An adaptive algorithm for determining the optimal degree of regression in constrained mock-Chebyshev least squares quadrature

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Abstract

In this paper we develop an adaptive algorithm for determining the optimal degree of regression in the constrained mock-Chebyshev least-squares interpolation of an analytic function to obtain quadrature formulas with high degree of exactness and accuracy from equispaced nodes. We numerically prove the effectiveness of the proposed algorithm by several examples.

1 Introduction

Let $f$ be a continuous function in the interval $[-1, 1]$ and let $w(x) \in L_1([-1, 1])$ be a nonnegative weight function. A widespread problem in applied mathematics consists in the approximation of the weighted integral

$$I[f] := \int_{-1}^{1} f(x) w(x) \, dx,$$

by using a quadrature formula. If the function $f$ is known in the whole interval $[-1, 1]$, one can use a Gauss–Christoffel quadrature rule \cite{10}

$$Q[f] := \sum_{i=1}^{N} w_i f(\xi_i),$$

where $\xi_1, \ldots, \xi_m \in (-1, 1)$ are the nodes and $w_1, \ldots, w_m \geq 0$ are the weights of the quadrature formula, which has algebraic degree of exactness $2m-1$, that is it integrates exactly polynomials of degree up to $2m-1$. In many practical applications, however, the function $f$ is not known on the whole interval $[-1, 1]$, but only on a finite set of points

$$X_n = \{x_0, \ldots, x_n\},$$

which are often equispaced, that is

$$x_i = -1 + \frac{2i}{n}, \quad i = 0, \ldots, n.$$

In these cases, composite trapezoidal or composite Simpson rules, of degree of exactness 1, 3, respectively, are widely used, since all Newton–Cotes rules of higher order (greater than 7 for $w(x) = 1$) have weights which differ in sign and become rapidly unstable \cite{14}. A possible approach to overcome this problem is based on the use of interpolation techniques which mitigate the Runge and the Gibbs phenomena and allow to obtain efficient quadrature formulas based on equidistant points (see \cite{6, 8, 14, 15, 16} and references therein). Two interesting approaches to get quadrature formulas from equispaced nodes have been presented in \cite{15, 16} and are based on the idea that it is possible to obtain quadrature rules by using Gauss–Christoffel formulas in combination with local polynomial interpolants or global rational interpolants, respectively. To describe the approach proposed by Majidian \cite{16}, we fix $s \in \mathbb{N}$ and we select the $s$–tubes

$$N_n^s = \{x_j, \ldots, x_{j+b-1}\} \subset X_n, \quad k = 1, \ldots, m,$$

such that $x_k \leq \xi_k \leq x_{k+b-1}, \ k = 1, \ldots, m$. The key point is to substitute the exact values $f(\xi_k)$ in (2) with the values, at $\xi_k$, of the Lagrange polynomial interpolants on the $s$–tube $N_n^s$. Consequently, the quadrature formula (2) becomes

$$Q[f] = \sum_{k=1}^{m} \sum_{i=0}^{b-1} w_i f(x_i) \ell_i(N_n^s, \xi_k),$$

where $\ell_i(N_n^s, \cdot)$ is the Lagrange fundamental polynomial relative to the nodes in $N_n^s$. Since the nodes are surrounding Christoffel abscissas $\xi_k$, accurate approximation of integral (1) by means of quadrature formula (3) are expected, at least for local polynomial interpolants of low degree. A different approach is proposed by De Marchi et al. in \cite{6} based on the so-called mapped bases or

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fake nodes [5], which requires the definition of a bijective differentiable function \( S : [-1, 1] \to \mathbb{R} \) which maps the set \( X_n \) onto the set of Chebyshev–Lobatto nodes

\[
X_n^{\text{CL}} = \left\{ x_i^{\text{CL}} = -\cos \left( \frac{i \pi}{n} \right), \; i = 0, \ldots, n \right\},
\]

that is \( S(x_i) = x_i^{\text{CL}} \), \( i = 0, \ldots, n \). Therefore, by using the function \( S^{-1} \) as a change of variable, the integral (1) can be approximated by the well-known Gauss–Chebyshev quadrature formula. This method allows to mitigate the Gibbs phenomenon without resampling the given function.

Among all techniques known to defeat the Runge phenomenon, it is included the mock-Chebyshev subset interpolation, which produces a more accurate quadrature formula \( \hat{r} \). More precisely, we propose an adaptive algorithm for numerically determining the “optimal” degree \( r^{\text{opt}} \) of the constrained mock-Chebyshev least squares interpolant.

The overview of the paper is the following. In Section 2 we develop the adaptive algorithm for determining the “optimal” degree of regression \( r^{\text{opt}} \) and we analyze its computational cost. In Section 3 we show how to generalize the algorithm for approximating integrals of bivariate analytic functions on the square \([-1, 1]^2\). Finally, in Section 4, numerical experiments demonstrate the effectiveness of the proposed algorithm.
We fix a suitable constant $\delta>0$ (for example $\delta=0.5$) and we search the outliers among the elements of non monotonic triples by assuming that:

- if $d_r<\delta$ and $d_{r+1}>\delta$, then $\hat{E}_{r+1}$ is outlier (see Figure 2 (left));
- if $d_r>\delta$ and $d_{r+1}<\delta$, then $\hat{E}_r$ is outlier (see Figure 2 (right)).

The process of determining new outliers ends when $r=2m-2$. The tolerance $tol$ is determined adaptively by initializing it with the epsilon machine $\varepsilon$ and by updating it by the rule

$$
\text{tol} := \max\{tol, \hat{E}_r\}
$$

as soon as we find a new outlier $\hat{E}_r$. To avoid under-estimation of the exact relative error we take into account the possible presence of outliers, by assuming that all approximate relative errors $\hat{E}_r$ less than or equal to the computed tolerance $tol$. The Algorithm 1 computes the tolerance $tol$. The Algorithm 2 detects the outliers and remove them, providing in output the increasing sequence $\{r_1, \ldots, r_p\} \subset \{m, \ldots, 2m-2\}$ of degrees $r$ such that $\hat{E}_r > tol$. In Figure 3 we display the effect of the Algorithm 2, in detecting outliers (left) and removing them (right).

Figure 1: Sequence of approximations $\hat{Q}_{mr}\{f\}$ (left) and sequence of approximate relative errors $\hat{E}_r\{f\}$ (center) versus sequence of exact relative errors $E_r\{f\}$ (right) with $f(x) = \frac{1}{1+x^2}$, $w(x)=1$, $n=100$ and $m=22$.  

2 Computing accurate quadrature formulas with high degree of exactness from equispaced nodes

The main goal of this section is the determination of a procedure for the choice of the “optimal” value of $r$ which guarantees the “best approximation accuracy” of the quadrature formula $Q_r\{f\}$, measured through the exact relative error

$$
E_r\{f\} = \frac{Q_r\{f\} - I\{f\}}{|I\{f\}|},
$$

where we assume $|I\{f\}|>0$. We denote this value by $r_{opt}^e = r_{opt}(f)$. To this aim, we analyze the trend of approximate relative errors

$$
\hat{E}_r\{f\} = \frac{|\hat{Q}_r\{f\} - \hat{Q}_{r+1}\{f\}|}{|\hat{Q}_{r+1}\{f\}|}, \quad m \leq r \leq 2m-2,
$$

computed by using quadrature formulas of subsequent degrees up to the maximum degree of exactness $2m-1$. At first sight, it might be thought to choose $r_{opt}^e$ as the value of $r \in \{m, m+1, \ldots, 2m-2\}$ which minimizes the approximate relative error $\hat{E}_r\{f\}$. Unfortunately, in general this choice could be misleading since, even for starting values of $r$, it could occur that two successive approximations $Q_{r}\{f\}$ and $Q_{r+1}\{f\}$ are so close to each other that the approximate relative error $\hat{E}_r\{f\}$ is very small, for example less than a tolerance $tol$, despite the exact relative error $E_r\{f\}$ is not, being much greater than $tol$. An example of this situation is well illustrated in Figure 1, where the sequence of approximate relative errors assumes values less than $10^{-14}$ despite all exact relative errors are not less than $10^{-6}$. The approximate relative errors $\hat{E}_r\{f\}$ less than $tol$ are then outliers and therefore they must be discarded in the process of the determination of $r_{opt}^e$. Instead of fixing a tolerance $tol$ a priori, we distinguish outliers from valid values of relative errors $\hat{E}_r\{f\}$ by analyzing the sequence of consecutive triples

$$
t_r = \{\hat{E}_r\{f\}, \hat{E}_{r+1}\{f\}, \hat{E}_{r+2}\{f\}\}, \quad r = m, \ldots, 2m-3.
$$

We call the triple $t_r$ monotonic if and only if

$$
\hat{E}_r\{f\} \geq \hat{E}_{r+1}\{f\} \geq \hat{E}_{r+2}\{f\} \quad \text{or} \quad \hat{E}_r\{f\} \leq \hat{E}_{r+1}\{f\} \leq \hat{E}_{r+2}\{f\},
$$

otherwise we call the triple $t_r$ non monotonic. Note that a triple $t_r$ is monotonic if and only if

$$
d_r, d_{r+1} \geq 0,
$$

where we set

$$
d_r = \log_{10}(\hat{E}_{r+1}\{f\}) - \log_{10}(\hat{E}_r\{f\}), \quad r = m, \ldots, 2m-2.
$$

We fix a suitable constant $\delta>0$ (for example $\delta=0.5$) and we search the outliers among the elements of non monotonic triples by assuming that:

- if $d_r<\delta$ and $d_{r+1}>\delta$, then $\hat{E}_{r+1}$ is outlier (see Figure 2 (left));
- if $d_r>\delta$ and $d_{r+1}<\delta$, then $\hat{E}_r$ is outlier (see Figure 2 (right)).
Figure 2: Approximate relative errors, with respect to the Gauss–Legendre scheme $(w(x) = 1)$, for the function $f(x) = \frac{1}{x^2 - 1.1}$ with $n = 1200$, $m = 76$ (left) and $n = 1280$, $m = 79$ (right) with related first non monotonic triples (■) and corresponding outliers (◦).

Algorithm 1 Tolerance determination

Require: $\hat{E}_m[f], \ldots, \hat{E}_{2m-2}[f]

Ensure: $tol$

$tol \leftarrow \text{eps}$

for $i = m, \ldots, 2m - 3$ do

$d_i \leftarrow \log_{10}(\hat{E}_{i+1}[f]) - \log_{10}(\hat{E}_i[f])$

end for

while $j \leq m - 2$ do

if $d_j \leq -\delta$ and $d_{j+1} \geq \delta$ then

$tol = \max\{tol, \hat{E}_{j+1}[f]\}$

$j \leftarrow j + 2$

else if $d_j \geq \delta$ and $d_{j+1} \leq -\delta$ then

$tol = \max\{tol, \hat{E}_j[f]\}$

$j \leftarrow j + 1$

else

$j \leftarrow j + 2$

end if

end while

Algorithm 2 Outlier detection

Require: $\hat{E}_m[f], \ldots, \hat{E}_{2m-2}[f]

Ensure: $r_1, \ldots, r_p$

$j \leftarrow 1$

for $i = m, \ldots, 2m - 2$ do

if $\hat{E}_i[f] > tol$ then

$r_j \leftarrow i$

$j \leftarrow j + 1$

end if

end for
We define $\sigma$. As well-known, the standard deviation tells us the typical amount by which the values $\{N_{i,\ell}\}$ deviate from their average value $\mu$. We write $\mu = \frac{1}{q} \sum_{i,\ell=1}^{q} N_{i,\ell}$, $\sigma = \sqrt{\frac{1}{q} \sum_{i,\ell=1}^{q} (N_{i,\ell} - \mu)^2}$. By definition, all outliers belong to

$$I = \bigcup_{j=0}^{p} I_j.$$

Let $q \geq 0$ be the number of intervals $I_j$ in (11) containing at least one outlier.

**Case $q > 0$.** We denote by $I_{j,\ell}$, $0 \leq j_1 < j_2 < \cdots < j_q \leq p$ the intervals containing at least one outlier and by $N_{i,\ell}$ the number of outliers in $I_{j,\ell}$. We set

$$\mu = \frac{1}{q} \sum_{\ell} N_{i,\ell}, \quad \sigma = \sqrt{\frac{1}{q} \sum_{\ell} (N_{i,\ell} - \mu)^2}.$$

As well-known, the standard deviation $\sigma$ tells us the typical amount by which the values $\{N_{i,\ell}\}$ deviate from their average value $\mu$. We define

$$R = \{r_{j,\ell} : \ell = 1, \ldots, q \wedge N_{i,\ell} > \mu + \sigma\},$$

and we set

$$r_{\text{opt}}^* = \begin{cases} \min R & \text{if } R \neq \emptyset, \\ 2m - 2 & \text{if } R = \emptyset. \end{cases}$$

By the nature of the constrained mock-Chebyshev least-squares interpolation [4, 7], the case $r_{\text{opt}}^* < 2m - 2$ frequently occurs as soon as the exact relative error $E_{r,\text{opt}}[f]$ is near to the machine precision already for values of $r \ll 2m - 2$. In such cases, by increasing the degree of the regression $r \geq r_{\text{opt}}^*$, it is also possible that the exact relative error $E_{r}[f]$ became worse. In fact, as $r$ approaches to $n$, the polynomial $P_{\text{opt}}[f](x)$ tends to the polynomial interpolant on the set of nodes $X_{r,\text{opt}}[f](x)$, which in its turn, can suffer the Runge phenomenon. If $R \neq \emptyset$, from the definition of $r_{\text{opt}}^*$, we expect a not increasing trend of significant data $E_{r,\text{opt}}[f], \ldots, E_{r,\text{opt}}[f]$ if $r_{\text{opt}}^* > r_1$ or a non decreasing trend of the significant data $E_{r_1,\text{opt}}[f], \ldots, E_{r,\text{opt}}[f]$ if $r_{\text{opt}}^* = r_1$. If $R = \emptyset$ nothing can be said on the trend of the significant data.

**Case $q = 0$.** We set $r_{\text{opt}}^* = 2m - 2$. In this case nothing can be said on the trend of the significant data.

We are now able to determine a value $r_{\text{opt}}^*$ of the degree of regression which produces more accurate quadrature formulas. The accuracy of $Q_{r_{\text{opt}}^*,w}[f]$ will depend on the quality of approximation of the constrained mock-Chebyshev least-squares to the function $f$. We distinguish the following cases:

**Figure 3:** Significant approximate relative errors (●) and outliers (○) for the function $f(x) = \frac{1}{x^{1.1}}$, $n = 1000$, $m = 70$ relative to the Gauss–Legendre scheme ($w(x) = 1$).
We determine the computational cost of the Algorithm 3 described above. The computational cost of the Adaptive algorithm for determining a quadrature formulas with high degree of exactness and accuracy from Algorithm 3 requires

\[ O \begin{pmatrix} \log_{10}(E_{\text{opt}}), \ldots, \log_{10}(\hat{E}_r) \end{pmatrix} \]

and we set

\[ r_{\text{opt}}^* = r_k, \]

where

\[ \log_{10}(\hat{E}_k) = \min_{s \in \{1, \ldots, p\}} \{ \log_{10}(\hat{E}_s) : \log_{10}(\hat{E}_s) - l(r_s) \geq 0 \}; \]

• if \( r_{\text{opt}}^* = r_1 \), we use a linear regression \( l(r) \) to model the trend of the data

\[ \{ \log_{10}(\hat{E}_{r_{\text{opt}}^*}/f), \ldots, \log_{10}(\hat{E}_{r_1}/f) \} \]

and we set

\[ r_{\text{opt}}^* = r_k, \]

where

\[ \log_{10}(\hat{E}_k) = \min_{s \in \{1, \ldots, j\}} \{ \log_{10}(\hat{E}_s) : \log_{10}(\hat{E}_s) - l(r_s) \geq 0 \}; \]

• if \( r_{\text{opt}}^* = r_j \) with \( j = 2, \ldots, p - 1 \), then if \( \log_{10}(\hat{E}_{r_j}) - \log_{10}(\hat{E}_{r_{j+1}}) > \delta \), we set \( r_{\text{opt}}^* = r_{j+1} \), else we use a linear regression \( l(r) \) to model the trend of the data

\[ \{ \log_{10}(\hat{E}_{r_{j+1}}/f), \ldots, \log_{10}(\hat{E}_{r_{\text{opt}}^*}/f) \} \]

and we set

\[ r_{\text{opt}}^* = r_k, \]

where

\[ \log_{10}(\hat{E}_k) = \min_{s \in \{1, \ldots, j\}} \{ \log_{10}(\hat{E}_s) : \log_{10}(\hat{E}_s) - l(r_s) \geq 0 \}. \]

Algorithm 3: Adaptive algorithm for determining a quadrature formulas with high degree of exactness and accuracy from equispaced nodes

\textbf{Require:} \( \xi = [x_0, \ldots, x_n]^T, f = [f_0, \ldots, f_n]^T \)

\textbf{Ensure:} \( Q_{m,n}[f] \)

1: Compute \( m \)
2: Compute the mock-Chebyshev subset \( \Lambda_m \)
3: Compute \( \Lambda_{m,m} = \Lambda_m \setminus \Lambda_{m-1} \)
4: Set \( \Lambda = \{ \Lambda_m, \Lambda_{m,m} \} \)
5: Compute the Gauss–Christoffel nodes and weights of order \( m \)
6: for \( r = m : 2m-1 \) do
7: \hspace{1cm} Compute \( P_{m,n}[f] \)
8: \hspace{1cm} Compute \( Q_{m,n}[f] \)
9: end for
10: Compute the approximate relative errors \( \hat{E}_m[f], \ldots, \hat{E}_{2m-2}[f] \)
11: Run Algorithm 1
12: Run Algorithm 2
13: Compute \( r_{\text{opt}} \)

Computational cost

We determine the computational cost of the Algorithm 3 described above. The computational cost of \( m = [\pi \sqrt{n}/\sqrt{2}] \) is negligible. By using the procedure proposed in [1], the selection of the mock-Chebyshev subset from the uniform grid \( \Lambda_n \) requires about \( O(mn) \) flops. The reordering of the set \( \Lambda_n \) involves a searching algorithm for the computation of the set \( \Lambda_{m,m}^* \), whose computational cost is \( O(m \log(n)) \) flops. Gauss–Christoffel nodes and weights can be computed through the Chebfun package [13]. The function \texttt{leastsq} to compute Legendre nodes and weights, used in the numerical experiments, requires \( O(\frac{m \log m n^2}{\log(nm)^2}) \) flops. For each \( r \in [m, 2m - 1] \), the computation of the coefficients of the polynomial \( \hat{P}_{m,n}[f] \) through the Lagrange multipliers method [7] requires \( O(m^2 n) \) flops for the construction of the KKT matrix and \( O(m n) \) for the solution of the linear system through the QR factorization [2]. In passing from \( r \) to \( r + 1 \) the new KKT matrix can be obtained from the previous one by a negligible cost, and then the computational cost for computing \( \hat{P}_{m,n}[f], r = m, \ldots, 2m - 1 \), is still \( O(m^2 n) \). Since \( Q_{m,n}[f] \) can be computed by \( O(m^2) \) flops, then the total cost of the for loop is \( O(m^2 n^2) \) flops. The tolerance is computed through the Algorithm 1 and the detection of the outliers is made by using the Algorithm 2, which both require \( O(m) \) flops. Finally, we determine the degree of regression \( r_{\text{opt}} \), which produces accurate quadrature formulas, by using the procedure described in the Section 2. In its turn this procedure requires \( O(n) \) flops. Since \( m = O(\sqrt{n}) \), the computational cost of the Algorithm 3 is \( O(n^2) \) flops.
3 Computing accurate cubature formulas with high degree of exactness from regular grids of nodes

Let \( f(x, y) \) be a sufficiently regular function in the square \([-1, 1]^2\) and let \( X_n \times Y_n \) be the grid of \((n_x + 1) \times (n_y + 1)\) equispaced points in the same domain. In analogy with the univariate case, we set \( m_x = [\pi \sqrt{n_x} / \sqrt{2}] \), \( m_y = [\pi \sqrt{n_y} / \sqrt{2}] \) and we fix \( r_{x,y} = (r_x, r_y) \in \mathbb{N} \times \mathbb{N} \) such that \( m_x \leq r_x \leq n_x \) and \( m_y \leq r_y \leq n_y \). We denote by
\[
\hat{P}_{r_{x,y}}[f] := \hat{P}_{(r_x,r_y)(n_x,n_y)}[f],
\]
the tensor product extension of the polynomial \( \hat{P}_{r_{x,y}}[f] \) and we consider the quadrature formulas
\[
I[f] := \int_{-1}^{1} \int_{-1}^{1} w(s, t) f(s, t) ds dt \approx \sum_{i=1}^{m_x} \sum_{j=1}^{m_y} w_i \kappa_i f(\xi_i, \eta_j),
\]
\[
\approx \sum_{i=1}^{m_x} \sum_{j=1}^{m_y} w_i \kappa_i \hat{P}_{r_{x,y}}(\xi_i, \eta_j) := Q_{r_{x,y}}[f],
\]
where \( \xi_1, \ldots, \xi_{m_x} \) and \( \eta_1, \ldots, \eta_{m_y} \) are nodes of a Gaussian quadrature formula with weights \( w_1, \ldots, w_{m_x} \) and \( \kappa_1, \ldots, \kappa_{m_y} \) of order \( m_x \) and \( m_y \), respectively. For a matter of simplicity, we restrict to the case \( n_x = n_y = n \), then \( m_x = m_y = m \), and we consider the family of quadrature formulas
\[
\hat{Q}_{r_{x,y}}[f], \quad m \leq r \leq 2m - 1,
\]
and the approximate relative errors
\[
\hat{E}_r[f] = \frac{\left| \hat{Q}_{r+1,r+1}[f] - \hat{Q}_{r,r+1}[f] \right|}{\hat{Q}_{r,r}[f]}, \quad m \leq r \leq 2m - 1.
\]
In analogy to the univariate case, by using the Algorithm 3, it is possible to determine a value \( r_{\text{opt}} \) of the degree of regression which produces accurate cubature formulas.

4 Numerical experiments

4.1 Univariate case

In this Section, we compute the approximation of the integral (1) by using the quadrature formula (7) with \( r = r_{\text{opt}} \), where the polynomial \( \hat{P}_{r_{\text{opt}}}^m[f] \) is expressed in the Chebyshev basis \( \mathcal{B}^C \). To this aim, we consider the grid of 1001 equispaced nodes in \([-1, 1]\), that is \( n = 1000 \), \( m = 70 \). The experiments are performed on the following functions
\[
\begin{align*}
 f_1(x) &= \frac{1}{1 + 8x^2}, \quad f_2(x) = \frac{1}{1 + 25x^2}, \quad f_3(x) = \frac{1}{(x + 4 + (2/5)2^2)}, \\
 f_4(x) &= \exp(-x^2), \quad f_5(x) = \frac{1}{x^4 + (\frac{2\pi}{5} - 1) x^2 + \left(\frac{\pi}{10}\right)^2}, \quad f_6(x) = \frac{1}{x + 1.01},
\end{align*}
\]
by using the Gauss–Legendre weight \( w(x) = 1 \).

In Table 1, from left to right, we compare the relative errors obtained by applying the trapezoidal composite rule \( (E_T) \), the Cavalieri–Simpson composite rule \( (E_C) \), the quadrature formula proposed in [16] with \( s = 6 \) \( (E_{MC}) \), the constrained mock-Chebyshev least squares quadrature formula proposed in [8] \( (E_{MC}) \) and the proposed here quadrature formula \( (E_{MC}) \). To appreciate the accuracy of the estimate of the exact relative error, obtained through the Algorithm 3, in the last column we report also \( \hat{E}_{r_{\text{opt}}} \). To show the efficacy of the Algorithm 3 in computing the optimal regression degree \( r_{\text{opt}} \), in Figures 4-9 we display the sequences of exact relative errors, approximate relative errors (with discarded outliers, in red) and the regression line of the significant data, for all test functions.

4.2 Bivariate case

We consider the grid of \( 151 \times 151 \) equispaced nodes in \([-1, 1]^2\), that is \( n_x = n_y = 150 \), \( m_x = m_y = 27 \). The experiments are performed by using the Gauss–Legendre weight and the well-known Franke’s function
\[
\begin{align*}
 f(x, y) &= 0.75 \exp\left(-\frac{(9(x + 1)/2 - 2)^2}{4} - \frac{(9(y + 1)/2 - 2)^2}{4}\right) \\
 &+ 0.75 \exp\left(-\frac{(9(x + 1)/2 + 1)^2}{49} - \frac{(9(y + 1)/2 + 1)}{10}\right) \\
 &+ 0.5 \exp\left(-\frac{(9(x + 1)/2 - 7)^2}{4} - \frac{(9(y + 1)/2 - 3)^2}{4}\right) \\
 &- 0.2 \exp\left(-\frac{(9(x + 1)/2 - 4)^2 - (9(y + 1)/2 - 7)^2}{4}\right),
\end{align*}
\]
Table 1: Comparisons among the relative errors in trapezoidal composite rule \(E_T\), Cavalieri–Simpson composite rule \(E_{CS}\), quadrature formula proposed in [16] \(E_M\), quadrature formula through the constrained mock-Chebyshev interpolant with optimal degree \(E_{r*opt}\) and the approximate relative error obtained through Algorithm 3 \(\hat{E}_{r*opt}\).

<table>
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<tr>
<th>(f_i(x))</th>
<th>(E_T)</th>
<th>(E_{CS})</th>
<th>(E_M)</th>
<th>(E_{r*opt})</th>
<th>(\hat{E}_{r*opt})</th>
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<tr>
<td>(f_1(x))</td>
<td>1.33e-04</td>
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<td>7.84e-13</td>
<td>2.55e-16</td>
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<td>(f_2(x))</td>
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<td>1.02e-12</td>
<td>4.13e-12</td>
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<td>(f_3(x))</td>
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<td>3.75e-13</td>
<td>1.59e-14</td>
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<tr>
<td>(f_4(x))</td>
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<td>8.77e-15</td>
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<td>(f_5(x))</td>
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Figure 4: From left to right. Exact relative errors, approximate relative errors (•) with discarded outliers (◦) and regression line of the significant data for the function \(f_1(x)\).

Figure 5: From left to right. Exact relative errors, approximate relative errors (•) with discarded outliers (◦) and regression line of the significant data for the function \(f_2(x)\).

Figure 6: From left to right. Exact relative errors, approximate relative errors (•) with discarded outliers (◦) and regression line of the significant data for the function \(f_3(x)\).
Figure 7: From left to right. Exact relative errors, approximate relative errors (•) with discarded outliers (◦) and regression line of the significant data for the function $f_4(x)$.

Figure 8: From left to right. Exact relative errors, approximate relative errors (•) with discarded outliers (◦) and regression line of the significant data for the function $f_5(x)$.

Figure 9: From left to right. Exact relative errors, approximate relative errors (•) with discarded outliers (◦) and regression line of the significant data for the function $f_6(x)$.
Algorithm 3 (\( \hat{E}_{opt} \)).

Table 2: Comparisons among the relative errors in trapezoidal composite rule (\( E_T \)), Cavalieri–Simpson composite rule (\( E_{CS} \)), cubature formulas through the constrained mock-Chebyshev tensor product interpolant with optimal degree (\( E_{opt}^{*} \)) and approximate relative error obtained through Algorithm 3 (\( \hat{E}_{opt}^{*} \)).

<table>
<thead>
<tr>
<th>( f(x,y) )</th>
<th>( E_T )</th>
<th>( E_{CS} )</th>
<th>( E_{opt}^{*} )</th>
<th>( \hat{E}_{opt}^{*} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.30e-05</td>
<td>1.71e-10</td>
<td>4.69e-12</td>
<td>2.46e-11</td>
<td></td>
</tr>
</tbody>
</table>

where the polynomial \( \hat{p}_{r,x} \) is expressed in the tensor product Chebyshev basis \( B_r \otimes B_C \). In Table 2, from left to right, we compare the relative errors obtained by applying the tensor product trapezoidal composite rule (\( E_T \)), the tensor product Cavalieri–Simpson composite rule (\( E_{CS} \)) and the proposed here quadrature formula (\( E_{opt}^{*} \)). To appreciate the accuracy of the estimate of the exact relative error, obtained through the Algorithm 3, in the last column we report also \( \hat{E}_{opt}^{*} \).

5 Conclusions

In this paper, we have developed an adaptive algorithm for determining accurate quadrature formulas with high degree of exactness from \( n + 1 \) equispaced nodes in the interval \([-1, 1]\). Starting from the mock-Chebyshev interpolant of an analytic function \( f(x) \), the increasing of the degree \( p \) of the simultaneous regression allows us to generate a family of quadrature formulas, with increasing degree of exactness \( r = m + p \). Infact, we approximate, in a Gauss–Cristoffel quadrature rule with \( m \) nodes, the values of \( f(x) \) at the Legendre nodes of order \( m = \lfloor \pi \sqrt{r}/\sqrt{2} \rfloor \) through the constrained mock-Chebyshev least-squares approximant. A data cleanup strategy and a linear regression on the significant relative errors allow us to determine the “optimal” degree of regression to obtain quadrature formulas as much accurate as possible. The procedure is generalized to obtain cubature formulas on the square \([-1,1]^2\) through the constrained mock-Chebyshev least-squares tensor product interpolation.

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