The Arason Invariant of Orthogonal Involutions of Degree 12 and 8, and Quaternionic Subgroups of the Brauer Group

To Sasha Merkurjev on his 60th birthday

Anne Quéguiner-Mathieu[1] and Jean-Pierre Tignol[2]

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Abstract. Using the Rost invariant for torsors under Spin groups one may define an analogue of the Arason invariant for certain hermitian forms and orthogonal involutions. We calculate this invariant explicitly in various cases, and use it to associate to every orthogonal involution σ with trivial discriminant and trivial Clifford invariant over a central simple algebra A of even co-index an element $f_3(\sigma)$ in the subgroup $F^\times \cdot [A]$ of $H^3(F, \mathbb{Q}/\mathbb{Z}(2))$. This invariant $f_3(\sigma)$ is the double of any representative of the Arason invariant $e_3(\sigma) \in H^3(F, \mathbb{Q}/\mathbb{Z}(2))/F^\times \cdot [A]$; it vanishes when $\deg A \leq 10$ and also when there is a quadratic extension of F that simultaneously splits A and makes σ hyperbolic. The paper provides a detailed study of both invariants, with particular attention to the degree 12 case, and to the relation with the existence of a quadratic splitting field.

As a main tool we establish, when $\deg(A) = 12$, an additive decomposition of (A, σ) into three summands that are central simple algebras of degree 4 with orthogonal involutions with trivial discriminant, extending a well-known result of Pfister on quadratic forms of dimension 12 in I^3F. The Clifford components of the summands generate a subgroup U of the Brauer group of F, in which every element is represented by a quaternion algebra, except possibly the class of A. We show that the Arason invariant $e_3(\sigma)$, when defined, generates the homology of a complex of degree 3 Galois cohomology groups, attached to the subgroup U, which was introduced and studied by Peyre. In the final section, we use the results on degree 12 algebras to extend the definition of the Arason invariant to trialitarian triples in which all three algebras have index at most 2.

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1. Introduction

In quadratic form theory, the Arason invariant is a degree 3 Galois cohomology class with μ_2 coefficients attached to an even-dimensional quadratic form with trivial discriminant and trivial Clifford invariant. Originally defined by Arason in [1], it can also be described in terms of the Rost invariant of a split Spin group, as explained in [26, § 31.B]. It is not always possible to extend this invariant to the more general setting of orthogonal involutions, see [7, §3.4]. Nevertheless, one may use the Rost invariant of some possibly non-split Spin groups to define *relative* and *absolute* Arason invariants for some orthogonal involutions (see [41] or section 2 below for precise definitions). This was first noticed by Bayer-Fluckiger and Parimala in [6], where they use the Rost invariant to prove classification theorems for hermitian or skew-hermitian forms, leading to a proof of the so-called Hasse Principle conjecture II.

Below, we will focus on the absolute Arason invariant, which we will refer to simply as the Arason invariant. For orthogonal involutions, it was considered by Garibaldi, who uses the notation e_3^{hyp}, in [15], and by Berhuy in the index 2 case in [8]. In particular, the latter covers the case of central simple algebras of degree $2m$ with m odd, since such an algebra has index 1 or 2 when it is endowed with an orthogonal involution.

Based on the Rost invariant for the exceptional group E_8, Garibaldi also defined, for orthogonal involutions on degree 16 central simple algebras, another invariant related to the Arason invariant of quadratic forms, denoted by e_3^{16}. Bermudez and Ruozzi [9] extended this definition to all degrees divisible by 16. It follows from the proof of Corollary 10.11 in [15], and Remark 4.10 in Barry's paper [2], that these invariants do *not* coincide with what we call Arason invariant in this paper.

A systematic study of the relative and absolute Arason invariants for orthogonal involutions was recently initiated in [35], where the degree 8 case is studied in detail. In this paper, we continue with an investigation of absolute invariants in degree 12.

Let (A, σ) be a central simple algebra with orthogonal involution over a field F of characteristic different from 2. The Arason invariant $e_3(\sigma)$, when defined, belongs to the quotient

$$H^3(F, \mathbb{Q}/\mathbb{Z}(2))/F^\times \cdot [A],$$

where $F^\times \cdot [A]$ denotes the subgroup consisting of cup products $(\lambda) \cdot [A]$, for $\lambda \in F^\times$, $[A]$ the Brauer class of A. In § 2 below, we give a general formula for computing the Arason invariant of an algebra with involution admitting a rank 2 factor. It follows from this formula that the Arason invariant is not always represented by a cohomology class of order 2. This reflects the fact that the

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Dynkin index of a non-split Spin group, in large enough degree, is equal to 4. We define a new invariant $f_3(\sigma) \in H^3(F, \mu_2)$, attached to any orthogonal involution for which the Arason invariant is defined, and which vanishes if and only if the Arason invariant is represented by a cohomology class of order 2. This invariant is zero if the algebra is split, or of degree ≤ 10; starting in degree 12, we produce explicit examples where it is non-zero. This is an important motivation for studying the degree 12 case in detail.

The main results of the paper are given in sections 3 to 5. First, we prove that a degree 12 algebra with orthogonal involution (A, σ), having trivial discriminant and trivial Clifford invariant, admits a non-unique decomposition as a sum—in the sense of algebras with involution—of three degree 4 algebras with orthogonal involution of trivial discriminant. This can be seen as a refinement of the main result of [17], even though our proof in index 4 relies on the open-orbit argument of [17], see Remark 3.5. Using this additive decomposition, we associate to (A, σ) in a non-canonical way some subgroups of the Brauer group of F, which we call decomposition groups of (A, σ), see Definition 3.6. Such subgroups $U \subset \mathrm{Br}(F)$ are generated by (at most) three quaternion algebras; they were considered by Peyre in [32], where the homology of the following complex is studied:

$$F^\times \cdot U \to H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(F_U, \mathbb{Q}/\mathbb{Z}(2)),$$

where $F^\times \cdot U$ denotes the subgroup generated by cup products $(\lambda) \cdot [B]$, for $\lambda \in F^\times$, $[B] \in U$, and F_U is the function field of the product of the Severi-Brauer varieties associated to the elements of U. Peyre's results are recalled in § 3.3.

In § 4, we restrict to those algebras with involution of degree 12 for which the Arason invariant is defined, and we prove $e_3(\sigma)$ detects isotropy of σ, and vanishes if and only if σ is hyperbolic. We then explore the relations between the decomposition groups and the values of the Arason invariant. Reversing the viewpoint, we also prove that the Arason invariant $e_3(\sigma)$ provides a generator of the homology of Peyre's complex, where (A, σ) is any algebra with involution admitting U as a decomposition group.

In § 5, we give a necessary and sufficient condition for the vanishing of $f_3(\sigma)$ in degree 12, in terms of decomposition groups of (A, σ). When there is a quadratic extension that splits A and makes σ hyperbolic, an easy corestriction argument shows that $f_3(\sigma) = 0$, see Proposition 2.5. We give in § 5.3 an explicit example to show that the converse does not hold. This also provides new examples of subgroups U for which the homology of Peyre's complex is non-trivial, which differ from Peyre's example in that the homology is generated by a Brauer class of order 2.

In the last section, we extend the definition of the Arason invariant in degree 8 to index 2 algebras with involution of trivial discriminant, and such that the two components of the Clifford algebra have index 2. In this case also, the algebra with involution has an additive decomposition, and the Arason invariant detects isotropy.

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NOTATION. Throughout this paper, we work over a base field F of characteristic different from 2. We use the notation $H^n(F,M) = H^n(\mathrm{Gal}(F_{\mathrm{sep}}/F), M)$ for any discrete torsion Galois module M. For every integer $n \geq 0$ we let

$$H^n(F) = H^n\big(F, \mathbb{Q}/\mathbb{Z}(n-1)\big),$$

(see [16, Appendix A, p. 151] for a precise definition). The cohomology classes we consider actually are in the 2-primary part of these groups, hence we shall not need the modified definition for the p-primary part when $\mathrm{char}(F) = p \neq 0$. For each integer $m \geq 0$ we let ${}_mH^n(F)$ denote the m-torsion subgroup of $H^n(F)$. Using the norm-residue isomorphism, one may check that

$${}_2H^n(F) = H^n(F, \mu_2) \qquad \text{and} \qquad {}_4H^3(F) = H^3(F, \mu_4^{\otimes 2}),$$

(see for instance [32, Remark 4.1]). In particular, ${}_2H^1(F) = F^\times/F^{\times 2}$. For every $a \in F^\times$ we let $(a) \in {}_2H^1(F)$ be the square class of a. For $a_1, \ldots, a_n \in F^\times$ we let $(a_1, \ldots, a_n) \in {}_2H^n(F)$ be the cup-product

$$(a_1, \ldots, a_n) = (a_1) \cdot \cdots \cdot (a_n).$$

We refer to [26] and to [28] for background information on central simple algebras with involution and on quadratic forms. However, we depart from the notation in [28] by letting $\langle\langle a_1, \ldots, a_n \rangle\rangle$ denote the n-fold Pfister form

$$\langle\langle a_1, \ldots, a_n \rangle\rangle = \langle 1, -a_1 \rangle \cdot \cdots \cdot \langle 1, -a_n \rangle \qquad \text{for } a_1, \ldots, a_n \in F^\times.$$

Thus, the discriminant, the Clifford invariant and the Arason invariant, viewed as cohomological invariants e_1, e_2 and e_3, satisfy:

$$e_1\big(\langle\langle a_1 \rangle\rangle\big) = (a_1), \quad e_2\big(\langle\langle a_1, a_2 \rangle\rangle\big) = (a_1, a_2), \quad e_3\big(\langle\langle a_1, a_2, a_3 \rangle\rangle\big) = (a_1, a_2, a_3).$$

For every central simple F-algebra A, we let $[A]$ be the Brauer class of A, which we identify to an element in $H^2(F)$. If L is a field extension of F, we let $A_L = A \otimes_F L$ be the L-algebra obtained from A by extending scalars.
Recall that the object function from the category Fields_F of field extensions of F to abelian groups defined by

$$L \mapsto \coprod_{n \geq 0} H^n(L)$$

is a cycle module over $\mathrm{Spec}F$ (see [37, Rem.1.11]). In particular, each group $H^n(L)$ is a module over the Milnor K-ring K_*L. The Brauer class $[A]$ of the algebra A generates a cycle submodule; we let M_A denote the quotient cycle module. Thus, for every field $L \supseteq F$, we have

$$M_A^n(L) = \begin{cases} H^n(L) & \text{if } n = 0 \text{ or } 1; \\ H^n(L)/\big(K_{n-2}L \cdot [A_L]\big) & \text{if } n \geq 2. \end{cases}$$

In particular, $M_A^2(L) = \mathrm{Br}(L)/\{0, [A_L]\}$ and $M_A^3(L) = H^3(L)/\big(L^\times \cdot [A_L]\big)$.

Let F_A denote the function field of the Severi–Brauer variety of A, which is a generic splitting field of A. Scalar extension from F to F_A yields group homomorphisms

$$M_A^2(F) \to M_A^2(F_A) = \mathrm{Br}(F_A) \qquad \text{and} \qquad M_A^3(F) \to M_A^3(F_A) = H^3(F_A).$$

The first map is injective by Amitsur's theorem, see [18, Th. 5.4.1]; the second one is injective if the Schur index of A divides 4 or if A is a division algebra that decomposes into a tensor product of three quaternion algebras, but it is not always injective (see [32], [23] and [24]).

2. Cohomological invariants of orthogonal forms and involutions

Most of this section recalls well-known facts on absolute and relative Arason invariants that will be used in the sequel of the paper. Since we will consider additive decompositions of algebras with involution, we need to state the results both for hermitian forms and for involutions. Some new results are also included. In Proposition 2.6 and Corollary 2.13, we give a general formula for the Arason invariant of an algebra with involution which has a rank 2 factor. In Definitions 2.4 and 2.15, we introduce a new invariant, called the f_3-invariant, which detects whether the Arason invariant is represented by a cohomology class of order 2. Finally, we state and prove in Proposition 2.7 a general formula for computing the f_3 invariant of a sum of hermitian forms, which is used in the proof of the main results of the paper.

Throughout this section, D is a central division algebra over an arbitrary field F of characteristic different from 2, and θ is an F-linear involution on D (i.e., an involution of the first kind). To any nondegenerate hermitian or skew-hermitian module (V,h) over (D,θ) we may associate the corresponding adjoint algebra with involution $\mathrm{Ad}_h = (\mathrm{End}_D V, \mathrm{ad}_h)$. Conversely, any central simple algebra A over F Brauer-equivalent to D and endowed with an F-linear involution σ can be represented as $(A,\sigma) \simeq \mathrm{Ad}_h$ for some nondegenerate hermitian or skew-hermitian module (V,h) over (D,θ). The hermitian or skew-hermitian module (V,h) is said to be a *hermitian module of orthogonal type* if the adjoint involution ad_h on $\mathrm{End}_D V$ is of orthogonal type. This occurs if and only if either h is hermitian and θ is of orthogonal type, or h is skew-hermitian and θ is of symplectic type, see [26, (4.2)]. Abusing terminology, we also say that h is a *hermitian form of orthogonal type* when (V,h) is a hermitian module of orthogonal type (even though h may actually be skew-hermitian if θ is symplectic).

2.1. Invariants of hermitian forms of orthogonal type. Let (V,h) be a hermitian module of orthogonal type over (D,θ); we call $r = \dim_D V$ the *relative rank* of h and $n = \deg \mathrm{End}_D V$ the *absolute rank* of h. These invariants are related by $n = r \deg D$. Cohomological invariants of h are defined in terms of invariants of the adjoint involution ad_h. Namely, if n is even, the *discriminant* of h, denoted $e_1(h) \in H^1(F,\mu_2)$, is the discriminant of ad_h;

the corresponding quadratic étale extension K/F is called the *discriminant extension*. If n is even and $e_1(h)$ is trivial, the *Clifford invariant* of h, denoted $e_2(h)$, is the class in $M_D^2(F)$ of any component of the Clifford algebra of ad_h.

Remark 2.1. It follows from the relations between the components of the Clifford algebra (see [26, (9.12)]) that the Clifford invariant is well-defined. However, since we do not assume n is divisible by 4, this invariant need not be represented by a cohomology class of order 2 in general.

Our definitions of rank and discriminant differ slightly from the definitions used by Bayer and Parimala in [5, §2], who call "rank" what we call the relative rank of h. The discriminant $d(h)$ of h in the sense of [5, §2.1] is related to $e_1(h)$ by

$$e_1(h) = d(h)\,\mathrm{disc}(\theta)^r,$$

where $\mathrm{disc}(\theta)$ is the discriminant of θ as defined in [26, §7], and $H^1(F,\mu_2)$ is identified with the group of square classes $F^\times/F^{\times 2}$. In particular, $e_1(h) = d(h)$ when h has even relative rank r. By [5, 2.1.3], the Clifford invariant $\mathcal{C}\ell(h)$ used by Bayer and Parimala coincides with our $e_2(h)$ when they are both defined, i.e., when h has even relative rank and trivial discriminant. Assume now that the hermitian form h has even relative rank, i.e., $\dim_D V$ is even. The vector space V then carries a hyperbolic hermitian form h_0 of orthogonal type, and the standard nonabelian Galois cohomology technique yields a canonical bijection between $H^1(F,\mathrm{O}(h_0))$ and the set of isomorphism classes of nondegenerate hermitian forms of orthogonal type on V, under which the trivial torsor corresponds to the isomorphism class of h_0, see [26, §29.D]. If $e_1(h)$ and $e_2(h)$ are trivial, the torsor corresponding to the isomorphism class of h has two different lifts to $H^1(F,\mathrm{O}^+(h_0))$, and one of these lifts can be further lifted to a torsor ξ in $H^1(F,\mathrm{Spin}(h_0))$. Bayer and Parimala consider the Rost invariant $R(\xi) \in H^3(F)$ and define in [6, §3.4, p. 664] an *Arason invariant* of h by the formula

$$e_3(h) = R(\xi) + F^\times \cdot [D] \in M_D^3(F).$$

This invariant satisfies the following properties:

LEMMA 2.2 (Bayer–Parimala [6, Lemma 3.7, Corollary 3.9]). *Let h and h' be two hermitian forms of orthogonal type over (D,θ) with even relative rank, trivial discriminant, and trivial Clifford invariant.*

(i) *If h is hyperbolic, then $e_3(h) = 0$;*
(ii) $e_3(h \perp h') = e_3(h) + e_3(h')$;
(iii) $e_3(\lambda h) = e_3(h)$ *for any* $\lambda \in F^\times$.

In particular, it follows immediately that $e_3(h)$ is a well-defined invariant of the Witt class of h. Moreover, we have:

COROLLARY 2.3. *The Arason invariant e_3 has order* 2.

Proof. Indeed, for any h as above, we have $2e_3(h) = e_3(h) + e_3(h) = e_3(h) + e_3(-h) = e_3(h \perp (-h)) = 0$, since $h \perp (-h)$ is hyperbolic. □

Using the properties of the Arason invariant, we may define a new invariant as follows. Assume h is as above, a hermitian form of orthogonal type with even relative rank, trivial discriminant, and trivial Clifford invariant. Let c, $c' \in H^3(F)$ be two representatives of the Arason invariant $e_3(h)$. Since $c - c' \in F^\times \cdot [D]$, we have $2c = 2c' \in H^3(F)$, hence $2c$ depends only on h and not on the choice of the representative c of $e_3(h)$. Because of Corollary 2.3, the image of $2c$ in $M^3_D(F)$ vanishes, hence $2c \in F^\times \cdot [D]$. These observations lead to the following definition:

Definition 2.4. Given an arbitrary representative $c \in H^3(F)$ of the Arason invariant $e_3(h) \in M^3_D(F)$, we let $f_3(h) = 2c \in F^\times \cdot [D] \subset {}_2H^3(F)$.

Thus, the invariant $f_3(h)$ is well-defined; it vanishes if and only if the Arason invariant $e_3(h)$ is represented by a class of order at most 2, or equivalently, if every representative of $e_3(h)$ is a cohomology class of order at most 2. It is clear from the definition that the f_3 invariant is trivial when D is split. Another case where the f_3 invariant vanishes is the following:

Proposition 2.5. *If there exists a quadratic extension K/F such that D_K is split and h_K is hyperbolic, then $f_3(h) = 0$.*

Proof. Assume such a field K exists, and let $c \in H^3(F)$ be any representative of $e_3(h) \in M^3_D(F)$. Since h_K is hyperbolic, we have $e_3(h_K) = c_K = 0 \in M^3_D(K) = H^3(K)$. Hence, $\operatorname{cor}_{K/F}(c_K) = 2c = 0$, that is $f_3(h) = 0$. □

In §5, we will see that the f_3 invariant is trivial if the absolute rank is ≤ 10. In addition, we will prove that the converse of Proposition 2.5 does not hold, even in absolute rank 12.
Since the Dynkin index of the group $\mathrm{Spin}(\mathrm{Ad}_{h_0})$ divides 4, the Arason invariant $e_3(h)$ is represented by a cohomology class of order dividing 4. Moreover, there are examples where it is represented by a cohomology class of order equal to 4. Therefore, $f_3(h)$ is nonzero in general. Explicit examples can be constructed by means of Proposition 2.6 below, which yields the e_3 and f_3 invariants of hermitian forms with a rank 2 factor. (See also Corollary 2.19 for examples in the lowest possible degree, which is 12.)

2.2. Hermitian forms with a rank 2 factor. Consider a hermitian form which admits a decomposition as $\langle 1, -\lambda\rangle \otimes h$ for some $\lambda \in F^\times$ and some hermitian form h. In this case, we have the following explicit formulae for the Arason and the f_3-invariant, when they are defined:

Proposition 2.6. *Let h be a hermitian form of orthogonal type with even absolute rank n, and let K/F be the discriminant quadratic extension. For any $\mu \in K^\times$, the hermitian form $\langle 1, -N_{K/F}(\mu)\rangle h$ has even relative rank, trivial discriminant and trivial Clifford invariant. Moreover,*

$$e_3\big(\langle 1, -N_{K/F}(\mu)\rangle h\big) = \operatorname{cor}_{K/F}\big(\mu \cdot e_2(h_K)\big)$$

and

$$f_3\big(\langle 1,-N_{K/F}(\mu)\rangle h\big) = \begin{cases} 0 & \text{if } n\equiv 0 \bmod 4,\\ N_{K/F}(\mu)\cdot[D] & \text{if } n\equiv 2 \bmod 4.\end{cases}$$

In particular, if h has trivial discriminant, then for $\lambda\in F^\times$ we have

$$e_3\big(\langle 1,-\lambda\rangle h\big)= \lambda\cdot e_2(h)$$

and

$$f_3\big(\langle 1,-\lambda\rangle h\big)= \begin{cases} 0 & \text{if } n\equiv 0 \bmod 4,\\ \lambda\cdot[D] & \text{if } n\equiv 2 \bmod 4.\end{cases}$$

Proof. We first need to prove that the hermitian form $\langle 1,-N_{K/F}(\mu)\rangle h$ has trivial discriminant and trivial Clifford invariant. This can be checked after scalar extension to a generic splitting field of D, since the corresponding restriction maps ${}_2H^1(F)\to {}_2H^1(F_D)$ and $M^2_D(F)\to H^2(F_D)$ are injective. In the split case, the result follows from an easy computation for the discriminant, and from [28, Ch. V, §3] for the Clifford invariant. Alternatively, one may observe that the algebra with involution $\mathrm{Ad}_{\langle 1,-N_{K/F}(\mu)\rangle h}$ decomposes as $\mathrm{Ad}_{\langle 1,-N_{K/F}(\mu)\rangle}\otimes\mathrm{Ad}_h$, and apply [26, (7.3)(4)] and [39]. This computation also applies to the trivial discriminant case, where $\lambda = N_{F\times F/F}(\lambda,1)$.
With this in hand, we may compute the Arason invariant by using the description of cohomological invariants of quasi-trivial tori given in [30]. Let us first assume h has trivial discriminant. Consider the multiplicative group scheme $\mathbb{G}_m$ as a functor from the category Fields_F to the category of abelian groups. For any field L containing F, consider the map

$$\varphi_L\colon \mathbb{G}_m(L)\to M^3_D(L)\quad\text{defined by}\quad \lambda\mapsto e_3\big(\langle 1,-\lambda\rangle h_L\big).$$

To see that φ_L is a group homomorphism, observe that in the Witt group of D_L we have for λ_1, $\lambda_2\in L^\times$

$$\langle 1,-\lambda_1\lambda_2\rangle h_L = \langle 1,-\lambda_1\rangle h_L + \langle\lambda_1\rangle\langle 1,-\lambda_2\rangle h_L.$$

Therefore, Lemma 2.2 yields

$$e_3\big(\langle 1,-\lambda_1\lambda_2\rangle h_L\big) = e_3\big(\langle 1,-\lambda_1\rangle h_L\big) + e_3\big(\langle 1,-\lambda_2\rangle h_L\big).$$

The collection of maps φ_L defines a natural transformation of functors $\mathbb{G}_m\to M^3_D$, i.e., a degree 3 invariant of $\mathbb{G}_m$ with values in the cycle module M_D. By [29, Prop. 2.5], there is an element $u\in M^2_D(F)$ such that for any L and any $\lambda\in L^\times$

$$\varphi_L(\lambda) = \lambda\cdot u_L\qquad\text{in } M^3_D(L).$$

To complete the computation of $e_3\big(\langle 1,-\lambda\rangle h\big)$, it only remains to show that $u = e_2(h)$. Since the restriction map $M^2_D(F)\to M^2_D(F_D)=H^2(F_D)$ is injective, it suffices to show that $u_{F_D} = e_2(h)_{F_D}$. Now, since F_D is a splitting field for D, there exists a quadratic form q over F_D, with trivial discriminant, such that $(\mathrm{Ad}_h)_{F_D}\simeq \mathrm{Ad}_q$. Let t be an indeterminate over F_D. We have

$$\mathrm{Ad}_{\langle 1,-t\rangle}\otimes(\mathrm{Ad}_h)_{F_D(t)}\simeq \mathrm{Ad}_{\langle 1,-t\rangle}\otimes(\mathrm{Ad}_q)_{F_D(t)}$$

hence $e_3(\langle 1,-t\rangle h_{F_D(t)})$ is the Arason invariant of the quadratic form $\langle 1,-t\rangle q_{F_D(t)}$, which is $t\cdot e_2(q) = t\cdot e_2(h)_{F_D(t)}$. Therefore, we have

$$t\cdot u_{F_D(t)} = t\cdot e_2(h)_{F_D(t)}.$$

Taking the residue $\partial\colon H^3(F_D(t)) \to H^2(F_D)$ for the t-adic valuation, we obtain $u_{F_D} = e_2(h)_{F_D}$, which completes the proof of the formula for $e_3\big(\langle 1,-\lambda\rangle h\big)$. To compute $f_3\big(\langle 1,-\lambda\rangle h\big)$, recall that $e_2(h)$ is represented by any of the two components C_+, C_- of the Clifford algebra of Ad_h. Therefore, $e_3\big(\langle 1,-\lambda\rangle h\big)$ is represented by $\lambda\cdot[C_+]$ or $\lambda\cdot[C_-]$, and

$$f_3\big(\langle 1,-\lambda\rangle h\big) = 2(\lambda\cdot[C_+]) = 2(\lambda\cdot[C_-]).$$

By [26, (9.12)] we have

$$2[C_+] = 2[C_-] = \begin{cases} 0 & \text{if } n\equiv 0 \bmod 4,\\ [D] & \text{if } n\equiv 2 \bmod 4.\end{cases}$$

The formula for $f_3\big(\langle 1,-\lambda\rangle h\big)$ follows.

Assume now h has nontrivial discriminant. The proof in this case follows the same pattern. Let K/F be the discriminant field extension. We consider the group scheme $R_{K/F}(\mathbb{G}_m)$, which is the Weil transfer of the multiplicative group. For every field L containing F, the map

$$\mu\in R_{K/F}(\mathbb{G}_m)(L) = (L\otimes_F K)^\times \mapsto e_3(\langle 1,-N_{L\otimes K/L}(\mu)\rangle h_L)\in M_D^3(L)$$

defines a degree 3 invariant of the quasi-trivial torus $R_{K/F}(\mathbb{G}_m)$ with values in the cycle module M_D. By [30, Th. 1.1], there is an element $u\in M_D^2(K)$ such that for any field L containing F and any $\mu\in(L\otimes K)^\times$,

$$e_3(\langle 1,-N_{L\otimes K/L}(\mu)\rangle h_L) = \mathrm{cor}_{L\otimes K/L}(\mu\cdot u_{L\otimes K}) \quad \text{in } M_D^3(L).$$

It remains to show that $u = e_2(h_K)$. To prove this, we consider the field $L = K(t)$, where t is an indeterminate. Since $e_1(h_{K(t)}) = 0$, the previous case applies. We thus get for any $\mu\in(K(t)\otimes_F K)^\times$

$$N_{K(t)\otimes K/K(t)}(\mu)\cdot e_2(h_{K(t)}) = \mathrm{cor}_{K(t)\otimes K/K(t)}(\mu\cdot u_{K(t)\otimes K}) \quad \text{in } M_D^3(K(t)).$$

Let ι be the nontrivial F-automorphism of K. The $K(t)$-algebra isomorphism $K(t)\otimes_F K \simeq K(t)\times K(t)$ mapping $\alpha\otimes\beta$ to $(\alpha\beta,\alpha\iota(\beta))$ yields an isomorphism $M_D^2(K(t)\otimes K)\simeq M_D^2(K(t))\times M_D^2(K(t))$ that carries $u_{K(t)\otimes K}$ to $(u_{K(t)},\iota(u)_{K(t)})$. Thus, for every $(\mu_1,\mu_2)\in K(t)^\times\times K(t)^\times$,

$$\mu_1\mu_2\cdot e_2(h_{K(t)}) = \mu_1\cdot u_{K(t)} + \mu_2\cdot\iota(u)_{K(t)} \quad \text{in } M_D^3(K(t)).$$

In particular, if $\mu_1 = t$ and $\mu_2 = 1$ we get $t\cdot e_2(h_{K(t)}) = t\cdot u_{K(t)}$, hence taking the residue for the t-adic valuation yields $e_2(h_K) = u$, proving the formula for $e_3\big(\langle 1,-N_{K/F}(\mu)\rangle h\big)$.

To complete the proof, we compute $f_3\big(\langle 1,-N_{K/F}(\mu)\rangle h\big)$. Let C be the Clifford algebra of Ad_h, so $[C]$ represents $e_2(h_K)$ and

$$f_3\big(\langle 1,-N_{K/F}(\mu)\rangle h\big) = 2\,\mathrm{cor}_{K/F}(\mu\cdot[C]).$$

By [26, (9.12)] we have

$$2[C] = \begin{cases} 0 & \text{if } n \equiv 0 \bmod 4, \\ [D_K] & \text{if } n \equiv 2 \bmod 4. \end{cases}$$

The formula for $f_3\big(\langle 1, -N_{K/F}(\mu)\rangle h\big)$ follows by the projection formula. □

2.3. HERMITIAN FORMS WITH AN ADDITIVE DECOMPOSITION. We now present another approach for computing the f_3-invariant, which does not rely on the computation of the Arason invariant. This leads to an explicit formula in a more general situation, which will be used in the proof of Theorem 5.4:

PROPOSITION 2.7. *Let (V_1, h_1), ..., (V_m, h_m) be hermitian modules of orthogonal type and even absolute rank n_1, ..., n_m over (D, θ), and let λ_1, ..., $\lambda_m \in F^\times$. Let also $h = \langle 1, -\lambda_1\rangle h_1 \perp \ldots \perp \langle 1, -\lambda_m\rangle h_m$. If $\sum_{i=1}^m \lambda_i \cdot e_1(h_i) = 0$, then h has trivial Clifford invariant, and*

$$f_3(h) = \lambda_1^{n_1/2} \ldots \lambda_m^{n_m/2} \cdot [D].$$

To prove this proposition, we need some preliminary results. Let (V, h_0) be a hyperbolic module of orthogonal type over (D, θ). Recall from [26, (13.31)] the canonical map ("vector representation")

$$\chi\colon \mathrm{Spin}(h_0) \to \mathrm{O}^+(h_0).$$

Since proper isometries have reduced norm 1, we also have the inclusion

$$i\colon \mathrm{O}^+(h_0) \to \mathrm{SL}(V).$$

LEMMA 2.8. *The following diagram, where R is the Rost invariant, is commutative:*

$$\begin{array}{ccc} H^1(F, \mathrm{Spin}(h_0)) & \xrightarrow{(i\circ\chi)_*} & H^1(F, \mathrm{SL}(V)) \\ \Big\downarrow{\scriptstyle R} & & \Big\downarrow{\scriptstyle R} \\ H^3(F) & \xrightarrow{2} & H^3(F) \end{array}$$

Proof. This lemma is just a restatement of the property that the Rost multiplier of the map $i \circ \chi$ is 2, see [16, Ex. 7.15, p. 124]. □

We next recall from [26, (29.27)] (see also [13]) the canonical description of the pointed set $H^1(F, \mathrm{O}^+(h_0))$. Define a functor $\mathrm{SSym}(h_0)$ from Fields_F to the category of pointed sets as follows: for any field L containing F, set

$$\mathrm{SSym}(h_0)(L) = \{(s, \lambda) \in \mathrm{GL}(V_L) \times L^\times \mid \mathrm{ad}_{h_0}(s) = s \text{ and } \mathrm{Nrd}(s) = \lambda^2\},$$

where the distinguished element is $(1, 1)$. Let F_s be a separable closure of F and let $\Gamma = \mathrm{Gal}(F_s/F)$ be the Galois group. We may identify $\mathrm{SSym}(h_0)(F_s)$ with the quotient $\mathrm{GL}(V_{F_s})/\,\mathrm{O}^+((h_0)_{F_s})$ by mapping a class $a \cdot \mathrm{O}^+((h_0)_{F_s})$ to $(a\,\mathrm{ad}_{h_0}(a), \mathrm{Nrd}(a))$ for $a \in \mathrm{GL}(V_{F_s})$. Therefore, we have an exact sequence of pointed Γ-sets

$$1 \to \mathrm{O}^+((h_0)_{F_s}) \to \mathrm{GL}(V_{F_s}) \to \mathrm{SSym}(h_0)(F_s) \to 1.$$

Since $H^1(F, \mathrm{GL}(V_{F_s})) = 1$ by Hilbert's Theorem 90, the induced exact sequence in Galois cohomology yields a canonical bijection between $H^1(F, \mathrm{O}^+(h_0))$ and the orbit set of $\mathrm{GL}(V)$ on $\mathrm{SSym}(h_0)(F)$. Abusing notation, we write simply $\mathrm{SSym}(h_0)$ for $\mathrm{SSym}(h_0)(F)$. The orbits of $\mathrm{GL}(V)$ on $\mathrm{SSym}(h_0)$ are the equivalence classes under the following relation:

$$(s, \lambda) \sim (s', \lambda') \quad \text{if } s' = as\,\mathrm{ad}_{h_0}(a) \text{ and } \lambda' = \lambda\,\mathrm{Nrd}(a) \text{ for some } a \in \mathrm{GL}(V).$$

Therefore, we may identify

$$H^1(F, \mathrm{O}^+(h_0)) = \mathrm{SSym}(h_0)/\!\sim.$$

LEMMA 2.9. *The composition* $H^1(F, \mathrm{O}^+(h_0)) \xrightarrow{i_*} H^1(F, \mathrm{SL}(V)) \xrightarrow{R} H^3(F)$ *maps the equivalence class of* (s, λ) *to* $\lambda \cdot [A]$.

Proof. Let $\pi\colon \mathrm{SSym}(h_0)(F_s) \to F_s^\times$ be the projection $(s, \lambda) \mapsto \lambda$. We have a commutative diagram of pointed Γ-sets with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{O}^+((h_0)_{F_s}) & \longrightarrow & \mathrm{GL}(V_{F_s}) & \longrightarrow & \mathrm{SSym}(h_0)(F_s) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle i} & & \| & & \downarrow{\scriptstyle \pi} & & \\
1 & \longrightarrow & \mathrm{SL}(V_{F_s}) & \longrightarrow & \mathrm{GL}(V_{F_s}) & \xrightarrow{\mathrm{Nrd}} & F_s^\times & \longrightarrow & 1
\end{array}$$

This diagram yields the following commutative square in cohomology:

$$\begin{array}{ccc}
\mathrm{SSym}(h_0) & \longrightarrow & H^1(F, \mathrm{O}^+(h_0)) \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle i_*} \\
F^\times & \longrightarrow & H^1(F, \mathrm{SL}(V))
\end{array}$$

On the other hand, the Rost invariant and the map $F^\times \to H^3(F)$ carrying λ to $\lambda \cdot [A]$ fit in the following commutative diagram (see [26, p. 437]):

$$\begin{array}{ccc}
F^\times & \longrightarrow & H^1(F, \mathrm{SL}(V)) \\
 & \searrow \quad \swarrow{\scriptstyle R} & \\
 & H^3(F) &
\end{array}$$

The lemma follows. □

For the next statement, let $\partial\colon H^1(F, \mathrm{O}^+(h_0)) \to {}_2H^2(F)$ be the connecting map in the cohomology exact sequence associated to

$$1 \to \mu_2 \to \mathrm{Spin}(h_0) \xrightarrow{\chi} \mathrm{O}^+(h_0) \to 1.$$

For any hermitian form h of orthogonal type on V, there exists a unique linear transformation $s \in \mathrm{GL}(V)$ such that $h(x, y) = h_0(x, s^{-1}(y))$ for all $x,\ y \in V$, hence $\mathrm{ad}_h = \mathrm{Int}(s) \circ \mathrm{ad}_{h_0}$ and $\mathrm{ad}_{h_0}(s) = s$. If the discriminant of h is trivial we have $\mathrm{Nrd}(s) \in F^{\times 2}$, hence there exists $\lambda \in F^\times$ such that $\lambda^2 = \mathrm{Nrd}(s)$, and we may consider (s, λ) and $(s, -\lambda) \in \mathrm{SSym}(h_0)$. By the main theorem of [13], $\partial(s, \lambda)$ and $\partial(s, -\lambda)$ are the Brauer classes of the two components of

the Clifford algebra of $\mathrm{Ad}_{h_0 \perp -h}$, so if the Clifford invariant of h is trivial we have

$$\{\partial(s,\lambda), \partial(s,-\lambda)\} = \{0, [D]\}.$$

LEMMA 2.10. *With the notation above, we have $f_3(h) = \lambda \cdot [D]$ if $\partial(s,\lambda) = 0$.*

Proof. By definition of s, the torsor in $H^1(F, \mathrm{O}(h_0))$ corresponding to h lifts to $(s,\lambda) \in H^1(F, \mathrm{O}^+(h_0))$. If $\partial(s,\lambda) = 0$, then (s,λ) lifts to some $\xi \in H^1(F, \mathrm{Spin}(h_0))$, and by definition of the invariants e_3 and f_3 we have

$$e_3(h) = R(\xi) + F^\times \cdot [D] \in M_D^3(F) \qquad \text{and} \qquad f_3(h) = 2R(\xi) \in H^3(F).$$

Lemma 2.8 then yields $f_3(h) = R \circ (i \circ \chi)_*(\xi) = R \circ i_*(s,\lambda)$, and by Lemma 2.9 we have $R \circ i_*(s,\lambda) = \lambda \cdot [D]$. □

In order to check the condition $\partial(s,\lambda) = 0$ in Lemma 2.10, the following observation is useful: Suppose (V_1, h_1) and (V_2, h_2) are hermitian modules of orthogonal type over (D,θ). The inclusions $V_i \hookrightarrow V_1 \perp V_2$ yield an F-algebra homomorphism $C(\mathrm{Ad}_{h_1}) \otimes_F C(\mathrm{Ad}_{h_2}) \to C(\mathrm{Ad}_{h_1 \perp h_2})$, which induces a group homomorphism $\mathrm{Spin}(h_1) \times \mathrm{Spin}(h_2) \to \mathrm{Spin}(h_1 \perp h_2)$. This homomorphism fits into the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mu_2 \times \mu_2 & \longrightarrow & \mathrm{Spin}(h_1) \times \mathrm{Spin}(h_2) & \xrightarrow{\chi_1 \times \chi_2} & \mathrm{O}^+(h_1) \times \mathrm{O}^+(h_2) & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \Pi} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \oplus} & & \\
1 & \longrightarrow & \mu_2 & \longrightarrow & \mathrm{Spin}(h_1 \perp h_2) & \xrightarrow{\chi} & \mathrm{O}^+(h_1 \perp h_2) & \longrightarrow & 1
\end{array}$$

The left vertical map is the product, and the right vertical map carries (g_1, g_2) to $g_1 \oplus g_2$. The induced diagram in cohomology yields the commutative square

$$\begin{array}{ccc}
H^1(F, \mathrm{O}^+(h_1)) \times H^1(F, \mathrm{O}^+(h_2)) & \xrightarrow{\partial_1 \times \partial_2} & {}_2H^2(F) \times {}_2H^2(F) \\
\Big\downarrow{\scriptstyle \oplus} & & \Big\downarrow{\scriptstyle \Pi} \\
H^1(F, \mathrm{O}^+(h_1 \perp h_2)) & \xrightarrow{\partial} & {}_2H^2(F)
\end{array}$$

The following additivity property of the connecting maps ∂ follows: for $(s_1,\lambda_1) \in H^1(F, \mathrm{O}^+(h_1))$ and $(s_2,\lambda_2) \in H^1(F, \mathrm{O}^+(h_2))$,

$$\partial_1(s_1,\lambda_1) + \partial_2(s_2,\lambda_2) = \partial(s_1 \oplus s_2, \lambda_1\lambda_2). \tag{1}$$

Proof of Proposition 2.7. Let $h_0 = \langle 1, -1\rangle h_1 \perp \ldots \perp \langle 1, -1\rangle h_m$, which is a hyperbolic form, and let $V = V_1^{\oplus 2} \oplus \cdots \oplus V_m^{\oplus 2}$ be the underlying vector space of h and h_0. The linear transformation $s \in \mathrm{GL}(V)$ such that $h(x,y) = h_0(x, s^{-1}(y))$ for all x, $y \in V$ is

$$s = 1 \oplus \lambda_1^{-1} \oplus 1 \oplus \lambda_2^{-1} \oplus \cdots \oplus 1 \oplus \lambda_m^{-1}.$$

By the additivity property (1), the connecting map

$$\partial\colon H^1(F, \mathrm{O}^+(h_0)) \to {}_2H^2(F)$$

satisfies

$$\partial(s, \lambda_1^{-n_1/2} \dots \lambda_m^{-n_m/2}) = \partial_1(\lambda_1^{-1}, \lambda_1^{-n_1/2}) + \dots + \partial_m(\lambda_m^{-1}, \lambda_m^{-n_m/2}).$$

A theorem of Bartels [4, p. 283] (see also [13]) yields $\partial_i(\lambda_i^{-1}, \lambda_i^{-n_i/2}) = \lambda_i^{-1} \cdot e_1(h_i)$ for all i. Therefore, if $\sum_{i=1}^m \lambda_i \cdot e_1(h_i) = 0$ we have $f_3(h) = \lambda_1^{n_1/2} \dots \lambda_m^{n_m/2} \cdot [D]$ by Lemma 2.10. □

2.4. RELATIVE ARASON INVARIANT OF HERMITIAN FORMS OF ORTHOGONAL TYPE. By using the Rost invariant, one may also define a relative Arason invariant, in a broader context:

DEFINITION 2.11. Let h_1 and h_2 be two hermitian forms of orthogonal type over (D, θ) such that their difference $h_1 + (-h_2)$ has even relative rank, trivial discriminant, and trivial Clifford invariant. Their relative Rost invariant is defined by

$$e_3(h_1/h_2) = e_3\big(h_1 \perp (-h_2)\big) \in M_D^3(F).$$

In particular, if both h_1 and h_2 have even relative rank, trivial discriminant, and trivial Clifford invariant, then $e_3(h_1/h_2) = e_3(h_1) + e_3(h_2) = e_3(h_1) - e_3(h_2)$.

Remark 2.12. Under the conditions of this definition, one may check that the involution ad_{h_2} corresponds to a torsor which can be lifted to a $\mathrm{Spin}(\mathrm{Ad}_{h_1})$ torsor (see [41, §3.5]). As explained in [6, Lemma 3.6], the relative Arason invariant $e_3(h_1/h_2)$ coincides with the class in $M_D^3(F)$ of the image of this torsor under the Rost invariant of $\mathrm{Spin}(\mathrm{Ad}_{h_1})$.

Combining the properties of the Arason invariant recalled in Lemma 2.2 and the computation of Proposition 2.6, we obtain:

COROLLARY 2.13. (i) *Let h be a hermitian form of orthogonal type with even absolute rank, and let K/F be the discriminant quadratic extension. For any $\mu \in K^\times$, the relative Arason invariant $e_3(\langle N_{K/F}(\mu)\rangle h/h)$ is well-defined, and*

$$e_3(\langle N_{K/F}(\mu)\rangle h/h) = \mathrm{cor}_{K/F}(\mu \cdot e_2(h_K)).$$

(ii) *Let h_1 and h_2 be two hermitian forms of orthogonal type with even absolute rank and trivial discriminant. We have*

$$e_3(h_1 \perp \langle\lambda\rangle h_2/h_1 \perp h_2) = e_3(\langle\lambda\rangle h_2/h_2) = \lambda \cdot e_2(h_2).$$

2.5. ARASON AND f_3 INVARIANTS OF ORTHOGONAL INVOLUTIONS. Let (A, σ) be an algebra with orthogonal involution, Brauer-equivalent to the division algebra D over F. We pick an involution θ on D, so that (A, σ) can be represented as the adjoint $(A, \sigma) \simeq \mathrm{Ad}_h$ of some hermitian module (V, h) over (D, θ). The co-index of A, which is the dimension over D of the module V, is equal to the relative rank of h. If the form h has even relative rank, trivial discriminant, and trivial Clifford invariant, then its Arason invariant is well-defined. Moreover, by Lemma 2.2, we have $e_3(h) = e_3(\lambda h)$ for any $\lambda \in F^\times$, and, as explained in [6, Prop 3.8], $e_3(h)$ does not depend on the choice of θ. Therefore, we get a

well-defined Arason invariant for the involution σ, provided the algebra A has even co-index, i.e. $\deg(A)/\operatorname{ind}(A) = \deg(A)/\deg(D) \in 2\mathbb{Z}$, and the involution σ has trivial discriminant and trivial Clifford invariant:

$$e_3(\sigma) = e_3(h) \in M_A^3(F) = M_D^3(F).$$

Remarks 2.14. (1) Under the assumptions above on (A, σ), one may also check that the algebra A carries a hyperbolic orthogonal involution σ_0, and the Arason invariant $e_3(\sigma)$ can be defined directly in terms of the Rost invariant of the group $\operatorname{Spin}(A, \sigma_0)$, see [41, §3.5].

(2) Similarly, we may also define a relative Arason invariant $e_3(\sigma_1/\sigma_2)$ if the involutions σ_1 and σ_2 both have trivial discriminant and trivial Clifford invariant. But we cannot relax those assumptions, as we did for hermitian forms. Indeed, if $e_2(h_2) = e_2(\mathrm{ad}_{h_2})$ is not trivial, then $e_3(\langle\lambda\rangle h_1/h_2)$ and $e_3(h_1/h_2)$ are generally different, as Corollary 2.13 shows.

In the setting above, we may also define an f_3-invariant by $f_3(\sigma) = f_3(h)$, or equivalently:

DEFINITION 2.15. Let (A, σ) be an algebra with orthogonal involution. We assume A has even co-index, and σ has trivial discriminant and trivial Clifford invariant. We define $f_3(\sigma) \in F^\times \cdot [A] \subset {}_2H^3(F)$ by $f_3(\sigma) = 2c$, where c is any representative of the Arason invariant $e_3(\sigma) \in M_A^3(F)$.

Remark 2.16. (i) If A is split, then $F^\times \cdot [A] = \{0\}$, and $f_3(\mathrm{ad}_\varphi) = 0$ for all quadratic forms $\varphi \in I^3(F)$. This also follows from the fact that $e_3(\varphi) \in {}_2H^3(F)$.
(ii) Using the same process, one may define an invariant f_3^{16} from Garibaldi's invariant e_3^{16}, and from Bermudez-Ruozzi's generalization (see [15], [9]). This invariant has values in ${}_2H^3(F)$, but need not have values in $F^\times \cdot [A]$ in general.

Example 2.17. Let Q be a quaternion algebra over F, and consider the algebra with involution $(A, \sigma) = (Q, \rho) \otimes \mathrm{Ad}_\varphi$, where ρ is an orthogonal involution with discriminant $\delta \cdot F^{\times 2} \in F^\times/F^{\times 2}$, and φ is a even-dimensional quadratic form with trivial discriminant. We have $e_3(\sigma) = \delta \cdot e_2(\varphi) \mod F^\times \cdot [Q]$, and $f_3(\sigma) = 0$. Indeed, since the restriction map $M_Q^3(F) \to M_Q^3(F_Q) = H^3(F_Q)$ is injective, it is enough to check the formula in the split case, where it follows from a direct computation.

The computation in Proposition 2.6 can be again rephrased as follows:

COROLLARY 2.18. *Let (A, σ) be a central simple F-algebra of even degree n with orthogonal involution, and let K/F be the discriminant quadratic extension. For any $\mu \in K^\times$, the algebra with involution $\mathrm{Ad}_{\langle 1, -N_{K/F}(\mu)\rangle} \otimes (A, \sigma)$ has even co-index, trivial discriminant and trivial Clifford invariant. Its Arason invariant is given by*

$$e_3(\mathrm{ad}_{\langle 1, -N_{K/F}(\mu)\rangle} \otimes \sigma) = \operatorname{cor}_{K/F}\big(\mu \cdot e_2(\sigma_K)\big),$$

and

$$f_3(\mathrm{ad}_{\langle 1,-N_{K/F}(\mu)\rangle}\otimes\sigma)=\begin{cases}0 & \text{if } n\equiv 0 \bmod 4,\\ N_{K/F}(\mu)\cdot[A] & \text{if } n\equiv 2 \bmod 4.\end{cases}$$

In particular, if σ has trivial discriminant, we have for any $\lambda\in F^\times$

$$e_3(\mathrm{ad}_{\langle 1,-\lambda\rangle}\otimes\sigma)=\lambda\cdot e_2(\sigma)$$

and

$$f_3(\mathrm{ad}_{\langle 1,-\lambda\rangle}\otimes\sigma)=\begin{cases}0 & \text{if } n\equiv 0 \bmod 4,\\ \lambda\cdot[A] & \text{if } n\equiv 2 \bmod 4.\end{cases}$$

Hence, the formula given in [35, Th. 5.5] for algebras of degree 8 is actually valid in arbitrary degree.

With this in hand, one may easily check that the f_3 invariant is trivial up to degree 10. Indeed, since the co-index of the algebra is supposed to be even, the algebra is possibly non-split only when its degree is divisible by 4. In degree 4, any involution with trivial discriminant and Clifford invariant is hyperbolic, hence has trivial invariants. In degree 8, any involution with trivial discriminant and Clifford invariant admits a decomposition as in Corollary 2.18 by [35, Th. 5.5], hence its f_3 invariant is trivial. In degree 12, one may construct explicit examples of (A,σ) with $f_3(\sigma)\neq 0$ as follows. Suppose E is a central simple F-algebra of degree 4. Recall from [26, §10.B] that the second λ-power λ^2E is a central simple F-algebra of degree 6, which carries a canonical involution γ of orthogonal type with trivial discriminant, and which is Brauer-equivalent to $E\otimes_F E$.

COROLLARY 2.19. *Let E be a central simple F-algebra of degree and exponent 4. Pick an indeterminate t, and consider the algebra with involution*

$$(A,\sigma)=\mathrm{Ad}_{\langle 1,-t\rangle}\otimes(\lambda^2E,\gamma)_{F(t)}.$$

We have

$$f_3(\sigma)=t\cdot[A]\neq 0\in H^3(F(t)).$$

Proof. The formula $f_3(A,\sigma)=t\cdot[A]$ readily follows from Corollary 2.18. The algebra E has exponent 4, therefore $[A]=[E\otimes_F E]\neq 0$. Since t is an indeterminate, we get $t\cdot[A]\neq 0$. □

3. Additive decompositions in degree 12

In the next three sections, we concentrate on degree 12 algebras (A,σ) with orthogonal involution of trivial discriminant and trivial Clifford invariant. The main result of this section is Theorem 3.2, which generalizes a theorem of Pfister on 12-dimensional quadratic forms.

3.1. ADDITIVE DECOMPOSITIONS. Given three algebras with involution, (A,σ), (A_1,σ_1) and (A_2,σ_2), we say that (A,σ) is a *direct sum* of (A_1,σ_1) and (A_2,σ_2), and we write

$$(A,\sigma) \in (A_1,\sigma_1) \boxplus (A_2,\sigma_2),$$

if there exist a division algebra with involution (D,θ) and hermitian modules (V_1,h_1) and (V_2,h_2) over (D,θ), which are both hermitian or both skew-hermitian, such that $(A_1,\sigma_1) = \mathrm{Ad}_{h_1}$, $(A_2,\sigma_2) = \mathrm{Ad}_{h_2}$ and $(A,\sigma) = \mathrm{Ad}_{h_1 \perp h_2}$. In particular, this implies A, A_1 and A_2 are all three Brauer-equivalent to D, and the involutions σ, σ_1 and σ_2 are of the same type. This notion of direct sum for algebras with involution was introduced by Dejaiffe in [10]. As explained there, the algebra with involution (A,σ) is generally not uniquely determined by the data of the two summands (A_1,σ_1) and (A_2,σ_2). Indeed, multiplying the hermitian forms h_1 and h_2 by a scalar does not change the adjoint involutions, so the adjoint of $\lambda_1 h_1 \perp \lambda_2 h_2$ also is a direct sum of (A_1,σ_1) and (A_2,σ_2) for any λ_1, $\lambda_2 \in F^\times$. If one of the two summands, say $(A_1,\sigma_1) = (A_1,\mathrm{hyp})$ is hyperbolic, then all hermitian forms similar to h_1 actually are isomorphic to h_1. Hence in this case, there is a unique direct sum, and we may write

$$(A,\sigma) = (A_1,\mathrm{hyp}) \boxplus (A_2,\sigma_2).$$

The cohomological invariants we consider, when defined, have the following additivity property:

PROPOSITION 3.1. *Suppose σ, σ_1, σ_2 are orthogonal involutions such that $(A,\sigma) \in (A_1,\sigma_1) \boxplus (A_2,\sigma_2)$. We have:*

(i) $\deg A = \deg A_1 + \deg A_2$.

(ii) *If* $\deg A_1 \equiv \deg A_2 \equiv 0 \bmod 2$, *then* $e_1(\sigma) = e_1(\sigma_1) + e_1(\sigma_2)$.

(iii) *If* $\deg A_1 \equiv \deg A_2 \equiv 0 \bmod 2$ *and* $e_1(\sigma_1) = e_1(\sigma_2) = 0$, *then*

$$e_2(\sigma) = e_2(\sigma_1) + e_2(\sigma_2).$$

(iv) *If the co-indices of A_1 and A_2 are even and $e_i(\sigma_1) = e_i(\sigma_2) = 0$ for $i = 1, 2$, then*

$$e_3(\sigma) = e_3(\sigma_1) + e_3(\sigma_2) \text{ and } f_3(\sigma) = f_3(\sigma_1) + f_3(\sigma_2).$$

Proof. Assertion (i) is clear by definition, and (ii) was established by Dejaiffe [10, Prop. 2.3]. Assertion (iii) follows from [10, § 3.3] (see also the proof of the "Orthogonal Sum Lemma" in [14, §3]). To prove (iv), let D be the division algebra Brauer-equivalent to A, A_1, and A_2, and let θ be an F-linear involution on D. We may find hermitian forms of orthogonal type h_1, h_2 over (D,θ) such that $(A_i,\sigma_i) \simeq \mathrm{Ad}_{h_i}$ for $i = 1, 2$, and $(A,\sigma) \simeq \mathrm{Ad}_{h_1 \perp h_2}$. By Lemma 2.2(ii) we have

$$e_3(h_1 \perp h_2) = e_3(h_1) + e_3(h_2).$$

By definition of the e_3-invariant of orthogonal involutions (see § 2.5), $e_3(\sigma)$ (resp. $e_3(\sigma_i)$ for $i = 1, 2$) is represented by $e_3(h_1 \perp h_2)$ (resp. $e_3(h_i)$), hence $e_3(\sigma) = e_3(\sigma_1) + e_3(\sigma_2)$. Likewise, the additivity of e_3 induces $f_3(\sigma) = f_3(\sigma_1) + f_3(\sigma_2)$, by definition of the f_3-invariant (see 2.15). □

By a result of Pfister [33, p.123-124], any 12-dimensional quadratic form φ in I^3F decomposes as $\varphi = \langle\alpha_1\rangle n_1 \perp \langle\alpha_2\rangle n_2 \perp \langle\alpha_3\rangle n_3$, where n_i is a 2-fold Pfister form and $\alpha_i \in F^\times$, for $1 \le i \le 3$. This can be rephrased as

$$\mathrm{Ad}_\varphi \in \mathrm{Ad}_{n_1} \boxplus \mathrm{Ad}_{n_2} \boxplus \mathrm{Ad}_{n_3},$$

where each summand Ad_{n_i} has degree 4 and discriminant 1. We now extend this result to the non-split case.

Theorem 3.2. *Let (A,σ) be a central simple F-algebra of degree 12 with orthogonal involution. Assume the discriminant and the Clifford invariant of σ are trivial. There is a central simple F-algebra A_0 of degree 4 and orthogonal involutions σ_1, σ_2, σ_3 of trivial discriminant on A_0 such that*

$$(A,\sigma) \in (A_0,\sigma_1) \boxplus (A_0,\sigma_2) \boxplus (A_0,\sigma_3).$$

Note that since $\deg A_0 = 4$ we have $e_2(\sigma_i) = 0$ if and only if σ_i is hyperbolic (see [41, Th. 3.10]); therefore, even when the index of A is 2 we *cannot* use Proposition 3.1(iv) to compute $e_3(\sigma)$ (unless each σ_i is hyperbolic).

Proof of Theorem 3.2. The index of A is a power of 2 since $2[A] = 0$ in $\mathrm{Br}(F)$, and it divides $\deg A = 12$, so $\mathrm{ind}\, A = 1$, 2 or 4. As we just pointed out, the index 1 case is Pfister's theorem. We consider separately the two remaining cases.

If $\mathrm{ind}\, A = 2$, we have $(A,\sigma) = \mathrm{Ad}_h$ for some skew-hermitian form h of relative rank 6 over a quaternion division algebra $(Q,\bar{\ })$ with its canonical involution. Let $q_1 \in Q$ be a nonzero pure quaternion represented by h, and write $h = \langle q_1\rangle \perp h'$. Over the quadratic extension $K_1 = F(q_1)$, the algebra Q splits and the form $\langle q_1\rangle$ becomes hyperbolic (because its discriminant becomes a square). Therefore, h_{K_1} and h'_{K_1} are Witt-equivalent, and $(\mathrm{ad}_{h'})_{K_1}$ is adjoint to a 10-dimensional form φ. The discriminant and Clifford invariant of σ are trivial, hence $\varphi \in I^3K_1$. Since there is no anisotropic 10-dimensional quadratic forms in I^3 (see [22, Th. 8.1.1]), it follows that h'_{K_1} is isotropic, hence by [34, Prop., p. 382], $h' = \langle -\lambda_1 q_1\rangle \perp k$ for some $\lambda_1 \in F^\times$ and some skew-hermitian form k of relative rank 4. We thus have

$$h = \langle q_1\rangle\langle 1, -\lambda_1\rangle \perp k,$$

and computation shows that $e_1\big(\langle q_1\rangle\langle 1,-\lambda_1\rangle\big) = 0$. Therefore, $e_1(k) = 0$, and $e_2(k) = e_2\big(\langle q_1\rangle\langle 1,-\lambda_1\rangle\big)$ because $e_2(h) = 0$. Now, let $q_2 \in Q$ be a nonzero pure quaternion represented by k, and let $K_2 = F(q_2)$, so $k = \langle q_2\rangle \perp k'$ for some skew-hermitian form k' of relative rank 3. The forms k_{K_2} and k'_{K_2} are Witt-equivalent, and $(\mathrm{ad}_{k'})_{K_2}$ is adjoint to a 6-dimensional form $\psi \in I^2K_2$, i.e., to an Albert form ψ. We have

$$e_2(\psi) = e_2(k')_{K_2} = e_2(k)_{K_2} = e_2\big(\langle q_1\rangle\langle 1,-\lambda_1\rangle\big)_{K_2},$$

hence the index of $e_2(\psi)$ is at most 2, and it follows that ψ is isotropic. Therefore, k'_{K_2} is isotropic, and $k' = \langle -\lambda_2 q_2\rangle \perp \ell$ for some $\lambda_2 \in F^\times$ and some

skew-hermitian form ℓ of relative rank 2. Thus, we have

$$h = \langle q_1\rangle\langle 1, -\lambda_1\rangle \perp \langle q_2\rangle\langle 1, -\lambda_2\rangle \perp \ell.$$

Since $e_1\big(\langle q_1\rangle\langle 1, -\lambda_1\rangle\big) = e_1\big(\langle q_2\rangle\langle 1, -\lambda_2\rangle\big) = 0$ and $e_1(h) = 0$, we also have $e_1(\ell) = 0$. We thus obtain the required decomposition, with

$$A_0 = M_2(Q), \qquad \sigma_1 = \mathrm{ad}_{\langle q_1\rangle\langle 1, -\lambda_1\rangle}, \qquad \sigma_2 = \mathrm{ad}_{\langle q_2\rangle\langle 1, -\lambda_2\rangle}, \qquad \sigma_3 = \mathrm{ad}_{\ell}\,.$$

Suppose now $\operatorname{ind} A = 4$, and let D be the division algebra of degree 4 Brauer-equivalent to A. By [17, Th. 3.1], there exists a quadratic extension K of F such that $(A, \sigma)_K$ is hyperbolic. The co-index of A_K is therefore even, so the index of A_K is 2, hence we may identify K with a subfield of D. The following construction is inspired by the Parimala–Sridharan–Suresh exact sequence in Appendix 2 of [5]. We have $D = \widetilde{D} \oplus D'$, where $\widetilde{D}$ is the centralizer of K in D and, writing ι for the nontrivial F-automorphism of K,

$$D' = \{x \in D \mid xy = \iota(y)x \text{ for all } y \in K\}.$$

Let θ be an orthogonal involution on D that fixes K (such involutions exist by [26, (4.14)]). We may represent $(A, \sigma) = (\operatorname{End}_D V, \mathrm{ad}_h)$ for some hermitian form h of relative rank 3 over (D, θ). In view of the decomposition $D = \widetilde{D} \oplus D'$, we have for x, $y \in V$

$$h(x, y) = \widetilde{h}(x, y) + h'(x, y) \qquad \text{with } \widetilde{h}(x, y) \in \widetilde{D} \text{ and } h'(x, y) \in D'.$$

Since h is a hermitian form over (D, θ), it follows that $\widetilde{h}$ is a hermitian form on V viewed as a $\widetilde{D}$-vector space, with respect to the restriction of θ to $\widetilde{D}$. Clearly, $\operatorname{End}_D V \subset \operatorname{End}_{\widetilde{D}} V$. We may also embed K into $\operatorname{End}_{\widetilde{D}} V$ by identifying $\alpha \in K$ with the scalar multiplication $x \mapsto x\alpha$ for $x \in V$. Thus, we have a K-algebra homomorphism

$$(\operatorname{End}_D V) \otimes_F K \to \operatorname{End}_{\widetilde{D}} V.$$

This homomorphism is injective because the left side is a simple algebra, hence it is an isomorphism by dimension count. For $f \in \operatorname{End}_D V$ we have $\mathrm{ad}_h(f) = \mathrm{ad}_{\widetilde{h}}(f)$, so the isomorphism preserves the involution, and therefore $(\mathrm{Ad}_h)_K = \mathrm{Ad}_{\widetilde{h}}$. Since σ becomes hyperbolic over K, the form $\widetilde{h}$ is hyperbolic. Therefore, there is an h-orthogonal base of V consisting of $\widetilde{h}$-isotropic vectors, which yields a diagonalization

$$h = \langle a_1, a_2, a_3\rangle \qquad \text{with } a_1,\, a_2,\, a_3 \in D' \cap \operatorname{Sym}(\theta).$$

We thus have $(A, \sigma) \in (D, \sigma_1) \boxplus (D, \sigma_2) \boxplus (D, \sigma_3)$ with $\sigma_i = \mathrm{Int}(a_i^{-1}) \circ \theta$ for $i = 1, 2, 3$. To complete the proof, we show that the discriminant of each σ_i is trivial. Recall from [26, (7.2)] that the discriminant is the square class of any skew-symmetric unit. Let $\alpha \in K^\times$ be such that $\iota(\alpha) = -\alpha$. Since $a_i \in D'$ we have $\sigma_i(\alpha) = -\alpha$, so $\operatorname{disc} \sigma_i = \operatorname{Nrd}_D(\alpha) = N_{K/F}(\alpha)^2$. □

Recall that a central simple algebra of degree 4 with orthogonal involution (A_0, σ_0) of trivial discriminant decomposes as $(A_0, \sigma_0) \simeq (Q, \bar{\ }) \otimes (H, \bar{\ })$ where the quaternion algebras Q, H are the two components of the Clifford algebra

$C(A_0, \sigma_0)$ (see [26, (15.12)]). Therefore, Theorem 3.2 can be rephrased as follows:

Corollary 3.3. *Let (A, σ) be a central simple algebra of degree 12 with orthogonal involution of trivial discriminant and Clifford invariant. There exist quaternion F-algebras Q_i, H_i for $i = 1, 2, 3$ such that $[A] = [Q_i] + [H_i]$ for $i = 1, 2, 3$, $[H_1] + [H_2] + [H_3] = 0$, and*

$$(A, \sigma) \in \big((Q_1, \bar{\ }) \otimes (H_1, \bar{\ })\big) \boxplus \big((Q_2, \bar{\ }) \otimes (H_2, \bar{\ })\big) \boxplus \big((Q_3, \bar{\ }) \otimes (H_3, \bar{\ })\big).$$

Proof. Theorem 3.2 yields orthogonal involutions σ_1, σ_2, σ_3 of trivial discriminant on the central simple F-algebra A_0 of degree 4 Brauer-equivalent to A such that

$$(A, \sigma) \in (A_0, \sigma_1) \boxplus (A_0, \sigma_2) \boxplus (A_0, \sigma_3).$$

Each (A_0, σ_i) has a decomposition

$$(A_0, \sigma_i) \simeq (Q_i, \bar{\ }) \otimes (H_i, \bar{\ })$$

for some quaternion F-algebras Q_i, H_i such that

$$e_2(\sigma_i) = [Q_i] + \{0, [A]\} = [H_i] + \{0, [A]\} \in M_A^2(F).$$

From this decomposition, it follows that

$$[Q_i] + [H_i] = [A_0] = [A] \qquad \text{for } i = 1, 2, 3.$$

Moreover, we have $\sum_{i=1}^3 e_2(\sigma_i) = e_2(\sigma)$ by Proposition 3.1, and $e_2(\sigma) = 0$, so

$$\sum_{i=1}^3 [Q_i] = \big(\sum_{i=1}^3 [H_i]\big) + [A] \in \{0, [A]\} \subset {}_2H^3(F).$$

Therefore, interchanging Q_i and H_i if necessary, we may assume $\sum_{i=1}^3 [H_i] = 0$. □

Example 3.4. For any quaternion algebra Q with norm form n_Q, we have $\mathrm{Ad}_{n_Q} \simeq (Q, \bar{\ }) \otimes (Q, \bar{\ })$, see for instance [26, (11.1)]. Therefore, if A is split, and σ is adjoint to the 12-dimensional form $\varphi = \langle \alpha_1 \rangle n_1 \perp \langle \alpha_2 \rangle n_2 \perp \langle \alpha_3 \rangle n_3$, where n_i is the norm form of a quaternion algebra Q_i for $1 \le i \le 3$, then

$$(A, \sigma) \in \boxplus_{i=1}^3 (Q_i, \bar{\ }) \otimes (Q_i, \bar{\ }).$$

Conversely, any decomposition of a split $(A, \sigma) = \mathrm{Ad}_\varphi$ as in the corollary corresponds to a decomposition $\varphi = \langle \alpha_1 \rangle n_1 \perp \langle \alpha_2 \rangle n_2 \perp \langle \alpha_3 \rangle n_3$, where n_i is the norm form of $Q_i \simeq H_i$.

Remark 3.5. In [17], it is proved that any (A, σ) of degree 12 with trivial discriminant and trivial Clifford invariant can be described as a quadratic extension of some degree 6 central simple algebra with unitary involution (B, τ), with discriminant algebra Brauer-equivalent to A. This algebra (B, τ) can be described from the above additive decomposition as follows. Since $\sum_{i=1}^3 [H_i] = 0$, the algebras H_i have a common quadratic subfield K, see [28, Th. III.4.13]. All three products $(Q_i, \bar{\ }) \otimes (H_i, \bar{\ })$ are hyperbolic over K, so σ_K is hyperbolic. Moreover, as observed in [17, Ex. 1.3], the tensor product $(Q_i, \bar{\ }) \otimes (H_i, \bar{\ })$

is a quadratic extension of $(Q_i, \overline{\ }) \otimes (K, \overline{\ })$. Therefore, (A, σ) is a quadratic extension of some $(B, \tau) \in \boxplus_{i=0}^{3}(Q_i, \overline{\ }) \otimes (K, \overline{\ })$, and the discriminant algebra of (B, τ) is Brauer-equivalent to $[Q_1] + [Q_2] + [Q_3] = [Q]$. Note that in the case where $\operatorname{ind} A = 4$, we use in our proof the main result of [17], which guarantees the existence of a quadratic extension K such that $(A, \sigma)_K$ is hyperbolic. But for $\operatorname{ind} A \neq 4$, our proof is independent, and does not use the existence of an open orbit of a half-spin representation as in [17, p. 1220].

3.2. DECOMPOSITION GROUPS OF (A, σ). Until the end of this section, (A, σ) denotes a central simple F-algebra of degree 12 with an orthogonal involution of trivial discriminant and trivial Clifford algebra.

DEFINITION 3.6. Given an additive decomposition as in Corollary 3.3

$$(A, \sigma) \in \boxplus_{i=1}^{3}\big((Q_i, \overline{\ }) \otimes_F (H_i, \overline{\ })\big) \qquad \text{with } \sum_{i=1}^{3}[H_i] = 0,$$

the subset

$$U = \{0, [A], [Q_1], [H_1], [Q_2], [H_2], [Q_3], [H_3]\} \subset {}_2\operatorname{Br}(F)$$

is called a *decomposition group* of (A, σ). It is indeed the subgroup of ${}_2\operatorname{Br}(F)$ generated by $[Q_1]$, $[Q_2]$, and $[Q_3]$, since $[A] = [Q_1] + [Q_2] + [Q_3]$ and $[H_i] = [A] + [Q_i]$ for $i = 1, 2, 3$.

As the following examples show, a given algebra with involution (A, σ) may admit several additive decompositions, corresponding to different decomposition groups, possibly not all of the same cardinality.

Example 3.7. Assume A is split. Since $[A] = 0$, we have $[H_i] = [Q_i]$ for all i. Hence all decomposition groups of (A, σ) have order dividing 4.
Consider three quaternion division algebras $[Q_1]$, $[Q_2]$ and $[Q_3]$ such that $[Q_1] + [Q_2] = [Q_3]$. By the "common slot lemma" [28, Th. III.4.13], there exist a, b_1, $b_2 \in F^\times$ such that $Q_i = (a, b_i)$ for $i = 1, 2$ and $Q_3 = (a, b_1 b_2)$. An easy computation then shows that the norm forms of Q_1, Q_2, Q_3, respectively denoted by n_1, n_2, n_3, satisfy $n_1 - n_2 = \langle b_2 \rangle n_3$ in the Witt group of F. Hence, extending scalars to a rational function fields in two variables over F, one may find scalars α_i for $1 \leq i \leq 3$ such that the form

$$\varphi = \langle \alpha_1 \rangle n_1 \perp \langle \alpha_2 \rangle n_2 \perp \langle \alpha_3 \rangle n_3$$

is either anisotropic, or isotropic and non-hyperbolic, or hyperbolic. By Example 3.4, in all three cases, $\{0, [Q_1], [Q_2], [Q_3]\}$ is a decomposition group of order 4 for the involution $\sigma = \operatorname{ad}_\varphi$.
On the other hand, the adjoint involution of an isotropic or a hyperbolic form also has smaller decomposition groups, as we now proceed to show. If the involution σ is isotropic, it is adjoint to a quadratic form φ which is Witt-equivalent to a 3-fold Pfister form π_3. Let Q be a quaternion algebra such that the norm form n_Q is a subform of π_3. There exists α_1, $\alpha_2 \in F^\times$ such that $\varphi = \langle \alpha_1, \alpha_2 \rangle \otimes n_Q \perp 2\mathbb{H}$, (where $\mathbb{H}$ denotes the hyperbolic plane) hence

$\{0, [Q]\}$ is a decomposition group of $\sigma = \operatorname{ad}\varphi$. If in addition σ, hence π_3, is hyperbolic, we may choose $[Q] = 0$.

Example 3.8. Assume now that $A = M_6(Q)$ has index 2. Since $0 \neq [Q] \in U$, all decomposition groups U have order 2, 4 or 8.
If σ is isotropic, then it is Witt-equivalent to a degree 8 algebra with involution $(M_4(Q), \sigma_0)$ that has trivial discriminant and trivial Clifford invariant, so

$$(A, \sigma) = (M_4(Q), \sigma_0) \boxplus (M_2(Q), \mathrm{hyp}) = (M_4(Q), \sigma_0) \boxplus \big((M_2(F), \bar{\ }) \otimes (Q, \bar{\ })\big).$$

Because $(M_4(Q), \sigma_0)$ has trivial discriminant and Clifford invariant, by [35, Th. 5.2] we may find λ, $\mu \in F^\times$ and an orthogonal involution ρ on Q such that

$$(M_4(Q), \sigma_0) \simeq \mathrm{Ad}_{\langle\langle \lambda, \mu \rangle\rangle} \otimes (Q, \rho).$$

Let Q_1 and H_1 be the two components of the Clifford algebra of $\mathrm{Ad}_{\langle\langle \mu \rangle\rangle} \otimes (Q, \rho)$. Then

$$\mathrm{Ad}_{\langle\langle \mu \rangle\rangle} \otimes (Q, \rho) \simeq (Q_1, \bar{\ }) \otimes (H_1, \bar{\ }).$$

Therefore,

$$\begin{aligned} \mathrm{Ad}_{\langle\langle \lambda, \mu \rangle\rangle} \otimes (Q, \rho) &\simeq \mathrm{Ad}_{\langle 1, -\lambda \rangle} \otimes (Q_1, \bar{\ }) \otimes (H_1, \bar{\ }) \\ &\in \big((Q_1, \bar{\ }) \otimes (H_1, \bar{\ })\big) \boxplus \big((Q_1, \bar{\ }) \otimes (H_1, \bar{\ })\big), \end{aligned}$$

and finally

$$(A, \sigma) \in \big(Q_1, \bar{\ }) \otimes (H_1, \bar{\ })\big) \boxplus \big((Q_1, \bar{\ }) \otimes (H_1, \bar{\ })\big) \boxplus \big((M_2(F), \bar{\ }) \otimes (Q, \bar{\ })\big).$$

It follows that $\{0, [Q], [Q_1], [H_1]\}$ is a decomposition group for (A, σ).
If in addition σ is hyperbolic, we may choose $\mu = 1$, so that $\{[Q_1], [H_1]\} = \{0, [Q]\}$. Hence $\{0, [Q]\}$ is a decomposition group of (A, σ) in this case.

Example 3.9. If A has index 4, then all decomposition groups of (A, σ) have order 8. Indeed, since $[A] = [Q_i] + [H_i]$, the quaternion algebras Q_i and H_i all are division algebras. Therefore $[Q_i] \neq 0$ for $i = 1, 2, 3$. Moreover, we have $[Q_1] + [Q_2] + [Q_3] = [A]$. Since A has index 4, this guarantees $[Q_i] + [Q_j]$ is non zero if $i \neq j$. Therefore, $[Q_1], [Q_2], [Q_3]$ are $\mathbb{Z}/2$ linearly independent, and they do generate a group of order 8.
One may also check that the involution σ is anisotropic in this case. Indeed, $A \simeq M_3(D)$ for some degree 4 division algebra D, hence A does not carry any hyperbolic involution. Moreover, its isotropic involutions with trivial discriminant are Witt-equivalent to $(Q_1, \bar{\ }) \otimes (H_1, \bar{\ })$, for some quaternion division algebras Q_1 and H_1 such that $D \simeq Q_1 \otimes H_1$. Hence isotropic involutions on A with trivial discriminant have non trivial Clifford invariant

$$[Q_1] + \{0, [D]\} = [H_1] + \{0, [D]\} \neq 0 \in H^2(F)/\{0, [D]\}.$$

From these examples, we easily get the following characterization of isotropy and hyperbolicity:

Lemma 3.10. *Let (A, σ) be a degree 12 algebra with orthogonal involution with trivial discriminant and trivial Clifford invariant.*

(i) *The involution σ is isotropic if and only if it admits a decomposition group generated by $[A]$ and $[Q_1]$ for some quaternion algebra Q_1.*
(ii) *The involution σ is hyperbolic if and only if it admits $\{0, [A]\}$ as a decomposition group.*
(iii) *The algebra with involution (A, σ) is split and hyperbolic if and only if it admits $\{0\}$ as a decomposition group.*

Proof. Assertion (iii) is clear from the definition of a decomposition group, since $(M_2(F), \overline{\ }) \otimes (M_2(F), \overline{\ })$ is hyperbolic. For (i) and (ii), the direct implications immediately follow from the previous examples. To prove the converse, let us first assume (A, σ) admits $\{0, [A]\}$ as a decomposition group. Since this group has order 1 or 2, A cannot have index 4 by Example 3.9. Therefore, it is Brauer-equivalent to a quaternion algebra Q. Moreover, by definition,

$$(A, \sigma) \in \boxplus_{i=1}^{3}\big((Q, \overline{\ }) \otimes (M_2(F), \overline{\ })\big).$$

Since each summand is hyperbolic, this proves σ is hyperbolic.
Assume now that U is generated by $[A]$ and $[Q_1]$. The order of U then divides 4, hence by Example 3.9, the algebra A is Brauer-equivalent to some quaternion algebra Q. Thus $U = \{0, [Q], [Q_1], [H_1]\}$, with $M_2(Q) \simeq Q_1 \otimes H_1$. If $[Q_1] = 0$ or $[H_1] = 0$, then $U = \{0, [Q]\}$, and by the previous case, σ is hyperbolic. Assume now that both H_1 and Q_1 are non split. We get

$$(A, \sigma) \in \boxplus_{i=1}^{3}(Q_i, \overline{\ }) \otimes (H_i, \overline{\ }),$$

with for $i = 2, 3$ $\{[Q_i], [H_i]\}$ equal either to $\{0, [Q]\}$ or to $\{[Q_1], [H_1]\}$. Picking an arbitrary element in $\{[H_1], [Q_1]\}$ for $1 \leq i \leq 3$, we get three quaternion algebras whose sum is never 0. Therefore, since by Corollary 3.3 we have $[H_1] + [H_2] + [H_3] = 0$, at least one summand must be $(Q, \overline{\ }) \otimes (M_2(F), \overline{\ })$, and this proves σ is isotropic. □

Remark 3.11. Reversing the viewpoint, note that any subgroup $U \subset {}_2\operatorname{Br}(F)$ of order 8 in which all the nonzero elements except at most one have index 2 is the decomposition group of some central simple algebra of degree 12 with orthogonal involution of trivial discriminant and trivial Clifford invariant. If all the nonzero elements in U have index 2, pick a quaternion algebra D representing a nonzero element in U; otherwise, let D be the division algebra such that $[D] \in U$ and $\operatorname{ind} D > 2$. In each case, we may organize the other nonzero elements in U in pairs $[Q_i]$, $[H_i]$ such that $[D] = [Q_i] + [H_i]$ for $i = 1, 2, 3$, and $\sum_{i=1}^{3}[H_i] = 0$. Any algebra with involution (A, σ) in $\boxplus_{i=1}^{3}\big((Q_i, \overline{\ }) \otimes (H_i, \overline{\ })\big)$ has decomposition group U. Modifying the scalars in the direct sum leads to several nonisomorphic such (A, σ). Moreover, when all the nonzero elements in U have index 2, we may select for D various quaternion algebras, and thus obtain various (A, σ) that are not Brauer-equivalent. Similarly, any subgroup $U \subset {}_2\operatorname{Br}(F)$ of order 4 containing at most one element $[D]$ with $\operatorname{ind} D > 2$, and any subgroup $\{0, [Q]\}$ where Q is a quaternion algebra, is the decomposition group of some central simple algebra of degree 12 endowed with an isotropic orthogonal involution of trivial discriminant and trivial Clifford invariant.

The decomposition groups of (A, σ) are subgroups of the Brauer group generated by at most three quaternion algebras. Those subgroups were considered by Peyre in [32]. His results will prove useful to study degree 12 algebras with involution. For the reader's convenience, we recall them in the next section.

3.3. A complex of Peyre. Let F be an arbitrary field, and let $U \subset \operatorname{Br} F$ be a finite subgroup of the Brauer group of F. We let $F^\times \cdot U$ denote the subgroup of $H^3(F)$ generated by classes $\lambda \cdot \alpha$, with $\lambda \in F^\times$ and $\alpha \in U$; any element in $F^\times \cdot U$ can be written as $\sum_{i=1}^r \lambda_i \cdot \alpha_i$ for some $\lambda_i \in F^\times$, where $\alpha_1, \ldots, \alpha_r$ is a generating set for the group U. Let F_U be the function field of the product of the Severi–Brauer varieties associated to elements of U. Clearly, F_U splits all the elements of U, hence the subgroup $F^\times \cdot U$ vanishes after scalar extension to F_U. Therefore, the following sequence is a complex, which was first introduced and studied by Peyre in [32, §4]:

$$F^\times \cdot U \to H^3(F) \to H^3(F_U).$$

We let $\mathcal{H}_U$ denote the corresponding homology group, that is

$$\mathcal{H}_U = \frac{\ker(H^3(F) \to H^3(F_U))}{F^\times \cdot U}.$$

We now return to our standing hypothesis that the characteristic of F is different from 2. Peyre considers in particular subgroups $U \subset \operatorname{Br} F$ generated by the Brauer classes of at most three quaternion algebras, and proves:

Theorem 3.12 (Peyre [32, Thm 5.1]). *If U is generated by the Brauer classes of two quaternion algebras, then $\mathcal{H}_U = 0$.*

In the next section, we need to consider only subgroups U such that all the elements of U are quaternion algebras; we call them *quaternionic subgroups* of the Brauer group. These subgroups have also been investigated by Sivatski [38]. We have:

Theorem 3.13 ([32, Prop. 6.1], [38, Cor. 11]). *If $U \subset \operatorname{Br} F$ is generated by the Brauer classes of three quaternion algebras, then $\mathcal{H}_U = 0$ or $\mathbb{Z}/2\mathbb{Z}$. Assume in addition U is quaternionic. Then the following conditions are equivalent:*

(a) *$\mathcal{H}_U = 0$;*
(b) *U is split by an extension of F of degree $2m$ for some odd m;*
(c) *U is split by a quadratic extension of F.*

The result that $\mathcal{H}_U = 0$ or $\mathbb{Z}/2\mathbb{Z}$ and the equivalence (a) $\iff$ (b) are due to Peyre [32, Prop. 6.1]. The equivalence (b) $\iff$ (c) was proved by Sivatski [38, Cor. 11].

We say that an extension K of F *splits* a subgroup $U \subset \operatorname{Br} F$ if it splits all the elements in U. If K splits a decomposition group of a central simple algebra with orthogonal involution (A, σ), then A_K is split because $[A] \in U$, and σ_K is hyperbolic by Lemma 3.10(iii) because (A_K, σ_K) has a trivial decomposition group. Therefore, Theorem 3.13 is relevant for the quadratic splitting of (A, σ), as we will see in § 5.2.

4. The Arason invariant and the homology of Peyre's complex

As in the previous section, (A, σ) is a degree 12 algebra with orthogonal involution of trivial discriminant and trivial Clifford invariant. From now on, we assume in addition that the Arason invariant $e_3(\sigma)$ is well defined. So the algebra A has even co-index, hence index 1 or 2. Under this assumption, any decomposition group of (A, σ) is quaternionic, that is consists only of Brauer classes of quaternion algebras. In this section, we relate the decomposition groups of (A, σ) with the values of the Arason invariant $e_3(\sigma)$. Reversing the viewpoint we then explain how one can use the Arason invariant to find explicit generators of the homology group $\mathcal{H}_U$ of Peyre's complex, for any quaternionic subgroup $U \subset \mathrm{Br}(F)$ of order dividing 8.

4.1. Arason invariant in degree 12.

For orthogonal involutions on a degree 12 algebra, isotropy and hyperbolicity can be detected via the Arason invariant as follows:

Theorem 4.1. *Let (A, σ) be a degree 12 and index 1 or 2 algebra with orthogonal involution of trivial discriminant and trivial Clifford invariant.*

(i) *The involution σ is hyperbolic if and only if $e_3(\sigma) = 0 \in M_A^3(F)$.*
(ii) *The involution σ is isotropic if and only if*

$$e_3(\sigma) = e_3(\pi) + F^\times \cdot [A] \in M_A^3(F)$$

for some 3-fold Pfister form π, i.e. $f_3(\sigma) = 0$ and $e_3(\sigma)$ is represented by a symbol.

Proof. Assume first that A is split, so that σ is adjoint to a 12-dimensional quadratic form φ, and $e_3(\sigma) = e_3(\varphi) \in {}_2H^3(F)$. Since the Arason invariant for quadratic forms has kernel the 4th power I^4F of the fundamental ideal of the Witt ring $W(F)$, the first equivalence follows from the Arason–Pfister Hauptsatz.

To prove (ii), note that there is no 10-dimensional anisotropic quadratic form in I^3F, see [28, Prop. XII.2.8]. So if φ is isotropic, then it has two hyperbolic planes, and it is Witt-equivalent to a multiple of some 3-fold Pfister form π. Hence, $e_3(\varphi) = e_3(\pi)$. Assume conversely that $e_3(\varphi) = e_3(\pi)$. By condition (i), φ becomes hyperbolic over the function field of π. Therefore, by [28, Th. X.4.11], the anisotropic kernel of φ is a multiple of π. In view of the dimensions, this implies $\varphi = \langle\alpha\rangle\pi + 2\mathbb{H}$ for some $\alpha \in F^\times$. In particular, φ is isotropic.

Assume now $A = M_6(Q)$ for some quaternion division algebra Q. By a result of Dejaiffe [11] and of Parimala–Sridharan–Suresh [31, Prop 3.3], the involution σ is hyperbolic if and only if it is hyperbolic after scalar extension to a generic splitting field F_Q of the quaternion algebra Q. Since the restriction map $M_Q^3(F) \to H^3(F_Q)$ is injective, the split case gives the result in index 2. If (A, σ) is isotropic, it is Witt-equivalent to a degree 8 and index 2 algebra with involution. The explicit description of the Arason invariant in degree 8 given in [35, Th. 5.2] shows it is equal to $e_3(\pi) \mod F^\times \cdot [Q] \in M_Q^3(F)$ for some

3-fold Pfister form π. Assume conversely that $e_3(\sigma) = e_3(\pi) + F^\times \cdot [Q]$. After scalar extension to F_Q, the split case shows σ_{F_Q} is isotropic. By Parimala–Sridharan–Suresh [31, Cor. 3.4], this implies σ itself is isotropic. □

Remark 4.2. In the split isotropic case, the involution can be explicitly described from its Arason invariant: the proof of Theorem 4.1(ii) shows that σ is adjoint to $\pi + 2\mathbb{H}$ if $e_3(\sigma) = e_3(\pi) \in {}_2H^3(F)$. In index 2, we also get an explicit description of (A, σ) in the isotropic case. Indeed, we have

$$(A, \sigma) = (M_2(Q), \mathrm{hyp}) \boxplus (M_4(Q), \sigma_0)$$

for some orthogonal involution σ_0 with trivial discriminant and trivial Clifford invariant, and $e_3(\sigma) = e_3(\sigma_0)$. If (a, b, c) is a symbol representing $e_3(\sigma)$, then by [35, Th. 5.2] we may assume that one of the slots, say a, is such that $F(\sqrt{a})$ splits Q, hence Q carries an orthogonal involution ρ with discriminant a. Theorem 5.2 of [35] further shows that

$$(M_4(Q), \sigma_0) \simeq (Q, \rho) \otimes \mathrm{Ad}_{\langle\langle b,c\rangle\rangle}\,.$$

Under some additional condition, we also have the following classification result:

Proposition 4.3. *Let $A = M_6(Q)$ be a degree 12 algebra of index at most 2, and let σ and σ' be two orthogonal involutions with trivial discriminant and trivial Clifford invariant. We assume either A is split, or σ is isotropic. The involutions σ and σ' are isomorphic if and only if $e_3(\sigma) = e_3(\sigma')$.*

Proof. It is already known that two isomorphic involutions have the same Arason invariant, so we only need to prove the converse. Assume first that A has index 2, in which case we assume in addition that σ is isotropic. Since $e_3(\sigma) = e_3(\sigma')$, by Theorem 4.1, the involution σ' also is isotropic. The result then follows from the explicit description given in Remark 4.2, or equivalently from [35, Cor. 5.3(2)], which shows that the anisotropic parts of σ and σ' are isomorphic.

Assume now A is split, and σ and σ' are adjoint to φ and φ' respectively. We have $e_3(\varphi) = e_3(\varphi')$. If there exists a 3-fold Pfister form π such that $e_3(\varphi) = e_3(\varphi') = e_3(\pi)$, then φ and φ' are both similar to $\pi + 2\mathbb{H}$. Otherwise, they are anisotropic, and the result in this case follows by combining Pfister's theorem (see for instance [22, Th. 8.1.1]), which asserts that φ and φ' can be decomposed as tensor products of a 1-fold Pfister form and an Albert form, with Hoffmann's result [19, Corollary], which precisely says that two such forms are similar if and only if their difference is in I^4F.

□

4.2. Arason invariant and decomposition groups. Recall from Example 3.7 that the decomposition groups corresponding to additive decompositions of (A, σ) are quaternionic subgroups of order at most 4 when A is split. Hence, by Peyre's Theorem 3.12, the corresponding homology group is trivial, $\mathcal{H}_U = 0$. Using this, we have:

PROPOSITION 4.4. *Let φ be a 12-dimensional quadratic form in I^3F, and let $U = \{0, [Q_1], [Q_2], [Q_3]\} \subset \mathrm{Br}(F)$ be a quaternionic subgroup of order at most 4. For $i = 1, 2, 3$, let n_i be the norm form of Q_i. The following are equivalent:*

(a) *There exists $\alpha_1, \alpha_2, \alpha_3 \in F^\times$ such that $\varphi = \langle\alpha_1\rangle n_1 \perp \langle\alpha_2\rangle n_2 \perp \langle\alpha_3\rangle n_3$;*
(b) *U is a decomposition group of Ad_φ;*
(c) *φ is hyperbolic over F_U;*
(d) *$e_3(\varphi) \in \ker\big(H^3(F) \to H^3(F_U)\big)$;*
(e) *$e_3(\varphi) \in F^\times \cdot U$.*

Proof. The equivalence between (a) and (b) follows from Example 3.4. Assume φ decomposes as in (a). Since the field F_U splits all three quaternion algebras Q_i, hence also their norm forms n_i, the form φ is hyperbolic over F_U, hence assertions (c) and (d) hold. By Peyre's result 3.12, we also get (e), and it only remains to prove that (e) implies (a).
Thus, assume now that $e_3(\varphi) \in F^\times \cdot U$. Since the subgroup U is generated by $[Q_1]$ and $[Q_2]$, there exists λ_1 and $\lambda_2 \in F^\times$ such that

$$e_3(\varphi) = (\lambda_1) \cdot [Q_1] + (\lambda_2) \cdot [Q_2].$$

The product $Q_1 \otimes Q_2 \otimes Q_3$ is split, so by the common slot lemma ([28, Th. III.4.13]), we may assume $Q_i = (a, b_i)_F$ for some a and $b_i \in F^\times$. A direct computation then shows that $n_1 - n_2 = \langle b_2\rangle n_3$. Hence the 12-dimensional quadratic form $\langle -\lambda_1\rangle n_1 + \langle\lambda_2\rangle n_2 + \langle b_2\rangle n_3$ is Witt-equivalent to $\langle 1, -\lambda_1\rangle n_1 + \langle -1\rangle\langle 1, -\lambda_2\rangle n_2$, which has the same Arason invariant as φ. By Proposition 4.3, this form is similar to φ, so that φ has an additive decomposition as required. □

Let us consider now the index 2 case. By Lemma 3.10, (A, σ) admits decomposition groups of order 4 if and only if it is isotropic. We prove:

PROPOSITION 4.5. *Let $A = M_6(Q)$ be an algebra of index ≤ 2, and consider an orthogonal involution σ on A, with trivial discriminant and trivial Clifford invariant. Pick a subgroup $U = \{0, [Q], [Q_1], [H_1]\} \subset \mathrm{Br}(F)$ containing the class of Q. The following are equivalent:*

(a) *(A, σ) admits an additive decomposition of the following type:*

$$(A, \sigma) \in \big((Q_1, \text{—}) \otimes (H_1, \text{—})\big) \boxplus \big((Q_1, \text{—}) \otimes (H_1, \text{—})\big) \boxplus \big((M_2(F), \text{—}) \otimes (Q, \text{—})\big);$$

(b) *U is a decomposition group of (A, σ);*
(c) *σ is hyperbolic over F_U;*
(d) *$e_3(\sigma) \in \ker\big(M_A^3(F) \to H^3(F_U)\big)$;*
(e) *There exists $\alpha \in F^\times$ such that*

$$e_3(\sigma) = (\alpha) \cdot [Q_1] \mod F^\times \cdot [Q] \in M_Q^3(F).$$

Proof. The proof follows the same line as for the previous proposition. By the definition of decomposition groups, (a) implies (b). Conversely, if (b) holds, then (A, σ) has an additive decomposition with summands isomorphic to $(Q_1, \text{—}) \otimes (H_1, \text{—})$ or to $(M_2(F), \text{—}) \otimes (Q, \text{—})$. If Q_1 or H_1 is split, then the two kinds of summands are isomorphic, hence (a) holds. If Q_1 and H_1 are not

split, then the number of summands isomorphic to $(Q_1, \bar{\ }) \otimes (H_1, \bar{\ })$ must be even because $e_2(\sigma) = 0$ (see Proposition 3.1), and it must be nonzero because U is the corresponding decomposition group. Therefore, (a) holds.
Now, assume (A, σ) satisfies (a). Since the field F_U splits Q, Q_1 and H_1, (c) holds. Assertion (d) follows since hyperbolic involutions have trivial Arason invariant. By Peyre's Proposition 3.12, we deduce assertion (e), and it only remains to prove that (e) implies (a). Hence, assume

$$e_3(\sigma) = (\alpha) \cdot [Q_1] \mod F^\times \cdot [Q] \in M_Q^3(F),$$

for some $\alpha \in F^\times$ and some quaternion algebra Q_1. By Theorem 4.1(ii), the involution σ is isotropic. Hence, in view of Proposition 4.3, it is enough to find an involution σ' satisfying (a) and having $e_3(\sigma') = (\alpha) \cdot [Q_1] \mod F^\times \cdot [Q]$. Since $Q \otimes Q_1 = H_1$ has index 2, the quaternion algebras Q and Q_1 have a common slot (see [28, Th. III.4.13]). Therefore, there exists a, b, $b_1 \in F^\times$ such that $Q = (a, b)$ and $Q_1 = (a, b_1)$. Let ρ be an orthogonal involution on Q with discriminant a, and let

$$(A, \sigma') = (M_2(Q), \mathrm{hyp}) \boxplus \big((Q, \rho) \otimes \mathrm{Ad}_{\langle\langle b_1, \alpha \rangle\rangle}\big).$$

One component of the Clifford algebra of $(Q, \rho) \otimes \mathrm{Ad}_{\langle\langle b_1 \rangle\rangle}$ is given by the cup product of the discriminants of ρ and $\mathrm{ad}_{\langle\langle b_1 \rangle\rangle}$, that is $(a, b_1) = Q_1$. By Remark 4.2, it follows that $e_3(\sigma') = e_3(\sigma)$, hence $(A, \sigma) \simeq (A, \sigma')$ by Proposition 4.3. Therefore (A, σ) satisfies (a) as required. □

4.3. Generators of the homology $\mathcal{H}_U$ of Peyre's complex. Let $A = M_6(Q)$ for some quaternion F-algebra Q, and let σ be an orthogonal involution on A with trivial discriminant and trivial Clifford invariant. Consider an additive decomposition of (A, σ) as in Theorem 3.2,

$$(A, \sigma) \in \coprod_{i=1}^{3} \big((Q_i, \bar{\ }) \otimes (H_i, \bar{\ })\big),$$

and let U be the corresponding decomposition group, which is a quaternionic subgroup of $\mathrm{Br}(F)$,

$$U = \{0, [Q], [Q_1], [H_1], [Q_2], [H_2], [Q_3], [H_3]\}.$$

Since $F^\times \cdot [Q] \subset F^\times \cdot U$, we may consider the canonical map

$$\bar{\ }^U \colon M_Q^3(F) \to H^3(F)/F^\times \cdot U.$$

As in §3.3, let F_U be the function field of the product of the Severi–Brauer varieties associated to elements of U. Since F_U splits U, Lemma 3.10 shows that A_{F_U} is split and σ_{F_U} is hyperbolic, hence $e_3(\sigma)_{F_U} = 0$. Therefore, $\overline{e_3(\sigma)}^U$ lies in the homology $\mathcal{H}_U$ of Peyre's complex.
As explained in Remark 3.11, for any quaternionic subgroup $U \subset \mathrm{Br}(F)$ of order dividing 8, we may find algebras with involution (A, σ) for which U is a decomposition group. The main result of this section is:

THEOREM 4.6. *Let U be a quaternionic subgroup of* $\operatorname{Br} F$ *of order dividing* 8. *For any (A,σ) admitting U as a decomposition group, the class of the Arason invariant $\overline{e_3(\sigma)}^U$ is a generator of the homology group $\mathcal{H}_U$ of Peyre's complex.*

The main tool in the proof is the following proposition:

PROPOSITION 4.7. *Let U be a quaternionic subgroup of* $\operatorname{Br} F$ *of order dividing* 8*, and pick an algebra $A = M_6(Q)$ with orthogonal involution σ, admitting U as a decomposition group.*

(i) *For all involutions σ' on A such that (A,σ') also admits U as a decomposition group, we have*
$$\overline{e_3(\sigma')}^U = \overline{e_3(\sigma)}^U.$$
(ii) *Conversely, for all $\xi \in M_A^3(F)$ such that $\overline{\xi}^U = \overline{e_3(\sigma)}^U$, there exists an involution σ' on A such that U is a decomposition group of (A,σ') and $e_3(\sigma') = \xi \mod F^\times \cdot [A]$.*
(iii) *There exists a hyperbolic involution σ' on A admitting U as a decomposition group if and only if $\overline{e_3(\sigma)}^U = 0$.*

Proof. (i) Since U is a decomposition group of (A,σ) and (A,σ'), we have
$$(A,\sigma) \text{ and } (A,\sigma') \in \coprod_{i=1}^{3} \big((Q_i,\bar{\ }) \otimes (H_i,\bar{\ })\big).$$
Therefore, σ and σ' are adjoint to some skew-hermitian forms h and h' over $(Q,\bar{\ })$ satisfying
$$h = h_1 \perp h_2 \perp h_3 \text{ and } h' = \langle\alpha_1\rangle h_1 \perp \langle\alpha_2\rangle h_2 \perp \langle\alpha_3\rangle h_3,$$
for some h_i such that $\operatorname{ad}_{h_i} \simeq (Q_i,\bar{\ }) \otimes (H_i,\bar{\ })$, and some $\alpha_i \in F^\times$. Therefore,
$$e_3(\sigma) - e_3(\sigma') = e_3(\perp_{i=1}^3 \langle 1, -\alpha_i\rangle h_i).$$
Since h_i has discriminant 1, Proposition 2.6 applies to each summand and shows $\langle 1,-\alpha_i\rangle \otimes h_i$ has trivial discriminant and trivial Clifford invariant, and
$$e_3(\langle 1,-\alpha_i\rangle \otimes h_i) = \alpha_i \cdot e_2(h_i) = \alpha_i \cdot [Q_i] \in M_Q^3(F).$$
Therefore, $e_3(\sigma) - e_3(\sigma')$ is represented modulo $F^\times \cdot \{0,[Q]\}$ by $\sum_{i=1}^3 \alpha_i \cdot [Q_i]$. Since this element lies in $F^\times \cdot U$, we have $\overline{e_3(\sigma)}^U = \overline{e_3(\sigma')}^U$.
(ii) Consider a skew hermitian form h over $(Q,\bar{\ })$ such that $\sigma = \operatorname{ad}_h$, and a decomposition $h = h_1 \perp h_2 \perp h_3$ as in the proof of (i). Since $\overline{\xi}^U = \overline{e_3(\sigma)}^U$, the difference $e_3(\sigma) - \xi \in M_Q^3(F)$ is represented by a cohomology class of the form $\sum_{i=1}^3 \alpha_i \cdot [Q_i]$ for some $\alpha_i \in F^\times$. The computation in (i) shows that $e_3(\operatorname{ad}_{h'}) = \xi$ for $h' = \langle\alpha_1\rangle h_1 \perp \langle\alpha_2\rangle h_2 \perp \langle\alpha_3\rangle h_3$.
(iii) It follows from (ii) that $\overline{e_3(\sigma)}^U = 0$ if and only if there exists an involution σ' with decomposition group U and $e_3(\sigma') = 0$. Theorem 4.1(i) completes the proof by showing σ' is hyperbolic. □

With this in hand, we can now prove Theorem 4.6.

Proof of Theorem 4.6. Since $\mathcal{H}_U$ is either 0 or $\mathbb{Z}/2\mathbb{Z}$, in order to prove that $\overline{e_3(\sigma)}^U$ generates $\mathcal{H}_U$ it is enough to prove that $\mathcal{H}_U$ is trivial as soon as $\overline{e_3(\sigma)}^U = 0$. If U has order at most 4, then $\mathcal{H}_U$ is trivial by Theorem 3.12. Hence, let us assume U has order 8, and $\overline{e_3(\sigma)}^U = 0$. By Proposition 4.7, replacing σ by σ', we may assume σ is hyperbolic. Recall σ is adjoint to a skew-hermitian form h, which admits a decomposition $h = h_1 \perp h_2 \perp h_3$ with $\mathrm{Ad}_{h_i} = (Q_i, \overline{\ })\otimes(H_i, \overline{\ })$. Since U has order 8, each summand h_i is anisotropic. The hyperbolicity of h says $h_1 \perp h_2 \simeq -h_3 \perp \mathbb{H}$ is isotropic. Therefore, there exists a pure quaternion q such that h_1 and h_2 represent q and $-q$ respectively. Over the quadratic extension $F(q)$ of F, the involutions ad_{h_1} and ad_{h_2} are isotropic. Since they are adjoint to 2-fold Pfister forms, they are hyperbolic. Hence $F(q)$ splits the Clifford algebra of h_1 and h_2, that is the quaternion algebras Q_1, Q_2. Since the Brauer classes of Q, Q_1 and Q_2 generate U, it follows that $F(q)$ is a quadratic splitting field of U. By Peyre's Theorem 3.13, we get $\mathcal{H}_U = 0$ as required. □

5. Quadratic splitting and the f_3 invariant

The f_3 invariant of an involution σ vanishes if the underlying algebra A is split, or of degree ≤ 10. We keep focusing on the case of degree 12 algebras, where we have explicit examples with $f_3(\sigma) \neq 0$, see Corollary 2.19. Thus, as in § 4, (A, σ) is a degree 12 algebra with orthogonal involution of trivial discriminant and trivial Clifford invariant for which the Arason and the f_3 invariants are defined. In particular, A has index at most 2.

Our first goal is to characterize the vanishing of $f_3(\sigma)$; this is done in Proposition 5.6 below. As pointed out in Proposition 2.5, $f_3(\sigma)$ vanishes if there exists a quadratic extension K/F over which (A, σ) is split and hyperbolic. Note that since A is Brauer-equivalent to a quaternion algebra, there exist quadratic extensions of the base field F over which A is split. Moreover, using the additive decompositions of Corollary 3.3, one may easily find quadratic extensions of the base field over which the involution is hyperbolic: it suffices to consider a common subfield of the quaternion algebras H_1, H_2, H_3, which exists by [28, Th. III.4.13] since $[H_1] + [H_2] + [H_3] = 0$. Yet, we give in Corollary 5.13 examples showing that the converse of Proposition 2.5 does not hold in degree 12: we may have $f_3(\sigma) = 0$ even when there is no quadratic extension that simultaneously splits A and makes σ hyperbolic.

First, we use quadratic forms to introduce an invariant of quaternionic subgroups of the Brauer group of F, which, as we next prove, coincides with the f_3-invariant of involutions admitting this subgroup as a decomposition group.

5.1. The invariants $f_3(U)$ and $f_3(\sigma)$.

To any quaternionic subgroup U of $\mathrm{Br}(F)$, we may associate in a natural way a quadratic form n_U by taking the sum of the norm forms n_H of the quaternion algebras H with Brauer class in U. We have:

LEMMA 5.1. *Let U be a quaternionic subgroup of* $\operatorname{Br} F$ *generated by the Brauer classes of three quaternion algebras. The quadratic form* $n_U = \sum_{[H]\in U} n_H$ *satisfies* $n_U \in I^3F$.

Proof. Pick three generators $[Q_1]$, $[Q_2]$ and $[Q_3]$ of U, and let H_1, H_2, H_3, Q be quaternion algebras with Brauer classes $[H_1] = [Q_2] + [Q_3]$, $[H_2] = [Q_1] + [Q_3]$, $[H_3] = [Q_1] + [Q_2]$, and $[Q] = [Q_1] + [Q_2] + [Q_3]$. We have $[H_1] + [H_2] + [H_3] = 0$, and

$$U = \{0, [Q], [Q_1], [H_1], [Q_2], [H_2], [Q_3], [H_3]\}.$$

Since the difference $n_{Q_i} - n_{H_i}$ is Witt-equivalent to an Albert form of $Q_i \otimes H_i$, which is Brauer-equivalent to Q, there exists $\lambda_i \in F^\times$ such that in the Witt group of F, we have $n_{Q_i} - n_{H_i} = \langle\lambda_i\rangle n_Q \in WF$. Therefore,

$$n_U = \langle 1, \lambda_1, \lambda_2, \lambda_3\rangle n_Q + \langle 1, 1\rangle(n_{H_1} + n_{H_2} + n_{H_3}). \tag{2}$$

Since the right side is in I^3F, the lemma is proved. □

In view of Lemma 5.1, we may associate to U a cohomology class of degree 3 as follows:

DEFINITION 5.2. For any quaternionic subgroup U generated by three elements, we let $f_3(U)$ be the Arason invariant of the quadratic form n_U:

$$f_3(U) = e_3(n_U) \in {}_2H^3(F).$$

We may easily compute $f_3(U)$ from formula (2): Since $[H_1] + [H_2] + [H_3] = 0$, we have $n_{H_1} + n_{H_2} + n_{H_3} \in I^3F$, hence $\langle 1, 1\rangle(n_{H_1} + n_{H_2} + n_{H_3}) \in I^4F$ and therefore

$$f_3(U) = (\lambda_1\lambda_2\lambda_3) \cdot [Q]. \tag{3}$$

With this in hand, we get:

PROPOSITION 5.3. *If $\mathcal{H}_U = 0$, then $f_3(U) = 0$.*

Proof. By Theorem 3.13, if $\mathcal{H}_U = 0$ then U admits a quadratic splitting field, i.e. the generators of U have a common quadratic subfield. So there exist a, b_1, b_2, and $b_3 \in F^\times$ such that $Q_i = (a, b_i)_F$ for $i = 1, 2, 3$. Thus, we have $H_1 = (a, b_2b_3)_F$ and

$$n_{Q_1} - n_{H_1} = \langle\langle a\rangle\rangle(\langle\langle b_1\rangle\rangle - \langle\langle b_2b_3\rangle\rangle) = \langle\langle a\rangle\rangle\langle -b_1, b_2b_3\rangle = \langle -b_1\rangle n_Q.$$

Similar formulas hold for $i = 2, 3$, and we get

$$f_3(U) = (-b_1b_2b_3) \cdot Q = (-b_1b_2b_3, a, b_1b_2b_3) = 0 \in {}_2H^3(F).$$

□

In [38], Sivatski asks about the converse[1], that is: if $f_3(U) = 0$, does the homology group $\mathcal{H}_U$ vanish, or equivalently by Peyre's Theorem 3.13, do the generators Q_1, Q_2, and Q_3 of the group U have a common quadratic subfield? Corollary 5.11 below shows that this is not the case.

[1]Sivatski's invariant has a different definition, but one may easily check the quadratic form he considers is equivalent to n_U modulo I^4F.

The relation between $f_3(U)$ and the f_3-invariant for involutions is given by the following:

Theorem 5.4. *Let (A,σ) be a central simple algebra of degree 12 and index ≤ 2, with orthogonal involution of trivial discriminant and trivial Clifford invariant. Let U be a quaternionic subgroup of the Brauer group, generated by three elements. If U is a decomposition group for (A,σ) then $f_3(\sigma) = f_3(U)$.*

Remark 5.5. (i) It follows that any two decomposition groups of a given algebra with involution have the same f_3-invariant, and any two algebras with involution having U as a decomposition group have the same f_3-invariant.
(ii) Let c be a generator of $\mathcal{H}_U$, and pick an arbitrary (A,σ) having U as a decomposition group. In view of Theorem 4.6, we have $e_3(\sigma) = c \mod F^\times \cdot U$. Hence $f_3(U) = f_3(\sigma) = 2c \in {}_2H^3(F)$. In particular, $f_3(U) = 0$ if and only if the homology group $\mathcal{H}_U$ is generated by a cohomology class of order 2.

Proof of Theorem 5.4. The result follows from the computation of $f_3(\sigma)$ in Proposition 2.7 and the computation of $f_3(U)$ in (3). We use the same notation as in Definition 3.6, and we let h_i be a rank 2 skew-hermitian form over $(Q,\bar{\ })$ such that

$$\mathrm{Ad}_{h_i} \simeq (Q_i,\bar{\ }) \otimes (H_i,\bar{\ }) \text{ and } \sigma = \mathrm{ad}_{h_1 \perp h_2 \perp h_3}\,.$$

For $i = 1, 2, 3$, let $q_i \in Q$ be a nonzero pure quaternion represented by h_i, and let $a_i = q_i^2 \in F^\times$. Let also $b_i \in F^\times$ be such that $Q = (a_i, b_i)_F$. Scalar extension to $F(q_i)$ makes h_i isotropic, hence hyperbolic since the discriminant of h_i is trivial. Therefore, we have $h_i \simeq \langle q_i\rangle\langle 1, -\lambda_i\rangle$ for some $\lambda_i \in F^\times$. The two components of the Clifford algebra of Ad_{h_i} are $(a_i,\lambda_i)_F$ and $(a_i,\lambda_i b_i)_F$, therefore

$$\{Q_i, H_i\} = \{(a_i,\lambda_i)_F, (a_i,\lambda_i b_i)_F\} \qquad \text{for } i = 1, 2, 3.$$

Since Q contains a pure quaternion which anticommutes with q_i and with square b_i, the form h_i is isomorphic to $\langle q_i\rangle\langle 1, -\lambda_i b_i\rangle$ for $i = 1, 2, 3$. Replacing some λ_i by $\lambda_i b_i$ if necessary, we may assume $H_i = (a_i,\lambda_i)_F$ for all i. Since $[H_1] + [H_2] + [H_3] = 0$, we get $\sum_{i=1}^3 (a_i,\lambda_i)_F = 0$. By Proposition 2.7 this implies $f_3(\sigma) = \lambda_1\lambda_2\lambda_3 \cdot [Q]$. On the other hand, since $n_{Q_i} - n_{H_i} = \langle\!\langle a_i, \lambda_i b_i\rangle\!\rangle - \langle\!\langle a_i,\lambda_i\rangle\!\rangle = \langle\lambda_i\rangle n_Q$, we have $f_3(U) = \lambda_1\lambda_2\lambda_2 \cdot [Q]$ by (3). □

5.2. Quadratic splitting, the f_3 invariant, and decomposition groups. By using Theorem 5.4 and Peyre's Theorem 3.13, we can now translate in terms of decomposition groups the two conditions we want to compare, as follows:

Proposition 5.6. *Let (A,σ) be a degree 12 and index ≤ 2 algebra with orthogonal involution of trivial discriminant and trivial Clifford invariant. The following conditions are equivalent:*

(a) *$f_3(\sigma) = 0$;*
(b) *(A,σ) has a decomposition group U with $f_3(U) = 0$;*
(c) *$f_3(U) = 0$ for all decomposition groups U of (A,σ).*

Likewise, the following conditions are equivalent:

(a') *there exists a quadratic extension K of F such that A_K is split and σ_K is hyperbolic;*

(b') *(A,σ) has a decomposition group U with $\mathcal{H}_U = 0$.*

Moreover, any of the conditions (a'), (b') *implies the equivalent conditions* (a), (b), (c).

Proof. The equivalence between conditions (a), (b), (c) follows directly from Theorem 5.4. Moreover, they can be deduced from (a'), (b') by Proposition 5.3 or Proposition 2.5. Hence, it only remains to prove that (a') and (b') are equivalent.

Assume first that (A,σ) has a decomposition group U with $\mathcal{H}_U = 0$. By Peyre's characterization of the vanishing of $\mathcal{H}_U$ for quaternionic groups, recalled in Theorem 3.13, U is split by a quadratic extension K of F. Hence, (A_K,σ_K) admits $\{0\}$ as a decomposition group. By Lemma 3.10, this implies (A_K,σ_K) is split and hyperbolic.

To prove the converse, let us assume there exists a quadratic field extension $K = F(d)$, with $d^2 = \delta \in F^\times$, such that A_K is split and σ_K is hyperbolic. If A is split, as explained in Example 3.7, all decomposition subgroups U of (A,σ) have order dividing 4, and therefore satisfy $\mathcal{H}_U = 0$ by Peyre's Theorem 3.12. Assume next $\operatorname{ind} A = 2$. Since A_K is split, we may identify $K = F(d)$ with a subfield of the quaternion division algebra Q Brauer-equivalent to A, and thus consider d as a pure quaternion in Q. Let h be a skew-hermitian form over $(Q,\overline{\ })$ such that $\sigma = \mathrm{ad}_h$. Since h_K is hyperbolic, it follows from [34, Prop., p. 382] that $h \simeq \langle d\rangle\varphi_0$ for some 6-dimensional quadratic form φ_0 over F. Decompose

$$\varphi_0 = \langle\alpha_1\rangle\langle 1,-\beta_1\rangle \perp \langle\alpha_2\rangle\langle 1,-\beta_2\rangle \perp \langle\alpha_3\rangle\langle 1,-\beta_3\rangle \qquad \text{for some } \alpha_i,\ \beta_i \in F^\times,$$

and let $Q_i = (\delta,\beta_i)_F$ be the quaternion F-algebra with norm $n_{Q_i} = \langle\!\langle \delta,\beta_i\rangle\!\rangle$ for $i = 1, 2, 3$. Computation shows that $e_2\big(\langle\alpha_i d\rangle\langle 1,-\beta_i\rangle\big)$ is represented by Q_i in $M_Q^2(F)$, hence (A,σ) decomposes as

$$(A,\sigma) \in \coprod_{i=1}^{3} \mathrm{Ad}_{\langle\alpha_i d\rangle\langle 1,-\beta_i\rangle}\,.$$

So, the subgroup $U \subset \operatorname{Br} F$ generated by $[Q_1]$, $[Q_2]$ and $[Q_3]$ is a decomposition group for (A,σ). Again, U is split by K, hence $\mathcal{H}_U = 0$. □

5.3. TRIVIAL f_3-INVARIANT WITHOUT QUADRATIC SPLITTING. We now construct an algebra with involution (A,σ), of degree 12 and index 2, such that $f_3(\sigma) = 0$, and yet, there is no quadratic extension K of F over which (A,σ) is both split and hyperbolic. In particular, by Peyre's Theorem 3.13, we have $\mathcal{H}_U \neq 0$ for all decomposition groups U of (A,σ). (See Remark 5.14 for an example where (A,σ) has a decomposition group U whith $\mathcal{H}_U \neq 0$ and another U' with $\mathcal{H}_{U'} = 0$.)

Remark 5.7. In his paper [32, §6.2], Peyre provides an example of a quaternionic subgroup $U \subset \operatorname{Br}(F)$ with $\mathcal{H}_U \neq 0$, but the way he proves $\mathcal{H}_U$ is nonzero is by

describing an element $c \in H^3(F)$ which is not of order 2, hence does not belong to $F^\times \cdot U$, and yet is in the kernel of the restriction map $H^3(F) \to H^3(F_U)$. Thus, the group U in Peyre's example satisfies $f_3(U) \neq 0$ (see Remark 5.5). In this section, we construct an example of a different flavor, namely a subgroup U with $\mathcal{H}_U \neq 0$, but $f_3(U) = 0$. Hence, the homology group in this case is generated by a cohomology class which is of order 2, and in the kernel of $H^3(F) \to H^3(F_U)$, but does not belong to $F^\times \cdot U$.

Notation 5.8. Until the end of this section, k is a field (of characteristic different from 2), M is a triquadratic field extension of k (of degree 8) and K is a quadratic extension of k in M,

$$M = k(\sqrt{a}, \sqrt{b}, \sqrt{c}) \supset K = k(\sqrt{a}).$$

We let C be a central simple k-algebra of exponent 2 split by M and we write

$$[C] \in \mathrm{Dec}(M/k)$$

to express the property that there exist α, β, $\gamma \in k^\times$ such that

$$[C] = (a, \alpha)_k + (b, \beta)_k + (c, \gamma)_k.$$

The existence of algebras C as above such that $[C] \notin \mathrm{Dec}(M/k)$ is shown in [12, §5]. By contrast, it follows from a theorem of Albert that every central simple algebra of exponent 2 split by a biquadratic extension has a decomposition up to Brauer-equivalence into a tensor product of quaternion algebras adapted to the biquadratic extension (see [27, Prop. 5.2]), so (viewing M as $K(\sqrt{bc}, \sqrt{c})$) there exist x, $y \in K^\times$ such that

$$[C_K] = (bc, x)_K + (c, y)_K.$$

By multiplying x and y by squares in K, we may—and will—assume x, $y \notin k$. We have $\mathrm{cor}_{K/k}[C_K] = 2[C] = 0$, hence letting N denote the norm map from K to k, the previous equation leads, by the projection formula, to the following: $(bc, N(x))_k + (c, N(y))_k = 0$. We may then consider the following quaternion k-algebra:

$$H = (bc, N(x))_k = (c, N(y))_k. \tag{4}$$

Since $N(x)$, $N(y)$ are norms from $K = k(\sqrt{a})$ to k, we have $(a, N(x))_k = (a, N(y))_k = 0$, hence we may also write

$$H = (abc, N(x))_k = (ac, N(y))_k. \tag{5}$$

Let $B = (bc, x)_K \otimes_K (c, y)_K$ be the biquaternion algebra Brauer-equivalent to C_K, and let ψ be the Albert form of B over K defined by

$$\psi = \langle bc, x, -bcx, -c, -y, cy \rangle.$$

Let $s\colon K \to k$ be a nontrivial linear map such that $s(1) = 0$, and let $s_\star$ denote the corresponding Scharlau transfer. Using the properties of $s_\star$ (see for

instance [28, p. 189, p. 198]), we can make the following computation in the Witt group $W(k)$:

$$\begin{aligned} s_\star(\psi) =& s_\star(\langle x, -bcx, -y, cy\rangle) = s_\star\big(\langle x\rangle\big)\langle\!\langle bc\rangle\!\rangle - s_\star\big(\langle y\rangle\big)\langle\!\langle c\rangle\!\rangle \\ =& \langle s(x)\rangle\langle\!\langle bc, N(x)\rangle\!\rangle - \langle s(y)\rangle\langle\!\langle c, N(y)\rangle\!\rangle. \end{aligned}$$

(Recall that we assume x, $y \notin k$, so $s(x)$, $s(y) \neq 0$.) In view of (4), the last equation yields

$$s_\star(\psi) = \langle s(x), -s(y)\rangle n_H,$$

where n_H is the norm form of H. Thus, $s_\star(\psi) \in I^3(k)$, and we may consider

$$e_3(s_\star(\psi)) = s(x)s(y)\cdot[H] \in {}_2H^3(k). \tag{6}$$

This class represents an invariant of B defined by Barry [2]. It is shown in [2, Prop. 4.4] that $e_3(s_\star(\psi)) \in N(K^\times)\cdot[C]$ if and only if the biquaternion algebra B has a descent to k, i.e., there exist quaternion k-algebras A_1, A_2 such that $B \simeq A_1 \otimes_k A_2 \otimes_k K$.

Finally, let t be an indeterminate over k, and let $F = k(t)$. Consider the subgroup $U \subset \mathrm{Br}(F)$ generated by the Brauer classes $(a,t)_F$, $(b,t)_F$ and $(c,t)_F + [H_F]$. In view of (4) and (5), one may easily check that U is a quaternionic subgroup of order 8:

$$\begin{aligned} U = \{0, (a,t)_F, (b,t)_F, (c, N(y)t)_F,& \\ (ab,t)_F, (ac, N(y)t)_F, (bc, N(x)t)_F, (abc, N(x)t)_F&\}. \end{aligned}$$

We set

$$\xi = t\cdot[C] + e_3(s_\star(\psi)) \in {}_2H^3(F).$$

This construction yields examples with trivial f_3 but with no quadratic splitting mentioned in the introduction to this section, as we proceed to show. First, we describe the group $\mathcal{H}_U$ and give a criterion for its vanishing:

THEOREM 5.9. *Use the notation 5.8. Denote by* $\overline{\xi}^U \in H^3(F)/F^\times\cdot U$ *the image of* $\xi \in H^3(F)$. *We have*

$$\mathcal{H}_U = \{0, \overline{\xi}^U\} \qquad \text{and} \qquad f_3(U) = 0.$$

Moreover, the following conditions are equivalent:

(a) $[C] \in \mathrm{Dec}(M/k)$;
(b) U *is split by some quadratic field extension* E/F;
(c) $\mathcal{H}_U = 0$;
(d) $\xi \in F^\times\cdot U$.

The core of the proof is the following technical lemma:

LEMMA 5.10. *With the notation 5.8, every field extension of* F *that splits* U *also splits* ξ.

Proof. Let L be an extension of F that splits U. We consider two cases, depending on whether $a \in L^{\times 2}$ or $a \notin L^{\times 2}$. Suppose first $a \in L^{\times 2}$, so we may identify K with a subfield of L, hence x, $y \in L^\times$ and $[C_L] = (bc, x)_L + (c, y)_L$. Since L splits $(b, t)_F$, we have $(t, bc, x)_L = (t, c, x)_L$, hence $t \cdot [C_L] = xy \cdot (t, c)_L$. Since L also splits $(t, c)_F + [H_F]$, we have

$$t \cdot [C_L] = xy \cdot [H_L].$$

Comparing with (6), we see that it suffices to show $xy \cdot [H_L] = s(x)s(y) \cdot [H_L]$ to prove that L splits ξ.
Let ι be the nontrivial automorphism of K over k. Writing $x = x_0 + x_1\sqrt{a}$ and $y = y_0 + y_1\sqrt{a}$ with $x_i, y_i \in k$, we have

$$s(x)s(y) = x_1y_1s(\sqrt{a})^2 \text{ and } (x - \iota(x))(y - \iota(y)) = 4x_1y_1a.$$

Hence $s(x)s(y) \equiv (x - \iota(x))(y - \iota(y)) \bmod L^{\times 2}$. We also have

$$(x - \iota(x), N(x))_K = (x, N(x))_K \text{ because } (x^2 - N(x), N(x))_K = 0.$$

From the expression $H = (bc, N(x))_k$ it then follows that $x \cdot [H_K] = (x - \iota(x)) \cdot [H_K]$. Similarly, from $H = (c, N(y))_k$ we have $y \cdot [H_K] = (y - \iota(y)) \cdot [H_K]$, hence

$$xy \cdot [H_L] = s(x)s(y) \cdot [H_L].$$

Thus, we have proved L splits ξ under the additional hypothesis that $a \in L^{\times 2}$. For the rest of the proof of (i), assume $a \notin L^{\times 2}$. Let $L' = L(\sqrt{a}) = L \otimes_k K$, and write again $s\colon L' \to L$ for the L-linear extension of s to L' and $N\colon L' \to L$ for the norm map. If $t \in L^{\times 2}$, then $\xi_L = e_3(s_\star(\psi))_L$. Moreover, L splits H because it splits U. Therefore, (6) shows that L splits $e_3(s_\star(\psi))$. For the rest of the proof, we may thus also assume $t \notin L^{\times 2}$.
Since $(a, t)_L = 0$, we may find $z_0 \in L'$ such that $t = N(z_0)$. Because L splits $(b, t)_F$, we have $(b, N(z_0))_L = 0$, so $\operatorname{cor}_{L'/L}(b, z_0)_{L'} = 0$. It follows that $(b, z_0)_{L'}$ has an involution of the second kind, hence also a descent to L by a theorem of Albert (see [26, (2.22)]). We may choose a descent of the form $(b, z_0)_{L'} = (b, \zeta)_{L'}$ for some $\zeta \in L^\times$; see [40, (2.6)]. Let $z = z_0\zeta \in {L'}^\times$. We then have $(b, z)_{L'} = 0$, hence after taking the corestriction to L

$$(b, N(z))_L = 0.$$

We also have $t = N(z_0) \equiv N(z) \bmod L^{\times 2}$. Since L splits $[H_F] + (c, t)_F$, we have

$$H_L = (c, N(z))_L.$$

Since $H = (bc, N(x))_k = (c, N(y))_k$ by (4), it follows that

$$(bc, N(xz))_L = (c, N(yz))_L = 0.$$

If $s(xz) = 0$ (i.e., $xz \in L$), then $s_\star(\langle xz\rangle\langle\langle bc\rangle\rangle)$ is hyperbolic. If $s(xz) \neq 0$, computation yields

$$s_\star(\langle xz\rangle\langle\langle bc\rangle\rangle) = \langle s(xz)\rangle\langle\langle bc, N(xz)\rangle\rangle;$$

but since the quaternion algebra $(bc, N(xz))_L$ is split, the form $s_\star(\langle xz\rangle\langle\langle bc\rangle\rangle)$ is also hyperbolic in this case. Therefore, we may find $\lambda \in L^\times$ represented by $\langle xz\rangle\langle\langle bc\rangle\rangle$; we then have

$$\langle xz\rangle\langle\langle bc\rangle\rangle = \langle\lambda\rangle\langle\langle bc\rangle\rangle, \qquad \text{hence also} \quad \langle x\rangle\langle\langle bc\rangle\rangle = \langle\lambda z\rangle\langle\langle bc\rangle\rangle. \tag{7}$$

Similarly, since the quaternion algebra $(c, N(yz))_L$ is split, the form $s_\star(\langle yz\rangle\langle\langle c\rangle\rangle)$ is hyperbolic, and we may find $\mu \in L^\times$ such that

$$\langle yz\rangle\langle\langle c\rangle\rangle = \langle\mu\rangle\langle\langle c\rangle\rangle, \qquad \text{hence also} \quad \langle y\rangle\langle\langle c\rangle\rangle = \langle\mu z\rangle\langle\langle c\rangle\rangle. \tag{8}$$

As a result of (7) and (8), we have $\langle x, -bcx\rangle = \langle\lambda z, -\lambda zbc\rangle$ and $\langle -y, cy\rangle = \langle -\mu z, \mu zc\rangle$, hence we may rewrite ψ over L' as

$$\psi_{L'} = \langle bc, -c, \lambda z, -\lambda zbc, -\mu z, \mu zc\rangle.$$

Note that $z \notin L$ since $t \notin L^{\times 2}$, hence $s(z) \neq 0$. Using the last expression for $\psi_{L'}$ we may now compute

$$s_\star(\psi)_L = s_\star(\psi_{L'}) = s_\star(\langle z\rangle)\langle\lambda, -\lambda bc, -\mu, \mu c\rangle = \langle s(z)\rangle\langle\langle N(z)\rangle\rangle\langle\lambda, -\lambda bc, -\mu, \mu c\rangle.$$

Since $(bc, N(z))_L = (c, N(z))_L = H_L$, we have $\langle\langle N(z), bc\rangle\rangle = \langle\langle N(z), c\rangle\rangle = (n_H)_L$, hence $s_\star(\psi)_L = \langle s(z)\rangle\langle\lambda, -\mu\rangle(n_H)_L$, and therefore

$$e_3(s_\star(\psi))_L = (\lambda\mu)\cdot[H_L]. \tag{9}$$

On the other hand, we have $[C_K] = (bc, x)_K + (c, y)_K$, hence since $(b, z)_{L'} = 0$

$$[C_{L'}] = (bc, xz)_{L'} + (c, yz)_{L'}.$$

In view of (7) and (8), we may rewrite the right side as follows:

$$[C_{L'}] = (bc, \lambda)_{L'} + (c, \mu)_{L'}.$$

Therefore, $[C_L] + (bc, \lambda)_L + (c, y)_L$ is split by L'. We may then find $\nu \in L^\times$ such that

$$[C_L] = (bc, \lambda)_L + (c, \mu)_L + (a, \nu)_L.$$

Since L splits U, we have $(t, a)_L = (t, b)_L = 0$ and $(t, c)_L = H_L$. It follows that

$$(t)\cdot[C_L] = (t, c, \lambda\mu)_L = (\lambda\mu)\cdot[H_L].$$

By comparing with (9), we see that ξ vanishes over L. The proof of the lemma is thus complete. □

Proof of Theorem 5.9. Since $2\xi = 0$, the assertion $f_3(U) = 0$ follows from $\mathcal{H}_U = \{0, \overline{\xi}^U\}$, see Remark 5.5(ii). Moreover, the field F_U splits U. Therefore, by Lemma 5.10, we have $\xi \in \ker\big(H^3(F) \to H^3(F_U)\big)$, so that $\overline{\xi}^U \in \mathcal{H}_U$. Since we know from Theorem 3.13 that the order of $\mathcal{H}_U$ is at most 2, it suffices to show that $\overline{\xi}^U \neq 0$ when $\mathcal{H}_U \neq 0$ to establish $\mathcal{H}_U = \{0, \overline{\xi}^U\}$. Therefore, proving the equivalence of (a), (b), (c) and (d) completes the proof.
Let us first prove (a) ⇒ (b). Suppose $[C] = (a, \alpha)_k + (b, \beta)_k + (c, \gamma)_k$ for some $\alpha, \beta, \gamma \in k^\times$. Since $[C_K] = (bc, x)_K + (c, y)_K$, it follows that $(b, \beta)_K + (c, \gamma)_K = (bc, x)_K + (c, y)_K$, hence

$$(bc, \beta x)_K = (c, \beta\gamma y)_K.$$

By the common slot lemma [28, Th. III.4.13], we may find $z \in K^\times$ such that

$$(bc, \beta x)_K = (bc, z)_K = (c, z)_K = (c, \beta\gamma y)_K. \tag{10}$$

Let $E = F\big(\sqrt{N(z)t}\big)$, a quadratic extension of F. We claim that E splits U. First, observe that $N(z)t$ is represented by the form $\langle t, -at\rangle$, hence the quaternion algebra $(a,t)_F$ contains a pure quaternion with square $N(z)t$. Therefore, E splits $(a,t)_F$. Likewise, from (10) we see that $(b,z)_K = 0$, hence by taking the corestriction to k we have $(b, N(z))_k = 0$. Therefore, $N(z)t$ is represented by the form $\langle t, -bt\rangle$, and it follows that E splits $(b,t)_F$. Finally, by taking the corestriction of each side of the rightmost equation in (10), we obtain $(c, N(z))_k = (c, N(y))_k$, so $N(y)N(z)$ is represented by $\langle 1, -c\rangle$ and therefore $N(z)t$ is represented by $\langle N(y)t, -cN(y)t\rangle$. It follows that E splits the quaternion algebra $(c, N(y)t)_F$. We have thus shown that E splits three generators of U, hence E splits U.

The implication (b) $\Rightarrow$ (c) follows immediately from Peyre's Theorem 3.13. Moreover, (c) $\Rightarrow$ (d) is clear since $\overline{\xi}^U \in \mathcal{H}_U$. To complete the proof, we show (d) $\Rightarrow$ (a). Suppose there exist λ_1, λ_2, $\lambda_3 \in F^\times$ such that

$$\xi = (\lambda_1, a, t) + (\lambda_2, b, t) + (\lambda_3, c, N(y)t). \tag{11}$$

Let $\partial\colon H^i(F) \to H^{i-1}(k)$ be the residue map associated to the t-adic valuation, for $i = 2, 3$. Since $e_3(s_\star(\psi)) \in H^3(k)$ we have $\partial(e_3(s_\star(\psi))) = 0$, hence $\partial(\xi) = [C]$. Therefore, taking the image of each side of (11) under the residue map yields

$$[C] = a \cdot \partial(\lambda_1, t) + b \cdot \partial(\lambda_2, t) + c \cdot \partial(\lambda_3, N(y)t),$$

so that $[C] \in \mathrm{Dec}(M/k)$. □

As a corollary, we get:

COROLLARY 5.11. *Use the notation 5.8, and assume* $[C] \notin \mathrm{Dec}(M/k)$. *Then* $U \subset \mathrm{Br}(F)$ *is a quaternionic subgroup of order* 8 *such that* $\sum_{[H]\in U} n_H \in I^4(F)$ *(i.e.,* $f_3(U) = 0$*), which is not split by any quadratic extension of* F *(i.e.,* $\mathcal{H}_U \neq 0$*).*

To obtain an example of a central simple algebra with orthogonal involution of degree 12 without quadratic splitting, we need a more stringent condition on C:

LEMMA 5.12. *With the notation 5.8, the following conditions are equivalent:*

(a) *there is a quadratic extension* E *of* F *that splits* $(a,t)_F$ *and* ξ*;*
(b) *the algebra* C *is Brauer-equivalent to a tensor product of quaternion* k*-algebras* $A_1 \otimes_k A_2 \otimes_k A_3$ *with* A_3 *split by* K*.*

Proof. (a) $\Rightarrow$ (b): Let $\widehat{F} = k((t))$ be the completion of F for the t-adic valuation. The field E does not embed in $\widehat{F}$ because $\widehat{F}$ does not split $(a,t)_F$. Therefore, E and $\widehat{F}$ are linearly disjoint over F and we may consider the field $\widehat{E} = E \otimes_F \widehat{F}$, which is a quadratic extension of $\widehat{F}$ that splits $(a,t)_{\widehat{F}}$ and $\xi_{\widehat{F}}$. Each square class in $\widehat{F}$ is represented by an element in $k^\times$ or an element of the

form ut with $u \in k^\times$, see [28, Cor. VI.1.3]. Therefore, we may assume that either $\widehat{E} = \widehat{F}(\sqrt{u})$ or $\widehat{E} = \widehat{F}(\sqrt{ut})$ for some $u \in k^\times$.
Suppose first $\widehat{E} = \widehat{F}(\sqrt{u})$ with $u \in k^\times$. Since the quaternion algebra $(a,t)_{\widehat{F}}$ is split by $\widehat{E}$, it must contain a pure quaternion with square u, hence u is represented by $\langle a, t, -at\rangle$ over $\widehat{F}$. Therefore, $u \equiv a \bmod k^{\times 2}$, and $\widehat{E} = K((t))$. From

$$\xi_{\widehat{E}} = t \cdot [C_{\widehat{E}}] + e_3(s_\star(\psi))_{\widehat{E}} = 0,$$

it follows by taking images under the residue map $H^3(\widehat{E}) \to H^2(K)$ associated to the t-adic valuation that $[C_K] = 0$. Then C is Brauer-equivalent to a quaternion algebra A_3 split by K, and (b) holds with A_1, A_2 split quaternion algebras.
Suppose next $\widehat{E} = \widehat{F}(\sqrt{ut})$ for some $u \in k^\times$. Since $\widehat{E}$ splits $(a,t)_{\widehat{F}}$, it follows as above that ut is represented by $\langle a, t, -at\rangle$ over $\widehat{F}$, hence u is represented by $\langle 1, -a\rangle$, which means that $u \in N(K^\times)$. Because ut is a square in $\widehat{E}$, we have $t \cdot [C_{\widehat{E}}] = u \cdot [C_{\widehat{E}}]$, hence the equation $\xi_{\widehat{E}} = 0$ yields

$$u \cdot [C_{\widehat{E}}] + e_3(s_\star(\psi))_{\widehat{E}} = \big(u \cdot [C] + e_3(s_\star(\psi))\big)_{\widehat{E}} = 0.$$

Since $\widehat{F} = k((t)) = k((ut))$ we have $\widehat{E} = k((\sqrt{ut}))$, hence the scalar extension map $H^3(k) \to H^3(\widehat{E})$ is injective. Therefore, the last equation yields

$$u \cdot [C] + e_3(s_\star(\psi)) = 0,$$

which shows that $e_3(s_\star(\psi)) \in N(K^\times) \cdot [C]$ because $u \in N(K^\times)$. By Barry's result [2, Prop. 4.4], it follows that the biquaternion algebra B has a descent to k: there exist quaternion k-algebras A_1, A_2 such that $B \simeq A_1 \otimes_k A_2 \otimes_k K$. Since C_K is Brauer-equivalent to B, it follows that $C \otimes A_1 \otimes_k A_2$ is split by K. It is therefore Brauer-equivalent to a quaternion algebra A_3 split by K, and C is Brauer-equivalent to $A_1 \otimes_k A_2 \otimes_k A_3$, proving (b).

(b) $\Rightarrow$ (a): Since B is Brauer-equivalent to C_K, condition (b) implies that $B \simeq A_1 \otimes_k A_2 \otimes_k K$. From Barry's result [2, Prop. 4.4], it follows that $e_3(s_\star(\psi)) = u \cdot [C]$ for some $u \in N(K^\times)$. Let $E = F(\sqrt{ut})$. Then $(a,t)_E \simeq (a,u)_E$, hence E splits $(a,t)_F$ because $u \in N(K^\times)$. Moreover, $\xi_E = \big(u \cdot [C] + e_3(s_\star(\psi))\big)_E$, so E also splits ξ. Therefore, (a) holds. □

Examples of algebras C for which condition (b) of Lemma 5.12 does not hold include indecomposable division algebras of degree 8 and exponent 2; other examples are given in [3]. Note that condition (b) is weaker than $[C] \in \mathrm{Dec}(M/k)$; it is in fact strictly weaker: see Remark 5.14.

COROLLARY 5.13. *Use the notation 5.8, and let $Q = (a,t)_F$. There exists an orthogonal involution ρ on $M_6(Q)$ with the following properties:*

(i) *ρ has trivial discriminant and trivial Clifford invariant;*
(ii) *$e_3(\rho) = \xi \mod F^\times \cdot [Q]$;*
(iii) *U is a decomposition group of ρ;*
(iv) *$f_3(\rho) = 0$.*

For any involution ρ satisfying (i) *and* (ii)*, there exists a quadratic extension of F over which Q is split and ρ is hyperbolic if and only if the equivalent conditions of Lemma 5.12 hold.*

Proof. By Remark 3.11, there is an orthogonal involution ρ on $M_6(Q)$ with trivial discriminant and trivial Clifford invariant, and with decomposition group U. By Theorems 4.6 and 5.4, $\overline{e_3(\rho)}^U$ generates $\mathcal{H}_U$, and $f_3(\rho) = f_3(U)$. Therefore, Theorem 5.9 yields $\overline{e_3(\rho)}^U = \overline{\xi}^U$ and $f_3(\rho) = 0$. By Proposition 4.7(ii), we may assume $e_3(\rho) = \xi \mod F^\times \cdot [Q]$. Thus, ρ satisfies conditions (i)–(iv).
Now, let ρ be any orthogonal involution on $M_6(Q)$ satisfying (i) and (ii). Because of (ii), condition (a) of Lemma 5.12 holds if and only if there is a quadratic extension E of F such that $[Q_E] = 0$ and $e_3(\rho)_E = 0$. By Theorem 4.1(i), the last equation holds if and only if ρ_E is hyperbolic. □

Suppose C does not satisfy condition (b) of Lemma 5.12 (e.g., C is an indecomposable division algebra of degree 8 and exponent 2 split by M). Then for any involution ρ on $M_6(Q)$ satisfying the properties (i)–(iv) of Corollary 5.13 we have $f_3(\rho) = 0$, and yet there is no quadratic extension of F over which Q is split and ρ is hyperbolic. From (b') ⇒ (a') in Proposition 5.6, it follows that $\mathcal{H}_{U'} \neq 0$ for every decomposition group U' of ρ.

Remark 5.14. By [40, Cor. 3.2], for any triquadratic extension M/k, any 2-torsion Brauer class in $\mathrm{Br}(k)$ split by M is represented modulo $\mathrm{Dec}(M/k)$ by a quaternion algebra. Therefore, if the triquadratic extension M/k is such that $\mathrm{Dec}(M/k)$ does not coincide with the subgroup of ${}_2\,\mathrm{Br}(k)$ split by M (see [12, §5] for examples of such extensions), we may find a quaternion k-algebra C split by M such that $[C] \notin \mathrm{Dec}(M/k)$. The algebra C obviously satisfies condition (b) of Lemma 5.12 (with A_2 and A_3 split), so for any involution ρ on $M_6(Q)$ satisfying the properties (i)–(iv) of Corollary 5.13 we may find a quadratic extension of F over which Q is split and ρ is hyperbolic. From (a') ⇒ (b') in Proposition 5.6, it follows that there exists a decomposition group U' of ρ such that $\mathcal{H}_{U'} = 0$. Yet, because $[C] \notin \mathrm{Dec}(M/k)$, the decomposition group U of ρ satisfies $\mathcal{H}_U \neq 0$ by Theorem 5.9.

6. Application to degree 8 algebras with involution

The Arason invariant in degree 8 was studied in [35] for orthogonal involutions with trivial discriminant and trivial Clifford algebra. In this section, we extend it to algebras of degree 8 and index 2, when the involution has trivial discriminant and the two components of the Clifford algebra also both have index 2. First, we prove an analogue of Theorem 3.2 on additive decompositions, for degree 8 algebras with orthogonal involution of trivial discriminant.

6.1. Additive decompositions in degree 8. Let (A, σ) be a degree 8 algebra with orthogonal involution of trivial discriminant. We let $(C^+(A,\sigma), \sigma^+)$ and $(C^-(A,\sigma), \sigma^-)$ denote the two components of the Clifford algebra of (A, σ), endowed with the involutions induced by the canonical involution of the Clifford

algebra. Both algebras have degree 8, both involutions have trivial discriminant, and by triality [26, (42.3)],

$$C(C^+(A,\sigma),\sigma^+) \simeq (C^-(A,\sigma),\sigma^-) \times (A,\sigma) \tag{12}$$

and

$$C(C^-(A,\sigma),\sigma^-) \simeq (A,\sigma) \times (C^+(A,\sigma),\sigma^+). \tag{13}$$

Assume (A,σ) decomposes as a sum $(A,\sigma) \in (A_1,\sigma_1) \boxplus (A_2,\sigma_2)$ of two degree 4 algebras with orthogonal involution of trivial discriminant. Each summand is a tensor product of two quaternion algebras with canonical involution, and we get

$$(A,\sigma) \in \big((Q_1,\bar{\ }) \otimes (Q_2,\bar{\ })\big) \boxplus \big((Q_3,\bar{\ }) \otimes (Q_4,\bar{\ })\big), \tag{14}$$

for some quaternion algebras Q_1, Q_2, Q_3 and Q_4 such that A is Brauer-equivalent to $Q_1 \otimes Q_2$ and $Q_3 \otimes Q_4$. By [36, Prop. 6.6], the two components of the Clifford algebra of (A,σ) then admit similar decompositions, namely, up to permutation of the two components, we have:

$$\big(C^+(A,\sigma),\sigma^+\big) \in \big((Q_1,\bar{\ }) \otimes (Q_3,\bar{\ })\big) \boxplus \big((Q_2,\bar{\ }) \otimes (Q_4,\bar{\ })\big), \tag{15}$$

and

$$\big(C^-(A,\sigma),\sigma^-\big) \in \big((Q_1,\bar{\ }) \otimes (Q_4,\bar{\ })\big) \boxplus \big((Q_2,\bar{\ }) \otimes (Q_3,\bar{\ })\big). \tag{16}$$

Mimicking the construction in §3, we associate to every decomposition of (A,σ) as above the subgroup W of the Brauer group of F generated by any three elements among the $[Q_i]$ for $1 \le i \le 4$. We call W a decomposition group of (A,σ). It consists of at most 8 elements, and can be described explicitly by

$$W = \{0, [A], [C^+(A,\sigma)], [C^-(A,\sigma)], [Q_1], [Q_2], [Q_3], [Q_4]\}.$$

In view of their additive decompositions, W also is a decomposition group of the two components $(C^+(A,\sigma),\sigma^+)$ and $(C^-(A,\sigma),\sigma^-)$ of the Clifford algebra. Note that, in contrast with the decomposition groups of algebras of degree 12 in Definition 3.6, the group W may contain three Brauer classes of index 4 instead of at most one. Nevertheless, it has similar properties; for instance, we prove:

PROPOSITION 6.1. *Let (A,σ) be a central simple algebra of degree 8 with an orthogonal involution of trivial discriminant.*

(i) *Suppose (A,σ) has an additive decomposition as in (14), with decomposition group W. For any extension K/F which splits W, the algebra with involution (A_K,σ_K) is split and hyperbolic.*
(ii) *The converse holds for quadratic extensions: if (A,σ) is split and hyperbolic over a quadratic extension K of F, then (A,σ) has an additive decomposition with decomposition group split by K.*

Proof. (i) If a field K splits W, then it splits A, and moreover each summand in (14) is split and hyperbolic over K, therefore σ_K is hyperbolic.
(ii) To prove the converse, suppose $K = F(d)$ with $d^2 = \delta \in F^\times$, and assume A_K is split and σ_K is hyperbolic, hence $\operatorname{ind} A \le 2$. If A is split, we have as in

the proof of Proposition 5.6 $(A,\sigma) \simeq \mathrm{Ad}_\varphi$ with φ an 8-dimensional quadratic form multiple of $\langle 1, -\delta\rangle$. We may then find quaternion F-algebras Q_1, Q_2 split by K and scalars α_1, $\alpha_2 \in F^\times$ such that $\varphi \simeq \langle\alpha_1\rangle n_{Q_1} \perp \langle\alpha_2\rangle n_{Q_2}$. As in Example 3.4, we obtain a decomposition

$$(A,\sigma) \in \big((Q_1,\bar{\ })\otimes(Q_1,\bar{\ })\big) \boxplus \big((Q_2,\bar{\ })\otimes(Q_2,\bar{\ })\big).$$

The corresponding decomposition group is $\{0, [Q_1], [Q_2], [Q_1]+[Q_2]\}$; it is split by K.

If $\operatorname{ind} A = 2$, let Q be the quaternion division algebra Brauer-equivalent to A. Since K splits A, we may, again as in the proof of Proposition 5.6, identify K with a subfield of Q and find a skew-hermitian form h of the form $\langle d\rangle\langle 1, \alpha, \beta, \gamma\rangle$ (with α, β, $\gamma \in F^\times$) such that $(A,\sigma) \simeq \mathrm{Ad}_h$. Then

$$(A,\sigma) \in \mathrm{Ad}_{\langle d\rangle\langle 1,\alpha\rangle} \boxplus \mathrm{Ad}_{\langle d\rangle\langle \beta,\gamma\rangle}$$

is a decomposition in which each of the summands becomes hyperbolic over K. The corresponding decomposition group is therefore split by K. □

There exist quadratic forms φ of dimension 8 with trivial discriminant and Clifford algebra of index 4 that do not decompose into an orthogonal sum of two 4-dimensional quadratic forms of trivial discriminant, see [21, Cor. 16.8] or [20, Cor 6.2]. For such a form, neither Ad_φ nor the components of its Clifford algebra have additive decompositions as in (14). The next proposition shows, by contrast, that such a decomposition always exist if at least two among the algebras A, $C^+(A,\sigma)$ and $C^-(A,\sigma)$ have index ≤ 2.

Proposition 6.2. *Let (A,σ) be a central simple algebra of degree 8 with orthogonal involution of trivial discriminant. We assume at least two among the algebras A, $C^+(A,\sigma)$ and $C^-(A,\sigma)$ have index ≤ 2. Then all three algebras with involution (A,σ), $(C^+(A,\sigma),\sigma^+)$ and $(C^-(A,\sigma),\sigma^-)$ admit an additive decomposition as a sum of two degree 4 algebras with orthogonal involution of trivial discriminant as in (14).*

Proof. Assume two among $\operatorname{ind} A$, $\operatorname{ind} C^+(A,\sigma)$, $\operatorname{ind} C^-(A,\sigma)$ are 1 or 2. By triality, see (12) to (16) above, it is enough to prove that one of the three algebras with involution, say (A,σ) has an additive decomposition. Since A, $C^+(A,\sigma)$, $C^-(A,\sigma)$ are interchanged by triality, we may also assume $\operatorname{ind} A \le 2$. If A is split, so $(A,\sigma) \simeq \mathrm{Ad}_\varphi$ for some 8-dimensional quadratic form φ with trivial discriminant and Clifford algebra of index at most 2, then (a) holds by a result of Knebusch [25, Ex. 9.12], which shows that φ is the product of a 2-dimensional quadratic form and a 4-dimensional quadratic form.

For the rest of the proof, assume $(A,\sigma) \simeq \mathrm{Ad}_h$ for some skew-hermitian form h over a quaternion division algebra $(Q,\bar{\ })$. Let $q \in Q$ be a nonzero quaternion represented by h, and let $h \simeq \langle q\rangle \perp h'$ for some skew-hermitian form h' of absolute rank 6. As we saw in the proof of Theorem 3.2, over the quadratic extension $K = F(q)$ the algebra Q splits and the form $\langle q\rangle$ becomes hyperbolic, hence h_K and h'_K are Witt-equivalent. In particular, it follows that $e_2((\mathrm{ad}_{h'})_K) = e_2((\mathrm{ad}_h)_K)$ has index at most 2. But $(\mathrm{ad}_{h'})_K = \mathrm{ad}_\psi$ for some

Albert form ψ over K, so ψ is isotropic. It follows by [34, Prop., p. 382] that h' represents some scalar multiple of q; thus $h \simeq \langle q\rangle\langle 1, -\lambda\rangle \perp h''$ for some $\lambda \in F^\times$ and some skew-hermitian form h'' of absolute rank 4. The discriminant of h'' must be trivial because h and $\langle q\rangle\langle 1, -\lambda\rangle$ have trivial discriminant, and we thus have the required decomposition for (A, σ). □

Remark 6.3. It follows that all trialitarian triples such that at least two of the algebras have index ≤ 2 have a description as in (14) to (16).

6.2. An extension of the Arason invariant in degree 8 and index 2. Throughout this section, (A, σ) is a central simple F-algebra of degree 8 and trivial discriminant. It is known that (A, σ) is a tensor product of quaternion algebras with involution if and only if $e_2(\sigma) = 0$, see [26, (42.11)]. In this case, the Arason invariant $e_3(\sigma) \in M^3_A(F)$ is defined when A has index at most 4 (see §2.5) , and represented by an element of order 2 in $H^3(F)$, see [35]. Here, we extend the definition of the e_3 invariant under the following hypothesis:

$$\operatorname{ind} A = \operatorname{ind} C^+(A, \sigma) = \operatorname{ind} C^-(A, \sigma) = 2. \tag{17}$$

By Proposition 6.2, this condition implies that (A, σ) decomposes into a sum of two central simple algebras of degree 4 with involutions of trivial discriminant. Moreover, the associated decomposition group W is a quaternionic subgroup of $\operatorname{Br}(F)$. Let Q, Q^+, Q^- be the quaternion division algebras over F that are Brauer-equivalent to A, $C^+(A, \sigma)$, and $C^-(A, \sigma)$ respectively. From the Clifford algebra relations [26, (9.12)], we know $[Q^+] + [Q^-] = [Q]$. Therefore, the following is a subgroup of the Brauer group:

$$V = \{0, [Q], [Q^+], [Q^-]\} \subset \operatorname{Br}(F).$$

Condition (17) implies that $|V| = 4$. Moreover, V also is a subgroup of every decomposition group of (A, σ).
To (A, σ), we may associate algebras of degree 12 with orthogonal involution with trivial discriminant and trivial Clifford invariant by considering any involution ρ of $M_6(Q)$ such that

$$(M_6(Q), \rho) \in (A, \sigma) \boxplus \big((Q^+, \bar{\ }) \otimes (Q^-, \bar{\ })\big). \tag{18}$$

Since the two components of the Clifford algebra of σ are Brauer-equivalent to Q^+ and Q^-, the involution ρ has trivial Clifford invariant. Therefore, we may consider its Arason invariant $e_3(\rho) \in M^3_Q(F)$. The following lemma compares the Arason invariant of two such involutions:

Lemma 6.4. *Let ρ and ρ' be two involutions of $M_6(Q)$ satisfying (18). There exists $\lambda \in F^\times$ such that*

$$e_3(\rho) - e_3(\rho') = (\lambda) \cdot [Q^+] = (\lambda) \cdot [Q^-] \in M^3_Q(F).$$

Moreover, $f_3(\rho) = f_3(\rho')$.

Proof. By definition of the direct orthogonal sum for algebras with involution, we may pick skew-hermitian forms h_1 and h_2 over $(Q, \bar{\ })$ such that $\sigma = \operatorname{ad}_{h_1}$,

$\overline{\ } \otimes \overline{\ } \simeq \mathrm{ad}_{h_2}$ and $\rho = \mathrm{ad}_{h_1 \perp h_2}$. Moreover, there exists $\lambda \in F^\times$ such that $\rho' = \mathrm{ad}_{h_1 \perp \langle\lambda\rangle h_2}$. Therefore, we have (see § 2.4 and 2.5):

$$e_3(\rho) - e_3(\rho') = e_3(\langle 1, -\lambda\rangle h_2) = (\lambda) \cdot [Q^+] \mod F^\times \cdot [Q],$$

by Proposition 2.6, since $e_2(h_2) = [Q^+] = [Q^-] \mod [Q]$. Moreover, if c and $c' \in H^3(F)$ are representatives of $e_3(\rho)$ and $e_3(\rho')$ respectively, then

$$c' - c \in (\lambda) \cdot [Q^+] + F^\times \cdot [Q].$$

So $2c = 2c'$, and this finishes the proof.

□

Let $F^\times \cdot V \subset H^3(F)$ be the subgroup consisting of the products $\lambda \cdot v$ with $\lambda \in F^\times$ and $v \in V = \{0, [Q], [Q^+], [Q^-]\}$. This subgroup contains $F^\times \cdot [Q]$, so we may consider the canonical map $\overline{\ }^V \colon M_Q^3(F) \to H^3(F)/F^\times \cdot V$. The previous lemma shows that the image $\overline{e_3(\rho)}^V$ of the Arason invariant of ρ does not depend on the choice of an involution ρ satisfying (18). This leads to the following:

DEFINITION 6.5. With the notation above, we set

$$e_3(\sigma) = \overline{e_3(\rho)}^V \in H^3(F)/F^\times \cdot V \qquad \text{and} \qquad f_3(\sigma) = f_3(\rho) \in F^\times \cdot [Q] \subset H^3(F)$$

where ρ is any involution satisfying

$$(M_6(Q), \rho) \in (A, \sigma) \boxplus \big((Q^+, \overline{\ }) \otimes (Q^-, \overline{\ })\big).$$

This definition functorially extends the definition of the Arason invariant. Indeed, if K is any extension of F that splits Q^+ or Q^- (or both), then the scalar extension map $\mathrm{Br}(F) \to \mathrm{Br}(K)$ carries V to $\{0, [Q]\}$, and any involution ρ as in (18) becomes Witt-equivalent to σ over K. Therefore, scalar extension carries $e_3(\sigma) \in H^3(F)/F^\times \cdot V$ defined above to $e_3(\sigma_K) \in M_Q^3(K)$ as defined in §2.5.

Example 6.6. Consider a central simple algebra $(M_6(Q), \rho)$ of degree 12 and index 2 with an orthogonal involution of trivial discriminant and trivial Clifford invariant. By Theorem 3.2 $(M_6(Q), \rho)$ admits additive decompositions

$$(M_6(Q), \rho) \in \coprod_{i=1}^{3} \big((Q_i, \overline{\ }) \otimes (H_i, \overline{\ })\big) \qquad \text{with } \sum_{i=1}^{3} [H_i] = 0,$$

so it contains symmetric idempotents e_1, e_2, e_3 such that

$$\big(e_i M_6(Q) e_i, \rho|_{e_i M_6(Q) e_i}\big) \simeq (Q_i, \overline{\ }) \otimes (H_i, \overline{\ }).$$

Consider the restriction of ρ to $(e_1 + e_2)M_6(Q)(e_1 + e_2)$; we thus obtain an algebra with involution $(M_4(Q), \sigma)$ such that

$$(M_6(Q), \rho) \in (M_4(Q), \sigma) \boxplus \big((Q_3, \overline{\ }) \otimes (H_3, \overline{\ })\big)$$

$$\text{and} \quad (M_4(Q), \sigma) \in \coprod_{i=1}^{2} \big((Q_i, \overline{\ }) \otimes (H_i, \overline{\ })\big).$$

It is clear that the discriminant of σ is trivial. Since $e_2(\rho) = 0$, we have

$$e_2(\sigma) = e_2\big((Q_3, \overline{\ }) \otimes (H_3, \overline{\ })\big) = \{[Q_3], [H_3]\}.$$

Therefore, Condition (17) holds for $(M_4(Q), \sigma)$ if Q_3 and H_3 are not split. In that case, we have $V = \{0, [Q], [Q_3], [H_3]\}$ and, by definition,

$$e_3(\sigma) = \overline{e_3(\rho)}^V \in H^3(F)/F^\times \cdot V \quad \text{and} \quad f_3(\sigma) = f_3(\rho) \in F^\times \cdot [Q] \subset H^3(F).$$

The condition that Q_3 and H_3 are not split holds in particular when the decomposition group U generated by $[Q_1]$, $[Q_2]$, $[Q_3]$ has order 8.

Example 6.7. Take for $(M_6(Q), \rho)$ the algebra $\mathrm{Ad}_{\langle 1, -t\rangle} \otimes (\lambda^2 E, \gamma)_{F(t)}$ of Corollary 2.19, with E a division algebra of degree and exponent 4. (Note that $\lambda^2 E$ is Brauer-equivalent to $E \otimes E$, hence it has index 2.) Since $f_3(\rho) \neq 0$, every decomposition group of ρ has order 8; indeed, quaternionic subgroups $U \subset \mathrm{Br}(F)$ of order dividing 4 have $\mathcal{H}_U = 0$ by Theorem 3.12, hence trivial f_3 by Proposition 5.3. The construction in the previous example yields an algebra with involution $(M_4(Q), \sigma)$ of degree 8 satisfying Condition (17), with $f_3(\sigma) = t \cdot [Q] \neq 0$.

Example 6.8. Also, we may take for $(M_6(Q), \rho)$ the algebra with involution of Corollary 5.13, and obtain an algebra with involution $(M_4(Q), \sigma)$ of degree 8 satisfying Condition (17) such that (with the notation 5.8) $e_3(\sigma) = \overline{\xi}^V \in H^3(F)/F^\times V$. Since $\xi \notin F^\times \cdot U$, we have $e_3(\sigma) \neq 0$. Yet, we have $f_3(\sigma) = f_3(\rho) = 0$ by Corollary 5.13. Moreover, there is no quadratic extension K of F such that Q_K is split and σ_K is hyperbolic. Indeed, over such a field, $(M_6(Q), \rho)_K$ would be Witt-equivalent to an algebra of degree 4, hence it would be hyperbolic because $e_2(\rho) = 0$. Corollary 5.13 shows that such quadratic extensions K do not exist.

The next proposition shows that the e_3 invariant detects isotropy, for any central simple algebra with involution (A, σ) satisfying Condition (17). As in §6.1, we let σ^+ and σ^- denote the canonical involutions on $C^+(A, \sigma)$ and $C^-(A, \sigma)$.

PROPOSITION 6.9. *Let (A, σ) be a central simple F-algebra of degree 8 with orthogonal involution of trivial discriminant satisfying* (17)*. With the notation above, we have $e_3(\sigma) = e_3(\sigma^+) = e_3(\sigma^-)$ and $f_3(\sigma) = f_3(\sigma^+) = f_3(\sigma^-)$. Moreover, the following conditions are equivalent:*

(a) *$e_3(\sigma) = 0$;*
(b) *σ is isotropic;*
(c) *(A, σ) is Witt-equivalent to $(Q^+, \overline{\ }) \otimes (Q^-, \overline{\ })$.*

Proof. As in §3.3, let F_V denote the function field of the product of the Severi–Brauer varieties associated to the elements of V. Extending scalars to F_V, we split Q and $e_2(\sigma)$, hence there is a 3-fold Pfister form π over F_V such that

$$\sigma_{F_V} \simeq \mathrm{ad}_\pi\,.$$

For Pfister forms, we have $\mathrm{ad}_\pi \simeq \mathrm{ad}_\pi^+ \simeq \mathrm{ad}_\pi^-$ (see [26, (42.2)]), hence $\sigma^+_{F_V} \simeq \sigma^-_{F_V} \simeq \sigma_{F_V}$, and therefore

$$e_3(\sigma)_{F_V} = e_3(\sigma^+)_{F_V} = e_3(\sigma^-)_{F_V} = e_3(\pi).$$

Since V is generated by the Brauer classes of two quaternion algebras, it follows from Theorem 3.12 that $F^\times \cdot V$ is the kernel of the scalar extension map $H^3(F) \to H^3(F_V)$, hence the preceding equations yield $e_3(\sigma) = e_3(\sigma^+) = e_3(\sigma^-)$. We then have $f_3(\sigma) = f_3(\sigma^+) = f_3(\sigma^-)$, since $f_3(\sigma)$ (resp. $f_3(\sigma^+)$, resp. $f_3(\sigma^-)$) is 2 times any representative of $e_3(\sigma)$ (resp. $e_3(\sigma^+)$, resp. $e_3(\sigma^-)$) in $H^3(F)$.

To complete the proof, we show that (a), (b), and (c) are equivalent. Clearly, (c) implies (b). The converse follows easily from [26, (15.12)] if A has index 2, and [26, (16.5)] if A is split. Moreover, in view of the definition of $e_3(\sigma)$, the equivalence between (a) and (c) follows from Proposition 4.5. □

As in §4, we may relate the e_3 invariant to the homology of the Peyre complex of any decomposition group, as follows:

PROPOSITION 6.10. *Let (A, σ) be a central simple algebra of degree 8 with an orthogonal involution of trivial discriminant satisfying* (17)*, and let W be a decomposition group of (A, σ). The image $\overline{e_3(\sigma)}^W$ of $e_3(\sigma) \in H^3(F)/F^\times \cdot V$ in $H^3(F)/F^\times \cdot W$ generates $\mathcal{H}_W$, and $f_3(\sigma) = f_3(W)$.*

Proof. As above, let Q be the quaternion division algebra Brauer-equivalent to A, so we may identify A with $M_4(Q)$. Let ρ be an involution on $M_6(Q)$ such that

$$(M_6(Q), \rho) \in (A, \sigma) \boxplus \big((Q^+, \bar{\ }) \otimes (Q^-, \bar{\ })\big).$$

By definition, we have $e_3(\sigma) = \overline{e_3(\rho)}^V$ and $f_3(\sigma) = f_3(\rho)$. Now, consider a decomposition of (A, σ) with decomposition group W (which necessarily contains V):

$$(A, \sigma) \in \big((C_1^+, \bar{\ }) \otimes (C_1^-, \bar{\ })\big) \boxplus \big((C_2^+, \bar{\ }) \otimes (C_2^-, \bar{\ })\big).$$

We have

$$(M_6(Q), \rho) \in \big((C_1^+, \bar{\ }) \otimes (C_1^-, \bar{\ })\big) \boxplus \big((C_2^+, \bar{\ }) \otimes (C_2^-, \bar{\ })\big) \boxplus \big((Q^+, \bar{\ }) \otimes (Q^-, \bar{\ })\big),$$

which is a decomposition of $(M_6(Q), \rho)$ with decomposition group W. Therefore, Theorem 4.6 shows that $\overline{e_3(\rho)}^W$ generates $\mathcal{H}_W$ and $f_3(\rho) = f_3(W)$. The proposition follows because $f_3(\sigma) = f_3(\rho)$ and $\overline{e_3(\sigma)}^W = \overline{e_3(\rho)}^W$ since $V \subset W$. □

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Anne Quéguiner-Mathieu
Université Paris 13,
Sorbonne Paris Cité
LAGA - CNRS (UMR 7539)
F-93430 Villetaneuse, France
queguin@math.univ-paris13.fr

Jean-Pierre Tignol
ICTEAM Institute, Box L4.05.01
Université catholique de Louvain
B-1348 Louvain-la-Neuve, Belgium
jean-pierre.tignol@uclouvain.be