The Combinatorics of the Garsia-Haiman Modules for Hook Shapes

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Abstract

Several bases of the Garsia-Haiman modules for hook shapes are given, as well as combinatorial decomposition rules for these modules. These bases and rules extend the classical ones for the coinvariant algebra of type A. We also give a decomposition of the Garsia-Haiman modules into descent representations.

1 Introduction

1.1 Outline

In [11], Garsia and Haiman introduced a module $\mathbf{H}_\mu$ for each partition μ, which we shall call the Garsia-Haiman module for μ. Garsia and Haiman introduced the modules $\mathbf{H}_\mu$ in

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attempt to prove Macdonald's q,t-Kostka polynomial conjecture and, in fact, the modules $\mathbf{H}_\mu$ played a major role in the resolution of Macdonald's conjecture [23]. When the shape μ has a single row, this module is isomorphic to the coinvariant algebra of type A. Our goal here is to understand the structure of this module when μ is a hook shape $(1^{k-1}, n-k+1)$.

A family of bases for the Garsia-Haiman module of hook shape $(1^{k-1}, n-k+1)$ is presented. This family includes the *k-th Artin basis*, the *k-th descent basis*, the *k-th Haglund basis* and the *k-th Schubert basis* as well as other bases. While the first basis appears in [13], the others are new and have interesting applications.

The k-th Haglund basis realizes Haglund's statistics for the modified Macdonald polynomials in the hook case. The k-th descent basis extends the well known Garsia-Stanton descent basis for the coinvariant algebra. The advantage of the k-th descent basis is that the S_n-action on it may be described explicitly. This description implies combinatorial rules for decomposing the bi-graded components of the module into Solomon descent representations and into irreducibles. In particular, a constructive proof of a formula due to Stembridge is deduced.

Contents

1.2 General Background

In 1988, I. G. Macdonald [27] introduced a remarkable new basis for the space of symmetric functions. The elements of this basis are denoted $P_\lambda(\overline{x};q,t)$, where λ is a partition, $\overline{x}$ is a vector of indeterminates, and q,t are parameters. The $P_\lambda(\overline{x};q,t)$'s, which are now called "*Macdonald polynomials*", specialize to many of the well-known bases for the symmetric functions, by suitable substitutions for the parameters q and t. In fact, we can obtain in this manner the Schur functions, the Hall-Littlewood symmetric functions, the Jack symmetric functions, the zonal symmetric functions, the zonal spherical functions, and the elementary and monomial symmetric functions.

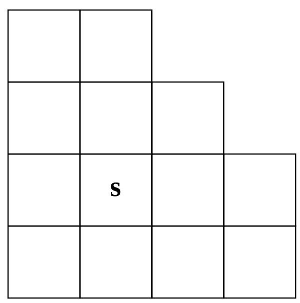


Figure 1: Diagram of a partition.

Given a cell s in the Young diagram (drawn according to the French convention) of a partition λ, let $leg_\lambda(s)$, $leg'_\lambda(s)$, $arm_\lambda(s)$, and $arm'_\lambda(s)$ denote the number of squares that lie above, below, to the right, and to left of s in λ, respectively. For example, when $\lambda=(2,3,3,4)$ and s is the cell pictured in Figure 1, $leg_\lambda(s)=2$, $leg'_\lambda(s)=1$, $arm_\lambda(s)=2$ and $arm'_\lambda(s)=1$. For each partition λ, define

$$h_\lambda(q,t) := \prod_{s\in\lambda}(1-q^{arm_\lambda(s)}t^{leg_\lambda(s)+1}) \tag{1}$$

For a partition $\lambda=(\lambda_1,\ldots,\lambda_k)$ where $0<\lambda_1\leq\ldots\leq\lambda_k$, let $n(\lambda):=\sum_{i=1}^{k}(k-i)\lambda_i$. Macdonald introduced the (q,t)-Kostka polynomials $K_{\lambda,\mu}(q,t)$ via the equation

$$J_\mu(\overline{x};q,t)=h_\mu(q,t)P_\mu(\overline{x};q,t)=\sum_\lambda K_{\lambda,\mu}(q,t)s_\lambda[X(1-t)], \tag{2}$$

and conjectured that they are polynomials in q and t with non-negative integer coefficients.

In an attempt to prove Macdonald's conjecture, Garsia and Haiman [11] introduced the so-called *modified Macdonald polynomials* $\tilde{H}_\mu(\overline{x};q,t)$ as

$$\tilde{H}_\mu(\overline{x};q,t) = \sum_\lambda \tilde{K}_{\lambda,\mu}(q,t)s_\lambda(\overline{x}), \tag{3}$$

where $\tilde{K}_{\lambda,\mu}(q,t) := t^{n(\mu)}K_{\lambda,\mu}(q,1/t)$. Their idea was that $\tilde{H}_\mu(\overline{x};q,t)$ is the Frobenius image of the character generating function of a certain bi-graded module $\mathbf{H}_\mu$ under the diagonal action of the symmetric group S_n. To define $\mathbf{H}_\mu$, assign $(row, column)$-coordinates to squares in the first quadrant, obtained by permuting the (x,y) coordinates of the upper right-hand corner of the square so that the lower left-hand square has coordinates (1,1), the square above it has coordinates (2,1), the square to its right has coordinates (1,2), etc. The first (row) coordinate of a square w is denoted $row(w)$, and the second (column) coordinate of w is the denoted $col(w)$. Given a partition $\mu \vdash n$, let μ also denote the corresponding Young diagram, drawn according to the French convention, which consists of all the squares with coordinates (i,j) such that $1 \leq i \leq \ell(\mu)$ and $1 \leq j \leq \mu_i$. For example, for $\mu = (2,2,4)$, the labelling of squares is depicted in Figure 2.

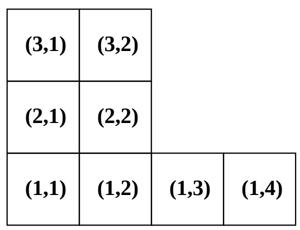


Figure 2: Labelling of the cells of a partition.

Fix an ordering $w_1,\ldots,w_n$ of the squares of μ, and let

$$\Delta_\mu(x_1,\ldots,x_n;y_1,\ldots,y_n) := \det\left(x_i^{row(w_j)-1}y_i^{col(w_j)-1}\right)_{i,j}. \tag{4}$$

For example,

$$\Delta_{(2,2,4)}(x_1,\ldots,x_8;y_1,\ldots,y_8) = \det\begin{pmatrix} 1 & y_1 & y_1^2 & y_1^3 & x_1 & x_1y_1 & x_1^2 & x_1^2y_1 \\ 1 & y_2 & y_2^2 & y_2^3 & x_2 & x_2y_2 & x_2^2 & x_2^2y_2 \\ \vdots & & & & & & & \vdots \\ 1 & y_8 & y_8^2 & y_8^3 & x_8 & x_8y_8 & x_8^2 & x_8^2y_8 \end{pmatrix}.$$

Now let $\mathbf{H}_\mu$ be the vector space of polynomials spanned by all the partial derivatives of $\Delta_\mu(x_1,\ldots,x_n;y_1,\ldots,y_n)$. The symmetric group S_n acts on $\mathbf{H}_\mu$ diagonally, where for any polynomial $P(x_1,\ldots,x_n;y_1,\ldots,y_n)$ and any permutation $\sigma \in S_n$,

$$P(x_1,\ldots,x_n;y_1,\ldots,y_n)^\sigma := P(x_{\sigma_1},\ldots,x_{\sigma_n};y_{\sigma_1},\ldots,y_{\sigma_n}).$$

The bi-degree (h,k) of a monomial $x_1^{p_1}\cdots x_n^{p_n}y_1^{q_1}\cdots y_n^{q_n}$ is defined by $h:=\sum_{i=1}^n p_i$ and $k:=\sum_{i=1}^n q_i$. Let $\mathbf{H}_\mu^{(h,k)}$ denote space of homogeneous polynomials of degree (h,k) in $\mathbf{H}_\mu$. Then

$$\mathbf{H}_\mu=\bigoplus_{(h,k)}\mathbf{H}_\mu^{(h,k)}.$$

The S_n-action clearly preserves the bi-degree so that S_n acts on each homogeneous component $\mathbf{H}_\mu^{(h,k)}$. The character of the S_n-action on $\mathbf{H}_\mu^{(h,k)}$ can be decomposed as

$$\chi^{\mathbf{H}_\mu^{(h,k)}}=\sum_{\lambda\vdash n}\chi_{\lambda,\mu}^{(h,k)}\chi^\lambda, \tag{5}$$

where χ^λ is the irreducible character of S_n indexed by the partition λ and the $\chi_{\lambda,\mu}^{(h,k)}$'s are non-negative integers. We then define the character generating function of $\mathbf{H}_\mu$ to be

$$\begin{aligned}\chi^{\mathbf{H}_\mu}(q,t) &= \sum_{h,k\geq 0}q^ht^k\sum_{\lambda\vdash|\mu|}\chi_{\lambda,\mu}^{(h,k)}\chi^\lambda \qquad (6)\\ &= \sum_{\lambda\vdash|\mu|}\chi^\lambda\sum_{h,k\geq 0}\chi_{\lambda,\mu}^{(h,k)}q^ht^k.\end{aligned}$$

The Frobenius map F, which maps the center of the group algebra of S_n to $\Lambda_n(\overline{x})$, is defined by sending the character χ^λ to the Schur function $s_\lambda(\overline{x})$. Garsia and Haiman conjectured that the Frobenius image of $\chi^{\mathbf{H}_\mu}(q,t)$ which they denoted by $F_\mu(q,t)$ is the modified Macdonald polynomial $\tilde{H}_\mu(\overline{x};q,t)$. That is, they conjectured that

$$\begin{aligned}F(\chi^{\mathbf{H}_\mu}(q,t)) &= \sum_{h,k\geq 0}q^ht^k\sum_{\lambda\vdash|\mu|}\chi_{\lambda,\mu}^{(h,k)}s_\lambda(\overline{x}) \qquad (7)\\ &= \sum_{\lambda\vdash|\mu|}s_\lambda(\overline{x})\sum_{h,k\geq 0}\chi_{\lambda,\mu}^{(h,k)}q^ht^k\\ &= \tilde{H}_\mu(\overline{x};q,t)\end{aligned}$$

so that

$$\tilde{K}_{\lambda,\mu}(q,t)=\sum_{h,k\geq 0}\chi_{\lambda,\mu}^{(h,k)}q^ht^k. \tag{8}$$

Since Macdonald proved that $K_{\lambda,\mu}(1,1)=f_\lambda$, the number of standard tableau of shape λ, equations (7) and (8) led Garsia and Haiman [11] to conjecture that as an S_n-module, $\mathbf{H}_\mu$ carries the regular representation. This conjecture was eventually proved by Haiman [23] using the algebraic geometry of the Hilbert Scheme.

The goal of this paper is to understand the structure of the modules $\mathbf{H}_\mu$ when μ is a hook shape $(1^{k-1},n-k+1)$. These modules were studied before by Stembridge [38], Garsia and Haiman [13], Allen [5] and Aval [6]. This paper suggests a detailed combinatorial analysis of the modules.

1.3 Main Results - Bases

Consider the inner product $\langle\ ,\ \rangle$ on the polynomial ring $\mathbf{Q}[\bar{x},\bar{y}] = \mathbf{Q}[x_1,\ldots,x_n,y_1,\ldots,y_n]$ defined as follows:

$$\langle f,g\rangle := \text{constant term of } f(\partial_{x_1},\ldots,\partial_{x_n};\partial_{y_1},\ldots,\partial_{y_n})g(x_1,\ldots,x_n;y_1,\ldots,y_n), \qquad (9)$$

where $f(\partial_{x_1},\ldots,\partial y_n)$ is the differential operator obtained by replacing each variable x_i (y_i) in f by the corresponding partial derivative $\frac{\partial}{\partial x_i}$ $(\frac{\partial}{\partial y_i})$. Let J_μ be the S_n-module dual to $\mathbf{H}_\mu$ with respect to $\langle\ ,\ \rangle$, and let $\mathbf{H}'_\mu := \mathbf{Q}[\bar{x},\bar{y}]/J_\mu$. It is not difficult to see that $\mathbf{H}_\mu$ and $\mathbf{H}'_\mu$ are isomorphic as S_n-modules.

1.3.1 The k-th Descent Basis

The *descent set* of a permutation $\pi \in S_n$ is

$$\mathrm{Des}(\pi) := \{i \,:\, \pi(i) > \pi(i+1)\}.$$

Garsia and Stanton [17] associated with each $\pi \in S_n$, the *descent monomial*

$$a_\pi := \prod_{i\in\mathrm{Des}(\pi)} (x_{\pi(1)}\cdots x_{\pi(i)}) = \prod_{j=1}^{n-1} x_{\pi(j)}^{|\mathrm{Des}(\pi)\cap\{j,\ldots,n-1\}|}.$$

Using Stanley-Reisner rings, Garsia and Stanton [17] showed that the set $\{a_\pi \,:\, \pi \in S_n\}$ forms a basis for the coinvariant algebra of type A. See also [39] and [4].

Definition 1.1. *For every integer $1 \le k \le n$ and permutation $\pi \in S_n$ define*

$$d_i^{(k)}(\pi) := \begin{cases} |\mathrm{Des}(\pi)\cap\{i,\ldots,k-1\}|, & \text{if } 1 \le i < k; \\ 0, & \text{if } i = k; \\ |\mathrm{Des}(\pi)\cap\{k,\ldots,i-1\}|, & \text{if } k < i \le n. \end{cases}$$

Definition 1.2. *For every integer $1 \le k \le n$ and permutation $\pi \in S_n$ define the k-*th descent monomial

$$\begin{aligned} a_\pi^{(k)} &:= \prod_{\substack{i\in\mathrm{Des}(\pi)\\ i\le k-1}} (x_{\pi(1)}\cdots x_{\pi(i)}) \cdot \prod_{\substack{i\in\mathrm{Des}(\pi)\\ i\ge k}} (y_{\pi(i+1)}\cdots y_{\pi(n)}) \\ &= \prod_{i=1}^{k-1} x_{\pi(i)}^{d_i^{(k)}(\pi)} \cdot \prod_{i=k+1}^{n} y_{\pi(i)}^{d_i^{(k)}(\pi)}. \end{aligned}$$

For example, if $n = 8$, $k = 4$, and $\pi = 8\ 6\ 1\ 4\ 7\ 3\ 5\ 2$, then $Des(\pi) = \{1,2,5,7\}$, $(d_1^{(4)}(\pi),\ldots,d_8^{(4)}(\pi)) = (2,1,0,0,0,1,1,2)$, and $a_\pi^{(4)} = x_1^2x_2y_6y_7y_8^2$.

Note that $a_\pi^{(n)} = a_\pi$, the Garsia-Stanton descent monomial.

Consider the partition $\mu = (1^{k-1}, n-k+1)$.

Theorem 1.3. *For every $1 \le k \le n$, the set of k-th descent monomials $\{a_\pi^{(k)} : \pi \in S_n\}$ forms a basis for the Garsia-Haiman module $\mathbf{H}_{(1^{k-1},n-k+1)}$.*

Two proofs of Theorem 1.3 are given in this paper. In Section 5 it is proved via a straightening algorithm. This proof implies an explicit description of the Garsia-Haiman hook module $\mathbf{H}'_{(1^{k-1},n-k+1)}$.

Theorem 1.4. *For $\mu = (1^{k-1}, n-k+1)$ the ideal $J_\mu = \mathbf{H}_\mu^\perp$ defined above is the ideal of $\mathbf{Q}[\overline{x}, \overline{y}]$ generated by*

(i) $\Lambda[\bar{x}]^+$ *and* $\Lambda[\bar{y}]^+$ *(the symmetric functions in* $\bar{x}$ *and* $\bar{y}$ *without a constant term),*

(ii) the monomials $x_{i_1} \cdots x_{i_k}$ $(i_1 < \cdots < i_k)$ *and* $y_{i_1} \cdots y_{i_{n-k+1}}$ $(i_1 < \cdots < i_{n-k+1})$*, and*

(iii) the monomials $x_i y_i$ $(1 \le i \le n)$*.*

This result has been obtained in a different form by J.-C. Aval [6, Theorem 2].

1.3.2 The k-th Artin and Haglund Bases

A second proof of Theorem 1.3 is given in Section 3. This proof applies a generalized version of the Garsia-Haiman kicking process. This construction is extended to a rich family of bases.

For every positive integer n, denote $[n] := \{1, \ldots, n\}$. For every subset $A = \{i_1, \ldots, i_k\} \subseteq [n]$ denote $\bar{x}_A := x_{i_1}, \ldots, x_{i_k}$ and $\bar{y}_A := y_{i_1}, \ldots, y_{i_k}$. Denote $\bar{x} := \bar{x}_{[n]} = x_1, \ldots, x_n$ and $\bar{y} := \bar{y}_{[n]} = y_1, \ldots, y_n$.

Let $k, c \in [n]$, let $A = \{a_1, \ldots, a_{k-1}\}$ be a subset of size $k-1$ of $[n] \setminus c$, and let $\bar{A} := [n] \setminus (A \cup \{c\})$. Let B_A be an arbitrary basis of the coinvariant algebra of S_{k-1} acting on $\mathbf{Q}[\bar{x}_A]$, and let $C_{\bar{A}}$ be a basis of the coinvariant algebra of S_{n-k} acting on $\mathbf{Q}[\bar{y}_{\bar{A}}]$. Finally define

$$m_{(A,c,\bar{A})} := \prod_{\{i \in A \,:\, i > c\}} x_i \prod_{\{j \in \bar{A} \,:\, j < c\}} y_j \in \mathbf{Q}[\bar{x}, \bar{y}].$$

Then

Theorem 1.5. *The set*

$$\bigcup_{A,c} m_{(A,c,\bar{A})}\ B_A\ C_{\bar{A}} := \bigcup_{A,c} \{m_{(A,c,\bar{A})} bc \ :\ b \in B_A, c \in C_{\bar{A}}\}$$

forms a basis for the Garsia-Haiman module $\mathbf{H}'_{(1^{k-1},n-k+1)}$.

Definition 1.6. *For every integer $1 \le k \le n$ and permutation $\pi \in S_n$ define*

$$\mathrm{inv}_i^{(k)}(\pi) := \begin{cases} |\{j \ :\ i < j \le k \ \textit{and}\ \pi(i) > \pi(j)\}|, & \textit{if}\ 1 \le i < k; \\ 0, & \textit{if}\ i = k; \\ |\{j \ :\ k \le j < i \ \textit{and}\ \pi(j) > \pi(i)\}|, & \textit{if}\ k < i \le n. \end{cases}$$

For every integer $1 \leq k \leq n$ and permutation $\pi \in S_n$ define the k-th Artin monomial

$$b_\pi^{(k)} \quad := \quad \prod_{i=1}^{k-1} x_{\pi(i)}^{\mathrm{inv}_i^{(k)}(\pi)} \cdot \prod_{i=k+1}^{n} y_{\pi(i)}^{\mathrm{inv}_i^{(k)}(\pi)}.$$

and the k-th Haglund monomial

$$c_\pi^{(k)} \quad := \quad \prod_{i=1}^{k-1} x_{\pi(i)}^{d_i^{(k)}(\pi)} \cdot \prod_{i=k+1}^{n} y_{\pi(i)}^{\mathrm{inv}_i^{(k)}(\pi)}.$$

For example, if $n = 8$, $k = 4$, and $\pi = 8\ 6\ 1\ 4\ 7\ 3\ 5\ 2$, then $Des(\pi) = \{1, 2, 5, 7\}$, $(inv_1^{(4)}(\pi), \ldots, inv_8^{(4)}(\pi)) = (3, 2, 0, 0, 0, 2, 1, 4)$, $b_\pi^{(4)} = x_8^3 x_6^2 y_3^2 y_5 y_2^4$, and $c_\pi^{(4)} = x_8^2 x_6 y_3^2 y_5 y_2^4$.

Interesting special cases of Theorem 1.5 are the following.

Corollary 1.7. *Each of the following sets :*

$$\{a_\pi^{(k)} \ :\ \pi \in S_n\}, \qquad \{b_\pi^{(k)} \ :\ \pi \in S_n\}, \qquad \{c_\pi^{(k)} \ :\ \pi \in S_n\}$$

forms a basis for the Garsia-Haiman module $\mathbf{H}'_{(1^{k-1},n-k+1)}$.

Remark 1.8.

1. Garsia and Haiman [12] showed that $\{b_\pi^{(k)} \ :\ \pi \in S_n\}$ is a basis for $\mathbf{H}'_{(1^{k-1},n-k+1)}$. Other bases of $\mathbf{H}_{(1^{k-1},n-k+1)}$ were also constructed by J-C Aval [6] and E. Allen [4, 5]. They used completely different methods. Aval constructed a basis of the form of an explicitly described set of partial differential operators applied to $\Delta_{(1^{k-1},n-k+1)}$ and Allen constructed a basis for $\mathbf{H}_{(1^{k-1},n-k+1)}$ out of his theory of bitableaux.

2. It should be noted that the last basis corresponds to Haglund's statistics for the Hilbert series of $\mathbf{H}_{(1^{k-1},n-k+1)}$ that is implied by his combinatorial interpretation for the modified Macdonald polynomial $\tilde{H}_{(1^{k-1},n-k+1)}(\bar{x}; q, t)$; see Section 8 below.

3. Choosing B_A and $C_{\bar{A}}$ in Theorem 1.5 to be the Schubert bases of the coinvariant algebras of S_{k-1} (acting on $\mathbf{Q}[\bar{x}_A]$) and of S_{n-k} (acting on $\mathbf{Q}[\bar{y}_{\bar{A}}]$), respectively, gives the *k-th Schubert basis.* One may study the Hecke algebra actions on this basis along the lines drawn in [2].

1.4 Main Results - Representations

1.4.1 Decomposition into Descent Representations

The set of elements in a Coxeter group having a fixed descent set carries a natural representation of the group, called a descent representation. Descent representations of Weyl groups were first introduced by Solomon [35] as alternating sums of permutation representations. This concept was extended to arbitrary Coxeter groups, using a different construction, by Kazhdan and Lusztig [25] [24, §7.15]. For Weyl groups of type A,

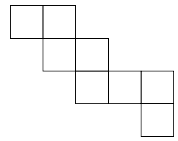


Figure 3: The zigzag shape corresponding to $n = 8$ and $A = \{2, 4, 7\}$.

these representations also appear in the top homology of certain (Cohen-Macaulay) rank-selected posets [37]. Another description (for type A) is by means of zig-zag diagrams [18, 16]. A new construction of descent representations for Weyl groups of type A, using the coinvariant algebra as a representation space, is given in [1].

For every subset $A \subseteq \{1, \ldots, n-1\}$, let

$$S_n^A := \{\pi \in S_n \,:\, \mathrm{Des}(\pi) = A\}$$

be the corresponding *descent class* and let ρ^A denote the corresponding *descent representation* of S_n. Given n and subset $A = \{a_1 < \cdots < a_k\} \subseteq \{1, \ldots, n-1\}$, we can associate a composition of n, $comp(A) = (c_1, \ldots, c_{k+1}) = (a_1, a_2 - a_1, \ldots, a_k - a_{k-1}, n - a_k)$ and zigzig (skew) diagram D_A which in French notation consists of rows of size $c_1, \ldots, c_{k+1}$, reading from top to bottom, which overlap by one square. For example, if $n = 8$ and $A = \{2, 4, 7\}$, then $comp(A) = (2, 2, 3, 1)$ and $D(A)$ is the diagram pictured in Figure (3).

Definition 1.9. *A bipartition (i.e., a pair of partitions) $\lambda = (\mu, \nu)$ where $\mu = (\mu_1 \geq \cdots \geq \mu_{k+1} \geq 0)$ and $\nu = (\nu_1 \geq \cdots \geq \nu_{n-k+1} \geq 0)$ is called an (n, k)-bipartition if*

1. *$\lambda_k = \nu_{n-k+1} = 0$ so that μ has at most $k-1$ parts and ν has at most $n-k$ parts, .*
2. *for $i = 1, \ldots, k-1$, $\lambda_i - \lambda_{i+1} \in \{0, 1\}$, and*
3. *for $i = 1, \ldots, n-k$, $\nu_i - \nu_{i+1} \in \{0, 1\}$.*

For a permutation $\pi \in S_n$ and a corresponding k-descent basis element $a_\pi^{(k)} = \prod_{i=1}^{k-1} x_{\pi(i)}^{d_i} \cdot \prod_{i=k+1}^{n} y_{\pi(i)}^{d_i}$, let

$$\lambda(a_\pi^{(k)}) := ((d_1, d_2, \ldots, d_{k-1}, 0), (d_n, d_{n-1}, \ldots, d_{k+1}, 0))$$

be its exponent bipartition.

For an (n, k)-bipartition $\lambda = (\mu, \nu)$ let

$$I_\lambda^{(k)\trianglelefteq} := \mathrm{span}_{\mathbf{Q}}\{a_\pi^{(k)} + J_{(1^{k-1}, n-k+1)} \,:\, \pi \in S_n,\ \lambda(a_\pi^{(k)}) \trianglelefteq \lambda\},$$

and

$$I_\lambda^{(k)\triangleleft} := \mathrm{span}_{\mathbf{Q}}\{a_\pi^{(k)} + J_{(1^{k-1}, n-k+1)} \,:\, \pi \in S_n,\ \lambda(a_\pi^{(k)}) \triangleleft \lambda\}$$

be subspaces of the module $\mathbf{H}'_{(1^{k-1}, n-k+1)}$, where $\trianglelefteq$ is the dominance order on bipartitions (see Definition 5.4.1), and let

$$R_\lambda^{(k)} := I_\lambda^{(k)\trianglelefteq} / I_\lambda^{(k)\triangleleft}.$$

Proposition 1.10. *$I_\lambda^{(k)\trianglelefteq}$, $I_\lambda^{(k)\triangleleft}$ and thus $R_\lambda^{(k)}$ are S_n-invariant.*

Lemma 1.11. *Let $\lambda = (\mu, \nu)$ be an (n, k)-bipartition. Then*

$$\{a_\pi^{(k)} + I_\lambda^{(k)\triangleleft} \,:\, \mathrm{Des}(\pi) = A_\lambda\} \tag{10}$$

is a basis for $R_\lambda^{(k)}$, where

$$A_\lambda := \{1 \le i < n \,:\, \mu_i - \mu_{i+1} = 1 \text{ or } \nu_{n-i} - \nu_{n-i+1} = 1 \ \}. \tag{11}$$

Theorem 1.12. *The S_n-action on $R_\lambda^{(k)}$ is given by*

$$s_j(a_\pi^{(k)}) = \begin{cases} a_{s_j\pi}^{(k)}, & \text{if } |\pi^{-1}(j+1) - \pi^{-1}(j)| > 1; \\ a_\pi^{(k)}, & \text{if } \pi^{-1}(j+1) = \pi^{-1}(j) + 1; \\ -a_\pi^{(k)} - \sum_{\sigma \in A_j(\pi)} a_\sigma^{(k)}, & \text{if } \pi^{-1}(j+1) = \pi^{-1}(j) - 1. \end{cases}$$

Here $s_j = (j, j+1)$ $(1 \le j < n)$ are the Coxeter generators of S_n and $\{a_\pi^{(k)} + I_\lambda^{(k)\triangleleft} : \mathrm{Des}(\pi) = A_\lambda\}$ is the descent basis of $R_\lambda^{(k)}$. For the definition of $A_j(\pi)$ see Theorem 6.4 below.

This explicit description of the action is then used to prove the following.

Theorem 1.13. *Let $\lambda = (\mu, \nu)$ be an (n, k)-bipartition. $R_\lambda^{(k)}$ is isomorphic, as an S_n-module, to the Solomon descent representation determined by the descent class $\{\pi \in S_n : \mathrm{Des}(\pi) = A_\lambda\}$.*

Let ${\mathbf{H}'}^{(t_1,t_2)}_{(1^{k-1},n-k+1)}$ be the (t_1, t_2)-th homogeneous component of $\mathbf{H}'_{(1^{k-1},n-k+1)}$.

Corollary 1.14. *For every $t_1, t_2 \ge 0$ and $1 \le k \le n$, the (t_1, t_2)-th homogeneous component of $\mathbf{H}'_{(1^{k-1},n-k+1)}$ decomposes into a direct sum of Solomon descent representations as follows:*

$${\mathbf{H}'}^{(t_1,t_2)}_{(1^{k-1},n-k+1)} \cong \bigoplus_\lambda R_\lambda^{(k)},$$

where the sum is over all (n, k)-bipartitions $\lambda = (\mu, \nu)$. such that

$$\sum_{i \ge k \text{ and } \nu_i > \nu_{i+1}} (n - i) = t_1, \qquad \sum_{i < k \text{ and } \mu_i > \mu_{i+1}} i = t_2. \tag{12}$$

For example, suppose that $k = 3$ and $n = 4$. Then if $\lambda = (\mu, \nu)$ is a $(4, 3)$ partition, then $\mu \in \{(0, 0, 0), (1, 0, 0), (1, 1, 0), (2, 1, 0)\}$ and $\nu \in \{(0, 0), (1, 0)\}$. Table 1 lists all the possible $(4, 3)$-partitions. Then for each $\lambda = (\mu, \nu)$, we list the corresponding weight $q^{t_1} t^{t_2}$ given by (12), the corresponding descent set A_λ, and the ribbon Schur function corresponding to A_λ.

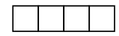
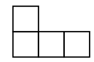
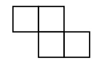
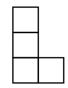
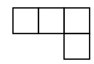
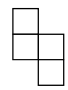
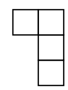
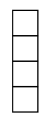
(μ,ν)	Weight	Descent Set	Shape
((0,0,0),(0,0))	**1**	**{}**	
((1,0,0),(0,0))	**t**	**{1}**	
((1,1,0),(0,0))	$\mathbf{t^2}$	**{2}**	
((2,1,0),(0,0))	$\mathbf{t^3}$	**{1,2}**	
((0,0,0),(1,0))	**q**	**{3}**	
((1,0,0),(1,0))	**qt**	**{1,3}**	
((1,1,0),(1,0))	$\mathbf{qt^2}$	**{2,3}**	
((2,1,0),(1,0))	$\mathbf{qt^3}$	**{1,2,3}**	

Figure 4: Table 1.

1.4.2 Decomposition into Irreducibles

A classical theorem of Lusztig and Stanley gives the multiplicity of the irreducibles in the homogeneous component of the coinvariant algebra of type A. Define $1 \le i < n$ to be a *descent* in a standard Young tableau T if $i+1$ lies strictly above and weakly to the left of i (in French notation). Denote the set of all descents in T by $\mathrm{Des}(T)$ and let the *major index* of T be

$$\mathrm{maj}(T) := \sum_{i \in \mathrm{Des}(T)} i.$$

Theorem 1.15. [36, Prop. 4.11] *The multiplicity of the irreducible S_n-representation S^λ in the k-th homogeneous component of the coinvariant algebra of type A is*

$$\#\{T \in SYT(\lambda) \,:\, \mathrm{maj}(T) = k\},$$

where $SYT(\lambda)$ is the set of all standard Young tableaux of shape λ.

In 1994, Stembridge [38] gave an explicit combinatorial interpretation of the (q,t)-Kostka polynomials for hook shape. Stembridge's result implies the following extension of the Lusztig-Stanley theorem.

For a standard Young tableau T define

$$\mathrm{maj}_{i,j}(T) := \sum_{\substack{r\in \mathrm{Des}(T)\\ i\le r<j}} r$$

and

$$\mathrm{comaj}_{i,j}(T) := \sum_{\substack{r\in \mathrm{Des}(T)\\ i\le r<j}} (n-r).$$

For example, for the column strict tableaux T pictured in Figure 5 and $k=4$, $Des(T) = \{2,3,5,7\}$, $\mathrm{maj}_{1,4}(T) = 2+3 = 5$, and $\mathrm{comaj}_{5,8} = (8-5)+(8-7) = 4$.

4	8		
3	6		
1	2	5	7

Figure 5: Des, $\mathrm{maj}_{1,k}$, and $\mathrm{comaj}_{k,n}$ for a standard tableau.

We can restate Stembridge's results [38] as follows.

Theorem 1.16.

$$\tilde{K}_{\lambda,(1^{k-1},n-k+1)}(q,t) = \sum_{T\in SYT(\lambda)} q^{\mathrm{maj}_{1,n-k+1}(T)} t^{\mathrm{comaj}_{n-k+1,n}(T)}. \tag{13}$$

Our next result is an immediate consequence of Haiman's result (8) and Theorem 1.16.

Theorem 1.17. *The multiplicity of the irreducible S_n-representation S^λ in the (h,h') level of $\mathbf{H}_{(1^{k-1},n-k+1)}$ (bi-graded by total degrees in the x's and y's) is*

$$\chi_\lambda^{(h,h')} = \#\{T\in SYT(\lambda)\ :\ \mathrm{maj}_{1,n-k+1}(T) = h,\ \mathrm{comaj}_{n-k+1,n}(T) = h'\},$$

where $SYT(\lambda)$ is the set of all standard Young tableaux of shape λ.

Haglund [19] gave another proof of Theorem 1.17 that used his conjectured combinatorial definition of $\tilde{H}_\mu(\overline{x};q,t)$. Haglund's conjecture has recently been proved by Haglund, Haiman and Loehr [20, 21].

We give two proofs of this decomposition rule. The first one, given in Section 6, follows from the decomposition into descent representations described in Corollary 1.14 above.

The second proof, given in Section 7, is more "combinatorial". It uses the mechanism of [21] but does not rely on Haglund's combinatorial interpretation of $\tilde{H}_{(1^{k-1},n-k+1)}(\overline{x};q,t)$.

2 The Garsia-Haiman Module $\mathbf{H}_\mu$

In this section, we shall provide some background on the Garsia-Haiman module, also known as the space of orbit harmonics. Proofs of the results in this section can be found in [13] and [14].

Let $m := 2n$ and $(z_1, \ldots, z_m) := (x_1, \ldots, x_n, y_1, \ldots, y_n)$. Given positive weights $w_1, \ldots, w_m$, define a grading of $\mathbf{R} = \mathbf{Q}[z_1, \ldots, z_m]$ by setting, for any monomial $\overline{z}^p = z_1^{p_1} \cdots z_m^{p_m}$,

$$d_w(\overline{z}^p) := \sum_{i=1}^{m} w_i p_i. \tag{14}$$

For any polynomial $g = \sum_p c_p \overline{z}^p$ and any k, set

$$\pi_k^w g = \sum_{d_w(\overline{z}^p)=k} c_p \overline{z}^p. \tag{15}$$

Call $\pi_k^w g$ the *w-homogeneous component of w-degree k*, and let $\mathrm{top}_w(g)$ denote the w-homogeneous component of highest w-degree in g. Let V be a subspace of $\mathbf{R}$. We say that V is *w-homogeneous* if $\pi_k^w V \subseteq V$ for all k. Let $V^\perp$ be the orthogonal complement of V with respect to the inner product defined at the beginning of Subsection 1.3. If V is w-homogeneous, then $V^\perp$ is also w-homogeneous.

If J is an ideal of $\mathbf{R}$ then

$$J^\perp = \{g \in \mathbf{Q}[\overline{z}] : f(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_m}) g(\overline{z}) = 0 \ (\forall f \in J)\}. \tag{16}$$

In particular, $J^\perp$ is closed under differentiation. We set $\mathbf{R}_J = \mathbf{R}/J$ and define the *associated w-graded ideal of J* to be the ideal

$$gr_w J := \langle \mathrm{top}_w(g) : g \in J \rangle. \tag{17}$$

The quotient ring $\mathbf{R}/gr_w J$ is referred to as the *w-graded version of* $\mathbf{R}_J$ and is denoted by $gr_w \mathbf{R}_J$.

Given a point $\rho = (\rho_1, \ldots, \rho_m)$, denote its S_n-orbit by

$$[\rho] = [\rho]_{S_n} = \{\sigma(\rho) : \sigma \in S_n\}. \tag{18}$$

We also set

$$\mathbf{R}_{[\rho]} := \mathbf{R}/J_{[\rho]} \tag{19}$$

where

$$J_{[\rho]} = \{g \in \mathbf{R} : g(\bar{z}) = 0 \ (\forall \bar{z} \in [\rho])\} \tag{20}$$

Considered as an algebraic variety, $\mathbf{R}_{[\rho]}$ is the coordinate ring of $[\rho]$. Since the ideal $J_{[\rho]}$ is S_n-invariant, S_n also acts on $\mathbf{R}_{[\rho]}$. In fact, it is easy to see that the corresponding representation is equivalent to the action of S_n on the left cosets of the stabilizer of ρ. Thus if the stabilizer of ρ is trivial, i.e. if ρ is a regular point, then $\mathbf{R}_{[\rho]}$ is a version of the left regular representation of S_n. If each $\sigma \in S_n$ preserves w-degrees, then we can associate two further S_n-modules with ρ; namely, $gr_w J_{[\rho]}$ and its orthogonal complement

$$\mathbf{H}_{[\rho]} = (gr_w J_{[\rho]})^{\perp}. \tag{21}$$

Garsia and Haiman refer to $\mathbf{H}_{[\rho]}$ as the space of *harmonics of the orbit* $[\rho]$. Clearly if $f(\overline{z})$ is an S_n-invariant polynomial, then $f(\overline{z}) - f(\rho) \in J_{[\rho]}$. In addition, if $f(\overline{z})$ is w-homogeneous, then $f(\overline{z}) \in gr_w J_{[\rho]}$. This implies that any element $g \in \mathbf{H}_{[\rho]}$ must satisfy the differential equation

$$f(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_m}) g(z_1, \ldots, z_m) = 0.$$

It is not difficult to show that $\mathbf{H}_{[\rho]}$ and $gr_w \mathbf{R}_{[\rho]}$ are equivalent w-graded modules and that these two spaces realize a graded version of the regular representations of S_n.

Given a partition $\mu = (\mu_1 \geq \cdots \geq \mu_k > 0)$ with k parts, let $h = \mu_k$ be the number of parts of the conjugate partition μ'. Let $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_h$ be distinct rational numbers. Alternatively, we can think of the α's and β's as indeterminates. An *injective tableau of shape* μ is a labelling of the cells of μ by the numbers $\{1, 2, \ldots, n\}$ so that no two cells are labelled by the same number. The collection of all such tableaux will be denoted by $\mathcal{IT}_\mu$. For each $T \in \mathcal{IT}_\mu$ and $\ell \in \{1, 2, \ldots, n\}$, let $s_T(\ell) = (i_T(\ell), j_T(\ell))$ denote the cell of μ which contains the number ℓ. We then construct a point $\rho(T) = (a(T), b(T))$ in $2n$-dimensional space by setting

$$a_\ell(T) := \alpha_{i_T(\ell)} \text{ and } b_\ell(T) = \beta_{j_T(\ell)} \qquad (1 \leq \ell \leq n). \tag{22}$$

For example, for the injective tableau T pictured in Figure 6,

$$\rho(T) = (\alpha_2, \alpha_1, \alpha_1, \alpha_1, \alpha_3, \alpha_2, \alpha_2, \alpha_1, \alpha_4; \beta_2, \beta_1, \beta_3, \beta_2, \beta_1, \beta_3, \beta_1, \beta_4, \beta_1)$$

$$\begin{array}{c|c|c|c|c|} \alpha_4 & 9 \\ \alpha_3 & 5 \\ \alpha_2 & 7 & 1 & 6 \\ \alpha_1 & 2 & 4 & 3 & 8 \\ & \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{array}$$

Figure 6: $\rho(T)$.

The collection $\{\rho(T) : T \in \mathcal{IT}_\mu\}$ consists of $n!$ points. In fact, it is simply the S_n-orbit under the diagonal action of any one of its elements, where the *diagonal action* of S_n on $2n$-dimensional space is defined by

$$\sigma(x_1, \ldots, x_n; y_1, \ldots, y_n) := (x_{\sigma_1}, \ldots, x_{\sigma_n}; y_{\sigma_1}, \ldots, y_{\sigma_n})$$

for any $\sigma \in S_n$. For the rest of this paper we let

$$\mathbf{R} = \mathbf{Q}[\overline{x}, \overline{y}] = \mathbf{Q}[x_1, \ldots, x_n; y_1, \ldots, y_n]$$

and use the grading $w_i = 1$ for all $i = 1, \ldots, 2n$. We shall let $\mathbf{R}_{[\rho_\mu]}$, $gr\mathbf{R}_{[\rho_\mu]}$, and $\mathbf{H}_{[\rho_\mu]}$ denote the corresponding coordinate ring, its graded version, and the space of harmonics for the S_n-orbit $\{\rho(T) : T \in \mathcal{IT}_\mu\}$.

Garsia and Haiman [13] proved the following.

Proposition 2.1. *If (i, j) is a cell outside of μ, then for any $s \in \{1, \ldots, n\}$, the monomial $x_s^{i-1} y_s^{j-1}$ belongs to the ideal $gr\,J_{[\rho_\mu]}$. In particular, if $x^p y^q = x_1^{p_1} \cdots x_n^{p_n} y_1^{q_1} \cdots y_n^{q_n} \notin J_{[\rho_\mu]}$, then all the pairs (p_s, q_s) must be cells of μ.*

Theorem 2.2. *Let $\mathbf{H}_\mu$ be the vector space of polynomials spanned by all the partial derivatives of $\Delta_\mu(x_1, \ldots, x_n; y_1, \ldots, y_n)$ (see (4) above). Then*

1. $\mathbf{H}_\mu \subseteq \mathbf{H}_{[\rho_\mu]}$.
2. *If* $\dim(\mathbf{H}_\mu) = n!$, *then* $\mathbf{H}_\mu = \mathbf{H}_{[\rho_\mu]}$ *and* $J_\mu = \mathbf{H}_\mu^\perp = gr\,J_{[\rho_\mu]}$.

In 2001, Haiman solved the $n!$ conjecture and proved that the dimension of $\mathbf{H}_\mu$ is indeed $n!$ [23]. Thus, Theorem 2.2 implies that for every partition μ, $J_\mu = \mathbf{H}_\mu^\perp$ is the ideal in $\mathbf{R}$ generated by the set

$$gr\,\{f \in \mathbf{R} \,:\, f(\rho(T)) = 0 \quad (\forall T \in \mathcal{IT}_\mu)\}.$$

We deduce

Corollary 2.3. *The following polynomials belong to the ideal $J_{(1^{k-1}, n-k+1)}$:*

(i) $\Lambda(x_1, \ldots, x_n)^+$ *and* $\Lambda(y_1, \ldots, y_n)^+$ *(the symmetric functions in x and y without a constant term),*

(ii) the monomials $x_{i_1} \cdots x_{i_k}$ $(i_1 < \cdots < i_k)$ *and* $y_{i_1} \cdots y_{i_{n-k+1}}$ $(i_1 < \cdots < i_{n-k+1})$*, and*

(iii) the monomials $x_i y_i$ $(1 \le i \le n)$.

Proof.

(i) The multiset of row-coordinates is the same for all injective tableaux of a fixed shape. Thus, for any $f \in \Lambda(x_1, \ldots, x_n)^+$, $g(\bar{z}) := f(x_1, \ldots, x_n) - f(\alpha_1, \ldots, \alpha_1, \alpha_2, \ldots, \alpha_k)$ is zero for all substitutions $\bar{z} = \rho(T)$, where T is an injective tableau of shape $(n - k + 1, 1^{k-1})$. Thus $f = \mathrm{top}(g) \in J_{(n-k+1, 1^{k-1})}$. Similarly, for any $f \in \Lambda(y_1, \ldots, y_n)^+$, $g(\bar{z}) := f(y_1, \ldots, y_n) - f(\beta_1, \ldots, \beta_1, \beta_2, \ldots, \beta_{n-k+1})$ is zero for all substitutions $\bar{z} = \rho(T)$. Thus $f = \mathrm{top}(g) \in J_{(n-k+1, 1^{k-1})}$.

(ii) In the shape $(1^{k-1}, n-k+1)$ there are only $k-1$ cells outside of row 1. Therefore $g(\bar{z}) := (x_{i_1} - \alpha_1) \cdots (x_{i_k} - \alpha_1)$ is zero for all $1 \le i_1 < \cdots < i_k \le n$ and all $\bar{z} = \rho(T)$. Thus $x_{i_1} \cdots x_{i_k} = \mathrm{top}(g) \in J_{(n-k+1,1^{k-1})}$. Similarly, $g(\bar{z}) := (y_{i_1} - \beta_1) \cdots (y_{i_{n-k+1}} - \beta_1)$ is zero for all $1 \le i_1 < \cdots < i_{n-k+1} \le n$ and all $\bar{z} = \rho(T)$. Thus $y_{i_1} \cdots y_{i_{n-k+1}} = \mathrm{top}(g) \in J_{(n-k+1,1^{k-1})}$.

(iii) Each cell of a hook shape is either in row 1 or in column 1. Thus $g(\bar{z}) := (x_i - \alpha_1)(y_i - \beta_1)$ is zero for all $1 \le i \le n$ and all $\bar{z} = \rho(T)$, so that $x_i y_i = \mathrm{top}(g) \in J_{(n-k+1,1^{k-1})}$.

$\square$

In the sequel we will show that the polynomials in Corollary 2.3 actually generate the ideal $J_{(1^{k-1},n-k+1)}$.

3 Generalized Kicking-Filtration Process

In this section we prove Theorem 1.5, which implies Corollary 1.7 as a special case and Theorem 1.4. The idea is to generalize the kicking process for obtaining a basis. The kicking process was used in an early paper of Garsia and Haiman [13] to prove the $n!$-conjecture for hooks. We combine ingredients of this process with a filtration.

3.1 Proof of Theorem 1.5

For every triple $(A, c, \bar{A})$, where $[n] = A \cup \{c\} \cup \bar{A}$ and $|A| = k-1, |\bar{A}| = n-k$, define an $(A, c, \bar{A})$-permutation $\pi_{(A,c,\bar{A})} \in S_n$, in which the letters of A appear in decreasing order, then c, and then the remaining letters in increasing order. For example, if $n = 9, k = 4, c = 5, A = \{1, 6, 7\}$ then $\pi_{(A,c,\bar{A})} = 761523489$.

For given n and k, order the $N := n\binom{n-1}{k-1}$ distinct $(A, c, \bar{A})$-permutations in reverse lexicographic order (as words): $\pi_1, \ldots, \pi_N$. If π_i corresponds to $(A, c, \bar{A})$, let $m_i := m_{(A,c,\bar{A})}$ (see Subsection 1.3.2 above).

For example, for $n = 4$ and $k = 3$, the list of $(A, c, \bar{A})$-permutations is

$$\pi_{(\{34\},2,\{1\})} = 4321,\ \pi_{(\{34\},1,\{2\})} = 4312,\ \pi_{(\{24\},3,\{1\})} = 4231, \ldots,$$

$$\pi_{(\{13\},2,\{4\})} = 3124,\ \pi_{(\{12\},4,\{3\})} = 2143,\ \pi_{(\{12\},3,\{4\})} = 2134.$$

and the order is

$$4321 <_L 4312 <_L 4231 <_L 4213 <_L 4132$$
$$<_L 4123 <_L 3241 <_L 3214 <_L 3142 <_L 3124 <_L 2143 <_L 2134.$$

Thus the permutations are indexed by $\pi_1 = 4321,\ \pi_2 = 4312,\ \pi_3 = 4231,\ \ldots, \pi_{11} = 2143,\ \pi_N = \pi_{12} = 2134$ and the corresponding monomials are $m_1 = x_4 x_3 y_1,\ m_2 = x_4 x_3,\ m_3 = x_4 y_1, \ldots,\ m_{11} = y_3,\ m_N = m_{12} = 1$.

Let

$$I_0 := J_{(1^{k-1},n-k+1)}$$

and define recursively

$$I_t := I_{t-1} + m_t\mathbf{Q}[\bar{x},\bar{y}] \qquad (1 \le t \le N).$$

Clearly,

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_N = \mathbf{Q}[\bar{x},\bar{y}].$$

The last equality follows from the fact that $m_N = 1$.

Observation 3.1.

$$\mathbf{H}'_{(1^{k-1},n-k+1)} = \mathbf{Q}[\bar{x},\bar{y}]/I_0 \cong \bigoplus_{t=1}^{N}(I_t/I_{t-1})$$

as vector spaces. In particular, a sequence of bases for the quotients I_t/I_{t-1}, $1 \le t \le N$, will give a basis for $\mathbf{H}'_{(1^{k-1},n-k+1)}$.

It remains to prove that the set $m_t \cdot B_A \cdot C_{\bar{A}}$, where B_A, $C_{\bar{A}}$ are bases of coinvariant algebras in $\bar{x}_A$ and $\bar{y}_{\bar{A}}$ respectively, is a basis for I_t/I_{t-1}.

Consider the natural projection

$$f_t : m_t\mathbf{Q}[\bar{x},\bar{y}] \longrightarrow I_t/I_{t-1}.$$

Clearly, f_t is a surjective map.

Lemma 3.2.

$$m_t \cdot \left(\sum_{i\notin A}\langle x_i\rangle + \sum_{j\notin \bar{A}}\langle y_j\rangle + \langle\Lambda[\bar{x}]^+\rangle + \langle\Lambda[\bar{y}]^+\rangle\right) \;\subseteq\; I_{t-1}\cap m_t\mathbf{Q}[\bar{x},\bar{y}] = \ker\,(f_t),$$

so that f_t is well defined on the quotient

$$m_t\mathbf{Q}[\bar{x},\bar{y}]/\left[m_t \cdot \left(\sum_{i\notin A}\langle x_i\rangle + \sum_{j\notin \bar{A}}\langle y_j\rangle + \langle\Lambda[\bar{x}]^+\rangle + \langle\Lambda[\bar{y}]^+\rangle\right)\right] \cong$$

$$m_t \cdot \mathbf{Q}[\bar{x}_A]/\langle\Lambda[\bar{x}_A]^+\rangle \cdot \mathbf{Q}[\bar{y}_{\bar{A}}]/\langle\Lambda[\bar{y}_{\bar{A}}]^+\rangle.$$

Proof of Lemma 3.2. First, let $i \notin A$. We shall show that $m_t x_i \in I_{t-1}$. Consider the four possible cases X1–X4:

X1. $a > c$ for all $a \in A$.
Then m_t has all the x_a ($a \in A$) as factors, so that $m_t x_i$ has exactly k distinct x-factors and therefore belongs to $I_0 \subseteq I_{t-1}$ (by Corollary 2.3(ii)).

X2. $i < c$ and there exists $a \in A$ such that $a < c$.
$i \notin A$ and $i \neq c$, hence $i \in \bar{A}$. As $i < c$, y_i is a factor of m_t. Hence $x_i y_i$ is a factor of $m_t x_i$ and therefore belongs to $I_0 \subseteq I_{t-1}$ (by Corollary 2.3(iii)).

X3. $i = c$ and there exists $a \in A$ such that $a < c$.

Let $a_0 := \max\{a \in A : a < c\}$. Let $t_{(c,a_0)}$ be the transposition interchanging c with a_0 and let $A' := (A \setminus \{a_0\}) \cup \{c\}$. By the definition of a_0, $m_{t'} = m_{(A',a_0,\bar{A}')}$ divides $m_t x_i$. To verify this, notice that for every j, such that x_j divides $m_{t'}$, $j \in A'$ and $j > a_0$. Hence $j \neq a_0$ and thus $j \in A$. Also, by the definition of a_0, for every $j \in A$, $j > a_0 \Rightarrow j > c$. It follows that x_j divides m_t. Finally, for every j, such y_j divides $m_{t'}$, $j \in \bar{A} = \bar{A}'$ and $j < a_0$. Hence $j < c$ and y_j divides m_t.

On the other hand, $\pi_{(A',a_0,\bar{A}')} = t_{(c,a_0)}\pi_{(A,c,\bar{A})} <_L \pi_{(A,c,\bar{A})}$, since $a_0 < c$. Hence $m_{t'} = m_{(A',a_0,\bar{A}')} \in I_{t-1}$ and $m_t x_i \in \langle m_{t'} \rangle \subseteq I_{t-1}$.

X4. $i > c$ and there exists $a \in A$ such that $a < c$.

Let $a_0 := \max\{a \in A : a < c\}$ as above. Let $t_{(a_0,i,c)}$ be the 3-cycle mapping a_0 to i, i to c and c to a_0. Let $A' := (A \setminus \{a_0\}) \cup \{i\}$ and $m_{t'} = m_{(A',a_0,\bar{A}')}$. Then $\pi_{(A',a_0,\bar{A}')} \leq_L t_{(a_0,i,c)}\pi_{(A,c,\bar{A})} <_L \pi_{(A,c,\bar{A})}$, since $a_0 < c < i$. Thus $t' < t$.

On the other hand, $m_{t'} = m_{(A',a_0,\bar{A}')}$ divides $m_t x_i$. This follows from the implications below:

$$i' \in A' \text{ and } i' > a_0 \Longrightarrow i' = i \text{ or } i' > c.$$

Hence, every $x_{i'}$ which divides $m_{t'}$ divides also $m_t x_i$. Also

$$j' \in \bar{A}' \text{ and } j' < a_0 \Longrightarrow j' \in \bar{A} \text{ and } j' < c.$$

Hence, every $y_{j'}$ which divides $m_{t'}$ divides also $m_t x_i$.

We conclude that $m_{t'} = m_{(A',a_0,\bar{A})} \in I_{t-1}$ and $\langle m_t x_i \rangle \subseteq \langle m_{(A',a_0,\bar{A})} \rangle \subseteq I_{t-1}$.

Similarly, by considering four analogous cases, one can show that if $j \notin \bar{A}$ then $m_t y_j \in I_{t-1}$.

Finally, observe that, by Corollary 2.3(i), $m_t\langle \Lambda[\bar{x}]^+ \rangle \subseteq I_0 \subseteq I_{t-1}$ and $m_t\langle \Lambda[\bar{y}]^+ \rangle \subseteq I_0 \subseteq I_{t-1}$.

This completes the proof of Lemma 3.2. $\square$

Now, recall that B_A and $C_{\bar{A}}$ are bases for the coinvariant algebras $\mathbf{Q}[\bar{x}_A]/\langle \Lambda[\bar{x}_A]^+ \rangle$ and $\mathbf{Q}[\bar{y}_{\bar{A}}]/\langle \Lambda[\bar{y}_{\bar{A}}]^+ \rangle$ respectively, so that $m_t \cdot B_A \cdot C_{\bar{A}}$ spans $m_t \cdot \mathbf{Q}[\bar{x}_A]/\langle \Lambda[\bar{x}_A]^+ \rangle \cdot \mathbf{Q}[\bar{y}_{\bar{A}}]/\langle \Lambda[\bar{y}_{\bar{A}}]^+ \rangle$.

In order to prove Theorem 1.5, it remains to show that for every $1 \leq t \leq N$,

(a) $m_t \cdot B_A \cdot C_{\bar{A}}$ is a basis for $m_t \cdot \mathbf{Q}[\bar{x}_A]/\langle \Lambda[\bar{x}_A]^+ \rangle \cdot \mathbf{Q}[\bar{y}_{\bar{A}}]/\langle \Lambda[\bar{y}_{\bar{A}}]^+ \rangle$, and

(b) f_t is one-to-one.

Indeed, by Lemma 3.2,

$$\begin{aligned} \dim\,(I_t/I_{t-1}) &\leq \dim\,(m_t \cdot \mathbf{Q}[\bar{x}_A]/\langle \Lambda[\bar{x}_A]^+ \rangle \cdot \mathbf{Q}[\bar{y}_{\bar{A}}]/\langle \Lambda[\bar{y}_{\bar{A}}]^+ \rangle) \\ &\leq \dim\,(\mathbf{Q}[\bar{x}_A]/\langle \Lambda[\bar{x}_A]^+ \rangle \mathbf{Q}[\bar{y}_{\bar{A}}]/\langle \Lambda[\bar{y}_{\bar{A}}]^+ \rangle) = (k-1)! \cdot (n-k)!\,. \end{aligned} \tag{23}$$

The last equality follows from the classical result that the coinvariant algebra of S_n carries the regular S_n-representation [24, §3].

If there exists $1 \le t \le N$ such that either (a) or (b) does not hold, then there exists t for which a sharp inequality holds in (23). Then

$$\dim \mathbf{H}'_{(1^{k-1},n-k+1)} = \dim \ (\mathbf{Q}[\bar{x},\bar{y}]/I_0) = \sum_{t=1}^{N} \dim \ (I_t/I_{t-1})$$

$$< N(k-1)!(n-k)! = n\binom{n-1}{k-1}(k-1)!(n-k)! = n!,$$

contradicting the $n!$ theorem. This completes the proof of Theorem 1.5. □

3.2 Applications

Proof of Corollary 1.7. By choosing B_A and $C_{\bar{A}}$ in Theorem 1.5 as Artin bases of the corresponding coinvariant algebras we get the k-th Artin basis $(b_\pi^{(k)})$.

By choosing B_A and $C_{\bar{A}}$ as descent bases we get the k-th descent basis $(a_\pi^{(k)})$.

Finally, by choosing B_A as a descent basis and $C_{\bar{A}}$ as an Artin basis we get the k-th Haglund basis $(c_\pi^{(k)})$. □

Proof of Theorem 1.4. Let J' be the ideal of $\mathbf{Q}[\bar{x},\bar{y}]$ generated by $\Lambda(x_1,\ldots,x_n)^+$, $\Lambda(y_1,\ldots,y_n)^+$ and the monomials $x_{i_1}\cdots x_{i_k}$ $(i_1 < \cdots < i_k)$, $y_{i_1}\cdots y_{i_{n-k+1}}$ $(i_1 < \cdots < i_{n-k+1})$, and $x_i y_i$ $(1 \le i \le n)$. By Corollary 2.3, $J' \subseteq J_{(1^{k-1},n-k+1)}$.

In the above proof of Theorem 1.5, if one replaces $I_0 := J_{(1^{k-1},n-k+1)}$ by $I_0 := J'$, the same conclusions hold:

$$\dim \ (\mathbf{Q}[\bar{x},\bar{y}]/J') = \sum_{t=1}^{N} \dim \ (I_t/I_{t-1}) \le N(k-1)!(n-k)! = n! \ .$$

On the other hand, by [13],

$$\dim \ (\mathbf{Q}[\bar{x},\bar{y}]/J') \ge \dim \ \left(\mathbf{Q}[\bar{x},\bar{y}]/J_{(1^{k-1},n-k+1)}\right) = \dim \mathbf{H}'_{(1^{k-1},n-k+1)} = n! \ .$$

Thus equality holds everywhere, and $J' = J_{(1^{k-1},n-k+1)}$. □

4 A k-th Analogue of the Polynomial Ring

The current section provides an appropriate setting for an extension of the straightening algorithm from the coinvariant algebra [4, 1] to the Garsia-Haiman hook modules. The algorithm will be given in Section 5 and will be used later to describe the S_n-action on $\mathbf{H}'_{(1^{k-1},n-k+1)}$ and resulting decomposition rules (see Section 6).

4.1 $\mathcal{P}_n^{(k)}$ and its Monomial Basis

Definition 4.1. *For every $1 \le k \le n$ let $\mathcal{I}_k$ be the ideal in $\mathbf{Q}[x_1,\ldots,x_n,y_1,\ldots,y_n]$ generated by*

(i) the monomials $x_{i_1}\cdots x_{i_k}$ $(1 \le i_1 < \cdots < i_k \le n)$,

(ii) the monomials $y_{i_1}\cdots y_{i_{n-k+1}}$ $(1 \le i_1 < \cdots < i_{n-k+1} \le n)$, and

(iii) the monomials x_iy_i $(1 \le i \le n)$.

Denote

$$\mathcal{P}_n^{(k)} := \mathbf{Q}[x_1,\ldots,x_n,y_1,\ldots,y_n]/\mathcal{I}_k.$$

Claim 4.2. *The ideal $\mathcal{I}_k$ is contained in the ideal $J_{(1^{k-1},n-k+1)}$.*

Proof. Immediate from Definition 4.1 and Corollary 2.3. $\square$

Definition 4.3. *For a monomial $m = \prod_{i=1}^{n} x_i^{p_i} \prod_{j=1}^{n} y_j^{q_j} \in \mathbf{Q}[x_1,\ldots,x_n,y_1,\ldots,y_n]$ define the* x-support *and the* y-support

$$\mathrm{Supp}_x(m) := \{i \,:\, p_i > 0\}, \qquad \mathrm{Supp}_y(m) := \{j \,:\, q_j > 0\}.$$

Definition 4.4. *Let $M_n^{(k)}$ be the set of all monomials in $\mathbf{Q}[x_1,\ldots,x_n,y_1,\ldots,y_n]$ with*

(i) $|\mathrm{Supp}_x(m)| \le k-1$,

(ii) $|\mathrm{Supp}_y(m)| \le n-k$, and

(iii) $\mathrm{Supp}_x(m) \cap \mathrm{Supp}_y(m) = \emptyset$.

Observation 4.5. *$\{m + \mathcal{I}_k \,:\, m \in M_n^{(k)}\}$ is a basis for $\mathcal{P}_n^{(k)}$.*

Example 4.6. *Let $k = n$. Then*

$$\mathcal{P}_n^{(n)} \cong \mathbf{Q}[x_1,\ldots,x_n]/\langle x_1\cdots x_n\rangle,$$

and $M_n^{(n)}$ consists of all the monomials in $\mathbf{Q}[x_1,\ldots,x_n]$ which do not involve all the x variables, i.e., are not divisible by $x_1\cdots x_n$. Similarly,

$$\mathcal{P}_n^{(1)} \cong \mathbf{Q}[y_1,\ldots,y_n]/\langle y_1\cdots y_n\rangle,$$

and $M_n^{(1)}$ consists of all the monomials in $\mathbf{Q}[y_1,\ldots,y_n]$ which do not involve all the y variables.

4.2 A Bijection

Next, define a map $\psi^{(k)} : M_n^{(k)} \to M_n^{(n)}$.

Every monomial $m \in M_n^{(k)}$ has the form $m = x_{i_1}^{p_{i_1}} \cdots x_{i_{k-1}}^{p_{i_{k-1}}} \cdot y_{j_1}^{q_{j_1}} \cdots y_{j_{n-k}}^{q_{j_{n-k}}}$ (with disjoint supports of x's and y's). Let $u := \max_j q_j$ and define

$$\psi^{(k)}(m) := x_{i_1}^{p_{i_1}} \cdots x_{i_{k-1}}^{p_{i_{k-1}}} \cdot x_{j_1}^{-q_{j_1}} \cdots x_{j_{n-k}}^{-q_{j_{n-k}}} \cdot (x_1 \cdots x_n)^u.$$

Notice that if $u = 0$, then $\psi^{(k)}(m) = m \in M_n^{(k)} \cap \mathbf{Q}[x_1, \ldots, x_n]$ has $|\mathrm{Supp}_x(m)| \leq k-1 \leq n-1$ so that $\psi^{(k)}(m) \in M_n^{(n)}$, and, if $u > 0$ is attained at y_{j_0}, then the exponent of x_{j_0} in $\psi^{(k)}(m)$ is zero so that again $\psi^{(k)}(m) \in M_n^{(n)}$.

Example 4.7. *Let $n = 8$, $k = 5$, and $m = x_3^6 x_4^6 x_7^5 y_6^2 y_2^3 \in M_8^{(5)}$. Then $u = 3$, $j_0 = 2$, and*

$$\psi^{(k)}(m) = x_3^6 x_4^6 x_7^5 x_6^{-2} x_2^{-3} \cdot (x_1 \cdots x_8)^3 = x_3^9 x_4^9 x_7^8 x_1^3 x_5^3 x_8^3 x_6^1 \in M_8^{(8)}.$$

We claim that the map $\psi^{(k)} : M_n^{(k)} \to M_n^{(n)}$ is a bijection. This will be proved by defining an inverse map $\phi^{(k)} : M_n^{(n)} \to M_n^{(k)}$.

Recall from [1] that the *index permutation* of a monomial $m = \prod_{i=1}^n x_i^{p_i} \in \mathbf{Q}[x_1, \ldots, x_n]$ is the unique permutation $\pi = \pi(m) \in S_n$ such that

$$(i) \qquad p_{\pi(i)} \geq p_{\pi(i+1)} \qquad (1 \leq i < n)$$

and

$$(ii) \qquad p_{\pi(i)} = p_{\pi(i+1)} \Longrightarrow \pi(i) < \pi(i+1).$$

In other words, π reorders the variables x_i by (weakly) decreasing exponents, where the variables with a given exponent are ordered by increasing indices. Also, let $\lambda = \lambda(m) = (\lambda_1, \ldots, \lambda_n)$ be the corresponding *exponent partition*, where λ_i is the exponent of $x_{\pi(i)}$ in m. Finally, let $\lambda' = (\lambda'_1, \ldots, \lambda'_\ell)$ be the partition conjugate to λ.

Every monomial $m = \prod_{i=1}^n x_{\pi(i)}^{\lambda_i} \in M_n^{(n)}$ can thus be written in the form

$$m = \prod_{t=1}^{\ell} (x_{\pi(1)} \cdots x_{\pi(\lambda'_t)}).$$

Note that, since m is not divisible by $x_1 \cdots x_n$, $\lambda_n = 0$ and therefore $\lambda'_t < n$ $(\forall t)$. Define

$$\phi^{(k)}(m) := \prod_{\lambda'_t \leq k-1} (x_{\pi(1)} \cdots x_{\pi(\lambda'_t)}) \cdot \prod_{\lambda'_t \geq k} (y_{\pi(\lambda'_t+1)} \cdots y_{\pi(n)}).$$

Proposition 4.8. *$\psi^{(k)} : M_n^{(k)} \to M_n^{(n)}$ and $\phi^{(k)} : M_n^{(n)} \to M_n^{(k)}$ are inverse maps.*

Proof. Take a monomial $m = \prod_{i=1}^{n} x_{\pi(i)}^{\lambda_i} \in M_n^{(n)}$ as above. Then

$$\begin{array}{rcl} \mathrm{Supp}_x(\phi^{(k)}(m)) & \subseteq & \{\pi(1), \ldots, \pi(k-1)\}, \\ \mathrm{Supp}_y(\phi^{(k)}(m)) & \subseteq & \{\pi(k+1), \ldots, \pi(n)\}, \end{array}$$

and therefore $\phi^{(k)}(m) \in M_n^{(k)}$.

Now apply $\psi^{(k)}$ to $\phi^{(k)}(m)$. Clearly, the highest exponent of a y-variable (say $y_{\pi(n)}$) in $\phi^{(k)}(m)$ is $u = \#\{t : \lambda'_t \geq k\}$ $(= \lambda_k)$. Thus

$$\begin{aligned} \psi^{(k)}(\phi^{(k)}(m)) &= \prod_{\lambda'_t \leq k-1} (x_{\pi(1)} \cdots x_{\pi(\lambda'_t)}) \cdot \prod_{\lambda'_t \geq k} (x_{\pi(\lambda'_t+1)} \cdots x_{\pi(n)})^{-1} \cdot (x_1 \cdots x_n)^u \\ &= \prod_{\lambda'_t \leq k-1} (x_{\pi(1)} \cdots x_{\pi(\lambda'_t)}) \cdot \prod_{\lambda'_t \geq k} (x_{\pi(1)} \cdots x_{\pi(\lambda'_t)}) = m. \end{aligned}$$

Conversely, take $m' \in M_n^{(k)}$. We want to show that $\phi^{(k)}(\psi^{(k)}(m')) = m'$. We can write

$$m' = \prod_{i=1}^{k-1} x_{\pi(i)}^{\mu_i} \cdot \prod_{i=k+1}^{n} y_{\pi(i)}^{\mu_i},$$

where $\mu_1 \geq \ldots \geq \mu_{k-1} \geq 0$ and $0 \leq \mu_{k+1} \leq \ldots \leq \mu_n$. Here π is the unique permutation that orders first the indices $i \in \mathrm{Supp}_x(m')$, then the indices $i \notin \mathrm{Supp}_x(m') \cup \mathrm{Supp}_y(m')$, and then the indices $i \in \mathrm{Supp}_y(m')$. The x-indices are ordered by weakly decreasing exponents, the y-indices are ordered by weakly increasing exponents, and indices with equal exponents are ordered in increasing (index) order. The variables with a given exponent are ordered by increasing indices. The highest exponent of a y-variable in m' is $u = \mu_n$, and therefore

$$\psi^{(k)}(m') = \prod_{i=1}^{k-1} x_{\pi(i)}^{u+\mu_i} \cdot x_{\pi(k)}^{u} \cdot \prod_{i=k+1}^{n} x_{\pi(i)}^{u-\mu_i} = \prod_{i=1}^{n} x_{\pi(i)}^{\lambda_i}$$

where

$$\lambda_i = \begin{cases} u + \mu_i, & \text{if } 1 \leq i \leq k-1; \\ u, & \text{if } i = k; \\ u - \mu_i, & \text{if } k+1 \leq i \leq n. \end{cases}$$

It follows that $\lambda = (\lambda_1, \ldots, \lambda_n)$ is the index partition of $\psi^{(k)}(m')$, and the conjugate partition λ' satisfies

$$\lambda'_t \geq k \iff t \leq u = \mu_n.$$

The map $\phi^{(k)}$ replaces, for each t with $\lambda'_t \geq k$, the product $x_{\pi(1)} \cdots x_{\pi(\lambda'_t)}$ by the product $y_{\pi(\lambda'_t+1)} \cdots y_{\pi(n)}$. It therefore reduces by u the exponent of each x_i with $i \leq k$, removes all the x_i with $i \geq k+1$, and replaces them by y_i with exponents $u - \lambda_i = \mu_i$. This implies that

$$\phi^{(k)}(\psi^{(k)}(m')) = \prod_{i=1}^{k-1} x_{\pi(i)}^{\mu_i} \cdot \prod_{i=k+1}^{n} y_{\pi(i)}^{\mu_i} = m',$$

as claimed. □

Corollary 4.9. *The map $\psi^{(k)} : M_n^{(k)} \to M_n^{(n)}$ is a bijection.*

4.3 Action and Invariants

Definition 4.10. *For $1 \le m \le n-1$ let*

$$e_m^{(k)} := \begin{cases} e_m(\bar{x}) = e_m(x_1, \ldots, x_n), & \text{if } 1 \le m \le k-1; \\ e_{n-m}(\bar{y}) = e_{n-m}(y_1, \ldots, y_n), & \text{if } k \le m \le n-1, \end{cases}$$

where $e_m(\bar{x})$ is the m-th elementary symmetric function.
For a partition $\mu = (0 < \mu_1 \le \cdots \le \mu_\ell)$ with $\mu_\ell < n$ let

$$e_\mu^{(k)} := \prod_{i=1}^{\ell} e_{\mu_i}^{(k)}.$$

For $e_\mu^{(n)}$ we shall frequently use the short notation e_μ.

Consider the natural S_n-action on $\mathcal{P}_n^{(k)}$, defined by:

$$\pi(x_i) := x_{\pi(i)}, \quad \pi(y_i) := y_{\pi(i)}, \qquad (\forall \pi \in S_n, 1 \le i \le n)$$

Proposition 4.11. *For every $\sigma \in S_n$ and $1 \le k \le n$,*

$$\sigma\psi^{(k)} = \psi^{(k)}\sigma.$$

Corollary 4.12. *$\mathcal{P}_n^{(k)}$ and $\mathcal{P}_n^{(n)}$ are isomorphic as S_n-modules.*

Let ${\mathcal{P}_n^{(k)}}^{S_n}$ be the algebra of S_n-invariants in $\mathcal{P}_n^{(k)}$. Proposition 4.11 implies

Corollary 4.13. *For any $1 \le k \le n$,*

1. *${\mathcal{P}_n^{(k)}}^{S_n}$ is generated, as an algebra, by $e_m^{(k)}$, $1 \le m < n$.*
2. *The set $\{e_\mu^{(k)} \; : \; \mu = (\mu_1 \le \cdots \le \mu_\ell) \text{ with } \mu_\ell < n\}$ forms a (vector space) basis for ${\mathcal{P}_n^{(k)}}^{S_n}$.*
3. *For every partition $\mu = (\mu_1 \le \cdots \le \mu_\ell)$ with $\mu_\ell < n$,*

$$\psi^{(k)}(e_\mu^{(k)}) = e_\mu.$$

Remark 4.14. By Proposition 4.11, $\psi^{(k)}$ and $\phi^{(k)}$ send invariants to invariants. Unfortunately, these maps are not multiplicative and they do not send the ideal generated by invariants (with no constant term) to its analogue. For example, $x_1x_3 + x_2x_3 + x_3^2 = x_3 \cdot e_1(x_1, x_2, x_3) \in I_{(3)}$ but $\phi^{(2)}(x_1x_3 + x_2x_3 + x_3^2) = y_2 + y_1 + x_3^2 \notin I_{(2,1)}$.

In order to define the "correct" map, recall the definitions of the Garsia-Stanton basis element a_π and the k-th descent basis element $a_\pi^{(k)}$ corresponding to $\pi \in S_n$ (see Definition 1.2).

Claim 4.15. *For every $\pi \in S_n$ and $1 \le k \le n$,*

$$\psi^{(k)}(a_\pi^{(k)}) = a_\pi.$$

In the next section it will be shown that $\{a_\pi^{(k)} e_\mu^{(k)} \ : \ \pi \in S_n, \mu = (\mu_1 \le \cdots \le \mu_\ell) \text{ with } \mu_\ell < n\}$ forms a basis for $\mathcal{P}_n^{(k)}$. Thus the map

$$\tilde{\psi}^{(k)}(a_\pi^{(k)} \cdot e_\mu^{(k)}) := \psi^{(k)}(a_\pi^{(k)}) \cdot \psi^{(k)}(e_\mu^{(k)}) = a_\pi \cdot e_\mu$$

extends to a linear map $\tilde{\psi}^{(k)} : \mathcal{P}_n^{(k)} \to \mathcal{P}_n^{(n)}$ that clearly sends the k-th ideal generated by invariants to its n-th analogue.

5 Straightening

5.1 Basic Notions

We now generalize several notions and facts which were used in [1] for straightening the coinvariant algebra of type A. Some of them were already introduced in the proof of Proposition 4.8 above. Proofs are similar to those in [1], and will be omitted.

Each monomial $m \in M_n^{(k)}$ can be written in the form

$$m = \prod_{i=1}^{k-1} x_{\pi(i)}^{p_i} \cdot \prod_{i=k+1}^{n} y_{\pi(i)}^{p_i}, \tag{24}$$

where $p_1 \ge \ldots \ge p_{k-1} \ge 0$ and $0 \le p_{k+1} \le \ldots \le p_n$. Here $\pi = \pi(m)$, the *index permutation* of m, is the unique permutation that orders first the indices $i \in \mathrm{Supp}_x(m)$, then the indices $i \notin \mathrm{Supp}_x(m) \cup \mathrm{Supp}_y(m)$, and then the indices $i \in \mathrm{Supp}_y(m)$. The x-indices are ordered by weakly decreasing exponents, the y-indices are ordered by weakly increasing exponents, and indices with equal exponents are ordered in increasing (index) order.

Observation 5.1. *The index permutation is preserved by $\psi^{(k)}$, i.e., for any monomial $m \in M_n^{(k)}$*

$$\pi(m) = \pi(\psi^{(k)}(m)).$$

Claim 5.2. *Let m be a monomial in $M_n^{(k)}$, $\pi = \pi(m)$ its index permutation, and $a_\pi^{(k)}$ the corresponding descent basis element:*

$$m \ = \ \prod_{i=1}^{k-1} x_{\pi(i)}^{p_i} \cdot \prod_{i=k+1}^{n} y_{\pi(i)}^{p_i},$$

$$a_\pi^{(k)} \quad = \quad \prod_{i=1}^{k-1} x_{\pi(i)}^{d_i^{(k)}(\pi)} \cdot \prod_{i=k+1}^{n} y_{\pi(i)}^{d_i^{(k)}(\pi)}.$$

Then: the sequence $(p_i - d_i^{(k)}(\pi))_{i=1}^{k-1}$ *of* x*-exponents in* $m/a_\pi^{(k)}$ *consists of nonnegative integers, and is weakly decreasing:*

$$p_1 - d_1^{(k)}(\pi) \geq \ldots \geq p_{k-1} - d_{k-1}^{(k)}(\pi) \geq 0,$$

and the sequence $(p_i - d_i^{(k)}(\pi))_{i=k+1}^{n}$ *of* y*-exponents* $m/a_\pi^{(k)}$ *consists of nonnegative integers, and is weakly increasing:*

$$0 \leq p_{k+1} - d_{k+1}^{(k)}(\pi) \leq \ldots \leq p_n - d_n^{(k)}(\pi).$$

For a monomial $m \in M_n^{(k)}$ of the form (24) with index permutation $\pi \in S_n$, let the associated pair of exponent partitions

$$\lambda(m) = (\lambda_x(m), \lambda_y(m)) := ((p_1, p_2, \ldots, p_{k-1}), (p_n, p_{n-1}, \ldots, p_{k+1}))$$

be its *exponent bipartition*. Note that $\lambda(m)$ is a bipartition of the total bi-degree of m.

Define the *complementary bipartition* $\mu^{(k)}(m) = (\mu_x(m), \mu_y(m))$ of m to be the pair of partitions conjugate to the partitions $(p_i - d_i^{(k)}(\pi))_{i=1}^{k-1}$ and $(p_i - d_i^{(k)}(\pi))_{i=n}^{k+1}$ respectively; namely,

$$(\mu_x)_j := |\{1 \leq i \leq k-1 \,:\, p_i - d_i^{(k)}(\pi) \geq j\}| \qquad (\forall j)$$

and

$$(\mu_y)_j := |\{k+1 \leq i \leq n \,:\, p_i - d_i^{(k)}(\pi) \geq j\}| \qquad (\forall j).$$

If $k = n$ then, for every monomial $m \in M_n^{(n)}$, $\mu_y(m)$ is the empty partition. In this case we denote

$$\mu(m) := \mu_x(m).$$

With each $m \in M_n^{(k)}$ we associate the *canonical complementary partition*

$$\nu(m) := \mu(\psi^{(k)}(m)).$$

Example 5.3. *Let* $m = x_1^2 y_2^4 x_3^2 y_5 x_6^3$ *with* $n = 7$ *and* $k = 5$*. Then*

$$m = x_6^3 x_1^2 x_3^2 y_5^1 y_2^4, \quad \lambda(m) = ((3,2,2,0),(4,1)), \quad \pi = 6134752 \in S_7,$$

$$a_\pi^{(5)} = x_6 y_5 y_2^2, \quad \lambda(a_\pi^{(5)}) = ((1,0,0,0),(2,1)),$$

$$\mu^{(5)}(m) = ((2,2,2,0)',(2,0)') = ((3,3),(1,1)),$$

$$\psi^{(5)}(m) = x_6^7 x_1^6 x_3^6 x_4^4 x_7^4 x_5^3, \quad a_\pi = x_6^3 x_1^2 x_3^2 x_4^2 x_7^2 x_5^1,$$

$$\nu(m) = \mu(\psi^{(5)}(m)) = (4,4,4,2,2,2)' = (6,6,3,3).$$

Definition 5.4. *1. For two partitions λ and μ, denote $\lambda \trianglelefteq \mu$ if λ is weakly smaller than μ in dominance order. For two bipartitions $\lambda^1 = (\mu^1, \nu^1)$ and $\lambda^2 = (\mu^2, \nu^2)$, denote $\lambda^1 \trianglelefteq \lambda^2$ if $\mu^1 \trianglelefteq \mu^2$ and $\nu^1 \trianglelefteq \nu^2$.*

2. For two monomials $m_1, m_2 \in M_n^{(k)}$ of the same total bi-degree (p, q), write $m_1 \preceq_k m_2$ if:

(1) $\lambda(m_1) \trianglelefteq \lambda(m_2)$; and

(2) if $\lambda(m_1) = \lambda(m_2)$ then $\mathrm{inv}(\pi(m_1)) > \mathrm{inv}(\pi(m_2))$.

5.2 The Straightening Algorithm

Lemma 5.5. *Let $m \in M_n^{(k)}$ be a monomial. For $1 \le d \le n-1$, let $S_{(d)}$ be the set of all monomials which appear (with coefficient 1) in the expansion of the polynomial $m \cdot e_d^{(k)} \in \mathcal{P}_n^{(k)}$. Let $\pi = \pi(m)$ be the index permutation of m and denote*

$$m_{(d)} := \begin{cases} m \cdot x_{\pi(1)} \cdots x_{\pi(d)}, & \text{if } 1 \le d \le k-1; \\ m \cdot y_{\pi(d+1)} \cdots y_{\pi(n)}, & \text{if } k \le d \le n-1. \end{cases}$$

Then:

(1) $m_{(d)} \in S_{(d)}$.

(2) $(m' \in S_{(d)}$ and $m' \neq m_{(d)}) \Longrightarrow m' \prec_k m_{(d)}$.

The proof of Lemma 5.5 is similar to the proof of [1, Lemma 3.2] and is omitted.

Corollary 5.6. *Let $m \in M_n^{(k)}$ be a monomial, $\pi = \pi(m)$ its index permutation, and $\nu = \mu(\psi^{(k)}(m))$ the canonical complementary partition of m defined in Subsection 5.1. Let S be the set of monomials which appear (with nonzero coefficient) in the expansion of $a_\pi^{(k)} \cdot e_\nu^{(k)} \in \mathcal{P}_n^{(k)}$. Then :*

(1) $m \in S$.

(2) $(m' \in S$ and $m' \neq m) \Longrightarrow m' \prec_k m$.

A straightening algorithm follows.

A Straightening Algorithm:
For a monomial $m \in \mathcal{P}_n^{(k)}$, let $\pi = \pi(m)$ be its index permutation, $a_\pi^{(k)}$ the corresponding descent basis element, and $\nu = \mu(\psi^{(k)}(m))$ the corresponding canonical complementary partition. Write (by Corollary 5.6)

$$m = a_\pi^{(k)} \cdot e_\nu^{(k)} - \Sigma,$$

where Σ is a sum of monomials $m' \prec_k m$. Repeat the process for each m'.

The algorithm implies

Lemma 5.7. (Straightening Lemma) *Each monomial $m \in \mathcal{P}_n^{(k)}$ has an expression*

$$m = a^{(k)}_{\pi(m)} e^{(k)}_{\nu(m)} + \sum_{m' \prec_k m} n_{m',m} a^{(k)}_{\pi(m')} e^{(k)}_{\nu(m')},$$

where $n_{m',m}$ are integers.

The Straightening Lemma yields a direct proof to the following special case of Theorem 1.5.

Corollary 5.8. *The set $\{a^{(k)}_\pi \,:\, \pi \in S_n\}$ forms a basis for $\mathbf{H}'_{(1^{k-1},n-k+1)}$.*

Proof. Recall that $\mathcal{P}_n^{(k)} := \mathbf{R}/\mathcal{I}_k$, where $\mathbf{R} := \mathbf{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ (Definition 4.1). Let I_k^0 be the ideal of $\mathbf{R}$ generated by the elementary symmetric functions $e_d^{(k)}$ $(1 \le d \le n-1)$. By definition, these functions are symmetric functions of either $\bar{x}$ or $\bar{y}$. Thus, Corollary 2.3 and Claim 4.2,

$$\mathcal{I}_k + I_k^0 \subseteq J_{(1^{k-1},n-k+1)}. \tag{25}$$

By Lemma 5.7, $\{a^{(k)}_\pi \,:\, \pi \in S_n\}$ spans $\mathbf{R}/(\mathcal{I}_k + I_k^0)$ and therefore also $\mathbf{R}/J_{(1^{k-1},n-k+1)}$ as vector spaces over $\mathbf{Q}$.

On the other hand, by [13], dim $(\mathbf{R}/J_{(1^{k-1},n-k+1)}) = |S_n|$, and therefore $\{a^{(k)}_\pi \,:\, \pi \in S_n\}$ is actually a basis for $\mathbf{H}'_{(1^{k-1},n-k+1)} = \mathbf{R}/J_{(1^{k-1},n-k+1)}$. $\square$

Second Proof of Theorem 1.4. From the proof of Corollary 5.8 it follows that

$$\dim\, (\mathbf{R}/(\mathcal{I}_k + I_k^0)) = \dim\, (\mathbf{R}/J_{(1^{k-1},n-k+1)}).$$

Combining this with inclusion (25), it follows that

$$\mathcal{I}_k + I_k^0 = J_{(1^{k-1},n-k+1)}.$$

This completes the proof. $\square$

6 Descent Representations

Recall the definition of the Solomon descent representation ρ^A from Subsection 1.4.1. The following theorem is well known. Recall the definition of Des(T) in Subsection 1.4.2.

Theorem 6.1. *For any subset $A \subseteq [n-1]$ and partition $\mu \vdash n$, the multiplicity in the descent representation ρ^A of the irreducible S_n-representation corresponding to μ is*

$$m_\mu^A := \#\,\{T \in SYT(\mu) \,:\, \mathrm{Des}(T) = A\},$$

the number of standard Young tableaux of shape μ with descent set A.

Recall the definition of a bipartition and the domination order on bipartitions from Definition 5.4.1. Recall the subspaces $I_\lambda^{(k)\trianglelefteq}$ and $I_\lambda^{(k)\triangleleft}$ of the S_n-module $\mathbf{H}'_{(1^{k-1},n-k+1)}$, and the quotient $R_\lambda^{(k)} := I_\lambda^{(k)\trianglelefteq}/I_\lambda^{(k)\triangleleft}$, defined in Subsection 1.4.1.

Proposition 6.2. *$I_\lambda^{(k)\trianglelefteq}$, $I_\lambda^{(k)\triangleleft}$ and thus $R_\lambda^{(k)}$ are S_n-invariant.*

Proof. Follows from the straightening algorithm. □

Recall that $\lambda = ((\mu_1 \geq \cdots \geq \mu_k = 0), (\nu_1 \geq \cdots \geq \nu_{n-k} = 0))$ is an (n,k) partition, if $\forall (1 \leq i < k-1) \quad (\mu_i - \mu_{i+1} \in \{0,1\})$ and $\forall (1 \leq i < n-k) \quad (\nu_i - \nu_{i+1} \in \{0,1\})$.

Lemma 6.3. *Let $\lambda = (\mu, \nu)$ be an (n,k)-bipartition. Then $R_\lambda^{(k)} \neq \{0\}$ and $\{a_\pi^{(k)} + I_\lambda^{(k)\triangleleft} : \mathrm{Des}(\pi) = A_\lambda\}$ is a basis for $R_\lambda^{(k)}$, where A_λ is defined as in (11).*

The proof is analogous to the proof of [1, Corollary 3.10].

Theorem 6.4. *The S_n-action on $R_\lambda^{(k)}$ is given by*

$$s_j(a_\pi^{(k)}) = \begin{cases} a_{s_j\pi}^{(k)}, & \text{if } |\pi^{-1}(j+1) - \pi^{-1}(j)| > 1; \\ a_\pi^{(k)}, & \text{if } \pi^{-1}(j+1) = \pi^{-1}(j) + 1; \\ -a_\pi^{(k)} - \sum_{\sigma \in A_j(\pi)} a_\sigma^{(k)}, & \text{if } \pi^{-1}(j+1) = \pi^{-1}(j) - 1. \end{cases}$$

Here $s_j = (j, j+1)$ $(1 \leq j < n)$ are the Coxeter generators of S_n; $\{a_\pi^{(k)} + I_\lambda^{(k)\triangleleft} : \mathrm{Des}(\pi) = A_\lambda\}$ is the descent basis of $R_\lambda^{(k)}$; for $\pi \in S_\lambda$ with $\pi^{-1}(j+1) = \pi^{-1}(j) - 1$ we define

$$\begin{aligned} t &:= \pi^{-1}(j+1), \\ m_1 &:= \max\{i \in \mathrm{Des}(\pi) \cup \{0\} : i \leq t-1\}, \\ m_2 &:= \min\{i \in \mathrm{Des}(\pi) \cup \{n\} : i \geq t+1\} \end{aligned}$$

(so that $\pi(t) = j+1$, $\pi(t+1) = j$, and $\{m_1+1, \ldots, m_2\}$ is the maximal interval containing t and $t+1$ on which $s_j\pi$ is increasing); and let $A_j(\pi)$ be the set of all $\sigma \in S_n$ satisfying

1. *$(i \leq m_1$ or $i \geq m_2+1) \Longrightarrow \sigma(i) = \pi(i)$;*
2. *the sequences $(\sigma(m_1+1), \ldots, \sigma(t))$ and $(\sigma(t+1), \ldots, \sigma(m_2))$ are increasing;*
3. *$\sigma \notin \{\pi, s_j\pi\}$ (i.e., $\{\sigma(t), \sigma(t+1)\} \neq \{j, j+1\}$).*

Example 6.5. *Let $\pi = 2416573 \in S_7$ and $j = 5$. Then:*

$$j = 5,\ j+1 = 6;\ t = 4,\ t+1 = 5;$$

$$\mathrm{Des}(\pi) = \{2,4,6\};\ m_1 = 2,\ m_2 = 6;\ s_j\pi = 24\underline{1567}3;$$

$$A_j(\pi) = \{24\underline{1756}3,\ 24\underline{5617}3,\ 24\underline{5716}3,\ 24\underline{6715}3\}.$$

Note that $|A_j(\pi)| = \binom{m_2-m_1}{t-m_1} - 2 = \binom{4}{2} - 2 = 4$.

Corollary 6.6. *The S_n-representation on $R_\lambda^{(k)}$ is independent of k.*

Proof of Theorem 6.4. If $|\pi^{-1}(j+1)-\pi^{-1}(j)|>1$ (i.e., if j and $j+1$ are not adjacent in the sequence $(\pi(1),\ldots,\pi(n))$), then $\mathrm{Des}(s_j\pi)=\mathrm{Des}(\pi)$ and therefore $s_j(a_\pi^{(k)})=a_{s_j\pi}^{(k)}$.

If $\pi^{-1}(j+1)=\pi^{-1}(j)+1$ then j immediately precedes $j+1$ in the sequence $(\pi(1),\ldots,\pi(n))$. Therefore $t:=\pi^{-1}(j)\notin\mathrm{Des}(\pi)$, so that: if $t+1\le k-1$ then x_j and x_{j+1} have the same exponent in $a_\pi^{(k)}$; if $t\ge k+1$ then y_j and y_{j+1} have the same exponent in $a_\pi^{(k)}$; and if $t\in\{k-1,k\}$ then none of the variables x_j, x_{j+1}, y_j and y_{j+1} appears in $a_\pi^{(k)}$. In all of these cases we have $s_j(a_\pi^{(k)})=a_\pi^{(k)}$.

We are left with the most involved case: $\pi^{-1}(j+1)=\pi^{-1}(j)-1$. Recall the notations t, m_1, m_2 and $A_j(\pi)$ from the statement of the theorem. In particular, recall that $\pi(t)=j+1$, $\pi(t+1)=j$, and $\{m_1+1,\ldots,m_2\}$ is the maximal interval containing t and $t+1$ on which $s_j\pi$ is increasing.

Assume first that $m_2\le k$. The variables $x_{\pi(m_1+1)},\ldots,x_{\pi(m_2)}$ are all the x-variables having the same exponent in $a_{s_j\pi}^{(k)}$ as x_j (and x_{j+1}). Since $t\in\mathrm{Des}(\pi)$, the exponents of $x_{\pi(1)},\ldots,x_{\pi(t)}$ in $a_\pi^{(k)}$ are 1 higher than the corresponding exponents in $a_{s_j\pi}^{(k)}$. Thus the product $a_{s_j\pi}^{(k)}\cdot e_t(x)$ (in $\mathcal{P}_n^{(k)}$) is a sum of $\binom{n}{t}$ monomials, one of which is $a_\pi^{(k)}$. In all of these monomials, t of the x-variables have their exponent increased by 1 (with respect to $a_{s_j\pi}^{(k)}$). If these t variables miss any one of $x_{\pi(1)},\ldots,x_{\pi(m_1)}$, or include any one of $x_{\pi(m_2+1)},\ldots,x_{\pi(n)}$, then the monomial belongs to $J_\lambda^{(k)\triangleleft}$ and contributes nothing to $R_\lambda^{(k)}$. The remaining $\binom{m_2-m_1}{t-m_1}$ monomials are: $a_\sigma^{(k)}$ for $\sigma\in A_j(\pi)$, $a_\pi^{(k)}=a_{s_j\pi}^{(k)}\cdot x_{\pi(1)}\cdots x_{\pi(t)}$, and $s_j(a_\pi^{(k)})=a_{s_j\pi}^{(k)}\cdot x_{s_j\pi(1)}\cdots x_{s_j\pi(t)}$. On the other hand, $a_{s_j\pi}^{(k)}\cdot e_t(x)\equiv 0$ in $R_\lambda^{(k)}$ since $1\le t\le m_2-1\le k-1$. This proves the claim of the theorem in this case.

If $m_1+1\ge k$, then an analogous argument holds for y- instead of x-variables, and here $a_{s_j\pi}^{(k)}\cdot e_{n-t}(y)\equiv 0$ in $R_\lambda^{(k)}$ since $1\le n-t\le n-m_1-1\le n-k$.

Finally, if $m_1+1<k<m_2$, then the variables $x_{\pi(m_1+1)},\ldots,x_{\pi(m_2)}$ (and the corresponding y-variables) do not appear at all in $a_{s_j\pi}^{(k)}$. If $t\le k-1$ (respectively, $t\ge k$) then $a_{s_j\pi}^{(k)}\cdot e_t(x)$ (respectively, $a_{s_j\pi}^{(k)}\cdot e_{n-t}(y)$) is, again, a sum of $\binom{m_2-m_1}{t-m_1}$ monomials: $a_\sigma^{(k)}$ for $\sigma\in A_j(\pi)$, $a_\pi^{(k)}=a_{s_j\pi}^{(k)}\cdot x_{\pi(1)}\cdots x_{\pi(t)}$ (respectively, $a_\pi^{(k)}=a_{s_j\pi}^{(k)}\cdot y_{\pi(t+1)}\cdots y_{\pi(n)}$), and $s_j(a_\pi^{(k)})=a_{s_j\pi}^{(k)}\cdot x_{s_j\pi(1)}\cdots x_{s_j\pi(t)}$ (respectively, $s_j(a_\pi^{(k)})=a_{s_j\pi}^{(k)}\cdot y_{s_j\pi(t+1)}\cdots y_{s_j\pi(n)}$). This completes the proof. $\square$

Theorem 6.7. *Let $\lambda=(\mu,\nu)$ be an (n,k)-bipartition. $R_\lambda^{(k)}$ is isomorphic, as an S_n-module, to the Solomon descent representation determined by the descent class $\{\pi\in S_n:\mathrm{Des}(\pi)=A_\lambda\}$.*

Proof. By Theorem 6.4 together with Lemma 6.3, for every Coxeter generator s_i, the representation matrices of s_i on $R_\lambda^{(k)}$ and on $R_\lambda^{(n)}$ with respect to the corresponding k-th and n-th descent monomials respectively are identical. By [1, Theorem 4.1], the multiplicity of the irreducible S_n-representation corresponding to μ in $R_\lambda^{(n)}$ is $m_{S,\mu}:=\#\{T\in$

$SYT(\mu)\,:\,\mathrm{Des}(T)=A_\lambda\,\}$, the number of standard Young tableaux of shape μ and descent set A_λ. Theorem 6.1 completes the proof. $\square$

Let $R^{(k)}_{t_1,t_2}$ be the (t_1,t_2)-th homogeneous component of $\mathbf{H}'_{(1^{k-1},n-k+1)}$.

Corollary 6.8. *For every $t_1,t_2\geq 0$ and $1\leq k\leq n$, the (t_1,t_2)-th homogeneous component of $\mathbf{H}'_{(1^{k-1},n-k+1)}$ decomposes into a direct sum of Solomon descent representations as follows:*

$$R^{(k)}_{t_1,t_2}\cong\bigoplus_{\lambda}R^{(k)}_{\lambda},$$

where the sum is over all (n,k)-bipartitions and

$$\sum_{\nu_i>\nu_{i+1}\text{ and }i\geq k}(n-i)=t_1,\qquad\sum_{\mu_i>\mu_{i+1}\text{ and }i<k}i=t_2\ .$$

Finally, Theorem 6.7 implies Stembridge's Theorem 1.16.

First Proof of Theorem 1.16. Combine Theorems 6.1 and 6.7 with Corollary 6.8. $\square$

7 The Schur Function Expansion of $\tilde{H}_{(1^{k-1},n-k+1)}(\overline{x};q,t)$

In this section we give a direct combinatorial proof of Theorems 1.16 and 1.17, using the axiomatic characterization of Macdonald polynomials and properties of the RSK algorithm.

7.1 Preliminaries

A *skew* Young diagram λ/μ is the difference of Young diagrams λ and $\mu\subseteq\lambda$. A skew diagram is a *horizontal strip* (resp. *vertical strip*) if it does not contain two cells in the same column (row). A *semistandard Young tableau* of (skew) shape λ is a function T from the diagram of λ to the ordered alphabet

$$A_+=\{1<2<\ldots\}$$

which is weakly increasing in each row of λ from left to right and strictly increasing in each column from bottom to top. A semistandard tableau is *standard* if it is a bijection form λ to $\{1,\ldots,n=|\lambda|\}$. More generally, we shall admit the ordered alphabet

$$A_\pm=A_+\cup A_-=\{1<\bar{1}<2<\bar{2}<\cdots\}$$

of *positive* letters $1,2,\ldots$ and *negative* letters $\bar{1},\bar{2},\ldots$. For any alphabet B, we let B^* denote the set of all words over B and B^n denote the set of all words $w\in B^*$ of length n.

A *super* tableau is a function $T:\lambda\to A_\pm$ which is weakly increasing in each row and column and is such that the entries equal to a occupy a horizontal strip if a is positive and a vertical strip if a is negative. Thus a semistandard tableaux is just a super tableau where all the entries are positive. We denote

$$\begin{aligned} SSYT(\lambda) &= \{\text{semistandard tableaux } T : \lambda \to A_+\}, \\ SSYT_\pm(\lambda) &= \{\text{super tableaux } T : \lambda \to A_\pm\}, \\ SSYT(\lambda,\mu) &= \{\text{semistandard tableaux } T : \lambda \to A_+ \text{ with entries } 1^{\mu_1}, 2^{\mu_2}, \ldots\}, \\ SSYT_\pm(\lambda,\mu,\nu) &= \{\text{semistandard super tableaux } T : \lambda \to A_+ \text{ with entries } 1^{\mu_1}, \bar{1}^{\nu_1}, \ldots\}, \\ SYT(\lambda) &= SSYT(\lambda, 1^n) = \{\text{standard tableaux } T : \lambda \to \{1, \ldots, n = |\lambda|\}\}. \end{aligned}$$

If T is any one of these types of tableaux, we shall let $sh(T) = \lambda$ denote the shape of T, $pos(T)$ denote the number of positive letters in the range of T and $neg(T)$ denote the number of negative letters in the range of T.

We write $\langle\ ,\ \rangle$ for the Hall inner product on symmetric functions, defined by either one of the identities

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu} = \langle s_\lambda, s_\mu \rangle. \tag{26}$$

We denote by ω the involution defined by either one of the identities

$$\omega(h_\lambda) = e_\lambda,\ \omega(e_\lambda) = h_\lambda,\ \omega(s_\lambda) = s_{\lambda'}. \tag{27}$$

We shall use square brackets $f[A]$ to denote the plethystic evaluation of a symmetric function f at a polynomial, rational function, or formal power series A. This is defined by writing f in terms of the power symmetric functions and then substituting $p_m[A]$ for $p_m(\overline{x})$, where $p_m[A]$ is the result of substituting $a \to a^m$ for every indeterminate a. The standard λ-ring identities hold for plethystic evaluation, e.g. $s_\lambda[X+Y] = \sum_{\mu\subseteq\lambda} s_\mu[X] s_{\lambda/\mu}[Y]$, etc. In particular, setting $Z = z_1 + z_2 + \cdots$, we have $f[Z] = f(z_1, z_2, \ldots)$.

If $W = w_1 + w_2 + \ldots$ and f is a symmetric function, we shall use the notation

$$\omega^W f[Z+W] \tag{28}$$

to denote the result of applying ω to $f[Z+W] = f(z_1, z_2, \ldots, w_1, w_2 \ldots)$ considered as a symmetric function in the w variables with functions of z as coefficients. Equations (27) and (28) then imply that the coefficient of a monomial $z^\mu w^\eta = z_1^{\mu_1} z_2^{\mu_2} \cdots w_1^{\eta_1} w_2^{\eta_2} \cdots$ in $\omega^W f[Z+W]$ is given by

$$\omega^W f[Z+W]|_{z^\mu w^\eta} = \langle f, e_\eta(z) h_\mu(w) \rangle. \tag{29}$$

If T is a semistandard tableau of (skew) shape λ, we set

$$z^T = \prod_{x\in\lambda} z_{T(x)}. \tag{30}$$

Then the usual combinatorial definition of the Schur function s_λ is

$$s_\lambda(z_1, z_2, \ldots) = \sum_{T \in SSYT(\lambda)} z^T. \tag{31}$$

Throughout what follows, we fix

$$Z = z_1 + z_2 + \cdots\,,\; W = w_1 + w_2 + \cdots\,,$$

and make the convention that

$$z_{\bar{a}}\ stands\ for\ w_a,\ for\ every\ negative\ letter\ \bar{a} \in A_-.$$

Then the "super" analogue of (31) is

$$HS_\lambda(z,w) = \omega^W s_\lambda[Z+W] = \sum_{T\in SSYT_\pm(\lambda)} z^T. \tag{32}$$

We shall refer to $HS_\lambda(z,w)$ as the super Schur function, but it is also called a hook Schur function $HS_\lambda(z,w)$ by Berele and Regev [8]. We note that the right hand side of (32) is actually independent of the relative order of positive letters A_+ and negative letters A_-. That is, if A_+ is ordered $1 < 2 < \cdots$ and A_- is ordered $\bar{1} < \bar{2} <$, then the RHS of (32) is independent of the relative order between the positive and negative letters. Thus, for example, if we let $1 < 2 < \cdots < \bar{1} < \bar{2} < \cdots$, the super tableaux T of shape λ will consists of a pair of tableau (T_1, T_2) where $T_1 : \mu \to A_+$ is a semistandard tableau of shape μ for some $\mu \subseteq \lambda$ and $T_2 : \lambda'/\mu' \to A_-$ is a semi-standard tableau of shape λ'/μ' or, equivalently, we can think of T_2 as a row strict tableau of shape λ/μ by conjugation. Thus we can think of T as a filling of λ with positive and negative numbers such that positive numbers form a column strict tableau of shape μ and the negative numbers form a row strict tableau of shape λ/μ. Given such a T, we can achieve a super tableau T' from T corresponding to any other relative order between A_+ and A_- by using jeu de taquin to move the negative numbers past the positive numbers as was done in [30, 31]. Another proof of the independence of the RHS of (31) of the relative order of the positive and negative letters can be found in [29]. It follows that

$$HS_\lambda(z,w) = \sum_{\mu\subseteq\lambda} s_\mu[Z]s_{\lambda'/\mu'}[W]. \tag{33}$$

Similarly, it follows that the RHS of (32) is also independent of the relative order of the positive letters among themselves and the relative order of the negative letters among themselves.

7.2 Second Proof of Theorem 1.16

As noted in [21], the set $\{\tilde{H}_\mu[Z;q,t]\}_{\mu\vdash n}$ can be characterized as the unique basis of the space of homogeneous symmetric polynomials of degree n over the field $F = \mathbf{Q}(q,t)$ of rational functions in q and t, that satisfies the following three properties:

A1 $\tilde{H}_\mu[Z(q-1);q,t] = \sum_{\rho\trianglelefteq\mu'} c_{\rho,\mu} m_\rho[Z]$.

A2 $\tilde{H}_\mu[Z(t-1);q,t] = \sum_{\rho\trianglelefteq\mu} d_{\rho,\mu} m_\rho[Z]$.

A3 $\tilde{H}_\mu[z_1,\ldots,z_n;q,t]|_{z_1^n}=1$.

where $\trianglelefteq$ denotes the dominance order.

The row insertion algorithm of the Robinson-Schensted-Knuth (RSK) correspondence has the property that if $a_1\cdots a_n$ is a sequence of positive letters and

$$a_1\cdots a_n \rightarrow_{RSK} (P,Q)$$

where P is a semistandard Young tableau, Q is a standard tableau, $sh(P)=sh(Q)$, and

1. if $a_i \le a_{i+1}$, then $i+1$ lies strictly to right and weakly below i in Q and
2. if $a_i > a_{i+1}$, then $i+1$ lies strictly above and weakly to the left of i in Q.

Given a sequence of positive letters $a=a_1\ldots a_n$, some $1\le k\le n$, and some linear order $\prec$ on A_+, we let (in analogy with the corresponding definitions for tableaux in Subsection 1.4.2 above)

$$\mathrm{maj}_{1,k}(a,\prec)=\sum_{1\le i<k, a_i\succ a_{i+1}} i \tag{34}$$

and

$$\mathrm{comaj}_{k,n}(a,\prec)=\sum_{k\le i<n, a_i\succ a_{i+1}} (n-i). \tag{35}$$

It then follows from Theorem 1.16 and the properties of the RSK algorithm that

$$\begin{aligned}\tilde{H}_{(1^{k-1},n-k+1)}[Z;q,t] &= \sum_{\lambda\vdash n} s_\lambda[Z]\tilde{K}_{\lambda,(1^{k-1},n-k+1)}(q,t) \qquad (36)\\ &= \sum_{\lambda\vdash n}\sum_{P\in SSYT(\lambda)} z^P \sum_{Q\in SYT(\lambda)} q^{\mathrm{maj}_{1,n-k+1}(Q)}t^{\mathrm{comaj}_{n-k+1,n}(Q)}\\ &= \sum_{a=a_1\cdots a_n\in A_\pm^n} z_{a_1}\cdots z_{a_n} q^{\mathrm{maj}_{1,n-k+1}(a,\prec)}t^{\mathrm{comaj}_{n-k+1,n}(a,\prec)}.\end{aligned}$$

To prove (A1) and (A2), we need to interpret $\tilde{H}_\mu[Z(q-1)]$ and $\tilde{H}_\mu[Z(t-1)]$. Note that

$$\begin{aligned}\tilde{H}_\mu[Z(q-1)] &= \sum_{\lambda\vdash|\mu|} s_\lambda[qZ-Z]\tilde{K}_{\lambda,\mu}(q,t) \qquad (37)\\ &= \sum_{\lambda\vdash|\mu|}\tilde{K}_{\lambda,\mu}(q,t)\sum_{\nu\subseteq\lambda} s_\nu[qZ](-1)^{|\lambda/\nu|}s_{\lambda'/\nu'}[Z].\end{aligned}$$

Here we have used the λ-ring identity

$$s_\lambda[X-Y]=\sum_{\nu\subseteq\lambda} s_\nu[X](-1)^{|\lambda/\nu|}s_{\lambda'/\nu'}[Y].$$

Our goal is to get an interpretation of $\tilde{H}_{(1^k,n-k)}[Z(1-q)]$ in terms of statistics on words which is similar to (36). To this end, we shall consider an extension of the RSK algorithm to words over $A_\pm$ where we row insert positive letters and dual row insert negative letters as in [30]. Recall that in the dual row insertion algorithm of the Robinson-Schensted-Knuth (DRSK) correspondence, an x bumps the first element which is greater than or equal to x in a row into which x is inserted as opposed to x bumping the first element which is greater than x in a row into which x inserted in row insertion algorithm. The DRSK algorithm has the property that that if $\bar{a}_1 \cdots \bar{a}_n$ is a sequence of negative letters and

$$\bar{a}_1 \cdots \bar{a}_n \rightarrow_{DRSK} (P, Q)$$

where the transpose of P, P^T, is a semistandard Young tableau, Q is a standard tableau, and $sh(P) = sh(Q)$, then

1. if $\bar{a}_i < \bar{a}_{i+1}$, then $i+1$ lies strictly to right and weakly below i in Q and
2. if $\bar{a}_i \geq \bar{a}_{i+1}$, then $i+1$ lies strictly above and weakly to the left of i in Q.

Given a word $w = w_1 \cdots w_n \in A_\pm^n$, $1 \leq k \leq n$, and some fixed linear order $\prec$ on $A_\pm$, we shall let

1. $neg(w)$ be the number of negative letters in w,
2. $pos(w)$ be the number of positive letters in w,
3. $Des(w, \prec)$ be the set of all i such that
   (i) $a_i \succeq a_{i+1}$ if both a_i and a_{i+1} are negative and
   (ii) $a_i \succ a_{i+1}$ otherwise;
4. $\mathrm{maj}_{1,k}(w, \prec) = \sum_{1 \leq i < k, i \in Des(a,\prec)} i$, and
5. $\mathrm{comaj}_{k,n} = \sum_{k \leq i < n, i \in Des(a,\prec)} (n-i)$.

This given, to prove Stembridge's Theorem 1.16 directly, we shall consider the following family of polynomials $\{\overline{H}_\mu[Z;q,t]\}_{\mu \vdash n}$ where

(a) $\overline{H}_\mu[Z;q,t] = \tilde{H}_\mu[Z;q,t]$ if $\mu \neq (1^{k-1}, n-k+1)$ and

(b)

$$\begin{aligned}
\overline{H}_{(1^{k-1},n-k+1)}[Z;q,t] &= \sum_{a=a_1\cdots a_n \in A_+^n} z_{a_1} \cdots z_{a_n} q^{\mathrm{maj}_{1,n-k+1}(a,\prec)} t^{\mathrm{comaj}_{n-k+1,n}(a,\prec)} \qquad (38) \\
&= \sum_{\lambda \vdash n} \sum_{P \in SSYT(\lambda)} \sum_{Q \in SYT(\lambda)} z^P q^{\mathrm{maj}_{1,n-k+1}(Q)} t^{\mathrm{comaj}_{n-k+1,n}(Q)} \\
&= \sum_{\lambda \vdash n} s_\lambda[Z] \sum_{Q \in SYT(\lambda)} q^{\mathrm{maj}_{1,n-k+1}(Q)} t^{\mathrm{comaj}_{n-k+1,n}(Q)}.
\end{aligned}$$

To prove that $\overline{H}_\mu[Z;q,t] = \tilde{H}_\mu[Z;q,t]$ for all μ, we need only prove that the family of polynomials $\{\overline{H}_\mu[Z;q,t]\}_{\mu\vdash n}$ satisfies the analogues of (A1), (A2), and (A3). Clearly from our definitions, we need only show that $\overline{H}_{(1^{k-1},n-k+1)}[Z;q,t]$ satisfies the analogues of (A1), (A2), and (A3).

It immediately follows from (38) that the analogue of (A3) holds since the only word contributing to the coefficient of z_1^n is the word $a = 1^n$ and clearly $\mathrm{maj}_{1,n-k+1}(a,\prec) = \mathrm{comaj}_{n-k+1,n}(a,\prec) = 0$ for all $\prec$ in this case. Thus $\overline{H}_{(1^{k-1},n-k+1)}[Z;q,t]|_{z_1^n} = 1$ as desired.

It follows from the properties of the row insertion and dual row insertion of the RSK correspondence described above that

$$\begin{aligned}
&\overline{H}_{(1^k,n-k)}[Z(q-1);q,t] \qquad (39)\\
&= \sum_{\lambda\vdash n}\left(\sum_{Q\in SYT(\lambda)} q^{\mathrm{maj}_{1,n-k+1}(Q)}t^{\mathrm{comaj}_{n-k+1,n}(Q)}\right)\sum_{\nu\subseteq\lambda} q^{|\nu|}s_\nu[Z](-1)^{|\lambda/\nu|}s_{\lambda'/\nu'}[Z]\\
&= \sum_{\lambda\vdash n}\sum_{P\in SSYT_\pm(\lambda)}\sum_{Q\in SYT(\lambda)} (-1)^{neg(P)}q^{pos(P)}z^P q^{\mathrm{maj}_{1,n-k+1}(Q)}t^{\mathrm{comaj}_{n-k+1,n}(Q)}\\
&= \sum_{a=a_1\cdots a_n\in A_\pm^n} q^{pos(a)}(-1)^{neg(a)}z_{|a_1|}\cdots z_{|a_n|}q^{\mathrm{maj}_{1,n-k+1}(a,\prec)}t^{\mathrm{comaj}_{n-k+1,n}(a,\prec)}
\end{aligned}$$

for any linear order $\prec$ on $A_\pm$. Similarly,

$$\begin{aligned}
&\overline{H}_{(1^k,n-k)}[Z(t-1);q,t] \qquad (40)\\
&= \sum_{\lambda\vdash n}\left(\sum_{Q\in SYT(\lambda)} q^{\mathrm{maj}_{1,n-k+1}(Q)}t^{\mathrm{comaj}_{n-k+1,n}(Q)}\right) s_\lambda[tZ-Z] \qquad (41)\\
&= \sum_{\lambda\vdash n}\left(\sum_{Q\in SYT(\lambda)} q^{\mathrm{maj}_{1,n-k+1}(Q)}t^{\mathrm{comaj}_{n-k+1,n}(Q)}\right)\sum_{\nu\subseteq\lambda} t^{|\nu|}s_\nu[Z](-1)^{|\lambda/\nu|}s_{\lambda'/\nu'}[Z]\\
&= \sum_{\lambda\vdash n}\sum_{P\in SSYT_\pm(\lambda)}\sum_{Q\in SYT(\lambda)} (-1)^{neg(P)}t^{pos(P)}z^P q^{\mathrm{maj}_{1,n-k+1}(Q)}t^{\mathrm{comaj}_{n-k+1,n}(Q)}\\
&= \sum_{a=a_1\cdots a_n\in A_\pm^n} t^{pos(a)}(-1)^{neg(a)}z_{|a_1|}\cdots z_{|a_n|}q^{\mathrm{maj}_{1,n-k+1}(a,\prec)}t^{\mathrm{comaj}_{n-k+1,n}(a,\prec)}
\end{aligned}$$

for any linear order $\prec$ on $A_\pm$.

Thus to prove (A1), we must prove that there exists a linear order $\prec$ on $A_\pm$ such that

$$\begin{aligned}
&\sum_{a=a_1\cdots a_n\in A_\pm^n} q^{pos(a)}(-1)^{neg(a)}z_{|a_1|}\cdots z_{|a_n|}q^{\mathrm{maj}_{1,n-k+1}(a,\prec)}t^{\mathrm{comaj}_{n-k+1,n}(a,\prec)}\\
&\qquad = \sum_{\rho\trianglelefteq(1^{n-k},k)} c_{\rho,(1^{k-1},n-k+1)}m_\rho. \qquad (42)
\end{aligned}$$

Similarly to prove (A2), we must prove that there exists a linear order $\prec$ on $A_{\pm}$ such that

$$\sum_{a=a_1\cdots a_n\in A_{\pm}^n} t^{pos(a)}(-1)^{neg(a)}z_{|a_1|}\cdots z_{|a_n|}q^{\mathrm{maj}_{1,n-k+1}(a,\prec)}t^{\mathrm{comaj}_{n-k+1,n}(a,\prec)}$$
$$= \sum_{\rho\trianglelefteq(1^{k-1},n-k+1)} d_{\rho,(1^{k-1},n-k+1)}m_\rho \tag{43}$$

for some $c_{\rho,(1^{k-1},n-k+1)}, d_{\rho,(1^{k-1},n-k+1)} \in Q(q,t)$.

We shall prove (42) and (43) via simple involutions. We shall start with proving (42). For any ordering $\prec$, define the weight $U(a)$ of a word $a = a_1\cdots a_n \in A_{\pm}^n$ by setting

$$U(a) = q^{pos(a)}(-1)^{neg(a)}z_{|a_1|}\cdots z_{|a_n|}q^{\mathrm{maj}_{1,n-k+1}(a,\prec)}t^{\mathrm{comaj}_{n-k+1,n}(a,\prec)}.$$

First we define $\prec$ so that

$$1 \prec 2 \prec \cdots \prec n \prec \overline{n} \prec \overline{n-1} \prec \cdots \prec \overline{1}.$$

Then we define an involution I_k on $A_{\pm}^n$ as follows . We let $|i| = |\overline{i}| = i$. Given a word $w = a_1\cdots a_n \in A_{\pm}^n$, we look for the smallest letter j which is repeated in $|a_1|\cdots|a_{n-k+1}|$. If there is no such letter, we let $I_k(w) = w$. Otherwise, let a_t be right most occurrence of either j or $\overline{j}$ in $a_1\cdots a_{n-k+1}$. Thus $1 < t \leq n-k+1$. Now let i be the smallest letter in $|a_1|\cdots|a_{t-1}|$ and let a_s be the left most occurrence of i or $\overline{i}$ in $a_1\ldots a_{t-1}$. Note that $1 \leq s \leq n-k$ since $t \leq n-k+1$. Then $I_k(a) = b = b_1\cdots b_n$ where (i) $b_r = a_r$ if $r \neq s$, (ii) $b_s = i$ if $a_s = \overline{i}$,and (iii) $b_s = \overline{i}$ if $a_s = i$. In other words, $I_k(a)$ is the result of changing a_s to $\overline{i}$ if $a_s = i$ or changing a_s to i if $a_s = \overline{i}$. Clearly I_k^2 is the identity. We claim that $U(a) = -U(b)$. Clearly $(-1)^{neg(a)} = -(-1)^{neg(b)}$ since we changed the sign of one letter. We did not change the absolute value of any letter so that $z_{|a_1|}\cdots z_{|a_n|} = z_{|b_1|}\cdots z_{|b_n|}$. Since we did not change any of the letters after place $n-k$, we have $t^{\mathrm{comaj}_{n-k+1,n}(a,\prec)} = t^{\mathrm{comaj}_{n-k+1,n}(b,\prec)}$. Thus we need only show that

$$q^{pos(a)}q^{\mathrm{maj}_{1,n-k+1}(a,\prec)} = q^{pos(b)}q^{\mathrm{maj}_{1,n-k+1}(b,\prec)} \tag{44}$$

Without loss of generality, we may assume that $a_s = i$. Then there are two cases.
Case 1. $s = 1$. Since i is the smallest element in $|a_1|\cdots|a_t|$, it follows $i \preceq a_2$ and hence $1 \notin Des(a,\prec)$. We claim that $1 \in Des(b,\prec)$. Now if $|a_2| > i$ or $a_2 = i$, then clearly $\overline{i} \succ a_2$ by our choice of the order $\prec$ so that $1 \in Des(b,\prec)$. Finally if $a_2 = \overline{i}$, then $1 \in Des(b,\prec)$ by our definition of $Des(a,\prec)$ since we have two consecutive equal negative numbers. Thus it follows that $\mathrm{maj}_{1,n-k+1}(b,\prec) - 1 = \mathrm{maj}_{1,n-k+1}(a,\prec)$ and hence

$$q^{pos(a)}q^{\mathrm{maj}_{1,n-k+1}(a,\prec)} = q^{pos(b)+1}q^{\mathrm{maj}_{1,n-k+1}(b,\prec)-1} = q^{pos(b)}q^{\mathrm{maj}_{1,n-k+1}(b,\prec)}$$

as desired.

Case 2. $s > 1$. We can argue exactly as in Case 1 that $s \notin Des(a,\prec)$ and $s \in Des(b,\prec)$.

So consider a_{s-1}. Our choice of i and s ensures that $|a_{s-1}|$ is strictly greater than i. Thus by our definition of $\prec$, $i \prec a_{s-1} \prec \bar{i}$. Hence $s-1 \in Des(a, \prec)$ and $s-1 \notin Des(b, \prec)$. Thus it again follows that $\mathrm{maj}_{1,n-k+1}(b, \prec) - 1 = \mathrm{maj}_{1,n-k+1}(a, \prec)$ and hence

$$q^{pos(a)} q^{\mathrm{maj}_{1,n-k+1}(a,\prec)} = q^{pos(b)+1} q^{\mathrm{maj}_{1,n-k+1}(b,\prec)-1} = q^{pos(b)} q^{\mathrm{maj}_{1,n-k+1}(b,\prec)}$$

as desired.

Thus our involution I_k shows that

$$\begin{aligned} &\sum_{a=a_1\cdots a_n \in A_{\pm}^n} q^{pos(a)} (-1)^{neg(a)} z_{|a_1|} \cdots z_{|a_n|} q^{\mathrm{maj}_{1,n-k+1}(a,\prec)} t^{\mathrm{comaj}_{n-k+1,n}(a,\prec)} \\ = &\sum_{a=a_1\cdots a_n \in A_{\pm}^n, I_k(a)=a} q^{pos(a)} (-1)^{neg(a)} z_{|a_1|} \cdots z_{|a_n|} q^{\mathrm{maj}_{1,n-k+1}(a,\prec)} t^{\mathrm{comaj}_{n-k+1,n}(a,\prec)}. \end{aligned}$$

But since the only words $a = a_1 \cdots a_n$ such that $I_k(a) = a$ are those where $|a_1| \cdots |a_{n-k+1}|$ are pairwise distinct, it follows that the largest possible type of a monomial $z_{|a_1|} \cdots z_{|a_n|}$ relative to the dominance order is $(1^{n-k}, k)$ since this is largest type of words with at least $k+1$ distinct letters. Thus (42) holds.

The proof of (43) is similar to the proof of (42). First we define $\prec^*$ so that

$$\bar{1} \prec^* \bar{2} \prec^* \cdots \prec^* \bar{n} \prec^* n \prec^* n-1 \prec^* \cdots \prec^* 1.$$

Then we define the weight $V(a)$ of a word $a = a_1 \cdots a_n \in A_{\pm}^n$ by setting

$$V(a) = t^{pos(a)} (-1)^{neg(a)} z_{|a_1|} \cdots z_{|a_n|} q^{\mathrm{maj}_{1,n-k+1}(a,\prec^*)} t^{\mathrm{comaj}_{n-k+1,n}(a,\prec^*)}.$$

We define an involution J_k on $A_{\pm}^n$ as follows. Given a word $a = a_1 \cdots a_n \in A_{\pm}^n$, we look for the smallest letter j which is repeated in $|a_{n-k+1}| \cdots |a_n|$. If there is no such letter, we let $J_k(a) = a$. Otherwise, let a_t be the left most occurrence of either j or $\bar{j}$ in $a_{n-k+1} \cdots a_n$. Now let i be the smallest letter in $|a_{t+1}| \cdots |a_n|$ and let s be the right most occurrence of either i or $\bar{i}$ in $a_{t+1} \ldots a_n$.

Thus $n-k+1 < t+1 \leq s$ since $n-k+1 \leq t$. Then $J_k(a) = b = b_1 \cdots b_n$ where (i) $b_r = a_r$ if $r \neq s$, (ii) $b_s = i$ if $a_s = \bar{i}$,and (iii) $b_s = \bar{i}$ if $a_s = i$. In other words, $J_k(a)$ is the result of changing a_s to $\bar{i}$ if $a_s = i$ or changing a_s to i if $a_s = \bar{i}$. Clearly J_k^2 is the identity. We claim that $V(a) = -V(b)$. Clearly $(-1)^{neg(a)} = -(-1)^{neg(b)}$ since we changed the sign of one letter. Again we did not change the absolute value of any letter so that $z_{|a_1|} \cdots z_{|a_n|} = z_{|b_1|} \cdots z_{|b_n|}$. We did not change any of the letters among $a_1 \cdots a_{n-k+1}$ so that $q^{\mathrm{maj}_{1,n-k+1}(a,\prec^*)} = q^{\mathrm{maj}_{1,n-k+1}(b,\prec^*)}$. Thus we need only show that

$$t^{pos(a)} t^{\mathrm{comaj}_{n-k+1,n}(a,\prec^*)} = t^{pos(b)} t^{\mathrm{comaj}_{n-k+1,n}(b,\prec^*)} \tag{45}$$

There is no loss of generality in assuming that $a_s = i$. Then there are two cases.

Case 1. $s = n$. Since i is the smallest element in $|a_t| \cdots |a_n|$, it follows $a_{n-1} \preceq^* i$. We claim that $n-1 \notin Des(a, \prec^*)$ and that $n-1 \in Des(b, \prec^*)$. Now if $|a_{n-1}| > i$, then clearly

$\bar{i} \prec^* a_{n-1} \prec^* i$ so that our claim holds. If $a_{n-1} = i$, then $n-1 \notin Des(a, \prec^*)$ because two equal positive letters do not cause a descent by our definitions. However, $\bar{i} \prec^* i$ so that $n-1 \in Des(b, \prec^*)$. Finally if $a_{n-1} = \bar{i}$, then $n-1 \in Des(b, \prec^*)$ since two equal negative letters cause a descent while $n-1 \notin Des(a, \prec^*)$ because $\bar{i} \prec^* i$. Thus it follows that $\mathrm{comaj}_{n-k+1,n}(b, \prec^*) - 1 = \mathrm{comaj}_{n-k+1,n}(a, \prec^*)$ and hence

$$t^{pos(a)} c^{\mathrm{comaj}_{n-k+1,n}(a,\prec^*)} = t^{pos(b)+1} t^{\mathrm{comaj}_{n-k+1,n}(b,\prec^*)-1} = t^{pos(b)} t^{\mathrm{comaj}_{n-k+1,n}(b,\prec^*)}$$

as desired.

Case 2. $s < n$. We can argue exactly as in Case 1 that $s-1 \notin Des(a, \prec^*)$ and $s-1 \in Des(b, \prec^*)$. So consider a_{s+1}. Our choice of i and s ensures that $|a_{s+1}|$ is strictly greater than i. Thus by our definition of $\prec^*$, $\bar{i} \prec^* a_{s+1} \prec^* i$. Hence $s \in Des(a, \prec^*)$ and $s \notin Des(b, \prec^*)$. Thus the only places were $Des(a, \prec^*)$ and $Des(a, \prec^*)$ differ is on the set $\{s-1, s\}$. We have $Des(a, \prec^*) \cap \{s-1, s\} = \{s\}$ and $Des(b, \prec^*) \cap \{s-1, s\} = \{s-1\}$. Since $n-(s-1) = (n-s)+1$, it follows that $\mathrm{comaj}_{n-k+1,n}(b, \prec^*) - 1 = \mathrm{comaj}_{n-k+1,n}(a, \prec^*)$ and hence

$$t^{pos(a)} t^{\mathrm{comaj}_{n-k+1,n}(a,\prec^*)} = t^{pos(b)+1} t^{\mathrm{comaj}_{n-k+1,n}(b,\prec^*)-1} = t^{pos(b)} t^{\mathrm{comaj}_{n-k+1,n}(b,\prec^*)}$$

as desired.

Thus our involution J_k shows that

$$\begin{aligned}
&\sum_{a=a_1\cdots a_n \in A_{\pm}^n} t^{pos(a)} (-1)^{neg(a)} z_{|a_1|} \cdots z_{|a_n|} q^{\mathrm{maj}_{1,n-k+1}(a,\prec^*)} t^{\mathrm{comaj}_{n-k+1,n}(a,\prec^*)} \\
&= \sum_{a=a_1\cdots a_n \in A_{\pm}^n, J_k(a)=a} t^{pos(a)} (-1)^{neg(a)} z_{|a_1|} \cdots z_{|a_n|} q^{\mathrm{maj}_{1,n-k+1}(a,\prec^*)} t^{\mathrm{comaj}_{n-k+1,n}(a,\prec^*)}.
\end{aligned}$$

But since the only words $a = a_1 \cdots a_n$ such that $J_k(a) = a$ must have $|a_{n-k+1}| \cdots |a_n|$ be pairwise distinct, it follows that the largest possible type of a monomial $z_{|a_1|} \cdots z_{|a_n|}$ relative to the dominance order is $(1^{k-1}, n-k+1)$ since this is largest type of words with at least k distinct letters. Thus (43) holds. □

8 Final Remarks

8.1 Haglund Statistics

Let ξ be a filling of the Ferrers diagram of a partition μ with the numbers $1, \ldots, n$. For any cell $u = (i,j) \in F_\mu$, let $\xi(u)$ be the entry in cell u. We say that $u = (i,j) \in F_\mu$ is a *descent* of ξ, written $u \in Des(\xi)$, if $i > 1$ and $\xi((i,j)) \geq \xi((i-1,j))$. Then $maj(\xi) = \sum_{u \in Des(\xi)} (leg(u) + 1)$. Two cells $u, v \in F_\mu$ *attack* each other if either

(a) they are in the same row, i.e. $u = (i,j)$ and $v = (i,k)$, or

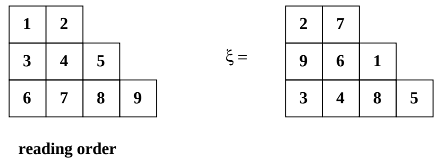


Figure 7: The reading order and a filling of $(4, 3, 2)$.

(b) they are in consecutive rows, with the cell in the upper row strictly to the right of the one in the lower row, i.e. $u = (i + 1, k)$ and $v = (i, j)$ where $j < k$.

The *reading order* is the total ordering on the cells of F_μ given by reading the cells row by row from top to bottom, and left to right within each row. For example, the reading order of $(2, 3, 4)$ is depicted on the left in Figure 7. An *inversion* of ξ is a pair of entries $\xi(u) > \xi(v)$ where u and v attack each other and u precedes v in the reading order. We then define $Inv(\xi) = \{\{u, v\} : \xi(u) > \xi(v) \text{ is an inversion}\}$ and $inv(\xi) = |Inv(\xi)| - \sum_{u \in Des(\xi)} arm(u)$.

For example, if ξ is the filling of shape $(2, 3, 4)$ depicted in Figure 7, then $Des(\xi) = \{(2, 1), (2, 2), (3, 2)\}$. There are four inversion pairs of type (a), namely $\{(2, 1), (2, 2)\}$, $\{(2, 1), (2, 3)\}$, $\{(2, 2), (2, 3)\}$, and $\{(1, 3), (1, 4)\}$, and one inversion pair of type (b), namely $\{(2, 2), (1, 1)\}$. Then one can check that $|Inv(\xi)| = 5$, $maj(\xi) = 5$ and $inv(\xi) = 2$. Finally, we can identify ξ with a permutation by reading the entries in the reading order. In the example of Figure 7, $\xi = 2\ 7\ 9\ 6\ 1\ 3\ 4\ 8\ 5$. Then we let $D(\xi) = Des(\xi^{-1})$. In our example, $\xi^{-1} = 5\ 1\ 6\ 7\ 9\ 4\ 2\ 8\ 3$ so that $D(\xi) = \{1, 5, 6, 8\}$.

Recently, Haglund, Haiman and Loehr [20, 21] proved Haglund's conjectured combinatorial interpretation [19] of $\tilde{H}_\mu(\overline{x}; q, t)$ in terms of quasi-symmetric functions. That is, given a non-negative integer n and a subset $D \subseteq \{1, \ldots, n-1\}$, Gessel's quasi-symmetric function of degree n in variables $x_1, x_2, \ldots$ is defined by the formula

$$Q_{n,D}(\overline{x}) := \sum_{\substack{a_1 \leq a_2 \leq \cdots \leq a_n \\ a_i = a_{i+1} \Rightarrow i \notin D}} x_{a_1} x_{a_2} \cdots x_{a_n}. \tag{46}$$

Then Haglund, Haiman and Loehr [21] proved

$$\tilde{H}_\mu(\overline{x}; q, t) = \sum_{\xi : \mu \simeq \{1, \ldots, n\}} q^{inv(\xi)} t^{maj(\xi)} Q_{n, D(\xi)}(\overline{x}). \tag{47}$$

Here the sum runs over all fillings ξ of the Ferrers diagram of μ with the numbers $1, \ldots, n$.

8.2 Relations with the Combinatorial Interpretation of Macdonald Polynomials

The Hilbert series of $\mathbf{H}_\mu$ is equal to the coefficient of $x_1 x_2 \cdots x_n$ in $\tilde{H}_\mu(\overline{x}; q, t)$. Since the coefficient of $x_1 x_2 \cdots x_n$ in any quasi-symmetric function $Q_{n,D}(\overline{x})$ is 1, it follows that the

Hilbert series of $\mathbf{H}_\mu$ is given by

$$\sum_{k,r} \dim \mathbf{H}_\mu^{(h,k)} q^h t^k = \tilde{H}_\mu(\overline{x}; q, t)|_{x_1 x_2 \cdots x_n} = \sum_{\xi:\mu \simeq \{1,\ldots,n\}} q^{inv(\xi)} t^{maj(\xi)},$$

where the sum runs over all fillings ξ of the Ferrers diagram of μ with the numbers $1, \ldots, n$. No known basis realizes this remarkable identity for general $\mathbf{H}_\mu$. The k-th Haglund basis described in Subsection 1.3.2 above provides such a basis when μ is of hook shape.

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