Analytic Structures on Representation Spaces of Reductive Groups

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Abstract. We show that every admissible representation of a real reductive group has a canonical system of Sobolev norms parametrized by positive characters of a minimal parabolic subgroup. These norms are compatible with morphisms of representations. Similar statement also holds for representations of reductive p-adic groups.

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1. Analytic structures on representation spaces

Let us fix a real reductive algebraic group G. We would like to study analytic structures which are naturally defined on a representation (π, G, V) of the group G.

The results which I discuss in this lecture hold for an arbitrary reductive group G but to explain the ideas and motivations I will discuss only the group $G = SL(2, \mathbb{R})$ and later on the more interesting case of the group $G = SL(3, \mathbb{R})$.

Historically, mathematicians first were interested only in unitary representations of G and for such representations analytic structure is rather clear.

But later it was realized that it is convenient to introduce and study also some continuous representations (π, G, V) . Here V is a complex topological vector space (we consider only Banach and Frechet spaces); representation π of the group G in the space V is called continuous if the corresponding map $G \times V \to V$ is continuous.

Here we immediately encounter a problem how we should think about a representation. In order to explain the problem consider the simplest case of the group $G = SL(2, \mathbb{R})$.

The typical representation of this group is described as follows. Fix a complex number λ and consider the space D_λ of even homogeneous functions on the punctured plane $\mathbb{R}^2 \setminus 0$ of homogeneous degree $\lambda - 1$. Then the group $G = SL(2, \mathbb{R})$ naturally acts on the space D_λ and we denote this representation as π_λ.

Now the problem with this description of the representation π_λ is that we have not specified the class of functions which we consider. We can take smooth functions, or functions which are locally L^2 or one of the many other classes of functions (L^p, Sobolev functions, Besov functions and so on).

It is easy to see that all these representations will be non isomorphic as topological representations. However it is intuitively clear that this are just different analytic realizations of the "same" representation or, in other words, all these representations are equivalent.

Harish Chandra proposed the following way to handle this problem. We say that a representation (π, G, V) is ADMISSIBLE if its restriction to a maximal compact subgroup $K \subset G$ has finite multiplicities and it has finite length as a topological representation.

Given such a representation we denote by V_f the space of K-finite vectors (a vector $v \in V$ is called K-FINITE if the subset $\pi(K)v$ lies in a finite dimensional subspace of V). The space V_f is not a G-module, but it has natural actions of the group K and of the Lie algebra $\mathfrak{g}$ of the group G. These two actions are compatible in a natural way.

Now we can abstractly define a purely algebraic notion of a $(\mathfrak{g}, K)$-module as a vector space E equipped with two actions, of the Lie algebra $\mathfrak{g}$ and of the group K, which satisfy these compatibility conditions. We say that a $(\mathfrak{g}, K)$-module E is a HARISH CHANDRA module if it is finitely generated as a $\mathfrak{g}$-module and has finite multiplicities as a K-module (see details in [1]).

Starting with an admissible topological representation (π, G, V) we have constructed a Harish Chandra module V_f. Now, following Harish Chandra, we say that two admissible topological representations are EQUIVALENT if the corresponding Harish Chandra modules are isomorphic.

I propose a slightly different point of view on this problem. Let us agree that AN ADMISSIBLE REPRESENTATION π of the group G is an equivalence class of admissible topological representations of G. Concrete topological representations in this class we consider as different "analytic realizations" of a given representation π.

With such an understanding we see that representations of G are parameterized by Harish Chandra modules (it is not difficult to show that every Harish Chandra module corresponds to some topological representation). In particular, the very difficult problem of classification of irreducible representations of G is reduced to a still difficult, but purely algebraic, problem of classification of irreducible Harish Chandra modules. This classification problem has been solved by several different algebraic methods.

1.1. WHAT WE WANT TO ACHIEVE. Let us come back to our analytic problem. Suppose we are given a representation (for example represented by a Harish Chandra module E). We would like to describe some NATURAL analytic structures which we can define on this representation.

In order to explain what we are after let us consider first a model case. Namely, suppose we are given a C^∞ manifold M and a C^∞ vector bundle E on M and we would like to study possible analytic structures on the space V of the sections of E.

We may consider many different analytic structures on V: smooth sections, C^n-sections, L^2-sections (or, more generally, L^p-sections), different kinds of Sobolev spaces of sections, of Besov spaces of sections and so on.

There is a convenient way to represent all these structures. Namely, we fix the space $V = V^\infty$ of smooth sections and study different topologies $\mathcal{T}$ on this space.

For a given topology $\mathcal{T}$ we denote by $L_\mathcal{T}$ the completion of the space V with respect to $\mathcal{T}$. It is convenient to consider the space $L_\mathcal{T}$ as a subspace of the space of distribution sections of the bundle E.

For example, while the space V does not have a canonical structure of a (pre)Hilbert space it clearly has a canonically defined Hilbert topology (i.e. topology defined by a norm N of Hermitian type , which means that the function $v \mapsto N(v)^2$ is a Hermitian form on V). The completion L of the space V with respect to this topology is a canonically defined Hilbertian space of sections of E.

More generally, for every real number s we can canonically define L^2 type Sobolev topology $\mathcal{T}_s$. It is defined by a Sobolev Hermitian norm S_s on the space V; the completion of the space V with respect to this norm is the Sobolev space of sections L_s.

The explicit description of Sobolev norms S_s is standard, but a little involved. The easiest way to define them is to use Fourier transform - but we are trying to avoid this since we will not be able to generalize this method.

A relatively simple description can be given when $s = k$ is a positive integer. In this case, if E is a trivial bundle and ϕ is a section of E (i.e. a function) supported in a small neighborhood with coordinates (x_i) we can define the Sobolev norm $S_k(\phi)$ to be $(\sum |\partial_\alpha \phi|^2)^{1/2}$, where the sum is over all multiindeces α of degree less or equal to k.

It is important that each analytic structure $\mathcal{T}$ on the space V which we considered has local description. Formally, this means that for any smooth function f on M the operator of multiplication by f is continuous in the corresponding topology; thus using the partition of unit we see that a distribution section v lies in the completion $L_\mathcal{T}$ if and only if this holds locally.

1.2. Analytic structures on representation spaces. Now let us come back to the case of an admissible representation (π, G, V). For every such representation we can consider its smooth part (π, G, V^∞), where V^∞ is the space of smooth vectors $v \in V$ (a vector $v \in V$ is called smooth if the corresponding function $G \to V$, $g \mapsto \pi(g)(v)$, is smooth).

By a remarkable theorem of Casselman and Wallach, for every two realizations of a given admissible representation π their smooth parts are canonically isomorphic as topological representations; in fact they have shown that the functor $V \mapsto V_f$ defines an equivalence of the category of smooth admissible representations of G and the category of Harish Chandra modules (see details in [2]).

In other words, any representation has a canonical "smooth model" (π, G, V) for which $V = V^\infty$.

My aim in this lecture is to discuss different analytic structures (in particular Sobolev structures) which can be canonically constructed on a given representation

π of the group G. As before, we will describe these structures as different topologies on the smooth model (π, G, V) of the given representation π.

2. Case of the group $SL(2, \mathbb{R})$

Consider as a simple example the group $G = SL(2, \mathbb{R})$ and its representation $(\pi_\lambda, G, D_\lambda)$ in the space of homogeneous functions described above. This representation is called a principle series representation; it is induced by some unramified character μ of the Borel subgroup $B \subset G$, $\pi_\lambda = Ind_B^G(\mathbb{C}_\mu)$.

Since the space D_λ is realized as the space of homogeneous functions on $\mathbb{R}^2 \setminus 0$, by restricting to the unit circle $S^1 \subset \mathbb{R}^2 \setminus 0$ we can identify the space V with the space $\mathcal{F}$, where $\mathcal{F} = C^\infty(S^1)_{even}$ is the space of even functions on the unit circle.

Thus, we can define s Sobolev structure on the space V using the norm N_s given in the realization D_λ by the formula $N_s(v) = S_s(v)$.

The problem with this definition is that for generic λ the space V has two natural realizations, as D_λ and as $D_{-\lambda}$, and the norms N_s obtained from these two realizations are not equivalent. Hence this approach does not work.

However, let us analyze these two realizations more carefully. Since the spaces D_λ and $D_{-\lambda}$ are both identified with the space $\mathcal{F}$ the equivalence between them is given by some operator $I_\lambda : \mathcal{F} \to \mathcal{F}$. This operator, which is usually called an INTERTWINING OPERATOR, can be explicitly described. It turns out that it is a pseudo-differential operator of order r, where $r = \operatorname{Re}\lambda$. In fact, it can be realized as a convolution operator with some distribution R_λ on S^1 which is smooth outside of the origin and near the origin is more or less homogeneous of degree $-\lambda - 1$.

So let us try to define the s Sobolev norm N_s on the space V to be $N_s = S_{s+r/2}$, where $r = \operatorname{Re}(\lambda)$ and where the Sobolev norm $S_{s+r/2}$ is computed using the realization D_λ. Then from the description above one can immedeatly see that the corresponding topology on the space V does not depend on the choice of the realization (at least for generic λ). This allows us to define a canonical s-Sobolev structure on the space V (for generic λ).

There are also other representations of principle series. They correspond to characters of the Borel subgroup which lie in a different component, i.e. have different discrete parameters compared with characters λ above. These representations are realized in the space of odd functions on $\mathbb{R}^2 \setminus 0$; since locally on S^1 they are exactly the same as representations D_λ they have the same analytic structure. Thus the Sobolev norm N_s on representations D_λ induces similar norm on these "odd" representations. In other words, discrete parameters do not affect the analytic structure.

We have described our Sobolev norm for generic point λ. For arbitrary λ this construction may not work - for example the operator I_λ may have pole, or, after normalization, it becomes bounded but not invertible. However there are standard algebraic methods which allow to reduce the study of these cases to the study of representations with generic λ.

Now we can use a deep algebraic theorem that every Harish Chandra module E can be imbedded into some generalized representation of principle series, i.e. a representation induced from some finite dimensional representation of the Borel subgroup B. We can extend our Sobolev norm N_s to these generalized principle series.

Then, again using deep algebraic results about Harish Chandra modules, we can show that the resulting norm N_s on the space E does not depend on the choice of a particular embedding.

Thus using these methods we can extend the definition of the Sobolev norm N_s to all admissible representations (π, G, V^∞) of the group G.

In particular the norm N_0 defines a CANONICAL Hilbertian structure on an admissible representation.

3. Construction of Sobolev norms for a general group G

Let us discuss the case of a general reductive group G. For simplicity assume that G is split (e.g. $G = SL(n, \mathbb{R})$). Then again we can consider a series of representations $(\pi_\lambda, G, D_\lambda)$ parameterized by unramified characters λ of the split Cartan subgroup $A \subset G$; each of these representations can be realized in the space $\mathcal{F}$ of functions on the flag variety $X = G/B$, where B is a Borel subgroup.

In this case it is not clear how to define Sobolev norms on V, since the usual family of Sobolev norms S_s on the space $\mathcal{F}$ depends on one parameter s while representations D_λ depend on several parameters λ.

However, it turns out that the flag variety X has a very special geometric structure. Using this structure we can equip the space $\mathcal{F}$ of functions on X with a canonical system of Sobolev norms S_s parameterized by points s of the $\mathbb{R}$-linear space $\mathfrak{a}^* = \mathrm{Mor}(A, \mathbb{R}^+)$.

This space $\mathfrak{a}^*$ is dual to the Cartan sub algebra $\mathfrak{a} = \mathrm{Lie}(A)$. Using the exponential map we will identify the space $\mathfrak{a}^*$ with the group of positive characters of the Cartan group A, or, equivalently, with the group of positive characters of the Borel group B. We will mostly think about the space $\mathfrak{a}^*$ in this realization; for example, in case of a general reductive group G the space $\mathfrak{a}^*$ is defined as $\mathfrak{a}^* = \mathrm{Mor}(P, \mathbb{R}^+) \simeq \mathrm{Mor}(P, \mathbb{R}^{+*})$, where P is the minimal parabolic subgroup of G.

Now, similarly to the $SL(2)$ case, we can define s-Sobolev norm N_s on the space V as $N_s(v) = S_{s+r/2}(v)$, where the positive character $r = \mathrm{Re}(\lambda) \in \mathfrak{a}^*$ is defined by $r(a) = |\lambda(a)|$ and the norm $S_{s+r/2}$ is computed in D_λ realization.

This construction defines a canonical system of the Sobolev norms N_s on any representation V of the group G which is isomorphic to one of the representations D_λ for generic λ. This system of Sobolev structures is parameterized by points $s \in \mathfrak{a}^*$.

Again, using algebraic methods we can reduce the case of an arbitrary admissible representation V to one of these non degenerate cases and using the definition described above we can define a canonical system of Sobolev topologies $\mathcal{T}_s$ on the space V.

4. Construction of the family of Sobolev norms S_s on the space $\mathcal{F}$

Let us describe how to construct the Sobolev norms S_s on the space $\mathcal{F}$ of functions on the flag variety X. For simplicity we consider only the case of the group $G = SL(3,\mathbb{R})$ – it shows all the ideas and all the difficulties.

In this case the space $\mathfrak{a}^*$ is two dimensional; it is best realized as a quotient space of the space $\{(x_1, x_2, x_3)\}$ modulo the subspace $\{(x,x,x)\}$. Unramified characters of the Borel group are parameterized by the points λ in the complexification $\mathfrak{a}^*_{\mathbb{C}}$. There are also some other characters which differ from the characters λ by some discrete parameters. As before we can ignore these discrete parameters and reduce all constructions to finding the family of Sobolev norms S_s on the space $\mathcal{F}$.

By definition, the space D_λ is unitarily induced from the character λ of the Borel subgroup B (which in this case is the subgroup of upper triangular matrices). This means that D_λ consists of smooth functions ϕ on G satisfying $\phi(bg) = \mu(b)\phi(g)$ for $g \in G$ and $b \in B$; here $\mu = \rho^{-1}\lambda$ is the character of the Borel subgroup B which differs from λ by the standard character ρ.

Restricting these functions to the maximal compact subgroup K we will identify all the spaces D_λ with the space $\mathcal{F}$ of smooth functions on the flag variety X.

We have the natural action of the Weyl group $W = S_3$ on $\mathfrak{a}^*$ and on $\mathfrak{a}^*_{\mathbb{C}}$ given by permutation of coordinates.

It is known that for generic λ all representations $D_{w\lambda}$ are isomorphic. Let us describe this more specifically for the case of a simple reflection σ; for example we consider the simple root $\alpha = (1,-1,0) \in \mathfrak{a}^*$ and the corresponding permutation $\sigma = \sigma_\alpha \in W$ of indeces 1 and 2. In this case $\sigma\lambda$ has the form $\sigma_\lambda = \lambda - \lambda_\alpha \cdot \alpha$ for some number λ_α and the equivalence between spaces D_λ and $D_\sigma\lambda$ can be described using an intertwining operator $I_{\sigma,\lambda_\alpha} : \mathcal{F} \to \mathcal{F}$, which depends only on σ and on the number λ_α.

In fact, in this case the operator I can be described quite explicitly. Namely, consider the natural fibration of the flag variety X over a Grassmannian $X_\alpha = \mathrm{G}r_{2,3}$. The fibers of this filtration are circles, and on each of these circles we can define an intertwining operator I_α as in the case of $SL(2,\mathbb{R})$. Together these operators represent the operator $I_{\sigma,\lambda_\alpha} : \mathcal{F} \to \mathcal{F}$.

The system of Sobolev spaces L_s for $s \in \mathfrak{a}^*$ should satisfy the following condition:

(*) $$I_{\sigma,\lambda_\alpha} L_s \subset L_{s - r_\alpha \alpha/2}, \text{ where } r_\alpha = \operatorname{Re}\lambda_\alpha.$$

Since any weight $s \in \mathfrak{a}^*$ is a linear combination of simple weights α and β we see that this condition, together with the similar condition for the root β and with the condition that the space L_0 is the space of L^2-functions, completely determine all the Sobolev norms S_s on $\mathcal{F}$ (up to topological equivalence). Namely if $s = a\alpha + b\beta$ we have to define L_s to be the image $I_{\sigma_\alpha,-2a} \circ I_{\sigma_\beta,-2b}(L^2(X))$ (for generic s). One can check that this definition gives a family of Sobolev norms satisfying (*).

5. Remarks

Remark 1. Let (π, G, V) be a unitary admissible representation of G. Then we have two Hilbertian structures on V - one canonical structure described above and another given by the unitary structure on V. It is natural to assume that these two structures always coincide (and in simplest cases this is true).

If this conjecture holds it may help in the description of unitary representations of the group G.

Remark 2. If we consider representations of a p-adic reductive group G then we will find exactly the same analytic structures. They are parameterized by points s of the real vector space $\mathfrak{a}^* = \mathrm{Mor}(A, \mathbb{R}^+)$, where A is the maximally split Cartan group of G. The proof in this case is different, since there are many representations of G which can not be realized on flag varieties (so called cuspidal representations).

Remark 3. In case of the group $SL(2,\mathbb{R})$ we can use Calderon-Zygmund theorem which implies that the intertwining operator I_r is continuous with respect to L^p Sobolev norms, i.e. it defines a continuous operator $I_r : (\mathcal{F}, S_{p,s}) \to (\mathcal{F}, S_{p,s-r})$. Using this we can canonically define L^p Sobolev structures on representations isomorphic to D_λ.

Thus, it is probable that if we fix a number $p \in [1, \infty)$ then for any group G and any admissible representation (π, G, V) of G we can define a canonical family of L^p type Sobolev norms $N_{p,s}$ on the space V which is parameterized by points $s \in \mathfrak{a}^*$.

Probably the same construction will also yield canonical Besov structures on the space V.

References

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