Lie Groups and p-Compact Groups

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Abstract. A p-compact group is the homotopical ghost of a compact Lie group; it is the residue that remains after the geometry and algebra have been stripped away. This paper sketches the theory of p-compact groups, with the intention of illustrating the fact that many classical structural properties of compact Lie groups depend only on homotopy theoretic considerations.

1 From compact Lie groups to p-compact groups

The concept of *p-compact group* is the culmination of a series of attempts, stretching over a period of decades, to isolate the key homotopical characteristics of compact Lie groups. It has been something of a problem, as it turns out, to determine exactly what these characteristics are. Probably the first ideas along these lines were due to Hopf [10] and Serre [31].

1.1. Definition. A *finite H-space* is a pair (X, m), where X is a finite CW–complex with basepoint $*$ and $m : X \times X \to X$ is a multiplication map with respect to which $*$ functions, up to homotopy, as a two-sided unit.

The notion of compactness in captured here in the requirement that X be a finite CW–complex. To obtain a structure a little closer to group theory, one might also ask that the multiplication on X be associative up to homotopy. Finite H-spaces have been studied extensively; see [18] and its bibliography. Most of the results deal with homological issues. There are a few general classification theorems, notably Hubbuck's theorem [11] that any path-connected homotopy commutative finite H-space is equivalent at the prime 2 to a torus; this is a more or less satisfying analog of the classical result that any connected abelian compact Lie group is a torus. Experience shows, though, that there is little hope of understanding the totality of all finite H-spaces, or even all homotopy associative ones, on anything like the level of detail that is achieved in the theory of compact Lie groups. The problem is that there are too many finite H-spaces; the structure is too lax.

Stasheff pointed out one aspect of this laxity [32] that is particularly striking when it comes to looking at finite H-spaces as models for group theory. He discovered a whole hierarchy of generalized associativity conditions, all of a homotopy theoretic nature, which are satisfied by a space with an associative multiplication but not necessarily by a finite H-space. These are called A_n-*conditions* ($n \geq 1$); a space is an H-space if it satisfies condition A_2 and homotopy associative if it satisfies condition A_3. Say that a space X is an A_∞-space if it satisfies condition A_n for all n. The following proposition comes from combining [32] with work of Milnor [22] [21] and Kan [17]; it suggests that that A_∞-spaces are very good models for topological groups. From now on we will use the term *equivalence* for spaces to mean *weak homotopy equivalence.*

1.2. Proposition. *If X is a path-connnected CW–complex, the following four conditions imply one another:*

1. *X is an A_∞-space,*
2. *X is equivalent to a topological monoid,*
3. *X is equivalent to a topological group, and*
4. *X is equivalent to the space ΩY of based loops on some 1-connected pointed space Y.*

In fact, there are bijections between homotopy classes of the four structures. There is a similar result for disconnected X, in which conditions 1 and 2 are expanded by requiring that an appropriate multiplication on $\pi_0 X$ make this set into a group.

If X is a topological group as in 1.2(3), then the space Y of 1.2(4) is the ordinary classifying space $\mathrm{B}X$. Proposition 1.2 leads to the following convenient formulation of the notion "finite A_∞-space" or "homotopy finite topological group". This definition appears in a slightly different form in work of Rector [30].

1.3. Definition. A *finite loop space* is a triple $(X, \mathrm{B}X, e)$, where X is a finite CW–complex, $\mathrm{B}X$ is a pointed space, and $e : X \to \Omega\, \mathrm{B}X$ is an equivalence.

Finite loop spaces appear as if they should be very good homotopy theoretic analogs of Lie groups, but one of the very first theorems about them was pretty discouraging. Rector proved in [29] that there are an *uncountable* number of distinct finite loop space structures on the three-sphere S^3. In other words, he showed that there are an uncountable number of homotopically distinct spaces Y with $\Omega Y \simeq S^3$. This is in sharp contrast to the geometric fact that up to isomorphism there is only *one* Lie group structure on S^3. It suggests that the theory of finite loop spaces is unreasonably complicated.

Rector's method was interesting. For any space X, Bousfield and Kan (also Sullivan) had constructed a *rationalization* $X_{\mathbb{Q}}$ of X, and $\mathbb{F}_p$-completions $X\hat{_p}$ (p a prime); if X is a simply connected space with finitely generated homotopy groups, then $\pi_i X_{\mathbb{Q}} \cong \mathbb{Q} \otimes \pi_i X$ and $\pi_i X\hat{_p} \cong \mathbb{Z}_p \otimes \pi_i X$. (Here $\mathbb{Z}_p$ is the ring of p-adic integers.) For such spaces there is a homotopy fibre square on the left

$$\begin{array}{ccc} X & \longrightarrow & \prod_p X\hat{_p} \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & (\prod_p X\hat{_p})_{\mathbb{Q}} \end{array} \qquad\qquad \begin{array}{ccc} \pi_i X & \longrightarrow & \prod_p \mathbb{Z}_p \otimes \pi_i X \\ \downarrow & & \downarrow \\ \mathbb{Q} \otimes \pi_i X & \longrightarrow & \mathbb{Q} \otimes (\prod_p \mathbb{Z}_p \otimes \pi_i X) \end{array}$$

which is a geometric reflection of the algebraic pullback diagrams on the right. This fibre square, called the *arithmetic square* [33] [3], amounts to a recipe for reconstituting X from its $\mathbb{F}_p$-completions by mixing in rational glue. Rector constructed an uncountable number of loop space structures on S^3 by taking the standard Lie group structure on S^3, $\mathbb{F}_p$-completing to get "standard" loop space structures on each of the spaces $(S^3)\hat{_p}$, and then regluing these standard structures over the rationals in an uncountable number of exotic different ways. In particular, all of his loop space structures become standard after $\mathbb{F}_p$-completion at any prime p. Later on [9] it became clear that this last behavior is unavoidable, since up to homotopy there is only *one* loop space structure on the space $(S^3)\hat{_p}$.

Apparently, then, the theory of finite loop spaces simplifies after $\mathbb{F}_p$-completion, and it is exactly this observation that leads to the definition of p-compact group. The definition uses some terminology. We will say that a space Y is $\mathbb{F}_p$*-complete* if the $\mathbb{F}_p$-completion map $Y \to Y\hat{_p}$ is an equivalence; if Y is simply connected and $\mathrm{H}_*(Y;\mathbb{F}_p)$ is of finite type, then Y is $\mathbb{F}_p$-complete if and only if the homotopy groups of Y are finitely generated modules over $\mathbb{Z}_p$. We will say that Y is $\mathbb{F}_p$*-finite* if $\mathrm{H}_i(Y;\mathbb{F}_p)$ is finite-dimensional for each i and vanishes for all but a finite number of i (in other words, if $\mathrm{H}_*(Y;\mathbb{F}_p)$ looks like the $\mathbb{F}_p$-homology of a finite CW–complex).

1.4. Definition. Suppose that p is a fixed prime number. A *p-compact group* is a triple $(X, \mathrm{B}X, e)$, where X is a space which is $\mathbb{F}_p$-finite, $\mathrm{B}X$ is a pointed space which is $\mathbb{F}_p$-complete, and $e : X \to \Omega\,\mathrm{B}X$ is an equivalence.

Here the idea of "compactness" is expressed in the requirement that X be $\mathbb{F}_p$-finite. Assuming in addition that $\mathrm{B}X$ is $\mathbb{F}_p$-complete is equivalent to assuming that X is $\mathbb{F}_p$-complete and that $\pi_0 X$ is a finite p-group.

1.5. Example. If G is a compact Lie group such that $\pi_0 G$ is a p-group, then the $\mathbb{F}_p$-completion of G is a p-compact group.

The definition of p-compact group is a homotopy theoretic compromise between between the inclination to stay as close as possible to the notion of Lie group, and the desire for an interesting and manageable theory. The reader should note that it is the remarkable machinery of Lannes [19] which makes p-compact groups accessible on a technical level. For instance, the machinery of Lannes lies behind the uniqueness result of [9] referred to above.

Organization of the paper. In section 2 we describe a general scheme for translating from group theory to homotopy theory. Sections 3 and 4 describe the main properties of p-compact groups; almost all of these are parallel to classical properties of compact Lie groups [4]. The final section discusses examples and conjectures.

It is impossible to give complete references or precise credit in a short paper like this one. The basic results about p-compact groups are in [6], [7], [8], [23], and [25]. There is a treatment of compact Lie groups based on homotopy theoretic arguments in [4]. The interested reader should look at the survey articles [20], [24], and [26], as well as their bibliographies, for additional information.

1.6. Terminology. There are a few basic topological issues which it is worth pointing out. We assume that all spaces have been replaced if necessary by equivalent CW–complexes. If $f : X \to Y$ is a map of spaces, then $\mathrm{Map}(X, Y)_f$ is the component containing f of the space of maps $X \to Y$. The space $\mathrm{Aut}_f(X)$ is the space of self-equivalences of X over Y; to obtain homotopy invariance, this is constructed by replacing f by an equivalent Serre fibration $f' : X' \to Y$ and forming the space of self homotopy equivalences $X' \xrightarrow{\sim} X'$ which commute with f'.

The notation $\mathrm{H}^*_{\mathbb{Q}_p}(Y)$ stands for $\mathbb{Q} \otimes \mathrm{H}^*(Y; \mathbb{Z}_p)$; this is a variant of rational cohomology which is better-behaved than ordinary rational cohomology for spaces Y which are $\mathbb{F}_p$-complete.

2 A dictionary between group theory and homotopy theory

We now set up a dictionary which will allow us to talk about p-compact groups in ordinary algebraic terms. We begin with concepts that apply to loop spaces in general (a *loop space* is a triple $(X, \mathrm{B}X, e)$ with $e : X \xrightarrow{\sim} \Omega\,\mathrm{B}X$) and then specialize to p-compact groups. From now on we will refer to a loop space or p-compact group $(X, \mathrm{B}X, e)$ as a space X with some (implicit) extra structure.

2.1. Definition. Suppose that X and Y are loop spaces.

- A *homomorphism* $f : X \to Y$ is a pointed map $\mathrm{B}f : \mathrm{B}X \to \mathrm{B}Y$. Two homomorphisms $f, f' : X \to Y$ are *conjugate* if $\mathrm{B}f$ and $\mathrm{B}f'$ are homotopic.
- The *homogeneous space* $Y/f(X)$ (denoted Y/X if f is understood) is the homotopy fibre of $\mathrm{B}f$.
- The *centralizer of* $f(X)$ *in* Y, denoted $\mathcal{C}_Y(f(X))$ or $\mathcal{C}_Y(X)$, is the loop space $\Omega\,\mathrm{Map}(\mathrm{B}X, \mathrm{B}Y)_{\mathrm{B}f}$.
- The *Weyl Space* $\mathcal{W}_Y(X)$ is the space $\mathrm{Aut}_{\mathrm{B}f}(\mathrm{B}X)$; this is in fact a a loop space, essentially because it is an associative monoid under composition (1.2). The *normalizer* $\mathcal{N}_Y(X)$ of X in Y is the loop space of the homotopy orbit space of the action of $\mathcal{W}_Y(X)$ on $\mathrm{B}X$ by composition.
- A *short exact sequence* $X \to Y \to Z$ of loop spaces is a fibration sequence $\mathrm{B}X \to \mathrm{B}Y \to \mathrm{B}Z$; Y is said to be an *extension* of Z by X.

2.2. Remark. If X and Y are discrete groups, treated as loop spaces via 1.2, and $f : X \to Y$ is an ordinary homomorphism, then the above definitions specialize to the usual notions of coset space, centralizer, normalizer, and short exact sequence, at least if $X \to Y$ is injective. It is not hard to see that in general there are natural loop space homomorphisms $\mathcal{C}_Y(X) \to \mathcal{N}_Y(X) \to Y$; the homomorphism $\mathcal{C}_Y(X) \to Y$, for instance, amounts to the map $\mathrm{Map}(\mathrm{B}X, \mathrm{B}Y)_{\mathrm{B}f} \to \mathrm{B}Y$ given by evaluation at the basepoint of $\mathrm{B}X$. There is always a short exact sequence $X \to \mathcal{N}_Y(X) \to \mathcal{W}_Y(X)$.

The key additional definitions for p-compact groups are the following ones.

2.3. Definition. A p-compact group X is *a p-compact torus* if X is the $\mathbb{F}_p$-completion of an ordinary torus, and *a p-compact toral group* if X is an extension of a finite p-group by a p-compact torus. If $f : X \to Y$ is a homomorphism of p-compact groups, then f is a *monomorphism* if $Y/f(X)$ is $\mathbb{F}_p$-finite.

3 Maximal tori and cohomology rings

If X is a p-compact group, a *subgroup Y of X* is a p-compact group Y and a monomorphism $i : Y \to X$ (i is called a *subgroup inclusion*). In general, if $f : Y \to X$ is a homomorphism of p-compact groups, the associated loop space homomorphism $g : \mathcal{C}_X(Y) \to X$ is not obviously a subgroup inclusion; it is not even clear that $\mathcal{C}_X(Y)$ is a p-compact group. For special choices of Y, though, the situation is nicer.

3.1. Proposition. *Suppose that $f : Y \to X$ is a homomorphism of p-compact groups, and that Y is a p-compact toral group. Then $\mathcal{C}_X(Y) \to X$ is a subgroup inclusion.*

A p-compact group is said to be *abelian* if the natural map $\mathcal{C}_X(X) \to X$ is an equivalence.

3.2. Proposition. *A p-compact group is abelian if and only if it is the product of a p-compact torus and a finite abelian p-group. If A is an abelian p-compact group and $f : A \to X$ is a homomorphism, then f naturally lifts over the subgroup inclusion $\mathcal{C}_X(A) \to X$ to a homomorphism $f' : A \to \mathcal{C}_X(A)$.*

A subgroup Y of X is said to be an *abelian subgroup* if Y is abelian, or a *torus in X* if Y is a p-compact torus. If Y' is another subgroup of X, Y' is said to be *contained in Y up to conjugacy* if the homomorphism $Y' \to X$ lifts up to conjugacy to a homomorphism $Y' \to Y$.

3.3. Definition. A torus T in X is said to be a *maximal torus* if any other torus T' in X is contained in T up to conjugacy.

We will say that an abelian subgroup A of X is *self-centralizing* if the map $A \to \mathcal{C}_X(A)$ is an equivalence. If Z is a space which is $\mathbb{F}_p$-finite, the *Euler characteristic* $\chi(Z)$ is the usual alternating sum of the ranks of the $\mathbb{F}_p$ homology groups of Z.

3.4. Proposition. *Suppose that X is a p-compact group and that T is a torus in X. Then T is maximal if and only if $\chi(X/T) \neq 0$. If X is connected, then T is maximal if and only if T is self-centralizing.*

3.5. Proposition. *Any p-compact group X has a maximal torus T, unique up to conjugacy.*

A space is said to be *homotopically discrete* if each of its components is contractible.

3.6. Proposition. *Suppose that X is a p-compact group with maximal torus T. Then the Weyl space $\mathcal{W}_X(T)$ is homotopically discrete, and $\pi_0\mathcal{W}_X(T)$, with the natural composition operation, is a finite group.*

If T is a maximal torus for X, the finitely generated free $\mathbb{Z}_p$-module $\pi_1 T$ is called the *dual weight lattice* L_X of X; its rank as a free module is the *rank* $\mathrm{rk}(X)$ of X. The finite group appearing in 3.6 is called the *Weyl group* of X and denoted W_X; by definition, W_X acts on L_X.

3.7. Definition. If M is a finitely generated free module over a domain R (such as $\mathbb{Z}_p$), an automorphism α of M is said to be a *reflection* (or sometimes a *pseudoreflection* or *generalized reflection*) if the endomorphism $(\alpha - \mathrm{Id})$ of M has rank one. A subgroup of $\mathrm{Aut}(M)$ is said to be *generated by reflections* if it is generated as a group by the reflections it contains.

3.8. Proposition. *Suppose that X is a connected p-compact group of rank r. Then the action of W_X on L_X is faithful and represents W_X as a finite subgroup of $\mathrm{GL}_r(\mathbb{Z}_p)$ generated by reflections.*

3.9. Proposition. *Suppose that X is a connected p-compact group with maximal torus T, Weyl group W, and rank r. Then the cohomology rings $\mathrm{H}^*_{\mathbb{Q}_p}(\mathrm{B}T)$ and $\mathrm{H}^*_{\mathbb{Q}_p}(\mathrm{B}X)$ are polynomial algebras over $\mathbb{Q}_p$ of rank r, and the natural restriction map $\mathrm{H}^*_{\mathbb{Q}_p}(\mathrm{B}X) \to \mathrm{H}^*_{\mathbb{Q}_p}(\mathrm{B}T)^W$ is an isomorphism.*

3.10. Proposition. *If X is a p-compact group, then the cohomology ring $\mathrm{H}^*(\mathrm{B}X; \mathbb{F}_p)$ is finitely generated as an algebra over $\mathbb{F}_p$.*

4 Centers and product decompositions

A *product decomposition* of a p-compact group X is a way of writing X up to homotopy as a product of two p-compact groups, or, equivalently, a way of writing $\mathrm{B}X$ up to homotopy as a product of spaces. The most general product theorem is the following one.

4.1. Proposition. *If X is a connected p-compact group, then there is a natural bijection between product decompositions of X and product decompositions of L_X as a module over W_X.*

In general, connected p-compact groups are constructed from indecomposable factors in much the same way that Lie groups are, by twisting the factors together over a finite central subgroup.

4.2. Definition. A subgroup Y of a p-compact group X is said to be *normal* if the usual map $\mathcal{N}_X(Y) \to X$ is an equivalence. The subgroup Y is *central* if the usual map $\mathcal{C}_X(A) \to X$ is an equivalence.

If the subgroup Y of X is normal, then there is a loop space structure on X/Y (because X/Y is equivalent to the Weyl space $\mathcal{W}_X(Y)$) and a short exact sequence $Y \to X \to X/Y$ of p-compact groups.

4.3. Proposition. *Any central subgroup of a p-compact group X is both abelian and normal; moreover, there exists up to homotopy a unique maximal central subgroup $\mathcal{Z}_X$ of X (called the* center *of X). The center of X can be identified as $\mathcal{C}_X(X)$.*

The center of X is maximal in the sense that up to conjugacy it contains *any* central subgroup A of X.

4.4. Proposition. *If X is a connected p-compact group, the quotient $X/\mathcal{Z}_X$ has trivial center.*

If X is connected, the quotient $X/\mathcal{Z}_X$ is called the *adjoint form* of X. For connected X there is a simple way to compute $\mathcal{Z}_X$ from what amounts to ordinary algebraic data associated to the normalizer $\mathcal{N}_X(T)$ of a maximal torus T in X.

4.5. Definition. A connected p-compact group X is said to be *almost simple* if the action of W_X on $\mathbb{Q}\otimes L_X$ affords an irreducible representation of W_X over $\mathbb{Q}_p$; X is *simple* if X is almost simple and $\mathcal{Z}_X = \{e\}$.

4.6. Proposition. *Any 1-connected p-compact group is equivalent to a product of almost simple p-compact groups. The product decomposition is unique up to permutation of factors.*

4.7. Proposition. *Any connected p-compact group with trivial center is equivalent to a product of simple p-compact groups. The product decomposition is unique up to permutation of factors.*

4.8. Proposition. *Any connected p-compact group is equivalent to a p-compact group of the form*

$$(T \times X_1 \times \cdots \times X_n)/A$$

where T is a p-compact torus, each X_i is a 1-connected almost simple p-compact group, and A is a finite abelian p-subgroup of the center of the indicated product.

5 Examples and conjectures

Call a connected p-compact group *exotic* if it is not equivalent to the $\mathbb{F}_p$-completion of a connected compact Lie group. The reader may well ask whether there are any exotic p-compact groups, or whether on the other hand the study of p-compact groups is just a way of doing ordinary Lie theory under artificially difficult circumstances. In fact, there are many exotic examples: Sullivan constructed loop space structures on the $\mathbb{F}_p$-completions of various odd spheres S^n $(n > 3)$ [33], and it is possible to do more elaborate things along the same lines, see, e.g., [1] and [5].

Conjecturally, the theory splits into two parts.

5.1. Conjecture. *Any connected p-compact group can be written as a product $X_1 \times X_2$, where X_1 is the $\mathbb{F}_p$-completion of a compact Lie group and X_2 is a product of exotic simple p-compact groups.*

In addition, all of the exotic examples are conjecturally known.

5.2. Conjecture. *The exotic simple p-compact groups correspond bijectively, up to equivalence, to the exotic p-adic reflection groups of Clark and Ewing [1].*

Here a p-adic reflection group is said to be *exotic* if it is not derived from the Weyl group of a connected compact Lie group. For example, it would follow from 5.2 there is up to equivalence only *one* exotic simple 2-compact group, the one constructed in [5]. Closely related to the above conjectures is the following one.

5.3. Conjecture. *Let X be a connected p-compact group with maximal torus T. Then X is determined up to equivalence by the loop space $\mathcal{N}_X(T)$.*

This would the analog for p-compact groups of a Lie-theoretic result of Curtis, Wiederhold and Williams [2]. It is easy to see that the loop space $\mathcal{N}_X(T)$ is determined by the Weyl group W_X, the p-adic lattice L_X, and an extension class in $\mathrm{H}^3(W_X; L_X)$. Explicit calculation with examples shows that if p is odd the extension class vanishes. It would be very interesting to find a simple, direct way to construct a connected p-compact group X from $\mathcal{N}_X(T)$. For a connected compact Lie group G, this would (according to [2]) give a direct way to constuct the homotopy type of $\mathrm{B}G$, or at least the $\mathbb{F}_p$-completion of this homotopy type, from combinatorial data associated to the root system of G. All constructions of this type which are known to the author involve building a Lie algebra and then exponentiating it; this kind of procedure does not generalize to p-compact groups.

The strongest results along the lines of 5.3 are due to Notbohm [27] [28].

The theory of homomorphisms between general p-compact groups is relatively undeveloped, though there is a lot of information available if the domain is a p-compact toral group or if the homomorphism is a rational equivalence [12] [14]. The general situation seems complicated [13], but it might be possible to find some analog for p-compact groups of the results of Jackowski and Oliver [15] on "homotopy representations" of compact Lie groups (see for instance [16]).

References

[1] A. Clark and J. Ewing, *The realization of polynomial algebras as cohomology rings*, Pacific J. Math. 50 (1974), 425–434.

[2] M. Curtis, A. Wiederhold, and B. Williams, *Normalizers of maximal tori*, (1974), 31–47. Lecture Notes in Math., Vol. 418.

[3] E. Dror, W. G. Dwyer, and D. M. Kan, *An arithmetic square for virtually nilpotent spaces*, Illinois J. Math. 21 (1977), no. 2, 242–254.

[4] W. G. Dwyer and C. W. Wilkerson, *The elementary geometric structure of compact Lie groups*, Bull. L. M. S., to appear.

[5] ______, *A new finite loop space at the prime two*, J. Amer. Math. Soc. 6 (1993), no. 1, 37–64.

[6] ______, *Homotopy fixed-point methods for Lie groups and finite loop spaces*, Ann. of Math. (2) 139 (1994), no. 2, 395–442.

[7] ______, *The center of a p-compact group*, The Čech centennial (Boston, MA, 1993), Contemp. Math., vol. 181, Amer. Math. Soc., Providence, RI, 1995, pp. 119–157.

[8] ______, *Product splittings for p-compact groups*, Fund. Math. 147 (1995), no. 3, 279–300.

[9] W. G. Dwyer, H. R. Miller, and C. W. Wilkerson, *The homotopic uniqueness of BS^3*, Algebraic topology, Barcelona, 1986, Lecture Notes in Math., vol. 1298, Springer, Berlin, 1987, pp. 90–105.

[10] H. Hopf, *Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen*, Ann. of Math. (2) 42 (1941), 22–52.

[11] J. R. Hubbuck, *On homotopy commutative H-spaces*, Topology 8 (1969), 119–126.

[12] S. Jackowski, J. McClure, and B. Oliver, *Homotopy classification of self-maps of BG via G-actions. I*, Ann. of Math. (2) 135 (1992), no. 1, 183–226.

[13] ______, *Maps between classifying spaces revisited*, The Čech centennial (Boston, MA, 1993), Contemp. Math., vol. 181, Amer. Math. Soc., Providence, RI, 1995, pp. 263–298.

[14] ______, *Self-homotopy equivalences of classifying spaces of compact connected Lie groups*, Fund. Math. 147 (1995), no. 2, 99–126.

[15] S. Jackowski and B. Oliver, *Vector bundles over classifying spaces of compact Lie groups*, Acta Math. 176 (1996), no. 1, 109–143.

[16] A. Jeanneret and A. Osse, *The K-theory of p-compact groups*, Comment. Math. Helv. 72 (1997), no. 4, 556–581.

[17] D. M. Kan, *A combinatorial definition of homotopy groups*, Ann. of Math. (2) 67 (1958), 282–312.

[18] R. M. Kane, *The homology of Hopf spaces*, North-Holland Mathematical Library, vol. 40, North-Holland Publishing Co., Amsterdam, 1988.

[19] J. Lannes, *Sur les espaces fonctionnels dont la source est le classifiant d'un p-groupe abélien élémentaire*, Inst. Hautes Études Sci. Publ. Math. (1992), no. 75, 135–244, With an appendix by Michel Zisman.

[20] ______, *Théorie homotopique des groupes de Lie (d'après W. G. Dwyer et C. W. Wilkerson)*, Astérisque (1995), no. 227, Exp. No. 776, 3, 21–45, Séminaire Bourbaki, Vol. 1993/94.

[21] J. Milnor, *Construction of universal bundles. I*, Ann. of Math. (2) 63 (1956), 272–284.

[22] ______, *Construction of universal bundles. II*, Ann. of Math. (2) 63 (1956), 430–436.

[23] J. M. Møller and D. Notbohm, *Centers and finite coverings of finite loop spaces*, J. Reine Angew. Math. 456 (1994), 99–133.

[24] J. M. Møller, *Homotopy Lie groups*, Bull. Amer. Math. Soc. (N.S.) 32 (1995), no. 4, 413–428.

[25] D. Notbohm, *Unstable splittings of classifying spaces of p-compact groups*, preprint (Göttingen) 1994.

[26] ———, *Classifying spaces of compact Lie groups and finite loop spaces*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 1049–1094.

[27] D. Notbohm, *Homotopy uniqueness of classifying spaces of compact connected Lie groups at primes dividing the order of the Weyl group*, Topology 33 (1994), no. 2, 271–330.

[28] ———, *Topological realization of a family of pseudoreflection groups*, Fund. Math. 155 (1998), no. 1, 1–31.

[29] D. L. Rector, *Loop structures on the homotopy type of S^3*, (1971), 99–105. Lecture Notes in Math., Vol. 249.

[30] ———, *Subgroups of finite dimensional topological groups*, J. Pure Appl. Algebra 1 (1971), no. 3, 253–273.

[31] J.-P. Serre, *Homologie singulière des espaces fibrés. Applications*, Ann. of Math. (2) 54 (1951), 425–505.

[32] J. D. Stasheff, *Homotopy associativity of H-spaces. I, II*, Trans. Amer. Math. Soc. 108 (1963), 275-292; ibid. 108 (1963), 293–312.

[33] D. Sullivan, *Genetics of homotopy theory and the Adams conjecture*, Ann. of Math. (2) 100 (1974), 1–79.

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