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Multicolored parallelisms of isomorphic spanning trees[†]

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It is proved that a complete graph on n (> 4) vertices can be properly edge-colored with $n-1$ colors in such a way that the edges can be partitioned into edge disjoint multicolored isomorphic spanning trees whenever n is a power of two or five times a power of two. A spanning tree is multicolored if all $n-1$ colors occur among its edges.

Keywords: Multicolored tree, orthogonal Latin squares, multicolored matching

1 Multicolored parallelisms in complete graphs

Throughout this paper K_s denotes the complete graph on s vertices. We color the edges of K_{2n} with $2n-1$ colors by assigning one color to each edge. Basic terminology and notation on graph theory is found in Berge (1971). The coloration is *proper* if whenever two edges that have one vertex in common carry different colors. A spanning tree is called *multicolored* if no two of its edges have the same color. Two trees are *edge disjoint* if they do not share common edges. Two graphs with colored edges are *isomorphic* if there exists a bijection σ between the sets of vertices and a bijection η between the sets of colors such that (i,j) is an edge of color c if and only if $(\sigma(i),\sigma(j))$ is an edge of color $\eta(c)$. We investigate the possibility of producing a proper edge-coloration of K_{2n} such that its edges can be partitioned into edge disjoint isomorphic multicolored spanning trees. [By isomorphic multicolored spanning trees we understand a set of spanning trees, each of which is multicolored, any two spanning trees of the set being isomorphic as uncolored spanning trees.] When this is possible to accomplish we obtain what we call a *multicolored tree parallelism* for K_{2n}.

When no coloring is involved, it is well-known, and a classical result of Euler, that the edges of K_{2n} can be partitioned into isomorphic spanning trees (paths, for example). Each of these spanning trees can easily be made multicolored, but the resulting edge coloration of K_{2n} usually fails to be proper. Indeed, there exists a proper coloration of K_8 that does not admit a multicolored path; see Buliga (2002), [8]. By an inductive construction we demonstrate that a partition of the edges of K_m into edge-disjoint isomorphic multicolored spanning trees that induce a proper coloration of K_m is possible whenever m (> 4) is a power of two, or five times a power of two.

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Such a partition of the edges of K_m can be viewed as a parallelism as defined in Cameron (1976), [9], with an additional property due to color. Specifically, finding a partition as described above corresponds to an arrangement of the edges of K_{2n} into an array of $2n-1$ rows and n columns such that each row contains all edges of some color (these edges form a perfect matching due to the fact that the graph is properly colored) and the edges in each column form a (necessarily multicolored) spanning tree the isomorphism type of which does not change from column to column. We ask, therefore, for a double parallelism of K_{2n}, one present in the rows of the array (perfect matchings) and the other in the columns that consist of edge disjoint isomorphic spanning trees.

The generating function of the multicolored spanning trees in any edge colored graph can be expressed as a sum of formal determinants; cf. [2] and [3]. These results have been used in constructing multicolored tree parallelisms for complete graphs on a small number of vertices. Algorithms for finding multicolored spanning trees are discussed in [6]. We applied the algorithm written by [8] to obtain tree parallelisms for complete graphs on a small number of vertices. An application of parallelisms of complete designs to population genetics data is found in [1]. Parallelisms are also useful in partitioning consecutive positive integers into sets of equal size with equal power sums; cf. [12]. Discussions of colored matchings and design parallelisms to parallel computing appear in [11].

2 A multicolored tree parallelism for powers of two

Our main result is the following

Theorem 1 *If $m \neq 1,3$ and a multicolored tree parallelism for K_{2m} exists, then a multicolored tree parallelism for $K_{2^r m}$ exists, for all $r \geq 1$.*

Proof: Starting with a multicolored tree parallelism for K_{2m} it suffices to prove that we can obtain a multicolored tree parallelism for K_{4m}. To complete the proof we simply iterate the process. Take a copy of the multicolored tree parallelism for K_{2m} and call it L. Take another copy of the multicolored tree parallelism for K_{2m}, on a disjoint set of vertices from those of L but using the same set of colors, and call it R. The graph having $L \cup R$ as vertices, with edges connecting any vertex of L with any vertex of R, is called B. It is apparent that we have thus constructed a graph K_{4m} on the vertex set $L \cup R$. Edges of B are still to be colored. Color the edges of B in accordance with a pair of orthogonal Latin squares. For a definition and basic properties of orthogonal Latin squares the reader is refered to [10] (p. 366). It is well known, cf. [5], that a pair of orthogonal Latin squares on n symbols exists for all $n \neq 2,6$. Further specificity on the type of such Latin squares appears in [7]. The rows of the Latin squares are indexed by the vertices of L, the columns by the vertices of R. Colors used are disjoint from those used on the edges of L and R. Entries in the first Latin square represent the assignment of colors to the edges. We have thus completed an edge coloration of K_{4m}. It is a proper coloration, since it is proper within L and R by assumption, and the distribution of colors in accordance with the entries of the first Latin square ensures that edges emanating from each vertex carry all possible colors. We now describe the spanning tree decomposition that produces a multicolored tree parallelism for K_{4m}. In general denote by $s(M)$ the set of multicolored spanning trees present in the multicolored tree parallelism of complete graph M. Let $B(i)$ be the set of edges of B associated to positions in which symbol i occurs in the second of the orthogonal Latin squares; $1 \leq i \leq 2m$. Consider any bijection α between the set of symbols in the second Latin square and the set $s(L) \cup s(R)$. The set $s(L \cup R)$ is now described as follows:

$$s(L \cup R) = \{B(i) \cup T_{\alpha(i)} : 1 \leq i \leq 2m\}.$$

Elements of $s(L \cup R)$ are spanning trees of $L \cup R$. Any one of them consists of a spanning tree of L (or R) appended with a set of pendant edges $B(i)$ for some i. They are therefore isomorphic as uncolored trees. By construction it is evident that they are multicolored. It follows that they are isomorphic multicolored spanning trees. Moreover, they are edge disjoint. The only possible overlap may occur among the edges in B. But the orthogonality of the Latin squares ensures that an edge occurs in precisely one such spanning tree. This completes the proof. □

Start out with the tree partition of K_8 written below. Rows represent colors, columns spanning trees. It may easily be verified that we have a proper coloration of K_8, the four columns representing edge disjoint isomorphic multicolored spanning trees.

18	34	56	27
17	36	28	45
16	38	47	25
26	15	48	37
23	14	58	67
46	78	35	12
57	24	13	68

Using this multicolored tree parallelism for K_8 as the K_{2m} in Theorem 1 we obtain

Corollary 1 For $n > 2$ the graph K_{2^n} admits a multicolored tree parallelism.

A proper coloration of K_{10}, with rows representing colors and columns representing isomorphic spanning trees, appears below:

12	*34*	*90*	*56*	*78*
24	*13*	*69*	*57*	*80*
60	*58*	*14*	*79*	*23*
37	*89*	*15*	*40*	*26*
49	*25*	*70*	*38*	*16*
50	*46*	*28*	*17*	*39*
67	*30*	*18*	*29*	*45*
68	*19*	*47*	*20*	*35*
59	*27*	*36*	*48*	*10*

Theorem 1 allows us now to conclude as follows.

Corollary 2 *For $n \geq 1$ the graph $K_{5 \cdot 2^n}$ admits a multicolored tree parallelism.*

3 Edge disjoint multicolored isomorphic spanning trees

Whereas the previous section offers a multicolored tree parallelism for powers of two, and five times powers of two, this section examines settings where we fall short of a multicolored tree parallelism but are able to construct, within the context of a proper edge coloration, a large number of edge disjoint isomorphic multicolored spanning trees. It is easy to see that for an odd number of vertices any proper edge coloration of K_{2m+1} using $2m+1$ colors can yield at most m edge disjoint multicolored spanning trees. Unlike in the case of an even number of vertices these spanning trees never partition the edge set of K_{2m+1}. At least m edges remain uncovered. By an $s-$*coloration* of K_n we understand a proper edge coloration on K_n with s colors, if one exists. By a method similar to that used in the previous section we demonstrate the following

Theorem 2 *If a $(2m+1)-$ coloration of K_{2m+1} that admits m edge disjoint isomorphic multicolored spanning trees on the same set of $2m$ colors exists, then there also exists a $(2m+1)2^r-$ coloration of $K_{(2m+1)2^r}$ that admits $2^r m$ edge disjoint isomorphic multicolored spanning trees, for all $(m,r) \neq (1,2)$. Furthermore, all the $2^r m$ spanning trees of $K_{(2m+1)2^r}$ in question involve the same set of $(2m+1)2^r - 1$ colors.*

Proof: Let A be a copy of the given K_{2m+1} and B another copy on a disjoint set of vertices with an obvious color preserving bijection γ between the vertices of A and B. Consider the graph A. Edges carrying the same color c form a matching of m edges and an isolated vertex; we call the isolated vertex a $c-vertex$. It is clear that if v is a $c-$vertex of A then $\gamma(v)$ is a $c-$vertex of B. The m spanning trees of A have edges colored with the same set of $2m$ colors. We denote by e the extra color that occurs in none of the spanning trees. Connect every vertex of A to every vertex of B and denote the set of resulting edges by C. We color the edges in C in accordance to a pair of orthogonal Latin squares, which exist since $(m,r) \neq (1,2)$, as follows. Label the rows of the Latin squares by vertices of A and columns by the corresponding vertices in B. We can select without loss the first Latin square to have all symbols different on the main diagonal. Symbols in the first Latin square are the colors assigned to the edges of C. The colors in the first Latin square are all different from the colors used on the edges of A (or B). The graph $A \cup B \cup C$, isomorphic to $K_{2(2m+1)}$, has all its edges colored with the $2m+1$ colors in A plus the $2m+1$ new colors in C. This coloration is not proper, however. The following change will render a proper edge coloration: recolor the diagonal entry in the first Latin square that corresponds to the vertex pair $(v_i, \gamma(v_i))$, with v_i a $c-$vertex, with color c. Do this for all diagonal entries in the first Latin square. A spanning tree of the type we want is obtained by pairing up one of the given spanning trees of A (or B) with nondiagonal edges in the first Latin square that carry the same symbol in the second Latin square. By construction it follows that these spanning trees are isomorphic, multicolored, edge disjoint and $2m$ in number. None of these trees have edges colored with the extra color e. We now iterate the construction to obtain the stated result. This ends the proof. □

The premise of Theorem 2 can be verified for small values of m. In particular, it is not hard to check that this is the case for $m = 2$ and 3. As a consequence we obtain

Corollary 3 *For a $(2p+1)2^r-$ coloration of $K_{(2p+1)2^r}$ there exist $p2^r$ edge disjoint isomorphic multicolored spanning trees each on the same set of $(2p+1)2^r - 1$ colors.*

Though a large number of spanning trees can be obtained by the construction of Theorem 2, in general this construction is suboptimal. We can see this even in the case of K_6. Our construction yields two edge disjoint isomorphic multicolored spanning trees. It is not hard to see that three such trees can actually be constructed.

A small case that poses some difficulty is K_{12}. We were able to produce a proper coloration of K_{12} that allows a partition of its edges into multicolored edge disjoint spanning trees but not all trees are isomorphic. It is listed below.

7-8	10-12	3-5	1-2	4-6	9-11
1-6	2-3	10-11	8-9	7-12	4-5
3-4	2-6	1-5	8-12	9-10	7-11
1-3	5-6	8-10	11-12	7-9	2-4
8-11	7-10	3-6	2-5	1-4	9-12
2-8	5-11	1-7	3-9	4-10	6-12
1-8	4-9	2-7	6-11	5-12	3-10
5-7	6-8	3-11	4-12	2-10	1-9
3-12	6-7	2-9	1-10	4-11	5-8
5-9	1-11	4-8	2-12	3-7	6-10
5-10	6-9	1-12	4-7	3-8	2-11

The construction given in this paper does not apply to K_{12} since it relies on orthogonal Latin squares of order 6; but Euler proved that they do not exist. Nevertheless, we state as follows:

Conjecture 1 *Any proper coloration of the edges of a complete graph on an even number of (more than four) vertices allows a partition of the edges into multicolored isomorphic spanning trees.*

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