Branching Processes, Random Trees and Superprocesses

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Abstract. We present some recent developments concerning the genealogy of branching processes, and their applications to superprocesses. We also discuss connections with partial differential equations, statistical mechanics and interacting particle systems.

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1 Discrete and continuous genealogical trees

(1.1) *Galton-Watson processes and trees.* A Galton-Watson branching process describes the evolution in discrete time of a population where each individual gives rise, independently of the others, to a random number of children distributed to a given offspring distribution. To be specific, consider an integer $k \geq 0$ representing the initial population, and a probability distribution ν on the set $\mathbb{N}$ of nonnegative integers. The corresponding Galton-Watson process is the Markov chain $(N_n, n \geq 0)$ in $\mathbb{N}$ such that, conditionally on N_n,

$$N_{n+1} \stackrel{\text{(d)}}{=} \sum_{i=1}^{N_n} U_i$$

where $U_1, U_2, \ldots$ are independent and distributed according to ν.

It is obvious that the genealogy of such a branching process can be described by k discrete trees. Take $k = 1$ for simplicity. Then the genealogical tree of the population is defined in the obvious way as a random subset $\mathcal{T}$ of $\bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n$, where $\mathbb{N}^* = \{1, 2, 3, \ldots\}$ and $(\mathbb{N}^*)^0 = \{\emptyset\}$ by convention (cf Fig.1 for an example). Here $\emptyset$ labels the ancestor of the population and, for instance, $(3, 2)$ corresponds to the second child of the third child of the ancestor.

Throughout this article, we will concentrate on the critical or subcritical case where $m = \sum_{j=0}^{\infty} j\nu(j) \leq 1$ and we also exclude the (trivial) case where $\nu(\{1\}) = 1$. Then the population becomes extinct in finite time and so the tree $\mathcal{T}$ is a.s. finite.

(1.2) *Continuous-state branching processes.* Continuous-state branching processes (in short, CSBP's) are the continuous analogues of Galton-Watson processes. Formally, a CSBP is a Markov process Y in $\mathbb{R}_+$ whose transition kernels $(P_t(x, dy); t \geq 0, x \in \mathbb{R}_+)$ satisfy the additivity or branching property

$P_t(x+x',\cdot) = P_t(x,\cdot) * P_t(x',\cdot)$. Lamperti [15] has shown that these processes are exactly the scaling limits of Galton-Watson processes. Start from a sequence N^n of Galton-Watson processes with initial values k_n and offspring distributions ν_n depending on n. Suppose that there exists a sequence of constants $a_n \uparrow \infty$ such that

$$\lim_{n\to\infty} \big(\frac{1}{a_n} N^n_{[nt]}, t \geq 0\big) = (Y_t, t \geq 0) \tag{1}$$

in the sense of weak convergence of the finite-dimensional marginals. Then the limiting process Y must be a CSBP, and conversely any CSBP can be obtained in this way.

The distribution of a CSBP can be described analytically as follows. Here again, we restrict our attention to the critical or subcritical situation where $\int y\, P_t(x,dy) \leq x$. Then, the Laplace functional of the kernels $P_t(x,dy)$ must be of the form $\int P_t(x,dy)\, e^{-\lambda y} = \exp(-x u_t(\lambda))$, and the function $u_t(\lambda)$ solves the ordinary differential equation

$$\frac{\partial u_t(\lambda)}{\partial t} = -\psi(u_t(\lambda)) \ , \qquad u_0(\lambda) = \lambda, \tag{2}$$

with a function ψ of the type

$$\psi(u) = \alpha u + \beta u^2 + \int_{(0,\infty)} \pi(dr)\, (e^{-ru} - 1 + ru), \tag{3}$$

where $\alpha, \beta \geq 0$ and π is a σ-finite measure on $(0,\infty)$ such that $\int (r \wedge r^2)\, \pi(dr) < \infty$. Conversely, for any choice of a function ψ of the type (3), there exists an associated CSBP, which we will call the ψ-CSBP.

The case when $\psi(u) = \beta u^2$ (quadratic branching mechanism) is of special importance. The associated process is called the Feller diffusion. It occurs as the limit in (1) when $\nu_n = \nu$ has mean 1 and finite variance, and $k_n \approx \lambda n$, $a_n = n$.

In contrast with the discrete setting, it is no longer straightforward to define the genealogical structure of a CSBP. At an informal level, one would like to answer questions of the following type. Suppose that we divide the population at time t in two parts, say green individuals and red individuals. Then which part of the population at time $t+s$ does consist of descendants of green individuals, resp. red individuals ? This should be answered in a consistent way when s and t vary.

(1.3) *The quadratic branching case.* It has been known for some time that the genealogical structure of the Feller diffusion can be coded by excursions of linear Brownian motion. To explain this coding, we will recall a result of Aldous [1].

Start from an offspring distribution ν on $\mathbb{N}$ with mean 1 and finite variance. Consider the Galton-Watson tree with offspring distribution ν, conditioned to have exactly n edges (some mild assumption on ν is needed here so that this conditioning makes sense). Then, provided we rescale each edge by the factor $1/\sqrt{n}$, this conditioned tree, denoted by $\mathcal{T}_{(n)}$, converges in distribution as $n \to \infty$ to the so-called Continuum Random Tree (CRT).

To give a precise meaning to the last statement, we need to say what the CRT is and to explain the meaning of the convergence. The easiest definition of the CRT is via the coding by a continuous function. Let $e = (e(s), s \geq 0)$ be a continuous function from $\mathbb{R}_+$ into $\mathbb{R}_+$ with compact support and let σ denote the supremum of the support of e. We can then think of this function as coding a "continuous tree" through the following prescriptions:

- Each $s \in [0, \sigma]$ labels a vertex of the tree at generation $e(s)$.
- The vertex s is an ancestor of the vertex s' if $e(s) = \inf_{r \in [s,s']} e(r)$. (In general, the quantity $\inf_{r \in [s,s']} e(r)$ is the generation of the last common ancestor to s and s'.)
- The distance on the tree is $d(s, s') = e(s) + e(s') - 2 \inf_{r \in [s,s']} e(r)$, and we identify s and s' if $d(s, s') = 0$.

According to these definitions, the set of ancestors (line of ancestors) of a given vertex s is isometric to the segment $[0, e(s)]$. The lines of ancestors of two vertices s and s' have a common part corresponding to the segment $[0, \inf_{r \in [s,s']} e(r)]$. More generally, for any finite set $s_1, \ldots, s_k$ of vertices, we can make sense of the reduced tree consisting of the lines of ancestors of $s_1, \ldots, s_k$ (see [1] and [17] for more details).

The CRT is the (random) continuous tree that corresponds in the previous coding to the case when the function e is a normalized Brownian excursion (positive Brownian excursion conditioned to have duration 1). Furthermore, the convergence of discrete trees towards the CRT should be understood as follows. Consider for each conditioned tree $\mathcal{T}_{(n)}$, the contour process of the tree (cf Fig.1). Provided that we rescale space by the factor $1/\sqrt{n}$ and space by the factor $1/(2n)$, the contour process of $\mathcal{T}_{(n)}$ converges in distribution towards the normalized Brownian excursion.

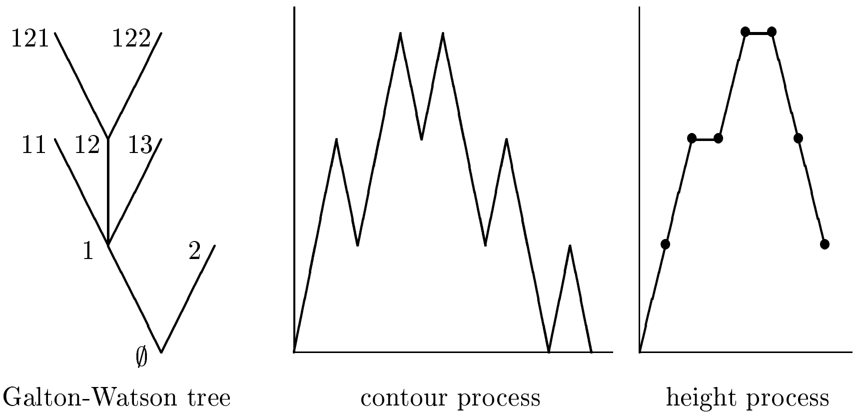

Figure 1

To summarize the previous considerations, we can say that the genealogical structure of the Feller diffusion ($\psi(u) = \beta u^2$) is coded by excursions of linear Brownian motion. This fact has appeared in different forms in many articles relating random walks or linear Brownian motion to branching processes (see in particular Harris [14], Dwass [9], Neveu-Pitman [22], etc.). It is also implicit in the Brownian snake construction of quadratic superprocesses [16], to which we will come back later.

In the next section, we will address the question of extending the previous coding to a general branching mechanism ψ.

2 Coding the genealogy of continuous-state branching processes

(2.1) *The discrete coding.* Consider a sequence $\mathcal{T}_1, \mathcal{T}_2, \dots$ of independent ν-Galton-Watson trees. Write σ_k for the number of vertices (or individuals) in the tree $\mathcal{T}_k$. Then suppose that we enumerate the vertices of the trees $\mathcal{T}_1, \mathcal{T}_2, \dots$ in lexicographical order: We write $\mathcal{T}_k = \{u_{\sigma_1+\dots+\sigma_{k-1}}, u_{\sigma_1+\dots+\sigma_{k-1}+1}, \dots, u_{\sigma_1+\dots+\sigma_k-1}\}$ where $u_{\sigma_1+\dots+\sigma_{k-1}}, u_{\sigma_1+\dots+\sigma_{k-1}+1}, \dots, u_{\sigma_1+\dots+\sigma_k-1}$ are the vertices of the tree $\mathcal{T}_k$ listed in lexicographical order.

Then for every $n \geq 0$, let H_n be the length (or generation) of the vertex u_n. The (random) process $(H_n, n \geq 0)$ is called the discrete height process (cf Fig.1 for an example with one tree). It is a variant of the contour process that was mentioned previously. It is easy to see that the data of the sequence $(H_n, n \geq 0)$ completely determines the sequence of trees and in this sense provides a coding of the trees. The interest of this coding comes from the following elementary lemma.

Lemma 2.1 *There exists a random walk $(S_n, n \geq 0)$ on $\mathbb{Z}$, with initial value $S_0 = 0$ and jump distribution $\mu(k) = \nu(k+1)$ for $k = -1, 0, 1, 2, \dots$, such that, for every $n \geq 0$,*

$$H_n = \mathrm{Card}\{j \in \{0, 1, \dots, n-1\}, S_j = \inf_{j \leq k \leq n} S_k\}. \tag{4}$$

Note that the random walk S is "left-continuous" in the sense that its negative jumps are of size -1 only. This lemma is taken from [19]. Closely related discrete constructions can be found in Borovkov-Vatutin [3] and Bennies-Kersting [2].

(2.2) *The continuous height process.* The previous lemma gives an explicit formula for the height process coding a sequence of Galton-Watson trees in terms of a random walk. Following [19], we will explain how this formula can be generalized to the continuous setting, thus yielding a coding of the genealogy of a CSBP in terms of a Lévy process with no negative jump (the continuous analogue of the left-continuous random walk S).

We start from a Lévy process X with no negative jump. We assume that $X_0 = 0$ and that that X does not drift to $+\infty$. Then the law of X is characterized by its "Laplace transform" $E[\exp(-\lambda X_t)] = \exp(t\psi(\lambda))$ (for $\lambda > 0$), where the possible functions ψ are exactly of the type (3), with the same assumptions on α, β and π. We assume in addition that $\beta > 0$ or $\int r\pi(dr) = \infty$ (or both these properties). This is equivalent to assuming that the paths of X are of infinite variation. (A simpler parallel theory can be developed in the finite variation case.) An important special case is the stable case $\psi(\lambda) = \lambda^{1+b}$, $0 < b < 1$.

Our first aim is to give a continuous analogue of the discrete formula (4). For every fixed $t \geq 0$, we let $X^{(t)} = (X_s^{(t)}, 0 \leq s \leq t)$ be the time-reversed process $X_s^{(t)} = X_t - X_{(t-s)-}$, and $M_s^{(t)} = \sup_{r \leq s} X_r^{(t)}$ be the associated maximum process. Note that $(X_s^{(t)}, 0 \le s \le t) \stackrel{\text{(d)}}{=} (X_s, 0 \le s \le t)$. The process $M^{(t)} - X^{(t)}$ is a Markov process in $\mathbb{R}_+$ and under our assumptions 0 is a regular point for this Markov process. This enables us to set the following definition.

Definition 2.2 *For every $t \ge 0$, let H_t denote the local time at level 0 and at time t of the process $M^{(t)} - X^{(t)}$. The process $(H_t, t \ge 0)$ is called the ψ-height process.*

A few comments are in order here. First, one needs to specify the normalization of local time. This can be achieved via the following approximation

$$H_t = P - \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t ds \, 1_{\{M_s^{(t)} - X_s^{(t)} < \varepsilon\}}.$$

Secondly, we have defined H_t for every fixed t, and the measurability properties of the process $(H_t, t \ge 0)$ are not obvious. One can in a canonical way construct a lower-semicontinuous modification of the process $(H_t, t \ge 0)$ (see [19]).

In one special case, namely when $\beta > 0$, one can give a much simpler formula for H_t: If $I_t^s = \inf\{X_r; s \le r \le t\}$, we have $H_t = \beta^{-1} m(\{I_t^s; 0 \le s \le t\})$, where m denotes Lebesgue measure on $\mathbb{R}$ (from this formula one immediately sees that H has continuous paths when $\beta > 0$). In the quadratic case $\psi(\lambda) = \beta\lambda^2$ (X is then a linear Brownian motion), we get that $H_t = \beta^{-1}(X_t - I_t^0)$ is a reflected linear Brownian motion, which agrees with the considerations in (1.3).

We now (informally) claim that H codes the genealogy of a ψ-CSBP "starting with an infinite mass". This should be understood in the sense of the coding of continuous trees via functions as explained previously. (Our present setting is slightly more general because the process H does not always have continuous sample paths.) Analogously to the discrete case, we get the genealogy of a ψ-CSBP starting at $\rho > 0$ by stopping H at $T_\rho = \inf\{t \ge 0, X_t = -\rho\}$.

In what follows, we will give several statements that provide a rigorous justification of the previous informal claim. We first state a "Ray-Knight theorem" that formalizes the naive idea that the number of visits of H at a level a corresponds to the population of the tree at that level.

Theorem 2.3 [19] *For every $a \ge 0$, the formula*

$$L_t^a = P - \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t ds \, 1_{\{a < H_s < a + \varepsilon\}}$$

defines a continuous increasing process $(L_t^a, t \ge 0)$. If $T_\rho = \inf\{t \ge 0, X_t = -\rho\}$, the process $(L_{T_\rho}^a, a \ge 0)$ is a ψ-CSBP started at ρ.

When $\psi(u) = \beta\, u^2$, Theorem 2.3 reduces to a classical Ray-Knight theorem for Brownian local times. In general, Theorem 2.3 can be applied to study the sample path continuity of H.

Theorem 2.4 [19] *The process H has a continuous modification if and only if $\int^\infty \frac{du}{\psi(u)} < \infty$.*

This condition holds in particular when $\beta > 0$ and in the stable case.

(2.3) *From discrete trees to continuous trees.* Our next result shows that if a sequence of rescaled Galton-Watson processes converges to a ψ-CSBP, the corresponding discrete height processes, suitably rescaled, also converge to the continuous height process H. This is analogous to Aldous' result in the quadratic branching case and proves in some sense that whenever rescaled Galton-Watson processes converge, their genealogical structure also converges to that of the limiting CSBP.

We consider a sequence (ν_n) of offspring distributions and a sequence (a_n) of positive numbers with $\lim a_n = \infty$. For every n let N^n be a Galton-Watson process with offspring distribution ν_n and initial value $N_0^n = [a_n]$.

THEOREM 2.5 [19],[7] *Suppose that the convergence* (1) *holds and that* Y *is a* ψ*-CSBP. For every* $n \geq 1$*, let* H^n *be the discrete height process associated with a sequence of independent* ν_n*-Galton-Watson trees. Then,*

$$\lim_{n\to\infty} \big(\frac{1}{n} H^n_{[na_n t]}, t \geq 0\big) = (H_t, t \geq 0) \tag{5}$$

in the sense of weak convergence of finite-dimensional marginals.

The last convergence can be shown to hold in a functional sense, provided that some regularity conditions are satisfied (Duquesne [7]). This reinforcement is important in various applications to invariance principles for functionals of Galton-Watson trees. For instance, one may want to look at the limiting behavior of the reduced tree that consists only of the ancestors of individuals alive at time p. The point is that this reduced tree can be written as an (almost) continuous functional of the discrete height process. Thus the (reinforced) convergence (5) allows one to pass to the limit and to obtain a limiting tree that is a simple functional of the height process H (see [7]).

3 SUPERPROCESSES

(3.1) *The snake construction.* Roughly speaking, superprocesses are obtained by combining a continuous branching mechanism with a Markovian spatial motion. To give a formal definition, consider a function ψ of the type (3) and a Borel right Markov process $(\xi_t, t \geq 0; \Pi_x, x \in E)$ with values in a Polish space E. Let $M_f(E)$ stand for the space of finite measures in E. The (ξ, ψ)-superprocess is the Markov process Z with values in $M_f(E)$ whose transition kernels are determined as follows. For every $0 \leq s < t$ and every bounded continuous function g on E, $E[\exp -\langle Z_t, g\rangle \mid Z_s] = \exp(-\langle Z_s, v_{t-s}\rangle)$, where $(v_t(x), t \geq 0, x \in E)$ is the unique nonnegative solution of the integral equation

$$v_t(x) + \Pi_x \left(\int_0^t ds\, \psi(v_{t-s}(\xi_s)) \right) = \Pi_x(g(\xi_t)). \tag{6}$$

(Compare with (2).) When ξ is a diffusion process with generator L, (6) is the integral form of the partial differential equation $\frac{\partial v}{\partial t} = Lv - \psi(v)$, $v_0 = g$. In the special case when ξ is Brownian motion in $\mathbb{R}^d$ and $\psi(u) = \beta\, u^2$, Z is called super-Brownian motion (see Perkins [23] for a discussion of super-Brownian motion and related processes).

We will now use our approach to the genealogy of the ψ-CSBP to give a construction of the (ξ, ψ)-superprocess. The idea is to use the height process H to construct in a Markovian way the individual spatial motions of the "particles" of the superprocess. To simplify the presentation, we assume that the condition of Theorem 2.4 holds, so that H has continuous sample paths.

Let us fix a starting point $x \in E$. Conditionally on $(H_s, s \geq 0)$, we define a path-valued (time-inhomogeneous) Markov process $(W_s, s \geq 0)$ whose law is characterized by the following properties:

- For every $s \geq 0$, $W_s = (W_s(t), 0 \leq t \leq H_s)$ is a finite cadlag path in E started at x and defined on the time interval $[0, H_s]$.
- If $s < s'$, $W_{s'}(t) = W_s(t)$ for every $t \leq m(s, s') := \inf_{[s,s']} H_r$, and, conditionally on $W_s(m(s, s'))$, $(W_{s'}(m(s, s') + t), 0 \leq t \leq H_{s'} - m(s, s'))$ is independent of W_s and distributed according to the law of ξ started at $W_s(m(s, s'))$.

Informally, W_s is a path of ξ started at x with length H_s. When H_s decreases, the path erases itself and when H_s increases the path extends itself by following the law of the spatial motion ξ. To summarize the previous properties, we will say that W is the snake driven by H with spatial motion ξ (and initial point x).

The connection with superprocesses is contained in the next theorem, which is essentially the main result of [20]. Recall the definition of L_t^a in Theorem 2.3.

Theorem 3.1 *For every $a \geq 0$, let Z_a be the random measure on E defined by*

$$\langle Z_a, g \rangle = \int_0^{T_\rho} d_s L_s^a \, g(W_s(a)).$$

Then $(Z_a, a \geq 0)$ is a (ξ, ψ)-superprocess started at $\rho\, \delta_x$.

To keep track of the dependence on the initial point x, we will use the notation $\mathbb{P}_x$ for the probability under which W is defined.

(3.2) *The Brownian snake and partial differential equations.* We now concentrate on the quadratic case $\psi(u) = \beta u^2$ and take $\beta = 1/2$ for definiteness. As pointed out previously, the process H is then a (scaled) reflected linear Brownian motion and in particular is Markovian. As a consequence, the process $(W_s, s \geq 0)$, which is now called the Brownian snake, is (time-homogeneous) Markov and indeed verifies the strong Markov property. This plays a crucial role in the applications that are outlined below.

From now on, we suppose that ξ is Brownian motion in $\mathbb{R}^d$. An easy application of the Kolmogorov criterion shows that W has a modification that is continuous with respect to the uniform topology on stopped (continuous) paths.

Our goal is to give some applications of the snake construction to connections between superprocesses and partial differential equations. These connections have been investigated by Dynkin in a series of important papers (see in particular [10], [11]). The Brownian snake turns out to be a useful tool in the quadratic branching case. The key to the connections with partial differential equations is the next theorem, which reformulates in terms of the Brownian snake a result of Dynkin [10] valid for superprocesses with a more general branching mechanism. We let D be a domain in $\mathbb{R}^d$ and for every path w, we denote by $\tau(w) = \inf\{t \geq 0, w(t) \notin D\}$ the first exit time of D by w (with the convention $\inf \emptyset = \infty$).

THEOREM 3.2 *Let $x \in D$. The limit*

$$\langle Z^D, g \rangle = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^{T_1} ds\, 1_{\{\tau(W_s) < H_s < \tau(W_s)+\varepsilon\}} g(W_s(\tau(W_s)))$$

exists $\mathbb{P}_x$-a.s. for every continuous function on ∂D, and defines a random measure Z^D on ∂D called the exit measure from D. If D is regular (in the classical potential-theoretic sense) and g is continuous and nonnegative on ∂D, the formula

$$u(x) = -\log \mathbb{E}_x(\exp - \langle Z^D, g \rangle) \qquad x \in D \tag{7}$$

defines the unique nonnegative solution of the equation $\Delta u = u^2$ in D with boundary value $u_{|\partial D} = g$.

A nice feature of the probabilistic representation formula (7) is that it can be used to produce many other solutions via suitable passages to the limit. In the setting of our next result, a generalized form of this representation holds for *any* nonnegative solution.

We denote by $\mathcal{R}^D$ the random set $\{W_s(t); 0 \leq s \leq T_1, t \leq \tau(W_s) \wedge H_s\}$.

THEOREM 3.3 [18] *Let D be a domain of class C^2 in $\mathbb{R}^2$. Then, for every $x \in D$, $\mathbb{P}_x$ a.s., the random measure Z^D has a continuous density $z_D(y), y \in \partial D$ with respect to Lebesgue measure on ∂D. Furthermore, the formula*

$$u(x) = -\log \mathbb{E}_x\left(1_{\{\mathcal{R}^D \cap K = \emptyset\}} \exp - \langle \gamma, z_D \rangle\right), \quad x \in D \tag{8}$$

gives a one-to-one correspondence between nonnegative solutions of $\Delta u = u^2$ in D and pairs (K, γ), where K is a closed subset of ∂D and γ is a Radon measure on $\partial D \backslash K$.

In the representation of Theorem 3.3, both K and γ can be determined analytically in terms of the boundary behavior of u: K is the set of points in ∂D where u blows up like the inverse of the squared distance to the boundary, and γ corresponds to the usual trace of u on $\partial D \backslash K$.

The analytic part of Theorem 3.3 has been extended by Marcus and Véron [21] to the equation $\Delta u = u^p$, $p > 1$ in a smooth domain of $\mathbb{R}^d$, provided that $d < \frac{p+1}{p-1}$. (see also Dynkin and Kuznetsov [12],[13]). In the supercritical case $d \geq \frac{p+1}{p-1}$, things become more complicated: One can still define the trace of a general nonnegative solution as a pair (K, γ), but a solution is in general not uniquely determined by its trace, and not all pairs (K, γ) are admissible traces (see [21], [13]). Recently, Dynkin and Kuznetsov [13] have proposed a finer definition of the trace that might lead to a one-to-one correspondence even in the supercritical case.

A remarkable feature of the connections between superprocesses or snakes and semilinear partial differential equations is the fact that almost all important probabilistic questions correspond to basic analytic problems, and conversely. We will give a last example involving on one hand a Wiener-type test for the Brownian snake and on the other hand solutions with boundary blow-up. We use the notation $c_{2,2}$ for the Sobolev capacity associated with the Sobolev space $W^{2,2}$.

THEOREM 3.4 [6] *Let D be a domain in $\mathbb{R}^d$. The following statements are equivalent.*

(i) *There exists a nonnegative solution of $\Delta u = u^2$ in D that blows up everywhere at the boundary.*

(ii) *Let $T = \inf\{s \geq 0, W_s(t) \notin D \text{ for some } t \in (0, H_s]\}$. Then $\mathbb{P}_y(T = 0) = 1$ for every $y \in \partial D$.*

(iii) *$d \leq 3$, or $d \geq 4$ and for every $y \in \partial D$,*

$$\sum_{n=1}^{\infty} 2^{n(d-2)} c_{2,2}(D^c \cap \{z \in \mathbb{R}^d, 2^{-n} \leq |z - y| < 2^{-n+1}\}) = \infty.$$

4 STATISTICAL MECHANICS AND INTERACTING PARTICLE SYSTEMS

(4.1) *Lattice trees.* A lattice tree with n bonds is a connected subgraph of $\mathbb{Z}^d$ with n edges in which there are no loops.

We are interested in a limit theorem that gives information on the typical shape of a lattice tree when n is large. To this end, let $Q_n(d\omega)$ be the uniform probability measure on the set of all lattice trees with n bonds that contain the origin of $\mathbb{Z}^d$. For every tree ω, let $X_n(\omega)$ be the probability measure on $\mathbb{R}^d$ obtained by putting mass $\frac{1}{n+1}$ to each vertex of the rescaled tree $cn^{-1/4}\omega$. Here $c > 0$ is a positive constant.

Provided that the dimension d is large enough, Derbez and Slade [5] proved that the limiting behavior of the law of X_n under Q_n involves a random measure which is closely related to Aldous' CRT. To define this random measure, consider the snake W driven by a normalized Brownian excursion $(e(s), 0 \leq s \leq 1)$, assuming again that the spatial motion is Brownian motion in $\mathbb{R}^d$ (and the initial point is 0). Then the formula

$$\langle \mathcal{I}, f \rangle = \int_0^1 ds\, f(W_s(e(s)))$$

defines a random measure in $\mathbb{R}^d$, sometimes called Integrated Super-Brownian Excursion (ISE).

THEOREM 4.1 [5] *For d sufficiently large and for a suitable choice of the constant $c = c(d) > 0$, the law of X_n under Q_n converges weakly to the law of $\mathcal{I}$.*

It is expected that the result holds when $d > 8$ (which is the condition needed to ensure that the topological support of $\mathcal{I}$ is a tree). This is true [5] if one considers "spread-out" trees rather than nearest-neighbor trees. A recent work of Hara and Slade indicates that ISE also appears as a scaling limit of the incipient infinite percolation cluster at the critical temperature, again in high dimensions ($d > 6$).

(4.2) *Interacting particle systems.* A number of recent papers explore the connections between the theory of superprocesses and some of the most classical interacting particle systems. Durrett and Perkins [8] show that the asymptotic behavior of the contact process in $\mathbb{Z}^d$ can be successfully analysed in terms of super-Brownian motion. Here we will concentrate on the classical voter model and follow a work in preparation in collaboration with M. Bramson and T. Cox. Closely related results can be found in a forthcoming article by Cox, Durrett and Perkins.

At each site of $\mathbb{Z}^d$ sits an individual who can have two possible opinions, say 0 or 1. At rate 1 each individual forgets his opinion and gets a new one by choosing uniformly at random one of his nearest neighbors and taking his opinion. Suppose that at the initial time all individuals have type 0, except for the individual at the origin who has type 1. For every $t > 0$, let $\mathcal{U}_t$ denote the set of individuals who have type 1 at time t, and let U_t be the random measure

$$U_t = \sum_{x \in \mathcal{U}_t} \delta_{x/\sqrt{t}}.$$

Then $P[\mathcal{U}_t \neq \emptyset] = P[U_t \neq 0]$ tends to 0 as $t \to \infty$, and the rate of this convergence is known [4]. One may then ask about the limiting behavior of U_t conditionally on $\{U_t \neq 0\}$.

The answer to this question can be formulated in terms of the snake W driven by a Brownian excursion conditioned to hit level 1, with spatial motion given by ($d^{-1/2}$ times) a standard Brownian motion in $\mathbb{R}^d$. We have the following result in dimension $d \geq 3$ (an analogous result holds for $d = 2$).

THEOREM 4.2 *The law of $t^{-1}U_t$ conditionally on $\{U_t \neq 0\}$ converges as $t \to \infty$ to the law of $c_d\mathcal{H}$, where $c_d > 0$ and the random measure $\mathcal{H}$ is defined by*

$$\langle \mathcal{H}, f \rangle = \int_0^\infty dL_s^1 \, f(W_s(1)),$$

where L_s^1 is as previously the local time of the excursion at level 1 and at time s.

To interpret this last theorem, one may say, for the voter model as well as for the (long-range) contact process [8], that the limiting behavior of the process depends on a pseudo-branching structure, which asymptotically comes close to the genealogical structure of the Feller diffusion.

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