Cubature rules with positive weights on union of disks

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Abstract

In this work we present a new algorithm that computes cubature formulas with positive weights, interior nodes and fixed algebraic degree of precision, over domains $\Omega$ that are arbitrary union of disks. This novel approach first determines the boundary $\partial\Omega$ and then defines a decomposition of $\Omega$ by means of nonoverlapping circular segments and polygons, where algebraic positive interior rules can be locally constructed. The resulting global Positive Interior (PI) formula is finally compressed by Carathéodory-Tchakaloff subsampling implemented via NonNegative Least-Squares.

1 Introduction

Numerical modelling by finite collections of arbitrary disks/balls is relevant in several different applications. Problems involving disk/ball intersection, union and difference arise for example in computational optics, wireless network analysis, computational chemistry (Van der Waals molecular modelling); see, e.g., [1, 3, 12, 14, 15, 26] with the references therein. A basic problem is the computation of areas and volumes of such sets, followed by the more difficult task of computing integrals on them by suitable cubature formulas, in particular algebraic formulas (i.e., with a given degree of polynomial exactness) having positive weights and interior nodes (PI-formulas).

In the recent paper [20] we have constructed low-cardinality algebraic PI-formulas on arbitrary disk intersections, the main tools being subdivision of the intersection into nonoverlapping symmetric circular sectors, PI algebraic cubature on such sectors via subperiodic trigonometric Gaussian quadrature [5], and cubature compression via Carathéodory-Tchakaloff subsampling implemented by NonNegative Least Squares [16]. This work was apparently the first systematic approach to the algebraic cubature problem on disk intersections, and contained also an approach for disk union, via a basic implementation of the inclusion-exclusion principle, which however suffers of exponential complexity and is prone to produce a huge number of nodes and negative weights.

The disk intersection cubature problem, though nontrivial, is somehow simplified by the fact that the intersection is a convex curvilinear polygon, whose sides are circular arcs. In this paper, we cope the more difficult disk union cubature problem, whose core is boundary tracking of the resulting intrinsically nonconvex (and possibly multiply connected) curvilinear polygon.

The main lines of the construction are the following. Let $\Omega = \cup_j B(p_j,r_j)$ an arbitrary finite union of closed planar disks centered at $p_j$ with radius $r_j$. First, we split $\Omega$ into its connected components (that are disk sub-unions), namely $\Omega = \cup_i \Omega_i$. Notice that some of the $\Omega_i$ can be multiply connected if such is $\Omega$. Then, by a boundary tracking algorithm that solves the delicate problem of detecting the arc components, we are able to split each $\Omega_i$ into the nonoverlapping union of circular segments $S_{k,j}$ (disk portions corresponding to a cut by a straight line) and of a single simple polygon $P_k$ (possibly multiply connected if such is $\Omega_k$), obtaining eventually

$$
\Omega = \cup_j B(p_j,r_j) = \cup_i (\cup_j S_{k,j}) \cup P_k, \tag{1}
$$

see Figure 1 and also Figure 2 to have an idea of the variety of possible configurations. From this splitting we obtain the algebraic cubature formulas exact for every polynomial $p \in \mathbb{P}_n$

$$
\int_{\Omega} p(x) \, dx = \sum_{k,j,h} \lambda_{k,j,h} p(\xi_{k,j,h}) + \sum_{k,l} \lambda_{k,l} p(\xi_{k,l}) = \sum_{i=1}^{M} \lambda_i p(\xi_i) = \sum_{i=1}^{M} w_i p(\xi_i), \quad \{\xi_i\} \subset \{\xi_j\} \subset \{\xi_{k,j}\} \subset \{\xi_{k,l}\} \subset \Omega, \quad \lambda_i, w_i > 0, \tag{2}
$$

where the first equality comes from the collection of PI-formulas on circular segments [7] and linear polygons (cf. e.g. [2]), the second one is simply a renumbering of the overall set of $M$ corresponding nodes, and the third one corresponds to cubature compression via Carathéodory-Tchakaloff subsampling implemented by NonNegative Least Squares, where a subset of nodes is extracted and re-weighted preserving the polynomial degree of exactness [16]. We stress that $m \leq \dim(\mathbb{P}_n) = (n+1)(n+2)/2 < M$ and $m \ll M$ for the union of a large number of disks (the ratio $M/m$ being roughly proportional to such a number).

In Section 2 we focus on the boundary tracking problem for arbitrary disk unions, that leads in Section 3 to the construction of high-cardinality PI-formulas by splitting into nonoverlapping circular segments and linear polygons. Such formulas can be conveniently compressed via NonNegative Least Squares applied to the underdetermined moment system, as shown in Section 4. Finally, in Section 5 we present some numerical experiments on disk unions with quite complex shape.

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2 Boundary tracking

In this section we intend to determine the boundary $\partial \Omega$, where $\Omega = \bigcup_{k=1}^{K} B(P_k, r_k)$. This result will be one of the key points for providing an algebraic cubature rule on $\Omega$ with positive weights and internal nodes. To understand the difficulty of this analysis, in Figure 2 we illustrate four very different domains $\Omega$, i.e., from the left to the right, a simply connected, a disconnected, a multiply connected domain and finally one with seven disks that are tangent.

To this purpose, the simplest case is given two disks $B_1 = B(P_1, r_1)$, $B_2 = B(P_2, r_2)$ respectively with centers $P_1$, $P_2$ and radii $r_1$, $r_2$. First define the interval of angular coordinates $I_{B_1, B_2} \subseteq [0, 2\pi)$ such that

$$\partial B_1 \cap \partial (B_1 \cup B_2) = \{(x, y) : x = P_1(1) + r_1 \cos(\theta), y = P_1(2) + r_1 \sin(\theta), \theta \in I_{B_1, B_2}\}.$$

In other words, $I_{B_1, B_2}$ is the set of angular coordinates in the range $[0, 2\pi)$ w.r.t $P_1$, center of the disk $B_1$, of the portion of the boundary of $\partial (B_1 \cup B_2)$ that is also in $\partial B_1$.

Observe that

- in general, $I_{B_1, B_2}$ is a pluri-interval (union of disconnected intervals): for example, in Figure 3 $I_{B_1, B_2} = [0, \pi/4] \cup [5\pi/4, 2\pi]$;
- if $B_1 \cap B_2 = \emptyset$ then $I_{B_1, B_2} = [0, 2\pi]$;
- if $\partial B_1$ and $\partial B_2$ are tangent in a point $T$ then if $B_1 \subset B_2$

$$\partial B_1 \cap \partial (B_1 \cup B_2) = \partial B_1 \cap \partial B_2 = T$$

and thus $I_{B_1, B_2}$ consists of the angular value of the polar coordinates of $T$ with respect to $P_1$, otherwise

$$\partial B_1 \cap \partial (B_1 \cup B_2) = \partial B_1$$
and \( I_{B_1, B_2} = [0, 2\pi] \).

- in general, in view of its definition, \( I_{B_1, B_2} \) may not be equal to \( I_{B_2, B_1} \), indeed in Figure 3 \( I_{B_2, B_1} = [\pi/4, 7\pi/4] \).

In the more general case of \( \Omega = \bigcup_{k=1}^K B_k \) where \( B_k := B(P_k, r_k) \), it is not difficult to see that the portion of \( \partial B_k \) that is in \( \partial \Omega \) corresponds to
\[
\partial B_k \cap \partial \Omega = \{(x, y) : x = P_k(1) + r_k \cos(\theta), y = P_k(2) + r_k \sin(\theta), \theta \in I_k := \cap_{b \neq k} I_{B_k, B_b} \}
\]

Figure 4: Left: Example of a disk, the lower one, with an isolated point \( Q \) belonging to the boundary of the union of disks. Right: Example of a disk, the inner one, with an isolated point \( Q \) belonging to the boundary of the union of disks.

Also notice that \( I_k \) is in general a pluri-interval, i.e. the union of some disconnected subintervals of \([0, 2\pi]\), some of which may actually be isolated points and in this case they can be dropped since they actually do not contribute to the determination of \( \partial \Omega \). In view of this last observation, we will define \( I_k^* \) as \( I_k \) without isolated points. It is easy to see that the set \( I_k^* \) may be
- equal to the empty set and in this case the disk \( B_k \) is completely in the interior of \( \Omega \) or has some point belonging to the boundary that does not add any contribution to the definition of \( \partial \Omega \);
- equal to \([0, 2\pi]\), i.e. the interior of the disk \( B_k \) does not intersect the interior of any other disk;
- union of disconnected intervals \( I_k^*, \ldots, I_{k, \mu_k} \) where we can suppose after a suitable reordering that max \( I_{k, j}^* \) < min \( I_{k, j+1}^* \) for \( j = 1, \ldots, \mu_k - 1 \), and in particular none of them are isolated points.

Applying the same procedure to all the disks \( B_k \), \( k = 1, \ldots, K \), we get all the sets \( \partial B_k \cap \partial \Omega \) in terms of \( I_k \) = \( \bigcap_{b=1}^\nu I_{B_k, B_b} \) and next \( I_k^* \) after purging \( I_k \) of possible isolated points. As previously stated, we can suppose that the pluri-interval \( I_k^* \) can be described as \( I_k^* = \bigcup_{b=1}^\nu I_{k, b}^* \) where max \( I_{k, j}^* \) < min \( I_{k, j+1}^* \) for \( j = 1, \ldots, \mu_k - 1 \).

At this point we can also require that all the disks \( B_k \) provide some not empty sets \( I_k^* \), otherwise they do not give any contribution to the determination of \( \partial \Omega = \partial (\bigcup_{k=1}^K B_k) \) and can be dropped without any consequence.

Now we intend to determine the boundary of each connected component of \( \Omega \) as a sequence of arcs, each one having intersection with the next one only on its final extremal. To help the reading, see Figure 4.

Let
\[
\gamma_{k,j} := \{(x, y) : x = P_k(1) + r_k \cos(\theta), y = P_k(2) + r_k \sin(\theta), \theta \in I_{k,j}^* \},
\]
for \( k = 1, \ldots, K, j = 1, \ldots, \mu_k \) be the arcs defining the boundary. Furthermore let \( I_{k,j}^* = [a_{k,j}, b_{k,j}] \) and set
\[
\gamma_{k,j}^{(1)} = (P_k(1) + r_k \cos(a_{k,j}), P_k(2) + r_k \sin(a_{k,j}))
\]
\[
\gamma_{k,j}^{(2)} = (P_k(1) + r_k \cos(b_{k,j}), P_k(2) + r_k \sin(b_{k,j}))
\]
i.e. the two extremal points of the arc \( \gamma_{k,j} \), for \( k = 1, \ldots, K, j = 1, \ldots, \mu_k \).

Letting \( \Gamma^{(1)}_1 = \gamma_{1,1} \), we have two possibilities:

- \( \gamma_{1,1}^{(1)} = \gamma_{1,1}^{(2)} \) in which case we have determined a closed arc \( \Gamma_1 = \gamma_{1,1}^{(1)} \) of \( \partial \Omega \), i.e. an arc where the first and final extrema coincide, that is the whole boundary of an isolated disk;

- \( \gamma_{1,1}^{(1)} \neq \gamma_{1,1}^{(2)} \) in which case, by construction, there is exactly one arc, say \( \Gamma_{1,2} = \gamma_{k,j} \) such that \( \gamma_{k,j}^{(1)} = \gamma_{k,j}^{(2)} \).

This procedure can be iterated until for a certain \( I_k \) the last extrema of the arc \( \Gamma_{k,1} \) is equal to the first extremal of \( \Gamma_{k,1} \).

At this point, if no arc is available, i.e. all the arcs \( \gamma_{k,j} \), \( k = 1, \ldots, K, j = 1, \ldots, \mu_k \) took part in the process, then we have determined the boundary of \( \partial \Omega \) and it corresponds to \( \Gamma_1 = \bigcup_{k=1}^K \Gamma_{k,1} \), otherwise we pick randomly one of the missing \( \gamma_{k,j} \) and repeat the procedure to compute \( \Gamma_2 \) and if necessary, \( \Gamma_3, \ldots, \Gamma_n \), until all the arcs \( \gamma_{k,j} \), \( k = 1, \ldots, K, j = 1, \ldots, \mu_k \) took part of the process.

Notice that, since the single circle arcs are counterclockwise tracked, as a result whatever is the order of the disks in the union, the outer boundaries are counterclockwise tracked, as well as in the multiply connected case the inner boundaries are clockwise tracked; see Figure 1 to have an idea.

When the procedure ends, we have determined the boundary of the domain, as \( \partial \Omega = \bigcup_{k=1}^n \Gamma_k \), i.e. of \( \nu \) possibly disconnected and closed curves, each being piecewise arcs.

Some of the main worries of this algorithm concern how it treats some pathological cases:
two connected closed arcs $\Gamma_i, \Gamma_j$, with $i \neq j$, are somewhere tangent: in this case, the boundary is still defined correctly, as consequence of the fact that if two disks are tangent then $I^{\ast}_{B_1B_2} = I^{\ast}_{B_2B_1} = \emptyset$;

• $\Omega$ is not a connected region: the algorithm detects correctly its boundaries since in two disconnected regions all the arcs in the first one are not connected to arcs in the second one, so providing two different $\Gamma_k$.

3 Construction of algebraic PI-formulas on union of disks

Once that the boundary $\partial \Omega$ is described via its closed curves $\Gamma_1, \ldots, \Gamma_n$, that are its outer and inner boundaries, we are ready to determine the cubature formula. For the sake of simplicity, we initially suppose that each of its possibly disconnected components $\Omega_k$ are simply connected, so that $\Gamma_k = \partial \Omega_k$ are outer boundaries.

We seek an algebraic PI-formula with Algebraic Degree of Exactness $ADE = n$. In view of the additivity of the integral, this is immediately obtained by collection of PI-formulas with $ADE = n$ on each $\Omega_k$, and in turn these can be obtained by nonoverlapping splitting of $\Omega_k$.

Indeed, we start observing that $\Omega_k$ is the union of say $L_k$ circular segments and a polygon $P_k$. More precisely, since $\Gamma_k = \bigcup_{j=1}^{n_k} \Gamma_{k,j}$ for some arcs $\Gamma_{k,1}, \ldots, \Gamma_{k,n_k}$, letting $I^{(1)}_{k,j}$, $I^{(2)}_{k,j}$ the extrema of each $\Gamma_{k,j}$ (ordered counterclockwise), we have that $\Omega_k$ is the union of:

1. the circular segments $S_{k,j}$, $j = 1, \ldots, L_k$ whose boundary is defined by $\Gamma_{k,j}$ and the linear segment connecting $I^{(2)}_{k,j}$ with $I^{(1)}_{k,j}$,

2. the polygon $P_k$ whose sides are the segments obtained by connecting $I^{(1)}_{k,j}$ with $I^{(2)}_{k,j}$ for $j = 1, \ldots, L_k$.

Notice that taken any two of these circular segments cannot overlap, otherwise there would be an arc portion of one segment, that is a portion of the boundary of $\Omega$, contained in the interior of another, that is in the interior of $\Omega$. This also implies that $P_k$ is a simple polygon, otherwise we would have two overlapping circular segments.

We are now ready to determine a cubature formula on $\Omega_k$. Again, by additivity of the integral, it is sufficient to have an algebraic PI-formula with $ADE = n$ on the circular segments $S_{k,j}$ and on the simple polygon $P_k$.

Concerning circular segments, by no loss of generality (up to a rotation and a translation) we can consider a circular segment, say $S$, of a disk centered at the origin with radius $r$, corresponding to a vertical cut with angular extension say $2\sigma$, $0 < \sigma < \pi$; see Figure 6. Then following [7], by the injective transformation $x(u, \theta) = (r \cos(\theta), ru \sin(\theta))$, $u \in [-1, 1]$, $\theta \in [0, \sigma]$, and the same with $\theta \in [-\sigma, 0]$, both with Jacobian $r^2 \sin^2(\theta)$, we get the cubature formula of product Gaussian type

$$\int \int_S p(x) \, dx = \frac{1}{2} \int_{-1}^{1} \int_{-\sigma}^{\sigma} p(x(u, \theta)) r^2 \sin^2(\theta) \, d\theta \, du = \sum_{j=1}^{n_k} \sum_{i=1}^{n_k} \lambda_{j,i} f(x_{j,i}) = \sum_{h=1}^{\nu} \lambda_h f(x_h), \ \forall p \in P_2(S)$$
\[ \lambda_{ij} = r^2 \sin^2(\phi_i) \omega_i z_i, \quad x_{ij} = (r \cos(\phi_i), ru_i \sin(\phi_i)), \]

where \(((u_i, \omega_i))\) are the nodes and weights of the algebraic Gauss-Legendre formula for degree \(n\) in \([-1, 1]\), and \((\{\phi_i, z_i\})\) are the angular nodes and weights of the subperiodic trigonometric Gaussian formula for degree \(n + 2\) in \([-\sigma, \sigma]\) developed in [6] (the last sum in \(n\) simply corresponds to a renumbering of the nodes). Indeed, the key points are that \(p(x(u, \theta)) \sin^2(\theta) \in P_a([-1, 1]) \otimes \mathbb{R}_{\sigma}[\mathbb{R}, -\sigma, \sigma])\) that is the tensor product space of univariate algebraic polynomials of degree not exceeding \(n\) and univariate trigonometric polynomials of degree not exceeding \(n + 2\) (where subperiodicity means that the trigonometric polynomials are restricted to a subinterval \([-\sigma, \sigma]\) of the period), and that the nodes are substantially repeated twice by symmetry.

Concerning the simple polygon \(P_k\), whose number of sides is \(L_k\), i.e., the number of circular segments pertaining to \(\Omega_k\), we have adopted the algebraic cubature rule with positive weights and internal points on general polygons, even not simply connected or disconnected, implemented in [2]. In such cases, the corresponding algorithm determines a minimal triangulation of the polygon (via the Matlab polyshape and triangulation routines), with a number of triangles equal to \(L_k - 2\), and then, by the best known rules over each triangle with \(ADE = n\), a PI cubature formula on the whole polygon.

![Figure 7: A multiply connected union and the definition of its boundary.](image)

The case in which a component \(\Omega_k\) is not simply connected is a little more complicated. Suppose that its outer boundary is \(\Gamma_k^+\) while \(\Gamma_{k,1}, \ldots, \Gamma_{k,\lambda_k}\) are the closed curves determining the inner boundaries, one for each possible hole (see Figure 7).

The component \(\Omega_k\) is the union of

1. the circular segments defined by the arcs in \(\Gamma_k^+\),
2. the circular segments defined by the arcs in each \(\Gamma_{k,1}, \ldots, \Gamma_{k,\lambda_k}\),
3. the not simply connected polygon \(P_k\) having as sides the segments connecting each of the previous arcs.

Observe that again \(\partial P_k\) does not cross itself, that is \(P_k\) is a simple polygon with holes.

One of the possible difficulties, in the case of a not simply connected domain, consists in detecting the boundary of a connected but not simply connected component of \(\Omega\). Consider two closed curves \(\Gamma_1, \Gamma_2\). Let \(P_1, P_2\) be respectively the polygons obtained by connecting the subsequent vertices of the arcs determining, respectively, \(\Gamma_1\) and \(\Gamma_2\). Then, if \(P_i\) contains in its interior a vertex of \(P_2\), necessarily \(\Gamma_2\) is the boundary of a hole inside the region spanned by \(\Gamma_1\).

Once we have determined \(\Gamma_1^+, \Gamma_{1,1}, \ldots, \Gamma_{1,\lambda_1}\), \(\Gamma_2^+, \Gamma_{2,1}, \ldots, \Gamma_{2,\lambda_2}\), we can obtain a PI cubature formula on this subdomain, by a PI algebraic rule on the not simply connected polygon \(P_k\) (having as outer vertices the extremal points of its ordered sequence of arcs, and as inner vertices the extremal points of the ordered sequence of arcs of each \(\Gamma_{k,j}, j = 1, \ldots, \lambda_k\)), and PI formulas constructed as described above on the remaining circular segments, whose union is the closure of \(\Omega_k \setminus P_k\).

The collection of all these PI cubature formulas provides eventually a PI cubature formula exact on \(\mathbb{P}_n^2\)

\[
\int \int_{\Omega} p(x) \, dx = \sum_{k,j} \lambda_{k,j} p(x_{k,j}) + \sum_{k,j} \lambda_{k,j} p(x_{k,j}) \equiv \sum_{i=1}^{M} \lambda_i p(x_i), \quad \forall p \in \mathbb{P}_n^2,
\]

where the first sum corresponds to the collection of circular segments \(S_{k,j}\) and the second to the collection of the (possibly multiply connected) polygons \(\{P_k\}\), whereas the final sum is simply a renumbering of the whole set of nodes. Notice that, denoting by \(L = \sum L_k\) the overall number of circular segments (that is typically proportional to the number of disks, or more precisely to the number of disks which contribute to the boundary of the union), and recalling the classical lower bound [23] for a cubature formula with \(ADE = n\) that is \(V_n \approx \text{dim}(\mathbb{P}_n^2) \approx (1 + n/2)(2 + n/2)/2 = (n + 2)(n + 4)/8\), we get that the cardinality of the polygon cubature formula is at least \((L - 2C)V_n \approx (L - 2C)(n + 2)(n + 4)/8\), where \(C\) denotes the overall number of connected components of the union. Then the overall cardinality \(M\) is at least of the order of \((n + 1)(n + 2)/4 + (L - 2C)(n + 2)(n + 4)/8 > LN/2 + (L - 2C)N/4 = (3L - C)/2)N\). This possibly large cardinality \(M\) can be reduced to at most \(N = \text{dim}(\mathbb{P}_n^2)\) by Caratheodory-Tchakaloff subsampling, as described in the next subsection, obtaining a compression ratio of at least \(3L/4 - C/2\).
3.1 Caratheodory-Tchakaloff subsampling

The possibility of reducing the cardinality of a PI cubature formula inserts in the more general problem of measure compression, that is finding a discrete representative with low-cardinality finite support of a given multivariate measure, keeping invariant a certain number of polynomial moments. Such a problem has a long history, dating back at least to V. Tchakaloff with his celebrated theorem in 1957 [24].

Technically, in the case of a starting discrete measure (like a cubature formula) with high-cardinality support $X = \{x_1, \ldots, x_M\} \subset \mathbb{R}^d$ and positive point-masses array $\lambda = (\lambda_1, \ldots, \lambda_M)$, the problem can be formulated as that of finding a subset of the support, say $\{\xi_1, \ldots, \xi_N\} \subset X$ such that

$$\sum_{i=1}^M \lambda_i p(x_i) = \sum_{\ell=1}^N w_\ell p(\xi_\ell), \quad \forall p \in \mathbb{P}_n^d,$$

or equivalently in matrix terms finding a sparse nonnegative solution $u \in \mathbb{R}^N$ of the underdetermined moment system

$$u \geq 0 : \quad V^+u = b = V\lambda, \quad V = [p_i(x_j)], \quad 1 \leq i \leq M, \quad 1 \leq j \leq N,$$

where $V$ is a Vandermonde-like matrix in any total-degree polynomial basis $\text{span}(p_1, \ldots, p_N) = \mathbb{P}_n^d$, and $b = (b_\ell) = (\sum_{i=1}^M \lambda_i p_i(x_\ell))$ is the corresponding moment array. We recall that existence of a nonnegative solution $u^*$ with a number of nonzeros $m \leq N$ is ensured by the well-known Caratheodory theorem [4] on conical linear combination of a set of $N < M$ vectors in $\mathbb{R}^d$, applied to the columns of $V$. The nonzero components of $u^*$ then are the new weights $\{w_\ell\}$ associated to a reduced support $\{\xi_\ell\} \subset X$.

Over the past decade, there has been a renewed interest in the numerical as well as in the probabilistic literature on the solution of (5) by optimization algorithms, namely by Linear or Quadratic Programming; cf. e.g. [11, 16, 25] with the references therein. Here we adopt the NNLS (NonNegative Least-Squares) approach developed in [22, 16] that consists in solving

compute $u^*$: $\|Vu^* - b\|_2 = \min_{u \geq 0} \|Vu - b\|_2,$

by Matlab implementations of the well-known Lawson-Hanson active-set algorithm [13], which automatically determines a sparse solution to (6). Its application gives a residual $\epsilon = \|Vu^* - b\|_2$ that is typically very small, say $< 10^{-14}$ for $n \leq 30$. We point out that there are several versions of NNLS codes available in Matlab. One is the built-in function $\text{lsqnonneg}$, based on the Lawson-Hanson algorithm while an open-source version is present in the package NNLSlab [18]. A new and promising acceleration of the Lawson-Hanson algorithm is implemented in the routine LHDM first discussed in [10], based on the concept of column selection by deviation maximization instead of standard column pivoting for QR factorizations (cf. also [8, 9] for a full theoretical and numerical analysis of the deviation maximization approach).

We can then conclude by stressing that by Caratheodory-Tchakaloff compression of the PI cubature formula for arbitrary disk union developed above, we are able to provide a final PI formula whose support is a subset of the original one, with cardinality not exceeding the dimension of the exactness polynomial space. The corresponding Matlab codes are freely available at [19].

Remark 1. It is worth obtaining an estimate of the convergence rate by the PI cubature formulas just derived, related to the integrand smoothness. Denoting by $p_n^*$ the best uniform approximation polynomial in $\mathbb{P}_n^d$ to a continuous integrand $f$ on $\Omega$, we get easily the estimate

$$\left| \int_\Omega f(x)dx - \sum_{\ell=1}^m w_\ell f(\xi_\ell) \right| \leq \int_\Omega \left( |f(x) - p_n^*(x)| dx + \int_\Omega p_n^*(x) dx - \sum_{\ell=1}^m w_\ell p_n^*(\xi_\ell) \right) + \sum_{\ell=1}^m w_\ell (p_n^*(\xi_\ell) - f(\xi_\ell))$$

$$\leq \left( \text{area}(\Omega) + \sum_{\ell=1}^m w_\ell \right) \|f - p_n^*\|_{\infty, \Omega} = 2 \text{area}(\Omega) \|f - p_n^*\|_{\infty, \Omega},$$

where we have used the fact that the formula is exact in $\mathbb{P}_n^d$ so that the second summand on the first row vanishes, and that the weights are positive. On the other hand

$$\|f - p_n^*\|_{\infty, \Omega} \leq c_k n^{-k} \left( \sum_{a_1, \ldots, a_k = 0}^k \|\partial^{a_1}_x \cdots \partial^{a_k}_x f\|_{\infty, \Omega} + \sum_{a_1, \ldots, a_k = 0}^k \text{osc}_\Omega(\partial^{a_1}_x \cdots \partial^{a_k}_x f; 1/n) \right), \quad \forall f \in C^k(\Omega),$$

where $\text{osc}_\Omega$ is the oscillation on $\Omega$ of a continuous function, i.e. $\text{osc}_\Omega(g; h) = \sup\{|g(u) - g(v)| : u, v \in \Omega, |u - v| \leq h\}$, and $c_k$ is a positive constant. The convergence rate shown in the last bound is a consequence of a classical Jackson-like estimate for multivariate euclidean balls by Ragozin [17, Thm. 3.4, p. 164], via the immediate property that the maximum uniform error on a finite union is the maximum of the uniform errors on the single components.

4 Numerical experiments

The purpose of this section is to numerically compare the cubature rules obtained with the present algorithm with that implemented in [20] by the inclusion-exclusion principle, also reporting for the latter the presence of negative weights. Our numerical tests have been performed on a Apple M1 CPU with 16 GB of RAM, using Matlab R2022a. The open source codes are available at [19].

We consider three different domains, having a complicated geometry:
• the simply connected domain $\Omega_1$ is the union of 15 random disks, with all the centers contained in the square $[0,1] \times [0,1]$ and random radii in $[0,1]$ (see Figure 8-left);
• the domain $\Omega_2$ is $\Omega_2^{(1)} \cup \Omega_2^{(2)}$, where $\Omega_2^{(j)}$, $j = 1, 2$, is the union of 19 disks with centers $P_k^{(j)} = (r_j \cos(\theta_k), r_j \sin(\theta_k))$, where $\theta_k = 2k\pi/19$, $k = 0, \ldots, 18$ and radius equal to $r_j/4$, with $r_1 = 2$ and $r_2 = 4$; notice that the set is the disconnected union of two multiply connected unions (see Figure 8-center);
• the domain $\Omega_3$ is the union of the sets $\Omega_3^{(1)}, \Omega_3^{(2)}$ defined as follows; letting $t_k = 5k/44$, $k = 0, \ldots, 44$, the set $\Omega_3^{(1)}$ is the union of the disks with centers $P_k^{(1)} = (2.5 \cos(2t_k), 2t_k)$ and radius $r_k^{(1)} = 0.3$, while $\Omega_3^{(2)}$ is the union of the disks with centers $P_k^{(2)} = (2.5 \sin(2t_k), 2t_k)$ and radius $r_k^{(2)} = 0.3$; the set is connected but not simply connected (see Fig. 8-right).

In the Tables, varying the domains, we display the quality of the cubature rules, named P(full), C(ompressed) and O(ld), on a sequence of exactness degrees. We may observe that:
• we get always $\text{card}_C = N = \text{dim}(P_2^2) < \text{card}_F \ll \text{card}_O$ and remarkable compression ratios $\text{card}_F/\text{card}_C$ varying in the examples from about 4 to more than 100 (indeed we expect, as discussed above after formula (4), a ratio size of at least $3/4$ the number of circular segments involved);
• the number $\text{negw}_O$ of negative weights of the old formula implemented in [20] is a consistent fraction of the overall weights, and the stability parameter $\sigma_0 = \sum |w_0|/\sum w_0$ (not reported for brevity), independently of the degree is $\sigma_0 \approx 213.0$ on $\Omega_1$, $\sigma_0 \approx 1.785$ on $\Omega_2$ and $\sigma_0 \approx 1.675$ on $\Omega_3$, while for both the new rules is always equal to 1 since all the weights are positive;
• we compare the moments with respect to the product Chebyshev basis of the smaller cartesian boxes containing the domains $\Omega_1$, via the new rule and the compressed one (using LHDM for solving the NNLS problem), by means of the Root Mean-Square Deviation $\text{RMSD}_{EC} = \|b_f - b_c\|_2/\sqrt{N}$ and $\text{RMSD}_{EO} = \|b_f - b_0\|_2/\sqrt{N}$, both being extremely small with the first not far from machine precision and the second at most of the order of $10^{-12}$;
• $\text{CPU}_F$, $\text{CPU}_C$, $\text{CPU}_O$ are the detected cpu-times and show that: (i) the new rule for mild degrees is faster than that proposed in [20]; (ii) boundary tracking time (not reported for brevity) requires on average respectively 6e-3s, 3e-2s, 2e-1s; (iii) as expected, the compression stage becomes relevant when one increases the ADE.

Table 1: Comparison of cubature rules with ADE $n = 5, 10, 15, 20, 25$, on the disk union $\Omega_1$ ($F$=Full, $C$=Compressed, $O$=Old).

<table>
<thead>
<tr>
<th>ADE</th>
<th>card$_F$</th>
<th>card$_C$</th>
<th>card$_O$</th>
<th>RMSD$_{EC}$</th>
<th>RMSD$_{EO}$</th>
<th>negw$_O$</th>
<th>CPU$_F$</th>
<th>CPU$_C$</th>
<th>CPU$_O$</th>
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<td>4e-03</td>
<td>5e-03</td>
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<td>136</td>
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<td>1e-12</td>
<td>1232082</td>
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<td>4e-02</td>
<td>2e+01</td>
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<tr>
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<td>231</td>
<td>3921247</td>
<td>1e-15</td>
<td>1e-12</td>
<td>1959716</td>
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<td>2e-01</td>
<td>3e+01</td>
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<td>6e-03</td>
<td>8e-01</td>
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5 Conclusion

We have implemented an algorithm that computes an algebraic Positive-Interior cubature formula on an arbitrary union of planar disks. The outcome of the algorithm gives several interesting information on such sets, that could be useful in applications where modelling by finite disk collections is adopted. In fact the algorithm:
• detects the connected components of the union;
Table 2: As in Table 1 for the disk union $\Omega_2$.

<table>
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<tr>
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<th>$\text{card}_C$</th>
<th>$\text{card}_O$</th>
<th>$\text{RMSD}_{F,\text{C}}$</th>
<th>$\text{RMSD}_{F,\text{O}}$</th>
<th>neg$\omega_0$</th>
<th>CPU$_F$</th>
<th>CPU$_C$</th>
<th>CPU$_O$</th>
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<td>66</td>
<td>4896</td>
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Table 3: As in Table 1 for the disk union $\Omega_3$.

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<th>$\text{card}_O$</th>
<th>$\text{RMSD}_{F,\text{C}}$</th>
<th>$\text{RMSD}_{F,\text{O}}$</th>
<th>neg$\omega_0$</th>
<th>CPU$_F$</th>
<th>CPU$_C$</th>
<th>CPU$_O$</th>
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<td>66</td>
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<td>6e−15</td>
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<td>9216</td>
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<td>3e−01</td>
<td>3e+00</td>
<td>3e+01</td>
</tr>
</tbody>
</table>

- detects the outer and possible inner boundaries of each connected component (i.e. detects also the possible holes providing their boundary);
- constructs nodes and weights of a PI-formula exact for polynomials of a given total-degree, with cardinality increasing proportionally to the overall number of disks times the degree squared;
- allows then to compute immediately at machine precision some relevant features in applications, such as the area of the union and its first and second monomial moments (i.e. its “center of mass” and “moment of inertia” for a constant density);
- finally, provides a compressed PI-formula with cardinality not exceeding the dimension of the exactness polynomial space, irrespectively of the overall number of disks.

Acknowledgements

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