# Constructions of Smooth 4-Manifolds

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Abstract. We describe a collection of constructions which illustrate a panoply of "exotic" smooth 4-manifolds.

1991 Mathematics Subject Classification: 57R55

## 1. Introduction

At the time of the previous (1994) International Congress of Mathematicians, steady, but slow, progress was being made on the classification of simply connected closed smooth 4-manifolds. In particular, the Donaldson invariants had begun to take a particularly nice form [13] (also [4]), their computations were becoming more routine [3], and their behavior under blowing up (i.e. taking connected sum with $\overline{\mathbf{CP}}^2$) was well understood [2]. Due to the complexity of the Donaldson invariants, great hope was held out that an even better understanding of these invariants would close the books on the classification of simply connected 4-manifolds.

A few short months after the 1994 ICM, the 4-manifold community was blind-sided by the introduction of the now famous Seiberg-Witten equations [28]. Most of the results obtained by using Donaldson theory were found to have quicker, and sometimes more general, counterparts using the Seiberg-Witten technology. The potential applications of the difficult Donaldson technology became much more transparent using these new equations. As of July 1998, there is good news as well as bad news. The good news is that many of the earlier focus problems have been solved. In particular, the Thom conjecture [14] and its natural generalizations have been verified [20, 21]; also the study of symplectic 4-manifolds has taken a more central role [23, 24, 25, 26]. The bad news is that recent constructions and computations indicate that the Seiberg-Witten and Donaldson theories are too weak to distinguish simply connected smooth 4-manifolds [6]. It is these latter constructions and computations that we will discuss at this 1998 International Congress of Mathematicians. It is becoming more apparent that we are seeing only a small constellation of 4-dimensional manifolds. More seriously, we are lacking a reasonable conjectural classification of simply connected closed smooth 4-manifolds.

Current technology has given us many more 4-manifolds than had been expected in 1994. The authors hope that during the 2002 ICM the construction of large classes of new 4-manifolds will be discussed; in particular, they hope that a sufficiently large collection of 4-manifolds will have been discovered so as to allow

[1]Partially supported by NSF Grant DMS9704927
[2]Partially supported by NSF Grant DMS9626330

for some general patterns to emerge and, at least, a conjectural classification to again be on the books.

## 2. The knot surgery construction

Let $X$ be a simply connected oriented smooth closed 4-manifold. Its most basic invariant is its intersection form

$$Q_X : H_2(X;\mathbf{Z}) \otimes H_2(X;\mathbf{Z}) \to \mathbf{Z}$$

defined by counting signed transverse intersections of embedded oriented surfaces representing given homology classes. It is a famous theorem of M. Freedman [10] that $Q_X$ determines the homeomorphism type of $X$, and an equally renowned theorem of S.K. Donaldson [1] that $Q_X$ is not sufficient to determine the diffeomorphism type of $X$. In this section we shall discuss geometric operations on a given smooth 4-manifold which preserve the underlying topological structure and alter its smooth structure. In particular, we shall consider the following construction: Let $X$ be a simply connected smooth 4-manifold which contains a smoothly embedded torus $T$ of self-intersection 0. Given a knot $K$ in $S^3$, we replace a tubular neighborhood of $T$ with $S^1 \times (S^3 \setminus K)$ to obtain the *knot surgery manifold* $X_K$.

More formally, this procedure is accomplished by performing 0-framed surgery on $K$ to obtain the 3-manifold $M_K$. The meridian $m$ of $K$ can be viewed as a circle in $M_K$; so in $S^1 \times M_K$ we have the smooth torus $T_m = S^1 \times m$ of self-intersection 0. Since a neighborhood of $m$ has a canonical framing in $M_K$, a neighborhood of the torus $T_m$ in $S^1 \times M_K$ has a canonical identification with $T_m \times D^2$. The knot surgery manifold $X_K$ is given by the fiber sum

$$X_K = X \#_{T=T_m} S^1 \times M_K = (X \setminus T \times D^2) \cup (S^1 \times M_K \setminus T_m \times D^2)$$

where the two pieces are glued together so as to preserve the homology class $[\mathrm{pt} \times \partial D^2]$. This latter condition does not, in general, completely determine the isotopy type of the gluing, and $X_K$ is taken to be any manifold constructed in this fashion.

Because $S^1 \times (S^3 \setminus K)$ has the same homology as a tubular neighborhood of $T$ in $X$ (and because the gluing preserves $[\mathrm{pt} \times \partial D^2]$) the homology and intersection form of $X_K$ will agree with that of $X$. If it is also assumed that $X \setminus T$ is simply connected, then $\pi_1(X_K) = 1$; so $X_K$ will be homeomorphic to $X$.

In order to distinguish the diffeomorphism types of the $X_K$, we rely on Seiberg-Witten invariants. We view the Seiberg-Witten invariant of a smooth 4-manifold as a multivariable (Laurent) polynomial. To do this, recall that the Seiberg-Witten invariant of a smooth closed oriented 4-manifold $X$ with $b_2^+(X) > 1$ is an integer-valued function which is defined on the set of *spin*$^c$ structures over $X$ (cf. [28]). In case $H_1(X,\mathbf{Z})$ has no 2-torsion (for example, as here where $X$ is simply connected) there is a natural identification of the *spin*$^c$ structures of $X$ with the characteristic elements of $H_2(X,\mathbf{Z})$ (i.e. those elements $k$ whose Poincaré duals $\hat{k}$ reduce mod 2 to $w_2(X)$). In this case we view the Seiberg-Witten invariant as

$$\mathrm{SW}_X : \{k \in H_2(X,\mathbf{Z}) | \hat{k} \equiv w_2(TX) \pmod 2)\} \to \mathbf{Z}.$$

The sign of $\mathcal{SW}_X$ depends on an orientation of $H^0(X,\mathbf{R}) \otimes \det H^2_+(X,\mathbf{R}) \otimes \det H^1(X,\mathbf{R})$. If $\mathrm{SW}_X(\beta) \neq 0$, then $\beta$ is called a *basic class* of $X$. It is a fundamental fact that the set of basic classes is finite. Furthermore, if $\beta$ is a basic class, then so is $-\beta$ with $\mathrm{SW}_X(-\beta) = (-1)^{(\mathrm{e}+\mathrm{sign})(X)/4}\,\mathrm{SW}_X(\beta)$ where $\mathrm{e}(X)$ is the Euler number and $\mathrm{sign}(X)$ is the signature of $X$.

Now let $\{\pm\beta_1,\dots,\pm\beta_n\}$ be the set of nonzero basic classes for $X$. Consider variables $t_\beta = \exp(\beta)$ for each $\beta \in H^2(X;\mathbf{Z})$ which satisfy the relations $t_{\alpha+\beta} = t_\alpha t_\beta$. We may then view the Seiberg-Witten invariant of $X$ as the Laurent polynomial

$$\mathcal{SW}_X = \mathrm{SW}_X(0) + \sum_{j=1}^{n} \mathrm{SW}_X(\beta_j)\cdot(t_{\beta_j} + (-1)^{(\mathrm{e}+\mathrm{sign})(X)/4}\, t_{\beta_j}^{-1}).$$

As an example of this notational device, consider the simply connected minimally elliptic surface $E(n)$ with holomorphic Euler characteristic $n$ and no multiple fibers. Its Seiberg-Witten invariant is $\mathcal{SW}_{E(n)} = (t - t^{-1})^{n-2}$ where $t = t_F$ for $F$ the fiber class. Thus, $\mathrm{SW}_{E(n)}((n-2m)F) = (-1)^{m-1}\binom{n-2}{m-1}$ for $m = 1,\dots,n-1$ and $\mathrm{SW}_{E(n)}(\alpha) = 0$ for any other $\alpha$. When $b^+(X) > 1$, the Laurent polynomial $\mathcal{SW}_X$ is a diffeomorphism invariant of $X$.

For our theorem, we need to place a mild hypothesis on the embedded torus $T$. We say that a smoothly embedded torus representing a nontrivial homology class $[T]$ is *c-embedded* if there is a neighborhood $N$ of $T$ in $X$ and a diffeomorphism $\varphi : N \to U$ where $U$ is a neighborhood of a cusp fiber in an elliptic surface and $\varphi(T)$ is a smooth elliptic fiber in $U$. Equivalently, $T$ is c-embedded if it contains two simple closed curves which generate $\pi_1(T)$ and which bound vanishing cycles in $X$. Note that a c-embedded torus has self-intersection 0.

THEOREM 2.1 ([6]). *Let $X$ be a simply connected oriented smooth 4-manifold with $b^+ > 1$. Suppose that $X$ contains a c-embedded torus $T$ with $\pi_1(X \setminus T) = 1$, and let $K$ be any knot in $S^3$. Then the knot surgery manifold $X_K$ is homeomorphic to $X$ and has Seiberg-Witten invariant*

$$\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(t)$$

*where $\Delta_K(t)$ is the symmetrized Alexander polynomial of $K$ and $t = \exp(2[T])$.*

For example, the theorem applies to the K3-surface $E(2)$ where $T$ is a smooth elliptic fiber, and since $\mathcal{SW}_{E(2)} = 1$, we have $\mathcal{SW}_{E(2)_K} = \Delta_K(t)$. It is a theorem of Seifert that any Laurent polynomial of the form $P(t) = a_0 + \sum_{j=1}^{n} a_j(t^j + t^{-j})$ with coefficient sum $P(1) = \pm 1$ is the Alexander polynomial of some knot in $S^3$. Call such a Laurent polynomial an *A-polynomial.* It follows that if $(X,T)$ satisfies the hypothesis of Theorem 2.1, then for any $A$-polynomial $P(t)$, there is a smooth simply connected 4-manifold $X_P$ which is homeomorphic to $X$ and has Seiberg-Witten invariant $\mathcal{SW}_{X_P} = \mathcal{SW}_X \cdot P(t)$ where $t = \exp(2[T])$. In particular, for each $A$-polynomial $P(t)$, there is a manifold homeomorphic to the K3-surface with $\mathcal{SW} = P(t)$.

The relationship between Seiberg-Witten type invariants and the Alexander polynomial was first discovered by Meng and Taubes. In [17] they showed that the 3-manifold Seiberg-Witten invariant is related to Milnor torsion.

If one starts with a fibered knot $K$, then $S^1 \times M_K$ is a surface bundle over a torus and thus carries a symplectic structure [27] for which $T_m$ is a symplectic submanifold. Thus if $X$ is a symplectic 4-manifold containing a c-embedded symplectic torus $T$, then $X_K = X\#_{T=T_m} S^1 \times M_K$ is also symplectic [11, 16]. In a fashion similar to the treatment of the Seiberg-Witten invariant as a Laurent polynomial, one can view the Gromov invariant of a symplectic 4-manifold $X$ as a polynomial $\mathcal{G}r_X = \sum \mathrm{Gr}_X(\beta)\, t_\beta$ where $\mathrm{Gr}_X(\beta)$ is the usual Gromov invariant of $\beta$. Let $\mathrm{A}_K(t) = t^d \Delta_K(t)$ denote the normalized Alexander polynomial, where $d$ is the degree of $\Delta_K(t)$. As a corollary to Theorem 2.1 and the theorems of Taubes relating the Seiberg-Witten and Gromov invariants of a symplectic 4$-$manifold [25, 26] we have:

Corollary 2.2 ([6]). *Let $X$ be a symplectic 4-manifold with $b^+ > 1$ containing a symplectic c-embedded torus $T$. If $K$ is a fibered knot, then $X_K$ is a symplectic 4-manifold whose Gromov invariant is $\mathcal{G}r_{X_K} = \mathcal{G}r_X \cdot A_K(\tau)$ where $\tau = \exp([T])$.*

This last calculation can also be made purely within the realm of symplectic topology [12, 15]. Our interest is directed more to the opposite situation. The Alexander polynomial of a fibered knot is monic; i.e. its top coefficient is $\pm 1$. On the other hand:

Corollary 2.3 ([6]). *If $\Delta_K(t)$ is not monic, then $X_K$ does not admit a symplectic structure. Furthermore, if $X$ contains a homologically nontrivial surface $\Sigma_g$ of genus $g$ disjoint from $T$ with $[\Sigma_g]^2 < 2-2g$ if $g>0$ or $[\Sigma_g]^2 < 0$ if $g=0$, then $X_K$ with the opposite orientation does not admit a symplectic structure.*

Until the summer of 1996, it was still a plausible conjecture (sometimes called the 'minimal conjecture') that each irreducible simply connected 4-manifold should admit a symplectic structure with one of its orientations. The first counterexamples to this conjecture were constructed by Z. Szabo [22]. The knot surgery construction gives a multitude of examples of simply connected irreducible 'nonsymplectic' 4-manifolds. In fact, if $X$ is simply connected with $\mathcal{SW}_X \neq 0$ and if $X$ contains a c-embedded torus $T$ with $\pi_1(X \setminus T) = 1$, then Theorem 2.1 and Corollary 2.3 imply that there are infinitely many distinct nonsymplectic smooth 4-manifolds $X_K$ homeomorphic to $X$.

If $K_1$ and $K_2$ have the same Alexander polynomial, Seiberg-Witten invariants are not able to distinguish $X_{K_1}$ from $X_{K_2}$. For example, take $X = E(2)$. Then $X_K$ has a self-intersection 0 homology class $\sigma$ satisfying $\sigma \cdot [T] = 1$ which is represented by an embedded surface of genus $g(K)+1$ where $g(K)$ is the genus of $K$. One might hope that these classes could be used to distinguish $X_{K_1}$ from $X_{K_2}$ when $g(K_1) \neq g(K_2)$.

Conjecture . *For $X = E(2)$, the manifolds $X_{K_1}$ and $X_{K_2}$ are diffeomorphic if and only if $K_1$ and $K_2$ are equivalent knots.*

The proof of Theorem 2.1 proceeds by successively simplifying the manifold $X_K$ in a fashion which mimics the calculation of the Alexander polynomial of $K$

via skein relations. Recall that $\Delta_K(t)$ can be calculated via the relation

$$\Delta_{K_+}(t) = \Delta_{K_-}(t) + (t^{1/2} - t^{-1/2}) \cdot \Delta_{K_0}(t) \tag{1}$$

where $K_+$ is an oriented knot or link, $K_-$ is the result of changing a single oriented positive (right-handed) crossing in $K_+$ to a negative (left-handed) crossing, and $K_0$ is the result of resolving the crossing as shown in Figure 1.

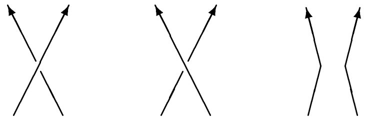

Figure 1

The point of using (1) to calculate $\Delta_K$ is that $K$ can be simplified to an unknot via a sequence of crossing changes. One builds a 'resolution tree' starting from $K$ and at each stage adding the bifurcation of Figure 2, where each $K_+$, $K_-$, $K_0$ is a knot or 2-component link, and so that at the bottom of the tree, there are only unknots, and split links. Then, because the Alexander polynomial of an unknot is 1, and is 0 for a split link (of more than one component) one can can work backwards using (1) to calculate $\Delta_K(t)$.

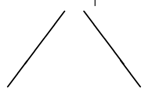

Figure 2

The manifold $X_{K_+}$ can be obtained from $X_{K_-}$ by means of a $(+1)$-log transform on a nullhomologous torus in $X_{K_-}$, and then the gluing theorems of [18] show that $\mathcal{SW}_{X_{K_+}}$ can be computed in terms of the Seiberg-Witten invariants of $X_{K_-}$ and a manifold $X_{K_-,0}$ obtained by a 0-log transform on $X_{K_-}$. With some work, this leads to a related resolution diagram of 4-manifolds where each knot $K'$ corresponds to $X_{K'}$, and this diagram can be used to prove Theorem 2.1.

We conclude this section by pointing out that the knot surgery construction can be generalized to manifolds with $b^+ = 1$ and to links in $S^3$ of more than one component in a more-or-less obvious way. One glues the complements of c-embedded tori in 4-manifolds to the product of $S^1$ with the link complement. See [6] for details. For example, if to each boundary component of $S^1 \times (S^3 \setminus N(L))$ we glue $E(1)$ minus the neighborhood of a smooth elliptic fiber, we obtain a manifold with $\mathcal{SW} = \Delta_L(t_1, \ldots, t_n)$, the multivariable Alexander polynomial of the link. Szabo's examples in [22] can be obtained from this construction.

## 3. Embeddings of surfaces in 4-manifolds

Knot surgery can also be used to change the embedding of a surface in a fixed 4-manifold. To motivate the construction, note that one can tie a knot in the core $\{0\}\times I$ of a cylinder $D^2\times I$ by removing a tubular neighborhood of a meridian circle and replacing it with a knot complement $S^3 \setminus K$. We shall perform a parametrized version of this construction in the 4-manifold setting. Consider an oriented surface $\Sigma$ of genus $g > 0$ which is smoothly embedded in a simply connected 4-manifold $X$. Let $\alpha$ be a simple closed curve on $\Sigma$ which is part of a symplectic basis, and let $\alpha \times I$ be an annular neighborhood of $\alpha$ in $\Sigma$. In $X$ we see the neighborhood $D^2 \times \alpha \times I$. For a fixed knot $K$ in $S^3$, we parametrize the above construction so as to perform it on each of the cylinders $D^2 \times \{y\} \times I$, $y \in \alpha$, to obtain an embedded surface $\Sigma_K$. This is equivalent to performing knot surgery on the (nullhomologous) rim torus $R = \partial D^2 \times \alpha$. We call this operation *rim surgery.*

Theorem 3.1 ([7]). *Let $X$ be a simply connected smooth 4-manifold with an embedded surface $\Sigma$ of positive genus. Suppose that $\pi_1(X \setminus \Sigma) = 1$. Then for each knot $K$ in $S^3$, rim surgery produces a surface $\Sigma_K$, and there is a homeomorphism $(X, \Sigma) \cong (X, \Sigma_K)$.*

The Seiberg-Witten invariant can be used to study these embeddings, but first, an auxilliary construction is needed. For each positive integer $g$, let $Y_g$ be the union of the Milnor fiber of the $(2, 2g + 1, 4g + 1)$ Brieskorn singularity and a generalized nucleus consisting of the 4-manifold obtained as the trace of the 0-framed surgery on $(2, 2g + 1)$ torus knot in $\partial B^4$ and a $-1$ surgery on a meridian. Then $Y_g$ is a Kahler surface and admits a holomorphic fibration over $\mathbf{CP}^1$ with generic fiber a surface $S_g$ of genus $g$.

Let $(X, \Sigma)$ be as in Theorem 3.1, and suppose that the self-intersection $\Sigma^2 = 0$. We call $(X, \Sigma)$ an *SW-pair* if satisfies the property that $\mathcal{SW}_{X\#_{\Sigma=S_g}Y_g} \neq 0$. (In general, if $\Sigma^2 = n > 0$, one makes this definition by first blowing up $n$ times.) For example, if $X$ is symplectic and $\Sigma$ is a symplectic submanifold (of square 0), then $X\#_{\Sigma=S_g}Y_g$ is symplectic, and it follows that $(X, \Sigma)$ is an SW-pair. In $X\#_{\Sigma=S_g}Y_g$, the rim torus $R$ becomes homologically essential and is c-embedded. We can use Theorem 2.1 to calculate Seiberg-Witten invariants:

$$\mathcal{SW}_{X\#_{\Sigma_K=S_g}Y_g} = \mathcal{SW}_{(X\#_{\Sigma=S_g}Y_g)_K} = \mathcal{SW}_{X\#_{\Sigma=S_g}Y_g} \cdot \Delta_K(r)$$

where $r = \exp(2[R])$, viewing $[R]$ as a class in the fiber sum. We have:

Theorem 3.2 ([7]). *Consider any SW-pair $(X, \Sigma)$ with $\Sigma^2 \geq 0$. If $K_1$ and $K_2$ are two knots in $S^3$ and if there is a diffeomorphism of pairs $(X, \Sigma_{K_1}) \cong (X, \Sigma_{K_2})$, then $\Delta_{K_1}(t) = \Delta_{K_2}(t)$.*

As a special case:

Theorem 3.3 ([7]). *Let $X$ be a simply connected symplectic 4-manifold and $\Sigma$ a symplectically embedded surface of positive genus and nonnegative self-intersection. Assume also that $\pi_1(X \setminus \Sigma) = 1$. If $K_1$ and $K_2$ are knots in $S^3$ and if $(X, \Sigma_{K_1}) \cong (X, \Sigma_{K_2})$, then $\Delta_{K_1}(t) = \Delta_{K_2}(t)$. Furthermore, if $\Delta_K(t) \neq 1$, then $\Sigma_K$ is not smoothly ambient isotopic to a symplectic submanifold of $X$.*

The second part of the theorem holds because if $\Sigma_K$ were symplectic, $X\#_{\Sigma_K=S_g}Y_g$ would be a symplectic manifold. The symplectic form $\omega$ on this manifold is inherited from the forms on $X$ and $Y_g$; so $\langle \omega, R \rangle = 0$. But $\mathcal{SW}_{X\#_{\Sigma_K=S_g}Y_g} = \mathcal{SW}_{X\#_{\Sigma=S_g}Y_g} \cdot \Delta_K(r)$, and it follows that the among the basic classes $k$ of $X\#_{\Sigma_K=S_g}Y_g$, more than one has $\langle\omega,k\rangle$ maximal. This contradicts the fact that, for a symplectic manifold, the maximality of $\langle\omega,K\rangle$ characterizes the canonical class among all basic classes [24].

## 4. Fiber sums of holomorphic Lefschetz fibrations

In this section we shall construct for every integer $g\geq 3$ a pair $(X_g, X'_g)$ of simply connected complex surfaces carrying holomorphic genus $g$ Lefschetz fibrations with the property that their fiber sum (along a regular fiber) is a symplectic 4-manifold $Z_g$ which supports no complex structure; in fact $Z_g$ is not even homeomorphic to a complex manifold.

Let $T(p,q)$ denote the $(p,q)$ torus knot in $S^3$ and let $N(p,q)$ denote the 4-manifold obtained by attaching a 2-handle to the 4-ball along $T(p,q)$ with 0-framing. It is well known that $N(p,q)$ is a Lefschetz fibration over $D^2$ with generic fiber a Riemann surface of genus $g(p,q)=(p-1)(q-1)/2$. Let $W(p,q)$ denote the canonical resolution of the Brieskorn singularity $\Sigma(p,q,pq)$, the Seifert-fibered 3-manifold with three exceptional fibers of order $p$, $q$, and $pq$, and with $H_1=\mathbf{Z}$. It is known that $W(p,q)$ also supports the structure of a genus $g(p,q)$ Lefschetz fibration over $D^2$ with a singular fiber over 0 which is a sequence of 2-spheres plumbed according to the resolution diagram of $\Sigma(p,q,pq)$. Finally, let

$$Z(p,q)=W(p,q)\cup N(p,q).$$

The manifold $Z(p,q)$ is a rational surface which is diffeomorphic to the connected sum of $\mathbf{CP}^2$ and $r(p,q)$ copies of $\overline{\mathbf{CP}}^2$ for some computable integer $r(p,q)$. Furthermore, $Z(p,q)$ supports the structure of a holomorphic Lefschetz fibration whose fiber has genus $g(p,q)$.

Now consider nontrivial torus knots $T(p,q)$ and $T(p',q')$ with the property that $g(p,q)=g(p',q')$. (This is possible for every $g(p,q)\geq 3$.) Let $F(p,q;p',q')$ denote the fiber sum along a regular fiber of $Z(p,q)$ with $Z(p',q')$. Then $F(p,q;p',q')$ is a simply connected symplectic 4-manifold with

$$c_1^2=10+8g(p,q)-r(p,q)-r(p',q'),\ \ \chi=(b^+ +1)/2=1+g(p,q).$$

Furthermore, $F(p,q;p',q')$ supports the structure of a Lefschetz fibration with fiber of genus $g(p,q)$. A computation of the Seiberg-Witten invariants of $F(p,q;p',q')$ shows that, up to sign, there is a unique Seiberg-Witten basic class. It follows that $F(p,q;p',q')$ is minimal.

Conjecture . *$F(p,q;p',q')$ supports the structure of a complex 4-manifold if and only if $\{p,q\}=\{p',q'\}$.*

As evidence, consider the pairs $(2,2n+1)$ and $(3,n+1)$, $n\not\equiv 2 \mod 3$. For $F(2,2n+1;3,n)$ one can show that $r(2,2n+1)=4n+4$ and $r(3,n+1)=3n+7$ so that

$$c_1^2=n-2,\ \ \chi=n+1.$$

Thus, $c_1^2=\chi-3$, which violates the Noether inequality $c_1^2\geq 2\chi-6$. This means that $F(2,2n+1;3,n)$ is a minimal symplectic 4-manifold that is not even homotopy equivalent to a complex manifold. In fact, it can be shown that the fiber sum of $Z(2,2n+1)$ with itself is the elliptic surface $E(n+1)$ and that the fiber sum of $Z(3, n+1)$ with itself is a Horikawa surface with $\chi = n+1$. Furthermore $F(2, 2n+1; 3, n)$ can be obtained from $E(n+1)$ by removing from $Z(2, 2n+1) \setminus F \subset E(n+1)$, $F$ a regular fiber, the regular neighborhood of the configuration of $(n-2)$ 2-spheres:

$$\overset{-(n+1)}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{-2}{\bullet}\!\!-\!\!-\!\!-\ \cdots\ -\!\!-\!\!-\!\!\overset{-2}{\bullet}$$

whose boundary is the lens space $L((n-1)^2, -n)$ and replacing it with the rational ball that this lens space bounds. (See [3] for all the details concerning this rational blowdown procedure.) Thus $F(2, 2n+1; 3, n)$ is the manifold $Y(n)$ constructed in Lemma 7.5 of [3].

## 5. Homeomorphic but non-diffeomorphic 4-manifolds with the same Seiberg-Witten invariants

In this section we construct examples of a pair $(X_1, X_2)$ of symplectic 4-manifolds with $X_1$ homeomorphic to $X_2$, $\mathcal{SW}_{X_1} = \mathcal{SW}_{X_2}$, but $X_1$ is not diffeomorphic to $X_2$. To do this choose a pair of fibered 2-bridge knots $K(\alpha, \beta_1)$ and $K(\alpha, \beta_2)$ with the same Alexander polynomials; for example $K_1 = K(105, 64)$ and $K_2 = K(105, 76)$ with Alexander polynomial

$$\Delta_K(t) = t^{-4} - 5t^{-3} + 13t^{-2} - 21t^{-1} + 25 - 21t + 13t^2 - 5t^3 + t^4.$$

Although these knots have the same Alexander polynomial, they can be distinguished by the fact that their branch covers are the lens spaces $L(\alpha, \beta_1)$ and $L(\alpha, \beta_2)$ which are distinct; in our specific case $L(105, 64)$ is not diffeomorphic to $L(105, 76)$. These knots are also distinguished by their dihedral linking numbers; let $S_{K_1}$ and $S_{K_2}$ denote the 2-fold covers of $S^3$ branched over $K_1$ and $K_2$, with lifted branched loci $\tilde{K}_1$ and $\tilde{K}_2$, respectively. Thus we have knots $\tilde{K}_i$ in $S_{K_i} = L(\alpha, \beta_i)$. Take the $\alpha$-fold covers of these lens spaces to obtain links $L_i = \{K_1^{(i)}, \dots, K_\alpha^{(i)}\}$ which are the lifts of the branch loci $\tilde{K}_i$. The linking numbers of the links $L_1$ and $L_2$ are known as the 'dihedral linking numbers' of the 2-bridge knot $K(\alpha, \beta)$.

Now perform the knot surgery construction of §2 on the $K3$ surface, replacing $T^2 \times D^2$ with $S^1 \times (S_{K_j} \setminus \tilde{K}_j)$. The resulting 4-manifolds are the manifolds $X_i$. Either by adapting the arguments of [6] or by using [12] or [15], it can be shown that $\mathcal{SW}_X = \mathcal{SW}_Y = \Delta_K(t) \cdot \Delta_K(-t)$. Unfortunately, the $X_i$ are not simply connected (but are homeomorphic). In particular, $\pi_1(X_1) = \pi_1(X_2) = \mathbf{Z}_\alpha$, and the $\alpha$-fold covers $\tilde{X}_1$ and $\tilde{X}_2$ of $X_1$ and $X_2$ are not diffeomorphic. To see this, observe that $\tilde{X}_i$ is obtained as our link construction in [6] (cf. § 2) by gluing one copy of $E(2)$ minus a neighborhood of a smooth elliptic fiber to every boundary component of $S^1 \times (S^3 \setminus L_i)$. It follows from [6] that

$$\mathrm{SW}_{\tilde{X}_i} = \Delta_{L_i}(t_1, \dots, t_\alpha) \cdot \prod_{j=1}^{\alpha} (t_j^{1/2} - t_j^{-1/2})$$

Since the linking numbers of the links $L_1$ and $L_2$ are different, it can be shown the Hosokawa polynomials of the links $L_1$ and $L_2$, when evaluated at 1 are distinct

[9]. Thus their Alexander polynomials are different and $\tilde{X}_1$ is not diffeomorphic to $\tilde{X}_2$.

There is a lesson to be learned from these examples. One must consider the Seiberg-Witten invariants of a 4-manifold $X$ together with those of all of its covers as the appropriate invariant for $X$.

## 6. Nonsymplectic 4-manifolds with one basic class

Recall from § 2, that if $k$ is a basic class of $X$, so is $-k$. Because of this, we say that $X$ *has n basic classes* if the set $\{k \mid \mathrm{SW}_X(k) \neq 0\}/\{\pm 1\}$ consists of $n$ elements. There are abundant examples of 4-manifolds with one basic class. Minimal nonsingular algebraic surfaces of general type have one basic class (the canonical class) [28]. The authors and others have constructed many examples of minimal symplectic manifolds with one basic class and $\chi - 3 \leq {c_1}^2 < 2\chi - 6$. (These manifolds cannot admit complex structures due to the geography of complex surfaces.) However, the examples described here are the first nonsymplectic manifolds with one basic class.

Let $X = E(2)$ and $T$ a smooth elliptic fiber. For a knot $K$ of genus $g$ form the knot surgery construction to obtain $X_K$. In $X_K$ there is a surface $\Sigma$ of genus $g+1$ with $[\Sigma]^2 = 0$ and $[\Sigma] \cdot [T] = 1$. Let $M$ be the 3-manifold obtained from 0-surgery on the trefoil knot. Then $S^1 \times M$ is a $T^2$-fiber bundle over $T^2$. The fiber sum of $g+1$ copies of the fiber bundle gives a 4-manifold $Y$ which is an $F = T^2$-bundle over a surface of genus $g+1$, and it is easily seen that there is a section $C$. Furthermore, $Y$ is a symplectic 4-manifold with $c_1(Y) = -2g[F]$. Our example, corresponding to the genus $g$ knot $K$ is $Z_K = X_K \#_{\Sigma = C} Y$. We perform this fiber sum so that $Z_K$ is a spin 4-manifold [11]. It can be seen to be simply connected.

Write the symmetrized Alexander polynomial of $K$ as $\Delta_K(t) = a_0 + \sum_{n=1}^{d} a_n(t^n + t^{-n})$, and call $d$ the *degree* of $\Delta_K(t)$. Since the genus of $K$ is $g$, we have $d \leq g$. If $K$ is an alternating knot, for example, then $d = g$. Say that the Alexander polynomial of $K$ has *maximal degree* if $d = g$. Using techniques of [20] we calculate:

Theorem 6.1 ([8]). *Let $K$ be a knot in $S^3$ whose Alexander polynomial has maximal degree. Then $Z_K$ has one basic class, $k$, with $|SW_{Z_K}(k)| = a_d$, the top coefficient of $\Delta_K(t)$. When $|a_d| > 1$, $Z_K$ is nonsymplectic.*

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