Within and Beyond the Reach of Brownian Innovation

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Abstract. Given a system whose time evolution is random, we often try to describe it as a deterministic system under independent random influences. Doing so, we reduce complicated statistical correlations to a complicated but deterministic mechanism, and a stochastic but uncorrelated noise. That is the idea of innovation. The corresponding mathematics is surprisingly interesting.

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1. The name of the game

An *innovation* is a *real-time transformation* of a *noise* into a given *random process.*

Out of the four terms, only one, "random process", is standard. The notion of a real-time transformation was introduced repeatedly, and used under various names: "lifting" (of a filtered probability space) [19, (7.1–7.3)], "Hypothèse $(\mathcal{H})$" [7, Sect. 2.4], "extension" (of a filtered probability space) [22, Chap. 2, Def. 7.1], [8, Def. 6.1], with no name [44, 17.3.1(a)], [2, Lemma 7(c)], "morphism" (from one filtration to another) [33, Def. 1.1], "immersion" (of one filtration into another) [4], "orthogonal factor" (of a reverse filtration) [15, Sect. 2]. My favorite "real-time transformation" appeared in [33].

A noise in the discrete-time framework amounts to an independent sequence (of random variables or σ-fields), or a product (of a sequence of probability spaces). For continuous time, the classical white noise is a special case of a noise as defined in [34, Def. 1.1]; see also "factored probability spaces" [13], "measure factorizations" [36, Def. 1.2], and "product measures" (on a factorized Borel space) [36, Def. 2.4].

Innovation processes are well-known in filtering theory (see [5, Sect. 8]). A far-reaching generalization is the "innovation" introduced here. In the discrete-time framework, innovation appeared as "standard extension" (of a reverse filtration) [8, p. 885], "generating parametrization" [28, Sect. 2], [26, Def. 2.1], "substandard representation" [15, Sect. 2]. My favorite "innovation" appeared in [26].

2. Trivial cases

Let μ be a probability measure on a space $\mathcal{X}$. (Usually $\mathcal{X} = \mathbb{R}$ or $\mathbb{R}^n$, but it may be a finite set, a complete separable metric space, a standard Borel space.) Every such μ can be represented as the image of the Lebesgue measure $\mathcal{U}(0,1)$ under a measurable map $f : (0,1) \to \mathcal{X}$. Let U be a random variable distributed uniformly on $(0,1)$ (in symbols $U \sim \mathcal{U}(0,1)$), then $f(U) \sim \mu$. Of course, μ does not determine f uniquely; $f(g(U)) \sim \mu$ for all measure preserving $g : (0,1) \to (0,1)$. So, every $\mathcal{X}$-valued random variable Y is distributed like some $X = f(U)$.

Consider a discrete time random process $Y = (Y_t)_{t\in T}$, assuming for now that T is finite, $T = \{1,\dots,n\}$; thus Y is just n random variables $Y_1,\dots,Y_n$, and its distribution is a measure μ on $\mathcal{X}^n$. Let $U_1,\dots,U_n$ be independent $\mathcal{U}(0,1)$ random variables. Choose $f_1 : (0,1) \to \mathcal{X}$ such that $f_1(U_1)$ is distributed like Y_1. For each $y_1 \in \mathcal{X}$ consider the conditional distribution of Y_2 given that $Y_1 = y_1$ (I omit trivial reservations) and choose $f_2(\cdot, y_1)$ accordingly. Introduce $X_1 = f_1(U_1)$, $X_2 = f_2(U_2, X_1)$, then the pair (X_1, X_2) is distributed like (Y_1, Y_2). Continuing the process, we get functions $f_1,\dots,f_n$ and random variables $X_1,\dots,X_n$ such that

$$\begin{aligned} &X_1 = f_1(U_1),\ X_2 = f_2(U_2, X_1),\ \dots,\ X_n = f_n(U_n, X_{n-1},\dots,X_1)\,, \\ &(X_1,\dots,X_n) \text{ is distributed like } (Y_1,\dots,Y_n)\,. \end{aligned} \tag{2.1}$$

That is the innovation: at a time $t \in T$ the process X takes on a value X_t produced by a deterministic mechanism f_t out of two sources: the past $(X_1,\dots,X_{t-1})$ of the process, and the current value U_t of a noise. Note that each U_t is used only once (formulas like $X_2 = f_2(U_2, U_1, X_1)$ are disallowed), and $U_1,\dots,U_n$ are independent. The uniform distribution of U_t is only conventional; in Sect. 4 we prefer the normal distribution. Note also the large choice available on each stage when constructing $f_1,\dots,f_n$.

Example. Let $(Y_t)_{t\in T}$ be a process with independent increments, having assumed that $\mathcal{X} = \mathbb{R}$ or another group. We may choose an innovation of the form

$$X_t = g_t(U_t) + X_{t-1}\,. \tag{2.2}$$

The simple form (2.2) seems to be decidedly preferable to (2.1) for such processes, which is a delusion, to be refuted in Sect. 3.

The distribution of $X = (X_1,\dots,X_n)$ is the given μ. Consider, however, the joint distribution of X and U. We have

$$\mathbb{E}\big(\,\varphi(X_1,\dots,X_n)\,\big|\,U_1,\dots,U_t\,\big) = \mathbb{E}\big(\,\varphi(X_1,\dots X_n)\,\big|\,X_1,\dots X_t\,\big) \tag{2.3}$$

for all $t = 1,\dots,n$ and all bounded Borel functions $\varphi : \mathcal{X}^n \to \mathbb{R}$. Forecasting the future of the process X, we want to know the past of X only, and not the past of U. In other words, $(X_{t+1},\dots,X_n)$ and $(U_1,\dots,U_t)$ are conditionally independent, given $(X_1,\dots,X_t)$. *

Consider the σ-field $\mathcal{F}_X(t)$ generated by $X_1,\dots,X_t$; clearly, $\mathcal{F}_X(t) \subset \mathcal{F}_U(t)$ for all t, that is, $\mathcal{F}_X \le \mathcal{F}_U$, where $\mathcal{F}_X = \big(\mathcal{F}_X(t)\big)_{t\in T}$ is the filtration generated by X. Writing (2.3) in the form $\mathbb{E}\big(\,\xi\,\big|\,\mathcal{F}_U(t)\,\big) = \mathbb{E}\big(\,\xi\,\big|\,\mathcal{F}_X(t)\,\big)$ for $\mathcal{F}_X(n)$-measurable ξ, note that $\mathbb{E}\big(\,\xi\,\big|\,\mathcal{F}_X(t)\,\big)$ is the general form of an $\mathcal{F}_X$-martingale; so,

$$\mathcal{M}(\mathcal{F}_X) \subset \mathcal{M}(\mathcal{F}_U)\,, \tag{2.4}$$

* Though, (2.1) stipulates more: $(X_{t+1}, U_{t+1},\dots,X_n,U_n)$ and $(U_1,\dots,U_t)$ are conditionally independent, given $(X_1,\dots,X_t)$.

where $\mathcal{M}(\mathcal{F})$ is the set of all $\mathcal{F}$-martingales. Relation (2.4) implies $\mathcal{F}_X \le \mathcal{F}_U$, and is much stronger; try $X_2 = f_2(U_2, U_1, X_1)$ instead of $f_2(U_2, X_1)$ and you'll find (2.4) violated but $\mathcal{F}_X \le \mathcal{F}_U$ is still valid.

The following definition is formulated in terms of processes, but only their distributions are relevant. Still, $T = \{1, \dots, n\}$.

2.6 DEFINITION. A *real-time transformation* of a random process $V = (V_1, \dots, V_n)$ into another process $W = (W_1, \dots, W_n)$ is a two-component process $(V', W') = \big((V_1', W_1'), \dots, (V_n', W_n')\big)$ such that V' is distributed like V, W' is like W, and for each $t = 1, \dots, n$, W_t' is equal to a function of $V_1', \dots, V_t'$, and two vectors $(V_1', \dots, V_t')$ and $(W_{t+1}', \dots, W_n')$ are conditionally independent given $(W_1', \dots, W_t')$.

Reformulations via (2.3), (2.4) and generalizations for infinite T are left to the reader. Nothing new emerges for an infinite *increasing* sequence of time moments, $t \in T = \mathbb{N} = \{1, 2, 3, \dots\}$. Still, an innovation is constructed step-by-step: f_1, then f_2, and so on *ad infinitum.* The same holds for every countable ordinal number, that is, every countable linearly ordered set T that contains no infinite strictly *decreasing* sequences.

3. DECREASING SEQUENCES ARE HIGHLY NON-TRIVIAL

The following two examples show an astonishing phenomenon: some information appears magically, from thin air; see [25, p. 156], [43, p. 136], [10] and references therein.

The first example: $X_t = \pm 1$ for $t \in \mathbb{Z}$ are i.i.d. equiprobable random signs, $U_t = X_t / X_{t-1}$; then U_t are i.i.d. equiprobable random signs, also. Thus, X is both a process with independent values, and a process with independent increments in the multiplicative group $\{-1, +1\}$. The equality $X_t = U_t X_{t-1}$ should be an innovation of the process X by the noise U. However, it is not; X contains more information than U, since U determines X only up to an overall sign. The missing information should be a kind of initial value, $X_{-\infty}$; however, any function of the germ (tail) of X at $-\infty$ is either constant almost sure, or nonmeasurable, which is the well-known *tail triviality.*

The second example is the "eternal" (stationary) Brownian motion in a circle (or any other compact Lie group). Let $\big(B(t)\big)_{t\in[0,\infty)}$ be the standard Brownian motion in $\mathbb{R}$, and α a random variable, uniform on $(0,1)$ and independent of $\big(B(t)\big)_{t\in[0,\infty)}$. Consider the complex-valued process $X(t) = \exp\big(2\pi i\alpha + iB(t)\big)$. The process $\big(X(t)\big)_{t\in[0,\infty)}$ is stationary. Therefore, it has a unique (in distribution) extension $\big(X(t)\big)_{t\in\mathbb{R}}$, the eternal motion. Multiplicative increments $U_t = X_t / X_{t-1}$ for $t \in \mathbb{Z}$ should innovate the process $\big(X(t)\big)_{t\in\mathbb{Z}}$. However, they do not, for the same reason as in the first example: they stay invariant under transformations of the form $\big(X(t)\big)_{t\in\mathbb{R}} \mapsto \big(e^{i\varphi}X(t)\big)_{t\in\mathbb{R}}$.

About notation: ergodic people, being more light-hearted toward the time arrow than probabilists, prefer $(X_1', X_2', \dots)$, where $X_1' = X_{-1}, X_2' = X_{-2}, \dots$, to $(\dots, X_{-2}, X_{-1})$. Accordingly, dependence on the past turns into dependence on *larger* indices t [8], [16], [28], [26], [15]. I adhere to the probabilistic school, [44], [4], [9], [10], choosing $T = (-\mathbb{N}) = \{\dots, -2, -1\}$.

Every process $Y = (Y_t)_{t\in T}$ is distributed like some process X satisfying $X_t = f_t(U_t; X_{t-1}, X_{t-2}, \dots)$ for some Borel functions f_t and independent U_t. It follows that $\mathcal{M}(\mathcal{F}_X) \subset \mathcal{M}(\mathcal{F}_{X,U})$, but we need $\mathcal{M}(\mathcal{F}_X) \subset \mathcal{M}(\mathcal{F}_U)$. The two-component process (U, X) is a real-time transformation of U into X if and only if $\mathcal{F}_X \le \mathcal{F}_U$. Chaining $f_t, f_{t-1}, \dots, f_{s+1}$ we get $f_{s,t}$ such that $X_t = f_{s,t}(U_t, \dots, U_{s+1}; X_s, X_{s-1}, \dots)$. However, we need $f_{-\infty,t}$ such that $X_t = f_{-\infty,t}(U_t, U_{t-1}, \dots)$. That is possible if and only if the influence of $X_s, X_{s-1}, \dots$ on $f_{s,t}(U_t, \dots, U_{s+1}; X_s, X_{s-1}, \dots)$ disappears in the limit $s \to -\infty$. Tail triviality is necessary but not sufficient. Both examples shown above are tail trivial, and satisfy $X_t = U_t \dots U_{s+1} X_s$. Given U, the influence of X_s on X_t is strong, irrespective of s. Thus, the equality $X_t = U_t X_{t-1}$ fails to give an innovation.

Despite the strong influence of X_s on X_t, these X_s, X_t are (statistically) independent in the first example, and asymptotically independent (for $s \to -\infty$) in the second example. The strong dependence characterizes the specific way of using U_t (namely, $X_t = U_t X_{t-1}$), that is, the parametrization $(f_t)_{t\in T}$ rather than the process X itself. Is there a better parametrization for the same process? For the first example, the answer is evidently positive. Here, the conditional distribution of X_t, given the past, does not depend on the past. The parametrization $X_t = U_t X_{t-1}$ is bad because it introduces an unnecessary dependence on the past. A good parametrization is simply $X_t = U_t$, which surely is an innovation. For the second example, restricted to $t \in \mathbb{Z}$, the conditional distribution of X_t, given the past, depends on X_{t-1}. However, such distributions (corresponding to different values of X_{t-1}) overlap. A good parametrization uses the overlap for reducing dependence on the past. In continuous time, an innovation for the eternal motion is constructed [10] by inventing a coupling for processes differing in remote past. They are forced to coalesce, which never happens under the bad parametrization $X_t = U_t X_{t-1}$ of the form (2.2). That is the refutation of the delusion mentioned after (2.2).

Is there an innovation for an arbitrary tail-trivial process $(X_t)_{t\in(-\mathbb{N})}$? The answer is negative, which fact is "highly non-trivial and remarkable" [26], "deep and surprising" [15]. The first example, admitting no innovation, was discovered in the context of ergodic theory [37]. There are more examples of ergodic flavor [38], [29], [39], [28], [21], and of probabilistic flavor [8], [17], [14], [26], [4], [9]. The example of [8], furthered in [17], [14], [26], [4], is strikingly close to the sequence of i.i.d. equiprobable random signs; namely, the product measure is replaced with an equivalent (that is, mutually absolutely continuous) measure.

Some criteria for existence of an innovation, outlined in [37], [39], are elaborated in [15]. There, "substandardness" is our "existence of innovation", while "product type" is stronger, stipulating that U_t is a function of $X_t, X_{t-1}, \dots$ In such a case one says that U_t is exactly the *new* information furnished by X at t (though it depends on the chosen innovation). "Substandardness" implies "product type" provided that the conditional distribution of X_t given the past, is nonatomic [15].

4. Cosiness

Cosiness is a useful necessary condition for existence of an innovation. (*Is it also sufficient? I do not know.*) Cosiness emerged in [33, Def. 2.4] for continuous time and in [4, Sect. 4] for discrete time, the latter with a reservation that "there is a whole range of possible variations" of the definition; one of the variations follows. Still, $T = (-\mathbb{N}) = \{\dots, -2, -1\}$, and processes are $\mathcal{X}$-valued.

4.1 DEFINITION. A random process $(X_t)_{t\in T}$ is *cosy,* if for each $\varepsilon > 0$ and each bounded Borel function $\varphi : \mathcal{X}^T \to \mathbb{R}$ there exists a two-component random process $(Y, Z) = \big((Y_t, Z_t)\big)_{t\in T}$ such that
(a) $\big((Y, Z), Y\big)$ and $\big((Y, Z), Z\big)$ are real-time transformations of (Y, Z) into X;
(b) $\mathbb{E}|\varphi(Y) - \varphi(Z)| < \varepsilon$;
(c) there exists $\delta \in (0, 1)$ such that for all bounded Borel functions $\psi, \chi : \mathcal{X}^T \to \mathbb{R}$,

$$\big(\mathbb{E}|\psi(Y)\chi(Z)|\big)^{2-\delta} \le \big(\mathbb{E}|\psi(Y)|^{2-\delta}\big)\big(\mathbb{E}|\chi(Z)|^{2-\delta}\big) \, .$$

Some comments. Condition (a) implies that each of the two processes Y, Z is distributed like X; thus, (Y, Z) is a joining of two copies of X, possessing the "real time" property $\mathcal{M}(Y) \subset \mathcal{M}(Y, Z)$, $\mathcal{M}(Z) \subset \mathcal{M}(Y, Z)$ (recall (2.4)). Condition (b) means that Y, Z are close, since φ may be one-one. Condition (c) means that Y, Z are "independent a little", since it is always satisfied for $\delta = 0$ and equivalent to independence of Y, Z for $\delta = 1$.

4.2 THEOREM. [4, Lemma 6 and Corollary 3] *A non-cosy process admits no innovation.*

The idea of a proof. Assume that X has an innovation; X is distributed like Y, $Y_t = f_{-\infty,t}(U_t, U_{t-1}, \dots)$, $U = (U_t)_{t\in T}$ being a sequence of independent $\mathcal{N}(0, 1)$ random variables. (This time we prefer the normal distribution $\mathcal{N}(0, 1)$ to $\mathcal{U}(0, 1)$.) Take another sequence $V = (V_t)_{t\in T}$ of independent $\mathcal{N}(0, 1)$ random variables such that U, V are independent. Introduce $W_t = U_t \cos\varepsilon + V_t \sin\varepsilon$, and let $Z_t = f_{-\infty,t}(W_t, W_{t-1}, \dots)$.* Condition (c) follows from the celebrated hypercontractivity theorem (pioneered by Nelson, see [24, Sect. 3])!

The first example of a non-cosy process in discrete time is given in [4, Th. 1]; it appears that the method of [8] produces non-cosy processes. It is interesting to know, whether "ergodic" examples [37], [38], [29], [39], [28], [21] are also non-cosy, or not. Another non-cosy discrete-time filtration [9] is the restriction of a continuous-time filtration to a discrete set on the time axis.

5. APPLICATIONS TO CONTINUOUS TIME

An $\mathcal{X}$-valued process $(X_t)_{t\in T}$, $T = (-\mathbb{N}) = \{\dots, -2, -1\}$, generates its filtration $\mathcal{F}_X = \big(\mathcal{F}_X(t)\big)_{t\in T}$. The family $\big(\mathcal{F}_X(2t)\big)_{t\in T}$ is also a filtration; it is generated by the $\mathcal{X}^2$-valued process $(Y_t)_{t\in T}$, $Y_t = (X_{2t-1}, X_{2t})$. If X admits an innovation, then the amalgamated process Y also does. The same applies for any infinite subset $T_1 \subset T$. If X is tail-trivial and T_1 is sparse enough, then Y admits an innovation, see [15, Th. 1.18] and references therein.

A continuous process $(X_t)_{t\in[0,\infty)}$ generates its filtration $\mathcal{F}_X = \big(\mathcal{F}_X(t)\big)_{t\in[0,\infty)}$. Choosing a sequence $(t_k)_{k\in(-\mathbb{N})}$, $t_k \in [0, \infty)$, $t_{k-1} < t_k$, $\inf t_k = 0$, we get a

* Which is anticipated in [23].

discrete-time filtration $\big(\mathcal{F}_X(t_k)\big)_{k\in(-\mathbb{N})}$, generated by the amalgamated process $(Y_k)_{k\in(-\mathbb{N})}$, $Y_k = (X_t)_{t\in[t_{k-1},t_k]}$. If Y admits no innovation, then X also admits no innovation, for any reasonable definition of continuous-time innovations. Some continuous-time problems are solved in that way.

The effect of "information from thin air" (see Sect. 3) can be reproduced by the stochastic differential equation

$$dX_t = dB_t + v\left(t,\ (X_s)_{s\in[0,t]}\right) dt \tag{5.1}$$

with a bounded drift v, if v is chosen properly. Then (5.1) fails to innovate X, which means that the equation has no strong solution. That is the "celebrated and mysterious" [25, V.3.18, p. 155] example, constructed in [32] and investigated in [5], [30], [43], [23], [10]. The eternal Brownian motion in a circle, mentioned in Sect. 3, can be obtained from X by a real-time transformation and a deterministic time change that maps $[0,\infty)$ onto $\mathbb{R}$ [10]. The same process X is a strong solution of the stochastic differential equation

$$dX_t = \sigma\left(t,\ (X_s)_{s\in[0,t]}\right) dB_t + v\left(t,\ (X_s)_{s\in[0,t]}\right) dt \tag{5.2}$$

for some $\sigma(\dots) = \pm 1$ [10] (see also [16]). Once again, a clever parametrization is better than the straightforward parametrization.

One of the processes admitting no innovation, mentioned in Sect. 3, leads to a more ingenious drift v in (5.1); the corresponding (continuous) process X has no innovation, which means that it cannot be the strong solution of any equation of the form (5.2) [8]; see also [17], [14], [26], [4]. The drift is not bounded, but I believe that it can be made bounded. "Dreadfully complicated, their construction is almost as incredible as the existence result itself" [4]. Is it really a complicated construction? In fact, the drift is not constructed "by hands", it is chosen at random. It is a *random drift;* here "random" is interpreted like the second "random" in the phrase "random walk in a *random environment*". Thus, it is a typical drift in the same sense as a nowhere differentiable Brownian sample path is a typical function. Few parameters are adjusted by authors, such as order of magnitude, and depth of dependence on the past, both depending on time in a simple prescribed way.

There exists a pure martingale admitting no innovation [9].

6. From stochastic analysis to stochastic topology

Some continuous-time phenomena have no (evident) discrete-time counterpart. For example, Brownian motion cannot be transformed in real time into a Poisson process. A non-Gaussian stable process cannot be transformed into Brownian motion. The m-dimensional Brownian motion can be transformed into the n-dimensional Brownian motion if and only if $m \ge n$, which may be treated as the starting point of *stochastic topology,* the theory of filtration invariants of random processes.* A diffusion process with smooth nondegenerate coefficients in an n-dimensional smooth manifold is equivalent to the n-dimensional Brownian motion

* A useful classification claimed in [27, Th. 7] appeared to be not exhaustive [8, Sect. 6].

in the sense that their filtrations are isomorphic; in other words, the two processes can be connected by an *invertible* real-time transformation. What happens in presence of singularities of the topology or the coefficients? Few results are available; they are based on stochastic analysis (Itô formula, local times, ...). All negative results are based on continuous-time cosiness [33, Def. 2.4]. Brownian motion of finite or countable dimension is cosy [33, Lemma 2.5]. A cosy process cannot be transformed in real time to a non-cosy process [33, Lemma 2.6]. Therefore, all non-cosy processes are beyond the reach of Brownian innovation.

Two well-known diffusion processes in $\mathbb{R}$ are singular at the origin ($x = 0$, not $t = 0$ as in Sect. 5). The skew Brownian motion (see [20]) has a singular drift at 0, and is equivalent to the usual Brownian motion [20]. The sticky Brownian motion (see [41]) is slowed down at 0; its filtration is non-cosy [42].

Consider n rays (say, on the plane) with a single common point, the origin. There is a natural diffusion process Z_n on the union of the rays; Z_2 is the usual Brownian motion, Z_1 is the reflecting Brownian motion; $Z_3, Z_4, \dots$ are so-called Walsh's Brownian motions [40], [3]. Such processes arise when considering small random perturbations of Hamiltonian dynamical systems [18] and some other topics [40], [3]. Processes Z_1 and Z_2 are equivalent (Lévy, Skorokhod). Nevertheless, Walsh's Brownian motions are non-cosy [33, Th. 4.13] (see also [11], [2]), which solves Problem 2 of [3].

Interestingly, stochastic topology can be of help to the classical (non-stochastic) analysis. Consider three non-intersecting domains in $\mathbb{R}^n$. If they are smoothly bounded, then points of trilateral contact are evidently rare among boundary points. It was conjectured for irregular domains, that the infimum of the three corresponding harmonic measures must vanish [6, Sect. 6], [12, Problem a]. In terms of the Martin boundary: its natural projection to the topological boundary is at most 2 to 1 almost everywhere. However, the best result of classical analysis is "at most 10 to 1" [6]. The final result "2 to 1" is achieved via stochastic topology [33, Th. 7.4]. A challenge for classical analysis!

So, some characteristic of $\mathbb{R}^n$ (or any smooth manifold) as a harmonic space, is equal to 2 irrespective of dimension, but exceeds 2 in presence of branching points. The nameless characteristic has its counterpart in stochastic topology, named *splitting multiplicity.* Introduced in [3, Def. 4.2], it was hibernating till the birth of cosiness. Every cosy process is of splitting multiplicity 2 (or 1, if it is degenerate) [2], while Walsh's Brownian motion Z_n, $n > 2$, is of splitting multiplicity n [2]. Splitting multiplicity is invariant under measure changes and time changes [2], while cosiness is not [4], [9].

7. White noise versus black noises

In discrete time we have no choice of noises for innovation; a noise is a sequence of independent random variables, each having a non-atomic distribution. In continuous time, the classical theory of processes with independent increments tells us that in general, a noise consists of a Gaussian component (a finite or countable collection of independent white noises) and a Poissonian component. The latter is useless for innovating diffusion processes. The former can innovate only cosy processes. Thus, Walsh's Brownian motion is beyond the reach of classical innovation.

We may turn to Brownian motions (defined as continuous processes with stationary independent increments) on more general groups. In that aspect, finite-dimensional Lie groups are equivalent to $\mathbb{R}^n$. The Polish group of all unitary operators on the (separable) Hilbert space, equipped with the strong operator topology, is equivalent to (the additive group of) the Hilbert space [34, Th. 1.6]. (Interestingly, the proof involves continuous tensor products and continuous quantum measurements.) A commutative Polish group cannot give more [34, Th. 1.8].

The system of *coalescing* independent one-dimensional Brownian motions [1], [31, Sect. 2], is a limiting case of a coalescing stochastic flow. The system generates a two-parametric family of σ-fields $(\mathcal{F}_{s,t})_{s<t}$ that shares with the white noise the following property:

$$\mathcal{F}_{r,s} \otimes \mathcal{F}_{s,t} = \mathcal{F}_{r,t} \quad \text{whenever } r < s < t\,; \tag{7.1}$$

that is, $\mathcal{F}_{r,s}$ and $\mathcal{F}_{s,t}$ are independent and, taken together, generate $\mathcal{F}_{r,t}$. Nevertheless, $(\mathcal{F}_{s,t})_{s<t}$ supports no white noise (nor a Poisson process); it means that there is no Brownian motion $(B_t)_{t\in[0,\infty)}$ such that $B_t - B_s$ is $\mathcal{F}_{s,t}$-measurable for all intervals $(s,t) \subset [0,\infty)$ [35]. Thus, $(\mathcal{F}_{s,t})_{s<t}$ is a *black noise* as defined in [34, Sect. 1]. It is predictable [34, Def. 1.12] in the sense that its filtration $(\mathcal{F}_{0,t})_{t\in[0,\infty)}$ supports only continuous martingales. In fact, the filtration is Brownian! Therefore, that black noise still cannot innovate Walsh's Brownian motion.

One more example of a black noise is available [36, Sect. 5]. Does it generate a cosy filtration? I do not know.

7.2 Problem. Can a predictable noise (see [34, Defs. 1.1, 1.12]) generate a non-cosy filtration?

If the answer is positive, another problem follows.

7.3 Problem. Can Walsh's Brownian motion be innovated by *some* predictable noise?

7.4 Problem. Can a noise generate a cosy but non-Brownian filtration?*

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* It was claimed in the abstract of my talk, that it never happens to Brownian motions in Polish groups. However, I withdraw the claim, since my proof was wrong. Sorry.

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