2005 International Conference on Analysis of Algorithms DMTCS proc. **AD**, 2005, 409–416

Analytic combinatorics for a certain well-ordered class of iterated exponential terms

Andreas Weiermann[†]

Mathematical Institute
P.O. Box 80010
3508 TA Utrecht
The Netherlands
e-mail: weierman@math.uu.nl

The aim of this paper is threefold: firstly, to explain a certain segment of ordinals in terms which are familiar to the analytic combinatorics community, secondly to state a great many of associated problems on resulting count functions and thirdly, to provide some weak asymptotic for the resulting count functions. We employ for simplicity Tauberian methods. The analytic combinatorics community is encouraged to provide (maybe in joint work) sharper results in future investigations.

Keywords: analytic combinatorics, Tauberian theorems orders of infinity, slow varying functions, ordinals

1 Introduction

It is usually difficult to attract the attention of mathematicians without background in logic to questions about ordinals. We hope to change this situation a bit by explaining a certain quite far reaching initial segment of these in terms of Hardy's 1910 [6] orders of infinity.

Ordinals reflect the process of counting, thus they start like $0, 1, 2, 3, \dots, n, \dots$. Then the first limit element ω appears and counting continues with $\omega, \omega+1, \omega+2, \omega+3, \dots, \omega+n, \dots$, and the longer the process lasts the more obscure the ordinals become.

Let us now switch the scene to the following subclass $\mathcal{E}$ of Hardy's order of infinity. Let $\mathcal{E}$ be the least set of functions $f : \mathbb{N} \to \mathbb{N}$ such that the constant zero function $x \mapsto 0$ is contained in $\mathcal{E}$ and such that with f and g also the function $x \mapsto x^{f(x)} + g(x)$ belongs to $\mathcal{E}$.

Define $f \prec g$ via eventual domination, i.e. $f \prec g$ holds if there exists an n_0 such that for all $n \geq n_0$ we have $f(n) < g(n)$. Let k_d denote the constant function with value n and then notice $k_0 \prec k_1 \prec k_2 \prec \dots \prec k_n \prec \dots$. The first limit element with respect to $\prec$ is then obviously given by the identity function id, i.e. $x \mapsto x$. Moreover $id \prec id + k_1 \prec id + k_2 \prec \dots \prec id + id \prec id \cdot id \prec id^{id} \prec id^{id^{id}} \dots$. As long as we stay within $\mathcal{E}$ all mysteriosity of the counting into the infinite disappears and we can consider the initial segment of ordinals provided by $\mathcal{E}$ as a natural mathematical structure for which no background in logic is necessary. To understand how $\mathcal{E}$ works one may verify that every polynomial function with non negative integer coefficients represents a function in $\mathcal{E}$. (Note that e.g. $k_1 = id^{k_0} + k_0$.)

Hardy proved already in 1910 that $\mathcal{E}$ is linearly ordered with respect to $\prec$, hence every non zero function f in $\mathcal{E}$ has a unique 'term' representation $f = id^{f_1} + \dots + id^{f_m}$ where $f_1 \succeq \dots \succeq f_m$. If further the non zero function g has a corresponding representation $g = id^{g_1} + \dots + id^{g_n}$ where $g_1 \succeq \dots \succeq g_n$ then we can decide $f \prec g$ using the corresponding exponents as follows; $f \prec g$ iff either $m < n$ and for all $i \leq m$ we have $f_i = g_i$ or there exists an $k \leq \min\{m, n\}$ such that $f_k \prec g_k$ and for all $l < k$ we have $f_l = g_l$.

Usually it is assumed that proving the well-foundedness of $\mathcal{E}$ with respect to $\prec$ is difficult to see. As a sidestep let us show how to resolve this. We show that every nonempty subset of $\mathcal{E}$ has a $\prec$-minimal element, or equivalently, there does not exist a strictly descending chain of elements in $\mathcal{E}$, or equivalently for every function $F : \mathbb{N} \to \mathcal{E}$ there exists an n such that $F(n) \preceq F(n+1)$. A non logical argument uses

[†]Research supported by a Heisenberg-Fellowship of the Deutsche Forschungsgemeinschaft.

either an appeal to Kruskal's tree theorem or a compactness argument which is familiar from the proof of the Bolzano Weierstraß theorem.

Indeed, let $id_1 := id$ and id_{n+1} be defined recursively as $x \mapsto x^{id_n(x)}$. Let $\mathcal{E}_n := \{f \in \mathcal{E} : f \prec id_n\}$. Then $\mathcal{E} = \bigcup_{n\in\mathbb{N}} \mathcal{E}_n$. We show by induction on m that for every function $F : \mathbb{N} \to \mathcal{E}_m$ there exists an n such that $F(n) \preceq F(n+1)$. Indeed this is clear for $m = 1$ since the natural numbers are well ordered. Now assume that there exists an infinitely descending chain $F(0) \succ F(1) \succ F(2) \succ \dots$ etc. in $\mathcal{E}_{m+1}$ Then the corresponding lists of exponents of $F(0), F(1), F(2)$ can be arranged in an infinite but finitely branching tree such that along any branch we obtain a strict descent of corresponding exponent functions. By compactness we would obtain an infinite strictly descending chain of functions appearing as exponents but this would give a strictly descending chain in $\mathcal{E}_m$ which is excluded by induction hypothesis.

Thus we can use the structure $\mathcal{E}$ for all sorts of transfinite recursion but in this paper we will not continue in exploring this further.

2 Some basic results

The elements of $\mathcal{E}$ come along with various natural norm functions. The most canonical choice is given as follows: $N(c_0) := 0$ and $N(id^f + g) := 1 + N(f) + N(g)$. (This is well defined as a moment reflection shows.) Then for every f in $\mathcal{E}$ and every natural number n there are finitely many $g \prec f$ such that $N(g) \le n$. We may thus consider

$$c_f(n) := \#\{g \prec f : N(g) = n\}.$$

For specific choices of f one re-obtains classical count functions, e.g. if $f = id^{id}$ then $c_f(n)$ is the number of partitions of n which has a well known and intriguing asymptotic.

For a proof of this correspondence simply observe that every function $id^{k_{i_1}} + \cdots id^{k_{i_m}} \prec id^{id}$ with $k_{i_1} \succeq \dots \succeq k_{i_m}$ corresponds uniquely to the partition $\langle i_1 - 1, \dots i_m - 1\rangle$.

The author has learned that there has been recently a lot of progress in classifying c_f (in the context of Lie algebras) and thus we will not pursue this issue further. We just quote (besides the standard ones) the results of Petrogradsky [9]. Let $id_0(f) := f$ and $id_{m+1}(f) := id^{id_m(f)}$. Moreover let $\ln^{(0)}(n) := n$ and $\ln^{(m+1)}(n) := \ln(\ln^{(m)}(n))$.

Theorem 1 *1.* $c_{id^{c_d}}(n) \sim \frac{1}{d!(d-1)!} n^{d-1}$.

2. $c_{id^{id}}(n) \sim \frac{exp(\pi\cdot\sqrt{\frac{2}{3}n})}{4\sqrt{3}n}$.

3. Let $\sigma := (1+\frac{1}{d})\big(\frac{1}{(d-1)!}\zeta(d+1)\big)^{\frac{1}{d+1}}$. *Then* $\ln(c_{id^{id^d}}(n)) \sim \sigma \cdot n^{\frac{d}{d+1}}$.

4. There is an explicitly calculable constant C *such that* $\ln(c_{id_{m+2}(k_d)}(n)) \sim C\frac{n}{\sqrt[d]{\ln^{(m)}(n)}}$.

Moreover it is known from [10] that $\lim \frac{c_f(n+1)}{c_f(n)} = 1$ for all $f \in \mathcal{E}$.

There is a multiplicative norm which is canonically associated with N. It is inferred by the indices of the enumeration function for the primes $(p_i)_{i\ge1}$. Let $I(k_0) := 1$ and $I(id^f + g) := p_{I(f)} \cdot I(g)$. The corresponding count function is

$$c^I_f(n) := \#\{g \prec f : I(g) \le n\}.$$

This norm is natural in a far as it provides a bijection between $\mathcal{E}$ and the positive integers using the theorem on unique prime factor decomposition for positive integers. (The commutativity of addition is reflected by the commutativity of multiplication.) By elementary calculations with Dirichlet functions following the advice provided in Burris one can prove the following Theorem.

Theorem 2 *1. There exists an explicitly calculable constant* C *such that* $c^I_{id^{c_d}}(n) \sim C(\ln(n))^d$.

2. $\ln(c^I_{id^{id}}(n)) \sim \pi \cdot \sqrt{\frac{2}{3\ln(2)}\ln(n)}$.

3. $\ln(c^I_{id^{id^{k_d}}}(n)) = \Theta((\ln(n))^{\frac{d}{d+1}})$.

4. $\ln(c^I_{id_{m+2}(k_d)}(n) = \Theta(\frac{\ln(n)}{\sqrt[d]{\ln^{(m)}(\ln(n))}})$.

Moreover it is known that c^I_f is slowly varying at infinity for each $f \in \mathcal{E}$. We conjecture that the Θ results can be sharpened to weak asymptotic similarly to Theorem 1 using $\sigma := (1+\frac{1}{d})\big(\frac{1}{(d-1)!\ln(2)}\zeta(d+1)\big)^{\frac{1}{d+1}}$.

3 Exponential norms

The main emphasis of this paper is put on a norm which arises naturally in the context of logic, in almost every book on recursion theory. This exponential coding norm E is defined by $E(k_0) := 1$ and $E(id^{f_1} + \cdots + id^{f_n}) := p_1^{E(f_1)} \cdots p_n^{E(f_n)}$ if $f_1 \succeq \ldots \succeq f_n$. Let

$$c_f^E(n) := \#\{g \prec f : E(g) \leq n\}.$$

An additive version of the exponential coding norm which leads to generalized Mahler partitions is as follows. Let the Mahler norm be defined by $M(0) := 0$ and $M(id^f + g) := 2^{M(f)} + M(g)$. Moreover let

$$c_f^M(n) := \#\{g \prec f : M(g) \leq n\}.$$

Note that for $f = id^{id}$ the number $c_f^M(n)$ is the number of sequences $\langle i_0, \ldots i_l \rangle$ such that $i_0 \geq \ldots \geq i_l$ and $2^{i_0} + \cdots + 2^{i_l} \leq n$, hence a version of the Mahler partition function. For other values of f one gets suitably generalized Mahler partitions. In particular we obtain the following standard partition identity for M which can be used to obtain the asymptotic for the resulting count functions: $\sum_{n=0}^{\infty} c_{id^f}^M(n) \cdot z^n = \prod_{i=1}^{\infty} \frac{1}{(1-z^{2^i})^{c_f^M(i)}}$. The treatment of weak asymptotic for c_f^M is very analogous to c_f^E and we therefore stick to the functions c_f^E from now on. (For better results the techniques of Dumas and Flajolet [5] seem appropriate here.)

Following Hardy and Ramanujan let $l_i := p_1 \cdot \ldots \cdot p_i$ where $l_0 := 1$. As a warm up exercise we indicate how the asymptotic for $c_{id^{c_d}}^E$ can be obtained by Karamata's Tauberian theorem which seems to be tailer made for asymptotic on bounded partitions. (The proof is very similar to one found in [11].)

Lemma 1 *Let $hr_d(x) := c_{id^{c_d}}^E(x)$. Then*

$$hr_d(x) = \sum_{e=1}^{d} \sum_{j_1 < \ldots j_e < d} \sum_{i_1 > \ldots > i_e} \#\{l_{i_1}^{l_{j_1}} \cdot l_{i_2}^{l_{j_2} - l_{j_1}} \cdot \ldots \cdot l_{i_e}^{l_{j_e} - l_{j_{e-1}}} \leq x\}.$$

Proof. It suffices to show

$$\#\{f \prec id^d : E(f) \leq x\} = \# \bigcup_{e=1}^{d} \bigcup_{j_1 < \ldots j_e < d} \bigcup_{i_1 > \ldots > i_e} \{l_{i_1}^{l_{j_1}} \cdot l_{i_2}^{l_{j_2} - l_{j_1}} \cdot \ldots \cdot l_{i_e}^{l_{j_e} - l_{j_{e-1}}} \leq x\}.$$

This is more or less obvious by grouping the factors appropriately together. (In some sense this is similar when one counts partitions and their conjugates. In terms of block diagrams this simply means that we are counting blocks at one time via columns and at the other time via rows.) □

Let $L(s) := \sum_{n=1}^{\infty} l_n^{-s}$.

Theorem 3 (Hardy and Ramanujan [7]) $L(s) \sim \frac{1}{s \ln(\frac{1}{s})}$ *for* $s \to 0^+$.

Recall that a (measurable) function $f : \mathbb{R} \to [0, \infty[$ is called slowly varying if $\lim_{t \to \infty} \frac{f(tx)}{f(t)} = 1$ for $x > 0$.

Theorem 4 (Karamata's Tauberian Theorem [2]) *Let U be a non decreasing right continuous function on the real numbers with $U(x) = 0$ for all $x < 0$. Let $LU(s) = \int_0^\infty exp(-sx)dU(x)$. If $f : \mathbb{R} \to [0, \infty[$ varies slowly and $c \geq 0$, $\rho \geq 0$ the following are equivalent*

1. $U(x) \sim \frac{cx^\rho f(x)}{\Gamma(1+\rho)}$ *for* $x \to \infty$,
2. $LU(s) \sim cs^{-\rho} f(\frac{1}{s})$ *as* $s \to 0^+$.

As a nice application we obtain the following result.

Theorem 5 $hr_d(x) \sim \frac{1}{(d!)^2 \prod_{e=1}^{d-1}(p_e - 1) \prod_{e=1}^{d-2} l_e} (\frac{\ln(x)}{\ln(\ln(x))})^d$ *for* $x \to \infty$.

Proof. Define natural numbers a_n by the equation

$$\sum_{n=1}^{\infty} a_n n^{-s} = \sum_{e=1}^{d} \sum_{j_1 < \ldots j_e < d} \sum_{i_1 > \ldots > i_e} (l_{i_1}^{l_{j_1}} \cdot l_{i_2}^{l_{j_2} - l_{j_1}} \cdot \ldots \cdot l_{i_e}^{l_{j_e} - l_{j_{e-1}}})^{-s}.$$

Then $\sum_{n\leq x} a_n = hr_d(x)$. Let $U(x) = \sum_{\ln(n)\leq x} a_n$. Then, as $s \to 0^+$,

$$\begin{aligned}
& \frac{1}{d!\prod_{e=1}^{d-1}(p_e-1)\prod_{e=1}^{d-2} l_e}\left(\frac{1}{s\ln(\frac{1}{s})}\right)^d \\
\sim\ & \sum_{e=1}^{d}\sum_{j_1<\ldots j_e<d}\frac{1}{e!}\frac{1}{l_{j_1}s\ln(l_{j_1}s)}\cdot\ldots\cdot\frac{1}{(l_{j_e}-l_{j_{e-1}})s\ln((l_{j_e}-l_{j_{e-1}})s)} \\
\sim\ & \sum_{e=1}^{d}\sum_{j_1<\ldots j_e<d}\frac{1}{e!}\sum(l_{i_1}^{l_{j_1}})^{-s}\cdot\ldots\cdot(l_{i_e}^{l_{j_e}-l_{j_{e-1}}})^{-s} \\
\sim\ & \sum_{e=1}^{d}\sum_{j_1<\ldots j_e<d}\sum_{i_1>\ldots>i_e}((l_{i_1}^{l_{j_1}})\cdot\ldots\cdot l_{i_e}^{l_{j_e}-l_{j_{e-1}}})^{-s}(l_{i_1}^{l_{j_1}})^{-s}\cdot\ldots\cdot(l_{i_e}^{l_{j_e}-l_{j_{e-1}}})^{-s} \\
=\ & \sum_{n=1}^{\infty} a_n n^{-s} \\
=\ & \int_0^{\infty} exp(-sx)dU(x) = LU(s).
\end{aligned}$$

The function $s \mapsto \frac{1}{(\ln(\frac{1}{s}))^d}$ is slowly varying. Theorem 4 yields

$$U(x) \sim \frac{1}{(d!)^2\prod_{e=1}^{d-1}(p_e-1)\prod_{e=1}^{d-2} l_e}\left(\frac{x}{\ln(x)}\right)^d$$

for $x \to \infty$. Now $\sum_{n\leq x} a_n = U(\ln(x))$ and the result follows. □

Now we consider count functions for functions id^f where f growth at least linearly. It turns out that tailor made Tauberian theorems are provided by Parameswaran [8].

Theorem 6 (de Bruijn [3]) *If M is slowly varying, then there is a (asymptotically uniquely determined) slowly varying function M^* such that $M^*(x\cdot M(x))\cdot M(x) \to 1$ as $x \to \infty$ and $M(x\cdot M^*(x))\cdot M^*(x) \to 1$ as $x \to \infty$.*

Theorem 7 (Parameswaran [8]) *Suppose that the following conditions hold.*

1. *$L(u)$ and $P(u)$ are functions on the non negative reals such that $\int_0^R L(u)du$ and $\int_0^R P(u)du$ exist in the Lebesgue sense for every positive R.*
2. *$\exp(s\int_0^\infty \frac{e^{-su}}{1-e^{-su}}L(u)du) = s\int_0^\infty P(u)e^{-su}du$ for all positive s,*
3. *$\langle M, M^*\rangle$ form a pair of conjugate slowly varying functions,*
4. *M is non decreasing,*
5. *$\int_0^u \frac{L(t)}{t}dt \sim M(u)$ as $u \to \infty$, and*
6. *$P(u)$ is non decreasing.*

Then $\log P(u) \sim \frac{1}{M^*(u)}$ as $u \to \infty$.

Theorem 8 *We have $c^E_{id^{id}}(n) = \#\{(l_{i_1},\ldots,l_{i_m}) : i_1 \geq \ldots \geq i_m \ \&\ p_1^{l_{i_1}}\cdot\ldots p_m^{l_{i_m}} \leq n\}$. Moreover, $\ln(c^E_{id^{id}}(n)) = \Theta(\frac{(\ln(\ln(n)))^2}{\ln(\ln(\ln(n)))})$.*

Proof. We have

$$\begin{aligned}
& c^E_{id^{id}}(n) \\
\leq\ & \#\{(l_{i_1},\ldots,l_{i_m}) : i_1 \geq \ldots \geq i_m \ \&\ 2^{l_{i_1}}\cdot\ldots\cdot 2^{l_{i_m}} \leq n\} \\
=\ & \#\{(l_{i_1},\ldots,l_{i_m}) : i_1 \geq \ldots \geq i_m \ \&\ l_{i_1}+\ldots+l_{i_m} \leq \frac{\ln(n)}{\ln(2)}\}.
\end{aligned}$$

Let

$$Q(n) = \#\{(l_{i_1},\ldots,l_{i_m}) : i_1 \geq \ldots \geq i_m \ \&\ l_{i_1}+\ldots+l_{i_m} \leq n\}$$

and

$$q(n) = \#\{l_i : l_i \le n\}.$$

Then $q(n) \sim \frac{\ln(n)}{\ln(\ln(n))}$. This follows from $l_i = \exp(\vartheta(p_i))$ and the well know facts (resulting from the prime number theorem) that $\vartheta(x) \sim x$ and $p_i \sim i\ln(i)$. Parameswaran's theorem now yields $\ln(Q(n)) \sim \frac{1}{2}\frac{\ln(n)^2}{\ln^{(2)}(n)}$. This yields $\ln(p(n) \le \frac{1}{2}\frac{\ln(\ln(n))^2}{\ln^{(3)}(n)}$. Moreover the prime number theorem gives a K such that $p_i \le Ki\ln(i)$ for all i. Hence $c^E_{id^{id}}(n) \ge \#\{(l_{i_1},\ldots,l_{i_m}) : i_1 \ge \ldots \ge i_m \ \& \ (Km\ln(m)^{l_{i_1}} \cdot \ldots \cdot (Km\ln(m)^{l_{i_m}})\}$. We claim that $p(n) \ge Q(\sqrt{\ln(n)})$ for large n. Indeed $(l_{i_1}+\ldots+l_{i_m})^2 \le \ln(n($ yields $(l_{i_1}+\ldots+l_{i_m})\cdot\ln(m\ln(m)K) \le \ln(n)$ for sufficiently large m and then $p_1^{l_{i_1}}\cdot\ldots\cdot p_m^{l_{i_m}} \le n$. Therefore $\ln(c^E_{id^{id}}(n)) \ge \ln(Q(n)) \sim \frac{1}{2}\frac{(\ln(\sqrt{\ln(n)}))^2}{\ln^{(2)}(\ln(n))}$. □

Recall that $\ln^{(m)}$ denotes the m-th iteration of the ln-function.

Theorem 9 *Let* $o_d(n) := c^E_{id^{id^{c_d}}}(n)$. *Then* $\ln(o_d(n)) = \Theta(\ln^{(2)}(n)(\frac{\ln^{(2)}(n)}{\ln^{(3)}(n)})^{d+1})$.

Proof. We have $o_d(n)$
$\le \#\{(Ef_1,\ldots,Ef_m) : c_d \succ f_1 \succeq \ldots \succeq f_m \ \& \ 2^{Ef_1}\cdot\ldots\cdot 2^{Ef_m} \le n\}$
$= \#\{(Ef_1,\ldots,Ef_m) : c_d \succ f_1 \succeq \ldots \succeq f_m \ \& \ Ef_1+\ldots+Ef_m \le \frac{n}{\ln(2)}\}$. Now

$$\#\{(Ef_1,\ldots,Ef_m) : c_d \succ f_1 \succeq \ldots \succeq f_m \ \& \ Ef_1+\ldots+Ef_m \le n\} \sim C(\frac{\ln(n)}{\ln(\ln(n))})^{d+1}.$$

Thus Parameswaran [8] yields $\ln(o_d(n)) \le C\ln^{(2)}(n)\cdot(\frac{\ln^{(2)}(n)}{\ln^{(3)}(n)})^{d+1}$. Similarly as in the proof of Theorem 4 we see that $o_d(n) \ge C\cdot\ln(\sqrt{\ln(n)})\cdot(\frac{\ln(\sqrt{\ln(n)}}{\ln^{(3)}(n)})^{d+1}$.

□

Recall that $id_0(f) := f$ and $id_{m+1}(f) := id^{id_m(f)}$. Moreover let $id_m := id_m(k_1)$.

Theorem 10 *Let* $c(n) := c^E_{id_{m+1}}(n)$. *Then* $\ln^{(m)}(c(n)) = \Theta(\frac{(\ln^{(m+1)}(n))^2}{\ln^{(m+2)}(n)})$.

Proof. By induction on m. Theorem 9 covers the case $m = 1$. Assume $m \ge 2$ and

$$\ln^{(m-1)}(\#\{g \in E : g \prec id_m \ \& \ Eg \le n\}) \sim \Theta(\frac{(\ln^{(m)}(n))^2}{\ln^{(m+1)}(n)}).$$

Then

$$\ln^{(m-1)}(\#\{g \in E : g \prec id_m \ \& \ \ln(2^{Eg}) \le n\}) \sim \Theta(\frac{(\ln^{(m)}(n))^2}{\ln^{(m+1)}(n)}).$$

By thinning out we can find a subset $S \subset E$ such that

$$\#\{g \in S : g \prec id_m \ \& \ \ln(2^{Eg}) \le n\} = \exp_{m-1}(C\cdot(\frac{(\ln^{(m)}(n))^2}{\ln^{(m+1)}(n)}))$$

for a suitable constant C. Let $L(u) = \exp_{m-1}(C\cdot(\frac{(\ln^{(m)}(u))^2}{\ln^{(m+1)}(u)}))$. Let $M(u) = \int_a^u \frac{L(u)}{u}du$. Then

$$M(u) \sim L(u)\cdot\frac{d}{du}(\exp_{m-1}(C\cdot(\frac{(\ln^{(m)}(u))^2}{\ln^{(m+1)}(u)})))$$

and $\frac{1}{M^*(u)} \sim M(u)$. Thus

$$\ln(\#\{\langle g_1,\ldots,g_m\rangle : g \in S \ \& \ 2^{Eg_1}\cdot\ldots\cdot 2^{Eg_1} \le n\}) \sim M(\ln(n))$$

and

$$\ln(\#\{\langle g_1,\ldots,g_k\rangle : g \in E \ \& \ g_1,\ldots,g_k \prec id_m \ \& \ p_1^{Eg_1}\cdot\ldots\cdot p_k^{Eg_k} \le \ln(n)\}) = \exp_{m-1}(\mathcal{O}(\frac{(\ln^{(m+1)}(n))^2}{\ln^{(m+2)}(n)})).$$

The lower bound is obtained similarly. Indeed, we have

$$\begin{aligned}
& \#\{\langle g_1,\ldots,g_k\rangle : id_m \succ g_1 \succeq \ldots \succeq g_k \;\&\; p_1^{Eg_1}\cdot\ldots\cdot p_k^{Eg_k} \le n\} \\
\ge\; & \#\{\langle g_1,\ldots,g_k\rangle : id_m \succ g_1 \succeq \ldots \succeq g_k \;\&\; (Kk\ln(k))^{Eg_1+\ldots+Eg_k} \le n\} \\
=\; & \#\{\langle g_1,\ldots,g_k\rangle : id_m \succ g_1 \succeq \ldots \succeq g_k \;\&\; Eg_1+\ldots+Eg_k \le \sqrt{\ln(n)}\} \\
=\; & \#\{\langle g_1,\ldots,g_m\rangle : id_m \succ g_1 \succeq \ldots \succeq g_k \;\&\; \ln(2^{Eg_1+\ldots+Eg_k}) \le \sqrt{\ln(n)}\} \\
\ge\; & \exp_{m-1}(C\cdot\frac{(\ln^{(m+1)}(n))^2}{\ln^{(m+2)}(n)})
\end{aligned}$$

since $m \ge 2$. □

The same proof yields the following refinement.

Theorem 11 *Then* $\ln^{(m)}(c^E_{id_{m+1}(c_d)}(n)) = \Theta(\ln^{(m+2)}(n)(\frac{(\ln^{(m+1)}(n)}{\ln^{(m+2)}(n)})^{d+1})$.

Investigations on count functions have applications in logic. Let us state one application to the phase transition for the Ackermann function. Let F be a number-theoretic function and let $count_f^E(F)(m)$ be the maximal possible number of $g_1,\ldots,g_k \in \mathcal{E}$ such that $f \succ g_1 \succ \ldots \succ g_k$ and $(\forall i \le k)[E(g_i) \le m + F(i)]$. This is well defined by a compactness argument for every function F. Then for $f = id_{m+2}$, d fixed, and functions F with $F(i) \ge 2\sqrt[d]{\ln^{(m)}(i)}$ for i large enough the function $count_f^E(F)$ will eventually dominate every primitive recursive function. But for $f = id_{m+2}$ and functions F with $F(i) \le \ln^{(m)}(i)$ (for i large enough) the function $count_f^E(F)$ will be bounded by a double exponential function.

We close with some conjectures.

Conjecture 1 *1.* $\ln^{(m)}(c^M_{id_{m+1}}(n)) = \Theta(\ln^{(m)}(n)^2)$.

2. c_f^E *is slowly varying for each* f *in* $\mathcal{E}$.

3. $n \mapsto \ln(c_f^M(n))$ *is slowly varying for each* f *in* $\mathcal{E}$.

Following Burris's philosophy on logical limit laws we conjecture that for the norm functions E and M there will be associated zero one laws for first order logic.

References

[1] T.M. Apostol: *Introduction to Analytic Number Theory*. Springer, 1976.

[2] N.H. Bingham, C.M. Goldie and J.L. Teugels: *Regular variation.* Encyclopedia of Mathematics and its applications 27. Cambridge University Press, 1989.

[3] N.G. de Bruijn: *Pairs of slowly oscillating functions occurring in asymptotic problems concerning the Laplace transform*, Nieuw Arch. Wiskunde 7 (19659), 20-26.

[4] S.N. Burris: *Number theoretic density and logical limit laws.* Mathematical Surveys and Monographs, 86. American Mathematical Society, Providence, RI, 2001. xx+289 pp.

[5] P. Dumas, P. Flajolet, Philippe: *Asymptotique des recurrences mahleriennes: le cas cyclotomique.* J. Theor. Nombres Bordeaux 8 (1996), no. 1, 1–30.

[6] G.H. Hardy: *Orders of infinity. The Infinitärcalcül of Paul du Bois-Reymond.* Reprint of the 1910 edition. Cambridge Tracts in Mathematics and Mathematical Physics, No. 12. Hafner Publishing Co., New York, 1971. v+62 pp.

[7] G.H. Hardy and S. Ramanujan. *Asymptotic formulae for the distribution of integers of various types.* Proceedings of the London Mathematical Society 1917.

[8] S. Parameswaran. *Partition functions whose logarithms are slowly oscillating.* Transactions of the American Mathematical Society 100 (1961), 217-240.

[9] V.M. Petrogradsky: *Growth of finitely generated polynilpotent Lie algebras and groups, generalized partitions, and functions analytic in the unit circle.* Internat. J. Algebra Comput. 9 (1999), no. 2, 179–212.

[10] A. Weiermann: *Zero-one law characterizations of* ϵ_0. Mathematics and computer science, II (Versailles, 2002), 527–539, Trends Math., Birkhäuser, Basel, 2002.

[11] A. Weiermann: *An application of results by Hardy, Ramanujan and Karamata to Ackermannian functions.* Discrete Math. Theor. Comput. Sci. 6 (2003), no. 1, 133–141 (electronic).