From Double Hecke Algebra to Analysis

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Abstract. We discuss q-counterparts of the Gauss integrals, a new type of Gauss-Selberg sums at roots of unity, and q-deformations of Riemann's zeta. The paper contains general results, one-dimensional formulas, and remarks about the current projects involving the double affine Hecke algebras.

Keywords and Phrases: Hecke algebra, Fourier transform, spherical function, Macdonald polynomial, Gauss integral, Gaussian sum, metaplectic representation, Verlinde algebra, braid group, zeta function.

Introduction.

The note is about the role of double affine Hecke algebras in the unification of the classical zonal and p-adic spherical functions and the corresponding Fourier transforms. The new theory contains one more parameter q and, what is important, dramatically improves the properties of the Fourier transform. In contrast to the real and p-adic theories, the q-transform is selfdual and has practically all other important properties of the classical Fourier transform. Here I will mainly discuss the Fourier-invariance of the Gaussian.

There are various applications. In combinatorics, they are via the Macdonald polynomials. As $q \to 1$, we complete the Harish-Chandra theory of the spherical transform. The limit $q \to \infty$ covers the p-adic Iwahori-Matsumoto-Macdonald theory. When q is a root of unity, we generalize the Verlinde algebras, directly related to quantum groups and Kac-Moody algebras, and come to a new class of Gauss-Selberg sums.

However the main applications could be of more analytic nature. The representation of the double affine Hecke algebra generated by the Gaussian and its Fourier transform can be described in full detail. So the next step is to examine the spaces generated by Gaussian-type functions. The Fourier transforms of the simplest examples lead to q-deformations of the classical zeta and L-functions.

Of course there are other projects involving the double Hecke algebras. I will mention at least some of them. The following is far from being complete.

1) *Macdonald's q-conjectures* [M1,M2]. Namely, the norm, duality, and evaluation conjectures [C1,C2]. My proof of the norm-formula is similar to that from [O1] in the differential case (the duality and evaluation conjectures collapse as $q \to 1$). I would add to this list the Pieri rules [C2]. As to the nonsymmetric Macdonald polynomials, see [O2,M3,C3]. See also [M3,DS,Sa] about the $C^\vee C$ (the Koornwinder polynomials), and [I,M4,C4] about the Aomoto conjecture.

2) *K-theoretic interpretation.* I mean the papers [KL1,KK] and more recent [GG,GKV]. Presumably it can lead to the Langlands-type description of irreducible representations of double Hecke algebras, but the answer can be more complicated than in [KL1] (see also recent Lusztig's papers on the representations of affine Hecke algebras with unequal parameters). The Fourier transform is misty in this approach. Let me add here the strong Macdonald conjecture (Hanlon).

3) *Induced and spherical representations.* The classification of the spherical representations is much simpler, as well as the irreducibility of the induced ones. I used the technique of intertwiners in [C4] following a similar theory for the affine Hecke algebras. The nonsymmetric polynomials form the simplest spherical representation. There must be connections with [HO1]. The intertwiners also serve as creation operators for the nonsymmetric Macdonald polynomials (the case of GL is due to [KS]).

4) *Radial parts via Dunkl operators.* The main references are [D1,H,C5]. In the latter it was observed that the trigonometric differential Dunkl operators form the degenerate (graded) affine Hecke algebra [L] ([Dr] for GL_n). The difference, elliptic, and difference-elliptic generalizations were introduced in [C6,C7,C8]. The nonsymmetric Macdonald polynomials are eigenfunctions of the difference Dunkl operators. The connections with the KZ-equation play an important role here. I mean Matsuo's and my theorems from [Ma,C5,C6]. See also [C9].

5) *Harmonic analysis.* In the rational-differential setup, the definition of the generalized Bessel functions is from [O3], the corresponding generalized Hankel transform was considered in [D2,J] (see also [He]). In contrast to the spherical transform, it is selfdual, as well as the difference generalization from [C2,C10]. The Mehta-Macdonald conjecture, directly related to the transform of the Gaussian, was checked in [M1,O1] in the differential case and generalized in [C10]. See [HO2,O2,C11] about applications to the Harish-Chandra theory.

6) *Roots of unity.* The construction from [C2] generalizes and, at the same time, simplifies the Verlinde algebras. The latter are formed by the so-called reduced representations of quantum groups at roots of unity. Another interpretation is via the Kac-Moody algebras [KL2] (due to Finkelberg for roots of unity). A valuable feature is the projective action of $PSL(2,\mathbf{Z})$ (cf. [K, Theorem 13.8]). In [C3] the nonsymmetric polynomials are considered, which establishes connections with the metaplectic (Weil) representations at roots of unity.

7) *Braids.* Concerning $PSL(2,\mathbf{Z})$, it acts projectively on the double Hecke algebra itself. The best known explanation (and proof) is based on the interpretation of this algebra as a quotient of the group algebra of the fundamental group of the elliptic configuration space [C6]. The calculation is mainly due to [B] in the GL-case. For arbitrary root systems, it is similar to that from [Le], but our configuration space is different. Switching to the roots of unity, there may be applications to the framed links including the Reshetikhin-Turaev invariants.

8) *Duality.* The previous discussion was about arbitrary root systems. In the case of GL, the theorem from [VV] establishes the duality between the double Hecke algebras and the q-toroidal (double Kac-Moody) algebras. It generalizes the classical Schur-Weyl duality, Jimbo's q-duality, and the affine analogues from [Dr,C12]. When the center charge is nontrivial it explains the results from [STU],

which were recently extended by Uglov to irreducible representations of the Kac-Moody gl_N of arbitrary positive integral levels.

Let me also mention the relations of the symmetric Macdonald polynomials (mainly of the GL-type) to: a) the spherical functions on q-symmetric spaces (Noumi and others), b) the interpolation polynomials (Macdonald, Lassalle, Knop and Sahi, Okounkov and Olshanski), c) the quantum gl_N (Etingof, Kirillov Jr.), d) the KZB-equation (—, —, Felder, Varchenko). There are connections with the affine Hecke algebra technique in the classical theory of GL_N and S_n. I mean, for instance, [C12], papers of Nazarov and Lascoux, Leclerc, Thibon, and recent results towards the Kazhdan-Lusztig polynomials.

The coefficients of the symmetric GL-polynomials have interesting combinatorial properties (Macdonald, Stanley, Garsia, Haiman, ...). These polynomials appeared in Kadell's work. Their norms are due to Macdonald, the evaluation and duality conjectures were checked by Koornwinder, the Macdonald operators were introduced independently by Ruijsenaars together with elliptic deformations.

Quite a few constructions can be extended to arbitrary finite groups generated by complex reflections. For instance, the Dunkl operators and the KZ-connection exist in this generality (Dunkl, Opdam, Malle). One can try the affine and even the hyperbolic groups (Saito's root systems).

1. One-dimensional formulas.

The starting point of many mathematical and physical theories is the formula:

$$2\int_0^\infty e^{-x^2}x^{2k}\mathrm{d}x \;=\; \Gamma(k+1/2),\ \Re k > -1/2. \tag{1}$$

Let us give some examples.

(a) Its generalization to the Bessel functions, namely, the invariance of the Gaussian e^{-x^2} with respect to the Hankel transform, is a cornerstone of the Plancherel formula.

(b) The following "perturbation" for the same $\Re k > -1/2$

$$\mathfrak{Z}(k) \stackrel{def}{=} 2\int_0^\infty (e^{x^2}+1)^{-1}x^{2k}\,\mathrm{d}x = (1-2^{1/2-k})\Gamma(k+1/2)\zeta(k+1/2) \tag{2}$$

is fundamental in the analytic number theory.

(c) The multi-dimensional extension due to Mehta with $\prod_{1\le i<j\le n}(x_i-x_j)^{2k}$ instead of x^{2k} gave birth to the theory of matrix models and the Macdonald theory with various applications in mathematics and physics.

(d) Switching to the roots of unity, the Gauss formula

$$\sum_{m=0}^{2N-1} e^{\frac{\pi m^2}{2N}i} \;=\; (1+i)\sqrt{N},\quad N\in\mathbf{N} \tag{3}$$

can be considered as a certain counterpart of (1) at $k=0$.

(e) Replacing x^{2k} by $\sinh(x)^{2k}$, we come to the theory of spherical and hypergeometric functions and to the spherical Fourier transform. The spherical transform of the Gaussian plays an important role in the harmonic analysis on symmetric spaces.

To employ modern mathematics at full potential, we do need to go from Bessel to hypergeometric functions. In contrast to the former, the latter can be studied, interpreted and generalized by a variety of methods in the range from representation theory and algebraic geometry to integrable models and string theory. However the straightforward passage $x^{2k} \to \sinh(x)^{2k}$ creates problems. The spherical transform is not selfdual anymore, the formula (1) has no sinh-counterpart, and the Gaussian looses its Fourier-invariance.

DIFFERENCE SETUP. It was demonstrated recently that these important features of the classical Fourier transform are restored for the kernel

$$\delta_k(x;q) \stackrel{def}{=} \prod_{j=0}^{\infty} \frac{(1-q^{j+2x})(1-q^{j-2x})}{(1-q^{j+k+2x})(1-q^{j+k-2x})},\ 0<q<1,\ k\in \mathbf{C}. \tag{4}$$

Actually the selfduality of the corresponding transform can be expected a priori because the Macdonald truncated theta-function δ is a unification of $\sinh(x)^{2k}$ and the Harish-Chandra function (A_1) serving the inverse spherical transform.

As to (1), setting $q = \exp(-1/a),\ a>0$,

$$(-i)\int_{-\infty i}^{\infty i} q^{-x^2}\delta_k \,\mathrm{d}x = 2\sqrt{a\pi}\prod_{j=0}^{\infty}\frac{1-q^{j+k}}{1-q^{j+2k}},\quad \Re k>0. \tag{5}$$

Here both sides are well-defined for all k except for the poles but coincide only when $\Re k > 0$, worse than in (1). This can be fixed as follows:

$$(-i)\int_{1/4-\infty i}^{1/4+\infty i} q^{-x^2}\mu_k \,\mathrm{d}x = \sqrt{a\pi}\prod_{j=1}^{\infty}\frac{1-q^{j+k}}{1-q^{j+2k}},\ \Re k > -1/2 \ \text{ for} \tag{6}$$

$$\mu_k(x;q) \stackrel{def}{=} \prod_{j=0}^{\infty} \frac{(1-q^{j+2x})(1-q^{j+1-2x})}{(1-q^{j+k+2x})(1-q^{j+k+1-2x})},\ 0<q<1,\ k\in \mathbf{C}. \tag{7}$$

The limit of (6) multiplied by $(a/4)^{k-1/2}$ as $a\to\infty$ is (1) in the imaginary variant.

Once we managed Γ, it would be unexcusable not to try (cf. (2))

$$\mathfrak{Z}_q(k) \stackrel{def}{=} (-i)\int_{1/4-\infty i}^{1/4+\infty i} (q^{x^2}+1)^{-1}\mu_k \,\mathrm{d}x \ \text{ for } \Re k > -1/2. \tag{8}$$

It has a meromorphic continuation to all k periodic in the imaginary direction. The limit of $(a/4)^{k-1/2}\mathfrak{Z}_q$ as $a\to\infty$ is $\mathfrak{Z}$ for all k except for the poles. The analytic continuation is based on the shift operator technique. It seems that all zeros of $\mathfrak{Z}_q(k)$ for $a>1, \Re k > -1/2$ are q-deformations of the zeros of $\mathfrak{Z}(k)$.

Jackson and Gauss sums. A most promising feature of special q-functions is a posiblility to replace the integrals by sums, the Jackson integrals.

Let $\int_\sharp$ be the integration for the path which begins at $z = \epsilon i + \infty$, moves to the left till ϵi, then down through the origin to $-\epsilon i$, and then returns down the positive real axis to $-\epsilon i + \infty$ (for small ϵ). Then for $|\Im k| < 2\epsilon, \Re k > 0$,

$$
\begin{aligned}
\frac{1}{2i}\int_\sharp q^{x^2}\delta_k \,\mathrm{d}x \; &= -\frac{a\pi}{2}\prod_{j=0}^{\infty}\frac{(1-q^{j+k})(1-q^{j-k})}{(1-q^{j+2k})(1-q^{j+1})}\times \mathfrak{g}_q^\sharp,\\
\mathfrak{g}_q^\sharp(k) &\stackrel{def}{=} \sum_{j=0}^{\infty} q^{\frac{(k-j)^2}{4}}\frac{1-q^{j+k}}{1-q^k}\prod_{l=1}^{j}\frac{1-q^{l+2k-1}}{1-q^l} \; = \\
& q^{\frac{k^2}{4}}\prod_{j=1}^{\infty}\frac{(1-q^{j/2})(1-q^{j+k})(1+q^{j/2-1/4+k/2})(1+q^{j/2-1/4-k/2})}{(1-q^j)}.
\end{aligned}
\tag{9}
$$

The sum $\mathfrak{g}_q^\sharp$ is the Jackson integral for a special choice $(k/2)$ of the starting point. The convergence of the sum (9) is for all k. Similarly,

$$
\begin{aligned}
\mathfrak{Z}_q^\sharp(k) &\stackrel{def}{=} -\frac{a\pi}{2}\prod_{j=0}^{\infty}\frac{(1-q^{j+k})(1-q^{j-k})}{(1-q^{j+2k})(1-q^{j+1})}\times \mathfrak{z}_q^\sharp,\\
\mathfrak{z}_q^\sharp(k) \; &= \; \sum_{j=0}^{\infty} q^{-kj}(q^{-\frac{(k+j)^2}{4}}+1)^{-1}\frac{1-q^{j+k}}{1-q^k}\prod_{l=1}^{j}\frac{1-q^{l+2k-1}}{1-q^l}.
\end{aligned}
\tag{10}
$$

For all k apart from the poles, $\lim_{a\to\infty}(\frac{a}{4})^{k-1/2}\mathfrak{Z}_q^\sharp(k) = \sin(\pi k)\mathfrak{Z}(k)$.

Numerically, it is likely that all zeros of $\mathfrak{Z}_q^\sharp$ in the strip

$$\{0 \le \Im k < \sqrt{2\pi a}-\epsilon,\ \Re k > -1/2\} \text{ for } a > 2/\pi \text{ and small } \epsilon$$

are deformations of the classical ones. Moreover there is a strong tendency for the deformations of the zeros of the $\zeta(k+1/2)$-factor to go to the right (big a). They are not expected in the left half-plane before $k = 1977.2714i$ (see [C13]).

When $q = \exp(2\pi i/N)$ and k is a positive integer $\le N/2$ we come to the Gauss-Selberg-type sums:

$$\sum_{j=0}^{N-2k} q^{\frac{(k-j)^2}{4}}\frac{1-q^{j+k}}{1-q^k}\prod_{l=1}^{j}\frac{1-q^{l+2k-1}}{1-q^l} = \prod_{j=1}^{k}(1-q^j)^{-1}\sum_{m=0}^{2N-1} q^{m^2/4}. \tag{11}$$

They resemble, for instance, [E,(1.2b)]. Substituting $k = [N/2]$ we arrive at (3).

Double Hecke algebras provide justifications and generalizations. In the A_1-case, $\mathcal{H}\!\!\!\mathcal{H} \stackrel{def}{=} \mathbf{C}[\mathcal{B}_q]/((T-t^{1/2})(T+t^{-1/2}))$ for the group algebra of the group $\mathcal{B}_q$ generated by $T, X, Y, q^{1/4}$ with the relations

$$TXT = X^{-1},\ T^{-1}YT^{-1} = Y^{-1},\ Y^{-1}X^{-1}YXT^2 = q^{-1/2} \tag{12}$$

for central $q^{1/4}, t^{1/2}$. Renormalizing $T \to q^{-1/4}T,\ X \to q^{1/4}X,\ Y \to q^{-1/4}Y$,

$$\mathcal{B}_q \cong \mathcal{B}_1 \mod q^{1/4},\ \mathcal{B}_1 \cong \pi_1(\{E \times E \setminus \mathrm{diag}\}/\mathbf{S}_2),\ E = \text{elliptic curve}, \quad (13)$$

a special case of the calculation from [B]. The T is the half-turn about the diagonal, X, Y correspond to the "periods" of E.

Thanks to the topological interpretation, the central extension $PSL_2^c(\mathbf{Z})$ of $PSL_2(\mathbf{Z})$ (Steinberg) acts on $\mathcal{B}_1$ and $\mathcal{H}$. The automorphisms corresponding to the generators $\binom{11}{01},\ \binom{10}{11}$ are as follows:

$$\tau_+ : Y \to q^{-1/4}XY,\ X \to X,\quad \tau_- : X \to q^{1/4}YX,\ Y \to Y, \quad (14)$$

fixing T, q, t. When $t = 1$ we get the well-known action of $SL_2(\mathbf{Z})$ on the Weyl and Heisenberg algebras (the latter as $q \to 1$). Formally, τ_+ is the conjugation by q^{x^2} for X represented here and later in the form $X = q^x$.

The Macdonald nonsymmetric polynomials are eigenfunctions of Y in the following $\mathcal{H}$-representation in the space $\mathcal{P}$ of the Laurent polynomials of q^x :

$$T \to t^{1/2}s + (q^{2x} - 1)^{-1}(t^{1/2} - t^{-1/2})(s - 1),\ Y \to spT \quad (15)$$

for the reflection $sf(x) = f(-x)$ and the translation $pf(x) = f(x + 1/2)$. It is nothing else but the representation of $\mathcal{H}$ induced from the character $\chi(T) = t^{1/2} = \chi(Y)$. The Fourier transform (on the generalized functions) is associated with the anti-involution $\{\varphi : X \to Y^{-1} \to X\}$ of $\mathcal{H}$ preserving T, t, q.

Combining τ_+ and φ, we prove that the Macdonald polynomials multiplied by q^{-x^2} are eigenfunctions of the q-Fourier transform and get (6) for $t = q^k$.

When q, k are from (11), let $q^x(m/2) = q^{m/2}$ for $m \in \mathbf{Z},\ -N < m \le N$, and

$$\bowtie \stackrel{def}{=} \{m \mid \mu_k(m/2) \ne 0\} = \{-N + k + 1, \ldots, -k,\ k + 1, \ldots, N - k\}.$$

The space $V_k = \mathrm{Funct}(\bowtie)$ has a unique structure of an (irreducible) $\mathcal{H}$-module making the evaluation map $\mathcal{P} \ni f \mapsto f(m/2) \in V_k$ a $\mathcal{H}$-homomorphism. Setting $V_k = V_k^+ \oplus V_k^-$ where $T = \pm t^{\pm 1/2}$ on $V_k^\pm$, the dimensions for $k < N/2$ are $2(N - 2k) = (N - 2k + 1) + (N - 2k - 1)$. The components $V_k^\pm$ are $PSL_2^c(\mathbf{Z})$-invariant. Calculating its action in V_k^+ (which is a subalgebra of V_k) we come to the formulas from [Ki,C2,C3]; V_k^- is $PSL_2^c(\mathbf{Z})$-isomorphic to V_{k+1}^+. For $k = 1$ it is the Verlinde algebra. Involving the the shift operator, we get (11).

We note that V_k may have applications to the arithmetic theory of coverings of elliptic curves ramified at one point thanks to (13).

2. General results.

Let $R = \{\alpha\} \subset \mathbf{R}^n$ be a root system of type $A, B, ..., F, G$ with respect to a euclidean form (z, z') on $\mathbf{R}^n \ni z, z'$, W the Weyl group generated by the reflections s_α, $\alpha_1, ..., \alpha_n$ simple roots, R_+ the set of positive roots, $\omega_1, ..., \omega_n$ the fundamental weights, $Q = \oplus_{i=1}^n \mathbf{Z}\alpha_i \subset P = \oplus_{i=1}^n \mathbf{Z}\omega_i$. We will also use coroots $\alpha^\vee = 2\alpha/(\alpha, \alpha)$

and the corresponding $Q^\vee$. The form will be normalized by the condition $(\theta,\theta)=2$ for the maximal coroot $\theta \in R_+^\vee$.

The affine Weyl group $\widetilde{W}$ acts on $\tilde{z}=[z,\zeta]\in \mathbf{R}^n\times\mathbf{R}$ and is generated by $s_i=s_{\alpha_i}$ and $s_0(\tilde{z}) = \tilde{z}+(z,\theta)\alpha_0$, for $\alpha_0=[-\theta,1]$. Setting $b(\tilde{z})=[z,\zeta-(z,b)]$ for $b\in P$, $\widetilde{W}=W\ltimes Q\subset \widehat{W}\stackrel{def}{=}W\ltimes P$. We call the latter the *extended affine Weyl group.* It is generated over $\widetilde{W}$ by the group $\pi\in\Pi\cong P/Q$ such that π leave the set $\alpha_0,\alpha_1^\vee,\ldots,\alpha_n^\vee$ invariant.

The length $l(\hat{w})$ of $\hat{w}=\pi\tilde{w}\in W^b,\ \pi\in\Pi,\tilde{w}\in W^a$ is by definition the length of the reduced decomposition of $\tilde{w}$ in terms of the simple reflections $s_i, 0\le i\le n$. Given $b\in P$, there is a unique decomposition

$$b=\pi_b w_b \ \text{ such that } \ w_b\in W,\ l(b)=l(\pi_b)+l(w_b) \text{ and } \ l(w_b)=\max. \quad (16)$$

Then $\Pi=\{\pi_{\omega_r}\}$ for the minuscule ω_r: $(\omega_r,\alpha^\vee)\le 1$ for all $\alpha\in R_+$.

Double Hecke algebras. Let $q_\alpha = q^{(\alpha,\alpha)/2}, t_\alpha = q_\alpha^{k_\alpha}$ for $\{k_\alpha\}$ such that $k_{w(\alpha)}=k_\alpha$ (all w), $t_i=t_{\alpha_i}$, $t_0=t_\theta$, $\rho_k=(1/2)\sum_{\alpha\in R_+}k_\alpha\alpha$,

$$X_{\tilde{b}} \ = \ \prod_{i=1}^n X_i^{l_i}q^l \ \text{ if } \ \tilde{b}=[b,l],\ b=\sum_{i=1}^n l_i\omega_i\in P,\ l\in(P,P)=(1/p)\mathbf{Z}$$

for $p\in\mathbf{N}$. By $\mathbf{C}_{q,t}^{\pm}[X]$ we mean the algebra of polynomials in terms of $X_i^{\pm 1}$ over the field $\mathbf{C}_{q,t}$ of rational functions of $q^{1/(2p)}, t_\alpha^{1/2}$. We will also use the evaluation $X_b(q^z)\stackrel{def}{=}q^{(b,z)}$.

The *double affine Hecke algebra* $\mathcal{H}$ is generated over the field $\mathbf{C}_{q,t}$ by the elements $\{T_j,\ 0\le j\le n\}$, pairwise commutative $\{X_i\}$, and the group Π where the following relations are imposed:

(o) $(T_j-t_j^{1/2})(T_j+t_j^{-1/2}) \ = \ 0, \ \ 0 \ \le \ j \ \le \ n;$
(i) $T_iT_jT_i\ldots \ = \ T_jT_iT_j\ldots,$ m_{ij} factors on each side;
(ii) $\pi T_i\pi^{-1}=T_j,\ \pi X_b\pi^{-1}=X_{\pi(b)}$ if $\pi\in\Pi,\ \pi(\alpha_i^\vee)=\alpha_j^\vee$;
(iii) $T_iX_bT_i \ = \ X_bX_{a_i}^{-1}$ if $(b,\alpha_i^\vee)=1,\ 1\le i\le n$;
(iv) $T_0X_bT_0 \ = \ X_{s_0(b)} \ = \ X_bX_\theta q^{-1}$ if $(b,\theta)=-1$;
(v) $T_iX_b \ = \ X_bT_i$ if $(b,\alpha_i^\vee)=0$ for $0\le i\le n$.

Here m_{ij} are from the corresponding Coxeter relations. Given $\tilde{w}\in\widetilde{W},\ \pi\in\Pi$, the product $T_{\pi\tilde{w}}\stackrel{def}{=}\pi T_{i_1}\cdots T_{i_l}$, where $\tilde{w}=s_{i_1}\cdots s_{i_l}$, $l=l(\tilde{w})$, does not depend on the choice of the reduced decomposition. In particular, we arrive at the pairwise commutative elements

$$Y_b \ = \ \prod_{i=1}^n Y_i^{l_i} \ \text{ if } \ b=\sum_{i=1}^n l_i\omega_i\in P, \ \text{ where } \ Y_i\stackrel{def}{=}T_{\omega_i}, \quad (17)$$

satisfying the relations $T_i^{-1}Y_bT_i^{-1} \ = \ Y_bY_{a_i}^{-1}$ if $(b,\alpha_i^\vee) \ = \ 1,$ $T_iY_b \ = \ Y_bT_i$ if $(b,\alpha_i^\vee)=0,\ 1\le i\le n.$

The Fourier transform is related to the anti-involution of $\mathcal{H\!H}$

$$\varphi:\ X_i \to Y_i^{-1},\ \ Y_i \to X_i^{-1},\ \ T_i \to T_i,\ t \to t,\ q \to q,\ 1 \le i \le n. \tag{18}$$

The "unitary" representations are defined for the anti-involution

$$X_i^* \ =\ X_i^{-1},\ Y_i^* \ =\ Y_i^{-1},\ T_i^* \ =\ T_i^{-1},\ t \to t^{-1},\ q \to q^{-1},\ 0 \le i \le n.$$

The next two automorphisms induce a projective action of $PSL_2(\mathbf{Z})$:

$$\begin{aligned} &\tau_+ :\ X_b \to X_b,\ \ Y_r \to X_r Y_r q^{-(\omega_r,\omega_r)/2},\ Y_\theta \to X_0^{-1} T_0^{-2} Y_\theta, \\ &\tau_- :\ Y_b \to Y_b,\ \ X_r \to Y_r X_r q^{(\omega_r,\omega_r)/2},\ X_\theta \to T_0 X_0 Y_\theta^{-1} T_0, \end{aligned} \tag{19}$$

where $b \in P$, ω_r are minuscule, $X_0 \ = qX_\theta^{-1}$. Obviously $\tau_- \ = \ \varphi\tau_+\varphi$. The projectivity means that $\tau_+^{-1}\tau_-\tau_+^{-1} = \tau_-\tau_+^{-1}\tau_-$.

POLYNOMIAL REPRESENTATION. Let $\hat{w}(X_{\tilde{b}}) \ = \ X_{\hat{w}(\tilde{b})}$ for $\hat{w} \in \widehat{W}$. Combining the action of the group Π, the multiplication by X_b, and the *Demazure-Lusztig operators*

$$T_j \ = \ t_j^{1/2} s_j \ + \ (t_j^{1/2} - t_j^{-1/2})(X_{\alpha_j} - 1)^{-1}(s_j - 1),\ 0 \le j \le n, \tag{20}$$

we get a representation of $\mathcal{H\!H}$ in $\mathbf{C}^{\pm}_{q,t}[X]$.

The coefficient of $X^0 = 1$ (*the constant term*) of a polynomilal $f \in \mathbf{C}^{\pm}_{q,t}[X]$ will be denoted by $\langle f \rangle$. Let

$$\mu \ = \ \prod_{a \in R_+^\vee} \prod_{i=0}^{\infty} \frac{(1 - X_\alpha q_\alpha^i)(1 - X_\alpha^{-1} q_\alpha^{i+1})}{(1 - X_\alpha t_\alpha q_\alpha^i)(1 - X_\alpha^{-1} t_\alpha q_\alpha^{i+1})}. \tag{21}$$

We will consider μ as a Laurent series with the coefficients in $\mathbf{C}[t][[q]]$. The form $\langle \mu_0 f g^* \rangle$ makes the polynomial representation unitary for

$$X_b^* \ = \ X_{-b},\ t^* \ = \ t^{-1},\ q^* \ = \ q^{-1},\ \mu_0 = \mu_0/\langle\mu\rangle = \mu_0^*.$$

The *Macdonald nonsymmetric polynomials* $\{e_b, b \in P\}$ are eigenvectors of the operators $\{L_f \stackrel{def}{=} f(Y_1, \cdots, Y_n), f \in \mathbf{C}^{\pm}_{q,t}[X]\}$:

$$L_f(e_b) \ = \ f(q^{-b_\sharp})e_b,\ \ \text{where}\ \ b_\sharp \stackrel{def}{=} b - w_b^{-1}(\rho_k)\ \ \text{for}\ \ w_b \text{ from (16)}. \tag{22}$$

They are pairwise orthogonal with respect to the above pairing and form a basis in $\mathbf{C}^{\pm}_{q,t}[X]$. The normalization $\epsilon_b \stackrel{def}{=} e_b/e_b(q^{-\rho_k})$ is the most convenient in the harmonic analysis. For instance, the duality relations become especially simple: $\epsilon_b(q^{c_\sharp}) = \epsilon_c(q^{b_\sharp})$ for all $b, c \in P$. The next formula establishes that ϵ_c multiplied by the Gaussian are eigenfunctions of the difference Fourier transform:

$$\begin{aligned} &\langle \epsilon_b \epsilon_c^* \tilde{\gamma}^{-1} \mu \rangle = q^{(b_\sharp,b_\sharp)/2 + (c_\sharp,c_\sharp)/2 - (\rho_k,\rho_k)} \epsilon_c^*(q^{b_\sharp}) \times \\ &\prod_{\alpha \in R_+} \prod_{j=1}^{\infty} \frac{1 - q_\alpha^{(\rho_k,\alpha^\vee)+j}}{1 - t_\alpha q_\alpha^{(\rho_k,\alpha^\vee)+j}} \quad \text{for}\ \ \tilde{\gamma}^{-1} \stackrel{def}{=} \sum_{b \in P} q^{(b,b)/2} X_b. \end{aligned} \tag{23}$$

When $b = c = 0$ we get (5). Indeed, the series for $\tilde{\gamma}^{-1}$ is nothing else but the expansion of γ^{-1} for $\gamma = q^{x^2/2}$, where we set $X_b = q^{x_b}$, $x^2 = \Sigma_{i=1}^n x_{\omega_i} x_{\alpha_i^\vee}$.

JACKSON AND GAUSS SUMS. We fix generic $\xi \in \mathbf{C}^n$ and set $\langle f\rangle_\xi \stackrel{def}{=} |W|^{-1} \sum_{w\in W, b\in B} f(q^{w(\xi)+b})$. Here f is a Laurent polynomial or any function well-defined on $\{q^{w(\xi)+b}\}$. We assume that $|q| < 1$. For instance, $\langle\gamma\rangle_\xi = \tilde{\gamma}^{-1}(q^\xi) q^{(\xi,\xi)/2}$. It is convenient to switch to $\mu^\circ(X,t) \stackrel{def}{=} \mu^{-1}(X, t^{-1})$. Given $b, c \in P$,

$$\begin{aligned}
&\langle \epsilon_b\, \epsilon_c^*\, \gamma\mu^\circ\rangle_\xi = q^{-(b_\sharp,b_\sharp)/2-(c_\sharp,c_\sharp)/2+(\rho_k,\rho_k)} \epsilon_c(q^{b_\sharp})\times \\
&|W|^{-1}\langle\gamma\rangle_\xi \prod_{\alpha\in R_+}\prod_{j=0}^{\infty} \frac{1 - t_\alpha^{-1} q_\alpha^{-(\rho_k,\alpha^\vee)+j}}{1 - q_\alpha^{-(\rho_k,\alpha^\vee)+j}}.
\end{aligned} \tag{24}$$

For $\xi = -\rho_k$, (24) generalizes (9). If $k \in \mathbf{Z}_+$, then $\mu^\circ = q^{\text{const}}\mu \in \mathbf{C}_q^\pm[X]$, the product in (24) is understood as the limit and becomes finite.

The proof of this formula and the previous one is based on the analysis of the anti-involution (18) in the corresponding representations of $\mathcal{H}$. Here it is the representation in $\mathcal{F} = \mathrm{Funct}(\widehat{W}, \mathbf{C}_{q,t}(q^{(\omega_i,\xi)}))$. For $a, b \in P$, $w \in W, \hat{v} \in \widehat{W}$, we set

$$X_a(bw) = X_a(q^{b+w(\xi)}),\ X_a g(bw) = (X_a g)(bw),\ \hat{v}(g)(bw) = g(\hat{v}^{-1}bw)$$

for $g \in \mathcal{F}$. It provides the action of X, Π. The T act as follows:

$$\begin{aligned}
T_i(g)(\hat{w}) \ &= \ \frac{t_i^{1/2} q^{(\alpha_i, b+w(\xi))} - t_i^{-1/2}}{q^{(\alpha_i,b+w(\xi))} - 1}\ g(s_i\hat{w}) \\
&- \frac{t_i^{1/2} - t_i^{-1/2}}{q^{(\alpha_i,b+w(\xi))} - 1}\ g(\hat{w}) \ \text{ for } \ 0 \le i \le n,
\end{aligned} \tag{25}$$

The formulas are closely connected with (20): the natural evaluation map from $\mathbf{C}_{q,t}^\pm[X]$ to $\mathcal{F}$ is a $\mathcal{H}$-homomorphism. The unitarity is for $\langle\mu_1 f g^*\rangle_\xi$, where the values of $\mu_1 = \mu/\mu(q^\xi) = \mu_1^\circ$ at $\hat{w}$ are $*$-invariant ($\xi^* = \xi$).

Dropping the X-action, we get a deformation of the regular representation of the affine Hecke algebra generated by T, Π. Indeed, taking ξ from the dominant affine Weyl chamber, (25) tend to the p-adic formulas from [Mat] when $q \to \infty$ and t are powers of p. For $\xi = -\rho_k$, the image of the restriction map from $\mathcal{F}$ to functions on the set $\{\pi_b, b \in P\}$, which is a $\mathcal{H}$-homomorphism, generalizes the spherical part of the regular representation. The limit to the Harish-Chandra theory is $q \to 1$ where k is fixed (the root multiplicity). See [He,C11].

Now q will be a primitive N-th root of unity, $P_N = P/(P \cap NQ^\vee)$; the evaluations of Laurent polynomials are functions on this set. Let $\langle f\rangle_N \stackrel{def}{=} \sum_{b\in P_N} f(q^b)$. We assume that $k_\alpha \in \mathbf{Z}_+$ for all $\alpha \in R$ and $\mu(q^{-\rho_k}) \ne 0$. We also pick q to ensure the existence of the Gaussian: $q^{(b,b)/2} = 1$ for all $b \in P \cap NQ^\vee$. It means that when N is odd and the root system is either B or C_{4l+2} one takes $q = \exp(4\pi i m/N)$ for $(m, N) = 1, 0 < 2m < N$. Otherwise it is arbitrary.

We claim that the formula (24) holds for $\langle\ \rangle_N$ instead of $\langle\ \rangle_\xi$ provided the existence of the nonsymmetric polynomials. It readily gives (11) for $b = 0 = c$.

Given $b' \in P_N$ such that $\mu(q^{b'}) \neq 0$, at least one ϵ_b exists with $b_\sharp$ equal to b' in P_N. Denoting the set of all such b' by P'_N, the space $\mathrm{Funct}(P'_N, \mathbf{Q}(q^{1/(2p)}))$ is an algebra and a $\mathcal{H}$-module isomorphic to the quotient of the polynomial representation by the radical of the pairing $\langle \mu f g^* \rangle_N$. The radical also coincides with the set of polynomials f such that $(g(Y)(f))(q^{-\rho_k}) = 0$ for all Laurent polynomials g. The evaluations of ϵ_b depend only on the images of $b_\sharp$ in P_N and form a basis of this module. The evaluations of the symmetric polynomials constitute the *generalized Verlinde algebra.*

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