Section 10. Partial Differential Equations

Estimates Near the Boundary for Solutions of Second Order Parabolic Equations

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Abstract. We discuss different forms of the Harnack inequality for second order, linear, uniformly parabolic differential equations, and their applications to the estimates of solutions near the boundary. These applications include some Gaussian estimates and doubling properties for the caloric measure, and estimates for the quotient of two positive solutions vanishing on a portion of the boundary of a Lipschitz cylinder. A genera approach to all these problems is demonstrated, which works for both the divergence and non-divergence equations and is based only on the "standard" Harnack inequality and elementary comparison arguments.

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1 Introduction. Preliminary results

In this paper, we deal with the estimates of solutions to second order parabolic equations, which do not depend on the smoothness of coefficients. Such estimates have many important applications, especially in the theory of nonlinear equations (see [K], [LSU], [T], [PE]). Here we treat simulteneously the equations in the *divergence* form

$$Lu = \sum_{i,j=1}^{n} D_i(a_{ij} D_j u) - u_t = 0, \tag{D}$$

and in the *non-divergence* form

$$Lu = \sum_{i,j=1}^{n} a_{ij} D_{ij} u - u_t = 0, \tag{ND}$$

where $D_j = \partial/\partial x_j$, $D_{ij} = D_i D_j$. We assume that the functions $u = u(X)$ and the coefficients $a_{ij} = a_{ij}(X)$ are defined and smooth for all $X = (x,t) \in \mathbb{R}^{n+1}$, and the operators L are uniformly parabolic, i.e. a_{ij} satisfy

$$\nu|\xi|^2 \le \sum_{i,j} a_{ij}\xi_i\xi_j \quad \text{for all } \xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n, \quad \max_{i,j} |a_{ij}(X)| \le \nu^{-1}, \tag{1.1}$$

with a constant $\nu \in (0,1]$. However, our estimates do not depend on the extra smoothness of u and a_{ij}, and by standard approximation procedures, they are extended to measurable a_{ij} in the divergence case (D) and to continuous a_{ij} in the non-divergence case (ND).

At present, the equation (D) are investigated much better than (ND). For example, under natural boundary conditions, the solution u of the equations (D) with measurable a_{ij} are well approximated by the solutions u^ε of equations with smooth $a_{ij}^\varepsilon \to a_{ij}$ as $\varepsilon \to 0$ (a.e.) A recent striking example by Nicolai Nadirashvili [N] (see also [S]) shows that this procedure fails to give a unique solution even for elliptic equations $\sum a_{ij} D_{ij} u = 0$ with measurable a_{ij} in the unit ball $B_1 \subset \mathbb{R}^n$, $n \ge 3$: different subsequences $\{u^{\varepsilon_k}\}$ may converge to different functions.

Nevertheless, some properties of solutions look similar for the equations (D) and (ND), though their proofs are essentially different in these two cases. It turns out that two such statements, the *comparison principle* (Theorem 1.1) and the *interior Harnack inequality* (Theorem 1.2), provide the background for many others. From this "unifying " point of view, we present different versions of the Harnack inequality, estimates of quotients of positive solutions, doubling properties for $L-$caloric measure, and other related results. The proofs of these results are very "compressed", for some statements we only give an outline of the main ideas. In the elliptic case, i.e. when a_{ij} and u in (D) or (ND) do not depend on t, most of these results are known from [CFMS], [B], [FGMS]. They were extended to the parabolic equation with time-independent coefficients in [S], [G], [FGS] , and to general parabolic equations (D), (ND) in recent papers [FS], [FSY], [SY].

For an arbitrary domain $V \subset \mathbb{R}^{n+1}$, we define its *parabolic boundary* $\partial_p V$ as the set of all the points $Y = (y,s) \in \partial V$, such that there is a continuous curve $X(t) = (x(t),t)$ lying in $V \cup \{Y\}$ with initial point Y, along which t is non-decreasing. In particular, for $Q = \Omega \times (t_0, T)$ we have

$$\partial_p Q = \partial_x Q \cup \partial_t Q, \text{ where } \partial_x Q = \partial\Omega \times (t_0, T), \ \ \partial_t Q = \overline{\Omega} \times \{t_0\}. \tag{1.2}$$

For $y \in \mathbb{R}^n$, $r > 0$, $Y = (y,s)$, $Q = \Omega \times (t_0, T)$, and small $\delta > 0$, we denote

$$\begin{aligned}
B_r &= B_r(y) = \{x \in \mathbb{R}^n : \ |x - y| < r\}, \quad C_r = C_r(Y) = B_r(y) \times (s - r^2, s + r^2), \\
Q_r &= Q_r(Y) = Q \cap C_r(Y), \quad \Delta_r = \Delta_r(Y) = (\partial_p Q) \cap C_r(Y), \\
\Omega^\delta &= \{x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) > \delta\}, \qquad Q^\delta = \Omega^\delta \times (t_0 + \delta^2, T).
\end{aligned}$$

THEOREM 1. *(Comparison principle). Let V be a bounded domain in $\mathbb{R}^{n+1}$, functions $u, v \in C^2(V) \cap C(\overline{V})$ and satisfy $Lu \le Lv$ in V, $u \ge v$ on $\partial_p V$. Then $u \ge v$ on $\overline{V}$.*

This theorem is well-known and its proof is elementary. The next one is far from obvious. In the divergence case, it was discovered by Moser [M] in 1964. In in the non-divergence case, it was proved in [KS] in 1978-79, see also [K], Chapter 4.

Theorem 2. *(Interior Harnack inequality). Let u be a nonnegative solution of $Lu = 0$ in a bounded cylinder $Q = \Omega \times (t_0, T)$, and let positive constants δ, λ be such that Ω^δ is a connected set, and $\operatorname{diam}\Omega + \sqrt{T - t_0} \le \lambda\delta$. Then for all $x, y \in \Omega^\delta$ and s, t satisfying $t_0 + \delta^2 \le s < s + \delta^2 \le t < T$, we have*

$$u(y, s) \le N u(x, t) \tag{1.3}$$

with a constant $N = N(n, \nu, \lambda)$.

From now on we assume that Ω is a bounded domain in $\mathbb{R}^n$ satisfying the following *Lipschitz* condition with some positive constants r_0, m : for each $y \in \partial\Omega$, there is an orthonormal coordinate system (centered at y), with coordinates $x = (x_1, \cdots, x_{n-1}, x_n) = (x', x_n)$, such that

$$\Omega \cap \{|x'| < r_0,\, |x_n| < (m+1)r_0\} = \{|x'| < r_0,\, \varphi(x') < x_n < (m+1)r_0\}, \tag{1.4}$$

and $|\nabla\varphi| \le m$ on the ball $\{|x'| < r_0\} \subset \mathbb{R}^{n-1}$. Then for any continuous function g on $\mathbb{R}^{n+1}$, there exists a unique solution $u \in C^2(Q) \cap C(\overline{Q})$ of the boundary value problem

$$Lu = 0 \quad \text{in} \quad \mathrm{Q} = \Omega \times (\mathrm{t}_0, \mathrm{T}), \qquad \mathrm{u} = \mathrm{g} \quad \text{on } \partial_p Q. \tag{1.5}$$

This is a well-known fact for smooth Ω, and it is easily extended to Lipschitz domains Ω by their approximation with smooth domains $\Omega^j \searrow \Omega$. From Theorem 1.1 it follows that $g \longrightarrow u(X)$ is a linear continuous functional on $C(\partial_p Q)$. By the Riesz representation theorem, there exists of a unique probability measure (*L-caloric measure*) $\omega^X = \omega^X_Q$ on $\partial_p Q$, such that the solution of the problem (1.5) has the form

$$u(X) = u(x, t) = \int_{\partial_p Q} g(Y)\, d\omega^X(Y). \tag{1.6}$$

The above representation is also valid for unbounded domains Q under some natural restrictions on the growth of solutions for $|x| \to \infty$. For example, if the function g is bounded, we restrict ourselves to the bounded solutions u.

Lemma 3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with constants r_0 and m, and let $Q = \Omega \times (t_0, \infty)$. Then for any $Y = (y, s) \in \partial_x Q = \partial\Omega \times (t_0, \infty)$, and $r \in (0, r_0]$, we have*

$$\omega^X(\triangle_r) \ge N^{-1} \quad \text{on} \quad Q_{r/2} \tag{1.7}$$

with a constant $N = N(n, \nu, m) > 1$. If $Y = (y, t_0) \in \partial_t Q = \overline{\Omega} \times \{t_0\}$, then the estimate (1.7) holds for all $r > 0$ with a constant $N = N(n, \nu) > 1$.

Proof. Without loss of generality, we may assume $t_0 = 0$. First we consider a simpler case $Y = (y, 0) \in \partial_t Q$. In this case,

$$C = C_r(Y) = B_r(y) \times (-r^2, r^2) \supset C^+ = B_r(y) \times (0, r^2) \supset Q_r = Q \cap C.$$

The function

$$u(X) = \omega^X_{C^+}(\partial_t C^+) \quad \text{on } C^+, \qquad u \equiv 1 \quad \text{on } C \setminus C^+$$

can be treated as a solution of the problem (1.5) in the cylinder C with $g \equiv 1$ for $t \le 0$, $g \equiv 0$ for $t > 0$. Therefore, applying Theorem 1.1 in $Q_r = Q \cap C$ and then Theorem 1.2 in C, we obtain the desired estimate with a constant $N = N(n, \nu) > 1$:

$$\omega^X(\triangle_r) \ge u(X) \ge N^{-1} u(y, -r^2/2) = N^{-1} \quad \text{on} \quad Q_{r/2}.$$

Now it remains to consider the case $Y = (y, s) \in \partial_x Q$, i.e. $y \in \partial\Omega$, $s \in (t_0, \infty)$. By the Lipschitz condition, the set $B_r(y) \setminus \Omega$ contains a ball $B_{\mu r}(z)$ with $\mu = \mu(m) > 0$. Then

$$Z = (z, s - r^2/2) \in C' = B_{\mu r}(z) \times (s - r^2, s + r^2) \subset C \setminus Q,$$

where $C = C_r(Y)$. We can apply Theorem 1.2 to the function $u(X) = \omega^X_C(\partial_t C')$ in C and to $u'(X) = \omega^X_{C'}(\partial_t C')$ in C' extended as $u' \equiv 1$ across $\partial_t C'$. This gives us

$$\omega^X(\triangle_r) \ge u(X) \ge N_1^{-1} u(Z) \ge N_1^{-1} u'(Z) \ge N_2^{-1} \quad \text{on} \quad Q_{r/2},$$

where the constants N_1 and N_2 depend only on n, ν, m. Lemma 1.1 is proved. □

Corollary 4. *Let $Lu = 0$, $u > 0$ in Q, and $u = 0$ on $\Delta_R(Y) = (\partial_p Q) \cap C_R(Y)$ for some $Y \in \partial_x Q$ and $R \in (0, r_0]$. Then*

$$\sup_{Q_{R/2}} u \le \theta \sup_{Q_R} u, \tag{1.8}$$

$$\sup_{Q_r} u \le (2r/R)^\alpha \sup_{Q_R} u \quad \text{for all} \quad r \in (0, R] \tag{1.9}$$

with constant $\theta = \theta(n, \nu, m) \in (0, 1)$, $\alpha = -\log_2 \theta > 0$. If $Y \in \partial_t Q$, then (1.8) and (1.9) hold for all $R > 0$ with θ, α depending only on n, ν.

Proof. Let ω^X denote the L-caloric measure on $\partial_p Q_R$. Then

$$\omega^X((\partial_p Q_R) \setminus \Delta_R) = 1 - \omega^X(\Delta_R) \le 1 - N^{-1} = \theta \quad \text{on} \quad Q_{R/2},$$

and since $u = 0$ on $\Delta_R(Y)$,

$$\sup_{Q_{R/2}} u = \sup_{Q_{R/2}} \int_{\partial_p Q_R} u \, d\omega^X \le \sup_{Q_{R/2}} \omega^X((\partial_p Q_R) \setminus \Delta_R) \cdot \sup_{\partial_p Q_R} u \le \theta \sup_{Q_R} u.$$

The estimate (1.8) is proved. Iterating this estimate, we also get (1.9). □

2 Gaussian Estimates for L-caloric Measure

For given cylinder $Q = \Omega \times (t_0, T) \subset \mathbb{R}^{n+1}$, introduce the functions $d(x) = \mathrm{dist}(x, \partial\Omega)$ on Ω, and

$$\rho(X) = \rho_Q(X) = \rho(x,t) = d(x)/\sqrt{t-t_0} \quad \text{on} \quad Q, \tag{2.1}$$

Theorem 5. *There exist positive constants N, β, depending only on n and ν, such that*

$$\omega^X(\partial_x Q) \le N e^{-\beta\rho^2(X)} \quad \text{on} \quad Q. \tag{2.2}$$

Proof. We fix $Y_0 = (0,1) \in \mathbb{R}^{n+1}$, and for $\rho > 0$ define $M(\rho) = \sup \omega_C^{Y_0}(\partial_x C)$, where $C = B_\rho(0) \times (0,1)$, and the supremum is taken with respect to all parabolic operators L with coefficients a_{ij} satisfying (1.1). It is easy to see that $M(\rho)$ decreases on $(0,\infty)$, and moreover, applying Corollary 1.1 to $u(X) = \omega_C^X(\partial_x C)$, we have $M(\rho) \searrow 0$ as $\rho \nearrow \infty$. This allows us to fix a constant $A = A(n,\nu)$ such that $M(A) \le 1/3$. By substitution $x \to (x-y)/\sqrt{h},\ t \to 1 + (t-s)/h$, we also have $\omega_C^Y(\partial_x C) \le M(\rho)$ for all $Y = (y,s) \in \mathbb{R}^{n+1}$ and $C = B_d(y) \times (s-h, s)$ with $d/\sqrt{h} \ge \rho$. If we take $Y = X = (x,t) \in Q = \Omega \times (t_0, T)$, $d = d(x)$, and $h = t - t_0$, then $C = B_d(x) \times (t_0, t) \subset Q$ and $\partial_t C \subset \partial_t Q$, hence

$$\omega_Q^X(\partial_x Q) \le \omega_C^X(\partial_x C) \le M(d/\sqrt{t-t_0}) = M(\rho(X)). \tag{2.3}$$

Further, for natural $j \ge 5$, set $\rho_j = 4A\sqrt{j},\ \varepsilon_j = 2/\sqrt{j},\ M_j = M(\rho_j)$, and consider the cylinders

$$Q_j = B_{\rho_j}(0) \times (0,1) \supset Q_j' = B_{\varepsilon_j A}(0) \times (1-\varepsilon_j^2, 1).$$

The function $\rho = \rho(X) = \rho(x,t) = (\rho_j - |x|)/\sqrt{t}$ corresponds by the equality (2.1) to $Q = Q_j$. One can easily verify the inequalities $\rho \ge \rho_{j-1}$ on $\partial_x Q_j'$ and $\rho \ge \rho_{j+1}$ on $\partial_t Q_j'$. Therefore, the caloric measure ω^X for L in Q_j satisfies

$$\omega^X(\partial_x Q_j) = \int_{\partial_p Q_j'} \omega^Y(\partial_x Q_j) d\omega^X(Y) \le M_{j-1} \cdot \omega^X(\partial_x Q_j') + M_{j+1} \cdot \omega^X(\partial_t Q_j')$$

for all $X \in \overline{Q_j'}$. By the choice of A, we have $\omega^{Y_0}(\partial_x Q_j') = 1 - \omega^{Y_0}(\partial_t Q_j') \le 1/3$, and the previous estimate yields

$$\begin{aligned} M_j &\le \frac{1}{3} M_{j-1} + \frac{2}{3} M_{j+1}, \\ M_j - M_{j+1} &\le 2^{-1}(M_{j-1} - M_j) \le \cdots \le 2^{5-j}(M_5 - M_6) \le 2^{5-j}. \end{aligned}$$

For arbitrary $\rho \ge \rho_5$, we choose $j \ge 5$ such that $\rho_j \le \rho < \rho_{j+1}$, so that

$$M(\rho) \le M(\rho_j) = M_j = \sum_{k \ge j} (M_k - M_{k+1}) \le 2^{6-j} \le N e^{-\beta\rho_{j+1}^2} \le N e^{-\beta\rho^2},$$

by appropriate choice of constants N, β, depending only on n and ν. If N is chosen large enough, the estimate $M(\rho) \le Ne^{-\beta\rho^2}$ also holds for $0 < \rho < \rho_5$. Together with (2.3), these estimates imply the desired estimate (2.2). $\square$

Remark 2.1. From Theorem 2.1 it follows immediately the uniqueness of the Cauchy problem

$$Lu = 0 \quad \text{in } \mathbb{R}^n \times (0,T), \qquad u(x,0) \equiv g(x) \tag{2.4}$$

in the class of functions satisfying $|u(x,t)| \le Ne^{N|x|^2}$, and the proof does not depend on the structure (divergence or non-divergence) of the operator L. Using some arguments in the papers by Moser [M] and Aronson [A1], one can prove a stronger statement: there is at most one solution of the problem (2.4) satisfying a one-sided inequality $u(x,t) \ge -Ne^{N|x|^2}$ for all $(x,t) \in \mathbb{R}^n \times (0,T)$.

Remark 2.2. In the divergence case, from Moser's Harnack inequality it follows the Hölder continuity of solutions, which was proved ealier by Nash [Ns]. Aronson [A2] also essentially used the Harnack inequality in the proof of the Gaussian estimates for the fundamental solution $\Gamma(x,t;y,s)$ of the divergence operator L: for $s < t$,

$$\begin{aligned} &\frac{1}{N}(t-s)^{-n/2}\exp\left(-\frac{N|x-y|^2}{t-s}\right) \\ &\qquad \le \Gamma(x,t;y,s) \le N(t-s)^{-n/2}\exp\left(-\frac{|x-y|^2}{N(t-s)}\right), \end{aligned} \tag{2.5}$$

with a constant $N = N(n,\nu)$. Fabes and Stroock [FS] gave another proof of the estimates (2.5) which is based on some ideas of Nash instead of the Harnack inequality, and they also showed that the Harnack inequality follows easily from (2.5). Thus all these facts are mutually related.

3 Harnack Inequalities

As before, let Ω be a bounded domain in $\mathbb{R}^n$ satisfying the Lipschitz condition with constants r_0, m, and let $Q = \Omega \times (t_0,\infty)$. For $y \in \overline{\Omega}$ and $r \in (0, r_0]$, the set $\Omega_r(y) = \Omega \cap B_r(y)$ contains a ball $B_{\mu r}(y_r)$, where $\mu = \mu(m) \in (0,1/2]$. We fix such y_r depending on y and r, and for $Y = (y,s)$, denote $Y_r^{\pm} = (y_r, s \pm 2r^2)$. The following result is often referred to as a *boundary Harnack inequality,* or *Carleson type estimate.* For parabolic equations, it was first proved by Salsa [S] (in the divergence case) and Garofalo [G] (in the non-divergence case), see also [FSY].

Theorem 6. *Let $Q = \Omega \times (t_0,\infty)$, $Y \in \partial_p Q$, $0 < r \le r_0/2$, and let u be a nonnegative solution of $Lu = 0$ in Q, satisfying $u = 0$ on $\Delta_{2r}(Y) = (\partial_p Q) \cap C_{2r}(Y)$. Then*

$$u \le N(n,\nu,m)u(Y_r^{+}) \quad \text{on } Q_r = Q_r(Y). \tag{3.1}$$

In the elliptic case, when a_{ij} and u do not depend on t, the interior Harnack inequality (1.3) is equivalent to

$$\sup_{\Omega^\delta} u \le N(n,\nu,\lambda) \inf_{\Omega^\delta} u, \tag{3.2}$$

provided $u \ge 0,\ Lu = 0$ in $\Omega,\ \Omega^\delta$ is a connected set, and $(\operatorname{diam}\Omega)/\delta \le \lambda$. An easy example of the function

$$u(x,t) = t^{-1/2}\exp[-(x-2)^2/4t] \text{ for } t>0, \quad u(x,t)\equiv 0 \text{ for } t \le 0,$$

which satisfies $u \ge 0,\ Lu = u_{xx} - u_t = 0$ in $Q = (-1,1)\times(-1,1)$, shows that we cannot simply replace Ω^δ by Q^δ in the parabolic case. However, this is possible under the additional assumption $u = 0$ on $\partial_x Q$. As in [G], [FGS], Theorem 3.1 yields the following *interior elliptic-type Harnack inequality.*

THEOREM 7. *Let $Lu = 0,\ u > 0$ in $Q = \Omega \times (t_0,T),\ u = 0$ on $\partial_x Q = \partial\Omega \times (t_0,T)$, and let positive constants $\delta \in (0,r_0)$ and $\lambda > 1$ be such that $(\operatorname{diam}\Omega + \sqrt{T-t_0})/\delta \le \lambda$. Then*

$$\sup_{Q^\delta} u \le N(n,\nu,m,\lambda) \inf_{Q^\delta} u. \tag{3.3}$$

Proof follows from Theorems 1.2 and 3.1 and the maximum principle:

$$\sup_{Q^\delta} u \le \sup_{x\in\Omega} u(x,\delta^2/4) \le N_1 \sup_{x\in\Omega^{\mu\delta}} u(x,\delta^2/2) \le N \inf_{Q^\delta} u,$$

where $N_1 = N_1(n,\nu,m)$, $\mu = \mu(m) > 0$. □

The next theorem is called a *boundary elliptic-type Harnack inequality*, because the constant N in (3.4) does not depend on the distance between $C_r(Y)$ and $\partial_x Q$. In equivalent forms, this result is contained in [FS], [FSY].

THEOREM 8. *Under the assumptions of the previous theorem, let $Y = (y,s) \in Q$ and $r > 0$ be such that $s - t_0 \ge 4\delta^2 > 0$ and $C_r(Y) \subset C_{2r}(Y) \subset Q$. Then*

$$\sup_{C_r(Y)} u \le N(n,\nu,m,\lambda) \inf_{C_r(Y)} u. \tag{3.4}$$

Proof. If $r > \delta$, from $C_{2r}(Y) \subset Q$ it follows $C_r = C_r(Y) \subset Q^\delta$, and (3.4) follows from the previous theorem. Therefore, we may restrict ourselves to the case $0 < r \le \delta$. Iterating the interior Harnack inequality, one can get the estimate

$$u(Y_R^-) \le N_0 (R/r)^\gamma \inf_{C_r} u \quad \text{for } 0 < r \le R \le \delta \tag{3.5}$$

with positive constants N_0, γ, depending only on n, ν, m. We take

$$R = \max\{\rho :\ r \le \rho \le \delta,\ \sup_{C_r} u \le (r/\rho)^\gamma \sup_{Q_\rho} u\},$$

where $Q_\rho = Q_\rho(Y) = Q \cap C_\rho(Y)$. By this choice of R and (3.5), the proof of the desired estimate (3.4) is now reduced to the following one:

$$M_R = \sup_{Q_R} u \le N u(Y_R^-). \tag{3.6}$$

For the proof of (3.6), we first consider the case $R \le \delta/K$, where $K = \mathit{const} \ge 2$. Introduce the cylinders

$$C' = B_{KR}(y) \times (s - 4R^2, s + 4R^2) \subset C_{KR}(Y), \quad Q' = Q \cap C' \subset Q_{KR}.$$

By definition of R,

$$\sup_{\partial_x Q'} u \le M_{KR} < (KR/r)^\gamma M_r = K^\gamma M_R.$$

Moreover, by Theorem 2.1, $\omega_{C'}^X(\partial_x Q') \le K^{-\gamma}/2$ on Q_R,provided $K = K(n, \nu, m)$ is large enough. Using the representation (1.6) in Q', we have

$$M_R = \sup_{Q_R} \int_{\partial_p Q'} u d\omega^X \le \sup_{\partial_x Q'} u \cdot \sup_{Q_R} \omega_{C'}^X(\partial_x Q') + \sup_{\partial_t Q'} u \le \frac{1}{2} M_R + \sup_{\partial_t Q'} u,$$

and $M_R \le 2u(Z)$ for some point $Z = (z, s - 4R^2) \in \partial_t Q'$, which lies strictly below Y_R^-. By Theorems 1.2 and 3.1, we get the estimate (3.6) in the case $R \le \delta/K$. If $\delta/K < R \le \delta$, then by the maximum principle $M_R \le u(Z)$ for some point $Z = (z, s - \delta^2)$, and since $\operatorname{diam} \Omega \le \lambda\delta < K\lambda R$, the previous argument is still valid. Thus we have (3.6) in any case, and so Theorem 3.3 is proved. □

4 ESTIMATES FOR QUOTIENTS OF SOLUTIONS

Let Ω be a bounded Lipschitz domain, and let $y \in \partial\Omega$ and $Y = (y, s)$ be fixed. We will use a local coordinate system which provides the representation (1.4) of a portion of Ω in r_0-neighborhood of $y = 0$. In this neighborhood, the distance function $d = d(x) = \operatorname{dist}\{x, \partial\Omega\}$ is equivalent to $d' = d'(x) = d'(x', x_n) = x_n - \varphi(x')$. For $r \in (0, r_0]$ and $K > 1$, we introduce the sets

$$\begin{aligned}
D_r &= \{x = (x', x_n) : |x'| < r,\ 0 < d'(x) < r\} \times (s - r^2, s + r^2),\\
S_r &= (\partial_p D_r) \cap \{d' = r\}, \quad \Gamma_r = (\partial_p D_r) \cap \{0 < d' < r\},\\
D_r^+ &= D_r \cap \{d' \ge r/K\}, \quad D_r^- = D_r \cap \{0 < d' < r/K\},\\
S_r' &= D_r \cap \{d' = r/K\}, \quad \Gamma_r' = (\partial_p D_r) \cap \{0 < d' < r/K\}.
\end{aligned}$$

For $K \gg 1$, S_r' is a "wide" portion of $\partial_p D_r'$ lying in $\{d' > 0\}$, Γ_r' is a "narrow" portion of $\partial_p D_r'$. Using Lemma 1.1 and Corollary 1.1, one can obtain the estimates

$$\inf_{D_{r/K}^+} \omega_{D_r^-}^X(S_r') \ge p_K = \frac{1}{N} K^{-\gamma}, \quad \sup_{D_{r/K}} \omega_{D_r^-}^X(\Gamma_r') \le q_K = N e^{-\beta K} \tag{4.1}$$

with some positive constants N, γ, β depending only on n, ν, m. The bounds p_K and q_K have different decay rates as $K \to \infty$, because one needs to apply the estimate (1.7) $O(\ln K)$ times in order to get the first inequality in (4.1), while the second one is obtained by application of the estimate (1.8) $O(K)$ times. We will fix $K = K(n,\nu,m) \geq 1$ large enough to guarantee the inequality $p_K \geq 2q_K$. These inequality helps to prove the following results.

Lemma 9. *Let ω^X be L-caloric measure in the domain D_{2r} for some $r \in (0, r_0/2]$. Then there exists a constant $N = N(n,\nu,m) \geq 1$, such that*

$$N^{-1}\omega^X(S_{2r}) \leq \omega^X(\Gamma_{2r}) \leq N\omega^X(S_{2r}) \quad \text{on } D_r. \tag{4.2}$$

Theorem 10. *Let $Q = \Omega \times (t_0, \infty)$ and $Y = (y,s) \in \partial_x Q$ be fixed. Let u_1 and u_2 be two positive solutions of $Lu = 0$ in Q, and $u_1 = 0$ on $\Delta_{4r}(Y) = (\partial_p Q) \cap C_{4r}(Y)$, where $0 < 4r \leq \min(r_0, \sqrt{s - t_0})$. Then*

$$\frac{u_1}{u_2} \leq N(n,\nu,m)\frac{u_1(Y_r^+)}{u_2(Y_r^-)} \quad \text{on } Q_r = Q_r(Y). \tag{4.3}$$

If also $u_2 = 0$ on $\Delta_{4r}(Y)$, we can interchange u_1 and u_2 in (4.3), and this yields a lower estimate for u_1/u_2 on Q_r. If $u_2 = 0$ on $\partial_x Q$, we can also use the elliptic-type Harnack inequality, which gives the estimate of oscillation and the Hölder continuity of u_1/u_2. For more details, see [FSY].

5 Doubling Properties

The following *doubling property* in the divergence case follows easily from Aronson's estimate (2.3). In the non-divergence case, this estimate is not valid. Our methods work for both the divergence and non-divergence cases.

Theorem 11. *Let a constant $\varepsilon \in (0, 1/2)$ be given. Then for all $r > 0$, we have*

$$\omega^X(\Delta_r) \leq N\omega^X(\Delta_{r/2}) \quad \text{on } P = \{\varepsilon|x|^2 \leq t\} \tag{5.1}$$

with a constant $N = N(n,\nu,\varepsilon)$, where $\Delta_r = B_r(0) \times \{0\} \subset \mathbb{R}^n \times \{0\}$, and ω^X is the L-caloric measure for $Q = \mathbb{R}^n \times (0,\infty)$.

Theorem 12. *Let $Q = \Omega \times (t_0,\infty)$, $Y = (y,s) \in \partial_p Q$, and constants $\varepsilon \in (0,1/2)$, $\lambda \geq 1$ be given. Then the estimate (5.1) holds for $\Delta_r = \Delta_r(Y)$ for all $r \in (0, \lambda r_0/4]$ and $X = (x,t) \in Q$ satisfying $\varepsilon|x-y|^2 \leq t - s$, $4r \leq \sqrt{t-s} \leq \lambda r_0$.*

These theorems are proved in [SY]. One of its applications is the *Fatou theorem* which states that any positive solution of $Lu = 0$ in Q has finite non-tangential limits at almost every (with respect to the L-caloric measure) point $Y \in \partial_p Q$. In the time-independent case, this result was proved in [FGS].

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