functor $t_* : \mathrm{DM}(-, R) \to HR$-Mod commutes with $\mathbf{R}f_*$, we get a trace morphism

$$Tr_f : \mathbf{R}f_* HR_Y \to HR_X$$

in HR_X-Mod. For any HR_X-module E, we obtain a trace morphism

$$Tr_f : \mathbf{R}f_*\mathbf{L}f^*(E) \to E$$

as follows. Since we have the projection formula

$$\mathbf{R}f_*(HR_Y) = \mathbf{R}f_*\mathbf{L}f^*(HR_X) \simeq \mathbf{R}f_*(\mathbb{1}_Y) \otimes^{\mathbf{L}} HR_X\,,$$

the unit $\mathbb{1}_X \to HR_X$ induces a map

$$\widetilde{Tr}_f : \mathbf{R}f_*(\mathbb{1}_Y) \to \mathbf{R}f_*(\mathbb{1}_Y) \otimes^{\mathbf{L}} HR_X \simeq \mathbf{R}f_*\mathbf{L}f^*(HR_X) \simeq \mathbf{R}f_*(HR_Y) \xrightarrow{Tr_f} HR_X.$$

For any HR_X-module E, tensoring the map $\widetilde{Tr}_f$ with identity of E and composing with the action $HR_X \otimes^{\mathbf{L}} E \to E$ leads to a canonical morphism in HR_X-Mod:

$$Tr_f : \mathbf{R}f_*\mathbf{L}f^*(E) \simeq \mathbf{R}f_*(\mathbb{1}_Y) \otimes^{\mathbf{L}} E \to E\,.$$

By construction of these trace maps, we have the following lemma.

Lemma 3.17. *Under the assumptions of paragraph 3.14, for any HR_X-module E, the composition of Tr_f with the unit of the adjunction between $\mathbf{L}f^*$ and $\mathbf{R}f_*$*

$$E \to \mathbf{R}f_*\mathbf{L}f^*(E) \xrightarrow{Tr_f} E$$

is d times the identity of E. If, moreover, f is radicial, then the composition

$$\mathbf{R}f_*\mathbf{L}f^*(E) \xrightarrow{Tr_f} E \to \mathbf{R}f_*\mathbf{L}f^*(E)$$

is also d times the identity of $\mathbf{R}f_*\mathbf{L}f^*(E)$.

This also has consequences when looking at the HR_K-modules associated to X and Y. To simplify the notations, we will write

$$HR(U) = HR_K \otimes^{\mathbf{L}} \Sigma^\infty(U_+)$$

for any smooth K-scheme U.

Lemma 3.18. *Under the assumptions of paragraph 3.14, if d is invertible in R, and if both X and Y are smooth over K, then $HR(X)$ is a direct factor of $HR(Y)$ in HR_K-Mod.*

Proof. Let $p : X \to \mathrm{Spec}(K)$ and $q : Y \to \mathrm{Spec}(K)$ be the structural maps of X and Y, respectively. Since $pf = q$, for any HR_K-module E, we have:

$$\mathrm{Hom}(HR(X), E) = \mathrm{Hom}(HR_X, p^*(E))$$
$$\mathrm{Hom}(HR(Y), E) = \mathrm{Hom}(HR_X, \mathbf{R}f_*\mathbf{L}f^*p^*(E))\,.$$

Therefore, this lemma is a translation of the first assertion of Lemma 3.17 and of the Yoneda Lemma. □

Proof of Proposition 3.13. We first consider the case of a perfect field K. The reference is [HKØ, 5.8]. We use here a slightly different theoretical setting than these authors so we give a proof to convince the reader.
Because t^* preserves the canonical compact generators of both categories, we need only to prove it is fully faithful on a family of compact generators of HR_K-Mod (see [CDa, Corollary 1.3.21]). For any HR_K-modules E, F belonging to a suitable generating family of HR_K-Mod, and and any integer n, we want to prove that the map

$$(3.18.1) \qquad \mathrm{Hom}_{HR_K\text{-Mod}}(E, F[n]) \xrightarrow{t^*} \mathrm{Hom}_{\mathrm{DM}(K,R)}(t^*(E), t^*(F)[n])$$

For this purpose, using the method of [Rio05, Sec. 1], with a small change indicated below, we first prove that HR_K-Mod is generated by objects of the form $HR(X)(i)$ for a smooth projective K-scheme X and an integer i. Since these are compact, it is sufficient to prove the following property: for any HR_K-module M such that

$$\mathrm{Hom}_{HR_K\text{-Mod}}(HR(X)(p)[q], M) = 0$$

for any integers p and q, we must have $M \simeq 0$. To prove the vanishing of M, it is sufficient to prove the vanishing of $M \otimes \mathbf{Z}_{(\ell)}$ for any prime $\ell \neq p$. On the other hand, for any compact object C, the formation of $\mathrm{Hom}(C, -)$ commutes with tensoring by $\mathbf{Z}_{(\ell)}$; therefore, we may assume R to be a $\mathbf{Z}_{(\ell)}$-algebra for some prime number $\ell \neq p$. Under this additional assumption, we will prove that, for any smooth connected K-scheme X, the object $HR(X) = HR_k \otimes^{\mathbf{L}} \Sigma^\infty(X_+)$ is in the thick subcategory $\mathcal{P}$ generated by Tate twists of HR_K-modules of the form $HR(W)$ for W a smooth projective K-scheme. Using the induction principle explained by Riou in *loc. cit.*, on the dimension d of X, we see that, given any couple (Y, V), where Y is a smooth K-scheme of dimension d, and V is a dense open subscheme of Y, the property that $HR(Y)$ belongs to $\mathcal{P}$ is equivalent to the property $HR(V)$ belongs to $\mathcal{P}$. Therefore, it is enough to consider the case of a dense open subscheme of X which we can shrink at will. In particular, applying Gabber's theorem [ILO14, IX, 1.1], we can assume there exists a flat, finite, and surjective morphism, $f : Y \to X$ which is of degree prime to ℓ, and such that Y is a dense open subscheme of a smooth projective k-scheme. Since $HR(Y) \in \mathcal{P}$, Lemma 3.18 concludes.
We now are reduced to prove that the map (3.18.1) is an isomorphism when $E = HR(X)(i)$ and $F = HR(Y)(j)$ for X and Y smooth and projective over K. Say d is the dimension of Y. Then according to [Dég08a, Sec. 5.4], $HR_K(Y)$ is strongly dualizable with strong dual $HR_K(Y)(-d)[-2d]$. Then the result follows from the fact that the two members of (3.18.1) compute the motivic cohomology group of $X \times_K Y$ in degree $(n - 2d, j - i - d)$ (in a compatible way, because the functor t^* is symmetric monoidal). This achieves the proof of Proposition 3.13 in the case where the ground field K is perfect.

Let us now consider the general case. Again, we are reduced to prove the map (3.18.1) is an isomorphism whenever E and F are compact (hence constructible). Let K be a finite extension of k, and let L/K be a finite totally inseparable extension of fields, with corresponding morphism of schemes $f : \mathrm{Spec}(L) \to \mathrm{Spec}(K)$. According to Lemma 3.15, the functor $\mathbf{L}f^* : \mathrm{DM}(K,R) \to \mathrm{DM}(L,R)$ is fully faithful. Moreover, the pullback functor $\mathbf{L}f^* : HR_K\text{-Mod} \to HR_L\text{-Mod}$ is fully faithful as well; see the last assertion of Lemma 3.17 (and recall that the degree of the extension L/K must be a power of p, whence is invertible in R). Thus, by continuity of the premotivic categories $\mathrm{DM}(-,R)$ and HR-Mod (see Examples 2.6(2) and 2.11), Proposition 2.7 gives the following useful lemma:

Lemma 3.19. *Let K^s be the inseparable closure of K (i.e. the biggest purely inseparable extension of K in some algebraic closure of K). Then the following pullback functors:*

$$\mathrm{DM}_c(K,R) \to \mathrm{DM}_c(K^s,R) \quad \textit{and} \quad HR_K\text{-Mod}_c \to HR_{K^s}\text{-Mod}_c$$

are fully faithful.

With this lemma in hands, to prove that (3.18.1) is an isomorphism for constructible HR_K-modules E and F, we can replace the field K by the perfect field K^s, and this proves Proposition 3.13 in full generality. □

3.c. Proof in the regular case. In the course of the proof of Theorem 3.1, we wil use the following lemma:

Lemma 3.20. *Let T and S be regular k-schemes and $f : T \to S$ be a morphism of k-schemes.*

(1) *If T is the limit of a projective system of S-schemes with dominant affine smooth transition morphisms, then t_* commutes with f^*.*
(2) *If f is a closed immersion, and if S is the limit of a projective system of smooth separated k-schemes of finite type with flat affine transition morphims, then t_* commutes with f^*.*
(3) *If f is an open immersion, then t_* commutes with $f_!$.*

Proof. The forgetful functor $\mathcal{O}_{HR} : HR\text{-Mod} \to \mathrm{SH}$ is conservative, and it commutes with f^* for any morphism f and with $j_!$ for any open immersion; see the last assertion of [CDa, Prop. 7.2.14]. Therefore, it is sufficient to check each case of this lemma after replacing t_* by φ_*.
Then, case (1) follows easily by continuity of DM and SH with respect to dominant maps, and from the case where f is a smooth morphism. Case (2) was proved in Proposition 3.12. (taking into account 3.5). Then case (3) finally follows from results of [CDa]: in fact φ_* is defined as the following composition:

$$\mathrm{DM}(S,R) \xrightarrow{\mathbf{L}\gamma_*} \mathrm{D}_{\mathbf{A}^1}(S,R) \xrightarrow{K} \mathrm{SH}(S)$$

with the notation of [CDa, 11.2.16] ($\Lambda = R$). The fact K commutes with $j_!$ is obvious and for $\mathbf{L}\gamma_*$, this is [CDa, 6.3.11]. □

To be able to use the refined version of Popescu's theorem proved by Spivakovsky (see [Spi99, Th. 10.1], "resolution by smooth sub-algebras"), we will need the following esoteric tool extracted from an appendix of Bourbaki (see [Bou93, IX, Appendice] and, in particular, Example 2).

Definition 3.21. Let A be a local ring with maximal ideal $\mathfrak{m}$.
We define the ∞-*gonflement* (resp. n-*gonflement*) of A as the localization of the polynomial A-algebra $A[(x_i)_{i\in\mathbf{N}}]$ (resp. $A[(x_i)_{0\leq i\leq n}]$) with respect to the prime ideal $\mathfrak{m}.A[x_i, i \in \mathbf{N}]$ (resp. $\mathfrak{m}.A[x_i, 0 \leq i \leq n]$).

3.22. Let B (resp. B_n) be the ∞-gonflement (resp. n-gonflement) of a local noetherian ring A. We will use the following facts about this construction, which are either obvious or follow from *loc. cit.*, Prop. 2:

(1) The rings B and B_n are noetherian.
(2) The A-algebra B_n is the localization of a smooth A-algebra.
(3) The canonical map $B_n \to B_{n+1}$ is injective.
(4) $B = \varinjlim_{n\in\mathbf{N}} B_n$, with the obvious transition maps.

We will need the following easy lemma:

Lemma 3.23. *Consider the notations above. Assume that A is a local henselian ring with infinite residue field. Then for any integer $n \geq 0$, the A-algebra B_n is a filtered inductive limit of its smooth and split sub-A-algebras.*

Proof. We know that B_n is the union of A-algebras of the form $A[x_1, \ldots, x_n][1/f]$ for a polynomial $f \in A[x_1, ..., x_n]$ whose reduction modulo $\mathfrak{m}$ is non zero. Let us consider the local scheme $X = \mathrm{Spec}(A)$, s be its closed point and put $U_n(f) = \mathrm{Spec}(A[x_1, \ldots, x_n][1/f])$ for a polynomial f as above. To prove the lemma, it is sufficient to prove that $U_n(f)/X$ admits a section. By definition, the fiber $U_n(f)_s$ of $U_n(f)$ at the point s is a non empty open subscheme. As $\kappa(s)$ is infinite by assumption, $U_n(f)_s$ admits a $\kappa(s)$-rational point. Thus $U_n(f)$ admits an S-point because X is henselian and $U_n(f)/X$ is smooth (see [EGA4, 18.5.17]). □

Combining properties (1)-(4) above with the preceding lemma, we get the following property:

(G) Let A be a noetherian local henselian ring with infinite residue field, and B be its ∞-gonflement. Then B is a noetherian A-algebra which is the filtering union of a family $(B_\alpha)_{\alpha\in I}$ of smooth split sub-A-algebras of B.

Lemma 3.24. *Consider the notations of property (G). Then the pullback along the induced map $p : X' = \mathrm{Spec}(B) \to X = \mathrm{Spec}(A)$ defines a conservative functor $\mathbf{L}p^* : \mathrm{SH}(X) \to \mathrm{SH}(X')$.*

Proof. Let E be an object of $\mathrm{SH}(X)$ such that $\mathbf{L}p^*(E) = 0$ in $\mathrm{SH}(X')$. We want to prove that $E = 0$. For this, it is sufficient to prove that, for any constructible object C of $\mathrm{SH}(X)$, we have

$$\mathrm{Hom}(C, E) = 0\,.$$

Given the notations of property (G), and any index $\alpha \in I$, let C_i and E_i be the respective pullbacks of C and E along the structural map $p_\alpha : \mathrm{Spec}(B_\alpha) \to \mathrm{Spec}(A)$. Then, by continuity, the map

$$\varinjlim_{\alpha} \mathrm{Hom}(C_\alpha, E_\alpha) \to \mathrm{Hom}(\mathbf{L}p^*(C), \mathbf{L}p^*(E))$$

is an isomorphism, and thus, according to property (G), the map

$$\mathrm{Hom}(C, E) \to \mathrm{Hom}(\mathbf{L}p^*(C), \mathbf{L}p^*(E))$$

is injective because each map p_α is a split epimorphism. □

In order to use ∞-gonflements in HR-modules without any restriction on the size of the ground field, we will need the the following trick, which makes use of transfers up to homotopy:

Lemma 3.25. *Let L/K be a purely transcendental extension of fields of transcendence degree* 1*, with K perfect, and let $p : \mathrm{Spec}(L) \to \mathrm{Spec}(K)$ be the induced morphism of schemes. Then, for any objects M and N of* $\mathrm{DM}(K, R)$*, if M is compact, then the natural map*

$$\mathrm{Hom}_{\mathrm{DM}(K,R)}(M, N) \to \\ \to \mathrm{Hom}_{\mathrm{DM}(K,R)}(M, \mathbf{R}p_*\, p^*(N)) = \mathrm{Hom}_{\mathrm{DM}(K,R)}(\mathbf{L}p^*(M), \mathbf{L}p^*(N))$$

is a split embedding. In particular, the pullback functor

$$\mathbf{L}p^* : \mathrm{DM}(K, R) \to \mathrm{DM}(L, R)$$

is conservative.

Proof. Let I be the cofiltering set of affine open neighbourhoods of the generic point of $\mathbf{A}^1_K$ ordered by inclusion. Obviously, $\mathrm{Spec}(L)$ is the projective limit of these open neighbourhoods. Thus, using continuity for DM with respect to dominant maps, we get that:

$$\mathrm{Hom}(M, \mathbf{R}p_*\, \mathbf{L}p^*(N)) = \varinjlim_{V \in I^{op}} \mathrm{Hom}(M(V), \underline{\mathrm{Hom}}(M, N))\,.$$

We will use the language of generic motives from [Dég08b]. Recall that $M(L) =$ "$\varprojlim M(V)$" is a pro-motive in $\mathrm{DM}(K)$, so that the preceding identification now takes the following form.

$$\mathrm{Hom}(M, \mathbf{R}p_*\, \mathbf{L}p^*(N)) = \mathrm{Hom}(M(L), \underline{\mathrm{Hom}}(M, N))\,.$$

Since, according to [Dég08b, Cor. 6.1.3], the canonical map $M(L) \to M(K)$ is a split epimorphism of pro-motives, this proves the first assertion of the lemma. The second assertion is a direct consequence of the first and of the fact that the triangulated category $\mathrm{DM}(K, R)$ is compactly generated. □

Proof of Theorem 3.1. We want to prove that for a regular noetherian k-scheme of finite dimension S, the adjunction:

$$t^* : HR_S\text{-Mod} \rightleftarrows \mathrm{DM}(S, R) : t_*$$

is an equivalence of triangulated categories. Since the functor t^* preserves compact objects, and since there is a generating family of compact objects of $\mathrm{DM}(S,R)$ in the essential image of the functor t^*, it is sufficient to prove that t^* is fully faithful on compact objects (see [CDa, Corollary 1.3.21]): we have to prove that, for any compact HR_S-module M, the adjunction map $\eta_M : M \to t_* t^*(M)$ is an isomorphism.

First case: We first assume that S is essentially smooth – *i.e.* the localization of a smooth k-scheme. We proceed by induction on the dimension of S. The case of dimension 0 follows from Proposition 3.13.

We recall that the category HR-Mod is continuous on Reg_k (3.10). Let x be a point of S and S_x be the localization of S at x, $p_x : S_x \to S$ the natural projection. Then it follows from [CDa, Prop. 4.3.9] that the family of functors:

$$p_x^* : HR_S\text{-Mod} \to HR_{S_x}\text{-Mod}, x \in S$$

is conservative.

Since p_x^* commutes with t^* (trivial) and with t_* (according to Lemma 3.20), we can assume that S is a local essentially smooth k-scheme.

To prove the induction case, let i (resp. j) be the immersion of the closed point x of S (resp. of the open complement U of the closed point of S). Since the localization property with respect to i is true in HR-Mod (because it is true in SH, using the last assertions of Proposition 3.10) and in DM (because of Proposition 3.12 that we can apply because we have assumed that S is essentially smooth), we get two morphisms of distinguished triangles:

$$\begin{tikzcd}
j_!j^*(M) \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & i_*i^*(M) \arrow[r] \arrow[d] & j_!j^*(M)[1] \arrow[d] \\
j_!j^*(t_*t^*(M)) \arrow[r] \arrow[d, "\wr"] & t_*t^*(M) \arrow[r] \arrow[d, equal] & i_*i^*(t_*t^*(M)) \arrow[r] \arrow[d, "\wr"] & j_!j^*(t_*t^*(M))[1] \arrow[d, "\wr"] \\
j_!t_*t^*j^*(M) \arrow[r] & t_*t^*(M) \arrow[r] & i_*t_*t^*i^*(M) \arrow[r] & j_!t_*t^*j^*(M)[1]
\end{tikzcd}$$

The vertical maps on the second floor are isomorphisms: both functors t^* and t_* commute with j^* (as t^* is the left adjoint in a premotivic adjunction, it commutes with $j_!$ and j^*, and this implies that t_* commutes with j^*, by transposition); the functor t^* commutes with i_* because it commutes with $j_!$, j^* and i^*, and because the localization property with respect to i is verified in HR-Mod as well as in DM); finally, applying the third assertion of Lemma 3.20 for $f = j$, this implies that the functor t_* commutes with i^*. To prove that the map η_M is an isomorphism, it is thus sufficient to treat the case of $j_!\eta_{j^*(M)}$ and of $i_*\eta_{i^*(M)}$. This means we are reduced to the cases of U and $\mathrm{Spec}(\kappa(x))$, which follow respectively from the inductive assumption and from the case of dimension zero.

General case: Note that the previous case shows in particular the theorem for any smooth k-scheme. Assume now that S is an arbitrary regular noetherian k-scheme. Using [CDa, Prop. 4.3.9] again, and proceeding as we already did above (but considering limits of Nisnevich neighbourhoods instead of Zariski ones), we may assume that S is henselian. Let $L = k(t)$ be the field of rational functions, and let us form the following pullback square.

$$\begin{tikzcd} S' \arrow[r, "q"] \arrow[d, "g"'] & S \arrow[d, "f"] \\ \mathrm{Spec}(k(t)) \arrow[r, "p"] & \mathrm{Spec}(k) \end{tikzcd}$$

Then the functor

$$\mathbf{R}p_* \, \mathbf{L}p^* : HR_k\text{-Mod} \to HR_k\text{-Mod}$$

is conservative: this follows right away from Lemma 3.25 and Proposition 3.13. This implies that the functor

$$\mathbf{L}q^* : HR_S\text{-Mod} \to HR_{S'}\text{-Mod}$$

is conservative. To see this, let us consider an object E of HR_S-Mod such that $\mathbf{L}q^*(E) = 0$. To prove that $E = 0$, it is sufficient to prove that $\mathrm{Hom}(M, E) = 0$ for any compact object M of HR_S-Mod. Formula

$$\mathrm{Hom}(HR_k, \mathbf{R}f_* \, \underline{\mathrm{Hom}}(M, E)) \simeq \mathrm{Hom}(M, E)$$

implies that it is sufficient to check that $\mathbf{R}f_* \, \underline{\mathrm{Hom}}(M, E) = 0$ for any compact object M (where $\underline{\mathrm{Hom}}$ is the internal Hom of HR_S-Mod).
Since the functor $\mathbf{R}p_* \, \mathbf{L}p^*$ is conservative, it is thus sufficient to prove that

$$\mathbf{R}p_* \, \mathbf{L}p^* \mathbf{R}f_* \, \underline{\mathrm{Hom}}(M, E) = 0 \, .$$

We thus conclude with the following computations (see [CDa, Propositions 4.3.11 and 4.3.14]).

$$\begin{aligned} \mathbf{R}p_* \, \mathbf{L}p^* \mathbf{R}f_* \, \underline{\mathrm{Hom}}(M, E) &\simeq \mathbf{R}p_* \, \mathbf{R}g_* \, \mathbf{L}q^* \, \underline{\mathrm{Hom}}(M, E) \\ &\simeq \mathbf{R}p_* \, \mathbf{R}g_* \, \underline{\mathrm{Hom}}(\mathbf{L}q^*(M), \mathbf{L}q^*(E)) = 0 \end{aligned}$$

In conclusion, since the functor $\mathbf{L}q^*$ commutes with t_* (see Lemma 3.20 (1)), we may replace S by S' and thus assume that the residue field of S is infinite. Let B be the ∞-gonflement of $A = \Gamma(S, \mathcal{O}_S)$ (Definition 3.21), and $f : T = \mathrm{Spec}(B) \to S$ be the map induced by the inclusion $A \subset B$. We know that the functor

$$\mathbf{L}f^* : HR_S\text{-Mod} \to HR_T\text{-Mod}$$

is conservative: as the forgetful functor HR-Mod $\to$ SH is conservative and commutes with $\mathbf{L}f^*$, this follows from Lemma 3.24 (or one can reproduce the proof of this lemma, which only used the continuity property of SH with respect to projective systems of schemes with dominant affine transition morphisms). Similarly, it follows again from Lemma 3.20 (1) that the functor t_* commutes with $\mathbf{L}f^*$. As the functor t^* commutes with $\mathbf{L}f^*$, it is sufficient to prove that the functor t^* is fully faithful over T, and it is still sufficient to check this property on compact objects. Since the ring B is noetherian and regular, and has a field of functions with infinite transcendance degree over the perfect field k (see 3.22), it follows from Spivakovsky's refinement of Popescu's Theorem [Spi99, 10.1] that B is the filtered union of its smooth subalgebras of finite type over k. In other terms, T is the projective limit of a projective system of smooth affine k-schemes of finite type (T_α) with dominant transition maps. Therefore, by continuity (see Examples 2.11 and 2.6(2)), we can apply Proposition 2.7 twice and see that the functor

$$2\text{-}\varinjlim_{\alpha} HR_{T_\alpha}\text{-Mod}_c \simeq HR_T\text{-Mod}_c \to 2\text{-}\varinjlim_{\alpha} \mathrm{DM}_c(T_\alpha, R) \simeq \mathrm{DM}_c(T, R)$$

is fully faithful, as a filtered 2-colimit of functors having this property. □

4. More modules over motivic Eilenberg-MacLane spectra

4.1. Given a scheme X, let $Mon(X)$ be the category of unital associative monoids in the category of symmetric Tate spectra Sp_X. The forgetful functor

$$U : Mon(X) \to \mathrm{Sp}_X$$

has a left adjoint, the free monoid functor:

$$F : \mathrm{Sp}_X \to Mon(X)\,.$$

Since the stable model category of symmetric Tate spectra satisfies the monoid axiom (see [Hoy15, Lemma 4.2]), by virtue of a well known theorem of Schwede and Shipley [SS00, Theorem 4.1(3)], the category $Mon(X)$ is endowed with a combinatorial model category structure whose weak equivalences (fibrations) are the maps whose image by U are weak equivalences (fibrations) in Sp_X; furthermore, any cofibrant monoid is also cofibant as an object of Sp_X.

4.2. We fix once and for all a cofibrant resolution

$$HR' \to HR_k$$

of the motivic Eilenberg-MacLane spectrum HR_k in the model category $Mon(k)$. Given a k-scheme X with structural map $f : X \to \mathrm{Spec}(k)$, we define

$$HR_{X/k} = f^*(HR')$$

(where f^* denotes the pullback functor in the premotivic model category Sp). The family $(HR_{X/k})_X$ is a cartesian section of the Sm-fibred category of monoids in Sp which is also homotopy cartesian (as we have an equality $\mathbf{L}f^*(HR_k) = HR_{X/k}$). We write $HR_{X/k}$-Mod for the homotopy category of (left) $HR_{X/k}$-modules.

This notation is in conflict with the one introduced in Definition 3.8. This conflict disappears up to weak equivalence[8]: when X is regular, the comparison

[8]In the proof of Theorem 3.1, we used the fact that the spectra $HR_{X/k}$, as defined in Definition 3.8, are *commutative* monoids of the model category of symmetric Tate spectra (because we used Poincaré duality in an essential way, in the case where X is the spectrum of a perfect field). This new version of motivic Eilenberg-MacLane spectra $HR_{X/k}$ is not required to be commutative anymore (one could force this property by working with fancier model categories of motivic spectra (some version of the 'positive model structure', as discussed in [Hor13] for instance), but these extra technicalities are not necessary for our purpose. We shall use Theorem 3.1 in a crucial way, though.

map
$$f^*(HR') \to f^*(HR_k)$$
is a weak equivalence (Proposition 2.8). For X regular, $HR_{X/k}$ is thus a cofibrant resolution of HR_X in the model category $Mon(X)$. In particular, in the case where X is regular, we have a canonical equivalence of triangulated categories:
$$HR_{X/k}\text{-Mod} \simeq HR_X\text{-Mod}\,.$$

Proposition 4.3. *The assignment $X \mapsto HR_{X/k}$-Mod defines a motivic category over the category of noetherian k-schemes of finite dimension which has the property of continuity with respect to arbitrary projective systems with affine transition maps. Moreover, when we let X vary, both the free $HR_{X/k}$-algebra (derived) functor*
$$L_{HR_{X/k}} : \mathrm{SH}(X) \to HR_{X/k}\text{-Mod}$$
and its right adjoint
$$\mathcal{O}_{HR_{X/k}} : HR_{X/k}\text{-Mod} \to \mathrm{SH}(X)$$
are morphisms of premotivic triangulated categories over the category of k-schemes. In other words both functors commute with $\mathbf{L}f^*$ for any morphism of k-schemes f, and with $\mathbf{L}g_\sharp$ for any separated smooth morphism of k-schemes g.

Proof. The first assertion comes from [CDa, 7.2.13 and 7.2.18], the one about continuity is a direct application of Lemma 2.10, and the last one comes from [CDa, 7.2.14]. □

Remark 4.4. Since the functor $\mathcal{O}_{HR_{X/k}} : HR_{X/k}\text{-Mod} \to \mathrm{SH}(X)$ is conservative and preserves small sums, the family of objects of the form $HR_{X/k} \otimes^{\mathbf{L}} \Sigma^\infty(Y_+)(n)$, for any separated smooth X-scheme Y and any integer n, do form a generating family of compact objects. In particular, the notions of constructible object and of compact object coincide in $HR_{X/k}$-Mod (see for instance [CDb, Remarks 5.4.10 and 5.5.11], for a context in which these two notions fail to coincide).

4.5. For any k-scheme X, we have canonical morphisms of monoids in Sp_X:
$$HR_{X/k} \to f^*(HR_k) \to HR_X\,.$$
In particular, we have a canonical functor
$$HR_{X/k}\text{-Mod} \to HR_X\text{-Mod}\;, \quad E \mapsto HR_X \otimes^{\mathbf{L}}_{HR_{X/k}} E\,.$$
If we compose the latter with the functor
$$HR_X\text{-Mod} \xrightarrow{t^*} \mathrm{DM}(X,R) \xrightarrow{\mathbf{L}\rho_!} \underline{\mathrm{DM}}(X,R) \xrightarrow{a^*_{\mathrm{cdh}}} \underline{\mathrm{DM}}_{\mathrm{cdh}}\,,$$
we get a functor
$$HR_{X/k}\text{-Mod} \to \underline{\mathrm{DM}}(X,R)$$

which defines a morphism a premotivic categories. In particular, this functor takes it values in $\mathrm{DM}_{\mathrm{cdh}}(X,R)$, and we obtain a functor

$$\tau^* : HR_{X/k}\text{-Mod} \to \mathrm{DM}_{\mathrm{cdh}}(X,R)\,.$$

As τ^* preserves small sums, it has a right adjoint τ_*, and we finally get a premotivc adjunction

$$\tau^* : HR_{(-)/k}\text{-Mod} \rightleftarrows \mathrm{DM}_{\mathrm{cdh}}(-,R) : \tau_*\,.$$

Moreover, the functor τ^* preserves the canonical generating families of compact objects. Therefore, the functor τ_* is conservative and commutes with small sums.

5. Comparison theorem: general case

The aim of this section is to prove:

Theorem 5.1. *Let k be a perfect field of characteristic exponent p. Assume that p is invertible in the ring of coefficients R. For any noetherian k-scheme of finite dimension X, the canonical functor*

$$\tau^* : HR_{X/k}\text{-Mod} \to \mathrm{DM}_{\mathrm{cdh}}(X,R)$$

is an equivalence of categories.

The proof will take the following path: we will prove this statement in the case where X is separated and of finite type over k. For this, we will use Gabber's refinement of de Jong's resolution of singularities by alterations, as well as descent properties for HR_k-modules proved by Shane Kelly to see that it is sufficient to consider the case of a smooth k-scheme. In this situation, Theorem 5.1 will be a rather formal consequence of Theorem 3.1. The general case will be obtained by a continuity argument.

5.2. Let ℓ be a prime number. Following S. Kelly [Kel12], one defines the ℓdh-topology on the category of noetherian schemes as the coarsest Grothendieck topology such that any cdh-cover is an ℓdh-cover and any morphism of the form $f : X \to Y$, with f finite, surjective, flat, and of degree prime to ℓ is an ℓdh-cover. For instance, if $\{U_i \to X\}_{i\in I}$ is a cdh-cover, and if, for each i one has a finite surjective flat morphism $V_i \to U_i$ of degree prime to ℓ, we get an ℓdh-cover $\{V_i \to X\}_{i\in I}$. In the case where X is noetherian, one can show that, up to refinement, any ℓdh-cover is of this form; see [Kel12, Prop. 3.2.5]. We will use several times the following non-trivial fact, which is a direct consequence of Gabber's theorem of uniformization prime to ℓ [ILO14, Exp. IX, Th. 1.1]: locally for the ℓdh-topology, any quasi-excellent scheme is regular. In other words, for any noetherian quasi-excellent scheme X (e.g. any scheme of finite type over field), there exists a morphism of finite type $p : X' \to X$ which is a covering for the ℓdh-topology and has a regular domain.

Proposition 5.3. *Let F be a* cdh*-sheaf with transfers over X which is $\mathbf{Z}_{(\ell)}$-linear. Then F is an ℓ*dh*-sheaf and, for any integer n, the map*

$$H^n_{\mathrm{cdh}}(X,F) \to H^n_{\ell\mathrm{dh}}(X,F)$$

is an isomorphism.

Proof. See [Kel12, Theorem 3.4.17]. □

Corollary 5.4. *Assume that X is of finite dimension, and let C be a complex of $\mathbf{Z}_{(\ell)}$-linear* cdh*-sheaves with transfers over X. Then the comparison map of hypercohomologies*

$$H^n_{\mathrm{cdh}}(X,C) \to H^n_{\ell\mathrm{dh}}(X,C)$$

is an isomorphism for all n.

Proof. Note that, for $t = \mathrm{cdh}$ or $t = \ell\mathrm{dh}$, the forgetful functor from $\mathbf{Z}_{(\ell)}$-linear t-sheaves with tranfers to $\mathbf{Z}_{(\ell)}$-linear t-sheaves on the big site of X is exact (this follows from the stronger results given by [Kel12, Prop. 3.4.15 and 3.4.16] for instance). Therefore, we have a canonical spectral sequence of the form

$$E_2^{p,q} = H^p_t(X, H^q(C)_t) \Rightarrow H^{p+q}_t(X,C)\,.$$

As the cohomological dimension with respect to the cdh-topology is bounded by the dimension, this spectral sequence strongly converges for $t = \mathrm{cdh}$. Proposition 5.3 thus implies that, for $t = \ell\mathrm{dh}$, the groups $E_2^{p,q}$ vanish for $p < 0$ or $p > \dim X$, so that this spectral sequence also converges in this case. Therefore, as these two spectral sequences agree on the E_2 term, we conclude that they induce an isomorphism on E_∞. □

Corollary 5.5. *For X of finite dimension and R an $\mathbf{Z}_{(\ell)}$-algebra, any object of the triangulated category $\underline{\mathrm{DM}}_{\mathrm{cdh}}(X,R)$ satisfies ℓ*dh*-descent (see* [CDa, Definition 3.2.5]*).*

Lemma 5.6. *Assume that X is of finite type over the perfect field k. Consider a prime ℓ which is distinct from the characteristic exponent of k. If R is a $\mathbf{Z}_{(\ell)}$-algebra, then any compact object of $HR_{X/k}$*-Mod *satisfies ℓ*dh*-descent.*

Proof. As X is allowed to vary, it is sufficient to prove that, for any constructible $HR_{X/k}$-modules M and any ℓdh-hypercover $p_\bullet : U_\bullet \to X$, the map

$$\mathbf{R}\Gamma(X,M) \to \mathbf{R}\varprojlim_{\Delta_n} \mathbf{R}\Gamma(U_n, p_n^* M) \tag{5.6.1}$$

is an isomorphism. The category of compact objects of HR_X-Mod is the thick subcategory generated by objects of the form $\mathbf{R}f_* HR_{Y/k}(p)$ for $f : Y \to X$ a projective map and p an integer (this follows right away from the fact that te analogous property is true in SH). We may thus assume that $M = \mathbf{R}f_* HR_{Y/k}(p)$. We can then form the following pullback in the category of simplicial schemes.

$$\begin{array}{ccc} V_\bullet & \xrightarrow{g} & U_\bullet \\ {\scriptstyle q_\bullet}\downarrow & & \downarrow{\scriptstyle p_\bullet} \\ Y & \xrightarrow{f} & X \end{array}$$

Using the proper base change formula for $HR_{(-)/k}$-modules, we see that the map (5.6.1) is isomorphic to the map

$$\mathbf{R}\Gamma(Y, HR_{Y/k}(p)) \to \mathbf{R}\varprojlim_{\Delta_n} \mathbf{R}\Gamma(V_n, HR_{V_n/k}(p))\,. \tag{5.6.2}$$

By virtue of Kelly's ℓdh-descent theorem [Kel12, Theorem 5.3.7], the map (5.6.2) is an isomorphism. □

Lemma 5.7. *Let X be a k-scheme of finite type. Assume that R is a $\mathbf{Z}_{(\ell)}$-algebra for ℓ a prime number distinct from the characteristic exponent of k. Let M be an object of $\underline{\mathrm{DM}}(X,R)$ satisfying ℓdh-descent on the site of smooth k-schemes over X: for any X-scheme of finite type Y which is smooth over k and any ℓdh-hypercover $p : U_\bullet \to Y$ such that U_n is smooth over k for any $n \geq 0$, the map*

$$\mathbf{R}\operatorname{Hom}_{\underline{\mathrm{DM}}(X,R)}(R(Y), M(p)) \to \mathbf{R}\varprojlim_{\Delta_n} \mathbf{R}\operatorname{Hom}_{\underline{\mathrm{DM}}(X,R)}(R(U_n), M(p))$$

is an isomorphism in the derived category of R-modules. Then, for any X-scheme Y which is smooth over k and any integer p, the canonical map

$$\mathbf{R}\operatorname{Hom}_{\underline{\mathrm{DM}}(X,R)}(R(Y), M(p)) \to \mathbf{R}\operatorname{Hom}_{\underline{\mathrm{DM}}_{\mathrm{cdh}}(X,R)}(R(Y), M_{\mathrm{cdh}}(p))$$

is an isomorphism.

Proof. Let us denote by $R\{1\}$ the complex

$$R\{1\} = R(1)[1] = \ker(R(\mathbf{A}^1_X - \{0\}) \to R)$$

induced by the structural map $\mathbf{A}^1 - \{0\} \times X \to X$. We may consider that the object M is a fibrant $R\{1\}$-spectrum in the category of complexes of R-linear sheaves with transfers on the category of X-schemes of finite type. In particular, M corresponds to a collection of complexes of R-linear sheaves with transfers $(M_n)_{\geq 0}$ together with maps $R\{1\} \otimes_R M_n \to M_{n+1}$ such that we have the following properties.

(i) For any integer $n \geq 0$ and any X-scheme of finite type Y, the map

$$\Gamma(Y, M_n) \to \mathbf{R}\Gamma(Y, M_n)$$

is an isomorphism in the derived category of R-modules (where $\mathbf{R}\Gamma$ stands for the derived global section with respect to the Nisnevich topology).

(ii) For any integer $n \geq 0$, the map

$$M_n \to \mathbf{R}\,\underline{\mathrm{Hom}}(R\{1\}, M_{n+1})$$

is an isomorphism in the derived category of Nisnevich sheaves with tranfers (where $\mathbf{R}\,\underline{\mathrm{Hom}}$ stands for the derived internal Hom).

We can choose another $R\{1\}$-spectum $N = (N_n)_{n\geq 0}$ of cdh-sheaves with transfers, together with a cofibration of spectra $M \to N$ such that $M_n \to N_n$ is a quasi-isomorphism locally for the cdh-topology, and such that each N_n satisfies cdh-descent: we do this by induction as follows. First, N_0 is any fibrant resolution of $(M_0)_{\mathrm{cdh}}$ for the cdh-local model structure on the category of complexes

of cdh-sheaves with transfers. If N_n is already constructed, we denote by E the pushout of M_n along the map $R\{1\} \otimes_R M_n \to R\{1\} \otimes_R N_n$, and we factor the map $E_{\mathrm{cdh}} \to 0$ into a trivial cofibration followed by a fibration in the cdh-local model structure.
Note that, for any X-scheme Y which is smooth over k, the map

$$H^i(Y, M_n) \to H^i(Y, N_n)$$

is an isomorphism of R-modules for any integers $i \in \mathbf{Z}$ and $n \geq 0$. Indeed, as, by virtue of Gabber's theorem of resolution of singularities by ℓdh-alterations [ILO14, Exp. IX, Th. 1.1], one can write both sides with the Verdier formula in the following way (because of our hypothesis on M and by construction of N):

$$H^i(Y, E) \simeq \varinjlim_{U_\bullet \to Y} H^i(\mathbf{R} \varprojlim_{\Delta_j} \Gamma(U_j, M_n)) \text{ for } E = M_n \text{ or } E = N_n,$$

where $U_\bullet \to Y$ runs over the filtering category of ℓdh-hypercovers of Y such that each U_j is smooth over k. It is also easy to see from this formula that each N_n is $\mathbf{A}^1$-homotopy invariant and that the maps

$$N_n \to \underline{\mathrm{Hom}}(R\{1\}, N_{n+1})$$

are isomorphisms. In other words, N satisfies the analogs of properties (i) and (ii) above with respect to the cdh-topology. We thus get the following identifications for $p \geq 0$:

$$\begin{aligned}\Gamma(Y, M_p) &= \mathbf{R}\,\mathrm{Hom}_{\underline{\mathrm{DM}}(X,R)}(R(Y), M(p))\\ \Gamma(Y, N_p) &= \mathbf{R}\,\mathrm{Hom}_{\underline{\mathrm{DM}}_{\mathrm{cdh}}(X,R)}(R(Y), M_{\mathrm{cdh}}(p))\,.\end{aligned}$$

The case where $p < 0$ follows from the fact that, for $d = -p$, $R(Y)(d)[2d]$ is then a direct factor of $R(Y \times \mathbf{P}^d)$ (by the projective bundle formula in $\underline{\mathrm{DM}}_{\mathrm{cdh}}(X, R)$). □

Lemma 5.8. *Let X be a smooth separated k-scheme of finite type. Assume that R is a $\mathbf{Z}_{(\ell)}$-algebra for ℓ a prime number distinct from the characteristic exponent of k. If M and N are two constructible objects of* $\mathrm{DM}(X, R)$*, then the comparison map*

$$\mathbf{R}\,\mathrm{Hom}_{\mathrm{DM}(X,R)}(M, N) \to \mathbf{R}\,\mathrm{Hom}_{\mathrm{DM}_{\mathrm{cdh}}(X,R)}(M, N)$$

is an isomorphism in the derived category of R-modules.

Proof. It is sufficient to prove this in the case where $M = R(Y)(p)$ for Y a smooth X-scheme and p any integer. By virtue of Lemma 5.7, it is sufficient to prove that any constructible object of $\mathrm{DM}(X, R)$ satisfies ℓdh-descent on the site of X-schemes which are smooth over k. By virtue of Theorem 3.1, it is thus sufficient to prove the analogous property for constructible HR_X-modules, which follows from Lemma 5.6. □

Proof of Theorem 5.1. It is sufficient to prove that the restriction of the comparison functor

$$HR_{X/k}\text{-Mod} \to \mathrm{DM}_{\mathrm{cdh}}(X,R)\ , \quad M \mapsto \tau^*(M) \tag{5.8.1}$$

to constructible $HR_{X/k}$-modules is fully faithful (by virtue of [CDa, Corollary 1.3.21], this is because both triangulated categories are compactly generated and because the functor (5.8.1) preserves the canonical compact generators). It is easy to see that this functor is fully faithful (on constructible objects) if and only if, for any prime $\ell \neq p$, its $R \otimes \mathbf{Z}_{(\ell)}$-linear version has this property (this is because the functor (5.8.1) preserves compact objects, which implies that its right adjoint commutes with small sums, hence both functors commute with the operation of tensoring by $\mathbf{Z}_{(\ell)}$). Therefore, we may assume that a prime number $\ell \neq p$ is given and that R is a $\mathbf{Z}_{(\ell)}$-algebra. We will then prove the property of being fully faithful first in the case where X is of finite type over k, and then, by a limit argument, in general.
Assume that X is of finite type over k, and consider constructible $HR_{X/k}$-modules M and N. We want to prove that, the map

$$\mathbf{R}\operatorname{Hom}_{HR_{X/k}\text{-Mod}}(M,N) \to \mathbf{R}\operatorname{Hom}_{\mathrm{DM}_{\mathrm{cdh}}(X,R)}(\tau^*(M),\tau^*(N)) \tag{5.8.2}$$

is an isomorphism (here all the $\mathbf{R}\operatorname{Hom}$'s take their values in the triangulated category of topological S^1-spectra; see [CDa, Theorem 3.2.15] for the existence (and uniqueness) of such an enrichment). By virtue of Gabber's theorem of resolution of singularities by ℓdh-alterations [ILO14, Exp. IX, Th. 1.1], we can choose an ℓdh-hypercover $p_\bullet : U_\bullet \to X$, with U_n smooth, separated, and of finite type over k for any non negative integer n. We then have the following chain of isomorphisms, justified respectively by ℓdh-descent for constructible $HR_{X/k}$-modules (Lemma 5.6), by the comparison theorem relating the category of HR-modules with DM over regular k-schemes (Theorem 3.1), by Lemma 5.8, and finally by the fact that any complex of R-modules with transfers on the category of separated X-schemes of finite type which satisfies cdh-descent must satisfy ℓdh-descent as well (Corollary 5.4):

$$\begin{aligned}
\mathbf{R}\operatorname{Hom}_{HR_{X/k}\text{-Mod}}(M,N) &\simeq \mathbf{R}\varprojlim_{\Delta_n} \mathbf{R}\operatorname{Hom}_{HR_{U_n}\text{-Mod}}(\mathbf{L}p_n^*M, \mathbf{L}p_n^*N)\\
&\simeq \mathbf{R}\varprojlim_{\Delta_n} \mathbf{R}\operatorname{Hom}_{\mathrm{DM}(U_n,R)}(\mathbf{L}p_n^*\, t^*(M), \mathbf{L}p_n^*\, t^*(N))\\
&\simeq \mathbf{R}\varprojlim_{\Delta_n} \mathbf{R}\operatorname{Hom}_{\mathrm{DM}_{\mathrm{cdh}}(U_n,R)}(\mathbf{L}p_n^*\, \tau^*(M), \mathbf{L}p_n^*\, \tau^*(N))\\
&\simeq \mathbf{R}\operatorname{Hom}_{\mathrm{DM}_{\mathrm{cdh}}(X,R)}(\tau^*(M),\tau^*(N))\,.
\end{aligned}$$

It remains to treat the case of an arbitrary noetherian k-scheme X. It is easy to see that the property that the functor (5.8.1) is fully faithful (on constructible objects) is local on X with respect to the Zariski topology. Therefore, we may assume that X is affine with structural ring A. We can then write A as a filtering colimit of k-algebras of finite type $A_i \subset A$, so that we obtain a projective system of k-schemes of finite type $\{X_i = \operatorname{Spec} A_i\}_i$ with affine and dominant transition maps, such that $X = \varprojlim_i X_i$. But then, by continuity (applying Proposition 2.7 twice, using Lemma 2.10 for $HR_{X/k}$-Mod, and Example 2.6(2) for $\mathrm{DM}_{\mathrm{cdh}}(X,R)$), we have canonical equivalences of categories at the level of constructible objects:

$$\begin{aligned}HR_{X/k}\text{-Mod}_c &\simeq 2\text{-}\varinjlim_i HR_{X_i/k}\text{-Mod}_c\\ &\simeq 2\text{-}\varinjlim_i \mathrm{DM}_{\mathrm{cdh}}(X_i,R)_c\\ &\simeq \mathrm{DM}_{\mathrm{cdh}}(X,R)_c\,.\end{aligned}$$

In particular, the functor (5.8.1) is fully faithful on constructible objects, and this ends the proof. □

Corollary 5.9. *Let X be a regular noetherian k-scheme of finite dimension. Then the canonical functor*

$$\mathrm{DM}(X,R) \to \mathrm{DM}_{\mathrm{cdh}}(X,R)$$

is an equivalence of symmetric monoidal triangulated categories.

Proof. This is a combination of Theorems 3.1 and 5.1, and of Proposition 3.10. □

Remark that we get for free the following result, which generalizes Kelly's ℓdh-descent theorem:

Theorem 5.10. *Let k be a field of characteristic exponent p, ℓ a prime number distinct from p, and R a $\mathbf{Z}_{(\ell)}$-algebra. Then, for any noetherian k-scheme of finite dimension X, any object of $HR_{X/k}$-*Mod *satisfies* ℓdh*-descent.*

Proof. This follows immediately from Theorem 5.1 and from Corollary 5.5. □

Similarly, we see that $\mathrm{DM}_{\mathrm{cdh}}$ is continuous is a rather general sense.

Theorem 5.11. *The motivic category $\mathrm{DM}_{\mathrm{cdh}}(-,R)$ has the properties of localization with respect to any closed immersion as well as the property of continuity with respect to arbitrary projective systems with affine transition maps over the category of noetherian k-schemes of finite dimension.*

Proof. Since $HR_{(-)/k}$-Mod has these properties, Theorem 5.1 allows to transfer it to $\mathrm{DM}_{\mathrm{cdh}}(-,R)$. □

6. Finiteness

6.1. In this section, all the functors are derived functors, but we will drop $\mathbf{L}$ or $\mathbf{R}$ from the notations. The triangulated motivic category $\mathrm{DM}_{\mathrm{cdh}}(-,R)$ is endowed with the six operations $\otimes_R$, $\underline{\mathrm{Hom}}_R$, f^*, f_*, $f_!$ and $f^!$ which satisfy the usual properties; see [CDa, Theorem 2.4.50] for a summary.
Recall that an object of $\mathrm{DM}_{\mathrm{cdh}}(X,R)$ is constructible if and only if it is compact. Here is the behaviour of the six operations with respect to constructible objects in $\mathrm{DM}_{\mathrm{cdh}}(-,R)$, when we restrict ourselves to k-schemes (see [CDa, 4.2.5, 4.2.6, 4.2.10, 4.2.12]):

(i) constructible objects are stable by tensor products;
(ii) for any morphism $f : X \to Y$, the functor $f^* : \mathrm{DM}_{\mathrm{cdh}}(Y,R) \to \mathrm{DM}_{\mathrm{cdh}}(X,R)$ preserves constructible objects;
(iii) The property of being constructible is local for the Zariski topology;
(iii) given a closed immersion $i : Z \to X$ with open complement $j : U \to X$, an object M of $\mathrm{DM}_{\mathrm{cdh}}(X,R)$ is constructible if and only if $i^*(M)$ and $j^*(M)$ are constructible;
(iv) the functor $f_! : \mathrm{DM}_{\mathrm{cdh}}(X,R) \to \mathrm{DM}_{\mathrm{cdh}}(Y,R)$ preserves constructible objects for any separated morphism of finite type $f : X \to Y$.

Proposition 6.2. *Let $i : Z \to X$ be a closed immersion of codimension c between regular k-schemes. Then there is a canonical isomorphism $i^!(R_X) \simeq R_Z(-c)[-2c]$ in $\mathrm{DM}_{\mathrm{cdh}}(Z,R)$.*

Proof. In the case where X and Z are smooth over k, this is a direct consequence of the relative purity theorem. For the general case, using the reformulation of the absolute purity theorem of [CDb, Appendix, Theorem A.2.8(ii)], we see that it is sufficient to prove this proposition locally for the Zariski topology over X. Therefore we may assume that X is affine. Since $\mathrm{DM}_{\mathrm{cdh}}(-,R)$ is continuous (5.11), using Popescu's theorem and [CDa, 4.3.12], we see that it is sufficient to treat the case where X is smooth of finite type over k. But then, this is a direct consequence of the relative purity theorem. □

Proposition 6.3. *Let $f : X \to Y$ be a morphism of noetherian k-schemes. Assume that both X and Y are integral and that f is finite and flat of degree d. Then, there is a canonical natural transformation*

$$Tr_f : \mathbf{R}f_*\mathbf{L}f^*(M) \to M$$

for any object M of $\mathrm{DM}_{\mathrm{cdh}}(X,R)$ such that the composition with the unit of the adjunction $(\mathbf{L}f^*, \mathbf{R}f_*)$

$$M \to \mathbf{R}f_*\mathbf{L}f^*(M) \xrightarrow{Tr_f} M$$

is d times the identity of M.

Proof. As in paragraphs 3.14 and 3.16 (simply replacing $\underline{\mathrm{DM}}(X,R)$ and $\mathrm{DM}(X,R)$ by $\underline{\mathrm{DM}}_{\mathrm{cdh}}(X,R)$ and $\mathrm{DM}_{\mathrm{cdh}}(X,R)$, respectively), we construct

$$Tr_f : \mathbf{R}f_*(R_X) = \mathbf{R}f_*\mathbf{L}f^*(R_Y) \to R_Y$$

such that the composition with the unit

$$R \to \mathbf{R}f_*(R_X) \xrightarrow{Tr_f} R_Y$$

is d. Then, since f is proper, we have a projection formula

$$\mathbf{R}f_*(R_X) \otimes^{\mathbf{L}}_R M \simeq \mathbf{R}f_*\mathbf{L}f^*(M)$$

and we construct

$$Tr_f : \mathbf{R}f_*\mathbf{L}f^*(M) \to M$$

as

$$M \otimes^{\mathbf{L}}_R \left(\mathbf{R}f_*(R_X) \xrightarrow{Tr_f} R_Y\right).$$

This ends the construction of Tr_f and the proof of this proposition. □

Theorem 6.4. *The six operations preserve constructible objects in $\mathrm{DM}_{\mathrm{cdh}}(-,R)$ over quasi-excellent k-schemes. In particular, we have the following properties.*

(a) *For any morphism of finite type between quasi-excellent k-schemes, the functor $f_* : \mathrm{DM}_{\mathrm{cdh}}(X,R) \to \mathrm{DM}_{\mathrm{cdh}}(Y,R)$ preserves constructible objects.*

(b) *For any separated morphism of finite type between quasi-excellent k-schemes $f : X \to Y$, the functor $f^! : \mathrm{DM}_{\mathrm{cdh}}(Y,R) \to \mathrm{DM}_{\mathrm{cdh}}(X,R)$ preserves constructible objects.*

(c) *If X is a quasi-excellent k-scheme, for any constructible objects M and N of $\mathrm{DM}_{\mathrm{cdh}}(M,N)$, the object $\underline{\mathrm{Hom}}_R(M,N)$ is constructible.*

Sketch of proof. It is standard that properties (b) and (c) are corollaries of property (a); see the proof of [CDb, Cor. 6.2.14], for instance. Also, to prove (a), the usual argument (namely [Ayo07a, Lem. 2.2.23]) shows that it is sufficient to prove that, for any morphism of finite type $f : X \to Y$, the object $f_*(R_X)$ is constructible. As one can work locally for the Zariski topology on X and on Y, one may assume that f is separated (e.g. affine) and thus that $f = pj$ with j an open immersion and p a proper morphism. As $p_! = p_*$ is already known to preserve constructible objects, we are thus reduced to prove that, for any dense open immersion $j : U \to X$, the object $j_*(R_U)$ is constuctible. This is where the serious work begins. First, using the fact that constructible objects are compact, for any prime $\ell \neq p$, the triangulated category $\mathrm{DM}_{\mathrm{cdh}}(X, R\otimes\mathbf{Z}_{(\ell)})$ is the idempotent completion of the triangulated category $\mathrm{DM}_{\mathrm{cdh}}(X,R)\otimes\mathbf{Z}_{(\ell)}$. Therefore, using [CDb, Appendix, Prop. B.1.7], we easily see that it is sufficient to consider the case where R is a $\mathbf{Z}_{(\ell)}$-algebra for some prime $\ell \neq p$. The rest of the proof consists to follow word for word a beautiful argument of Gabber: the very proof of [CDb, Lem. 6.2.7]. Indeed, the only part of the proof of *loc. cit.* which is not meaningful in an abstract motivic triangulated category is the proof of the sublemma [CDb, 6.2.12], where we need the existence of trace maps for flat finite surjective morphisms satisfying the usual degree formula. In the case of $\mathrm{DM}_{\mathrm{cdh}}(X,R)$, we have such trace maps natively: see Proposition 6.3. □

7. Duality

In this section, we will consider a field K of exponential characteristic p, and will focus our attention on K-schemes of finite type. As anywhere else in this article, the ring of coefficients R is assumed to be a $\mathbf{Z}[1/p]$-algebra.

Proposition 7.1. *Let $f : X \to Y$ be a surjective finite radicial morphism of noetherian K-schemes of finite dimension. Then the functor*

$$\mathbf{L}f^* : \mathrm{DM}_{\mathrm{cdh}}(Y,R) \to \mathrm{DM}_{\mathrm{cdh}}(X,R)$$

is an equivalence of categories and is canonically isomorphic to the functor $f^!$.

Proof. By virtue of [CDa, Prop. 2.1.9], it is sufficient to prove that pulling back along such a morphism f induces a conservative functor $\mathbf{L}f^*$ (the fact that $\mathbf{L}f^* \simeq f^!$ come from the fact that if $\mathbf{L}f^*$ is an equivalence of categories, then so is its right adjoint $f_! \simeq \mathbf{R}f_*$, so that $\mathbf{L}f^*$ and $f^!$ must be quasi-inverses of the same equivalence of categories). Using the localization property as well as a suitable noetherian induction, it is sufficient to check this property generically on Y. In particular, we may assume that Y and X are integral and that f is moreover flat. Then the degree of f must be some power of p, and Proposition 6.3 then implies that the functor $\mathbf{L}f^*$ is faithful (and thus conservative). □

Proposition 7.2. *Let X be a scheme of finite type over K, and Z a fixed nowhere dense closed subscheme of X. Then the category of constructible motives $\mathrm{DM}_{\mathrm{cdh,c}}(X,R)$ is the smallest thick subcategory containing objects of the form $f_!(R_Y)(n)$, where $f : Y \to X$ is a projective morphism with Y smooth over a finite purely inseparable extension of K and such that $f^{-1}(Z)$ is either empty, the whole scheme Y itself, or the support of a strict normal crossing divisor, while n is any integer.*

Proof. Let $\mathcal{G}$ be the family of objects of the form $f_!(R_Y)(n)$, with $f : Y \to X$ a projective morphism, Y smooth over a finite purely inseparable extension of K, $f^{-1}(Z)$ either empty or the support of a strict normal crossing divisor, and n any integer. We already know that any element of $\mathcal{G}$ is constructible. Since the constructible objects of $\mathrm{DM}_{\mathrm{cdh}}(X,R)$ precisely are the compact objects, which do form a generating family of the triangulated category $\mathrm{DM}_{\mathrm{cdh}}(X,R)$, it is sufficient to prove that the family $\mathcal{G}$ is generating. Let M be an object of $\mathrm{DM}_{\mathrm{cdh}}(X,R)$ such that $\mathrm{Hom}(C,M[i]) = 0$ for any element C of $\mathcal{G}$ and any integer i. We want to prove that $M = 0$. For this, it is sufficient to prove that $M \otimes \mathbf{Z}_{(\ell)} = 0$ for any prime ℓ which not invertible in R (hence, in particular, is prime to p). Since, for any compact object C of $\mathrm{DM}_{\mathrm{cdh}}(X,R)$, we have

$$\mathrm{Hom}(C,M[i]) \otimes \mathbf{Z}_{(\ell)} \simeq \mathrm{Hom}(C, M \otimes \mathbf{Z}_{(\ell)}[i])\,,$$

and since $f_!$ commutes with tensoring with $\mathbf{Z}_{(\ell)}$ (because it commutes with small sums), we may assume that R is a $\mathbf{Z}_{(\ell)}$-algebra for some prime number $\ell \neq p$. Under this extra hypothesis, we will prove directly that $\mathcal{G}$ generates the thick category of compact objects. Let T be the smallest thick subcategory of $\mathrm{DM}_{\mathrm{cdh}}(X,R)$ which contains the elements of $\mathcal{G}$.
For Y a separated X-scheme of finite type, we put

$$M^{BM}(Y/X) = f_!(R_Y)$$

with $f : Y \to X$ the structural morphism. If Z is a closed subscheme of Y with open complement U, we have a canonical distinguished triangle

$$M^{BM}(U/X) \to M^{BM}(Y/X) \to M^{BM}(Z/X) \to M^{BM}(Z/X)[1]\,.$$

We know that the subcategory of constructible objects of $\mathrm{DM}_{\mathrm{cdh}}(X,R)$ is the smallest thick subcategory which contains the objects of the form $M^{BM}(Y/X)(n)$ for $Y \to X$ projective, and $n \in \mathbf{Z}$; see [Ayo07a, Lem. 2.2.23].

By cdh-descent (as formulated in [CDa, Prop. 3.3.10 (i)]), we easily see that objects of the form $M^{BM}(Y/X)(n)$ for $Y \to X$ projective, Y integral, and $n \in \mathbf{Z}$, generate the thick subcategory of constructible objects of $\mathrm{DM}_{\mathrm{cdh}}(X,R)$. By noetherian induction on the dimension of such a Y, it is sufficient to prove that, for any projective X-scheme Y, there exists a dense open subscheme U in Y such that $M^{BM}(U/X)$ belongs to T. By virtue of Gabber's refinement of de Jong's theorem of resolution of singularities by alterations [ILO14, Exp. X, Theorem 2.1], there exists a projective morphism $Y' \to Y$ which is generically flat, finite surjective of degree prime to ℓ, such that Y' is smooth over a finite purely inseparable extension of K, and such that the inverse image of Z in Y' is either empty, the whole scheme Y', or the support of a strict normal crossing divisor. Thus, by induction, for any dense open subscheme $V \subset Y'$, the motive $M^{BM}(V/X)$ belongs to T. But, by assumption on $Y' \to Y$, there exists a dense open subscheme U of Y such that, if V denote the pullback of U in Y', the induced map $V \to U$ is a finite, flat and surjective morphism between integral K-schemes and is of degree prime to ℓ. By virtue of Proposition 6.3, the motive $M^{BM}(U/X)$ is thus a direct factor of $M^{BM}(V/X)$, and since the latter belongs to T, this shows that $M^{BM}(Y/X)$ belongs to T as well, and this achieves the proof. □

THEOREM 7.3. *Let X be a separated K-scheme of finite type, with structural morphism $f : X \to \mathrm{Spec}(K)$. Then the object $f^!(R)$ is dualizing. In other words, for any constructible object M in $\mathrm{DM}_{\mathrm{cdh}}(X,R)$, the natural map*

$$M \to \mathbf{R}\,\underline{\mathrm{Hom}}_R(\mathbf{R}\,\underline{\mathrm{Hom}}_R(M, f^!(R), f^!(R))) \tag{7.3.1}$$

is an isomorphism. In particular, the natural map

$$R_X \to \mathbf{R}\,\underline{\mathrm{Hom}}_R(f^!(R), f^!(R)) \tag{7.3.2}$$

is an isomorphism in $\mathrm{DM}_{\mathrm{cdh}}(X,R)$.

Proof. By virtue of Proposition 7.2, it is sufficient to prove that the map (7.3.1) is an isomorphism for $M = p_!(R_Y)$ with $p : Y \to X$ projective and Y smooth over a finite purely inseparable extension of K. We then have

$$\mathbf{R}\,\underline{\mathrm{Hom}}_R(M, f^!(R)) \simeq p_!\mathbf{R}\,\underline{\mathrm{Hom}}_R(R_Y, p^!f^!(R)) \simeq p_!p^!(f^!(R))\,,$$

hence

$$\begin{aligned}\mathbf{R}\,\underline{\mathrm{Hom}}_R(\mathbf{R}\,\underline{\mathrm{Hom}}_R(M, f^!(R), f^!(R)) &\simeq \mathbf{R}\,\underline{\mathrm{Hom}}_R(p_!p^!(f^!(R)), f^!(R))\\ &\simeq p_!\mathbf{R}\,\underline{\mathrm{Hom}}_R(p^!f^!(R), p^!f^!(R))\,.\end{aligned}$$

The map (7.3.1) is thus, in this case, the image by the functor $p_!$ of the map $R_Y \to \mathbf{R}\,\underline{\mathrm{Hom}}_R(p^!f^!(R), p^!f^!(R))$. In other words, it is sufficient to prove that the map (7.3.2) is an isomorphism in the case where X is projective over K, and smooth over a finite purely inseparable field extension L/K. In particular we get the following factorization of f

$$X \xrightarrow{g} \mathrm{Spec}(L) \xrightarrow{h} \mathrm{Spec}(K)$$

such that g is smooth and h surjective finite radicial. By virtue of Proposition 7.1, $h^!(R) = R_L$. Moreover, if d is the dimension of X, since $\mathrm{DM}_{\mathrm{cdh}}$ is oriented, we have a purity isomorphism $g^!(R_L) \simeq R_X(d)(2d]$, Thus we get an isomorphism $f^!(R) \simeq R_X(d)[2d]$. Since we obviously have the identification, $R_X \simeq \mathbf{R}\,\underline{\mathrm{Hom}}_R(R_X(d), R_X(d))$, this achieves the proof. □

Remark 7.4. The preceding theorem means that, if we restrict to separated K-schemes of finite type, the whole formalism of Grothendieck-Verdier duality holds in the setting of R-linear cdh-motives. In other words, for a separated K-scheme of finite type X with structural map $f : X \to \mathrm{Spec}(K)$, we define the functor D_X by

$$D_X(M) = \mathbf{R}\,\underline{\mathrm{Hom}}_R(M, f^!(R))$$

for any object M of $\mathrm{DM}_{\mathrm{cdh}}(X, R)$. We already know that D_X preserves constructible objects and that the natural map $M \to D_X(D_X(M))$ is invertible for any constructible object M of $\mathrm{DM}_{\mathrm{cdh}}(X, R)$. For any objects M and N of $\mathrm{DM}_{\mathrm{cdh}}(X, R)$, if N is constructible, we have a natural isomorphism

$$\mathbf{R}\,\underline{\mathrm{Hom}}_R(M, N) \simeq D_X(M \otimes^{\mathbf{L}}_R D_X(N))\,. \tag{7.4.1}$$

For any K-morphism between separated K-schemes of finite type $f : Y \to X$, and for any constructible objects M and N in $\mathrm{DM}_{\mathrm{cdh}}(X, R)$ and $\mathrm{DM}_{\mathrm{cdh}}(Y, R)$, respectively, we have the following natural identifications.

$$D_Y(f^*(M)) \simeq f^!(D_X(M)) \tag{7.4.2}$$

$$f^*(D_X(M)) \simeq D_Y(f^!(M)) \tag{7.4.3}$$

$$D_X(f_!(N)) \simeq f_*(D_Y(N)) \tag{7.4.4}$$

$$f_!(D_Y(N)) \simeq D_X(f_*(N)) \tag{7.4.5}$$

8. Bivariant cycle cohomology

PROPOSITION 8.1. *Let K be a field of characteristic exponent p, and K^s its inseparable closure.*

(a) *The map $u : \mathrm{Spec}(K^s) \to \mathrm{Spec}(K)$ induces fully faithful functors*

$$u^* : \underline{\mathrm{DM}}^{\mathit{eff}}(K, R) \to \underline{\mathrm{DM}}^{\mathit{eff}}(K^s, R) \text{ and } u^* : \underline{\mathrm{DM}}^{\mathit{eff}}_{\mathrm{cdh}}(K, R) \to \underline{\mathrm{DM}}^{\mathit{eff}}_{\mathrm{cdh}}(K^s, R)\,.$$

(b) *We have a canonical equivalence of categories*

$$\mathrm{DM}^{\mathit{eff}}(K^s, R) \simeq \underline{\mathrm{DM}}^{\mathit{eff}}_{\mathrm{cdh}}(K^s, R)\,.$$

(c) *At the level of non-effective motives, we have canonical equivalences of categories*

$$\mathrm{DM}(K, R) \simeq \mathrm{DM}_{\mathrm{cdh}}(K, R) \simeq \underline{\mathrm{DM}}_{\mathrm{cdh}}(K, R)\,.$$

(d) *The pullback functor*

$$u^* : \mathrm{DM}(K, R) \to \mathrm{DM}(K^s, R)$$

is an equivalence of categories.

Proof. In all cases, u^* has a right adjoint $\mathbf{R}u_*$ which preserves small sums (because u^* preserves compact objects, which are generators).
Let us prove that the functor

$$u^* : \underline{\mathrm{DM}}^{\mathit{eff}}(K) \to \underline{\mathrm{DM}}^{\mathit{eff}}(K^s)$$

is fully faithful. By continuity (see [CDa, Example 11.1.25]), it is sufficient to prove that, for any finite purely inseparable extension L/K, the pullback functor along the map $v : \mathrm{Spec}(L) \to \mathrm{Spec}(K)$,

$$v^* : \mathrm{DM}^{\mathit{eff}}(K, R) \to \mathrm{DM}^{\mathit{eff}}(L, R)\,,$$

is fully faithful. As, for any field E, we have a fully faithful embedding

$$\mathrm{DM}^{\mathit{eff}}(E, R) \to \underline{\mathrm{DM}}^{\mathit{eff}}(E, R)$$

which is compatible with pulbacks (see [CDa, Prop. 11.1.19]), it is sufficient to prove that the pullback functor

$$v^* : \underline{\mathrm{DM}}^{\mathit{eff}}(K, R) \to \underline{\mathrm{DM}}^{\mathit{eff}}(L, R)$$

is fully faithful. In this case, the functor v^* has a left adjoint $v_\sharp$, and we must prove that the co-unit

$$v_\sharp\, v^*(M) \to M$$

is fully faithful for any object M of $\underline{\mathrm{DM}}^{\mathit{eff}}(K)$. The projection formula $v_\sharp\, v^*(M) = v_\sharp(R) \otimes^{\mathbf{L}}_R M$ reduces to prove that the co-unit $v_\sharp\, v^*(R) \to R$ is an isomorphism, which follows right away from [CDa, Prop. 9.1.14]. The same arguments show that the functor

$$u^* : \underline{\mathrm{DM}}^{\mathit{eff}}_{\mathrm{cdh}}(K, R) \to \underline{\mathrm{DM}}^{\mathit{eff}}_{\mathrm{cdh}}(K^s, R)$$

is fully faithful.
The canonical functor

$$\mathrm{DM}^{\mathit{eff}}(L, R) \to \underline{\mathrm{DM}}^{\mathit{eff}}_{\mathrm{cdh}}(L, R)$$

is an equivalence of categories for any perfect field L of exponent characteristic p by a result in Kelly's thesis (more precisely the right adjoint of this functor is an equivalence of categories; see the last assertion of [Kel12, Cor. 5.3.9]).
The fact that the functor

$$u^* : \mathrm{DM}_c(K, R) \to \mathrm{DM}_c(K^s, R)$$

is an equivalence of categories follows by continuity from the fact that the pullback functor

$$\mathrm{DM}_c(K, R) \to \mathrm{DM}_c(L, R)$$

is an equivalence of categories for any finite purely inseparable extension L/K (see [CDa, Prop. 2.1.9 and 2.3.9]). As the right adjoint of u^* preserves small sums, this implies that $u^* : \mathrm{DM}(K, R) \to \mathrm{DM}(K^s, R)$ is fully faithful. Since any compact object of $\mathrm{DM}(K^s, R)$ is in the essential image and since $\mathrm{DM}(K^s, R)$ is compactly generated, this proves that $u^* : \mathrm{DM}(K, R) \to \mathrm{DM}(K^s, R)$ is an equivalence of categories; see [CDa, Corollary 1.3.21].

As we already know that the functor

$$\mathrm{DM}(K, R) \to \mathrm{DM}_{\mathrm{cdh}}(K, R)$$

is an equivalence of categories (Cor. 5.9), it remains to prove that the functor

$$\mathrm{DM}_{\mathrm{cdh}}(K, R) \to \underline{\mathrm{DM}}_{\mathrm{cdh}}(K, R)$$

is an equivalence of categories (or even an equality). Note that we have

$$\mathrm{DM}_{\mathrm{cdh}}(L, R) = \underline{\mathrm{DM}}_{\mathrm{cdh}}(L, R)$$

for any perfect field of exponent characteristic p. This simply means that motives of the form $M(X)(n)$, for X smooth over L and $n \in \mathbf{Z}$, do form a generating family of $\underline{\mathrm{DM}}(L, R)$. To prove this, let us consider an object C of $\mathrm{DM}_{\mathrm{cdh}}(L, R)$ such that

$$\mathrm{Hom}(M(X)(n), C[i]) = 0$$

for any smooth L-scheme X and any integers n and i. To prove that $C = 0$, since, for any compact object E and any localization A of the ring $\mathbf{Z}$, the functor $\mathrm{Hom}(E, -)$ commutes with tensoring by A, we may assume that R is a $\mathbf{Z}_{(\ell)}$-algebra for some prime number $\ell \neq p$. Under this extra assumption, we know that the object C satisfies ℓdh-descent (see Corollary 5.5). Since, by Gabber's theorem, any scheme of finite type over L is smooth locally for the ℓdh-topology, this proves that $C = 0$.
Finally, let us consider an object C of $\underline{\mathrm{DM}}_{\mathrm{cdh}}(K, R)$ such that $\mathrm{Hom}(M, C) = 0$ for any object M of $\mathrm{DM}_{\mathrm{cdh}}(K, R)$. Then, for any object N of $\mathrm{DM}_{\mathrm{cdh}}(K^s, R)$, we have $\mathrm{Hom}(N, u^*(C)) = 0$: indeed, such an N must be of the form $u^*(M)$ for some M in $\mathrm{DM}_{\mathrm{cdh}}(K, R)$, and the functor u^* is fully faithful on $\underline{\mathrm{DM}}_{\mathrm{cdh}}(-, R)$. Since K^s is a perfect field, this proves that $u^*(C) = 0$, and using the fully faithfulness of u^* one last time implies that $C = 0$. This proves that $\mathrm{DM}_{\mathrm{cdh}}(K, R) = \underline{\mathrm{DM}}_{\mathrm{cdh}}(K, R)$ and achieves the proof of the proposition. □

Corollary 8.2. *Let K be a field of exponent characteristic p. Then the infinite suspension functor*

$$\Sigma^\infty : \underline{\mathrm{DM}}^{\mathit{eff}}_{\mathrm{cdh}}(K, R) \to \underline{\mathrm{DM}}_{\mathrm{cdh}}(K, R) = \mathrm{DM}_{\mathrm{cdh}}(K, R)$$

is fully faithful.

Proof. Let K^s be the inseparable closure of K. The functor

$$\Sigma^\infty : \underline{\mathrm{DM}}^{\mathit{eff}}_{\mathrm{cdh}}(K^s, R) \to \underline{\mathrm{DM}}_{\mathrm{cdh}}(K^s, R) = \mathrm{DM}_{\mathrm{cdh}}(K^s, R)$$

is fully faithful: this follows from the fact that the functor

$$\Sigma^\infty : \mathrm{DM}^{\mathit{eff}}(K^s, R) \to \mathrm{DM}(K^s, R)$$

is fully faithful (which is a reformulation of Voevodsky's cancellation theorem [Voe10]) and from assertions (b) and (c) in Proposition 8.1.

Pulling back along the map $u : \mathrm{Spec}(K^s) \to \mathrm{Spec}(K)$ induces an essentially commutative diagram of the form

$$\begin{tikzcd}
\underline{\mathrm{DM}}^{\mathit{eff}}_{\mathrm{cdh}}(K) \arrow[r, "\Sigma^\infty"] \arrow[d, "u^*"'] & \underline{\mathrm{DM}}_{\mathrm{cdh}}(K) \arrow[r, equal] \arrow[d, "u^*"] & \mathrm{DM}_{\mathrm{cdh}}(K,R) \arrow[d, "u^*"] \\
\underline{\mathrm{DM}}^{\mathit{eff}}_{\mathrm{cdh}}(K^s) \arrow[r, "\Sigma^\infty"] & \underline{\mathrm{DM}}_{\mathrm{cdh}}(K^s) \arrow[r, equal] & \mathrm{DM}_{\mathrm{cdh}}(K^s,R)
\end{tikzcd}$$

and thus, Proposition 8.1 allows to conclude. □

8.3. The preceding proposition and its corollary explain why it is essentially harmless to only work with perfect ground fields[9]. From now on, we will focus on our fixed perfect field k of characteristic exponent p, and will work with separated k-schemes of finite type.
Let X be a separated k-scheme of finite type and $r \geq 0$ an integer. Let $z_{equi}(X,r)$ be the presheaf with transfers of equidimensional relative cycles of dimension r over k (see [VSF00, Chap. 2, page 36]); its evaluation at a smooth k-scheme U is the free group of cycles in $U \times X$ which are equidimensionnal of relative dimension r over k; see [VSF00, Chap. 2, Prop. 3.3.15]. If $\Delta^\bullet$ denotes the usual cosimplicial k-scheme,

$$\Delta^n = \mathrm{Spec}\left(k[t_0,\dots,t_n]/(\sum_i t_i = 1)\right),$$

then, for any presheaf of ablian groups F, the Suslin complex $\underline{C}_*(F)$ is the complex associated to the simplicial presehaf of abelian groups $F((-)\times\Delta^\bullet)$.
Let Y be another k-scheme of finite type. After Friedlander and Voevodsky, for $r \geq 0$, the (R-linear) *bivariant cycle cohomology of Y with coefficients in cycles on X* is defined as the following cdh-hypercohomology groups:

$$A_{r,i}(Y,X)_R = H^{-i}_{\mathrm{cdh}}(Y, \underline{C}_*(z_{equi}(X,r))_{\mathrm{cdh}} \otimes^{\mathbf{L}} R)\,. \tag{8.3.1}$$

Since $\mathbf{Z}(Y)$ is a compact object in the derived category of cdh-sheaves of abelian groups, we have a canonical isomorphism

$$\mathbf{R}\Gamma(Y, \underline{C}_*(z_{equi}(X,r))_{\mathrm{cdh}} \otimes^{\mathbf{L}} R) \simeq \mathbf{R}\Gamma(Y, \underline{C}_*(z_{equi}(X,r))_{\mathrm{cdh}}) \otimes^{\mathbf{L}} R \tag{8.3.2}$$

in the derived category of R-modules. We also put $A_{r,i}(Y,X)_R = 0$ for $r < 0$.
Recall that, for any separated k-scheme of finite type X, we have its motive $M(X)$ and its motive with compact support $M^c(X)$. Seen in $\mathrm{DM}(k,R)$, they are the objects associated to the presheaves with transfers $R(X)$ and $R^c(X)$ on smooth k-schemes: for a smooth k-scheme U, $R(X)(U)$ (resp. $R^c(X)(U)$) is the free R-module on the set of cycles in $U \times X$ which are finite (resp. quasi-finite) over U and dominant over an irreducible component of U. We will also denote by $M(X)$ and $M^c(X)$ the corresponding objects in $\mathrm{DM}_{\mathrm{cdh}}(k,R)$ through the equivalence $\mathrm{DM}(k,R) \simeq \mathrm{DM}_{\mathrm{cdh}}(k,R)$.

[9]Note however that the recent work of Suslin [Sus13] should provide explicit formulas such as the one of Theorem 8.11 for separated schemes of finite type over non-perfect infinite fields.

Theorem 8.4 (Voevodsky, Kelly). *For any integers $r, i \in \mathbf{Z}$, there is a canonical isomorphism of R-modules*

$$A_{r,i}(Y,X)_R \simeq \mathrm{Hom}_{\mathrm{DM}(k,R)}(M(Y)(r)[2r+i], M^c(X)) .$$

Proof. For $R = \mathbf{Z}$, in view of Voevodsky's cancellation theorem, this is a reformulation of [VSF00, Chap. 5, Prop.4.2.3] in characteristic zero; the case where the exponent characteristic is p, with $R = \mathbf{Z}[1/p]$, is proved by Kelly in [Kel12, Prop. 5.5.11]. This readily implies this formula for a general $\mathbf{Z}[1/p]$-algebra R as ring of coefficients, using (8.3.2). □

Remark 8.5. Let $g : Y \to \mathrm{Spec}(k)$ be a separated morphism of finite type. The pullback functor

$$\mathbf{L}g^* : \mathrm{DM}_{\mathrm{cdh}}(k,R) \to \mathrm{DM}_{\mathrm{cdh}}(Y,R) \tag{8.5.1}$$

has a left adjoint

$$\mathbf{L}g_\sharp : \mathrm{DM}_{\mathrm{cdh}}(Y,R) \to \mathrm{DM}_{\mathrm{cdh}}(k,R) . \tag{8.5.2}$$

Indeed, this is obviously true if we replace $\mathrm{DM}_{\mathrm{cdh}}(-,R)$ by $\underline{\mathrm{DM}}_{\mathrm{cdh}}(-,R)$. Since we have $\mathrm{DM}(k,R) \simeq \mathrm{DM}_{\mathrm{cdh}}(k,R) = \underline{\mathrm{DM}}_{\mathrm{cdh}}(k,R)$ (8.1 (c)), the restriction of the functor

$$\mathbf{L}g_\sharp : \underline{\mathrm{DM}}_{\mathrm{cdh}}(Y,R) \to \underline{\mathrm{DM}}_{\mathrm{cdh}}(k,R)$$

to $\mathrm{DM}_{\mathrm{cdh}}(Y,R) \subset \underline{\mathrm{DM}}_{\mathrm{cdh}}(Y,R)$ provides the left adjoint of the pullback functor $\mathbf{L}g^*$ in the fibred category $\mathrm{DM}_{\mathrm{cdh}}(-,R)$. This construction does not only provide a left adjoint, but also computes it: the motive of Y is the image by this left adjoint of the constant motive on Y:

$$M(Y) = \mathbf{L}g_\sharp(R_Y) . \tag{8.5.3}$$

We also deduce from this description of $\mathbf{L}g_\sharp$ that, for any object M of $\mathrm{DM}_{\mathrm{cdh}}(k,R)$, we have a canonical isomorphism

$$\mathbf{R}g_*\mathbf{L}g^*(M) \simeq \mathbf{R}\,\underline{\mathrm{Hom}}_R(M(Y),M) \tag{8.5.4}$$

(where $\underline{\mathrm{Hom}}_R$ is the internal Hom of $\mathrm{DM}_{\mathrm{cdh}}(k,R)$): again, this readily follows from the analogous formula in $\underline{\mathrm{DM}}_{\mathrm{cdh}}(-,R)$).
If we wite $z(X,r)$ for the cdh-sheaf asociated to $z_{equi}(X,r)$ (which is compatible with the notations of Suslin and Voevodsky, according to [VSF00, Chap. 2, Thm. 4.2.9]), we thus have another way of expressing the preceding theorem.

Corollary 8.6. *With the notations of Remark 8.5, we have a canonical isomorphism of R-modules:*

$$A_{r,i}(Y,X)_R \simeq \mathrm{Hom}_{\mathrm{DM}_{\mathrm{cdh}}(Y,R)}(R_Y(r)[2r+i], \mathbf{L}g^*(M^c(X))) .$$

8.7. The preceding corollary is not quite the most natural way to express bivariant cycle cohomology $A_{r,i}(Y,X)$. Keeping track of the notations of Remark 8.5, we can see that there is a canonical isomorphism

$$g_!\, g^!(R) \simeq M(Y) . \tag{8.7.1}$$

Indeed, we have:

$$\mathbf{R}\,\underline{\mathrm{Hom}}_R(g_!g^!(R),R) = \mathbf{R}g_*\mathbf{R}\,\underline{\mathrm{Hom}}_R(g^!(R),g^!(R))\,.$$

But Grothendieck-Verdier duality (7.3) implies that

$$R_Y = \mathbf{R}\,\underline{\mathrm{Hom}}_R(g^!(R),g^!(R))\,,$$

and thus (8.5.4) gives:

$$\mathbf{R}\,\underline{\mathrm{Hom}}_R(g_!g^!(R),R) \simeq \mathbf{R}g_*\mathbf{L}g^*(R) \simeq \mathbf{R}\,\underline{\mathrm{Hom}}_R(M(Y),R)\,.$$

Since the natural map

$$M \to \mathbf{R}\,\underline{\mathrm{Hom}}_R(\mathbf{R}\,\underline{\mathrm{Hom}}_R(M,R),R)$$

is invertible for any constructible motive M in $\mathrm{DM}_{\mathrm{cdh}}(k,R)$, we obtain the identification (8.7.1) (note that $M(Y)$ is constructible; see [Kel12, Lemma 5.5.2]).

Corollary 8.8. *With the notations of Remark 8.5, we have a canonical isomorphism of R-modules:*

$$A_{r,i}(Y,X)_R \simeq \mathrm{Hom}_{\mathrm{DM}_{\mathrm{cdh}}(Y,R)}(g^!(R)(r)[2r+i],g^!(M^c(X)))\,.$$

8.9. Let $f: X \to \mathrm{Spec}(k)$ be a separated morphism of finite type. We want to describe $M^c(X)$ in terms of the six operations in $\mathrm{DM}_{\mathrm{cdh}}(-,R)$.

Proposition 8.10. *With the notations of 8.9, there are canonical isomorphisms*

$$M^c(X) \simeq \mathbf{R}f_*\,f^!(R) \simeq \mathbf{R}\,\underline{\mathrm{Hom}}_R(f_!(R_X),R)$$

in the triangulated category $\mathrm{DM}_{\mathrm{cdh}}(k,R)$.

Proof. If f is proper, then $f_!(R_X) = \mathbf{R}f_*(R_X)$, while $M^c(X) = M(X)$ (we really mean equality here, in both cases). Therefore, we also have

$$\mathbf{R}\,\underline{\mathrm{Hom}}_R(M^c(X),R) = \mathbf{R}\,\underline{\mathrm{Hom}}_R(M(X),R) \simeq \mathbf{R}f_*(R_X) = f_!(R_X)$$

in a rather canonical way: the identification $\mathbf{R}\,\underline{\mathrm{Hom}}_R(M(X),R) \simeq \mathbf{R}f_*(R_X)$ can be constructed in $\underline{\mathrm{DM}}_{\mathrm{cdh}}(K,R)$, in which case it can be promoted to a canonical weak equivakence at the level of the model category of symmetric Tate spectra of complexes of (R-linear) cdh-sheaves with transfers over the category of separated K-schemes of finite type. In particular, for any morphism $i: Z \to X$ with $g = fi$ proper, we have a commutative diagram of the form

$$\begin{tikzcd}
\mathbf{R}\,\underline{\mathrm{Hom}}_R(M(X),R) \arrow[r, "\sim"] \arrow[d, "i^*"'] & \mathbf{R}f_*(R_X) \arrow[d, "i^*"] \\
\mathbf{R}\,\underline{\mathrm{Hom}}_R(M(Z),R) \arrow[r, "\sim"] & \mathbf{R}g_*(R_Z)
\end{tikzcd}$$

in the (stable model category underlying the) triangulated category $\mathrm{DM}_{\mathrm{cdh}}(X,R)$.

In the general case, let us choose an open embedding $j: X \to \bar{X}$ with a proper k-scheme $q: \bar{X} \to \mathrm{Spec}(k)$, such that $f = qj$. Let $\partial\bar{X}$ be a closed subscheme of $\bar{X}$ such that $\bar{X} \setminus \partial\bar{X}$ is the image of j, and write $r: \partial\bar{X} \to \mathrm{Spec}(k)$ for the structural map. What precedes means that there is a canonical identification between the homotopy fiber of the restriction map

$$\mathbf{R}q_*(R_{\bar{X}}) \to \mathbf{R}r_*(R_{\partial\bar{X}})$$

and the homotopy fiber of the restriction map

$$\mathbf{R}\,\underline{\mathrm{Hom}}_R(M(\bar{X}), R) \to \mathbf{R}\,\underline{\mathrm{Hom}}_R(M(\partial\bar{X}), R)\,.$$

But, by definition of $f_!(R_X)$, and by virtue of [VSF00, Chap. 5, Prop. 4.1.5] in characteristic zero, and of [Kel12, Prop. 5.5.5] in general, this means that we have a canonical isomorphism

$$\mathbf{R}\,\underline{\mathrm{Hom}}_R(M^c(X), R) \simeq f_!(R_X)\,.$$

By duality (7.3), taking the dual of this identification leads to a canonical isomorphism $\mathbf{R}f_*\,f^!(R) \simeq M^c(X)$. □

Theorem 8.11. *Let Y and X be two separated k-schemes of finite type with structural maps $g : Y \to \mathrm{Spec}(k)$ and $f : X \to \mathrm{Spec}(k)$. Then, for any $r \geq 0$, there is a natural identification*

$$A_{r,i}(Y,X)_R \simeq \mathrm{Hom}_{\mathrm{DM}_{\mathrm{cdh}}(k,R)}(g_!g^!(R)(r)[2r+i], \mathbf{R}f_*f^!(R))\,.$$

Proof. We simply put Corollary 8.8 and Proposition 8.10 together. □

Corollary 8.12. *Let X be an equidimensional quasi-projective k-scheme of dimension n, with structural morphism $f : X \to \mathrm{Spec}(k)$, and consider any subring $\Lambda \subset \mathbf{Q}$ in which the characteristic exponent of k is invertible. Then, for any integers i and j, we have a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{cdh}}(X,\Lambda)}(\Lambda_X(i)[j], f^!\Lambda) \simeq \mathrm{CH}^{n-i}(X, j-2i) \otimes \Lambda$$

(where $\mathrm{CH}^{n-i}(X, j-2i)$ is Bloch's higher Chow group.

Proof. In the case where k is of characteristic zero, this is a reformulation of the preceding theorem and of [VSF00, Chap. 5, Prop. 4.2.9]. For the proof of *loc. cit.* to hold *mutatis mutandis* for any perfect field k of characteristic $p > 0$ (and with $\mathbf{Z}[1/p]$-linear coefficients), we see that apart from Proposition 8.1 and Theorem 8.4 above, the only ingredient that we need is the $\mathbf{Z}[1/p]$-linear version of [VSF00, Theorem 4.2.2], which is provided by results of Kelly [Kel12, Theorems 5.4.19 and 5.4.21]. □

Corollary 8.13. *Let X be a separated k-scheme of finite type, with structural morphism $f : X \to \mathrm{Spec}(k)$. For any subring $\Lambda \subset \mathbf{Q}$ in which p is invertible, there is a natural isomorphism*

$$\mathrm{CH}_n(X) \otimes \Lambda \simeq \mathrm{Hom}_{\mathrm{DM}_{\mathrm{cdh}}(X,\Lambda)}(\Lambda_X(n)[2n], f^!\Lambda)$$

for any integer n (where $\mathrm{CH}_n(X)$ is the usual Chow group of cycles of dimension n on X, modulo rational equivalence).

Proof. Thanks to [VSF00, Chap. 4, Theorem 4.2] and to [Kel12, Theorem 5.4.19], we know that

$$\mathrm{CH}_n(X) \otimes \Lambda \simeq A_{n,0}(\mathrm{Spec}(k), X)_\Lambda \, .$$

We thus conclude with Theorem 8.11 for $r = n$ and $i = 0$. □

9. Realizations

9.1. Recall from paragraph 1.3 that, for a noetherian scheme X, and a ring a coefficients Λ, one can define the Λ-linear triangulated category of mixed motives over X associated to the h-topology $\mathrm{DM}_{\mathrm{h}}(X, \Lambda)$. The latter construction is the subject of the article [CDb], in which we see that $\mathrm{DM}_{\mathrm{h}}(X, \Lambda)$ is a suitable version of the theory of étale mixed motives. In particular, we have a natural functor induced by the h-sheafification functor:

(9.1.1) $$\mathrm{DM}_{\mathrm{cdh}}(X, \Lambda) \to \mathrm{DM}_{\mathrm{h}}(X, \Lambda) \ , \quad M \mapsto M_{\mathrm{h}} \, .$$

These functors are part of a premotivic adjunction in the sense of [CDa, Def. 1.4.6].

From now on, we assume that the schemes X are defined over a given field k and that the characteristic exponent of k is invertible in Λ. Since both $\mathrm{DM}_{\mathrm{cdh}}$ and DM_{h} are motivic categories over k-schemes in the sense of [CDa, Def. 2.4.45] (see Theorem 5.11 above and [CDb, Theorem 5.6.2], respectively), we have the following formulas (see [CDa, Prop. 2.4.53]):

(9.1.2)
$$(M \otimes^{\mathbf{L}}_\Lambda N)_{\mathrm{h}} \simeq M_{\mathrm{h}} \otimes^{\mathbf{L}}_\Lambda N_{\mathrm{h}}$$

(9.1.3) $(\mathbf{L}f^*(M))_{\mathrm{h}} \simeq \mathbf{L}f^*(M_{\mathrm{h}})$ (for any morphism f)

(9.1.4) $(\mathbf{L}f_\sharp(M))_{\mathrm{h}} \simeq \mathbf{L}f_\sharp(M_{\mathrm{h}})$ (for any smooth separated morphism f)

(9.1.5) $(f_!(M))_{\mathrm{h}} \simeq f_!(M_{\mathrm{h}})$ (for any separated morphism of finite type f)

Note finally that the functor (9.1.1) has fully faithful right adjoint; its essential image consists of objects of $\mathrm{DM}_{\mathrm{cdh}}$ which satisfy the property of cohomological h-descent (see [CDa, Def. 3.2.5]).

Lemma 9.2. *Let $f : X \to \mathrm{Spec}\, k$ be a separated morphism of finite type. Then the natural morphism*

$$(\mathbf{R}f_*(\Lambda_X))_{\mathrm{h}} \to \mathbf{R}f_*((\Lambda_X)_{\mathrm{h}})$$

is invertible in $\mathrm{DM}_{\mathrm{h}}(k, \Lambda)$.

Proof. We may assume that k is a perfect field (using Prop. 8.1 (d) as well as its analogue for the h-topology (which readily follows from [CDb, Prop. 6.3.16])). We know that $\mathrm{DM}_{\mathrm{cdh}}(k, \Lambda) = \underline{\mathrm{DM}}_{\mathrm{cdh}}(k, \Lambda)$ by Prop. 8.1 (c), and similarly that $\mathrm{DM}_{\mathrm{h}}(k, \Lambda) = \underline{\mathrm{DM}}_{\mathrm{h}}(k, \Lambda)$ (since, by virtue of de Jong's theorem of resolution of singularities by alterations, locally for the h-topology, any k-scheme of finite type is smooth). The functor

$$\underline{\mathrm{DM}}_{\mathrm{cdh}}(k, \Lambda) \to \underline{\mathrm{DM}}_{\mathrm{h}}(k, \Lambda) \ , \quad M \mapsto M_{\mathrm{h}}$$

is symmetric monoidal and sends $\mathbf{L}f_\sharp(\Lambda_X)$ to $\mathbf{L}f_\sharp((\Lambda_X)_{\mathrm{h}})$. On the other hand, the motive $\mathbf{L}f_\sharp(\Lambda_X) \simeq f_!(f^!(\Lambda))$ is constructible (see (8.7.1) for $g = f$ and Theorem 6.4), whence has a strong dual in $\mathrm{DM}_{\mathrm{cdh}}(k,\Lambda)$ (since objects with a strong dual form a thick subcategory, this follows from Proposition 7.2, by Poincaré duality; see [CDa, Theorems 2.4.42 and 2.4.50]). The functor $M \mapsto M_{\mathrm{h}}$ being symmetric monoidal, it preserves the property of having a strong dual and preserves strong duals. Since $\mathbf{R}f_*(\Lambda_X)$ is the (strong) dual of $\mathbf{L}f_\sharp(\Lambda_X)$ both in $\underline{\mathrm{DM}}_{\mathrm{cdh}}(k,\Lambda)$ and in $\underline{\mathrm{DM}}_{\mathrm{h}}(k,\Lambda)$, this proves this lemma. □

Lemma 9.3. *Let $f : X \to Y$ be a k-morphism between separated k-schemes of finite type. Then the functors*

$$\mathbf{R}f_* : \mathrm{DM}_{\mathrm{cdh}}(X,\Lambda) \to \mathrm{DM}_{\mathrm{cdh}}(Y,\Lambda) \quad \textit{and} \quad \mathbf{R}f_* : \mathrm{DM}_{\mathrm{h}}(X,\Lambda) \to \mathrm{DM}_{\mathrm{h}}(Y,\Lambda)$$

commute with small sums.

Proof. In the case of cdh-motives follows from the fact that the functor

$$\mathbf{L}f^* : \mathrm{DM}_{\mathrm{cdh}}(Y,\Lambda) \to \mathrm{DM}_{\mathrm{cdh}}(X,\Lambda)$$

sends a family of compact generators into a family of compact objects. The case of h-motives is proven in [CDb, Prop. 5.5.10]. □

Proposition 9.4. *Let $f : X \to Y$ be a k-morphism between separated k-schemes of finite type. Then, for any object M of $\mathrm{DM}_{\mathrm{cdh}}(X,\Lambda)$, the natural map*

$$\mathbf{R}f_*(M)_{\mathrm{h}} \to \mathbf{R}f_*(M_{\mathrm{h}})$$

is invertible in $\mathrm{DM}_{\mathrm{h}}(Y,\Lambda)$.

Proof. The triangulated category $\mathrm{DM}_{\mathrm{cdh}}(X,\Lambda)$ is compactly generated by objects of the form $\mathbf{R}g_*(\Lambda_{X'}(n)$ for $g : X' \to X$ a proper morphism and n any integer; see [CDa, Prop. 4.2.13], for instance. Since the lemma is already known in the case of proper maps (see equation (9.1.5)), we easily deduce from Lemma 9.3 that we may assume M to be isomorphic to the constant motive Λ_X. In this case, we conclude with Lemma 9.2. □

Corollary 9.5. *Under the assumptions of paragraph 9.1, the restriction of the motivic functor $M \mapsto M_{\mathrm{h}}$ (9.1.1) to constructible objects commutes with the six operations of Grothendieck over the category of separated k-schemes of finite type.*

Proof. After Proposition 9.4, we see that it is sufficient to prove the compatibility with internal Hom and with operations of the form $g^!$ for any morphism g between separated k-schemes of finite type.

Let us prove that, for any separated k-scheme of finite type Y and any constructible objects A and N of $\mathrm{DM}_{\mathrm{cdh}}(Y,\Lambda)$, the natural map

$$\mathbf{R}\,\underline{\mathrm{Hom}}(A,N)_{\mathrm{h}} \to \mathbf{R}\,\underline{\mathrm{Hom}}(A_{\mathrm{h}},N_{\mathrm{h}})$$

is invertible in $\mathrm{DM}_{\mathrm{h}}(Y, \Lambda)$. We may assume that $A = f_\sharp(\Lambda_X)$ for some smooth morphism $f : X \to Y$. Since we have the canonical identification

$$\mathbf{R}\,\underline{\mathrm{Hom}}(\mathbf{L}f_\sharp(\Lambda_X), N) \simeq \mathbf{R}f_*\, f^*(N)\,,$$

we conclude by using the isomorphism provided by Proposition 9.4 in the case where $M = f^*(N)$.

Consider now a separated morphism of finite type $f : X \to \operatorname{Spec} k$. For any constructible objects M and N of $\mathrm{DM}_{\mathrm{cdh}}(X, \Lambda)$ and $\mathrm{DM}_{\mathrm{cdh}}(k, \Lambda)$, respectively, we have:

$$\begin{aligned}
\mathbf{R}f_*(\mathbf{R}\,\underline{\mathrm{Hom}}(M_{\mathrm{h}}, f^!(N)_{\mathrm{h}})) &\simeq \mathbf{R}f_*(\mathbf{R}\,\underline{\mathrm{Hom}}(M, f^!(N))_{\mathrm{h}}) \\
&\simeq (\mathbf{R}f_*\mathbf{R}\,\underline{\mathrm{Hom}}(M, f^!(N)))_{\mathrm{h}} \\
&\simeq \mathbf{R}\,\underline{\mathrm{Hom}}(f_!(M), N)_{\mathrm{h}} \\
&\simeq \mathbf{R}\,\underline{\mathrm{Hom}}(f_!(M_{\mathrm{h}}), N_{\mathrm{h}}) \\
&\simeq \mathbf{R}f_*(\mathbf{R}\,\underline{\mathrm{Hom}}(M_{\mathrm{h}}, f^!(N_{\mathrm{h}})))\,.
\end{aligned}$$

Therefore, for any object C of $\mathrm{DM}_{\mathrm{h}}(k, \Lambda)$, there is an isomorphism:

$$\mathbf{R}\operatorname{Hom}(\mathbf{L}f^*(C) \otimes^{\mathbf{L}}_\Lambda M_{\mathrm{h}}, f^!(N)_{\mathrm{h}}) \simeq \mathbf{R}\operatorname{Hom}(\mathbf{L}f^*(C) \otimes^{\mathbf{L}}_\Lambda M_{\mathrm{h}}, f^!(N_{\mathrm{h}}))\,.$$

Since the constructible objects of the form M_{h} are a generating family of $\mathrm{DM}_{\mathrm{h}}(k, \Lambda)$, this proves that the natural map

$$f^!(N)_{\mathrm{h}} \to f^!(N_{\mathrm{h}})$$

is an isomorphism. The functor $M \mapsto M_{\mathrm{h}}$ preserves internal Hom's of constructible objects, whence it follows from Formula (7.4.1) that it preserves duality. Therefore, Formula (7.4.2) shows that it commutes with operations of the form $g^!$ for any morphism g between separated k-schemes of finite type. $\square$

Remark 9.6. In the case where Λ is of positive characteristic, the trianguated category $\mathrm{DM}_{\mathrm{h}}(X, \Lambda)$ is canonically equivalent to the derived category $\mathrm{D}(X_{\text{ét}}, \Lambda)$ of the abelian category of sheaves of Λ-modules on the small étale site of X; see [CDb, Cor. 5.4.4]. Therefore, Corollary 9.5 then provides a system of triangulated functors

$$\mathrm{DM}_{\mathrm{cdh}}(X, \Lambda) \to \mathrm{D}(X_{\text{ét}}, \Lambda)$$

which preserve the six operations when restricted to constructible objects. Moreover, constructible objects of $\mathrm{DM}_{\mathrm{h}}(X, \Lambda)$ correspond to the full subcategory $\mathrm{D}^b_{ctf}(X_{\text{ét}}, \Lambda)$ of the category $\mathrm{D}(X_{\text{ét}}, \Lambda)$ which consists of bounded complexes of sheaves of Λ-modules over $X_{\text{ét}}$ with constructible cohomology, and which are of finite tor-dimension; see [CDb, Cor. 5.5.4 (and Th. 6.3.11)]. Therefore, for $\ell \neq p$, using [CDb, Prop. 7.2.21], we easily get ℓ-adic realizations which are compatible with the six operations (on constructible objects) over separated k-schemes of finite type:

$$\mathrm{DM}_{\mathrm{cdh},c}(X, \mathbf{Z}[1/p]) \to \mathrm{D}^b_c(X_{\text{ét}}, \mathbf{Z}_\ell) \to \mathrm{D}^b_c(X_{\text{ét}}, \mathbf{Q}_\ell)\,.$$

For instance, this gives an alternative proof of some of the results of Olsson (such as [Ols15, Theorem 1.2]).

Together with Theorem 8.11, Corollary 9.5 is thus a rather functorial way to construct cycle class maps in étale cohomology (and in any mixed Weil cohomology, since they define realization functors of $\mathrm{DM}_{\mathrm{h}}(-,\mathbf{Q})$ which commute with the six operations on constructible objects; see [CDa, 17.2.5] and [CDb, Theorem 5.2.2]). This provides a method to prove independence of ℓ results as follows. Let X be a separated k-scheme of finite type, with structural map $a : X \to \operatorname{Spec} k$, and $f : X \to X$ any k-morphism. Then f induces an endomorphism of $\mathbf{R}a_*(\mathbf{Z}[1/p]_X)$ in $\mathrm{DM}_{\mathrm{cdh}}(k,\mathbf{Z}[1/p])$. Since the latter object is constructible (by Theorem 6.4 (a)), it has a strong dual (as explained in the proof of Lemma 9.2), and thus one can define the trace of the morphism induced by f, which is an element of $\mathbf{Z}[1/p]$ (since one can identify $\mathbf{Z}[1/p]$ with the ring of endomorphisms of the constant motive $\mathbf{Z}[1/p]$ in $\mathrm{DM}_{\mathrm{cdh}}(k,\mathbf{Z}[1/p])$ using Corollary 8.13). Let ℓ be a prime number distinct from the characteristic exponent of k. Since the ℓ-adic realization functor is symmetric monoidal, it preserves the property of having a strong dual and preserves traces of endomorphisms of objects with strong duals. Therefore, if $\bar{k}$ is any choice of an algebraic closure of k, and if $\bar{X} = \bar{k} \otimes_k X$, the number

$$\sum_i (-1)^i \mathrm{Tr}\big[f^* : H^i_{\text{ét}}(\bar{X},\mathbf{Q}_\ell) \to H^i_{\text{ét}}(\bar{X},\mathbf{Q}_\ell)\big]$$

is independent of ℓ and belongs to $\mathbf{Z}[1/p]$: Corollary 9.5 implies that it is the image through the unique morphism of rings $\mathbf{Z}[1/p] \to \mathbf{Q}_\ell$ of the trace of the endomorphism of the motive $\mathbf{R}a_*(\mathbf{Z}[1/p]_X)$ induced by f. This might be compared with Olsson's proof in the case where f is finite; see [Ols, Theorem 1.2]. One may also replace $H^i(\bar{X},\mathbf{Q}_\ell)$ with the evaluation at X of any mixed Weil cohomology defined on smooth k-schemes, and still use the same argument.

Remark 9.7. If the ring Λ is a $\mathbf{Q}$-algebra, the functor $M \mapsto M_{\mathrm{h}}$ defines an equivalence of categories $\mathrm{DM}_{\mathrm{cdh}}(X,\Lambda) \simeq \mathrm{DM}_{\mathrm{h}}(X,\Lambda)$ (so that Corollary 9.5 becomes a triviality). This is because, under the extra hypothesis that $\mathbf{Q} \subset \Lambda$, the abelian categories of cdh-sheaves of Λ-modules with transfers and of h-sheaves of Λ-modules are equivalent: by a limit argument, it is sufficient to prove this when X is excellent, and then, this is an exercise which consists to put together [CDa, Prop. 10.4.8, Prop. 10.5.8, Prop. 10.5.11 and Th. 3.3.30].

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