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Lyndon factorization of generalized words of Thue

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The i-th symbol of the well-known infinite word of Thue on the alphabet $\{0,1\}$ can be characterized as the parity of the number of occurrences of the digit 1 in the binary representation of i. Generalized words of Thue are based on counting the parity of occurrences of an arbitrary word $w \in \{0,1\}^* - 0^*$ in the binary representation of i. We provide here the standard Lyndon factorization of some subclasses of this class of infinite words.

Keywords: Lyndon word, Lyndon factorization, automatic sequence

1 Introduction

When we are interested in getting a better insight into the structure of some object, whether in mathematics or in computer science, one of the basic approaches how to tackle the problem is decomposition of the object into smaller "canonical" objects. The classical example is the factorization of a natural number into prime numbers being a powerful tool in the number theory. Natural numbers coded in unary notation correspond to finite words (strings) over a single-letter alphabet. Words over arbitrary alphabets can be considered to be a generalization of the concept of the natural number. A canonical decomposition of words to concatenation factors being Lyndon words is provided by the Lyndon factorization theorem ([10]). Lyndon words are primitive words minimal in their conjugacy classes. They are easy to deal with, e.g., there exists a simple linear algorithm, introduced by Duval ([8], [2]) for generating Lyndon words, up to a given length, in lexicographic order. A structural relationship of Lyndon words to other important classes of words has been established ([1]). Application of the Lyndon factorization leads to important results in the investigation of finite factorizations of the free monoid A^* and to discovery of new unavoidable regularities in words ([20]). Similar investigations of factorizations were performed for the case of free partially commutative monoids ([7]). The Lyndon factorization can be generalized to Viennot factorization by proceeding from the lexicographic ordering to Viennot orderings ([13]).

The Lyndon-factorization theorem has been further extended to infinite words (infinite sequences of symbols - [17]). Investigation of infinite words is crucial in the area of combinatorics on words, since properties of infinite words very often imply regularities in infinite sets of finite words. It is quite natural

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that, after the existence of the unique Lyndon factorization of infinite words had been proved, the factorization of some important sequences was investigated. In particular, the factorization of the sequence of Thue, of the Fibonacci sequence, of the regular paperfolding sequence and of the characteristic Sturmian sequences was characterized in [9],[11], [12].

The sequence of Thue was historically the first sequence where the structural properties were systematically studied ([18], [19]; a construction of that sequence was mentioned, probably for the first time, in [14]). The Thue sequence is one of the simplest non-periodic sequences in the important class of automatic sequences (originally called uniform tag sequences - [6]). Automatic sequences can be equivalently characterized by, among others, finite state automata, uniform morphisms or substitutions ([6], [5]).

In the current paper we investigate the Lyndon factorization of several subclasses of the class of generalized sequences of Thue. The generalized Thue sequences were first introduced in [5]. Each word w on the alphabet $\{0,1\}$, which contains at least one symbol 1, describes, as a parameter, a sequence t_w in this class. The i-th symbol of t_w corresponds to the parity of occurrences of w as a factor of the binary representation for i. The Thue sequence corresponds to the single-letter parameter word 1. As is well known, the Thue sequence does not contain cubic factors. Similarly, the sequence t_w does not contain a factor, which is the $2^{|w|}$-th power of a non-empty word ([3]). In fact, t_w does not contain any cubic factors except powers of a single letter ([15],[16]).

The present work is, in a sense, of an experimental character. We believe that studying the factorization pattern of a whole class of structurally related infinite sequences may bring some deeper insight to the relationship between the structure of a sequence and its Lyndon decomposition. Though we provide here a description of the Lyndon factorization of just a few subclasses of the generalized Thue sequences, it seems to be apparent that the Lyndon factors follow the block structure of the sequence determined by the iteration underlying uniform morphism or substitution. On the other hand, there does not seem to be a straightforward way leading to the unique description of the factorization of the whole class of the generalized words of Thue.

2 Notation and basic notions

2.1 Preliminaries

We denote as $\mathbb{N} = \{0,1,\ldots\}$ the set of all natural numbers and, for $p \in \mathbb{N}$, we denote $[p] = \{i \in \mathbb{N}; i < p\}$. An *alphabet* is any finite non-empty set Σ (its elements are called symbols). In particular, we will use the set $[2] = \{0,1\}$ as an alphabet consisting of 2 symbols. A *finite word* w of *length* $|w| = n \in \mathbb{N}$ on Σ is a finite sequence $w = a_0a_1\cdots a_{n-1}$ of symbols from Σ. The *empty word* (of length 0) is denoted by λ. An *infinite word* (a *sequence* for short) on Σ is an infinite sequence $w = a_0a_1\cdots$ of symbols from Σ. The symbol in a word or sequence w at the position i is denoted as $w(i)$ (the position numbering starts from 0). By $\Sigma^*, \Sigma^p, (\Sigma^p)^*, \Sigma^\omega$ we denote the sets of all words, of all words of length p, of all words of length being a multiple of p and of all sequences on Σ, respectively. Further we denote $\Sigma^\infty = \Sigma^* \cup \Sigma^\omega$ and $(\Sigma^p)^\infty = (\Sigma^p)^* \cup \Sigma^\omega$.

For a number $i \in [2^k]$ we denote as $i_{[2],k} \in [2]^k$ the word being the binary representation of i of length $k \geq 0$. For a word $w \in [2]^k$ we denote as $[w]_2$ the unique number from $[2^k]$ with binary representation w.

The *concatenation* of the words $x \in \Sigma^*$ and $y \in \Sigma^\infty$ is $xy \in \Sigma^\infty$. For $x,y \in \Sigma^*$, the words xy and yx are called *conjugates.* If $w = xyz \in \Sigma^\infty$, then x,y,z are called a *prefix*, a *factor at position* $|x|$, and a *suffix* of w, respectively. In this case, we further denote $x^{-1}w = yz$ and $wz^{-1} = xy$. A prefix, factor, or suffix is called *proper* if the remaining part of w is not empty. A word is *primitive* if it is not a concatenation of two or more equal factors. If $(w_i)_{i=0}^{\infty}$ is a sequence of finite words of unlimited length such that, for each $i \geq 0$, w_i is a prefix w_{i+1}, then $\lim_{i\to\infty} w_i$ will denote the unique infinite word having each w_i as a prefix.

Let $r \geq 1, s \geq 0$ and let $(\Sigma^r)^* = \bigcup_{k\in\mathbb{N}} \Sigma^{kr}$. A (r,s)-*substitution* (introduced in [5]) is a mapping $\mu : (\Sigma^r)^\infty \to \Gamma^\infty$, satisfying $\mu(\Sigma^r) \subset \Gamma^s$, and $\mu(xy) = \mu(x)\mu(y)$ for $x \in (\Sigma^r)^*, y \in (\Sigma^r)^\infty$. Thus μ is completely determined by its values on Σ^r. The substitution μ is *prolongable* in a word $u \in \Sigma^r$ if u is a proper prefix of $\mu(u)$. Then, clearly, $s > r \geq 1$ and the sequence $(\mu^i(u))_{i=0}^{\infty}$ of words obtained by repeated application of μ starting from u yields a limit $s = \lim_{i\to\infty} \mu^i(u)$ being a fixed point of μ, i.e., satisfying $\mu(s) = s$. A $(1,p)$-substitution is called *p-uniform morphism*.

Automatic sequences are described by finite state automata ([6], where these sequences are called uniform tag sequences). We will limit here our considerations to 2-automatic sequences on the alphabet $[2]$ only. A 2-*finite-state automaton* (2-*fsa*) is a tuple $\mathfrak{A} = (\Sigma, [2], \delta, a_0, F)$ where Σ is a (*state*) *alphabet*, $\delta : \Sigma \times [2] \to \Sigma$ is the *transition function*, $a_0 \in \Sigma$ is the *initial state* and $F \subset \Sigma$. We will assume that $\mathfrak{A}$ satisfies $\delta(a_0, 0) = a_0$. We will consider the usual extension $\delta^* : \Sigma \times [2]^* \to \Sigma$ defined for $a \in \Sigma, w \in [2]^*, x \in [2]$ inductively as $\delta^*(a,\lambda) = a, \delta^*(a,wx) = \delta(\delta^*(a,w),x)$. A sequence $s = b_0 b_1 \cdots \in [2]^\omega$ is 2-*automatic* if there exists a 2-fsa $\mathfrak{A}$ as above such that, for $i \in \mathbb{N}$, $b_i = 1$ iff $\delta^*(a_0, i_{[2],\lceil \log(i+1)\rceil}) \in F$. We say that s is *described by* $\mathfrak{A}$. We will use here several equivalent characterizations of 2-automatic sequences - see [6], [4] for the proof of Theorem 2.1.1. (Remark 2.1.2 is based on this proof.)

Theorem 2.1.1 *Let $s = b_0 b_1 \cdots \in [2]^\omega$ be a sequence. The following conditions i, ii, iii are equivalent.*

(i) s is a 2-automatic sequence

(ii) there is an alphabet Σ, a 2-uniform morphism $\varphi : \Sigma^\infty \to \Sigma^\infty$ and a 1-uniform morphism $\psi : \Sigma^\infty \to [2]^\infty$ such that $s = \psi(\lim_{n\to\infty} \varphi^n(a))$

(iii) s is a fixed point of some (k, kp^m)-substitution $\mu : [2]^\infty \to [2]^\infty$, $k, m \geq 1$.

Remark 2.1.2 *Let a 2-automatic sequence be described by the fsa $\mathfrak{A} = (\Sigma, [2], \delta, a_0, F)$. The morphisms φ, ψ from ii of Theorem 2.1.1 are given, for $a \in \Sigma$, as $\varphi(a) = \delta(a,0)\delta(a,1)$, $\psi(a) =$ (if $a \in F$ then 1 else 0). The symbol of $\psi(\varphi^r(a))$ at position $i \in [2^r]$, $r \geq 0$, is then $\psi(\delta^*(a, i_{[2],r}))$.*

2.2 Lyndon words

Assume a totally ordered alphabet Σ with the corresponding lexicographic order on Σ^∞. *Lyndon words* are defined as primitive words, which are minimal in the class of all their conjugates. The proof of the following equivalent characterization of Lyndon words and the proof of the Lyndon's theorem can be found in [10].

Proposition 2.2.1 *A non-empty word is a Lyndon word iff it is strictly smaller than any of its non-empty proper suffixes.*

Theorem 2.2.2 (Lyndon) *Any non-empty finite word can be written uniquely as a concatenation of a non-increasing sequence of Lyndon words.*

In [17] the authors defined *infinite Lyndon words* as those sequences, which have infinitely many prefixes being Lyndon words. They proved the following generalization of the Lyndon theorem.

Theorem 2.2.3 *Any infinite word s can be uniquely factorized in one of the following forms:*

$s = l_0 l_1 \cdots,$ *where* $l_0 \geq l_1 \geq \cdots$ *are finite Lyndon words*

or

$s = l_0 l_1 \cdots l_k l_{k+1}$ *where* $l_0 \geq l_1 \geq \cdots \geq l_k$ *are finite Lyndon words and* $l_{k+1} < l_k$ *is an infinite Lyndon word.*

2.3 Generalized words of Thue

The infinite word of Thue t can be defined as $t = \lim_{i \to \infty} \varphi^i(0)$ where φ is the 2-uniform morphism on the alphabet $\{0,1\}$ described as $\varphi(0) = 01$, $\varphi(1) = 10$. Hence $t = 0110100110010110\cdots$. The digit at the i-th position of t can be determined as the parity of the number of occurrences of the digit 1 in the binary representation of i. In [5] *generalized words of Thue* were introduced. They were further investigated in [3], [15] and [16]. Each such infinite word $t_w = t_w(0)t_w(1)t_w(2)\cdots \in \{0,1\}^{\omega}$ is based on a word $w \in \{0,1\}^* - \{0\}^*$. The symbol $t_w(i)$ at the i-th position of t_w is defined as $t_w(i) = \#_w 0^{|w|} i_{[2]} \bmod 2$ where $\#_w 0^{|w|} i_{[2]}$ denotes the number of factors w occurring in the binary representation of i padded with at least $|w|$ leading zeroes. Every occurrence of a factor is counted, e.g., $t_{010}(10) = 0$, since 0001010 (the binary representation of 10 padded with 3 leading zeroes) contains two overlapping factors 010. In particular, $t = t_1$.

Let for the remaining part of the paper $w \in \{0,1\}^* - \{0\}^*$ be an arbitrary but fixed word. Denote $k = |w|$, $\Sigma_w = \{\langle \alpha, m\rangle ; \alpha \text{ is a proper prefix of } w, m \in [2]\}$. We will suppose that $k \geq 2$ since the case $k = 1$ means t_w is the sequence of Thue. For the purpose of the next theorem (the proof of the theorem can be found in [3]), for each $x, y \neq \lambda$, let $sp(x,y)$ denote the longest suffix of x that is a proper prefix of y.

Theorem 2.3.1 *The sequence* t_w *is described by the fsa* $\mathfrak{A}_w = (\Sigma_w, [2], \delta_w, a_{0,w}, F_w)$ *where, for* $\langle \alpha, m\rangle \in \Sigma_w$, $j \in [2]$,

$$\delta_w(\langle \alpha, m\rangle, j) = \langle sp(\alpha j, w), \mathit{if}\ \alpha j = w\ \mathit{then}\ 1 - m\ \mathit{else}\ m\rangle, \tag{1}$$

$a_{0,w} = \langle sp(0^k, w), 0\rangle$ *and* $F_w = \{\langle \alpha, 1\rangle ; \langle \alpha, 1\rangle \in \Sigma_w\}$. *Moreover ,* $\mathfrak{A}_w$ *is the minimal fsa, with respect to the size of the state alphabet, describing* t_w.

We will denote as φ_w, ψ_w the morphisms corresponding to $\mathfrak{A}_w$ according to Remark 2.1.2 (it is worth to observe that $\psi_w(\langle \alpha, m\rangle) = m$). The sequence t_w is a fixed point of a substitution described by the following theorem from [5].

Theorem 2.3.2 *Let* μ_w *be the* $(2^{k-1}, 2^k)$*-substitution defined on* $[2]^{k-1}$ *as*

$$\mu_w(x_0 x_1 \cdots x_{2^{k-1}-1}) = y_0 y_1 \cdots y_{2^k - 1} \tag{2}$$

where, for $i \in [2^k]$, $y_i = (x_{\lfloor i/2 \rfloor} + \chi_w(i)) \bmod 2$, *and* $\chi_w(i) = \mathit{if}\ w = i_{[2],k}\ \mathit{then}\ 1\ \mathit{else}\ 0$. *Then* $\mu_w(t_w) = t_w$.

Property 2.3.3 *For* $i \geq 0$ *and* $a \in \Sigma_w$, $\mu_w^i(\psi_w(\varphi_w^{k-1}(a))) = \psi_w(\varphi_w^{k-1+i}(a))$.

Proof. Let $a \in \Sigma_w$. Since the automaton $\mathfrak{A}_w$ is minimal, the state a can be reached from the initial state $a_{0,w}$. Remark 2.1.2 then implies that a occurs in the sequence $\lim_{n \to \infty} \varphi^n(a_{0,w})$. Let it be at position j.

Then Theorem 2.3.2 implies that

$$\begin{aligned} \mu_w(\psi_w(\varphi_w^{k-1}(a))) &= \mu_w(t_w(2^{k-1}j)\cdots \\ t_w(2^{k-1}j+2^{k-1}-1)) &= t_w(2^k j)\cdots \\ t_w(2^k j+2^k-1) &= \psi_w(\varphi_w^k(a)) \\ &= \psi w(\varphi_w^{k-1}(\varphi(a))). \end{aligned} \tag{3}$$

The required assertion is obtained by a simple inductive argument. □

Example 2.3.4 *Let $w = 010$. The sequence (spaces are inserted for better readability)*

$$t_{010} = 0010\ 1100\ 1101\ 0000\ 1101\ 0011\ 0010\ 0000\ 1101\ \cdots \tag{4}$$

is described by the fsa $\mathfrak{A}_{010} = (\Sigma_{010}, [2], \delta_{010}, \langle 0,0\rangle, F_{010})$ where

$$\begin{aligned} \Sigma_{010} &= \{\langle\lambda,0\rangle, \langle 0,0\rangle, \langle 01,0\rangle, \langle\lambda,1\rangle, \langle 0,1\rangle, \langle 01,1\rangle\}, \\ F_{010} &= \{\langle\lambda,1\rangle, \langle 0,1\rangle, \langle 01,1\rangle\} \end{aligned}$$

and the transition function is given by the following table

δ_{010}	0	1
$\langle\lambda,0\rangle$	$\langle 0,0\rangle$	$\langle\lambda,0\rangle$
$\langle 0,0\rangle$	$\langle 0,0\rangle$	$\langle 01,0\rangle$
$\langle 01,0\rangle$	$\langle 0,1\rangle$	$\langle\lambda,0\rangle$

δ_{010}	0	1
$\langle\lambda,1\rangle$	$\langle 0,1\rangle$	$\langle\lambda,1\rangle$
$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 01,1\rangle$
$\langle 01,1\rangle$	$\langle 0,0\rangle$	$\langle\lambda,1\rangle$

(5)

The morphism φ_{010} is described by the rows of the table, e.g.,

$$\varphi_{010}(\langle 0,1\rangle) = \langle 0,1\rangle\langle 01,1\rangle,$$

and

$$\psi_w(\langle\alpha,m\rangle) = m \quad \textit{for} \quad \alpha \in \{\lambda,0,01\}, m \in \{0,1\}.$$

The sequence t_{010} is a fixed point of the substitution

$$\begin{array}{ll} 0000 \mapsto 00100000 & 1111 \mapsto 11011111 \\ 0010 \mapsto 00101100 & 1101 \mapsto 11010011 \\ 1100 \mapsto 11010000 & 0011 \mapsto 00101111 \end{array} \tag{6}$$

We will further use the following two easy observations without explicitly referring to them.

Observation 2.3.5 *Let $r \geq 1$. $x \in \Sigma_w^\infty$. No symbol from Σ can occur in $\varphi_w^r(x)$ both in an even and in an odd position.*

Observation 2.3.6 *For each $x \in \Sigma_w^\infty$ let $\bar{x}$ denote the word obtained by replacing each symbol $\langle\alpha,m\rangle$ by $\langle\alpha,1-m\rangle$. Similarly, for each word $x \in [2]^\infty$ let $\bar{x}$ denote the word obtained by replacing each symbol m by $1-m$. Then*

1. *For each* $a \in \Sigma_w$ *and* $r \geq 0$, $\varphi_w^r(\bar{a}) = \overline{\varphi_w^r(a)}$

2. *For each* $a \in \Sigma_w$ *and* $r \geq 0$, $\psi_w(\varphi_w^r(\bar{a})) = \overline{\psi_w(\varphi_w^r(a))}$

3. *For each* $x \in ([2]^{k-1})^\infty$, $\mu_w(\bar{x}) = \overline{\mu_w(x)}$.

In the remaining part of this text we will omit the subscript w from $\mathfrak{A}_w$, Σ_w, δ_w, δ_w^*, $a_{0,w}$, F_w, φ_w, ψ_w, μ_w and χ_w.

3 Generalized words of Thue and the lexicographic order

3.1 Lexicographic order on the set Σ^∞

We consider the set $[2]^\infty$ to be ordered lexicographically based on the ordering $0 < 1$ of $[2]$. We will investigate the factors of the sequence t_w and their properties related to this lexicographic ordering. The structure of the sequence t_w closely matches the structure of the sequence $\tau_w = \lim_{n\to\infty} \varphi^n(a_0) \in \Sigma^\omega$. It seems to be useful to look for an order relation on Σ (and the corresponding lexicographic order on Σ^∞) related to the order of the block factors of Σ of the form $\psi(\varphi^r(a))$, such that the morphism φ is growing on Σ^∞. Since t_w is a fixed point of the $(2^{k-1}, 2^k)$-substitution μ, one would expect that the relative order of two symbols $a, b \in \Sigma$ should be implied by the order of the words $\psi(\varphi^{k-1}(a)), \psi(\varphi^{k-1}(b))$. This choice need not be (always) suitable as illustrated by the following example.

Example 3.1.1 *Consider* $w = 10^s$, $s \geq 2$, *hence* $k-1 = s$, $k-1+s = 2s$. *Let* $a = \langle 1, 1\rangle$, $b = \langle 10^{s-1}, 1\rangle$. *Then*

$$\psi(\delta^*(a, i_{[2],2s-1})) = \begin{cases} 0 & \textit{for } i < 2^{s-1}; \\ 1 & \textit{for } i = 2^{s-1}; \end{cases} \qquad \psi(\delta^*(a, i_{[2],2s})) = 0 \quad \textit{for } i \leq 2^s;$$

$$\psi(\delta^*(b, i_{[2],2s-1})) = 0 \quad \textit{for } i \leq 2^{s-1}; \qquad \psi(\delta^*(b, i_{[2],2s})) = \begin{cases} 0 & \textit{for } i < 2^s; \\ 1 & \textit{for } i = 2^s. \end{cases}$$

Therefore $\psi(\varphi^{k-1+s-1}(a)) > \psi(\varphi^{k-1+s-1}(b))$ *and* $\psi(\varphi^{k-1+s}(a)) < \psi(\varphi^{k-1+s}(b))$.

For the rest of the paper, denote by s the length of the longest suffix of w belonging to 0^*. Example 3.1.1 documented that choosing for the definition of the order relation on Σ an exponent of φ smaller than $k-1+s$ results in an order relation, with respect to which φ is not a growing function. Therefore we will consider the ordering of Σ based on the blocks $\psi(\varphi^{k-1+s}(a))$.

Definition 3.1.2 *The (total) order relation* $<$ *on* Σ *is defined, for* $a, b \in \Sigma$, *as*
$a < b$ *iff* $\psi(\varphi^{k-1+s}(a)) < \psi(\varphi^{k-1+s}(b))$.

The relation $<$ is indeed an order relation due to the following Lemma.

Lemma 3.1.3 *Let* $a, b \in \Sigma$, $a \neq b$. *Then, for* $r \geq 0$, $\psi(\varphi^{k-1+r}(a)) \neq \psi(\varphi^{k-1+r}(b))$.

Proof. Let $a = \langle \alpha, m\rangle$, $b = \langle \beta, n\rangle$, $a \neq b$ and let $d \in [2]$ be different from the last symbol of w. We will use Remark 2.1.2. If $m = n$ then $\alpha \neq \beta$, assume $|\alpha| < |\beta|$. Let $w = \beta\gamma$. Then $\psi(\delta^*(a, \gamma d^{k-1+r-|\gamma|})) = m$, $\psi(\delta^*(b, \gamma d^{k-1+r-|\gamma|})) = 1 - n \neq m$. If $m \neq n$ then $\psi(\delta^*(a, d^{k-1+r})) = m \neq n = \psi(\delta^*(b, \gamma d^{k-1+r-|\gamma|}))$. □

3.2 Monotonicity of φ and μ

The total order of Σ induces the lexicographic order on Σ^∞. We will denote $\Psi = \psi \circ \varphi^{k-1+s}$. Clearly, the $(k-1+s)$-uniform morphism $\Psi : \Sigma^\infty \to [2]^\infty$ is a strictly growing function. Property 2.1.2 implies

Property 3.2.1 *For $i \geq 0$ and $x \in \Sigma^*$, $\mu^i(\Psi(x)) = \Psi(\varphi^i(x))$.*

We want to show that the morphism φ is strictly growing, as well. To prove it we need Property 3.2.2, which was proved in [3] and one more technical lemma.

Property 3.2.2 *φ is a one-to-one function.*

Lemma 3.2.3 *If $a, b \in \Sigma$, $a \neq b$, and $r \geq 0$ is the first position where $\Psi(a)$ and $\Psi(b)$ contain a different symbol then r is a multiple of 2^s.*

Proof. Assume r is not a multiple of 2^s, i.e., $r \geq 1$ and 0^s is not a suffix of $r_{[2],k-1+s}$. Denote ρ the word obtained from $r_{[2],k-1+s}$ by removing the last symbol. Let $a = \langle \alpha, m \rangle$, $b = \langle \beta, n \rangle$. Since neither of $\alpha r_{[2],k-1+s}$, $\beta r_{[2],k-1+s}$ contains the suffix w, we have

$$\psi(\delta^*(a, r_{[2],k-1+s})) = \psi(\delta^*(a, \rho)) \textit{and}$$
$$\psi(\delta^*(b, r_{[2],k-1+s})) = \psi(\delta^*(b, \rho)),$$

hence $\psi(\varphi^{k-1+s}(a))$ and $\psi(\varphi^{k-1+s}(b))$ differ in position $\lfloor r/2 \rfloor < r$ - a contradiction to the minimality of r. □

Lemma 3.2.4 *For $a, b \in \Sigma$, if $a < b$ then $\varphi(a) < \varphi(b)$.*

Proof. Let $a < b$ (i.e., $\Psi(a) < \Psi(b)$) and let r be the first position in which $\Psi(a)$ and $\Psi(b)$ differ. Then $\Psi(a)(r) = 0$ and $\Psi(b)(r) = 1$. Assume by contrary that $\varphi(a) \geq \varphi(b)$, i.e., $\varphi(a) > \varphi(b)$ as implied by Property 3.2.2. Then $\Psi(\varphi(a)) > \Psi(\varphi(b))$. Considering Proposition 3.2.1, this means $\mu(\Psi(a)) > \mu(\Psi(b))$. Theorem 2.3.2 implies that $\mu(\Psi(a))$ and $\mu(\Psi(b))$ have a common prefix of length $2r$. Moreover, the "correction" caused by the function χ must take place at position $2r$, since otherwise $\mu(\Psi(a))(2r) = \Psi(a)(r) < \Psi(b)(r) = \mu(\Psi(b))(2r)$ and, consequently, $\mu(\Psi(a)) < \mu(\Psi(b))$. Therefore $2r = 2^k t + [w]_2$ for some $t \geq 0$ and Lemma 3.2.3 implies that $[w]_2$ is a multiple of 2^{s+1}- a contradiction to the maximality of s. □

Corollary 3.2.5 *φ is strictly growing on Σ^∞.*

The following Corollary 3.2.6 follows from Property 3.2.1.

Corollary 3.2.6 *μ is strictly growing on $\{\Psi(x); x \in \Sigma^\infty\}$.*

In the remaining part of the paper we will provide the characterization of the Lyndon factorization of the sequence t_w for four different shapes of the word w.

4 Factorization of four types of generalized words of Thue.

We consider here four types of generalized Thue sequences, based on the shape of the underlying word w. For each type we provide a theorem describing the Lyndon factorization of the sequence t_w. Typically, the Lyndon factorization of t_w starts by an initial finite sequence of Lyndon factors (we will call this part the "preamble" of the factorization) , followed by a product of a recurrently described infinite sequence of Lyndon factors (the "body" of the factorization). The length of the preamble may grow with the increasing length of the word w. The recurrent description of the body is based on the generating substitution μ_w. Proofs of the theorems from this section can be found in Section 5.

Each theorem is followed by an example of factorization of one sequence t_w of the particular type, together with the corresponding mappings μ_w and φ_w. In the examples, we include spaces to increase readability of t_w. The first symbol of each Lyndon factor of t_w is underlined. The morphism φ_w is given for arguments of the form $\langle\alpha,0\rangle$ only. The values for arguments of the form $\langle\alpha,1\rangle$ can be obtained by replacing each 0 or 1 in the second position by the complementary value.

4.1 Case $w = 0^{k-1}1$

Theorem 4.1.1 *Let $w = 0^{k-1}1$. Then $t_w = \prod_{i=0}^{\infty} w_i$ where $w_0 = 01^{2^k}$, $w_1 = 01^{2^k-1}$, $w_{i+1} = 1^{-1}\mu(1w_i1^{-1})1$ for $i > 1$. For $i \geq 0$, the word w_i is a Lyndon word and $w_i > w_{i+1}$.*

Example 4.1.2 $w = 001 \ \ (k = 3)$

t_{001}

$\underline{0}$1111111 1$\underline{0}$111111 1$\underline{0}$001111 10111111 1$\underline{0}$000000 10111111 10001111 10111111
1$\underline{0}$000000 01000000 10001111 10111111 10000000 10111111 10001111 10111111
1$\underline{0}$000000 01000000 01110000 01000000 10000000 10111111 10001111 10111111
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1$\underline{0}$000000 01000000 01110000 01000000 01111111 01000000 01110000 01000000
01111111 10111111 01110000 01000000 01111111 01000000 01110000 $\cdots$

μ_{001}	
0000 $\mapsto$ 0100 0000	1111 $\mapsto$ 1011 1111
0100 $\mapsto$ 0111 0000	1011 $\mapsto$ 1000 1111
0111 $\mapsto$ 0111 1111	1000 $\mapsto$ 1000 0000

φ_{001}
$\langle\lambda,0\rangle \mapsto \langle 0,0\rangle\langle\lambda,0\rangle$
$\langle 0,0\rangle \mapsto \langle 00,0\rangle\langle\lambda,0\rangle$
$\langle 00,0\rangle \mapsto \langle 00,0\rangle\langle\lambda,1\rangle$

4.2 Case $w = 10^{k-2}1$

Theorem 4.2.1 *Let $w = 10^{k-2}1$, $k \geq 3$ [‡]. Then $t_w = \prod_{i=0}^{\infty} w_i$ where*

$$
\begin{aligned}
w_0 &= 0^{2^{k-1}+1}10^{2^{k-1}}1^20^{2^{k-1}-3}1;\\
w_1 &= \begin{cases} 0^6101^20^21^201 & \text{if } k = 3;\\ 0^{2^{k-1}+2}1^40^{2^{k-1}-7}10^{2^{k-1}}1^20^{2^{k-1}-3}1 & \text{if } k > 3;\end{cases}\\
w_{i+1} &= 0^{2^{k-1}-2}\mu((0^{2^{k-1}-2})^{-1}w_i0^{2^{k-1}-2})(0^{2^{k-1}-2})^{-1} \quad \text{for} \quad i \geq 1.
\end{aligned}
$$

For $i \geq 0$, the word w_i is a Lyndon word and $w_i > w_{i+1}$.

Example 4.2.2 $w = 1001 \ \ (k = 4)$

‡ The case $k = 2$ yields a different Lyndon factorization—see Section 4.3.

t_{1001}

00000000 01000000 00110000 01000000 00001111 01000000 00110000 01000000
00000000 10111111 00110000 01000000 00001111 01000000 00110000 01000000
00000000 01000000 11001111 10111111 00001111 01000000 00110000 01000000
00000000 10111111 00110000 01000000 00001111 01000000 00110000 01000000
00000000 01000000 00110000 01000000 11110000 10111111 11001111 10111111
00000000 10111111 00110000 01000000 00001111 01000000 00110000 01000000
00000000 01000000 11001111 10111111 00001111 01000000 00110000 01000000
00000000 10111111 00110000 01000000 00001111 01000000 00110000 01000000
00000000 01000000 00110000 01000000 00001111 01000000 00110000 01000000
11111111 01000000 11001111 10111111 11110000 10111111 11001111 $\cdots$

μ_{1001}

00000000 $\mapsto$ 00000000 01000000	11111111 $\mapsto$ 11111111 10111111
01000000 $\mapsto$ 00110000 01000000	10111111 $\mapsto$ 11001111 10111111
00110000 $\mapsto$ 00001111 01000000	11001111 $\mapsto$ 11110000 10111111
00001111 $\mapsto$ 00000000 10111111	11110000 $\mapsto$ 11111111 01000000

φ_{1001}

$\langle\lambda,0\rangle \mapsto \langle\lambda,0\rangle\langle 1,0\rangle$
$\langle 1,0\rangle \mapsto \langle 10,0\rangle\langle 1,0\rangle$
$\langle 10,0\rangle \mapsto \langle 100,0\rangle\langle 1,1\rangle$
$\langle 100,0\rangle \mapsto \langle\lambda,0\rangle\langle 1,1\rangle$

4.3 Case $w = 1^k$.

Theorem 4.3.1 *Let* $w = 1^k$,. $k \geq 2$. *Then* $t_w = \prod_{i=0}^{\infty} w_i$ *where* $w_0 = 0^{2^k-1}10^{2^{k-2}}1$, $w_1 = w_0^{-1}\mu^2(w_0 0)0^{-1}$ *and* $w_{i+1} = 0\mu^2(0^{-1}w_i 0)0^{-1}$ *for* $i \geq 1$.
For $i \geq 0$, *the word* w_i *is a Lyndon word and* $w_i > w_{i+1}$.

Example 4.3.2 $w = 11 \ \ (k = 2)$

t_{11}

00010010 00011101 00010010 11100010 00010010 00011101 11101101 00011101
00010010 00011101 00010010 11100010 11101101 11100010 00010010 11100010
00010010 00011101 00010010 11100010 00010010 00011101 11101101 00011101
11101101 11100010 11101101 00011101 00010010 00011101 11101101 00011101
00010010 00011101 00010010 11100010 00010010 00011101 11101101 00011101
00010010 00011101 00010010 11100010 11101101 11100010 00010010 11100010
11101101 11100010 11101101 00011101 11101101 11100010 00010010 11100010
00010010 00011101 00010010 11100010 11101101 11100010 00010010 11100010
00010010 00011101 00010010 11100010 00010010 00011101 11101101 00011101
00010010 00011101 00010010 11100010 11101101 11100010 00010010 $\cdots$

μ_{11}

00 $\mapsto$ 00 01	11 $\mapsto$ 11 10
01 $\mapsto$ 00 10	10 $\mapsto$ 11 01

φ_{11}

$\langle\lambda,0\rangle \mapsto \langle\lambda,0\rangle\langle 1,0\rangle$
$\langle 1,0\rangle \mapsto \langle\lambda,0\rangle\langle 1,1\rangle$

4.4 Case $w = 10^{k-1}$

Theorem 4.4.1 *Let $w = 10^{k-1}, k \geq 2$. Denote*

$$\begin{aligned} \rho &= \mu^{2k-2}(1^{2^{k-1}})\mu^{k-1}(0^{2^{k-1}})(0^{2^{k-1}-1})^{-1}, \\ j_0 &= 0, \\ j_1 &= \left[10^{k-1}10^{2k-2}\right]_2 \\ j_2 &= \left[10^{k}10^{2k-2}\right]_2, \end{aligned}$$

and, for $1 \leq r \leq k-1$, denote

$$\begin{aligned} j_{2r+1} &= \left[10^{k-1}10^{k-1}1^r0^{k-2}1\right]_2, \\ j_{2r+2} &= \left[10^{k}10^{k-1}1^r0^{k-2}1\right]_2. \end{aligned}$$

For $0 \leq i \leq 2k-1$, let w_i be the factor of t_w of length $j_{i+1} - j_i$ at the position j_i and, for $i \geq 2k$, let $w_i = \rho^{-1}\mu(\rho w_{i-1}\rho^{-1})\rho$.
Then $t_w = \prod_{i=0}^{\infty} w_i$ and, for $i \geq 0$, $w_i > w_{i+1}$, and w_i is a Lyndon word.

Example 4.4.2 $w = 100$ $(k = 3)$

t_{100}
00001000 11001000 11111000 11001000 11110111 11001000 11111000 11001000
11110111 00110111 11111000 11001000 11110111 11001000 11111000 11001000
11110111 00110111 00000111 00110111 11110111 11001000 11111000 11001000
11110111 00110111 11111000 11001000 11110111 11001000 11111000 11001000
11110111 00110111 00000111 00110111 00001000 00110111 00000111 00110111
11110111 00110111 11111000 11001000 11110111 11001000 11111000 11001000
11110111 00110111 00000111 00110111 11110111 11001000 11111000 11001000
11110111 00110111 11111000 11001000 11110111 11001000 11111000 11001000
11110111 00110111 00000111 00110111 00001000 00110111 00000111 00110111
00001000 11001000 00000111 00110111 00001000 00110111 00000111 $\cdots$

μ_{100}		φ_{100}
0000 $\mapsto$ 0000 1000	1111 $\mapsto$ 1111 0111	$\langle\lambda,0\rangle \mapsto \langle\lambda,0\rangle\langle 1,0\rangle$
1000 $\mapsto$ 1100 1000	0111 $\mapsto$ 0011 0111	$\langle 1,0\rangle \mapsto \langle 10,0\rangle\langle 1,0\rangle$
1100 $\mapsto$ 1111 1000	0011 $\mapsto$ 0000 0111	$\langle 10,0\rangle \mapsto \langle\lambda,1\rangle\langle 1,0\rangle$

5 Proofs of the factorization results.

The present part contains proofs of the four theorems from Section 3. In each of the four cases, we start by observing some specific properties of φ_w, Ψ_w, $\mathfrak{A}_w$and μ_w for the particular shape of w. The proof of each of the theorems consists of the following parts:

Part A. We prove the consistency of the recurrent formula for factors w_i and we show that the sequence t_w can be written as a concatenation of the factors w_i.

Part B. We prove that $w_0, w_1, \ldots$ is a decreasing sequence.

Part C. We show that each word w_i is a Lyndon word—separately for the preamble and for the body of the factorization.

5.1 Case $w = 0^{k-1}1$

Let $w = 0^{k-1}1$, $k \geq 2$[§]. In this case $s = 0$, $\Psi = \psi \circ \varphi^{k-1}$ and each element of Σ is of the form $\langle 0^r, m\rangle$ for some $r \in [k]$, $m \in [2]$. The initial state of $\mathfrak{A}$ is $\langle 0^{k-1}, 0\rangle$.

We have a simple equivalent characterization of the order relation on Σ for this case.

Property 5.1.1 *Let $a, b \in \Sigma$, $a = \langle 0^p, m\rangle$, $b = \langle 0^q, n\rangle$. Then $a < b$ iff one of the following is true:*

1. $m < n$ or

2. $m = n = 0$ and $p < q$ or

3. $m = n = 1$ and $p > q$.

Proof. If $0 = m < n = 1$ then $\psi(\delta^*(a, 0^{k-1})) = 0$, $\psi(\delta^*(b, 0^{k-1})) = 1$, and $a < b$. Now let $m = n$. For $r \in \{p, q\}$, $i \in [2^{k-1}]$ we have

$$\Psi(\langle 0^r, m\rangle)(i) = \psi(\delta^*(\langle 0^r, m\rangle, i_{[2],k-1})) = \begin{cases} m & \text{if } i = 0 \text{ or } i \geq 2^r; \\ 1 - m & \text{for } 1 \leq i < 2^r. \end{cases} \tag{7}$$

Thus $\Psi(\langle 0^r, m\rangle)$ has the prefix $m(1-m)^{2^r-1}$. If $m = 0$ then $a < b$ iff $p < q$, if $m = 1$ then $a < b$ iff $p > q$. □

The following complete description of the mapping φ can be easily obtained using Remark 2.1.2.

Property 5.1.2 $\varphi(\langle 0^r, 0\rangle) = \langle 0^{r+1}, 0\rangle \langle \lambda, 0\rangle$ *for* $0 \leq r < k-1$, *and* $\varphi(\langle 0^{k-1}, 0\rangle) = \langle 0^{k-1}, 0\rangle \langle \lambda, 1\rangle$.

The first of the next two properties may be obtained, considering Remark 2.1.2, by observing the shapes of the words $i_{[2],k-1}$, $i \in [2^{k-1}]$. The second is a consequence of the trivial fact that the binary representation of a position number has suffix 0 iff the position number is even.

Property 5.1.3 *Let $a \in \Sigma$. If 0^2 is a prefix of $\Psi(a)$ then $a = \langle \lambda, 0\rangle$. If $0^{2^{k-1}-1}$ is a suffix of $\Psi(a)$ then $a = \langle \lambda, 0\rangle$ or $a = \langle 0^{k-1}, 1\rangle$.*

Property 5.1.4 *Let $x \in \Sigma^*$. Then $\langle \lambda, 0\rangle$ does not occur in any even position of $\varphi(x)$ and $\langle 0^{k-1}, 1\rangle$ does not occur in any odd position of $\varphi(x)$.*

[§] The case $k = 1$ corresponds to the sequence of Thue. Though the factorization in this case (see [9]) has the same shape as for $k \geq 2$, the proof for $k \geq 2$ will use the fact that w contains at least one symbol 0.

Proof of Theorem 4.1.1

Part A

Lemma 5.1.5 *w_0w_1 is a prefix of t_w.*

Proof. Observing the shape of the words $r_{[2],k+2}$, for $0 \le r \le 2^{k+1}$, we find

$$\psi(\delta^*(\langle 0^{k-1},0\rangle, r_{[2],k+2})) = \begin{cases} 0 & \textit{if } r = 0; \\ 1 & \textit{if } 0 < r \le 2^k; \\ 0 & \textit{if } r = 2^k+1; \\ 1 & \textit{if } 2^k+1 < r \le 2^{k+1}. \end{cases} \tag{8}$$

Therefore t_w has the prefix $01^{2^k}01^{2^k-1}$. □

Lemma 5.1.6 *For $i \ge 1$, 11 is a suffix of w_i and $1w_i1^{-1} = \Psi(\varphi^i(\langle\lambda,1\rangle))$.*

Proof. The assertion is true for $i = 1$ since $w_1 = 01^{2^k-1}$ and $1w_11^{-1}$ is a factor of t_w at position 2^k, as implied by Lemma 5.1.5. Assume that the assertion is true for some $i \ge 1$. Then 1 is both a prefix and a suffix of $1w_i1^{-1}$. Theorem 2.3.2 implies that 1 is both a prefix and a suffix of $\mu(1w_i1^{-1})$ since $w_{[2]}$ is neither a suffix of $0_{[2],k+i}$ nor of $(2^{k+i}-1)_{[2],k+i}$. Hence 11 is a suffix of $w_{i+1} = 1^{-1}\mu(1w_i1^{-1})1$. Using Property 3.2.1 we obtain

$$1w_{i+1}1^{-1} = 11^{-1}\mu(1w_i1^{-1})11^{-1} = \mu(1w_i1^{-1}) = \mu(\Psi(\varphi^i(\langle\lambda,1\rangle))) = \Psi(\varphi^{i+1}(\langle\lambda,1\rangle)). \tag{9}$$

□

Lemma 5.1.7 $t_w = \prod_{i=0}^{\infty} w_i$.

Proof. Since t_w is a fixed point of μ, the image in μ of the prefix 01^{2^k-1} of length 2^k of t_w is the prefix of length 2^{k+1} of t_w. Then

$$\begin{aligned} \mu\left(\prod_{i=0}^{\infty} w_i\right) &= \mu\left(01^{2^k}\prod_{i=1}^{\infty} w_i\right) \\ &= \mu\left(01^{2^k-1}\prod_{i=1}^{\infty}(1w_i1^{-1})\right) \\ &= \mu(01^{2^k-1})\prod_{i=1}^{\infty}\mu(1w_i1^{-1})) \\ &= (w_0w_11^{-1})\prod_{i=1}^{\infty}\mu(1w_i1^{-1})) \\ &= (w_0w_11^{-1})\prod_{i=1}^{\infty}(1w_{i+1}1^{-1}) \\ &= \prod_{i=0}^{\infty} w_i. \end{aligned} \tag{10}$$

Both sequences t_w and $\prod_{i=0}^{\infty} w_i$ are fixed points of μ starting by the same prefix of length 2^{k-1}, therefore $t_w = \prod_{i=0}^{\infty} w_i$. □

Part B

Lemma 5.1.8 *$(w_i)_{i=0}^{\infty}$ is a decreasing sequence of words.*

Proof. Clearly $w_0 > w_1$. Property 5.1.1 implies that $\langle\lambda,1\rangle > \langle 0,1\rangle\langle\lambda,1\rangle = \varphi(\langle\lambda,1\rangle)$. Since φ an Ψ are strictly growing, we get, for $i \ge 1$, $\varphi^i(\langle\lambda,1\rangle) > \varphi^{i+1}(\langle\lambda,1\rangle)$, and, consequently,

$$w_i = 1^{-1}\Psi(\varphi^i(\langle\lambda,1\rangle))1 > 1^{-1}\Psi(\varphi^{i+1}(\langle\lambda,1\rangle))1 = w_{i+1}$$

□

Part C

We start by observing the fact that w_0 is strictly smaller than any of its non-empty proper suffixes.

Observation 5.1.9 *The word w_0 is a Lyndon word.*

Lemma 5.1.10 *For $1 \leq i \leq k$, $w_i = 0^{2^i-1}1^{2^k-2^i}x_i$, where x_i does not contain a factor 0^{2^i-1}*

Proof. For $0 \leq r \leq k-2$, $\varphi(\langle 0^r, 0\rangle) = \langle 0^{r+1}, 0\rangle \langle \lambda, 0\rangle$ and (for $r = k-1$) $\varphi(\langle 0^{k-1}, 0\rangle) = \langle 0^{k-1}, 0\rangle \langle \lambda, 1\rangle$. An easy induction yields, for $0 \leq i \leq k-1$,

$$\psi(\varphi^i(\langle \lambda, 0\rangle)) = \psi(\langle 0^i, 0\rangle \langle \lambda, 0\rangle \varphi(\langle \lambda, 0\rangle)\varphi^2(\langle \lambda, 0\rangle) \cdots \varphi^{i-1}(\langle \lambda, 0\rangle)) = 0^{2^i}.$$

For $1 \leq i \leq k$,

$$\psi(\varphi^{k-1+i}(\langle \lambda, 1\rangle)) = xy \tag{11}$$

where

$$\begin{aligned} x &= \psi(\langle 0^{k-1}, 1\rangle \langle \lambda, 0\rangle \varphi(\langle \lambda, 0\rangle) \cdots \varphi^{i-1}(\langle \lambda, 0\rangle)\varphi^i(\langle \lambda, 1\rangle) \cdots \varphi^{k-1}(\langle \lambda, 1\rangle)) \\ &= 10^{2^0+2^1+\cdots 2^{(i-1)}}1^{2^i+2^{i+1}+\cdots 2^{k-1}} \end{aligned}$$

and

$$\begin{aligned} y &= \psi(\varphi^{k-1+1}(\langle \lambda, 1\rangle) \cdots \varphi^{k-1+i-1}(\langle \lambda, 1\rangle)) \\ &= \Psi(\varphi(\langle \lambda, 1\rangle)) \cdots \Psi(\varphi^{i-1}(\langle \lambda, 1\rangle)). \end{aligned}$$

Hence, following Lemma 5.1.6,

$$\begin{aligned} w_i &= 1^{-1}\psi(\varphi^{k-1+i}(\langle \lambda, 1\rangle))1 \\ &= 0^{2^i-1}1^{2^k-2^i}(1w_1 1^{-1}) \cdots (1w_{i-1}1^{-1})1 \\ &= 0^{2^i-1}1^{2^k-2^i}w_1 \cdots w_{i-1}, \end{aligned} \tag{12}$$

where $w_1 \cdots w_{i-1}$ does not contain a factor 0^{2^i-1}, the latter fact follows by induction. □

Corollary 5.1.11 *$w_1, \ldots, w_k$ are Lyndon words*

Lemma 5.1.12 *For $i \geq k+1$, 0^{2^k} is a prefix of w_i.*

Proof. It is enough to prove the assertion for w_{k+1}, since, by Lemma 5.1.8, for $i > k+1$, $w_{k+1} > w_i$. From Lemma 5.1.6 we obtain

$$\begin{aligned} w_{k+1} &= 1^{-1}\Psi(\varphi^{k+1}(\langle \lambda, 1\rangle))1 \\ &= 1^{-1}\Psi(\langle 0^{k-1}, 1\rangle \langle \lambda, 0\rangle \langle 0, 0\rangle x)1 \\ &= 1^{-1}10^{2^{k-1}-1}0^{2^{k-1}}010^{2^{k-1}-2}\Psi(x) = 0^{2^k}y \end{aligned} \tag{13}$$

where $x \in \Sigma^*, y \in [2]^*$ are suitable words. □

Lemma 5.1.13 *For $i \geq 1$, w_1 is a primitive word.*

Proof. Denote $\omega = \Psi(\varphi^i(\langle\lambda,1\rangle))$. By Lemma 5.1.6, $w_i = 1^{-1}\omega 1$. Assume that w_i is not primitive. Then neither is ω. Since the length of ω is a positive power of 2, ω is a concatenation of two equal factors, implying $\varphi(b_1) = aa$ for some symbol $a \in \Sigma$ - a contradiction to Proposition 2.3.5. □

Lemma 5.1.14 *For $i \geq k$, w_i is a Lyndon word.*

Proof. By induction. For $i = k$ the assertion follows from Corollary 5.1.11. Let $i > k$. Assume that w_{i-1} is a Lyndon word. By Lemma 5.1.13, w_i is not primitive. Assume that w_i is not minimal in its conjugacy class. Let $w_i = uv$ where $vu < w_i$ is the smallest conjugate of w_i. The minimality of vu implies that the last symbol of u is 1. The word vu, as a conjugate of ω, is of the form $\xi_0 x_1 x_2 y \xi_0'$, $\xi_0 \neq \lambda$, where $\xi_0'\xi_0 = \Psi(c_0)$, $x_1 = \Psi(c_1)$, $x_2 = \Psi(c_2)$ for some $c_0, c_1, c_2 \in \Sigma$ and $\xi_0'\xi_0 x_1 x_2 y = \Psi(c_0 c_1 c_2 z)$ for some conjugate $c_0 c_1 c_2 z$ of $\varphi^i(\langle\lambda,1\rangle)$. Since 0^{2^k} is a prefix of w_i, 0^{2^k} must be a prefix of vu. The last symbol of v is 1 (the last symbol of w_i) therefore 0^{2^k} is a prefix of v. Hence $\xi_0 x_1 \in 0^*$, and, consequently, $x_1 = 0^{2^{k-1}}$ and $c_1 = \langle\lambda,0\rangle$.

Property 5.1.4 implies that $c_0 \neq \langle\lambda,0\rangle$, $c_2 \neq \langle\lambda,0\rangle$, otherwise $\langle\lambda,1\rangle$ occurs in two neighbor positions. As a consequence of Property 5.1.3, $x_1 x_2$ has the prefix $0^{2^{k-1}+1}1$ or $0^{2^{k-1}}1$. Therefore ξ_0 has the suffix $0^{2^{k-1}-1}$ and, again by Property 5.1.3, $c_0 = \langle 0^{k-1},1\rangle$ and $\xi_0' = 1$. We have

$$\Psi(c_0 c_1 c_2 z) = \xi_0'\xi_0 x_1 x_2 y = 1vu1^{-1} < 1w_i 1^{-1} = \Psi(\varphi^i(\langle\lambda,1\rangle)) \tag{14}$$

yielding $c_0 c_1 c_2 z < \varphi^i(\langle\lambda,1\rangle)$.

The word $c_0 c_1 c_2 z = \langle 0^{k-1},1\rangle\langle\lambda,0\rangle c_2 z$ is a conjugate of $\varphi^i(\langle\lambda,1\rangle)$, hence either $\langle 0^{k-1},1\rangle\langle\lambda,0\rangle c_2 z$ or $\langle\lambda,0\rangle c_2 z\langle 0^{k-1},1\rangle$ is an image in φ of some word from Σ^*, the latter case contradicting Property 5.1.4.

Thus $\langle 0^{k-1},1\rangle\langle\lambda,0\rangle c_2 z = \varphi(z')$ for some conjugate z' of $\varphi^{i-1}(\langle\lambda,1\rangle)$ where the initial symbol of z' is either $\langle 0^{k-1},1\rangle$ or $\langle 0^{k-2},1\rangle$, in both cases the first symbol of $\Psi(z')$ is 1. Since $\varphi(z') < \varphi^i(\langle\lambda,1\rangle)$, we have $z' < \varphi^{i-1}(\langle\lambda,1\rangle)$ and $\Psi(z') < \Psi(\varphi^{i-1}(\langle\lambda,1\rangle))$ implying

$$1^{-1}\Psi(z')1 < 1^{-1}\Psi(\varphi^{i-1}(\langle\lambda,1\rangle))1 = w_{i-1} \tag{15}$$

where the left-hand side is a conjugate of w_{i-1} - a contradiction to the inductive hypothesis that w_{i-1} is a Lyndon word. □

Theorem 4.1.1 now follows from Lemma 5.1.7, Lemma 5.1.8, Observation 5.1.9, Corollary 5.1.11 and Lemma 5.1.14. □

5.2 Case $w = 10^{k-2}1$

Let $w = 10^{k-2}1$, $k \geq 3$. In this case $s = 0$, $\Psi = \psi \circ \varphi^{k-1}$ and each element of Σ is of the form $\langle 10^r, m\rangle$ for some $r \in [k-1]$, $m \in [2]$. The initial state of $\mathfrak{A}$ is $\langle\lambda,0\rangle$.

The first of the following two properties may be observed applying Remark 2.1.2. The second one follows from the fact that the only proper prefix of w finished by 1 is 1.

Property 5.2.1 *for $r \in [k-1]$,*

$$\psi(\varphi^r(\langle\lambda,0\rangle)) = \psi(\varphi^r(\langle 1,0\rangle)) = 0^{2^r},$$

and

$$\Psi(\langle\lambda,0\rangle) = 0^{2^{k-1}},$$
$$\Psi(\langle 10^r,0\rangle) = 0^{2^r}1^{2^r}0^{2^{k-1}-2^{r+1}}.$$

Property 5.2.2 *If a symbol $a \in \Sigma$ occurs in an odd position of a word of the form $\varphi(x)$, $x \in \Sigma^*$, then $a = \langle 1,0\rangle$ or $a = \langle 1,1\rangle$.*

Proof of Theorem 4.2.1

Part A

Lemma 5.2.3 *w_0w_1 is a prefix of t_w and $w_1 = 0^{2^{k-1}-2}\Psi(\varphi^2(\langle 1,0\rangle))(0^{2^{k-1}-2})^{-1}$.*

Proof. The sequence t_w has the prefix

$$\Psi(\varphi^3(\langle\lambda,0\rangle)) = \begin{cases} \Psi(\langle\lambda,0\rangle\langle 1,0\rangle\langle 10,0\rangle\langle 1,0\rangle\langle\lambda,0\rangle\langle 1,1\rangle\langle 10,0\rangle\langle 1,0\rangle) & \text{if } k=3; \\ \Psi(\langle\lambda,0\rangle\langle 1,0\rangle\langle 10,0\rangle\langle 1,0\rangle\langle 100,0\rangle\langle 1,0\rangle\langle 10,0\rangle\langle 1,0\rangle) & \text{if } k>3. \end{cases} \tag{16}$$

In both cases Property 5.2.1 implies

$$\Psi(\varphi^2(\langle\lambda,0\rangle\langle 1,0\rangle)) = w_0w_10^{2^{k-1}-2} \quad \text{and} \tag{17}$$

$$(0^{2^{k-1}-2})^{-1}w_10^{2^{k-1}-2} = \Psi(\varphi^2(\langle 1,0\rangle)) \tag{18}$$

since the latter word is a factor of t_w of the length 2^{k+1} starting at position 2^{k+1}. □

Lemma 5.2.4 *For $i \geq 1$, $0^{2^{k-1}-2}$ is a suffix of $\mu((0^{2^{k-1}-2})^{-1}w_i0^{2^{k-1}-2})$ and*

$$w_i = 0^{2^{k-1}-2}\Psi(\varphi^{i+1}(\langle 1,0\rangle))(0^{2^{k-1}-2})^{-1}.$$

Proof. Theorem 2.3.2 implies that the string $\mu(x0^{2^{k-1}-2})$, where $x0^{2^{k-1}-2}$ is a string of a length, which is a multiple of 2^{k-1}, ends in a string of length 2^k-4 containing 0 in all positions i except those where $i_{[2],\lceil\log i\rceil}$ ends in w. This is not true for the last $2^{k-1}-2$ position numbers, therefore 0 occurs in these positions. The proof of the latter part of the assertion of the lemma is a simple induction where Lemma 5.2.3 provides the basic step and Property 3.2.1 is applied in the inductive step. □

Lemma 5.2.5 *$t_w = \prod_{i=0}^{\infty} w_i$.*

Proof. Since t_w is a fixed point of μ, the image in μ of the prefix $w_00^{2^{k-1}-2}$ of length 2^{k+1} of t_w is the prefix $w_0w_10^{2^{k-1}-2}$of length 2^{k+2} of t_w. Applying Lemma 5.2.4 and Property 3.2.1 we obtain

$$\begin{aligned} \mu(\textstyle\prod_{i=0}^{\infty} w_i) &= \mu(w_00^{2^{k-1}-2}\textstyle\prod_{i=1}^{\infty}(0^{2^{k-1}-2})^{-1}w_i0^{2^{k-1}-2}) \\ &= \mu(w_00^{2^{k-1}-2}\textstyle\prod_{i=1}^{\infty}\Psi(\varphi^{i+1}(\langle 1,0\rangle))) \\ &= \mu(w_00^{2^{k-1}-2})\textstyle\prod_{i=1}^{\infty}\mu(\Psi(\varphi^{i+1}(\langle 1,0\rangle))) \\ &= w_0w_10^{2^{k-1}-2}\textstyle\prod_{i=1}^{\infty}\Psi(\varphi^{i+2}(\langle 1,0\rangle)) \\ &= (w_0w_10^{2^{k-1}-2})\textstyle\prod_{i=1}^{\infty}(0^{2^{k-1}-2})^{-1}w_{i+1}0^{2^{k-1}-2}) \\ &= \textstyle\prod_{i=0}^{\infty} w_i \end{aligned} \tag{19}$$

Both sequences t_w and $\prod_{i=0}^{\infty} w_i$ are fixed points of μ starting by the same prefix of length 2^{k-1}, therefore $t_w = \prod_{i=0}^{\infty} w_i$. □

Part B

Lemma 5.2.6 *For $i \geq 0$, $w_i > w_{i+1}$.*

Proof. Clearly, $w_0 > w_1$. Let $i \geq 1$. Property 5.2.1 implies that

$$\Psi(\langle 1,0\rangle) > \Psi(\langle 10,0\rangle),$$

therefore

$$\langle 1,0\rangle > \langle 10,0\rangle\langle 1,0\rangle = \varphi(\langle 1,0\rangle),$$

and Lemma 5.2.4 implies

$$w_i = 0^{2^{k-1}-2}\Psi(\varphi^{i+1}(\langle 1,0\rangle))(0^{2^{k-1}-2})^{-1} > 0^{2^{k-1}-2}\Psi(\varphi^{i+2}(\langle 1,0\rangle))(0^{2^{k-1}-2})^{-1} = w_{i+1}.$$

□

Part C

Lemma 5.2.7 *For $i \in [k-1]$, w_i is a Lyndon word.*

Proof. Applying Property 5.2.1, for $2 \leq i \leq k-2$, we obtain

$$\begin{aligned}
(0^{2^{k-1}-2})^{-1}w_i0^{2^{k-1}-2} &= \Psi(\varphi^{i+1}(\langle 1,0\rangle)) \\
&= \psi(\varphi^{k+i}(\langle 1,0\rangle)) \\
&= \psi(\varphi^{i+2}(\langle 10^{k-2},0\rangle\langle 1,0\rangle\varphi(\langle 1,0\rangle)\cdots\varphi^{k-3}(\langle 1,0\rangle))) \\
&= \psi(\varphi^{i+1}(\langle\lambda,0\rangle\langle 1,1\rangle\varphi(\langle 1,0\rangle)\cdots\varphi^{k-2}(\langle 1,0\rangle))).
\end{aligned} \tag{20}$$

If $i < k-2$ then

$$\begin{aligned}
(0^{2^{k-1}-2})^{-1}w_i0^{2^{k-1}-2} &= 0^{2^{i+1}}1^{2^{i+1}}\psi(\varphi^{i+2}(\langle 1,0\rangle)\cdots\varphi^{k+i-1}(\langle 1,0\rangle)) \\
&= 0^{2^{i+1}}1^{2^{i+1}}xyz
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
x &= \psi(\varphi^{i+2}(\langle 1,0\rangle)\cdots\varphi^{k-2}(\langle 1,0\rangle)) \\
&= 0^{2^{i+2}+\cdots+2^{k-2}} \\
&= 0^{2^{k-1}-2^{i+2}} \\
y &= \psi(\varphi^{k-1}(\langle 1,0\rangle)\varphi^{k}(\langle 1,0\rangle)) \\
&= \Psi(\langle 1,0\rangle\langle 10,0\rangle\langle 1,0\rangle) \\
&= (010^{2^{k-1}-2})(0^21^20^{2^{k-1}})(010^{2^{k-1}-2}) \\
z &= \psi(\varphi^{k+1}(\langle 1,0\rangle)\cdots\varphi^{k+i-1}(\langle 1,0\rangle)) \\
&= (0^{2^{k-1}-2})^{-1}w_1\cdots w_{i-1}0^{2^{k-1}-2}
\end{aligned} \tag{22}$$

and, consequently, $w_i = 0^{2^{k-1}+2^{i+1}-2}1^{2^{i+1}}0^{2^{k-1}}10^{2^{k-1}}1^20^{2^{k-1}+1}1w_1\cdots w_{i-1}$.
If $i = k-2$ then

$$\begin{aligned}
(0^{2^{k-1}-2})^{-1}w_i0^{2^{k-1}-2} &= \Psi(\langle\lambda,0\rangle\langle 1,1\rangle\varphi(\langle 1,0\rangle)\varphi^2(\langle 1,0\rangle)\cdots\varphi^{k-2}(\langle 1,0\rangle)) \\
&= xy
\end{aligned} \tag{23}$$

where

$$
\begin{array}{rcl}
x & = & \Psi(\langle\lambda,0\rangle\,\langle 1,1\rangle\,\Psi(\varphi(\langle 1,0\rangle))) \\
 & = & (0^{2^{k-1}})(101^{2^{k-1}-2})(0^2 1^2 0^{2^{k-1}-4} 010^{2^{k-1}-2}) \\
y & = & \Psi(\varphi^2(\langle 1,0\rangle))\cdots\Psi(\varphi^{k-2}(\langle 1,0\rangle)) \\
 & = & (0^{2^{k-1}-2})^{-1} w_1 \cdots w_{i-1} 0^{2^{k-1}-2}
\end{array}
\tag{24}
$$

and, consequently, $w_i = 0^{2^k-2}101^{2^{k-1}-2}0^2 1^2 0^{2^{k-1}-3}1 w_1\cdots w_{i-1}$.

In both cases, denote as α the prefix of w_i satisfying $w_i = \alpha w_1 \cdots w_{i-1}$. Let us now prove the assertion of the lemma by induction. Each of the words w_0, w_1 is smaller than any of its non-empty proper suffixes, therefore they are Lyndon words. Let us now assume that $w_0, \ldots, w_{i-1}$, $2 \leq i \leq k-2$, are Lyndon words. If $\beta \neq \lambda$ is a proper suffix of α then $0^{2^k+2^{i+1}-2}$ is not a factor of β, hence $w_i < \beta < \beta w_1 \cdots w_{i-1}$. If $\beta \neq \lambda$ is a proper suffix of some w_j, $j \in [i]$, then Lemma 5.4.7 implies $w_i < w_j < \beta < \beta w_1 \cdots w_{i-1}$ since w_j is a Lyndon word. Moreover, $w_i < w_j < w_j \cdots w_{i-1}$ for $j \in [i]$. Therefore w_i is a Lyndon word, since it is smaller than any of its non-empty proper suffixes. □

Lemma 5.2.8 *If $\langle\lambda,0\rangle x \in \Sigma^*$ is a conjugate of a word $\varphi^2(y)$ for some $y \in \Sigma^*$ then $\langle\lambda,0\rangle x = \varphi^2(y')$ for some conjugate y' of y and the last symbol of $\varphi(y')$ is $\langle 1,0\rangle$.*

Proof. Assume the smallest number r, $0 \leq r \leq 2$, such that $\langle\lambda,0\rangle x = \varphi^{2-r}(z)$ for some conjugate z of $\varphi^r(y)$. If $r > 0$ then, as implied by Property 5.2.2, z starts either by $\langle 1,0\rangle$ or by $\langle 1,1\rangle$. Consequently, $\langle\lambda,0\rangle x = \varphi^{2-r}(z)$ starts by $\left\langle 10^{2-r},0\right\rangle$ or by $\left\langle 10^{2-r},1\right\rangle$ - a contradiction. The second assertion of the lemma follows from the observation, that if, for some $a \in \Sigma$, $\varphi(a)$ ends in $\langle 1,0\rangle$ then either $a = \left\langle 10^{k-2},1\right\rangle$ or $a = \langle 1,0\rangle$. As follows from Property 5.2.2, the former case is not possible if a is the last symbol of $\varphi(y')$. □

Lemma 5.2.9 *For $i \geq 1$, w_1 is a primitive word.*

Proof. Analogous to the proof of Lemma 5.1.13. □

Lemma 5.2.10 *For $i \geq k-2$, w_i is a Lyndon word.*

Proof. We proceed by induction. For $i = k-2$ the assertion follows from Lemma 5.2.7.

Let $i \geq k-1$ and let w_{i-1} be a Lyndon word. Lemma 5.2.9 states that w_i is not primitive. Assume that w_i is not minimal in its conjugacy class. Let $w_i = uv$ where $vu < w_i$ is the smallest conjugate of w_i (and of ω). The minimality of vu implies that the last symbol of u is 1. The word vu, as a conjugate of w_i, is of the form $\xi_0 x_1 x_2 y \xi_0'$, $\xi_0 \neq \lambda$, where $\xi_0'\xi_0 = \Psi(c_0)$, $x_1 = \Psi(c_1)$, $x_2 = \Psi(c_2)$ for some $c_0, c_1, c_2 \in \Sigma$ and $\xi_0'\xi_0 x_1 x_2 y = \Psi(c_0 c_1 c_2 z)$ for some conjugate $c_0 c_1 c_2 z$ of $\varphi^{i+1}(\langle 1,0\rangle)$. The prefix of $(0^{2^{k-1}-2})^{-1} w_i$ is $\Psi(\varphi^{i+1-(k-1)}(\langle\lambda,0\rangle))$ being a prefix of length $2^{i+1} \geq 2^k$ of t_w. Therefore w_i has the prefix $0^{2^{k-1}-2}0^{2^{k-1}+1}1 = 0^{2^k-1}1$. This must be a prefix of vu, as well.

Either $\xi_0 x_1 \in 0^*$ or $x_1 = 0^{2^{k-1}-1}1$, the latter case is not possible due to Property 5.2.1, hence $x_1 = 0^{2^{k-1}}$ and $c_1 = \langle\lambda,0\rangle$. Property 5.2.2 implies that $c_0 \neq \langle\lambda,0\rangle$, and, again using Property 5.2.1, $c_0 = \langle 1,0\rangle$, $\xi_0 = 0^{2^{k-1}-2}$, $\xi_0' = 01$, and $c_2 = \langle 1,0\rangle$.

Following Lemma 5.2.8, $c_1 c_2 z c_0 = \varphi^2(y')$ for some conjugate $y'$of $\varphi^{i-1}(\langle 1,0\rangle)$ where the last symbol of $\varphi^i(y')$ is $\langle 1,0\rangle$. We have

$$
\Psi(c_1 c_2 z c_0) = x_1 x_2 y \xi_0' \xi_0 = \xi_0^{-1} v u \xi_0 < \xi_0^{-1} w_i \xi_0 = \Psi(\varphi^{i+1}(\langle 1,0\rangle))
$$

yielding $\varphi^2(y') = c_1c_2zc_0 < \varphi^{i+1}(\langle 1,0\rangle)$ and $\varphi(y') < \varphi^i(\langle 1,0\rangle)$. Since the suffix $\langle 1,0\rangle$ is preserved by φ, the strings on both sides of the last inequality are finished by $\langle 1,0\rangle$, therefore $\Psi(\varphi(y')) < \Psi(\varphi^i(\langle 1,0\rangle))$ and the strings on both sides of the last inequality are finished by $\Psi(\langle 1,0\rangle) = 010^{2^{k-1}-2}$.

Hence

$$0^{2^{k-1}-2}\Psi(\varphi(y'))(0^{2^{k-1}-2})^{-1} < 0^{2^{k-1}-2}\Psi(\varphi^i(\langle 1,0\rangle))(0^{2^{k-1}-2})^{-1} = w_{i-1}$$

- a contradiction, since the left-hand side is a conjugate of w_{i-1} and w_{i-1} is a Lyndon word. □

Theorem 4.1.1 now follows from Lemma 5.2.5, Lemma 5.2.6, Lemma 5.2.7 and Lemma 5.2.10.

5.3 Case $w = 1^k$.

Let $w = 1^k$, $k \geq 2$. In this case $s = 0$, $\Psi = \psi \circ \varphi^{k-1}$ and each element of Σ is of the form $\langle 1^r, m\rangle$ for some $r \in [k]$, $m \in [2]$. The initial state of $\mathfrak{A}$ is $a_0 = \langle \lambda, 0\rangle$.

The first of the following two properties follows from Remark 2.1.2. The second is based on the fact that the last symbol of each non-empty prefix of w is 1.

Property 5.3.1 *for $r \in [k-1]$,*

$$\psi(\varphi^r(\langle \lambda,0\rangle)) = \psi(\varphi^r(\langle 1,0\rangle)) = 0^{2^r},$$

and

$$\Psi(\langle \lambda,0\rangle) = 0^{2^{k-1}},$$
$$\Psi(\langle 1,0\rangle) = 0^{2^{k-1}-1}1,$$

and, for $k \geq 3$, $r \geq 2$,

$$\Psi(\langle 1^r,0\rangle) = 0^{2^{k-1}-2^{r-1}}1^{2^{r-2}}0^{2^{r-3}}1^{2^{r-4}}0^{2^{r-5}}\cdots d^{2^0}(1-d)$$

where $d = 0$ if r is odd and $d = 1$ if r is even.

Property 5.3.2 *If a symbol $a \in \Sigma$ occurs in an even position of a word of the form $\varphi(x)$, $x \in \Sigma^*$, then $a = \langle \lambda, 0\rangle$ or $a = \langle \lambda, 1\rangle$. No symbol from Σ can occur both in an even and in an odd position of $\varphi(x)$.*

Proof of Theorem 4.3.1

Part A

Lemma 5.3.3 *w_0w_1 is a prefix of t_w, $w_1 = 0\Psi(\varphi^2(\langle 1,0\rangle)\varphi^3(\langle 1,0\rangle))0^{-1}$ and $0^{2^k}1$ is a prefix of w_1.*

Proof. Property 5.3.1 implies that

$$w_0 = \Psi(\langle \lambda,0\rangle\langle 1,0\rangle\langle \lambda,0\rangle\langle 1,1\rangle)0^{-1}i \quad \text{if } k = 2$$

and

$$w_0 = \Psi(\langle \lambda,0\rangle\langle 1,0\rangle\langle \lambda,0\rangle\langle 11,0\rangle)0^{-1} \quad \text{if } k \geq 3.$$

In both cases, $w_0 = .\Psi(\varphi^2(\langle\lambda,0\rangle))0^{-1}$. The sequence t_w has the prefix $\Psi(\varphi^2(\langle\lambda,0\rangle)) = w_0 0$. Using Property 3.2.1 we obtain

$$\begin{aligned}
w_0 w_1 0 &= w_0[(w_0)_0^{-1}\mu^2(w_0 0)] \\
&= \mu^2(w_0 0) \\
&= \Psi(\varphi^4(\langle\lambda,0\rangle))) \\
&= \Psi(\varphi^2(\langle\lambda,0\rangle\langle1,0\rangle))\Psi(\varphi^3(\langle1,0\rangle)) \\
&= w_0 0\Psi(\varphi^2\langle1,0\rangle))\Psi(\varphi^3(\langle1,0\rangle)).
\end{aligned} \tag{25}$$

Thus $w_0 w_1 is$ a prefix of t_w,

$$w_1 = 0\Psi(\varphi^2(\langle1,0\rangle\,\varphi^3(\langle1,0\rangle)0^{-1}$$

and w_1 has the prefix

$$0\Psi(\langle\lambda,0\rangle\langle1,0\rangle) = 00^{2^{k-1}}0^{2^{k-1}-1}1 = 0^{2^k}1,$$

as implied by Property 5.3.1. □

Lemma 5.3.4 *For $i \geq 1$,*

$$w_i = 0\Psi(\varphi^{2i}(\langle1,0\rangle)\varphi^{2i+1}(\langle1,0\rangle))0^{-1} \tag{26}$$

and the last symbol of the word $\mu^2(0^{-1}w_{i-1}0)$ is 0.

Proof. Theorem 2.3.2 implies, that the last symbol of $\mu(x)$ is complementary to the last symbol of x. Therefore μ^2 preserves the last symbol of its argument.
The assertion of the lemma is true for $i = 1$ as follows from Lemma 5.3.3.
Now let the assertion be true for some $i-1 \geq 1$. Then

$$\begin{aligned}
w_i &= 0\mu^2(0^{-1}w_{i-1}0)0^{-1} \\
&= 0\mu^2(0^{-1}(0\Psi(\varphi^{2i-2}(\langle1,0\rangle)\varphi^{2i-1}(\langle1,0\rangle))0^{-1})0)0^{-1} \\
&= 0\Psi(\varphi^2(\varphi^{2i-2}(\langle1,0\rangle)\varphi^{2i-1}(\langle1,0\rangle)))0^{-1} \\
&= 0(\Psi(\varphi^{2i}(\langle1,0\rangle)\varphi^{2i+1}(\langle1,0\rangle)))0^{-1}.
\end{aligned} \tag{27}$$

□

Lemma 5.3.5 $t_w = \prod_{i=0}^{\infty} w_i$

Proof. It is enough to prove that the sequence $\prod_{i=0}^{\infty} w_i$, having the same prefix $w_0 0$ of length 2^{k+1} as t_w, is a fixed point of μ^2. Applying Lemma 5.3.3 we obtain

$$\begin{aligned}
\mu^2(\prod_{i=0}^{\infty} w_i) &= \mu^2(w_0 0\prod_{i=1}^{\infty}(0^{-1}w_i 0)) \\
&= w_0(w_0^{-1}\mu^2(w_0 0)\prod_{i=1}^{\infty}(0^{-1}w_{i+1}0) = \prod_{i=0}^{\infty} w_i.
\end{aligned} \tag{28}$$

□

Part B

Lemma 5.3.6 *For $i \geq 0$, $w_i > w_{i+1}$.*

Proof. By Lemma 5.3.3, w_1 has the prefix $0^{2^k}1$, therefore $w_0 > w_1$. Let $i \geq 1$. Property 5.3.1 implies that

$$\Psi(\langle 1,0\rangle) > \Psi(\langle \lambda,0\rangle), \tag{29}$$

therefore

$$\langle 1,0\rangle > \langle \lambda,0\rangle \tag{30}$$

and

$$\langle 1,0\rangle > \varphi^2(\langle 1,0\rangle) \tag{31}$$

since the first symbol of $\varphi^2(\langle 1,0\rangle)$ is $\langle \lambda,0\rangle$. Corollary 3.2.5 and Lemma 5.3.3 imply

$$w_i = 0\Psi(\varphi^{2i}(\langle 1,0\rangle)\varphi^{2i+1}(\langle 1,0\rangle))0^{-1} > 0\Psi(\varphi^{2i+2}(\langle 1,0\rangle)\varphi^{2i+3}(\langle 1,0\rangle))0^{-1} = w_{i+1}. \tag{32}$$

□

Part C

Lemma 5.3.7 *For $i \geq 2$, 0^{2^k} is a prefix of w_i.*

Proof. An easy application of Theorem 2.3.2. □

Lemma 5.3.8 *For $i \geq 1$, w_i is a primitive word.*

Proof. Denote $\omega = 0^{-1}w_1 0 = \Psi(\varphi^{2i}(\langle 1,0\rangle)\varphi^{2i+1}(\langle 1,0\rangle)) = \Psi(\varphi^{2i-1}(\langle \lambda,0\rangle\, a\, \langle \lambda,0\rangle\, \langle 1,0\rangle\, bc))$

$$\text{where } \begin{cases} a = \langle 1,1\rangle,\, b = \langle \lambda,1\rangle,\, c = \langle 1,0\rangle & \text{if } k = 2; \\ a = \langle 11,0\rangle,\, b = \langle \lambda,0\rangle,\, c = \langle 11,1\rangle & \text{if } k = 3; \\ a = \langle 11,0\rangle,\, b = \langle \lambda,0\rangle,\, c = \langle 111,0\rangle & \text{if } k > 3. \end{cases} \tag{33}$$

If w_i is not primitive then neither is ω. The length of ω is $3 \cdot 2^{2i}$, therefore ω is a concatenation of either two or three identical factors. Since both Ψ and φ are strictly growing, this implies $\langle \lambda,0\rangle = \langle 1,0\rangle$ in the former case and $a = \langle 1,0\rangle$ in the latter case. □

Lemma 5.3.9 *For $i \geq 1$, w_i is a Lyndon word.*

w_0 is clearly a Lyndon word, since it is smaller than any of its non-empty proper suffixes. It is easy to check, that w_1 is of the form

$$w_1 = \begin{cases} 0\Psi(x\langle \lambda,0\rangle\langle 1,0\rangle x\langle \lambda,0\rangle\langle 1,1\rangle x\langle \lambda,1\rangle\langle 1,1\rangle)0^{-1} = 0^{-1}0^4 10^3 10^3 10^2 10^4 10^2 1 & \textit{if } k = 2; \\ 0\Psi(x\langle \lambda,0\rangle\langle 1^2,1\rangle x\langle \lambda,0\rangle\langle 1^2,0\rangle x\langle \lambda,1\rangle\langle 1^2,1\rangle)0^{-1} = 0^8 10^4 1^2 010^7 10^6 10^8 1^5 0^2 1 & \textit{if } k = 3; \\ 0\Psi(x\langle \lambda,0\rangle\langle 1^3,0\rangle x\langle \lambda,0\rangle\langle 1^2,0\rangle x\langle \lambda,0\rangle\langle 1^3,1\rangle)0^{-1} = & \\ \qquad 0^{16}10^{12}1^2 010^{15}10^{14}10^{16}10^8 1^4 0^2 1 & \textit{if } k = 4; \\ 0\Psi(x\langle \lambda,0\rangle\langle 1^3,0\rangle x\langle \lambda,0\rangle\langle 1^2,0\rangle x\langle \lambda,0\rangle\langle 1^4,0\rangle)0^{-1} = & \\ \qquad 0^{2^k}10^{2^k-4}1^2 010^{2^k-1}10^{2^k-2}10^{2^k}10^{2^k-8}1^4 0^2 1 & \textit{if } k > 4. \end{cases} \tag{34}$$

In each case, w_i is smaller than any of its non-empty proper suffix.

Let now $i \geq 2$. By Lemma 5.3.4,

$$w_i = 0\Psi(\varphi^{2i}(\langle 1,0\rangle)\varphi^{2i+1}(\langle 1,0\rangle))0^{-1} = 0\Psi(\varphi^{2i-1}(\langle \lambda,0\rangle\langle\langle 1,0\rangle)\varphi^{2i+1}(\langle 1,0\rangle))0^{-1}, \tag{35}$$

therefore $0^{-1}w_i$ has the same prefix $0^{2^k-1}1$ as t_w and w_i has the prefix $0^{2^k}1$. Lemma 5.3.8 implies that w_i is a primitive word. The word w_i is a conjugate of $\Psi(\varphi^{2i}(\langle 1,0\rangle)\varphi^{2i+1}(\langle 1,0\rangle))$. Assume that w_i is not minimal in its conjugacy class. Let $w_i = uv$ where $vu < w_i$ is the smallest conjugate of w_i. The minimality of vu implies that the last symbol of u is 1. The word vu is of the form $\xi_0 x_1 x_2 y \xi_0'$, $\xi_0 \neq \lambda$, where $\xi_0'\xi_0 = \Psi(c_0)$, $x_1 = \Psi(c_1)$, $x_2 = \Psi(c_2)$ for some $c_0, c_1, c_2 \in \Sigma$ and $\xi_0'\xi_0 x_1 x_2 y = \Psi(c_0 c_1 c_2 z)$ for some conjugate $c_0 c_1 c_2 z$ of $\varphi^{2i}(\langle 1,0\rangle)\varphi^{2i+1}(\langle 1,0\rangle)$. Since 0^{2^k} is a prefix of w_i, 0^{2^k} must be a prefix of vu. Therefore $\xi_0 x_1 \in 0^*$, $x_1 = 0^{2^{k-1}}$ and, consequently, $c_1 = \langle \lambda,0\rangle$. Property 5.3.2 then implies that $c_0 \neq \langle \lambda,0\rangle$, $c_2 \neq \langle \lambda,0\rangle$ and from Property 5.3.1 we obtain that $\Psi(c_2)$ does not have the prefix $0^{2^{k-1}}$. Hence $\Psi(c_0)$ has the suffix 10, therefore $\xi_0 = 0$.

This further implies that $x_2 = 0^{2^{k-1}-1}1$ and $c_2 = \langle 1,0\rangle$. Now either $c_0c_1c_2z$ or $c_1c_2zc_0$ is of the form $\varphi(z')$ where z' is a conjugate of $\varphi^{2i-1}(\langle 1,0\rangle)\varphi^{2i}(\langle 1,0\rangle)$.

The former case is not possible, since, by Property 5.3.2, $c_1 = \langle \lambda,0\rangle$ can occur only at even positions.

If $z' = az''$ where $a \in \Sigma$ then $\varphi(a) = c_1c_2 = \langle \lambda,0\rangle\langle\langle 1,0\rangle$, hence $a = \langle \lambda,0\rangle$. Now either $az'' = \varphi(y')$ or $z''a = \varphi(y')$ for some $y' \in \Sigma^*$ being a conjugate of $\varphi^{2i-2}(\langle 1,0\rangle)\varphi^{2i-1}(\langle 1,0\rangle)$. Considering Property 5.3.2 again, a must appear in $\varphi(y')$ at an even position, thus the former case takes place and

$$0^{-1}vu0 = x_1x_2y\xi_0'\xi_0 = \Psi(\varphi(z')) = \Psi(\varphi^2(y')). \tag{36}$$

Moreover, since $a = \langle \lambda,0\rangle$, the first symbol of $\varphi^{k-1}(y')$ is $\langle \lambda,0\rangle$, and the first symbol of $\Psi(y')$ is 0. Since $vu < w_i$,

$$\Psi(\varphi^2(y')) = 0^{-1}vu0 < 0^{-1}w_i0 = \Psi(\varphi^{2i}(\langle 1,0\rangle)\varphi^{2i+1}(\langle 1,0\rangle)). \tag{37}$$

Thus

$$\varphi^2(y') < \varphi^{2i}(\langle 1,0\rangle)\varphi^{2i+1}(\langle 1,0\rangle), \tag{38}$$

$$y' < \varphi^{2i-2}(\langle 1,0\rangle)\varphi^{2i-1}(\langle 1,0\rangle) \tag{39}$$

and

$$0^{-1}\Psi(y')0 < 0^{-1}\Psi(\varphi^{2i-2}(\langle 1,0\rangle)\varphi^{2i-1}(\langle 1,0\rangle))0 = w_{i-1}. \tag{40}$$

However, the left-hand side is a conjugate of the right-hand side and we have a contradiction to the inductive hypothesis that w_{i-1} is a Lyndon word. This concludes the proof of Theorem 4.1.1

Theorem 4.3.1 follows from Lemma 5.3.5, Lemma 5.3.6, and Lemma 5.3.9..

5.4 Case $w = 10^{k-1}$

Let $w = 10^{k-1}$, $k \geq 2$. In this case $s = k-1$, $\Psi = \psi \circ \varphi^{2k-2}$ and each element of Σ is of the form $\langle \lambda, m\rangle$ or $\langle 10^r, m\rangle$ for some $r \in [k-1]$, $m \in [2]$. The initial state of $\mathfrak{A}$ is $a_0 = \langle \lambda,0\rangle$.

The first of the next two properties may be observed applying Remark 2.1.2. The second follows from the fact that the only prefix of w, which ends in 1 is 1.

Property 5.4.1

(i) $\varphi(\langle\lambda,0\rangle) = \langle\lambda,0\rangle\langle 1,0\rangle$, $\varphi(\langle 1,0\rangle) = \langle\lambda,1\rangle\langle 1,0\rangle$,

(ii) $\psi(\varphi^{k-1}(\langle\lambda,0\rangle)) = 0^{2^{k-1}}$ *and, for* $r \in [k-1]$, $\psi(\varphi^{k-1}(\langle 10^r,0\rangle)) = 1^{2^r}0^{2^{k-1}-2^r}$.

Property 5.4.2 *If a symbol* $a \in \Sigma$ *occurs in an odd position of a word of the form* $\varphi(x)$, $x \in \Sigma^*$, *then* $a = \langle 1,0\rangle$ *or* $a = \langle 1,1\rangle$.

Proof of Theorem 4.4.1

Part A

Lemma 5.4.3

(i) $\mu^{k-1}(0^{2^{k-1}})$ *has the suffix* $0^{2^{k-1}-1}$.

(ii) $\rho = \Psi(\varphi^{k-1}(\langle\lambda,1\rangle))\Psi(\langle\lambda,0\rangle)(0^{2^{k-1}-1})^{-1}$.

(iii) For $i \geq 0$, $\Psi(\varphi^{2k-1+i}(\langle 1,0\rangle))$ *has the prefix* $\rho 0^{2^k}1$.

(iv) w_{2k-2} *has the suffix* ρ.

(v) For $i \geq 2k-1$, $w_i = \rho^{-1}\Psi(\varphi^i(\langle 1,0\rangle))\rho$ *and* $\mu(\rho w_i \rho^{-1})$ *has the prefix* ρ.

Proof.

(i) Property 2.3.3 implies

$$\mu^{k-1}(0^{2^{k-1}}) = \mu^{k-1}(\psi(\varphi^{k-1}(\langle\lambda,0\rangle))) = \psi(\varphi^{2k-2}(\langle\lambda,0\rangle)) = \psi(\varphi^{k-1}(\varphi^{k-1}(\langle\lambda,0\rangle))). \tag{41}$$

Using Remark 2.1.2 it is easy to observe that the last symbol of $\varphi^{k-1}(\langle\lambda,0\rangle)$ is $\langle 1,0\rangle$. Hence $\mu^{k-1}(0^{2^{k-1}})$ has the suffix $\psi(\varphi^{k-1}(\langle 1,0\rangle)) = 10^{2^{k-1}}$.

(ii) Again applying Property 2.3.3 we obtain $\mu^{2k-2}(1^{2^{k-1}}) = \Psi(\varphi^{k-1}(\langle\lambda,0\rangle))$ and $\mu^{k-1}(0^{2^{k-1}}) = \Psi(\langle\lambda,0\rangle)$ and, consequently, $\rho = \Psi(\varphi^{k-1}(\langle\lambda,1\rangle))\Psi(\langle\lambda,0\rangle)(0^{2^{k-1}-1})^{-1}$.

(iii) Let $i \geq 0$. Denote $v = \Psi(\varphi^{2k-1+i}(\langle 1,0\rangle)) = \Psi(\varphi^{k-1}(\varphi^i(\varphi(\varphi^{k-1}(\langle 1,0\rangle)))))$. The word $\varphi^{k-1}(\langle 1,0\rangle)$ has the prefix $\langle\lambda,1\rangle$ therefore v has the prefix

$$\Psi(\varphi^{k-1}(\varphi^i(\varphi(\langle\lambda,1\rangle)))) = \Psi(\varphi^{k-1}(\varphi^i(\langle\lambda,1\rangle\langle 1,1\rangle))).$$

Now either

$$i = 0 \text{ and } \varphi^i(\langle\lambda,1\rangle\langle 1,1\rangle) = \langle\lambda,1\rangle\langle 1,1\rangle \tag{42}$$

or

$$i > 0 \text{ and } \varphi^i(\langle\lambda,1\rangle)\text{has the prefix } \langle\lambda,1\rangle\langle 1,1\rangle, \tag{43}$$

in either case v has the prefix $= \Psi(\varphi^{k-1}(\langle\lambda,1\rangle))\Psi(\varphi^{k-1}(\langle 1,1\rangle))$. Thus v has the prefix

$$\Psi(\varphi^{k-1}(\langle\lambda,1\rangle))\Psi(\langle\lambda,0\rangle\langle 1,1\rangle)$$

having the prefix $(\rho 0^{2^{k-1}-1})0^{2^{k-1}+1}1$, since $\varphi^{k-1}(\langle 1,1\rangle)$ starts by $\langle\lambda,0\rangle\langle 1,1\rangle$.

(iv) w_{2k-2} is the factor of t_w of length $j_{2k-1} - j_{2k-2}$ at position j_{2k-2}. The length of ρ is

$$2^{3k-3} + 2^{2k-2} - 2^{k-1} + 1.$$

The suffix of w_{2k-2} of length $|\rho|$ is a factor of t_w at position

$$j_{2k-2} + |w_{2k-1}| - |\rho| = j_{2k-1} - |\rho| = 2^{4k-3}.$$

This is the starting position of the factor $\Psi(\varphi^{2k-1}(\langle 1,0\rangle))$, which by (iii) has the prefix ρ.

(v) We will prove the assertion by induction. Let first $i = 2k-1$. The factor of length $|\rho|$ preceding the factor w_{2k-1} in t_w starts at position $j_{2k-1} - |\rho| = 2^{4k-3}$. This is the starting position of the factor $\Psi(\varphi^{2k-1}(\langle 1,0\rangle))$, which by (iii) has the prefix ρ. Thus the first part of the assertion has been proved in (iii).

Since $|w_{2k-1}| = j_{2k-1} - |\rho| = 2^{4k-3}$, the word $\rho w_{2k-1}\rho^{-1}$ is identical with $\Psi(\varphi^{2k-1}(\langle 1,0\rangle))$ and $w_{2k-1} = \rho^{-1}\Psi(\varphi^{2k-1}(\langle 1,0\rangle))\rho$. Now

$$\mu(\rho w_{2k-1}\rho^{-1}) = \mu(\Psi(\varphi^{2k-1}(\langle 1,0\rangle))) = \Psi(\varphi^{2k}(\langle 1,0\rangle)) \tag{44}$$

having the prefix ρ by (iii).

Assume now that the assertion is true for some $i \geq 2k-1$. Then

$$w_{i+1} = \rho^{-1}\mu(\rho w_i \rho^{-1})\rho = \rho^{-1}\mu(\Psi(\varphi^i(\langle 1,0\rangle)))\rho = \rho^{-1}\Psi(\varphi^{i+1}(\langle 1,0\rangle))\rho \tag{45}$$

and

$$\mu(\rho w_{i+1}\rho^{-1}) = \mu(\Psi(\varphi^{i+1}(\langle 1,0\rangle))) = \Psi(\varphi^{i+2}(\langle 1,0\rangle)) \tag{46}$$

has the prefix ρ according to (iii).

□

Lemma 5.4.4 $t_w = \prod_{i=0}^{\infty} w_i$

Proof. Since t_w is a fixed point of μ with the prefix w_0 of length greater than 2^{k-1}, it is enough to prove that the sequence $\prod_{i=0}^{\infty} w_i$ having the same prefix is a fixed point of μ as well. From the definition of the words w_0, w_{2k-1}, ρ we can observe that $w_0 w_1 \cdots w_{2k-2} w_{2k-1}$ is a prefix of t_w, the length of $w_0 w_1 \cdots w_{2k-2}\rho^{-1}$ is twice the length of $w_0 w_1 \cdots w_{2k-2}\rho^{-1}$ and the length of the latter word is a multiple of 2^{k-1}.We obtain:

$$\begin{aligned}\mu(\textstyle\prod_{i=0}^{\infty} w_i) &= \mu(w_0 w_1 \cdots (w_{2k-2}\rho^{-1})\rho w_{2k-1}\rho^{-1}) \textstyle\prod_{i=2k}^{\infty}(\rho w_i \rho^{-1})) \\ &= \mu(w_0 w_1 \cdots w_{2k-2}\rho^{-1})\mu(\rho w_{2k-1}\rho^{-1}) \textstyle\prod_{i=2k}^{\infty}\mu(\rho w_i \rho^{-1}) \\ &= (w_0 w_1 \cdots w_{2k-2} w_{2k-1}\rho^{-1})\rho w_{2k}\rho^{-1} \textstyle\prod_{i=2k}^{\infty}(\rho w_{i+1}\rho^{-1}) \\ &= \textstyle\prod_{i=0}^{\infty} w_i.\end{aligned} \tag{47}$$

since $|w_0 w_1 \cdots (w_{2k-2}\rho^{-1})| = 2^{4k-3}$ and $|w_0 w_1 \cdots w_{2k-2}(w_{2k-1}\rho^{-1})| = 2^{4k-2}$. □

Part B

Lemma 5.4.5 *For $0 \le i \le 2k-1$ the last symbol in w_i is* 1.

Proof. The last symbol of w_i occurs at position $j_{i+1}-1$ in t_w. The binary representation of each $j_{i+1}-1$ contains an odd number of occurrences of w. □

Lemma 5.4.6

w_0 has the prefix $0^{2^{k-1}}1$,

w_1, w_2, have the prefix $0^{2^{k-1}+1}1$, for $1 \le r \le k-2$,

w_{2r+1} and w_{2r+2} have the prefix $0^{2^{k-1}-1+2^r}1$ and

w_{2k-1} has the prefix $0^{2^k}1$.

Proof. A simple observation of the shape of binary representation of the position numbers following j_i, $0 \le i \le 2k$. □

Lemma 5.4.7 *For $0 \le i \le 2k-2$, $w_i > w_{i+1}$.*

Proof. Lemma 5.4.6 implies that $w_0 > w_1$ and, for $2 \le r \le k-1$, $w_{2r} > w_{2r+1}$.

Let $0 \le r \le k-2$. The word w_{2r+1} occurs as a factor of length 2^{3k-2+r} in t_w at position

$$j_{2r+1} = 2^{3k-2+r} + 2^{2k-2+r} + 2^{k-1+r} - 2^{k-1} + d$$

where $d = 0$ if $r = 0$ and $d = 1$ otherwise.

Therefore the suffix of
$$v = \psi(\varphi^{3k-2+r}(\langle 1,0\rangle)) = \Psi(\varphi^r(\varphi^k(\langle 1,0\rangle)))$$
at position
$$2^{2k-2+r} + 2^{k-1+r} - 2^{k-1} + 1$$
is a prefix of w_{2r+1}.

Since $\varphi^k(\langle 1,0\rangle)$ starts by $\langle \lambda,1\rangle\langle 1,1\rangle b$ where $b = \langle 10,0\rangle$ if $k \ge 3$ and $b = \langle \lambda,1\rangle$ if $k = 2$, the suffix of $\Psi(\varphi^r(\langle 1,1\rangle b))$ at position $2^{k-1+r} - 2^{k-1} + d$ is a prefix of w_{2r+1}.

In a similar way, the suffix of

$$v = \psi(\varphi^{3k-2+r+1}(\langle 1,0\rangle)) = \Psi(\varphi^r(\varphi^{k+1}(\langle 1,0\rangle)))$$

at position $2^{2k-2+r} + 2^{k-1+r} - 2^{k-1} + d$ is a prefix of w_{2r+2}.

Since $\varphi^{k+1}(\langle 1,0\rangle)$ starts by $\langle \lambda,1\rangle\langle 1,1\rangle c$ where $c = \langle 10,1\rangle$ if $k \ge 3$ and $c = \langle \lambda,0\rangle$ if $k = 2$, the suffix of $\Psi(\varphi^r(\langle 1,1\rangle c))$ at position $2^{k-1+r} - 2^{k-1} + d$ is a prefix of w_{2r+2}. In any case, $\Psi(b)$ starts by 1 and $\Psi(c)$ starts by 0, hence $w_{2r+1} > w_{2r+2}$.

We still have to prove that $w_2 > w_3$ for $k \ge 3$.

Since $\varphi^{k-1}(\langle 1,1\rangle)$ has the prefix $\langle \lambda,0\rangle\langle 1,1\rangle\langle 10,1\rangle\langle 1,1\rangle$, the suffix of $\Psi(\langle 1,1\rangle\langle 10,1\rangle)$ at position $2^{k-1+0} - 2^{k-1} + 0 = 0$ has the prefix

$$(0^{2^{k-1}})(01^{2^{k-1}-1})(0^2 1^{2^{k-1}-2})(01^{2^{k-1}-1})$$

and w_2 has the prefix $0^{2^{k-1}+1}1^{2^{k-1}-1}$.

The word $\varphi(\langle 1,1\rangle\langle 10,0\rangle)$ has the prefix $\langle 10,1\rangle\langle 1,1\rangle$ and $\varphi^{k-1}(\langle 10,1\rangle\langle 1,1\rangle)$ has the prefix

$$\langle\lambda,0\rangle\langle 1,0\rangle\langle 10,1\rangle\langle 1,1\rangle.$$

Therefore the suffix of

$$\Psi(\varphi(\langle 1,1\rangle\langle 10,0\rangle))$$

at position

$$2^{k-1+1}-2^{k-1}+1=2^{k-1}+1$$

has the prefix

$$(0^{2^{k-1}})(10^{2^{k-1}-1})(0^2 1^{2^{k-1}-2})(01^{2^{k-1}-1})$$

and w_3 has the prefix

$$0^{2^{k-1}+1}1^{2^{k-1}-2}01^{2^{k-1}-1}.$$

Clearly, $w_2 > w_3$. □

Lemma 5.4.8 *For $i \geq 2k-1$, $w_i > w_{i+1}$*

Proof. Using Remark 2.1.2, it is not difficult to observe that $\Psi(\langle 1,0\rangle)$ has the prefix $1^{2^{k-1}+1}$ while $\Psi(\langle\lambda,1\rangle)$ has the prefix $1^{2^{k-1}}0$. Hence $\Psi(\langle 1,0\rangle) > \Psi(\langle\lambda,1\rangle)$ and $\langle 1,0\rangle > \langle\lambda,1\rangle$. Property 5.4.1 implies that $\langle 1,0\rangle > \varphi(\langle 1,0\rangle)$ and v of Lemma 5.4.3 implies $w_i > w_{i+1}$. □

Part C

Lemma 5.4.9 *Let $\tau=\lim_{n\to\infty}\varphi^n(\langle\lambda,0\rangle)$.*

(i) The first two occurrences of the factor $\langle\lambda,0\rangle$ in τ are at positions $\lfloor j_0/2^{k-1}\rfloor$ and $\lfloor j_1/2^{k-1}\rfloor$.

(ii) The first two occurrences of the factor $\langle\lambda,0\rangle\langle 1,1\rangle$ in τ are at positions $\lfloor j_1/2^{k-1}\rfloor$ and $\lfloor j_2/2^{k-1}\rfloor$.

(iii) For $1\leq r\leq k-2$, the first two occurrences of the factor $\langle 1,0\rangle\langle 10^r,1\rangle$ are at positions $\lfloor j_{2r+1}/2^{k-1}\rfloor$ and $\lfloor j_{2r+2}/2^{k-1}\rfloor$.

(iv) The first two occurrences of the factor $\langle 1,0\rangle\langle\lambda,0\rangle\langle 1,1\rangle$ in τ are at positions $\lfloor j_{2k-1}/2^{k-1}\rfloor$ and $\lfloor j_{2k}/2^{k-1}\rfloor$.

(v) The first occurrence of the factor $\langle 1,0\rangle\langle\lambda,0\rangle\langle 1,0\rangle$ in τ is at a position greater than $\lfloor j_{2k-1}/2^{k-1}\rfloor$.

Proof.

(i) The first occurrence of $\langle\lambda,0\rangle$ after position $0=\lfloor j_0/2^{k-1}\rfloor$ is at the position $(10^{k-1}10^{k-1})_{[2]}=\lfloor j_1/2^{k-1}\rfloor$.

(ii) If the factor $\langle\lambda,0\rangle\langle 1,1\rangle$ occurs in τ at (an even) position j then the symbol at position $j/2$ is $\langle 10^{k-2},1\rangle$. This must be a prefix of a factor $\varphi^{k-2}(\langle 1,1\rangle)$. Since the first two occurrences of $\langle 1,1\rangle$ in τ are at positions $(10^{k-1}1)_{[2]}=2^k+1$ and $(10^k1)_{[2]}=2^{k+1}+1$, the first two occurrences of the factor $\langle\lambda,0\rangle\langle 1,1\rangle$ in τ are at positions $(2^k+1).2^{k-2}.2=2^{2k-1}+2^{k-1}=\lfloor j_1/2^{k-1}\rfloor$ and $(2^{k+1}+1).2^{k-2}.2=2^{2k}+2^{k-1}=\lfloor j_2/2^{k-1}\rfloor$.

(iii) If the factor $\langle 1,0\rangle\langle 10^r,1\rangle$ occurs in τ then $\langle 10^r,1\rangle$ is the prefix of a factor $\varphi^r(\langle 1,1\rangle)$. In that case $\langle 1,0\rangle$ is a suffix of some $\varphi^r(b)$, $b\in\Sigma$. Then $b=\langle\alpha,0\rangle$ where $\alpha\neq 1$, since $b\langle 1,1\rangle$ is a factor of τ and $\langle 1,1\rangle$ must be in an odd position. Then $b\langle 1,1\rangle=\varphi(c)$ for some $c\in\Sigma$. This is only possible if $c=\langle 10^{k-2},1\rangle$ and c is a prefix of a factor $\varphi^{k-2}(\langle 1,1\rangle)$. Since the first two occurrences of $\langle 1,1\rangle$ in τ are at positions $(10^{k-1}1)_{[2]}=2^k+1$ and $(10^k1)_{[2]}=2^{k+1}+1$, the first two occurrences of the factor $\langle 1,0\rangle\langle 10^r,1\rangle$ in τ are at positions

$$(2^k+1).2^{k-2}.2.2^r+2^r-1=2^{2k-1+r}+2^{k-1+r}+2^r-1=\lfloor j_{2r+1}/2^{k-1}\rfloor \tag{48}$$

and

$$(2^{k+1}+1).2^{k-2}.2.2^r+2^r-1=2^{2k+r}+2^{k-1+r}+2^r-1=\lfloor j_{2r+2}/2^{k-1}\rfloor. \tag{49}$$

(iv) Let the factor $\langle 1,0\rangle\langle\lambda,0\rangle\langle 1,1\rangle$ occur in τ at (an odd) position j. Then the factor of length 2 at position $\lfloor j/2\rfloor$ is $\langle 1,0\rangle\langle 10^{k-2},1\rangle$ and in the same way (taking $r=k-2$) as in the proof of iii we deduce that $\lfloor j/2\rfloor$corresponds to the position $2(2^{k-2}-1)+1$ of a factor $\varphi^{1+(k-2)+1+(k-2)}(\langle 1,1\rangle)$, hence the first two occurrences of the factor $\langle 1,0\rangle\langle\lambda,0\rangle\langle 1,1\rangle$ in τ are at positions

$$(2^k+1).2.2^{k-2}.2.2^{k-2}+2^{k-1}-1=2^{3k-2}+2^{2k-2}+2^{k-1}-1=\lfloor j_{2k+1}/2^{k-1}\rfloor \tag{50}$$

and

$$(2^{k+1}+1).2.2^{k-2}.2.2^{k-2}+2^{k-1}-1=2^{3k-1}+2^{2k-2}+2^{k-1}-1=\lfloor j_{2k+2}/2^{k-1}\rfloor. \tag{51}$$

(v) Let the factor $\langle 1,0\rangle\langle\lambda,0\rangle\langle 1,0\rangle$ occur in τ for the first time at (an odd) position j. Then the factor of length 3 at position $\lfloor j/2\rfloor$ is either $\langle 1,0\rangle\langle\lambda,0\rangle\langle 1,0\rangle$ or $\langle 1,0\rangle\langle\lambda,0\rangle\langle 1,1\rangle$. The former case would mean an earlier occurrence of the factor, hence the latter case take place. The factor $\langle 1,0\rangle\langle\lambda,0\rangle\langle 1,0\rangle$ hence occurs later than $\langle 1,0\rangle\langle\lambda,0\rangle\langle 1,1\rangle$.

□

Lemma 5.4.10 *For $0\leq i\leq 2k-1$, w_i is a Lyndon word.*

Proof. The binary representation of each $j_{i+1}-1$ contains an odd number of occurrences of 10^{k-1}, and therefore the last symbol of each w_i is 1. To prove that w_i is a Lyndon word, we will show that w_i besides in its prefix 0^p1 (as described by Lemma 5.4.6), does not contain any occurrence of the factor 0^p. We will use Lemma 5.4.9, which will imply in each particular case that the next occurrence of 0^p in t, after the prefix of w_i, is in some of the following factors w_j.

1. $i=0$. Property 5.4.1 implies that a factor $0^{2^{k-1}}$ occurs in t_w at some position j only if either $\langle\lambda,0\rangle$ or, for some $r\geq s\geq 0$, $\langle 10^s,0\rangle\langle 10^r,1\rangle$ occurs at position $\lfloor j/2^{k-1}\rfloor$ of $\tau=\lim_{n\to\infty}\varphi^n(\langle\lambda,0\rangle)$. Property 5.4.2 implies that in the latter case exactly one of r,s must be equal to 0, hence $r>s=0$ and the factor has the form $\langle 1,0\rangle\langle 10^r,1\rangle$ for some $r\geq 1$. We apply i and iii of Lemma 5.4.9.

2. $i=1$ or $i=2$. A factor $0^{2^{k-1}+1}$ occurs in t_w at position j only if either $\langle\lambda,0\rangle\langle 1,1\rangle$ or $\langle 1,0\rangle\langle 10^r,1\rangle$, for some $r\geq 1$, or $\langle 1,0\rangle\langle\lambda,0\rangle$ occurs at position $\lfloor j/2^{k-1}\rfloor$ of τ. We apply ii and iii of Lemma 5.4.9.

3. $i = 3$ or $i = 4$, $k \geq 3$. As shown in the proof of Lemma 5.4.7, w_3 has the prefix $0^{2^{k-1}+1}1^{2^{k-1}-2}01^{2^{k-1}-1}$ and w_4 the same or smaller prefix. If a factor $0^{2^{k-1}+1}$ occurs in w_3 or w_4 at the position j of t_w then the factor $\langle\lambda,0\rangle\langle 1,1\rangle$ occurs at position $\lfloor j/2^{k-1}\rfloor$ of τ. Hence $0^{2^{k-1}+1}$ occurs in w_3 or w_4 as a prefix of the factor $0^{2^{k-1}}01^{2^{k-1}-1} > 0^{2^{k-1}+1}1^{2^{k-1}-2}01^{2^{k-1}-1}$.

4. $i = 2r+1$ or $i = 2r+2$, $2 \leq r \leq k-2$. A factor $0^{2^{k-1}-1+2^r}$, occurs in t_w at position j only if either $\langle 1,0\rangle\langle\lambda,0\rangle$ or $\langle 1,0\rangle\langle 10^r,1\rangle$occurs at position $\lfloor j/2^{k-1}\rfloor$ of τ. If the factor $\langle 1,0\rangle\langle\lambda,0\rangle$ occurs in τ, then it is a prefix of the factor $\langle 1,0\rangle\langle\lambda,0\rangle\langle 1,1\rangle$ or $\langle 1,0\rangle\langle\lambda,0\rangle\langle 1,0\rangle$. We apply iii and iv and v of Lemma 5.4.9.

5. $i = 2k-1$. A factor 0^{2^k} occurs in t_w at position j only if $\langle 1,0\rangle\langle\lambda,0\rangle\langle 1,1\rangle$ occurs at position $\lfloor j/2^{k-1}\rfloor$ of τ. We apply iv of Lemma 5.4.9.

□

Lemma 5.4.11 *For $i \geq 2k-1$, w_1 is a primitive word.*

Proof. Analogous to the proof of Lemma 5.1.13. □

Lemma 5.4.12 *For $i \geq 2k-1$, w_i is a Lyndon word.*

Proof. We proceed by induction. The assertion for $i = 2k-1$ is implied by Lemma 5.4.10.

Let the assertion be true for some $i-1 \geq 2k-1$. Lemma 5.4.11 implies that w_i is a primitive word. By Lemma 5.4.3, w_i is a conjugate of $\Psi(\varphi^i(\langle 1,0\rangle))$. Assume that w_i is not minimal in its conjugacy class. Let $w_i = uv$ where $vu < w_i$ is the smallest conjugate of w_i. According to Lemma 5.4.8, $w_i < w_{2k-1}$, hence w_i, and vu have the same prefix $0^{2^k}1$ as w_{2k-1} because no conjugate of $\Psi(\varphi^i(\langle 1,0\rangle))$ may contain the factor 0^{2^k+1}, as one easily observes using Property 5.4.1. The word vu is of the form $\xi_0x_1x_2x\xi_0'$, $\xi_0 \neq \lambda$, where $\xi_0'\xi_0 = \psi(\varphi^{k-1}(c_0))$, $x_1 = \psi(\varphi^{k-1}(c_1))$, $x_2 = \psi(\varphi^{k-1}(c_2))$ for some $c_0,c_1,c_2 \in \Sigma$ and $\xi_0'\xi_0x_1x_2x = \psi(\varphi^{k-1}(c_1c_2x'c_0)$ for some conjugate $c_1c_2x'c_0$ of $\varphi^{k-1+i}(\langle 1,0\rangle)$.

Property 5.4.1 in combination with Property 5.4.2 implies that the only possibility is $\xi_0 = 0^{2^{k-1}-1}$, $\xi_0' = 1$, $x_1 = 0^{2^{k-1}}$, $x_2 = 01^{2^{k-1}-1}$, $c_0 = \langle 1,0\rangle$, $c_1 = \langle\lambda,0\rangle$ and $c_2 = \langle 1,1\rangle$.

A repeated application of Property 5.4.2 leads to the conclusion that $c_1c_2x'c_0$ is a conjugate of

$$\varphi^{k-1+i}(\langle 1,0\rangle) \quad \text{of the form } \varphi^{k-1}(\langle 1,1\rangle y\langle\alpha,0\rangle),$$

where $\langle 1,1\rangle y\langle\alpha,0\rangle$ is a conjugate of $\varphi^i(\langle 1,0\rangle)$, since c_1c_2 may occur only as a prefix of $\varphi^{k-1}(\langle 1,1\rangle)$.

Hence $\xi_0^{-1}vu\xi_0 = x_1x_2z\xi_0'\xi_0 = \Psi(\langle 1,1\rangle y\langle\alpha,0\rangle)$ for a conjugate $\langle 1,1\rangle y\langle\alpha,0\rangle$ of $\varphi^i(\langle 1,0\rangle)$. Property 5.4.2 now implies that $\langle\alpha,0\rangle\langle 1,1\rangle y = \varphi(y')$ for some conjugate y' of $\varphi^{i-1}(\langle 1,0\rangle)$.

Again, the only possibility is that $\alpha = \lambda$ and $\langle\lambda,0\rangle\langle 1,1\rangle y = \varphi^{k-1}(\langle 1,1\rangle by'')$ for some conjugate $\langle 1,1\rangle by''$ of $\varphi^{i-(k-1)}(\langle 1,0\rangle)$ where b is some symbol from Σ occurring at an even position and therefore different from $\langle 1,0\rangle$ and $\langle 1,1\rangle$. We will use the fact that, since $b \neq \langle 1,0\rangle$ and $b \neq \langle 1,1\rangle$, $\varphi^{k-2}(b)$ starts by $c \in \Sigma$ where either $c = \langle\lambda,0\rangle$ or $c = \langle\lambda,1\rangle$. Hence $y' = \varphi^{k-2}(\langle 1,1\rangle)cz$ for some $z \in \Sigma^*$.

We have $\langle\lambda,0\rangle\langle 1,1\rangle y = \varphi^{k-1}(\langle 1,1\rangle)\varphi(cz)$, where either $c = \langle\lambda,0\rangle$ or $c = \langle\lambda,1\rangle$ and $\varphi^{k-2}(\langle 1,1\rangle)cz$ is a conjugate of $\varphi^{i-1}(\langle 1,0\rangle)$.

Furthermore, $\xi_0^{-1} vu \xi_0 = \Psi(\langle 1,1\rangle\, y \langle\lambda,0\rangle)$ and, finally, $\rho = \Psi(\varphi^{k-1}(\langle\lambda,1\rangle)\,\langle\lambda,0\rangle)\xi_0^{-1}$ as stated by ii of Lemma 5.4.3. We get

$$\begin{array}{rcl} vu & < & w_i = \rho^{-1}\Psi(\varphi^i(\langle 1,0\rangle))\rho \\ \rho vu & < & \Psi(\varphi^i(\langle 1,0\rangle))\rho \\ (\rho\xi_0)(\xi_0^{-1} vu\xi_0) & < & \Psi(\varphi^i(\langle 1,0\rangle))\rho\xi_0 \\ \Psi(\varphi^{k-1}(\langle\lambda,1\rangle)\,\langle\lambda,0\rangle)\Psi(\langle 1,1\rangle\, y\,\langle\lambda,0\rangle) & < & \Psi(\varphi^i(\langle 1,0\rangle)))\Psi(\varphi^{k-1}(\langle\lambda,1\rangle)\,\langle\lambda,0\rangle) \\ \varphi^{k-1}(\langle\lambda,1\rangle)\,\langle\lambda,0\rangle\,\langle 1,1\rangle\, y\,\langle\lambda,0\rangle & < & \varphi^i(\langle 1,0\rangle)\varphi^{k-1}(\langle\lambda,1\rangle)\,\langle\lambda,0\rangle \\ \varphi^{k-1}(\langle\lambda,1\rangle)\varphi^{k-1}(\langle 1,1\rangle)\varphi(cz) & < & \varphi^i(\langle 1,0\rangle)\varphi^{k-1}(\langle\lambda,1\rangle) \\ \varphi^{k-2}(\langle\lambda,1\rangle\,\varphi^{k-2}(\langle 1,1\rangle)cz & < & \varphi^{i-1}(\langle 1,0\rangle)\varphi^{k-2}(\langle\lambda,1\rangle) \\ \varphi^{k-2}(\langle\lambda,1\rangle\,\langle 1,1\rangle)cz & < & \varphi^{i-1}(\langle 1,0\rangle)\varphi^{k-2}(\langle\lambda,1\rangle) \\ \varphi^{k-1}(\langle\lambda,1\rangle)cz & < & \varphi^{i-1}(\langle 1,0\rangle)\varphi^{k-2}(\langle\lambda,1\rangle) \\ \varphi^{k-1}(\langle\lambda,1\rangle)cz & < & \varphi^{i-1}(\langle 1,0\rangle)\varphi^{k-2}(\langle\lambda,1\rangle) \\ \Psi(\varphi^{k-1}(\langle\lambda,1\rangle)cz) & < & \Psi(\varphi^{i-1}(\langle 1,0\rangle)\varphi^{k-2}(\langle\lambda,1\rangle)) \end{array}$$

The word $\Psi(\varphi^{i-1}(\langle 1,0\rangle)) = \rho w_{i-1}\rho^{-1}$ has the prefix $\rho = \Psi(\varphi^{k-1}(\langle\lambda,1\rangle))\Psi(\langle\lambda,0\rangle\,\xi_0^{-1}$. The latter inequality therefore implies that the first symbol of $\Psi(c)$ is not greater than 0 being the first symbol of $\Psi(\langle\lambda,0\rangle)\xi_0^{-1}$. The word $\Psi(\langle\lambda,0\rangle)$ starts by 0 and $\Psi(\langle\lambda,1\rangle)$ starts by 1, hence $c = \langle\lambda,0\rangle$. Then

$$\begin{array}{rcl} \Psi(\varphi^{k-1}(\langle\lambda,1\rangle)\,\langle\lambda,0\rangle\, z) & < & \Psi(\varphi^{i-1}(\langle 1,0\rangle)\varphi^{k-2}(\langle\lambda,1\rangle)) \\ \Psi(\varphi^{k-1}(\langle\lambda,1\rangle))\Psi(\langle\lambda,0\rangle)\Psi(z) & < & \Psi(\varphi^{i-1}(\langle 1,0\rangle))\Psi(\varphi^{k-2}(\langle\lambda,1\rangle)) \\ \rho\xi_0\Psi(z(\varphi^{k-2}(\langle 1,1\rangle)\,\langle\lambda,0\rangle)\xi_0^{-1} & < & \Psi(\varphi^{i-1}(\langle 1,0\rangle))\nu \end{array}$$

where

$$\begin{array}{rcl} \nu & = & \Psi(\varphi^{k-2}(\langle\lambda,1\rangle))\Psi(\varphi^{k-2}(\langle 1,1\rangle)\,\langle\lambda,0\rangle)\xi_0^{-1} \\ & = & \Psi(\varphi^{k-1}(\langle\lambda,1\rangle)\,\langle\lambda,0\rangle)\xi_0^{-1} \\ & = & \rho. \end{array}$$

Therefore

$$\begin{array}{rcl} \xi_0\Psi(z\varphi^{k-2}(\langle 1,1\rangle)c))\xi_0^{-1} & < & \rho^{-1}\Psi(\varphi^{i-1}(\langle 1,0\rangle))\rho \\ \xi_0\Psi(z\varphi^{k-2}(\langle 1,1\rangle)c)\xi_0^{-1} & < & w_{i-1} \end{array}$$

where the left-hand side is a conjugate of $\Psi(\varphi^{i-1}(\langle 1,0\rangle))$ and hence of w_{i-1}. This contradicts to the inductive hypothesis that w_{i-1} is a Lyndon word. □

Theorem 4.4.1 follows from Lemma 5.4.4, Lemma 5.4.7, Lemma 5.4.8, Lemma 5.4.10 and Lemma 5.4.12.

6 Concluding remarks

Out of the four subclasses of the generalized Thue sequences, for three subclasses the canonical Lyndon factors grow proportionally with applications of the substitution μ. In the only case, when $w = 1^k$, the factors are growing with μ^2. This fact prevents us from trying to make any kind of conjecture on the general shape of the Lyndon factors for the whole class of generalized sequences of Thue. A study of the Lyndon factorization of further subclasses will be necessary to have a kind of general result. We hope that Definition 3.1.2 together with Lemma 3.2.4 may be equally useful as they proved to be in the investigation of the four subclasses in the present work.

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