Edge and total choosability of near-outerplanar graphs

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Abstract

It is proved that, if G is a K_4-minor-free graph with maximum degree $\Delta \geqslant 4$, then G is totally $(\Delta+1)$-choosable; that is, if every element (vertex or edge) of G is assigned a list of $\Delta+1$ colours, then every element can be coloured with a colour from its own list in such a way that every two adjacent or incident elements are coloured with different colours. Together with other known results, this shows that the List-Total-Colouring Conjecture, that $\mathrm{ch}''(G) = \chi''(G)$ for every graph G, is true for all K_4-minor-free graphs. The List-Edge-Colouring Conjecture is also known to be true for these graphs. As a fairly straightforward consequence, it is proved that both conjectures hold also for all $K_{2,3}$-minor free graphs and all $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs.

Keywords: Outerplanar graph; Minor-free graph; Series-parallel graph; List edge colouring; List total colouring.

1 Introduction

We use standard terminology, as defined in the references: for example, [8] or [11]. We distinguish *graphs* (which are always simple) from *multigraphs* (which may have multiple edges); however, our theorems are only for graphs. For a graph (or multigraph) G, its edge chromatic number, total (vertex-edge) chromatic number, edge choosability (or list edge chromatic number), total choosability, and maximum degree, are denoted by $\chi'(G)$, $\chi''(G)$, $\mathrm{ch}'(G)$, $\mathrm{ch}''(G)$, and $\Delta(G)$, respectively. So $\mathrm{ch}''(G)$ is the smallest k for which G is totally k-choosable.

There is great interest in discovering classes of graphs H for which the choosability or list chromatic number $\mathrm{ch}(H)$ is equal to the chromatic number $\chi(H)$. The *List-Edge-Colouring Conjecture* (*LECC*) and *List-Total-Colouring Conjecture* (*LTCC*) [1, 5, 6] are that, for every multigraph G, $\mathrm{ch}'(G) = \chi'(G)$ and $\mathrm{ch}''(G) = \chi''(G)$, respectively; so the conjectures are that $\mathrm{ch}(H) = \chi(H)$ whenever H is the line graph or the total graph of a multigraph G.

For an outerplanar (simple) graph G, Wang and Lih [9] proved that $\mathrm{ch}'(G) = \chi'(G) = \Delta(G)$ if $\Delta(G) \geqslant 3$ and $\mathrm{ch}''(G) = \chi''(G) = \Delta(G) + 1$ if $\Delta(G) \geqslant 4$. For the larger class of K_4-minor-free (series-parallel) graphs, the first of these results had already been proved by Juvan, Mohar and Thomas [7], and we will prove the second in Section 2, following an incomplete outline proof by Zhou, Matsuo and Nishizeki [13].

Woodall [12] filled in the missing case by proving that every K_4-minor-free graph with maximum degree 3 is totally 4-choosable. Incorporating obvious results for $\Delta = 1$ and known results [4, 6] for $\Delta = 2$, we can summarize the situation for both edge and total colourings as follows.

Theorem 1.1. *The LECC and LTCC hold for all K_4-minor-free graphs. In fact, if G is a K_4-minor-free graph with maximum degree Δ, then* $\mathrm{ch}'(G) = \chi'(G) = \Delta$ *and* $\mathrm{ch}''(G) = \chi''(G) = \Delta + 1$, *apart from the following exceptions*:

(i) *if* $\Delta = 1$ *then* $\mathrm{ch}''(G) = \chi''(G) = 3 = \Delta + 2$;

(ii) *if* $\Delta = 2$ *and G has an odd cycle as a component, then* $\mathrm{ch}'(G) = \chi'(G) = 3 = \Delta + 1$;

(iii) *if* $\Delta = 2$ *and G has a component that is a cycle whose length is not divisible by* 3, *then* $\mathrm{ch}''(G) = \chi''(G) = 4 = \Delta + 2$.

It is well known that a graph is outerplanar if and only if it is both K_4-minor-free and $K_{2,3}$-minor-free. By a *near-outerplanar* graph we mean one that is either K_4-minor-free or $K_{2,3}$-minor-free. In fact, in the following theorem we will replace the class of $K_{2,3}$-minor-free graphs by the slightly larger class of $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs, where $\bar{K}_2 + (K_1 \cup K_2)$ is the graph obtained from $K_{2,3}$ by adding an edge joining two vertices of degree 2, or, equivalently, it is the graph obtained from K_4 by adding a vertex of degree 2 subdividing an edge. We will prove the following result in Section 3.

Theorem 1.2. *The LECC and LTCC hold for all $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs. In fact, if G is a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph with maximum degree Δ, then* $\mathrm{ch}'(G) = \chi'(G) = \Delta$ *and* $\mathrm{ch}''(G) = \chi''(G) = \Delta + 1$, *apart from the following exceptions*: (i)–(iii) *as in Theorem* 1.1, *and*

(iv) *if* $\Delta = 3$ *and G has K_4 as a component, then* $\mathrm{ch}''(G) = \chi''(G) = 5 = \Delta + 2$.

We will make use of the following simple results. Theorem 1.3 is a slight extension of a theorem of Dirac [2]. Part (a) of Theorem 1.4 is contained in Theorem 1.1, and follows from the well-known result [4] that a cycle of even length is 2-choosable (or, equivalently, edge-2-choosable). Part (b) is an easy exercise (using part (a)), but it also follows from the result of Ellingham and Goddyn [3] that a d-regular edge-d-colourable planar graph is edge-d-choosable.

Theorem 1.3. [10] *A K_4-minor-free graph G with $|V(G)| \geqslant 4$ has at least two nonadjacent vertices with degree at most* 2. *Hence a K_4-minor-free graph with no vertices of degree* 0 *or* 1 *has at least two vertices with degree (exactly)* 2.

Theorem 1.4. (a) $\mathrm{ch}'(C_4) = \chi'(C_4) = 2$.
(b) $\mathrm{ch}'(K_4) = \chi'(K_4) = 3$.

For brevity, when considering total colourings of a graph G, we will sometimes say that a vertex and an edge incident to it are *adjacent* or *neighbours*, since they correspond to adjacent or neighbouring vertices of the total graph $T(G)$ of G. As usual, $d(v) = d_G(v)$ will denote the degree of the vertex v in the graph G.

2 K_4-minor-free graphs with $\Delta \geqslant 4$

In this section we prove the following theorem. Our method of proof follows that outlined by Zhou, Matsuo and Nishizeki [13], which in turn is based on the proof of Juvan, Mohar and Thomas [7] for edge-choosability.

Theorem 2.1. *Let G be a K_4-minor-free graph with maximum degree $\Delta \geqslant 4$. Then* $\mathrm{ch}''(G) = \chi''(G) = \Delta + 1$.

Proof. Clearly $\mathrm{ch}''(G) \geqslant \chi''(G) \geqslant \Delta+1$, and so it suffices to prove that $\mathrm{ch}''(G) \leqslant \Delta+1$. Fix the value of $\Delta \geqslant 4$, and suppose if possible that G is a minimal K_4-minor-free graph with maximum degree *at most* Δ such that $\mathrm{ch}''(G) > \Delta + 1$. Assume that every edge e and vertex v of G is given a list $L(e)$ or $L(v)$ of $\Delta + 1$ colours such that G has no proper total colouring from these lists. We will prove various statements about G. Clearly G is connected.

Claim 2.1. *There is no vertex of degree* 1 *in G.*

Proof. Suppose u is a vertex of G with only one neighbour, v. By the definition of G, $G - u$ has a proper total colouring from its lists. The edge uv has at most Δ coloured neighbours, and so it can be given a colour from its list that is used on none of its neighbours; the vertex u is now easily coloured. These contradictions prove Claim 2.1. □

Claim 2.2. *G does not contain two adjacent vertices of degree* 2.

Proof. Suppose $xuvy$ is a path (or cycle, if $x = y$), where u and v both have degree 2. Then $G - \{u, v\}$ has a proper total colouring from its lists. The edges xu and vy can now be coloured as in Claim 2.1, followed by uv; and the vertices u and v now have only 3 coloured neighbours each and $\Delta + 1 \geqslant 5$ colours in their lists, and so they can both be coloured. These contradictions prove Claim 2.2. □

Claim 2.3. *G does not contain a 4-cycle with two opposite vertices of degree* 2 *in G.*

(a) (b)

Fig. 1

Proof. Suppose $xuyvx$ is a 4-cycle such that u and v have degree 2 in G. Then $G-\{u,v\}$ has a proper total colouring from its lists. The edges xu, uy, yv, vx each have at least two usable colours (i.e., colours not already used on any neighbour) in their lists, and so can be coloured by Theorem 1.4(a). The vertices u and v now each have 4 coloured neighbours and $\Delta+1 \geqslant 5$ colours in their lists, and so they can be coloured. □

Claim 2.4. *G does not contain the configuration in Fig.* 1(a), *in which only x and y are incident with edges not shown.*

Proof. Suppose it does. Then $G-w$ has a proper total colouring from its lists. The edge wy can now be coloured, since it has at least one usable colour in its list. Now we can colour uw and then w, since each of them has 4 coloured neighbours at the time of its colouring and a list of $\Delta+1 \geqslant 5$ colours. □

Claim 2.5. *G does not contain the configuration in Fig.* 1(b), *in which only x and y are incident with edges not shown.*

Proof. Suppose it does. Then $G-\{u,v,w\}$ has a proper total colouring from its lists. For each uncoloured element z, let $L'(z)$ denote the residual list of usable colours for z, comprising the colours in $L(z)$ that are not used on any neighbour of z in the colouring of $G-\{u,v,w\}$. The elements

$$vx,\ ux,\ uy,\ wy,\ u,\ uw,\ uv \tag{1}$$

have usable lists of at least 2, 2, 2, 2, 3, 5 and 5 colours, respectively, since $\Delta+1 \geqslant 5$. (The vertices v and w can be coloured last, since each has four neighbours and a list of $\Delta+1 \geqslant 5$ colours.) If we try to colour the elements in the order given in (1), we will succeed except possibly with uv. If $L'(uv) \cap L'(uy) = \emptyset$ then we will succeed with uv as well; so we may suppose that $L'(uv) \cap L'(uy) \neq \emptyset$, and similarly (by symmetry) that there exists some colour $c_1 \in L'(ux) \cap L'(uw)$. If vx and uy can be given the same colour, then the remaining elements can be coloured in the order (1); so we may suppose that $L'(vx) \cap L'(uy) = \emptyset$. If ux can be given a colour that is not in the list of vx, then we can colour the elements in the order (1) except that vx is coloured last; so we may suppose that $L'(ux) \subseteq L'(vx)$, which means that $L'(ux) \cap L'(uy) = \emptyset$, and also that $c_1 \in L'(vx) \cap L'(uw)$. If $c_1 \in L'(u)$, then give colour c_1 to vx and u, and then colour the remaining elements in the order (1), which is possible since $c_1 \notin L'(uy)$ and uv has two neighbours with the same colour. If however $c_1 \notin L'(u)$, then give colour c_1 to vx and uw, and then colour wy, uy (which is possible since $c_1 \notin L'(uy)$), then ux (since the colour of uy is not in its list), then u (since $c_1 \notin L'(u)$), and finally uv. In all cases the colouring can be completed, which is a contradiction. This completes the proof of Claim 2.5. $\square$

However, Claims 2.1–2.5 give a contradiction, since Juvan, Mohar and Thomas [7] proved that every K_4-minor-free graph contains at least one of the configurations that is proved to be impossible in these Claims (and we will prove a slightly stronger result than this at the end of the proof of Theorem 1.2 in the next section). This completes the proof of Theorem 2.1. $\square$

3 Extension to $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graphs

In this section we use Theorem 1.1 to prove Theorem 1.2. We will need the following two simple lemmas.

Lemma 3.1. *Let G be a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph. Then each block of G is either K_4-minor-free or isomorphic to K_4.*

Proof. If some block B of G is not K_4-minor-free then it has a K_4 minor. Since K_4 has maximum degree 3, it follows that B has a subgraph H homeomorphic to K_4. Since any graph obtained by subdividing an edge of K_4, or by adding a path joining two vertices of K_4, has a $\bar{K}_2 + (K_1 \cup K_2)$ minor, it follows that $H \cong K_4$ and $B = H$. $\square$

Lemma 3.2. $\mathrm{ch}''(K_4) = \chi''(K_4) = 5$. *In fact, if one vertex z_0 of K_4 is precoloured, each edge incident with z_0 is given a list of three colours not including the colour of z_0, and every other vertex and edge of K_4 is given a list of five colours, then the given colouring of z_0 can be extended to all the remaining vertices and edges.*

Proof. It is clear that $\mathrm{ch}''(K_4) \geqslant \chi''(K_4) \geqslant 5$, since there are ten elements (four vertices and six edges) to be coloured, and no colour can be used on more than two of them. We must prove that $\mathrm{ch}''(K_4) \leqslant 5$. To do this, suppose that z_0 is coloured, and lists are assigned, as in the second part of the lemma. Then the edges incident with z_0 can be coloured from their lists. The remaining uncoloured vertices and edges form a K_3, and each of them has a residual list of at least three usable colours. Since $\mathrm{ch}''(K_3) = 3$ by Theorem 1.1, these elements can all be coloured from their lists. (This argument is taken from the proof of Theorem 3.1 in [6].) $\square$

We can now prove Theorem 1.2.

Proof of Theorem 1.2. Let G be a $(\bar{K}_2 + (K_1 \cup K_2))$-minor-free graph with maximum degree Δ. If $\Delta \leqslant 2$ then the result follows from Theorem 1.1, since every graph with maximum degree $\leqslant 2$ is K_4-minor-free. If $\Delta = 3$ then the result again follows from Theorem 1.1, since by Lemma 3.1 and the value of Δ every component of G is either K_4-minor-free or isomorphic to K_4, and $\mathrm{ch}'(K_4) = \chi'(K_4) = 3$ by Theorem 1.4(b), and $\mathrm{ch}''(K_4) = \chi''(K_4) = 5$ by Lemma 3.2. So we may assume that $\Delta \geqslant 4$.

Clearly $\mathrm{ch}'(G) \geqslant \chi'(G) \geqslant \Delta$ and $\mathrm{ch}''(G) \geqslant \chi''(G) \geqslant \Delta+1$, and so it suffices to prove that $\mathrm{ch}'(G) \leqslant \Delta$ and $\mathrm{ch}''(G) \leqslant \Delta+1$. Let G be a minimal counterexample to either of these results. Clearly G is connected. By Lemma 3.1, every block of G is either K_4-minor-free or isomorphic to K_4. If G is 2-connected, then G is K_4-minor-free, since its maximum degree is too large for it to be isomorphic to K_4, and so the result follows from Theorem 1.1. So we may suppose that G is not 2-connected. Let B be an end-block of G with cut-vertex z_0.

Claim 3.1. *$B \ncong K_4$.*

Proof. Suppose $B \cong K_4$. Suppose first that G is a minimal counterexample to the statement that $\mathrm{ch}'(G) \leqslant \Delta$, and suppose that every edge of G is given a list of Δ colours. Then the edges of $G-(B-z_0)$ can be properly coloured from these lists. Since each edge of B still has a residual list of at least 3 usable colours, and since $\mathrm{ch}'(K_4)=3$ by Theorem 1.4(b), this colouring can be extended to the edges of B. This shows that $\mathrm{ch}'(G) \leqslant \Delta$, contradicting the choice of G.

So suppose now that G is a minimal counterexample to the statement that $\mathrm{ch}''(G) \leqslant \Delta+1$, and suppose that every vertex and edge of G is given a list of $\Delta+1$ colours. Then the vertices and edges of $G-(B-z_0)$ can be properly coloured from these lists. Each edge of B incident with z_0 has a residual list of at least $(\Delta+1)-(\Delta-3)-1=3$ usable colours, not including the colour of z_0, and each other vertex and edge of B has a list of at least 5 colours. By Lemma 3.2 this colouring can be extended to all the remaining vertices and edges of B. This shows that $\mathrm{ch}''(G) \leqslant \Delta+1$, again contradicting the choice of G. This completes the proof of Claim 3.1. $\Box$

In view of Claim 3.1 and Lemma 3.1, B must be K_4-minor-free. By the proof of Claim 2.1, $B \ncong K_2$, so that B is 2-connected and $d_G(z_0) \geqslant 3$. (Note that Claims 2.1–2.5 were proved in [7] in the edge-colouring case, in which G is a minimal K_4-minor-free graph such that $\mathrm{ch}'(G) > \Delta$; the proofs are essentially easier versions of the proofs in Theorem 2.1.) Let B_1 be the graph whose vertices consist of all vertices of B with degree at least 3 in G, where two vertices are adjacent in B_1 if and only if they are connected in G by an edge or a path whose internal vertices all have degree 2. By the proofs of Claims 2.2 and 2.3, B does not contain two adjacent vertices of degree 2 that are both different from z_0, nor a 4-cycle $xuyvx$ such that u and v both have degree 2 and are different from z_0. It follows that B_1 has no vertex with degree 0 or 1. Moreover, any vertex with degree 2 in B_1, other than z_0, must occur in B as vertex u in Fig. 1(a) or 1(b), where only x and y are incident with edges of G that are not shown (so that w, and v if present, have degree 2 in G and not just in B; that is, $z_0 \notin \{u,w\}$ in Fig. 1(a) and $z_0 \notin \{u,v,w\}$ in Fig. 1(b)). However, this is impossible by the proof of Claim 2.4 or Claim 2.5. This means that B_1 has no vertex of degree 2 other than z_0. But clearly B_1 is a minor of B, and so is K_4-minor-free, and this means that B_1 contains at least two vertices of degree 2, by Theorem 1.3. This contradiction completes the proof of Theorem 1.2. $\Box$

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