The Turán Density of the Hypergraph $\{abc, ade, bde, cde\}$

Zoltán Füredi*

Department of Mathematics, University of Illinois

Urbana, Illinois 61801

z-furedi@math.uiuc.edu

Rényi Institute of Mathematics, Hungarian Academy of Sciences

furedi@renyi.hu

Oleg Pikhurko†

Centre for Mathematical Sciences, Cambridge University

Cambridge CB3 0WB, England

o.pikhurko@dpmms.cam.ac.uk

Miklós Simonovits

Rényi Institute of Mathematics, Hungarian Academy of Sciences

PO Box 127, H-1364, Budapest, Hungary

miki@renyi.hu

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Abstract

Let $\mathbf{F}_{3,2}$ denote the 3-graph $\{abc, ade, bde, cde\}$. We show that the maximum size of an $\mathbf{F}_{3,2}$-free 3-graph on n vertices is $(\frac{4}{9} + o(1))\binom{n}{3}$, proving a conjecture of Mubayi and Rödl [*J. Comb. Th. A*, **100** (2002), 135–152].

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1 Introduction

Let $[n] := \{1, \ldots, n\}$ and let $\binom{[n]}{k}$ denote the family of k-element subsets of $[n]$. The *Turán function* $\mathrm{ex}(n, F)$ of a k-graph F is the maximum size of $H \subset \binom{[n]}{k}$ not containing a subgraph isomorphic to F. It is well known [5], that the ratio $\mathrm{ex}(n, F)/\binom{n}{k}$ is non-increasing with n. In particular, the limit

$$\pi(F) := \lim_{n\to\infty} \frac{\mathrm{ex}(n, F)}{\binom{n}{k}}$$

exists. See [4] for a survey on the Turán problem for hypergraphs. The value of $\pi(F)$, for $k \geq 3$, is known for very few F and any addition to this list is of interest.

In this note we consider the 3-graph

$$\mathbf{F}_{3,2} = \{\, \{1, 2, 3\},\ \{1, 4, 5\},\ \{2, 4, 5\},\ \{3, 4, 5\}\, \}.$$

The notation $\mathbf{F}_{3,2}$ comes from [7] where, more generally, the 3-graph $\mathbf{F}_{p,q}$ consists of those edges in $\binom{[p+q]}{3}$ which intersect $[p]$ in either 1 or 3 vertices. Note that we shall use both $\mathbf{F}_{3,2}$ and $\mathbf{F}_{2,3}$ and they are different.

The extremal graph problem of $\mathbf{F}_{3,2}$ originates from a Ramsey-Turán hypergraph paper of Erdős and T. Sós [2]. They investigated examples where the Turán function and the Ramsey-Turán number essentially differ from each other. They observed that $\mathrm{ex}(n, \mathbf{F}_{3,2}) > cn^3$, while, if $\mathcal{H}_n$ is a 3-uniform hypergraph without $\mathbf{F}_{3,2}$ and the independence number of $\mathcal{H}_n$ is $o(n)$ then $e(\mathcal{H}_n) = o(n^3)$. A more general theorem is proved in [3].

Mubayi and Rödl [7, Theorem 1.5] showed that

$$\frac{4}{9} \leq \pi(\mathbf{F}_{3,2}) \leq \frac{1}{2},$$

and conjectured [7, Conjecture 1.6] that the lower bound is sharp. An $\mathbf{F}_{3,2}$-free hypergraph of density $\frac{4}{9} + o(1)$ can be obtained by taking those 3-subsets of $[n]$ which intersect $[a]$ in precisely two vertices, $a = (\frac{2}{3} + o(1))\, n$.

Here we verify this conjecture.

Theorem 1. $\pi(\mathbf{F}_{3,2}) = 4/9$.

In a forthcoming paper we will present a different argument showing that the above construction with $a = \lceil 2n/3 \rceil$ gives the *exact* value of $\mathrm{ex}(n, F)$ for all sufficiently large n.

2 Preliminary Observations

We frequently identify a hypergraph with its edge set but write $V(H)$ for its vertex set. For a 3-graph H the *link graph* of a vertex $x \in V(H)$ is

$$H_x := \{\{y, z\} \mid \{x, y, z\} \in H\}.$$

Suppose, to the contrary to Theorem 1, that $\delta := \pi(\mathbf{F}_{3,2}) > 4/9 + \varepsilon$ for some $\varepsilon > 0$. Let n be sufficiently large and let $\mathcal{H} \subset \binom{[n]}{3}$ be a maximum $\mathbf{F}_{3,2}$-free hypergraph.

The degrees of any two vertices of $\mathcal{H}$ differ by at most $n-2$. Indeed, otherwise we can delete the vertex with the smaller degree and duplicate the other, strictly increasing the size of $\mathcal{H}$. (This is a variant of Zykov's symmetrization.) Hence, $e(\mathcal{H}_v) = (\delta + o(1))\binom{n}{2}$ for every $v \in [n]$.

For distinct $x, y \in V(\mathcal{H})$ let

$$\mathcal{H}_{x,y} := \{z \in V(\mathcal{H}) \mid \{x,y,z\} \in \mathcal{H}\}.$$

Let $|\mathcal{H}_{x,y}|$ attain its maximum for (x_0, y_0). Put $A := \mathcal{H}_{x_0,y_0}$, $\alpha := |A|/n$, and $\overline{A} := [n] \setminus A$. Equivalently, αn is the maximum of $\Delta(\mathcal{H}_x)$ over $x \in V(\mathcal{H})$, where Δ stands for the maximum degree. As $\mathcal{H}$ is $\mathbf{F}_{3,2}$-free, no edge of $\mathcal{H}$ lies inside A.

For $v \in V(\mathcal{H})$ let $e_v := e(G_v[A, \overline{A}])$ be the number of edges in $\mathcal{H}_v$ connecting A to $\overline{A}$.

$$e_v = 2e(\mathcal{H}_v) - \sum_{x \in \overline{A}} |\mathcal{H}_{x,v}| \geq (\delta - \alpha(1-\alpha) + o(1))\, n^2, \quad v \in A. \tag{1}$$

The assumption $v \in A$ is essential in (1) as we use the fact that A is an independent vertex-set in G_v.

By (1), the average degree of $G_v[A, \overline{A}]$ over $x \in A$ is

$$\frac{e_v}{|A|} \geq \left(\frac{\delta}{\alpha} - 1 + \alpha + o(1)\right) n =: \gamma n. \tag{2}$$

Thus we can find a set $C \subset \overline{A}$ of size $|C| = \gamma n$ covered in G_v by some $x \in A$, i.e., $C \subseteq \mathcal{H}_{v,x}$. Let $B := \overline{A} \setminus C$ and

$$\beta := \frac{|B|}{n} = 1 - \alpha - \gamma = 2 - 2\alpha - \frac{\delta}{\alpha} + o(1). \tag{3}$$

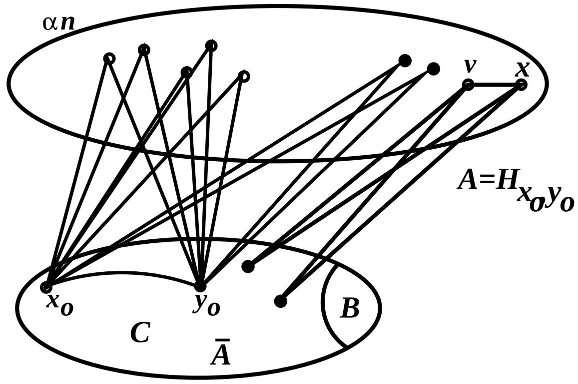
Let $c_v := e(G_v[A, C])$ and $b_v := e(G_v[A, B])$. Obviously, $e_v = b_v + c_v$ for every $v \in [n]$. The nonnegativity of β and γ together with (2) and (3) imply

$$\frac{4}{9} + \varepsilon < \delta \leq \alpha + o(1) \leq \frac{2}{3}, \qquad \frac{1}{3} \leq \gamma, \qquad 0 \leq \beta < 0.12$$

Concerning the edge densities we obtain by (1) for $v \in A$ that

$$\begin{aligned} \frac{c_v}{|A||C|} &= \frac{e_v - b_v}{\alpha\gamma n^2} \geq \frac{e_v - \alpha\beta n^2}{\alpha\gamma n^2} \\ &\geq \frac{\delta - \alpha(1-\alpha) - \alpha\beta}{\delta - \alpha(1-\alpha)} + o(1) = \frac{2\delta - 3\alpha(1-\alpha)}{\delta - \alpha(1-\alpha)} + o(1) > \frac{5}{7}. \end{aligned} \tag{4}$$

Here the last step is implied by $9\delta > 4 \geq 16\alpha(1-\alpha)$.



Note that no edge $E \in \mathcal{H}$ can lie inside C, otherwise $E \cup \{v, x\}$ would span a forbidden subhypergraph. The independence properties of A and C will play a crucial role in our proof.

Following [7] we make the following definitions. Let $\mathcal{F}_2 = \{\mathbf{F}_{2,3}\}$ consist of the single 3-graph $\mathbf{F}_{2,3}$. Recall that

$$\mathbf{F}_{2,3} = \{\,\{1,3,4\},\,\{1,3,5\},\,\{1,4,5\},\,\{2,3,4\},\,\{2,3,5\},\,\{2,4,5\}\,\}.$$

For $t \geq 3$ let $\mathcal{F}_t$ be the family of all 3-graphs obtained by adding to each $F \in \mathcal{F}_{t-1}$ two new vertices x, y and any set of t edges of the form $\{x, y, z\}$ with $z \in V(F)$. It is easy to show (see [7, Proposition 4.2]) that each $F \in \mathcal{F}_t$ has $2t + 1$ vertices and any $t + 2$ vertices of F span at least one edge.

Why is this family useful in our study of $\pi(\mathbf{F}_{3,2})$? A straightforward attempt to find $\mathbf{F}_{3,2} \subset \mathcal{H}$ is to pick an arbitrary edge $E = \{x, y, z\} \in \mathcal{H}$ and to prove that $\mathcal{H}_x \cap \mathcal{H}_y \cap \mathcal{H}_z \neq \emptyset$. To guarantee the last property, it is enough to require that each $\mathcal{H}_x$, $x \in V(\mathcal{H})$, has more than $\frac{2}{3}\binom{n}{2}$ edges. This leads to $\pi(\mathbf{F}_{3,2}) \leq 2/3$. But suppose that we have $F \subset \mathcal{H}$ with $F \in \mathcal{F}_t$. To find a copy of $\mathbf{F}_{3,2}$ in $\mathcal{H}$, it is enough to find a $(t+2)$-set $X \subset V(F)$ with $\cap_{x \in X} \mathcal{H}_x \neq \emptyset$. The condition that for every $x \in X$, $e(\mathcal{H}_x) > \frac{t+1}{2t+1}\binom{n}{2}$ is sufficient for this. So, if we can find $\mathcal{F}_t$-subgraphs for sufficiently large t, then we can show $\pi(\mathbf{F}_{3,2}) \leq 1/2$.

This idea is due to Mubayi and Rödl [7]. Here, we take it one step further by trying to find an $\mathcal{F}_t$-subgraph which lies "nicely" with respect to A and C. Then we exploit the fact that each link graph has a large independent set, so its edge density is relatively large between A and C. Here is the crucial definition.

Definition 2. An $\mathcal{F}_t$-subgraph $F \subset \mathcal{H}$ is *well-positioned* if $V(F) \subset A \cup C$ and

$$|V(F) \cap A| = t + 1 \text{ and } |V(F) \cap C| = t. \tag{5}$$

3 Proof of Theorem 1

The proof consists of three steps. First, in a lemma, we show that there are well-positioned $\mathcal{F}_t$-subhypergraphs in $\mathcal{H}$, namely we can take $t = 2$. In this step we do not use our assumption that $\delta > \frac{4}{9} + \varepsilon$, only that $n > n_0$. Next we show that there is no well-positioned $\mathcal{F}_t$-subhypergraph with $t = \lceil 1/\varepsilon \rceil$. In the last step we consider a well-positioned $\mathcal{F}_t$ subgraph F, which is not contained in any well-positioned $\mathcal{F}_{t+1}$-subhypergraph, and $t < 1/\varepsilon$.

Lemma 3. $\mathbf{F}_{2,3} \subset \mathcal{H}$.

Proof. Denote the number of hyperedges of $\mathcal{H}$ of type AAC, i.e., those having two vertices in A and one in C, by Δ_{AAC}. Let $a_w := e(G_w[A])$ and recall that $c_v = e(G_v[A, C])$. Then

$$\sum_{w \in C} a_w = \Delta_{AAC} = \frac{1}{2} \sum_{v \in A} c_v.$$

By (4) we have

$$\sum_{w \in C} a_w > \frac{5}{14} |A|^2 |C|.$$

Count the 4-vertex 3-edge subhypergraphs $\mathbf{F}_{1,3}$ of the form $\{wxy, wxz, wyz\}$, $w \in C$, $x, y, z \in A$. For a given w they are obtained from the triangles in $G_w[A]$. So we may apply the Moon-Moser's extension of Turán's theorem [6], that the number of triangles $k_3(G)$ of an n-vertex e-edge graph G is at least $e(4e - n^2)/(3n)$. The convexity of this function implies for $n > n_0$,

$$\begin{aligned} \#\, \mathbf{F}_{1,3} \;&=\; \sum_{w \in C} k_3(G_w[A]) \;\geq\; \sum_{w \in C} \frac{|A|^3}{3} \frac{a_w}{|A|^2} \left(\frac{4a_w}{|A|^2} - 1 \right) \\ &\geq\; |C| \times \frac{|A|^3}{3} \frac{5}{14} \left(\frac{20}{14} - 1 \right) \;>\; \binom{|A|}{3}. \end{aligned}$$

So at least two of these triangles coincide, giving a well-positioned $\mathcal{F}_2$-subgraph. □

Lemma 4. *Let $t = \lceil 1/\varepsilon \rceil$. Then $\mathcal{H}$ contains no well-positioned $\mathcal{F}_t$-subgraph.*

Proof. Suppose, to the contrary, that such an $F \subset \mathcal{H}$ exists and consider the link graphs G_v, $v \in V(F)$. As $\mathcal{H}$ is $\mathbf{F}_{3,2}$-free, any pair of vertices belongs to at most $t + 1$ links. For the edges between A and B we have

$$(t + 1)\alpha\beta n^2 \geq \sum_{v \in V(F)} b_v. \qquad (6)$$

Recall that $b_v = e(G_v[A, B])$.

We need the following analogue of (1) for $w \in C$:

$$\begin{aligned} e_w &= 2e(G_w) - \sum_{v \in A} |\mathcal{H}_{v,w}| - 2e(G_w[\overline{A}]) \\ &\geq (\delta - \alpha^2 - 2\beta\gamma - \beta^2 + o(1))\, n^2, \quad w \in C. \end{aligned} \tag{7}$$

For the edges connecting A to C, we obtain by (5), (1), (7), and (6) that

$$\begin{aligned} (t+1)\alpha\gamma &\geq \frac{1}{n^2} \sum_{v \in V(F)} c_v = \frac{1}{n^2} \left(\sum_{v \in V(F) \cap A} e_v + \sum_{v \in V(F) \cap C} e_v - \sum_{v \in V(F)} b_v \right) \\ &\geq (t+1)(\delta - \alpha(1-\alpha)) + t(\delta - \alpha^2 - 2\beta\gamma - \beta^2) \\ &\qquad\qquad -(t+1)\alpha\beta + o(t). \end{aligned}$$

Rearranging, we get

$$\begin{aligned} &\alpha\gamma - (\delta - \alpha(1-\alpha)) + \alpha\beta \\ &\quad \geq t\Big(-\alpha\gamma + (\delta - \alpha(1-\alpha)) + (\delta - \alpha^2 - 2\beta\gamma - \beta^2) - \alpha\beta + o(1)\Big). \end{aligned} \tag{8}$$

Here the left hand side equals to $2\alpha(1-\alpha) - \delta$. We have $\alpha(1-\alpha) \leq 1/4$, $\delta > 4/9$, therefore

$$\text{the left hand side of (8)} < \frac{1}{2} - \frac{4}{9} = \frac{1}{18}.$$

Substituting the values of γ and β given by (2) and (3) into the right hand side of (8) we obtain after routine transformations that the coefficient of t equals $\alpha^2 - 2\alpha + 4\delta - \frac{2\delta}{\alpha} + \frac{\delta^2}{\alpha^2} + o(1)$, which equals

$$\frac{1}{\alpha^2}\left(\alpha - \frac{2}{3}\right)^2 \left((\alpha - \frac{1}{3})^2 + \frac{1}{3}\right) + \frac{1}{\alpha^2}\left(\delta - \frac{4}{9}\right)\left(\delta + \frac{4}{9} + 4\alpha^2 - 2\alpha\right) + o(1).$$

Here the first term is non-negative, and in the second term $\delta + \frac{4}{9} + 4\alpha^2 - 2\alpha > 2\alpha^2$ since $\delta > \frac{4}{9}$. Thus (8) implies that $1/18 \geq 2\varepsilon t$ which is impossible. $\square$

Let t be the largest integer such that well-positioned $\mathcal{F}_2, \mathcal{F}_3, \ldots, \mathcal{F}_t$-subhypergraphs exist. By our above arguments we have $2 \leq t < 1/\varepsilon$. We are going to use the maximality of t, which tells us that any pair connecting $A \setminus V(F)$ to $C \setminus V(F)$ belongs to at most t graphs $\mathcal{H}_v$, $v \in V(F)$. We obtain

$$t(|A| - t - 1)(|C| - t) + |V(F)|^2 n \geq \sum_{v \in V(F)} c_v.$$

Note that we cannot make the same claim about the edges between A and B because a well-positioned subgraph must lie inside $A \cup C$ by definition. However, we can use the weaker inequality (6). We obtain

$$\begin{aligned} t\alpha\gamma + O(t^2/n) &\geq \frac{1}{n^2}\sum_{v\in V(F)} c_v = \frac{1}{n^2}\left(\sum_{v\in V(F)} e_v - \sum_{v\in V(F)} b_v\right) \\ &\geq (t+1)(\delta - \alpha(1-\alpha)) \\ &\quad + t(\delta - \alpha^2 - 2\gamma\beta - \beta^2) - (t+1)\alpha\beta + o(1), \end{aligned}$$

leading to

$$\begin{aligned} &-(\delta - \alpha(1-\alpha)) + \alpha\beta \\ &\quad \geq t\Big(-\alpha\gamma + (\delta - \alpha(1-\alpha)) + (\delta - \alpha^2 - 2\beta\gamma - \beta^2) - \alpha\beta\Big) + o(1) \end{aligned} \tag{9}$$

Here the left hand side is negative

$$-(\delta - \alpha(1-\alpha)) + \alpha\beta = 3\alpha(1-\alpha) - 2\delta + o(1) \leq 3 \times \frac{1}{4} - 2 \times \frac{4}{9} + o(1) < 0,$$

and the right hand side of (9) is the same as in inequality (8), so it is at least $2\varepsilon t$. This contradiction proves Theorem 1.

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