Products of Trees, Lattices and Simple Groups

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The group of automorphisms of a locally finite tree, denoted $\operatorname{Aut} T$, is a locally compact group which exhibits behavior analogous to that of a rank one simple Lie group. This analogy has motivated many recent works, in particular the study of lattices in $\operatorname{Aut} T$ by Bass, Kulkarni, Lubotzky and others. Recall that in the case of semisimple Lie groups, irreducible lattices in higher rank groups have a very rich structure theory and one encounters many deep and interesting phenomena such as (super)rigidity and arithmeticity. Motivated by this we study in several joint works with Marc Burger and with Robert J. Zimmer cocompact lattices in the group of automorphisms of a product of trees, or rather in groups of the form $\operatorname{Aut} T_1 \times \operatorname{Aut} T_2$ where each of the trees T_i, $i = 1, 2$, is (bi-)regular. The results obtained concerning the structure of lattices in $\operatorname{Aut} T_1 \times \operatorname{Aut} T_2$ enable us to construct the first examples of finitely presented torsion free simple groups, see [BM2].

One Tree. There is a close relation between certain simple Lie groups and groups of tree automorphisms. Let G be a simple algebraic group of rank one over a non-archimedean local field K. Considering the action of G on its associated Bruhat-Tits tree T, we have a continuous embedding of G in $\operatorname{Aut} T$ with cocompact image. In [Tit], Tits has shown that if T is a locally finite tree and its automorphism group $\operatorname{Aut} T$ acts minimally (i.e. without an invariant proper subtree and not fixing an end) on it, then the subgroup $\operatorname{Aut}^+ T$ generated by edge stabilizers is a simple group. In particular the automorphism group of a regular tree is virtually simple. These results motivated the study of $\operatorname{Aut} T$ taking this analogy with rank one Lie groups as a guideline, see [Lu1], [Lu3].

When T is a locally finite tree its automorphism group is locally compact. Recall that a subgroup Γ of a locally compact group G is called a lattice when it is discrete and the quotient $\Gamma \backslash G$ carries a finite invariant measure. In case the quotient is compact the lattice is called uniform. Observe that a subgroup of $\operatorname{Aut} T$ is discrete if and only if it acts with finite stabilizers. One may determine whether a discrete subgroup is a lattice by checking the finiteness of the sum $\sum_{v \in F} 1/|\Gamma_v|$, where Γ_v is the stabilizer of the vertex v and the set F is a fundamental domain for the action of Γ on some $\operatorname{Aut} T$ orbit in T. Of particular interest is the case when $\operatorname{Aut} T$ acts with finitely many orbits on T; in this case a lattice is uniform if and only if the quotient $\Gamma \backslash T$ is finite. Such lattices, called "uniform tree lattices", correspond to finite graphs of groups in which all vertex and edge groups are finite. These were extensively studied by Bass and Kulkarni, [BK]. By a result of Leighton, cf. [BK], any two uniform tree lattices in $\mathrm{Aut}\, T$ are commensurable up to conjugation. A key role in the study of lattices in semisimple Lie groups is played by their commensurators (cf. [AB]): $\mathrm{Comm}_G(\Gamma) = \{g \in G : g^{-1}\Gamma g \cap \Gamma$ is of finite index in both Γ and $g^{-1}\Gamma g\}$. In particular, Margulis has shown that an irreducible lattice $\Gamma < G$ in a semisimple Lie group is arithmetic if and only if its commensurator is dense in G.

It was shown by Liu [Liu] (confirming a conjecture by Bass and Kulkarni) that the commensurator of a uniform tree lattice is dense. The situation concerning non-uniform lattices is much more involved and not well understood. There are examples, by Bass and Lubotzky [BL2], cf. also [BM1], of non-uniform lattices in the automorphism groups of regular trees whose commensurators are discrete. At the other extreme it was shown by the author that the commensurator of the Nagao lattice $\mathrm{SL}_2(F_p[t])$ in the full automorphism group of the $(p+1)$-regular tree is dense. An example of a cocompact lattice with dense commensurator which is not a uniform tree lattice appears in [BM1].

Note that as all uniform tree lattices of a given tree are commensurable up to conjugation, the isomorphism class of the commensurator of a uniform tree lattice is determined by the tree. In the other direction it is shown in [LMZ] that for regular trees the commensurator determines the tree. In proving this we use a superrigidity theorem for the commensurators of lattices in the automorphism groups of regular trees. In a much more general setting of divergence groups in the isometry group of CAT(-1) spaces we have shown in [BM1] (see also [Bur]) that:

THEOREM 1. *Let X, Y be proper CAT(-1)-spaces, $\Gamma < Is(X)$ a discrete divergence group, $\Lambda < Is(X)$ a subgroup such that $\Gamma < \Lambda < Comm_{\mathrm{Is}(X)}(\Gamma)$ and $\pi : \Lambda \to Is(Y)$ a homomorphism such that $\pi(\Lambda)$ acts convex-minimally and $\pi(\Gamma)$ is not elementary. Then π extends to a continuous homomorphism*

$$\pi_{\mathrm{ext}} : \overline{\Lambda} \to Is(Y) \ .$$

PRODUCTS OF TREES AND LOCALLY PRIMITIVE GROUPS. Among the most striking results concerning lattices in semisimple Lie groups are the arithmeticity and superrigidity theorems established by G.A. Margulis (cf. [Mar], [Zim], [AB]). These assert that:

1. An irreducible lattice in a higher rank (i.e. ≥ 2) semisimple Lie group is arithmetic.

2. Any linear representation of such a lattice with unbounded image essentially extends to a continuous representation of the ambient Lie group.

Recall that a lattice $\Gamma < G$ in a semisimple Lie group is called *reducible* if the following equivalent conditions hold:

1. There exists a decomposition of G (up to isogeny) as a product $G = G_1 \times G_2$ with both G_i non compact semisimple Lie groups and Γ projects discretely on each G_i, $i = 1, 2$.

2. Γ contains a finite index subgroup of the form $\Gamma_1 \times \Gamma_2$ where $\Gamma_i < G_i$ is a lattice and $G = G_1 \times G_2$ a decomposition as above.

Using Borel's density theorem ([Bor], cf. [Fur], [Dan]) we have that a lattice Γ in a semisimple Lie group G is irreducible if it satisfies the following equivalent conditions:

- (Ir1) The projection of Γ on any factor G_i of G with $\ker : G \to G_i$ noncompact has non-discrete image.
- (Ir2) A projection as above has dense image.

Pursuing further the analogy between $\operatorname{Aut} T$ and rank one Lie groups, it is natural to ask for a structure theory for lattices in groups of the form $\operatorname{Aut} T_1 \times \operatorname{Aut} T_2$ with T_i trees. In particular one would like to have "rigidity-" and "arithmeticity-" like results. Some steps in this direction were taken jointly with M. Burger and R.J. Zimmer [BMZ], [BM2], [BM3] Let us assume henceforth (unless explicitly stated otherwise) that our trees are (bi)-regular. A lattice $\Gamma < \operatorname{Aut} T_1 \times \operatorname{Aut} T_2$ is reducible when its projections on each factor are discrete. Restricting our attention to uniform (i.e., cocompact) lattices observe that the projection $\mathrm{pr}_i(\Gamma) < \operatorname{Aut} T_i$ of a lattice $\Gamma < \operatorname{Aut} T_1 \times \operatorname{Aut} T_2$ is never dense. This follows by observing that the compact open subgroup $K = \mathrm{Stab}_{\operatorname{Aut} T_i}(x)$, $x \in T_i$ a vertex, maps onto $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ and hence is not topologically finitely generated, whereas the intersection of Γ with the product of K with $\operatorname{Aut} T_{3-i}$, being a uniform lattice in this product, is finitely generated. Thus the intersection of K with the projection of Γ to $\operatorname{Aut} T_i$ cannot be dense in K (see [BM3]). Given a lattice $\Gamma < \operatorname{Aut} T_1 \times \operatorname{Aut} T_2$ denote by $H_i = \overline{\mathrm{pr}_i(\Gamma)}$, $i = 1, 2$. Thus $\Gamma < H_1 \times H_2$. Clearly the representation theory of Γ cannot be more rigid than that of $H_1 \times H_2$. Indeed, one can construct irreducible lattices $\Gamma < \operatorname{Aut} T_1 \times \operatorname{Aut} T_2$ such that the corresponding groups H_i surject onto free groups. Requiring various conditions on the projections of Γ leads to interesting structure theory.

DEFINITION 2. *A subgroup $H < AutT$ is called* locally primitive *if for every vertex $x \in T$ its stabilizer in H induces a primitive permutation group, denoted $\underline{H}(x)$, on the set $E(x)$ of neighbouring edges.*

The class of closed locally primitive subgroups of $\operatorname{Aut} T$ has a structure theory reminiscent in some ways to that of semisimple Lie groups. A key role in the study of these groups is played by the following lemma which shows that normal subgroups of a locally primitive group are either free or very large.

LEMMA 3. *Let T be a tree and let $H < AutT$ be a closed locally primitive subgroup. Any normal subgroup $N \triangleleft H$ either acts freely on T, or is cocompact and has a fundamental domain which is either a ball of radius 1 or an edge in T.*

For a locally compact, totally disconnected group H, let $H^{(\infty)} := \bigcap_{L<H} L$, where the intersection is taken over all open subgroup $L < H$ of finite index, and $QZ(H) := \bigcup_{U<H} Z_H(U)$, where the union is taken over all open subgroups $U < H$.

Thus $QZ(H) = \{h \in H : Z_H(h) \text{ is open}\}$. Observe that $H^{(\infty)} = \bigcap_{N \lhd H} N$, where the intersection is taken over all closed, cocompact normal subgroups $N \lhd H$. Note also that any discrete normal subgroup of H is contained in $QZ(H)$. Using Lemma 3 one shows that when H is non-discrete, $H^{(\infty)}$ is cocompact in H.

With the (limited) analogy between closed locally primitive subgroups $H < \operatorname{Aut} T$ and algebraic groups G in mind, we may view $H^{(\infty)} < H$ as playing the role of the subgroup $G^+ < G$ generated by all one parameter unipotent subgroups. We have the following structure theorem:

THEOREM 4. *(Burger-Mozes) Let $H < AutT$ be a closed, non-discrete, locally primitive subgroup. Then $H^{(\infty)}/QZ(H^{(\infty)})$ decomposes as a finite direct product*

$$H^{(\infty)}/QZ(H^{(\infty)}) = M_1 \cdot M_2 \cdot \cdots \cdot M_r$$

Where each M_i, $1 \leq i \leq r$ is a topologically simple group.

Various examples of closed subgroups of $\operatorname{Aut} T$ may be obtained via the following construction: Let $d \geq 3$, and $F < S_d$ be a permutation group. Let $T_d = (X, Y)$ be the d-regular tree and $i : Y \to \{1, 2, \ldots, d\}$ a legal (edge) coloring, that is, a map such that $i(y) = i(\bar{y}), \forall y \in Y$, and $i|_{E(x)} : E(x) \to \{1, 2, \ldots, d\}$ is a bijection, $\forall x \in X$. Define $U(F) = \{g \in \operatorname{Aut} T_d : i|_{E(gx)} g i^{-1}|_{E(x)} \in F, \ \forall x \in X\}$. Observe that $U(F)$ is a closed subgroup of $\operatorname{Aut} T_d$; the group $U(F)$ acts transitively on X; the finite group $\underline{U(F)}(x) < \operatorname{Sym} E(x)$ is permutation isomorphic to $F < S_d$, and hence, when F is a primitive permutation group, $U(F)$ is locally primitive. We notice also the following:

1. Using Tits' theorem, [Tit], it follows that $U(F)^+$ (the subgroup generated by edge stabilizers) is simple.

2. The subgroup $U(F)^+$ is of finite index in $U(F)$ if and only if $F < S_d$ is transitive and F is generated by its subgroups $\operatorname{Stab}_F(j)$, $1 \leq j \leq d$. In this case, $U(F)^+ = U(F) \cap \operatorname{Aut}^+ T_d$ and is of index 2 in $U(F)$.

3. Let $F < S_d$ be a transitive subgroup and $H < \operatorname{Aut} T_d$ be a vertex-transitive subgroup such that, for some $x \in X$, $\underline{H}(x) < \operatorname{Sym} E(x)$ is permutation isomorphic to $F < S_d$. Then, for some suitable legal coloring, we have $H < U(F)$.

We are especially interested at those subgroups $U(F)$ which arise as closures of projections of irreducible uniform lattices. As these must be topologically finitely generated we note:

PROPOSITION 5. *[BM3] Let $F < S_d$ be a transitive permutation group. Then $\underline{U(F)}(x)$ is topologically finitely generated if and only if $F_1 = \operatorname{Stab}_F(1)$ is perfect and equal to its normalizer in F.*

NOTATION: Denote by $S(x,n)$ the sphere of radius n around a vertex $x \in T$. For $H < \operatorname{Aut} T$, $x \in X$, $n \geq 1$, $H_n(x) = \{h \in H : h|_{S(x,n)} = id\}$, $\underline{H}_n(x) = H_n(x)/H_{n+1}(x)$.

PROPOSITION 6. *Let $F < S_d$ be a 2-transitive permutation group such that $F_1 = \mathrm{Stab}_F(1)$ is non-abelian simple and $H < Aut T_d$ a closed vertex transitive subgroup such that $\underline{H}(x) < \mathrm{Sym}\, E(x)$ is permutation isomorphic to $F < S_d$. Then $\underline{H}_1(x) \simeq F_1^a$ where $a \in \{0, 1, d\}$. Moreover*

$$\begin{array}{ll} H \text{ is discrete} & \Leftrightarrow a \in \{0,1\} \\ H = U(F) & \Leftrightarrow a = d. \end{array}$$

In the proof of the above proposition one needs to show that when $a = d$ the group H is not discrete. This is established using the Thompson-Wielandt theorem (see [Tho], [Wie], [Fan]).

THEOREM 7. *(Thompson-Wielandt) Let $T = \mathfrak{T}_n$ be the n-regular tree. Let $U < Aut(T)$ be the pointwise stabilizer of a ball of radius 1 around an edge e. (Note that U is an open compact neighborhood of the identity.) Then for every vertex transitive locally primitive lattice $\Gamma < Aut(T)$ the group $\Gamma \cap U$ is an l-group for some prime $l < n$.*

In the context of lattices $\Gamma < \mathrm{Aut}\, T_1 \times \mathrm{Aut}\, T_2$ one would like to verify for a given lattice whether it is reducible or not, namely whether its projections are discrete or not. The Thompson-Wielandt theorem may be used to verify non-discreteness and hence irreducibility in certain cases but we do not know a general algorithm for deciding this question.

RIGIDITY. The following result may be viewed as an analog of the Mostow rigidity theorem:

THEOREM 8. *[BMZ] Let $\Gamma < AutT_1 \times AutT_2$ and $\Gamma' < AutT_1' \times AutT_2'$ be uniform lattices. Assume that the subgroups $H_i = \overline{pr_i(\Gamma)} < AutT_i$, $i = 1, 2$ are locally primitive. If we have an isomorphism $\Gamma \cong \Gamma'$, then it is induced by an isometry between $T_1 \times T_2$ and $T_1' \times T_2'$.*

Note that we do not assume that the lattices are irreducible. Bass and Lubotzky [BL1] have shown that a certain class of closed (non-discrete) subgroups of $\mathrm{Aut}\, T$ determines the tree T (up to some natural modifications). We note that any isomorphism between locally primitive lattices $\Lambda < \mathrm{Aut}\, T$ and $\Lambda' < \mathrm{Aut}\, T'$ acting without inversion on the corresponding trees is induced by an isometry between T and T'. (To establish this one notes that the tree structure may be reconstructed from, say, Λ using the correspondence between the vertices and maximal finite subgroups of Λ and between the edges and pairs of such maximal finite groups which generate the group and whose intersection is a maximal subgroup in each.)

THEOREM 9. *[BMZ] Let $\Gamma < AutT_1 \times AutT_2$ be a uniform lattice with $H_i = \overline{pr_i(\Gamma)} < AutT_i$, $i = 1, 2$ locally primitive. Let Y be a proper CAT(-1) space and $\pi : \Gamma \to Is(Y)$ a homomorphism such that $\pi(\Gamma)$ is not elementary and acts convex-minimally on Y. Then π extends to a continuous homomorphism $\pi_{ext} : H_1 \times H_2 \to Is(Y)$ which factors via a proper homomorphism of one of the H_i, $i = 1, 2$.*

In the Lie groups setting, Margulis' superrigidity theorem plays a key role in showing that irreducible lattices in higher rank groups are arithmetic. In the context of lattices in $\mathrm{Aut}\, T_1 \times \mathrm{Aut}\, T_2$ we have:

THEOREM 10. *[BMZ] Let $\Gamma < AutT_1 \times AutT_2$ be an irreducible cocompact lattice. Assume that each $H_i = \overline{pr_i(\Gamma)} < AutT_i$ is locally primitive. Then one of the following possibilities holds:*

1. *Every linear image of Γ is finite.*
2. *Γ has an infinite linear image over a field of characteristic 0. Then H_i is a p_i-adic analytic group for some prime p_i. The adjoint map, which we denote by $\varphi = \varphi_1 \times \varphi_2 : H_1 \times H_2 \to Aut(Lie(H_1)) \times Aut(Lie(H_2))$, yields a continuous surjection from $H_1 \times H_2$ onto a semisimple Lie group over some local fields and the image $\varphi(\Gamma)$ is an arithmetic lattice. Moreover, the kernel of this homomorphism is a torsion free discrete subgroup of $H_1 \times H_2$.*
3. *Γ has an infinite linear image over a field of positive characteristic p. Then there is a continuous map with unbounded image from $H_1 \times H_2$ into a simple Lie group over $F_p((t))$.*

Let us remark that:

- It seems reasonable to expect in case 3 of the theorem a result similar to that of 2.
- In case 2:
  - We do not claim that the image $\varphi(\Gamma)$ is an irreducible arithmetic lattice. Indeed, one can construct examples where this lattice is reducible.
  - The algebraic groups $\varphi_i(H_i)$ need not be of rank one. In fact they are of rank one if and only if $\ker \varphi = \{e\}$. Moreover, $\varphi_i(H_i)$ is of rank one and $\ker \varphi_i$ is trivial exactly when the action of H_i on the corresponding tree T_i is locally infinitely transitive, i.e., the stabilizer of each vertex acts transitively on simple paths of arbitrary length starting at the vertex.

An example of an irreducible lattice in $\mathrm{Aut}\, T_1 \times \mathrm{Aut}\, T_2$ which is an extension of an arithmetic lattice in a semisimple algebraic group $G = G_1 \times G_2$, where each G_i is a semisimple group over some local field k_i, may be obtained as follows, cf. [BM3]. Associated with each G_i one has an affine building Δ_i on which the group G_i acts. "Draw" on Δ_i a graph $\mathcal{G}_i$ defined in an equivariant way (for example let $G_i = SL_3(\mathbb{Q}_p)$, the associated Bruhat-Tits building is a simplicial complex whose set of vertices has a natural 3-coloring (see [Bro]); consider the graph consisting of the vertices belonging to two fixed colors and the corresponding edges). The group G acts on $\mathcal{G}_1 \times \mathcal{G}_2$ and hence an extension $H_1 \times H_2$ of G by $\pi_1(\mathcal{G}_1 \times \mathcal{G}_2)$ acts on the universal covering space $T_1 \times T_2$ of $\mathcal{G}_1 \times \mathcal{G}_2$. Taking a lattice $\Lambda < G$, its extension by $\pi_1(\mathcal{G}_1 \times \mathcal{G}_2)$ is a lattice in $H_1 \times H_2$.

We examine next the normal subgroups structure of lattices in $\mathrm{Aut}\, T_1 \times \mathrm{Aut}\, T_2$.

Proposition 11. *[BM3] Let T_1, T_2 be locally finite trees, $\Gamma < \operatorname{Aut} T_1 \times \operatorname{Aut} T_2$ a discrete subgroup such that $\Gamma \setminus (T_1 \times T_2)$ is finite and $N < \Gamma$ a normal subgroup such that the quotient graphs $pr_i(N) \setminus T_i$, $i = 1, 2$, are finite trees. Then Γ/N has property (T).*

Proposition 12. *[BM3] Let T_1, T_2, Γ be as in Proposition 11 and $H_i := \overline{pr_i(\Gamma)} < \operatorname{Aut} T_i$.*

(a) The homomorphism $\operatorname{Hom}_c(H_1 \times H_2, \mathbb{C}) \to \operatorname{Hom}(\Gamma, \mathbb{C})$ mapping χ to $\chi |_\Gamma$ is an isomorphism.

(b) Let (π, V) be an irreducible finite dimensional unitary representation of Γ with $H^1(\Gamma, \pi) \neq 0$.

Then π extends continuously to $H_1 \times H_2$, factoring via one of the projections.

The following result, obtained in [BM3], is an analog of Margulis' normal subgroup theorem:

Theorem 13. *Let $\Gamma < AutT_1 \times AutT_2$ be a cocompact lattice such that $H_i := \overline{pr_i(\Gamma)}$ is locally ∞-transitive and $H_i^{(\infty)}$ is of finite index in H_i, $i = 1, 2$. Then, any non-trivial normal subgroup of Γ has finite index.*

The proof of this theorem follows the lines of the proof by Margulis of the corresponding result in the context of Lie groups. In particular one uses the following analog of the Howe-Moore theorem concerning vanishing of matrix coefficients. In the context of $\operatorname{Aut}^+ T$ with T a regular tree this was shown in [LM], see also [FTN].

Theorem 14. *[BM3] Let $H < AutT$ be a closed locally ∞-transitive subgroup and $(\pi, \mathcal{H})$ be a continuous unitary representation of H with no nonzero $H^{(\infty)}$ invariant vectors. Then for every $u, v \in \mathcal{H}$, $\lim \langle \pi(g)u, v\rangle \to 0$ as $g \in H$ tends to ∞.*

However there is an interesting application of Theorem 13 which is based on a fundamental difference between cocompact lattices in $\operatorname{Aut} T_1 \times \operatorname{Aut} T_2$ and cocompact lattices in Lie groups. Whereas any finitely generated subgroup of a linear group is residually finite, finitely generated subgroups, and even cocompact lattices, in $\operatorname{Aut} T_1 \times \operatorname{Aut} T_2$ need not be residually finite. A criterion for establishing that certain lattices are not residually finite is provided by the following:

Proposition 15. *[BM3] Let G_i, $i = 1, 2$ be closed locally compact groups. Let $\Gamma < G_1 \times G_2$ be a discrete subgroup. Assume that for $i = 1, 2$, $G_i^{(\infty)} < \overline{pr_i(\Gamma)} < G_i$. Then*

$$\Gamma^{(\infty)} > [G_1^{(\infty)}, \Lambda_1] \cdot [G_2^{(\infty)},\ \Lambda_2],$$

where $\Lambda_1 := pr_1((G_1 \times e) \cap \Gamma)$, $\Lambda_2 := pr_2((e \times G_2) \cap \Gamma)$. In particular, if each G_i has trivial centralizer and $\Lambda_1 \times \Lambda_2 \neq e$, then Γ is not residually finite.

In particular, let $\Gamma < \operatorname{Aut} T_1 \times \operatorname{Aut} T_2$ be an irreducible lattice such that each $H_i = \overline{\mathrm{pr}_i(\Gamma)}$ is locally primitive. Each $H_i^{(\infty)}$ acts on T_i with finite quotient and hence has trivial centralizer. Thus if in addition the projection of Γ to one of the factors $\operatorname{Aut} T_i$ is not injective then Γ is not residually finite. The construction described following Theorem 10 provides a non residually finite lattice.

Combining Proposition 15 and Theorem 13 allows us to construct (see [BM2] and [BM3]) examples of finitely presented torsion free simple groups. One constructs first a non residually finite lattice $\tilde{\Gamma}$ which satisfies the conditions of Theorem 13. Given $\tilde{\Gamma}$ let $\Gamma = \tilde{\Gamma}^{(\infty)}$. It follows that Γ is the minimal finite index subgroup of $\tilde{\Gamma}$. Verifying that Γ satisfies the conditions of Theorem 13, one deduces that Γ is simple. It should be observed that a lattice Γ which satisfies the conditions of Proposition 15 for not being residually finite must have a non trivial normal subgroup of infinite index! namely either Λ_1 or Λ_2. Thus in order to produce a non residually finite lattice which satisfies the conditions of Theorem 13 one uses the geometric description of lattices in $\operatorname{Aut} T_1 \times \operatorname{Aut} T_2$ (see below) to embed a non residually finite lattice obtained using Proposition 15 in a lattice as in Theorem 13.

THEOREM 16. *For every pair (n, m) of sufficiently large even integers there exists a group $\Gamma_{n,m}$ such that:*

1. *The group $\Gamma_{n,m}$ is simple, finitely presented, torsion free and isomorphic to a free amalgam $F *_G F$ where F, G are finitely generated free groups.*
2. *The group $\Gamma_{n,m}$ has cohomological dimension 2.*
3. *$\Gamma_{n,m}$ is automatic.*

The question of existence of simple groups which are amalgams of free groups was raised by P.M. Neumann ([Neu], see also the Kourovka notebook [MK] problem 4.45). M. Bhattacharjee [Bha] constructed examples of amalgams of free groups which do not have any finite quotients. The groups $\Gamma_{n,m}$ are constructed as lattices in $\operatorname{Aut} \mathfrak{T}_n \times \operatorname{Aut} \mathfrak{T}_m$ (where $\mathfrak{T}_k$ denotes the k-regular tree). Considering the action of $\Gamma_{n,m}$ on each of the trees $\mathfrak{T}_k$, $k = n, m$, we obtain two decompositions of $\Gamma_{n,m}$ as amalgams $A *_C B$. The groups A, B and C, being torsion free lattices in $\operatorname{Aut} \mathfrak{T}_k$, are free groups (note that $[A : C] = [B : C] \in \{n, m\}$). Moreover, using the superrigidity theorem 9 it follows that these are the only nontrivial decompositions of $\Gamma_{n,m}$ as amalgamated products. This implies also that the groups $\Gamma_{n,m}$ are mutually non isomorphic. These also form the first examples of finitely presented simple groups of finite cohomological dimension. We refer to [Sco] for a survey and discussion of various families of finitely presented simple groups constructed by R. Thompson, Higman, Brown and Scott.

GEOMETRICAL DESCRIPTION. When a subgroup $D < \operatorname{Aut} T_1 \times \operatorname{Aut} T_2$ acts freely on $T_1 \times T_2$, it may be identified with the fundamental group of the quotient space $\mathcal{Y} = D \backslash (T_1 \times T_2)$. Note that $\mathcal{Y}$ is a square complex whose universal covering space is $T_1 \times T_2$. Square complexes whose universal covering space is a product of trees are characterized as those square complexes in which the link of every vertex is a complete bipartite graph. (Again under the analogy with semisimple groups consider the geometric characterization of locally symmetric spaces.) More generally, when the action is not free, D may be reconstructed as the fundamental group of a certain complex of groups ([Hae] and see [Ser] and [Bas] for the corresponding theory of graph of groups). This geometric way of considering subgroups and in particular cocompact lattices in $\mathrm{Aut}\,T_1 \times \mathrm{Aut}\,T_2$ allows one to explicitly construct and modify such lattices. Note that any such finite square complex gives a finite presentation of its fundamental group. In [BM3] we give an explicit construction of the complexes associated with the groups $\Gamma_{n,m}$ thus providing an explicit finite presentation of these torsion free simple groups.

MINIMAL VOLUMES. Kazhdan and Margulis [KM] have shown that for any semisimple Lie group there is a positive lower bound on the volume of $\Gamma\backslash G$ for any lattice $\Gamma < G$. In this vein let us mention the works of Lubotzky and Weigel, [Lu2], [LW], who determined the lattices of minimal covolume in the groups of the form $\mathrm{SL}_2(K)$, where K is a non archimedean local field.

In contrast, there is no lower bound for arbitrary lattices in either $\mathrm{Aut}\,T$ or $\mathrm{Aut}\,T_1 \times \mathrm{Aut}\,T_2$. Bass and Kulkarni, [BK], constructed cocompact lattices in $\mathrm{Aut}\,T$, where T is a k-regular tree, $k \geq 3$, with arbitrarily small covolume. I. Levitz determined precisely the (dense) set of (positive) rational numbers appearing as covolumes of uniform lattices in $\mathrm{Aut}\,T$. Moreover, Bass and Lubotzky have shown in [BL2] that given any real number $\alpha > 0$ there exists a non-uniform lattice $\Gamma < \mathrm{Aut}\,T$ such that $\mathrm{Vol}(\Gamma\backslash\mathrm{Aut}\,T) = \alpha$. Considering lattices in $\mathrm{Aut}\,T_1 \times \mathrm{Aut}\,T_2$, Y. Glasner has constructed examples of irreducible lattices $\Gamma < \mathrm{Aut}\,T_1 \times \mathrm{Aut}\,T_2$ of arbitrarily small covolume. In these examples the subgroups $H_i = \overline{\mathrm{pr}_i(\Gamma)} < \mathrm{Aut}\,T_i$ are not locally primitive.

However, a well known conjecture of Goldschmidt and Sims, translated into the language of lattices, asserts that for any given tree T there are only finitely many locally primitive lattices in $\mathrm{Aut}\,T$. Thus in particular there is a lower bound on the covolume of such lattices. The Goldschmidt-Sims conjecture is usually stated as saying that for any $n, m \geq 3$ there are only finitely many effective amalgams $A *_C B$ of finite groups with $[A : C] = n$, $[B : C] = m$ and C is maximal in both A and B. This was established by D. Goldschmidt [Gol] for the case $n = m = 3$. In view of the above results concerning the analogy between semisimple Lie groups and locally primitive groups of tree automorphisms, one is led to ask whether there is a positive lower bound on the covolume of lattices $\Gamma < \mathrm{Aut}\,T_1 \times \mathrm{Aut}\,T_2$ having locally primitive projections (note that for reducible such lattices this would follow from the Goldschmidt-Sims conjecture). Studying this question Y. Glasner [Gla] has proved the following analog of the Goldschmidt-Sims conjecture:

THEOREM 17. *(Glasner) For any fixed primes p, q there are only finitely many effective complexes of groups consisting of a single square whose universal covering space is a product of regular trees $\mathfrak{T}_p \times \mathfrak{T}_q$ and whose fundamental group $\Gamma < Aut\mathfrak{T}_p \times Aut\mathfrak{T}_q$ is an irreducible lattice.*

A central role in the proof of the above theorem is played by the Thompson-Wielandt theorem (Theorem 7). Recall that in establishing the lower bound on the covolume of a lattice in a semisimple Lie group G one may use ([KM], cf. [Rag]) the existence of a "Zassenhaus neighbourhood" $U < G$ such that for every discrete subgroup $\Gamma < G$ the elements of $\Gamma \cap U$ are contained in some connected nilpotent Lie subgroup of G. The Thompson-Wielandt theorem gives a neighbourhood whose intersection with locally primitive discrete groups is an l-group, and hence, in particular, nilpotent.

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