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Well-spread sequences and edge-labellings with constant Hamilton-weight

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A sequence (a_i) of integers is *well-spread* if the sums $a_i + a_j$, for $i < j$, are all different. For a fixed positive integer r, let $W_r(N)$ denote the maximum integer n for which there exists a well-spread sequence $0 \leq a_1 < \cdots < a_n \leq N$ with $a_i \equiv a_j \pmod r$ for all i, j. We give a new proof that $W_r(N) < (N/r)^{1/2} + O((N/r)^{1/4})$; our approach improves a bound of Ruzsa [*Acta. Arith.* **65** (1993), 259–283] by decreasing the implicit constant, essentially from 4 to $\sqrt{3}$. We apply this result to verify a conjecture of Jones et al. from [*Discuss. Math. Graph Theory* **23** (2003), 287–307]. The application concerns the growth-rate of the maximum label $\Lambda(n)$ in a 'most-efficient' metric, injective edge-labelling of K_n with the property that every Hamilton cycle has the same length; we prove that $2n^2 - O(n^{3/2}) < \Lambda(n) < 2n^2 + O(n^{61/40})$.

Keywords: Well-spread, weak Sidon, graph labelling, Hamilton cycle

1 Introduction

Ostensibly our purpose is to prove a conjecture from [JKMW03] concerning the growth-rate of the maximum label in a certain edge-labelling of K_n. The essential ingredient in the proof, Theorem 4, determines asymptotically the maximum 'density' of a finite, well-spread sequence of nonnegative integers. This result was first proved (explicitly) by Ruzsa [Ruz93]; our proof improves upon his bound and as such may be of independent interest.

Sets and sequences

We write $\mathbb{Z}^+$ and $\mathbb{N}$, respectively, for the sets of positive and nonnegative integers. Kotzig [Kot72] called a sequence (a_i) of integers *well-spread* if the sums $a_i + a_j$, for $i < j$, are all different; *weak Sidon* is used synonymously, e.g., in [Ruz93]. He studied finite, well-spread sequences in part due to their relationship with 'magic valuations'—now called 'edge-magic total labellings'—of graphs; see [PW99] for further details. If we strengthen the condition and require that all the sums $a_i + a_j$, for $i \leq j$, be distinct, then (a_i) is called a *Sidon sequence*. In connection with his studies in Fourier theory, Sidon [Sid32, Sid35] considered these sequences under the name B_2*-sequence*; see [HR83] for a basic reference. Every Sidon sequence is well-spread, but it is easy to construct examples to show that the converse is false: e.g., $(1,2,3)$. We shall fix a modulus $r \in \mathbb{Z}^+$ and consider *constant-residue* integral sequences (a_i), i.e., ones for which $a_i \equiv a_j \pmod{r}$ for all i, j; our application depends on the case $r = 2$, viz., the *constant-parity* sequences.

Our main number-theoretic contribution (Theorem 4) concerns the asymptotic behaviour of the following functions from $\mathbb{N}$ onto $\mathbb{Z}^+$:

$$W(N) \quad := \quad \max\{n : \text{ there is a well-spread sequence } 0 \le a_1 < \cdots < a_n \le N\};$$
$$W_r(N) \quad := \quad \max\{n : \text{ there is a constant-residue, well-spread sequence } 0 \le a_1 < \cdots < a_n \le N\}.$$

We use S, S_r, respectively, for the functions defined by replacing 'well-spread' by 'Sidon' in these definitions. Several basic inequalities follow at once:

$$S_r(N) \le W_r(N), S(N) \le W(N) \quad \text{for each } N \in \mathbb{N}. \tag{1}$$

Since the well-spread and Sidon properties are invariant under (integral) affine transformations, the maximum length of either type of sequence contained in an $(N+1)$-term arithmetic progression is the same as among an initial segment of $N+1$ nonnegative integers. Thus,

$$S_r(N) = S\left(\left\lfloor \frac{N}{r} \right\rfloor\right) \quad \text{and} \quad W_r(N) = W\left(\left\lfloor \frac{N}{r} \right\rfloor\right). \tag{2}$$

Though we need only W_2 for our graph labelling application, we shall state our number-theoretic results in terms of W_r since we prefer to display explicitly the dependence on the modulus r.

Graphs and labellings

Since we employ standard graph-theoretic notation, we refer the reader to any basic text—e.g. [Wes01]—for omitted definitions. We use $[n] := \{1, \ldots, n\}$ for the vertex set of a complete graph K_n. If A is an edge with ends i, j, then we write $A = ij$. An *edge-labelling* of K_n is a function $\lambda : E(K_n) \to \mathbb{Z}^+$. We say that λ has *constant Hamilton-weight* whenever the value of $\sum_{A \in E(H)} \lambda(A)$ is independent of the Hamilton cycle H, and is *metric* if it satisfies the triangle-inequality: $\lambda(ik) \le \lambda(ij) + \lambda(jk)$ for every triple $i, j, k \in [n]$.

Our main graph-theoretic contribution (Theorem 6) verifies a conjecture from [JKMW03] by determining the asymptotic growth-rate of the following function from $\mathbb{Z}^+$ into $\mathbb{Z}^+$:

$$\Lambda(n) := \min_{\lambda} \max_{A \in E(K_n)} \lambda(A),$$

the minimum being taken over all metric, injective edge-labellings λ of K_n having constant Hamilton-weight.

Background

Let us begin with a celebrated result of Erdős and others on the 'density' of finite Sidon sequences. Here and throughout this paper, all asymptotic assertions are contingent on the relevant parameter (N or n) tending to infinity.

Theorem 1 *$S(N) \sim N^{1/2}$; i.e.,*

$$\Big(1 - o(1)\Big)N^{1/2} < S(N) < \Big(1 + o(1)\Big)N^{1/2}. \tag{3}$$

The upper bound in (3)—in the form $N^{1/2}+O(N^{1/4})$—was proved by Erdős and Turán [ET41], who also established the lower bound $(1/\sqrt{2}-o(1))N^{1/2}$; later Chowla [Cho44a, Cho44b] and independently Erdős (1944, unpublished) applied a result of Singer [Sin38] (Theorem 5 below) to improve the lower bound to that in (3). Bose and Chowla [BC63] proved a generalization of (3) to 'B_r-sequences'; this reference also provides a more accessible discussion of Chowla's result. Eventually Lindström [Lin69] improved the upper bound to $N^{1/2}+N^{1/4}+O(1)$. It remains open—and was given a price tag by Erdős—to decide whether, for every $\varepsilon>0$, the inequality $S(N)<N^{1/2}+o(N^{\varepsilon})$ holds. See [BS85, Sós91] for further discussion and references. See [AKS81, Guy94, Ruz98, Sid32, Sid35] for a precise statement and related progress on the corresponding infinite problem.

The following theorem from [JKMW03] provides a connection between sequences and labellings; see also [KP03] and the references therein for antecedents of this result.

Theorem 2 *For $n\geq 3$, a metric, injective edge-labelling λ of K_n has constant Hamilton-weight if and only if there is a constant-parity, well-spread $\mathbb{N}$-sequence $(a_i)_{i=1}^n$ such that*

$$\lambda(ij)=\frac{a_i+a_j}{2}\ \textit{for each edge } ij \textit{ of } K_n.$$

The sequence (a_i) is uniquely determined by λ.

Theorem 2 shows that if we define $\psi_{cp}:\mathbb{Z}^+\to\mathbb{N}$ by

$$\psi_{cp}(n):=\min\{a_{n-1}+a_n:\ \text{there exists a constant-parity, well-spread } \mathbb{N}\text{-sequence } a_1<\cdots<a_n\},$$

then

$$\Lambda(n)=\frac{\psi_{cp}(n)}{2}\quad \text{for every } n\geq 3. \tag{4}$$

We note in passing that for finite Sidon sequences (a_i), similar 'sum-sets' $\{a_i+a_j\mid i\leq j\}$ have been investigated considerably; see [Ruz96] for recent results and further references. For our study of Λ, we additionally introduce the function $\sigma_{cp}:\mathbb{Z}^+\to\mathbb{N}$, defined by

$$\sigma_{cp}(n):=\min\{a_n:\ \text{there exists a constant-parity, well-spread } \mathbb{N}\text{-sequence } a_1<\cdots<a_n\}.$$

Packing with 2-sums

The definition of ψ_{cp} exhibits a 'packing flavour'; indeed, a variant of ψ_{cp} using this terminology was studied by Graham and Sloane [GS80]. They defined $v_\alpha(n)$ to be the smallest nonnegative integer N such that there exists an integral sequence $0=a_1<\cdots<a_n$ with the property that the sums a_i+a_j, for $i<j$, belong to $[0,N]$ and represent each element of this set at most once. If ψ denotes our ψ_{cp} without the constant-parity condition, then $\psi=v_\alpha$. Graham and Sloane tabulated the values $v_\alpha(n)$ for $n\leq 10$, gave exemplary sequences, and outlined a proof of

$$2n^2-O(n^{3/2})<v_\alpha(n)<2n^2+O(n^{36/23}). \tag{5}$$

They also considered the three functions that arise when $i<j$ is changed to $i\leq j$ (giving the Sidon version of v_α) or when the arithmetic is done modulo N, and the four functions resulting from changing *smallest* to *largest* and *at most* to *at least* (giving the covering analogues of the four packing functions). By now these eight functions enjoy a vast literature, much of which was cited already in [GS80].

After proving our main graph-theoretic result (Theorem 6), we shall indicate a slight improvement to the upper bound in (5). Similar improvements are possible in the bounds for the other packing functions.

2 Well-spread sequences

Theorem 1 and (2) show that $S_r(N) \sim (N/r)^{1/2}$. The functions W_r exhibit the same asymptotic behaviour, since Ruzsa [Ruz93] proved that a well-spread sequence contained in the set $\{1,\ldots,N\}$ contains at most $N^{1/2}+4N^{1/4}+11$ terms. An upper bound for $W(N)$ of the form $N^{1/2}+O(N^{1/4})$ is also implicit in the work of Graham and Sloane [GS80] and was probably known to these authors. Presently, we shall derive this result again, in terms of W_r.

To get started, we need a cruder estimate:

Lemma 3 *If N is sufficiently large, then $W_r(N) < 2.001(N/r)^{1/2}$.*

Proof. Let $n = W_r(N)$ and $0 \le a_1 < \cdots < a_n \le N$ be a well-spread sequence with each $a_i \equiv k \pmod{r}$, for some $0 \le k < r$. The sums $a_i + a_j$, for $i < j$, are distinct, at most $2N-r$, congruent modulo r to $2k$, and hence lie in the set $\{2k+r, 2k+2r, \ldots, 2k+\ell r\}$, where $\ell := \lfloor (2N-r-2k)/r \rfloor$. Thus $\binom{n}{2} \le \ell$, from which $n < (2\ell)^{1/2}+1$, and hence the assertion, follow easily. □

Theorem 4 $W_r(N) < (N/r)^{1/2} + O((N/r)^{1/4})$.

Proof. Let N be large enough to invoke Lemma 3, and set $n := W_r(N)$. Then there exists a constant-residue, well-spread sequence $0 \le a_1 < \cdots < a_n \le N$.

For $1 \le i < j \le n$, Lindström [Lin69] called $j-i$ the *order* of the difference $a_j - a_i$. He observed that the differences of order $\nu > 0$ can be arranged into sequences of the form

$$a_\alpha - a_\beta, a_\beta - a_\gamma, a_\gamma - a_\delta, \ldots,$$

where $\alpha-\beta = \beta-\gamma = \gamma-\delta = \cdots = \nu$. Because of 'telescoping', the sum of all these differences is at most νN (and less than νN for $\nu > 1$). Thus, for $m \ge 2$, the sum $\mathcal{S}$ of all the positive differences of order at most m is less than $m(m+1)N/2$.

Let us call a_i a *mean-point* if $2a_i = a_j + a_k$ for some $j,k \in [n]$; notice that in this case $a_i - a_k = a_j - a_i$. Except for the values $a_j - a_i$, for mean points a_i (or a_j), the differences $a_k - a_\ell$, for $1 \le \ell < k \le n$, are all different since (a_i) is well-spread. As the only candidates for mean-points are $a_2, \ldots, a_{n-1}$, we have at most $t := n-2$ differences occurring with higher multiplicity, and the well-spread property implies that this multiplicity is 2. Since (a_i) has constant-residue, the differences are all multiples of r. If $1 \le m < n$ and $s := n - (m+1)/2$, then the number of positive differences of order at most m is $mn - m(m+1)/2 = ms$. Therefore,

$$\mathcal{S} \ge \sum_{i=1}^{t}(ri+ri) + \sum_{j=1}^{ms-2t}(rt+rj) = \frac{rms(ms+1)}{2} - rt(ms-t).$$

For $1 < m < n$, it follows that

$$\frac{rms(ms+1)}{2} - rt(ms-t) < \frac{m(m+1)}{2}N,$$

so that

$$\frac{r(ms)^2}{2} < \frac{m(m+1)}{2}N + rmst.$$

Since $s,t < n$, the second term on the right side is less than rmn^2, which by Lemma 3 is at most $(2.001)^2 mN < 4.5mN$. Thus, $s^2 < N(1+10/m)/r$, and since $(1+x)^{1/2} < 1+x/2$ for $x = 10/m$, we have

$$n = \frac{m+1}{2} + s < \frac{m+1}{2} + \left(\frac{N}{r}\right)^{1/2}\left(1+\frac{5}{m}\right). \tag{6}$$

With $m := \lceil (N/r)^{1/4} \rceil$, this gives the bound in the statement of the theorem. □

Remarks Our proof of Theorem 4 adapts the main idea of Lindström [Lin69] to well-spread, constant-residue sequences. Ruzsa [Ruz93] also based his proof on the idea of studying the 'small' differences $a_j - a_i$, though in a "somewhat hidden" fashion (quote from [Ruz93]). Here we compare the resulting implicit constants.

To optimize ours, we iterate the proof once again. Instead of applying Lemma 3 (to bound rmn^2 from above), we apply Theorem 4 itself. This allows us to replace '10' by '$3 + O((N/r)^{-1/4})$'. To minimize the right side of (the adjusted) inequality (6), we now choose m to be $\lceil \sqrt{3}(N/r)^{1/4} \rceil$. These modifications replace the big-oh term in Theorem 4 by $\sqrt{3}(N/r)^{1/4} + O(1)$. Ruzsa's proof essentially produces the value 4 in place of our $\sqrt{3}$.

While we're comparing bounds, we should mention that the upper bound for $S_r(N)$ implied by (1) and Theorem 4 does *not* improve on earlier results. For example, Lindström's bound [Lin69] together with (2) gives the implied constant 1 in place of our $\sqrt{3}$ □

3 Edge-labellings with constant Hamilton-weight

We turn to verifying the main conjecture from [JKMW03]. Proofs of the following basic connections are left to the reader (or see [JKMW03]):

$$W_2(N) \geq \sigma_{cp}^{-1}(N) \quad \text{for every } N \in \text{range}(\sigma_{cp}); \tag{7}$$

$$\psi_{cp}(n) \geq \sigma_{cp}(n) + \sigma_{cp}(n-1) \quad \text{for every } n \geq 2. \tag{8}$$

We also need a simple upper bound on $\sigma_{cp}(n)$, a theorem on the density of primes, and Singer's theorem on difference sets. The first of these follows immediately from our work in [JKMW03]:

$$\sigma_{cp}(n) < 2n^2(1+o(1)). \tag{9}$$

For the second, we opt for the present state-of-the-art, due to Baker et al. [BHP01]: if x is sufficiently large, then there is a prime p with

$$x < p \leq x + x^{21/40}. \tag{10}$$

For the third, we have

Theorem 5 ([Sin38]) *If q is a prime power, then there are integers $b_0, b_1, \ldots, b_q \in [0, q^2+q]$ such that the differences $b_i - b_j$, for $i \neq j$, are congruent, modulo q^2+q+1, to the integers $1, 2, \ldots, q^2+q$. In particular, $(b_i)_{i=0}^{q}$ forms a Sidon sequence, hence is well-spread.*

Finally, we state and prove our main graph-theoretic result:

Theorem 6 $\Lambda(n) \sim 2n^2$*; more precisely,*

$$2n^2 - O(n^{3/2}) < \Lambda(n) < 2n^2 + O(n^{61/40}). \tag{11}$$

Proof. For the upper bound, consider an integer n, large enough to apply (10) with $x = n-1$; then we can find a prime p so that

$$n-1 < p < n+n^{21/40}.$$

Theorem 5 delivers a well-spread sequence $0 \le b_0 < b_1 < \cdots < b_p \le p^2+p$. Now $a_i := 2b_{i-1}$, for $i = 1,2,\ldots,n$, defines a constant-parity, well-spread sequence with

$$a_{n-1}+a_n = 2(b_{n-2}+b_{n-1}) \le 4p^2+4p-6 < 4n^2+O(n^{61/40}).$$

By definition, $\psi_{cp}(n) \le a_{n-1}+a_n$, and since $\Lambda(n) = \psi_{cp}(n)/2$ (see (4)), the upper bound in (11) follows.

For the lower bound, let $n \in \mathbb{N}$ and $N = \sigma_{cp}(n)$. Then (7) and Theorem 4 imply that

$$n = \sigma_{cp}^{-1}(N) \le W_2(N) < \left(\frac{N}{2}\right)^{1/2} + O\left(\left(\frac{N}{2}\right)^{1/4}\right),$$

so that

$$2n^2 < N + O(N^{3/4}).$$

Now (9) shows that

$$\sigma_{cp}(n) = N > 2n^2 - O(n^{3/2}).$$

Thus (8) gives $\psi_{cp}(n) > 4n^2 - O(n^{3/2})$, and again applying (4) yields the desired bound. $\square$

Closing remarks

We first elaborate on the lower bound in (3). The idea in the proof of the upper bound in Theorem 6 can be used to show that $S_r(N) > (N/r)^{1/2}$ for infinitely many integers N and that $S_r(N) > (N/r)^{1/2} - (N/r)^{21/80}$ for sufficiently large N. Absent the modulus r, these observations have been made elsewhere; cf. [PS95]. The slight improvement here over previously published bounds—e.g., in [PS95], the fraction $5/16$ replaces $21/80$—results from our use of a more recent prime density theorem.

Baker and Harman [BH96] sketch the history of such theorems, i.e., those of the form

$$[x, x+x^{\vartheta}] \text{ contains a prime whenever } x \text{ is sufficiently large}$$

for a specified constant ϑ; cf. (10).

An alternate approach to Theorem 6 is to reduce the problem to one considered in [GS80]. It is not difficult to see that $\psi_{cp}(n)$ is achieved when $a_1 = 0$, so that the constant parity is even. Then $\Lambda(n)$ can be identified with Graham and Sloane's $v_\alpha(n)$, so that (5) also gives $\Lambda(n) \sim 2n^2$.

Turning this observation around shows that our (11) improves (5). This stems from the decrease in the minimum ϑ since [GS80] appeared. The present value $\vartheta = 21/40$ (cf. $13/23$ available to Graham and Sloane) improves not only (5), but also the upper bounds for the other three packing functions considered in [GS80].

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