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Minimum survivable graphs with bounded distance increase

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We study in graphs a property related to fault-tolerance in case a node fails. A graph G is *k-self-repairing*, where k is a non-negative integer, if after the removal of any vertex no distance in the surviving graph increase by more than k. We give upper and lower bounds on the minimum number of edges of a k-self-repairing graph for prescribed k and n, where n is the order of the graph. We also prove that the problem of finding, in a k-self-repairing graph, a spanning k-self-repairing subgraph of the minimum size is *NP-Hard*.

Keywords: distance, fault-tolerance, spanning subgraph

1 Introduction

We consider communication networks modeled by graphs in which the vertices represent the nodes and the edges represent the direct communication links. This model allows to measure performances of the network by different graph theoretic parameters. For instance, the delay of communication in the network can be measured by the diameter of the underlying graph that is the maximum distance over all pairs of nodes.

When designing reliable communication networks, the least that we must guarantee is that, after failure of some nodes or links, the surviving network still allows communication between all no-faulty nodes. This implies constraints on the connectivity of the corresponding graph. The k-connectivity (resp. k-edge-connectivity) is associated to the capability of a network to resist to the failure of any subset of $(k-1)$ nodes (resp. links). If in addition, the surviving network must allow communication with a reasonably increased delay, constraints of bounded length for disjoint paths must be involved. The k-diameter of a k-connected graph and other variants of this parameter give means to study such properties (see [Hsu94]).

In a network, every communication link has a cost. The cost of a network is the sum of the costs of all its communication links. When all costs are the same, their value can be considered equal to 1 and, in this case, the cost of the network is simply the number of its communication links. In the rest of this paper, communication links are all considered of cost 1. Networks with small cost are desirable because they are "easy" to maintain. In particular, they do not require large capacity storage in a computer. The problem of designing communication networks which satisfy certain requirement with small cost has been extensively studied. One can see [Cac89] for an account on such work and [CHH01] if the requirement is a given diameter and a small cost.

One approach consists in finding spanning subgraphs of a graph with the same good properties as the original graph and fewer edges. When the property dealing with is 2-connectivity, the problem is known as the 2-connected Steiner subgraph problem. This problem has been extensively studied in series of papers from the early one by Steiglitz et al. [SWK69] to the more recent ones [BM95], [CRRW93], [GM90], [GMS91], [GMS92a], [GMS92b]. Applications of this problem can be found in [BBM90], [CW81], [EMV87], [MMP90], [Win85], [Win86].

Fortz et al. in [FLM00] consider an additive condition to the 2-connected Steiner problem. The condition is that in the spanning subgraph every edge must be on a cycle whose length is bounded by a given constant k. The authors called this problem *the two-connected network with bounded meshes (rings) problem*. Note that in such a spanning subgraph the failure of any edge does not yield any distance increase by more than $(k-2)$ in the surviving network. This property is a desirable feature because in case a link is broken, the traffic is rerouted with a limited alteration of the communication delay. Constraints involving distance conditions are also considered by M. Elkin and D. Peleg who introduced what they called *an* (α,β)*-spanner* of a graph [EP01]. Given a graph $G=(V,E)$ and a subset H of E, $G'=(V,H)$ is an (α,β)-spanner of G if $d_H(u,v) \leq \alpha . d_G(u,v)+\beta$ for every pair of vertices u, v.

Farley and Proskurowski [FP97] consider a related problem concerning node failure. The requirement is that no penalty occur on any distance after the removal of a node. That is no distance increase in the surviving graph. The authors called such graphs *self-repairing* and called *minimum self-repairing* self-repairing graphs with the minimum number of edges given an order n (the number of vertices). In their paper, they give the exact number of edges in a minimum self-repairing graph (given an order n) and characterize the class of such minimum graphs.

In this paper, we consider graphs in which the removal of any node makes no distance in the surviving graph increase by more than k.

Let $G=(V,E)$ be a connected graph with m edges (m is the size of the graph) and n vertices (n is the order of the graph). For a vertex x, $G-\{x\}$ denotes the subgraph of G induced by $V-\{x\}$. For x and y in V, we denote by $d_G(x,y)$ the length of a shortest path between x and y in G. We say that G is k*-self-repairing*, where k is a non-negative integer, if after the removal of any vertex the distances in the surviving graph increase by at most k. That is :

$$\forall x \in V, \forall \{y,z\} \subset V-\{x\}, d_{G-\{x\}}(y,z) \leq d_G(y,z)+k.$$

Note that such a graph is 2-connected and every pair of adjacent edges (a pair of distinct edges sharing a common extremity) is on a cycle of length bounded by $k+4$. Every pair $[u,v]$, $[u,w]$, $v \neq w$ of adjacent edges is called a *transition*.

Let $n \geq 4$ and $0 \leq k \leq n-4$. We define $g(n,k)$ as the minimum size of a k-self-repairing graph of order n. Biconnectivity implies that $g(n,k) \geq n$. For a fixed n, we have :

$$g(n,0) \geq g(n,1) \geq g(n,2) \geq \ldots \geq g(n,n-4).$$

It is easy to see that C_n, $n \geq 4$, is a $(n-4)$-self-repairing graph of size n. Hence, $g(n,n-4)=n$.

Self-repairing graphs studied in [FP97] are those satisfying the definition above for $k=0$. The authors proved that $g(n,0)=2n-4$ and characterized such minimum graphs.

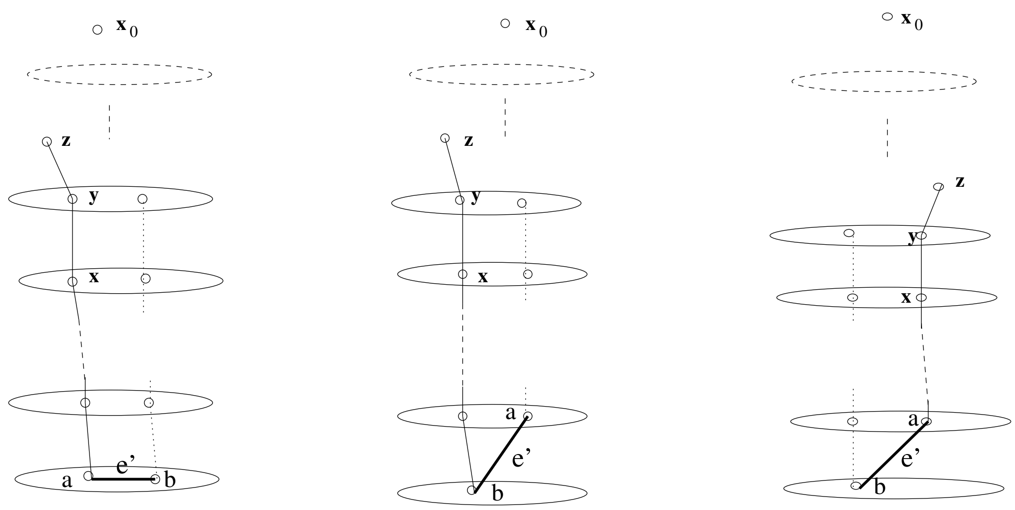

Fig. 1: $e = [x,y]$. Edges shown in bold are *e-backedges*. Dotted edges are not necessarily in T.

2 Bounds on the size of k-self-repairing graphs

Proposition 2.1 *: For $n \geq 6$ and $k \geq 1$,*

$$(n-1) + \frac{n-3}{k+1} \leq g(n,k) \leq (n+4) + 2\left\lfloor \frac{n-2}{k+2} \right\rfloor .$$

Proof : The lower bound needs to be proved only for k-self-repairing graphs having at least one vertex of degree 2 since otherwise the size of the graph is at least $3n/2$ which is better than the lower bound given in the proposition above.

Since the graph is k-self-repairing , we fix for each transition (recall that a transition is a pair of adjacent edges), a cycle of length at most $k+4$ containing this transition. This induces a map h from the set of transitions to the set of cycles. Let x_0 be a vertex of degree 2. Consider a BFS tree T rooted at x_0. For an edge $[u,v]$ of G define $lmax([u,v])$ as $max\{L(u),L(v)\}$, where $L(u)$ is the *level* of vertex u in T, that is its distance from x_0. Let $e = [x,y]$ be an edge of T such that $lmax(e) \geq 2$. We assume that $L(x) > L(y)$. Let z be the parent of y. The transition defined by edges $[y,z],[y,x]$ is denoted by $tr(e)$. Starting from y, the cycle $h(tr(e))$ goes down through at most $\lceil (k+1)/2 \rceil$ levels using edges of T before using an edge of $E-E(T)$. The first such edge, say e', is called an *e-backedge* (see Figure 1).

Conversely, each edge of $e' \in E-E(T)$ is a backedge for at most $(k+1)$ edges e ($lmax(e) \geq 2$) of $E(T)$. Indeed, let $e' = [a,b]$. If $L(a) < L(b)$,then, e' is the backedge of at most $\lceil (k+1)/2 \rceil$ edges contained in the subpath $P[x_0,b]$ and at most $k/2$ edges contained in the subpath $P[x_0,a]$ of T. If $L(a) = L(b)$, e' is the backedge of at most twice $(k+1)/2$ edges of T. In all cases, e' is the backedge of at most (k+1) edges of T. Since $d_G(x_0) = 2$, the lower bound follows.

To prove the upper bound, we give constructions of families of k-self-repairing graphs with a given number of vertices n.

The first construction :

Two vertices u and v are said to be *related* if the following conditions are satisfied :

1. $d_G(u,v) = \lfloor (k+4)/2 \rfloor$.
2. For every edge e incident to u, there exists a path of length at most $\lceil (k+4)/2 \rceil$, containing e and connecting u and v.
3. For every edge e incident to v, there exists a path of length at most $\lceil (k+4)/2 \rceil$, containing e and connecting u and v.

Let q and r be the two integers such that $n-(k+4) = q(\lfloor (k+4)/2 \rfloor - 1) + r$, where $0 \le r < \lfloor (k+4)/2 \rfloor - 1$. Start with a cycle G_0 of length $(k+4)$. Iterate q times the following step. Choose two related vertices in G_i. Connect them by a new path of $(\lfloor (k+4)/2 \rfloor - 1)$ new vertices. This gives a graph G_{i+1}. If r is non-zero, then, after the q iterations, choose again two related vertices u and v in G_q. Connect them by a new path of r new vertices. Let G be the resulting graph.

The k-self-repairing graph G so obtained is of order n and has a size of at most $n+q+1$. More precisely,

$$|E(G)| \le \begin{cases} (n-1) + 2\frac{n-2}{k+2} & \text{if } k \text{ is even} \\ (n-1) + 2\frac{n-3}{k+1} & \text{otherwise} \end{cases}$$

The second construction :

We give a second construction which is more complicated than the previous one but provides a better upper bound when k is odd and bounded by $O(\sqrt{n})$.

For $k \ge 1$, and $n > (k+4)$, let t and r be the non-negative integers such that $\lfloor n/2 \rfloor - 1 = t(k+2) + r$, with $0 \le r \le (k+1)$. Set $l = t(k+2)$ and $\alpha = \lceil (k+2)/2 \rceil$. Consider the following family of k-self-repairing graphs of order n denoted $\mathcal{G}(n,k)$ constructed as follows (Figure 2) :

First take two vertex-disjoint paths P_1 and P_2 each of length l. Set $P_1 = [x_0, x_1, \ldots, x_l]$ and $P_2 = [y_0, y_1, \ldots, y_l]$. Connect x_0 and y_0 by an edge.

Case k is at least 2 :

- For all $i, 1 \le i \le t$, connect $x_{i(k+2)}$ to $y_{i(k+2)}$ and $x_{i(k+2)-1}$ to $y_{i(k+2)-1}$.
- If k is even, connect $x_{i(k+2)+\alpha}$ to $y_{i(k+2)+\alpha}$ and $x_{i(k+2)+\alpha-1}$ to $y_{i(k+2)+\alpha-1}$, for all $i, 0 \le i \le t-1$. If k is odd connect $x_{i(k+2)+\alpha}$ to $y_{i(k+2)+\alpha-1}$ and $x_{i(k+2)+\alpha-1}$ to $y_{i(k+2)+\alpha-2}$, for all $i, 0 \le i \le t-1$.
- Set $n' = n - |V(P_1) \cup V(P_2)|$. We have $n' = n - 2\lfloor n/2 \rfloor + 2r$ and then, $0 \le n' \le 2k+2$ if n is even and $1 \le n' \le 2k+3$ otherwise. Now, we add n' new vertices and finish the construction by adding necessary edges in order to obtain a k-self-repairing graph.
- If $n' \le k+1$, place n' new vertices along a new path, say P and connect one of its extremities to x_l, the other one to y_l. If $n' \le k$, no more edge is needed. The k-self-repairing graph G so obtained satisfies $|E(G)| = n + 4t$. If $n' = k+1$, add a chord in P as shown in Figure 2c. This suffices to obtain a k-self-repairing graph G and we have $|E(G)| = n+1+4t$. If $n' = k+2$, place k vertices

along a path as done for P. Connect the two remaining vertices, say v_0 and w_0 by an edge and, connect v_0 to x_0 and w_0 to y_0 (Figure 2d). No more edge is needed and the k-self-repairing graph G so obtained satisfies $|E(G)| = n+2+4t$. If $k+3 \le n' \le 2k+2$, start with a construction as in the previous case (Figure 2d). This places $k+2$ vertices from the n'. Then, place the remaining vertices along a new path P' and connect its extremities one to v_0, the other to w_0 (Figure 2e). No more edge is needed and the k-self-repairing graph G so obtained satisfies $|E(G)| = n+3+4t$. If $n' = 2k+3$, start with a construction as in the previous case (Figure 2e). This places $2k+2$ vertices from the n'. Then, place the remaining vertex on P or P' and add a new chord in the path where the remaining vertex has just been placed, as done in case c). We obtain a k-self-repairing graph G that satisfies $|E(G)| = n+4+4t$.

Case $k = 1$ *:*

Connect x_1 to y_2, x_2 to y_1, x_l to y_l and, for all $i, 1 \le i \le (t-1)$, connect x_{3i} to y_{3i} and x_{3i+1} to y_{3i-1} and y_{3i+2} and x_{3i+2} to y_{3i+1} (Figure 3) . If $n' = n - |V(P_1) \cup V(P_2)|$ is non-zero, place n' new vertices along a new path, connected as done above for P. If $n' = 1$ no more edge is needed. If $n' \ge 2$, let z_1 be the extremity of P connected to x_l and $z_{n'}$ be the one connected to y_l. Add the chord $[z_1, y_{l-1}]$. If $n' \le 3$, no more edge is needed. If $n' = 4$, add the chord $[z_1, z_3]$ and then, no more edge is needed. If $n' = 5$, add the chord $[z_2, z_5]$ and then, no more edge is needed. The resulting graph is k-self-repairing and has a size of at most $n+2+4t$.

A graph in $\mathcal{G}(n,k)$ is k-self-repairing and has a size of at most :

$$n+4+4t = n+4+4\left\lfloor \frac{\frac{n}{2}-1}{k+2} \right\rfloor \le (n+4) + 2\frac{n-2}{k+2}.$$

□

3 Complexity of the problem of finding a minimum k-self-repairing spanning subgraph

In this section, we consider minimum k-self-repairing spanning subgraphs of a given graph. We denote by $N_G(x)$ the open neighborhood of x in G, that is, the set of vertices of G adjacent to x in G.

First note that checking whether a graph is k-self-repairing can be done in polynomial time using the simple following algorithm :

Algorithm 3.1 *:*

Input : *A graph G, a positive integer k.*

Output : *Yes* if G is k-self-repairing , *No* if it is not.

begin
For each vertex x of G do
begin
  $H \leftarrow G - x$;
  $S \leftarrow N_G(x)$;

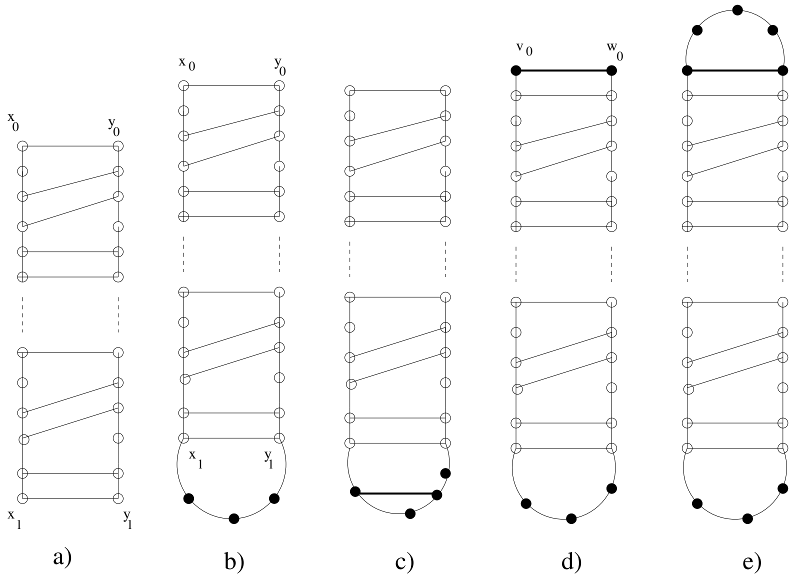

Fig. 2: a) $n' = 0$. b) $1 \leq n' \leq k$. c) $n' = k+1$. d) $n' = k+2$. e) $k+3 \leq n' \leq 2k+2$.

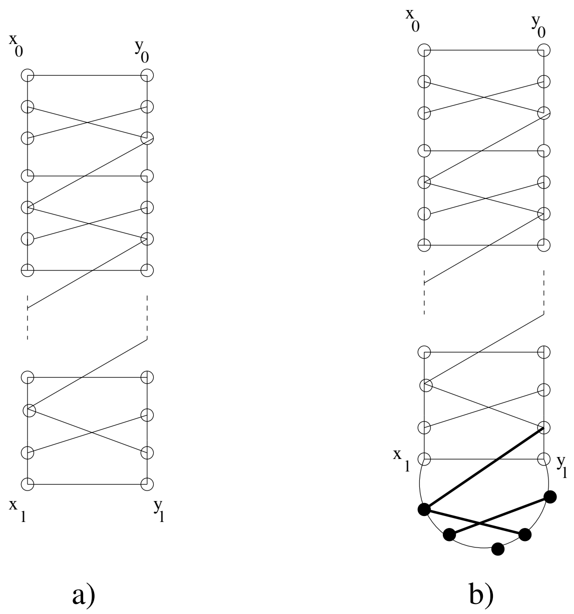

Fig. 3: a) $n' = 0$. b) $n' = 5$.

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      For each pair of vertices {u,v} ⊆ S do
      begin
         compute d_H(u,v) ;
         if d_H(u,v) > k+2, return No ;
      end ;
   end ;
   return Yes ;
   end ;
```

Computing the distance between two vertices is done in polynomial time. For each vertex x, we perform at most $d_G(x)(d_G(x)-1)/2$ such computations. Therefore, the time of Algorithm 3.1 is bounded by a polynomial of the size of G.

Problem 3.2 *: The minimum spanning subgraph problem with distance increase constraints* (MSSPDistIC).

Instance : *A positive integer $K \geq 2$, a K-self-repairing graph $\mathcal{G}$, a positive integer bound B.*

Question : *Does there exist a K-self-repairing spanning subgraph of $\mathcal{G}$ of size no more than B ?*

We shall prove that MSSPDistIC is NP-complete by proving a polynomial reduction from the following NP-complete *minimum spanning subgraph problem with diameter constraints* (MSSPDC) ([CHH01]).

Problem 3.3 *:* MSSPDC.

Instance : *A positive integer $k \geq 2$, a graph G of diameter k, a positive integer bound b.*

Question : *Does there exist a spanning subgraph of G of diameter k and of size no more than b ?*

Theorem 3.4 *: MSSPDistIC is NP-complete.*

Proof : MSSPDistIC is in NP since, given a spanning subgraph of $\mathcal{G}$, one can verify, in polynomial time, whether it is K-self-repairing (using Algorithm 3.1, for example) and then compare its size to B.

Let I be an instance of MSSPDC that is positive integers k and b and a graph G of diameter k. Let $V = \{x_1, x_2, \ldots, x_n\}$ be the set of vertices of G. Consider the following instance $\mathcal{J}$ of MSSPDistIC :

1. The graph $\mathcal{G}$ of $\mathcal{J}$ is obtained by taking a copy of G and adding, from each vertex x_i a path P_i joining x_i and $t = k-1$ new vertices. Let y_i be the second extremity of P_i. Add a new vertex z connected to all y_i (see Figure 4).
2. $K = 3k-4$ and $B = b + kn$, where n is the order of G.

The size of I is $O(n^2 \log^2 n)$. Since $k \leq n$ and $b \leq n^2$, the size of $\mathcal{J}$ is bounded by a polynomial in $n^2 \log^2 n$.

Now, we have to prove that I is a *yes-instance* of MSSPDC if and only if $\mathcal{J}$ is a *yes-instance* of MSSPDistIC.

Suppose there exists a spanning subgraph H of G of diameter k and of size bounded by b. Consider the spanning subgraph $\mathcal{H}$ of $\mathcal{G}$ induced by all edges of H and all edges external to G. It is easy to see that $\mathcal{H}$ is K-self-repairing and of size at most B. Therefore, $\mathcal{J}$ is a *yes-instance* of MSSPDistIC.

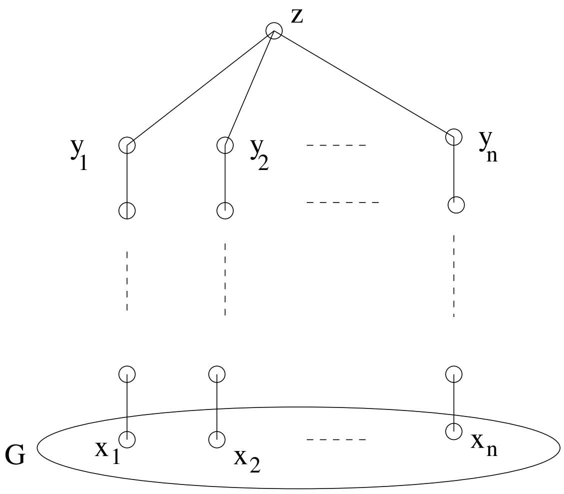

Fig. 4:

Now, let $\mathcal{J}$ be an instance of MSSPDistIC constructed from an instance I of MSSPDC as described above. If $\mathcal{J}$ is a *yes-instance* of MSSPDistIC, there exists a spanning K-self-repairing graph $\mathcal{H}$ of $\mathcal{G}$ of size no more than B. The minimum degree in $\mathcal{H}$ is at least 2. This means that all edges external to G are in $\mathcal{H}$. Let H be the spanning subgraph of G induced by all the edges in $E(\mathcal{H}) \cap E(G)$. Let $\{x_i, x_j\}$ be a pair of vertices of H. The transition of $\mathcal{H}$ defined by edges $[z, y_i]$, $[z, y_j]$ is in a cycle of length at most $K+4$. Every cycle containing y_i and y_j must include all the vertices of the paths P_i and P_j. Therefore, the distance in H between x_i and x_j is at most $K+4-2k = k$ and then, H is a spanning subgraph of G of diameter k. The size of H is $|E(H)| = |E(\mathcal{H})| - kn \leq B - kn = b$ and, therefore I is a *yes-instance* of MSSPDC.

The previous reduction shows that MSSPDistIC is NP-complete for $K = 3p+2$, $p \geq 0$. In fact, if we consider the same reduction from intances of MSSPDC, with $k \geq 3$, by setting $t = k-2$, $K = 3k-6$ and $B = b+n(k-1)$, we show that MSSPDistIC is NP-complete for $K = 3p, p \geq 1$. In the same way, considering instances of MSSPDC with $k \geq 4$, setting $t = k-3$, $K = 3k-8$ and $B = b+n(k-2)$, we show that MSSPDistIC is also NP-complete for $K = 3p+1, p \geq 1$. Therefore MSSPDistIC is NP-complete for all $K \geq 2$. □

References

[BBM90] D. Bienstock, E.F. Brickell, and C.L. Monma. On the structure of minimum weight k-connected spanning networks. *SIAM Journal on Discrete Mathematics*, 3:245–259, 1990.

[BM95] F. Barahona and A.R. Mahjoub. On two-connected subgraph polytopes. *Discrete Mathematics*, 147:19–34, 1995.

[Cac89] L. Caccetta. Graph theory in network design and analysis. In VR. Kulli, editor, *Recent Studies in Graph Theory*, pages 29–63. Vishwa International, India, 1989.

[CHH01] L. Caccetta, I.B. Hartman, and J. Huang. Concerning the number of edges in a graph with diameter constraints. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 36:161–173, 2001.

[CRRW93] C.R. Coullard, A. Rais, R.L. Rardin, and D.K. Wagner. Linear-time algorithms for the 2-connected Steiner subgraph problem on special classes of graphs. *Networks*, 23:195–206, 1993.

[CW81] N. Christofides and C.A. Whitlock. Network synthesis with connectivity constraints : a survey. In J.P. Brans, editor, *Operational Research*, pages 705–723, North Holland, 1981.

[EMV87] R.E. Erikson, C.L. Monma, and A.F. Veinott, Jr. Send and split method for a minimum-concave-cost network flows. *Mathematics of Operations Research*, 12:634–664, 1987.

[EP01] M.L. Elkin and D. Peleg. $(1+\varepsilon,\beta)$-spanner constructions for general graphs. In *Proceedings of* 33*rd* *Annual ACM Symp. on Theory of Computing*, pages 173–182, Greece, Crete, July 2001. To appear in SIAM J. of Computing.

[FLM00] B. Fortz, M. Labbé, and F. Maffioli. Solving the two-connected network with bounded meshes problem. *Operations Research*, 48(6):866–877, 2000.

[FP97] A.M. Farley and A. Proskurowski. Minimum self-repairing graphs. *Graphs Combin.*, 13:345–351, 1997.

[GM90] M. Grötschel and C. Monma. Integer polyhedra arising from certain network design problem with connectivity constraints. *SIAM Journal of Discrete Mathematics*, 3:502–523, 1990.

[GMS91] M. Grötschel, C. Monma, and M. Stoer. Polyhedral approaches to network survivability. In F. Roberts et al., editors, *Reliability of Computer and Communication Networks*, pages 121–141, 1991. Volume 5, Series in Discrete Mathematics and Computer Science, AMS/ACM.

[GMS92a] M. Grötschel, C. Monma, and M. Stoer. Computational results with a cutting plane algorithm for designing communication networks with low-connectivity constraints. *Operations Research*, 40/2:309–330, 1992.

[GMS92b] M. Grötschel, C. Monma, and M. Stoer. Facets for polyhedra arising in the design of communication networks with low-connectivity constraints. *SIAM Journal on Optimization*, 2:474–504, 1992.

[Hsu94] D.F. Hsu. On container width and length in graphs, groups and networks. *IEICE. Trans. Fundam E77*, 4:668–680, 1994.

[MMP90] C. Monma, B.S. Munson, and W.R. Pulleyblank. Minimum-weight two-connected spanning networks. *Mathematical Programming*, 46:153–171, 1990.

[SWK69] K. Steiglitz, P. Weiner, and D. Kleitman. The design of minimum-cost survivable networks. *IEEE Transactions on Circuit Theory*, CT-16:455–460, 1969.

[Win85] P. Winter. Generalized Steiner problem in Halin graphs. In *Proceedings of the* 12^{th} *International Symposium on Mathematical Programming*. MIT, 1985.

[Win86] P. Winter. Generalized Steiner problem in series-parallel networks. *Journal of Algorithms*, 7:549–566, 1986.