A Note on Orthogonal Dirichlet Polynomials with Rational Weight

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Abstract

Let \( \lambda_j \to \infty \) be a strictly increasing sequence of positive numbers with \( \lambda_1 > 0 \). We find an explicit formula for the orthogonal Dirichlet polynomials \( \{ \phi_n \} \) formed from linear combinations of \( \{ \lambda_j^{-it} \}_{j=1}^n \), associated with rational weights

\[
w(t) = \sum_{j=1}^L \frac{c_j}{\pi (1 + (b_j t)^2)},
\]

where \( 0 < b_1 < b_2 < ... \), and the \( \{ c_j \} \) are appropriately chosen. Only \( \{ \lambda_j^{-it} \}_{j=1}^n \) appear in the formula. In the case \( L = 2 \), we show that the weight can always be taken positive in \( \mathbb{R} \).

Keywords: Dirichlet polynomials, orthogonal polynomials.

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1 Introduction

Throughout, let

\[
0 < \lambda_1 < \lambda_2 < \lambda_3 < ... .
\]

Let \( L_n \) denote the set of Dirichlet polynomials

\[
\sum_{j=1}^n c_j \lambda_j^{-it}
\]

with complex coefficients \( \{ c_j \} \).

In a 2014 paper [5], we showed that

\[
\phi_n(t) = \frac{\lambda_n^{1-it} - \lambda_{n-1}^{1-it}}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}} = \frac{-1}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}} \det \begin{pmatrix} \lambda_n^{1-it} \\ \lambda_{n-1}^{1-it} \end{pmatrix}
\]

is the \( n \)th orthogonal Dirichlet polynomial for the arctan density, that is

\[
\int_{-\infty}^{\infty} \phi_n(t) \phi_m(t) \frac{dt}{\pi (1 + t^2)} = \delta_{mn}, \ n,m \geq 1.
\]

We also estimated the Christoffel functions, convergence of associated orthonormal expansions, and universality limits. These orthonormal polynomials have been applied and provided in a variety of questions by Weber and Dimitrov as well as the author [4], [6], [8], [10], [11], [12]. In a follow up paper [7], the author considered orthogonal Dirichlet polynomials for the Laguerre weight, though it turned out that much of the material there was already subsumed by Müntz orthogonal polynomials [3].

In this note, we consider rational densities

\[
w(t) = \sum_{m=1}^L \frac{c_m}{\pi (1 + (b_m t)^2)}
\]

with appropriately chosen \( \{ c_j \} \). Here \( L \geq 1 \), and

\[
1 = b_1 < b_2 < ... < b_L.
\]

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Define, for \( n \geq L \),
\[
\psi_n(t) = \det \begin{bmatrix}
\lambda_{n-1}^{(t)} & \lambda_{n-1}^{(t)} & \cdots & \lambda_{n-1}^{(t)} \\
\lambda_{n-1}^{(t)} & \lambda_{n-2}^{(t)} & \cdots & \lambda_{n-2}^{(t)} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{n-1}^{(t)} & \lambda_{n-2}^{(t)} & \cdots & \lambda_{n-2}^{(t)} \\
\lambda_{n-1}^{(t)} & \lambda_{n-2}^{(t)} & \cdots & \lambda_{n-2}^{(t)} \\
\end{bmatrix}.
\] (5)

Observe that \( \psi_n(t) \) is a linear combination of only \( \{\lambda_j^{(t)}\}_{n-L \leq j \leq n} \). Also define for a given fixed \( n \), and \( j \geq 1, 1 \leq m \leq L \),
\[
d_{jm} = \int_{-\infty}^{\infty} \psi_n(t) \frac{\lambda_j^{(t)}}{\pi(1 + (b_m t)^2)} \, dt
\] (6)
and let \( B \) be the \((L-1) \times L\) matrix
\[
B = \begin{bmatrix}
d_{n-L+1,1} & d_{n-L+1,2} & \cdots & d_{n-L+1,L} \\
d_{n-L+2,1} & d_{n-L+2,2} & \cdots & d_{n-L+2,L} \\
\vdots & \vdots & \ddots & \vdots \\
d_{n-1,1} & d_{n-1,2} & \cdots & d_{n-1,L} \\
\end{bmatrix}.
\] (7)
and
\[
D = \det \begin{bmatrix}
d_{n-L+1,1} & d_{n-L+1,2} & \cdots & d_{n-L+1,L} \\
d_{n-L+2,1} & d_{n-L+2,2} & \cdots & d_{n-L+2,L} \\
\vdots & \vdots & \ddots & \vdots \\
d_{n,1} & d_{n,2} & \cdots & d_{n,L} \\
\end{bmatrix}.
\] (8)

**Theorem 1**

Let \( n \geq L \geq 1 \). Let \( 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \) and \( \psi_n \) be given by (5).

(a) \( \mathbf{c} = [c_1, c_2, \ldots, c_L]^T \) be taken as any non-trivial solution of \( B \mathbf{c} = \mathbf{0} \). Let
\[
w(t) = \sum_{m=1}^{L} \frac{c_m}{\pi(1 + (b_m t)^2)}. \]
(9)

Then for \( 1 \leq j \leq n-1 \),
\[
\int_{-\infty}^{\infty} \psi_n(t) \lambda_j^{(t)} w(t) \, dt = 0.
\] (10)

(b) If \( D \) defined by (8) is non-0, then we can take
\[
w(t) = A \det \begin{bmatrix}
d_{n-L+1,1} & d_{n-L+1,2} & \cdots & d_{n-L+1,L} \\
d_{n-L+2,1} & d_{n-L+2,2} & \cdots & d_{n-L+2,L} \\
\vdots & \vdots & \ddots & \vdots \\
d_{n,1} & d_{n,2} & \cdots & d_{n,L} \\
\end{bmatrix}
\]
\[
\frac{1}{\pi(1 + (b_m t)^2)} \
\] (11)
for any \( A \neq 0 \), while
\[
\int_{-\infty}^{\infty} \psi_n(t) \lambda_j^{(t)} w(t) \, dt = AD.
\] (12)

(c)
\[
\psi_n(t) = \sum_{j=1}^{n} \alpha_j \lambda_j^{(t)}
\]
(13)
where for \( n-L \leq j \leq n \),
\[
\alpha_j (-1)^{n+1} > 0.
\] (14)

**Remarks**

(a) Note that as \( \left\{ \frac{1}{\pi(1 + (b_m t)^2)} \right\}_{m=1}^{L} \) are linearly independent, \( w \) above is not identically 0. As an even rational function with numerator degree at most \( 2L-2 \) and denominator degree \( 2L \), \( w \) has at most \( L-1 \) sign changes in \((0, \infty)\). It seems to be an interesting problem to investigate the positivity of \( w \).

(b) In addition to the orthogonality relation above, we note that for any \( 1 \leq m \leq L \), and \( 0 < \lambda \leq \lambda_{n-L} \),
\[
\int_{-\infty}^{\infty} \psi_n(t) \frac{\lambda^{(t)}}{\pi(1 + (b_m t)^2)} \, dt = 0.
\]
This does not require anything of the \( \{c_i\} \) above.

In the case \( L = 2 \), we can prove positivity of the weight:

**Theorem 2**

Assume the notation of Theorem 1 with \( L = 2 \). Then we can choose \( c_1 < 0 < c_2 \) such that if

\[
 w(t) = \sum_{k=1}^{2} \frac{c_k}{\pi (1 + (b_k t)^2)}
\]

then

\[
 w(t) > 0, \; t \in \mathbb{R},
\]

and \( w \) is given by the determinant (11), with

\[
 A = \frac{c_2}{d_{n-1,1}^2} < 0.
\]

**Remark**

In the proof of Theorem 2, we show that one can take

\[
 c_1 = -c_2 \frac{g \left( \frac{1}{a} \right)}{g \left( \frac{1}{b} \right)}
\]

where

\[
 g(s) = s \left( \frac{\lambda_{m-2}}{\lambda_{m-1}} \right)^s - \left( \frac{\lambda_{m-2}}{\lambda_{m-1}} \right)^{-s}.
\]

We prove the theorems in the next section.

## 2 Proofs

**Proof of Theorem 1**

(a) We use the following simple consequence of the residue theorem: for real \( \mu \),

\[
 \int_{-\infty}^{\infty} e^{\mu t} \frac{1}{1 + t^2} dt = e^{-|\mu|}.
\]

Then if \( 0 < \lambda \leq \lambda_{m-L} \), and \( n-L \leq k \leq n \),

\[
 \int_{-\infty}^{\infty} \frac{(\lambda/\lambda_k)^{it}}{\pi (1 + (b_n t)^2)} dt = \frac{1}{b_n} \int_{-\infty}^{\infty} e^{it \log(\lambda/\lambda_k)} \frac{1}{\pi (1 + s^2)} ds = \frac{1}{b_n} \frac{\lambda}{\lambda_k} \lambda_k^{1/b_n}.
\]

Then for such \( \lambda \),

\[
 \int_{-\infty}^{\infty} \psi_n(t) \frac{\lambda^{iz}}{\pi (1 + (b_n t)^2)} dt = \text{det} \begin{bmatrix} \lambda_k^{-1/b_n} & \cdots & \lambda_k^{-1/b_n} \\ \cdots & \cdots & \cdots \\ \lambda_k^{-1/b_n} & \cdots & \lambda_k^{-1/b_n} \end{bmatrix} = 0,
\]

by taking \( \frac{1}{b_n} \lambda^{1/b_n} \) times row \( m+1 \) from the first row. So we have the orthogonality relation (10) for \( \lambda = \lambda_j \), all \( j \leq n-L \). Next, the equations

\[
 \int_{-\infty}^{\infty} \psi_n(t) \lambda_{m-L-j}^{iz} w(t) dt = 0, \; 1 \leq j \leq L - 1
\]

are equivalent to (recall (3) and (6))

\[
 \sum_{n=1}^{L} c_m d_{n-L-j,m} = \sum_{n=1}^{L} c_m \int_{-\infty}^{\infty} \psi_n(t) \frac{\lambda_{m-L-j}^{iz}}{\pi (1 + (b_n t)^2)} dt = 0, \; 1 \leq j \leq L - 1
\]
which in turn is equivalent to \( Bc = 0 \), recall (7). This is a system of \( L - 1 \) homogeneous linear equations in \( L \) variables, so there is a non-trivial solution for \( c \).

(b) First observe that \( w \) defined by (11) is indeed a linear combination of \( \frac{1}{\pi(1 + (b_k t)^2)} \) for \( k = 1, \ldots, L \). Next, we see from (11) that

\[
\int_{-\infty}^{\infty} \psi_n(t) \lambda_i^n w(t) \, dt = A \det \begin{bmatrix}
  d_{n-L+1,1} & d_{n-L+1,2} & \cdots & d_{n-L+1,L} \\
  d_{n-L+2,1} & d_{n-L+2,2} & \cdots & d_{n-L+2,L} \\
  \vdots & \vdots & \ddots & \vdots \\
  d_{n-1,1} & d_{n-1,2} & \cdots & d_{n-1,L} \\
  d_{k,1} & d_{k,2} & \cdots & d_{k,L}
\end{bmatrix} = 0,
\]

if \( n - L + 1 \leq k \leq n - 1 \). If \( k = n \), we instead obtain the non-0 number \( AD \). It also then follows that \( w \) cannot be the zero function.

(c) Let \( E \) be the \( L \times (L + 1) \) matrix

\[
E = \begin{bmatrix}
  \lambda_{n-L}^{-1/b_1} & \lambda_{n-L+1}^{-1/b_1} & \cdots & \lambda_n^{-1/b_1} \\
  \vdots & \vdots & \ddots & \vdots \\
  \lambda_{n-L}^{-1/b_{k-1}} & \lambda_{n-L+1}^{-1/b_{k-1}} & \cdots & \lambda_n^{-1/b_{k-1}} \\
  \lambda_{n-L}^{-1/b_k} & \lambda_{n-L+1}^{-1/b_k} & \cdots & \lambda_n^{-1/b_k}
\end{bmatrix}.
\]

Thus \( E \) consists of the last \( L \) rows of the matrix used to define \( \psi_n \). For \( 1 \leq k \leq L + 1 \), let \( E(k) \) denote the \( L \times L \) matrix obtained from \( E \) by deleting its \( k \)th column. Then with the notation (13), we see that

\[
\alpha_j = (-1)^{j+n+1} \det(E(j-n+L+1)).
\]

To show that each \( \det (E(k)) > 0 \), we use the fact that the kernel \( K(s, t) = e^{it} \) is totally positive for \( s, t \in \mathbb{R} \) [1, p. 212] or [9]. If we set \( s_j = -\frac{1}{b_j} \), while \( t_j = \log \lambda_{n-L+j-1} \), then \( s_1 < s_2 < \ldots < s_L \) and \( t_1 < t_2 < \ldots < t_{L+1} \), then

\[
\det(E(k)) = \det \begin{bmatrix}
  K(s_1, t_1) & \cdots & K(s_1, t_{k-1}) & K(s_1, t_{k+1}) & \cdots & K(s_1, t_{L+1}) \\
  K(s_2, t_1) & \cdots & K(s_2, t_{k-1}) & K(s_2, t_{k+1}) & \cdots & K(s_2, t_{L+1}) \\
  \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\
  K(s_L, t_1) & \cdots & K(s_L, t_{k-1}) & K(s_L, t_{k+1}) & \cdots & K(s_L, t_{L+1})
\end{bmatrix} > 0.
\]

**Proof of Theorem 2**

From (5) for \( L = 2 \),

\[
\psi_n(t) = \det \begin{bmatrix}
  \lambda_{n-1}^{-it} & \lambda_n^{-it} \\
  \lambda_{n-2}^{-it} & \lambda_{n-1}^{-it}
\end{bmatrix}.
\]

Let

\[
w(t) = \sum_{k=1}^{2} \frac{c_k}{\pi \left( 1 + (b_k t)^2 \right)},
\]

where for the moment we do not specify the choice of \( c_1, c_2 \). Then we already have for \( k = 1, 2, \ldots, n - 2 \),

\[
\int_{-\infty}^{\infty} \psi_n(t) \lambda_i^n w(t) \, dt = 0
\]

no matter what is the choice of \( c_1, c_2 \) - as follows from the proof of Theorem 1(a). So let us investigate the remaining condition in (10), namely

\[
\int_{-\infty}^{\infty} \psi_n(t) \lambda_i^{-1} w(t) \, dt = 0.
\]

This is equivalent to

\[
0 = \sum_{k=1}^{2} c_k \int_{-\infty}^{\infty} \psi_n(t) \lambda_i^{-1} \frac{dt}{\pi \left( 1 + (b_k t)^2 \right)} = c_1 d_{n-1,1} + c_2 d_{n-1,2}.
\]

(17)
Now for \( k = 1, 2 \), we see from the determinant expression (16) and then from (15) that

\[
d_{n-1,k} = \frac{1}{b_k} \det \left[ \begin{array}{cccc} f_0 & \cdots & f_{n-1} & f_n \\ \lambda_{n-1}^{-1/2} & \cdots & \lambda_{n-2}^{-1/2} & \lambda_{n-1}^{-1/2} \\ \lambda_{n-1}^{-1/2} & \cdots & \lambda_{n-2}^{-1/2} & \lambda_{n-1}^{-1/2} \\ \vdots & \cdots & \vdots & \vdots \\ \lambda_{n-1}^{-1/2} & \cdots & \lambda_{n-2}^{-1/2} & \lambda_{n-1}^{-1/2} \\ \end{array} \right] \
\]

\[
= \frac{1}{b_k} \lambda_{n-1}^{-1/2} \det \left[ \begin{array}{cccc} (\lambda_{n-2}^{-1/2})^{1/2} & \cdots & (\lambda_{n-2}^{-1/2})^{1/2} & (\lambda_{n-1}^{-1/2})^{1/2} \\ \lambda_{n-1}^{-1/2} & \cdots & \lambda_{n-2}^{-1/2} & \lambda_{n-1}^{-1/2} \\ \lambda_{n-1}^{-1/2} & \cdots & \lambda_{n-2}^{-1/2} & \lambda_{n-1}^{-1/2} \\ \vdots & \cdots & \vdots & \vdots \\ \lambda_{n-1}^{-1/2} & \cdots & \lambda_{n-2}^{-1/2} & \lambda_{n-1}^{-1/2} \\ \end{array} \right] \
\]

\[
= \frac{1}{b_k} \lambda_{n-1}^{-1/2} \det \left[ \begin{array}{cccc} \lambda_{n-1}^{-1/2} & \cdots & \lambda_{n-2}^{-1/2} & 0 \\ \lambda_{n-1}^{-1/2} & \cdots & \lambda_{n-2}^{-1/2} & 0 \\ \vdots & \cdots & \vdots & \vdots \\ \lambda_{n-1}^{-1/2} & \cdots & \lambda_{n-2}^{-1/2} & 0 \\ \end{array} \right] \
\]

\[
= \frac{1}{b_k} \lambda_{n-1}^{-1/2} \lambda_{n-2}^{-1/2} - \lambda_{n-1}^{-1/2} \lambda_{n-2}^{-1/2} < 0, \\
\]

as \( \frac{\lambda_{n-2}}{\lambda_{n-1}} \in (0, 1) \), \( \frac{1}{b_1} - \frac{1}{b_2} > 0 \), and

\[
\lambda_{n-1}^{-1/2} \lambda_{n-2}^{-1/2} - \lambda_{n-1}^{-1/2} \lambda_{n-1}^{-1/2} = \lambda_{n-1}^{-1/2} \lambda_{n-2}^{-1/2} \left[ 1 - \left( \frac{\lambda_{n-2}}{\lambda_{n-1}} \right)^{1/2} \right] > 0.
\]

In summary,

\[
d_{n-1,k} < 0, \quad k = 1, 2.
\]

Next, let \( r = \frac{\lambda_{n-2}}{\lambda_{n-1}} \in (0, 1) \), and

\[
g(s) = s \left[ r^t - r^{-s} \right].
\]

From (18) and (17) and cancelling a common factor of \( \lambda_{n-1}^{-1/2} \lambda_{n-2}^{-1/2} - \lambda_{n-1}^{-1/2} \lambda_{n-2}^{-1/2} \), we have

\[
c_1 g \left( \frac{1}{b_1} \right) + c_2 g \left( \frac{1}{b_2} \right) = 0.
\]

Here

\[
g' (s) = (r^t - r^{-s}) + (s \ln r) (r^t + r^{-s}) < 0,
\]

as \( r = \frac{\lambda_{n-2}}{\lambda_{n-1}} < 1 \) so \( \ln r < 0 \). Then \( g \) is decreasing and negative, and

\[
0 > g \left( \frac{1}{b_2} \right) > g \left( \frac{1}{b_1} \right)
\]

so (20) gives

\[
c_1 = -c_2 \frac{g \left( \frac{1}{b_1} \right)}{g \left( \frac{1}{b_2} \right)} \quad \text{and} \quad |c_1| < |c_2|.
\]

To ensure that \( w(0) = \frac{1}{T} \left( c_1 + c_2 \right) > 0 \), we then need to choose \( c_1 < 0 < c_2 \). To ensure that \( w(t) > 0 \) for all \( t \), we need for all such \( t \).

\[
|c_1| \leq c_2 \frac{1 + (b_1 t)^2}{1 + (b_2 t)^2}.
\]

As

\[
\min_{t \in \mathbb{R}} \frac{1 + (b_1 t)^2}{1 + (b_2 t)^2} = \left( \frac{b_1}{b_2} \right)^2,
\]

this is equivalent to

\[
g \left( \frac{1}{b_2} \right) \leq \left( \frac{b_1}{b_2} \right)^2.
\]
that is, (recall $g < 0$),
\[ b_2 \left[ r^{-1/b_2} - r^{-1/b_1} \right] \leq b_1 \left[ r^{-1/b_1} - r^{-1/b_1} \right]. \]

Now let
\[ h(s) = \frac{1}{s} \left[ r^{-s} - r^{-s} \right], \]
so that we want
\[ h \left( \frac{1}{b_2} \right) \leq h \left( \frac{1}{b_1} \right). \quad (22) \]

This would be true if $h$ is increasing over the range $\left[ \frac{1}{b_2}, \frac{1}{b_1} \right]$. Now
\[ h'(s) = -\frac{1}{s^2} \left[ r^{-s} - r^{-s} \right] - \frac{1}{s} \left( \ln r \right) \left[ r^{-s} + r^{-s} \right] \]
\[ = -\frac{r^{-s}}{s^2} \left[ 1 - r^{-s} + \frac{1}{2} \left( \ln r^2 \right) \left[ 1 + r^{-s} \right] \right] = -\frac{r^{-s}}{s^2} G(x) \]
where
\[ x(s) = r^{s} \in (0, 1) \] decreases as $s$ increases
and
\[ G(x) = 1 - x + \frac{1}{2} (\ln x) (1 + x). \]

Here $G(0+) = -\infty$ and $G(1) = 0$ while for $x \in (0, 1)$,
\[ G'(x) = \frac{1}{2} + \frac{1}{2x} + \frac{1}{2} \ln x \]
\[ \Rightarrow G''(x) = \frac{1}{2x} \left( 1 - \frac{1}{x} \right) < 0. \]

Thus $G$ is concave in $(0, 1)$ and $G'$ is a decreasing function of $x$ with $G'(0+) = \infty$ and $G'(1) = 0 = G(1)$. It follows that $G'(x) > 0$ for $x \in (0, 1)$, so
\[ G(x) < G(1) = 0 \text{ for } x \in (0, 1). \]

So, indeed,
\[ h'(s) = -\frac{r^{-s}}{s^2} G(x) > 0 \text{ for } s > 0, \]
and as desired, we have (22). Then with $c_1$ and $c_2$ given by (21), and $c_2 > 0$, we do have
\[ w(t) > 0, \quad t \in (-\infty, \infty). \]

It remains to show that this $w$ is also given by (11) with $L = 2$. We know that $c_1, c_2$ are non-0 so
\[
\begin{align*}
\det \begin{bmatrix}
\frac{d_{n-1,1}}{\pi(1+t^2/r^2)} & \frac{d_{n-1,2}}{\pi(1+t^2/r^2)} \\
\frac{d_{n-1,1}}{\pi(1+t^2/r^2)} & \frac{d_{n-1,2} + \frac{1}{c_2}d_{n-1,1}}{\pi(1+t^2/r^2)}
\end{bmatrix}
&= \det \begin{bmatrix}
\frac{d_{n-1,1}}{\pi(1+t^2/r^2)} & \frac{0}{\pi(1+t^2/r^2)} \\
\frac{d_{n-1,1}}{\pi(1+t^2/r^2)} & \frac{1}{c_2} w(t)
\end{bmatrix}
&= \frac{d_{n-1,1}}{c_2} w(t).
\end{align*}
\]
Thus the determinant is of one sign. Choosing $A = \frac{c_2}{d_{n-1,1}} < 0$ gives the result. ■

References