The Existence of FGDRP$(3, g^u)'$s *

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Abstract

By an FGDRP$(3, g^u)$, we mean a uniform frame $(X, \mathcal{G}, \mathcal{A})$ of block size 3, index 2 and type g^u, where the blocks of $\mathcal{A}$ can be arranged into a $gu/3 \times gu$ array. This array has the properties: (1) the main diagonal consists of u empty subarrays of sizes $g/3 \times g$; (2) the blocks in each column form a partial parallel class partitioning $X \setminus G$ for some $G \in \mathcal{G}$, while the blocks in each row contain every element of $X \setminus G$ 3 times and no element of G for some $G \in \mathcal{G}$. The obvious necessary conditions for the existence of an FGDRP$(3, g^u)$ are $u \geq 5$ and $g \equiv 0 \pmod 3$. In this paper, we show that these conditions are also sufficient with the possible exceptions of $(g, u) \in \{(6, 15), (9, 18), (9, 28), (9, 34), (30, 15)\}$.

1 Introduction

In this paper, we use [1] and [2] as our standard design-theoretic references. A group divisible design, or a (K, λ)-GDD in short, is a triple $(X, \mathcal{G}, \mathcal{A})$, where X is a finite set of v points, $\mathcal{G}$=$\{G_0, G_1, \cdots, G_{u-1}\}$ is a partition of X into u subsets (called groups), and $\mathcal{A}$ is a collection of subsets (called blocks) of X with $|A| \in K$ for any $A \in \mathcal{A}$, such that every pair of points from distinct groups occurs in exactly λ blocks and no pair of points from the same group occurs in any block. The group type or the type of a (K, λ)-GDD is the multiset $T = \{|G_0|, |G_1|, \cdots, |G_{u-1}|\}$ which is often described by an exponential notation. When K consists of a single number k, the notation (k, λ)-GDD is used. Further, we denote $(k, 1)$-GDD as k-GDD. A (K, λ)-GDD of type 1^v is known as a pairwise balanced design (PBD), or a (v, K, λ)-PBD. In this case, the group set $\mathcal{G}$ is the same as the point set, and hence the symbol $\mathcal{G}$ is often omitted from the notation $(X, \mathcal{G}, \mathcal{A})$. Remark that a transversal design (TD), or a TD(k, n) is defined as a k-GDD of type n^k.

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Motivated by the construction of constant composition codes, the present author [8] defined a frame generalized doubly resolvable packing, or an FGDRP in short. Consider a $(k, k-1)$-GDD of type g^u, $(X, \{G_0, G_1, \cdots, G_{u-1}\}, \mathcal{A})$ with $u \geq k+2$ and $k \mid g$. Define $C_j = \{s + jg : s = 0, 1, \cdots, g-1\}$ and $R_i = \{w + (ig/k) : w = 0, 1, \cdots, (g/k) - 1\}$ for $0 \leq i,\ j \leq u-1$. The GDD $(X, \{G_0, G_1, \cdots, G_{u-1}\}, \mathcal{A})$ is called an FGDRP(k, g^u) if the blocks of $\mathcal{A}$ can be arranged into a $\frac{|X|}{k} \times |X|$ array satisfying the properties listed below. We index the rows and columns of the array by the elements of $R_0, R_1, \cdots, R_{u-1}$ and $C_0, C_1, \cdots, C_{u-1}$ in turn.

(1) Suppose that F_x is the subarray indexed by the elements of R_x and C_x for $0 \leq x \leq u-1$. Then F_x is empty. (These u subarrays of sides $(g/k) \times g$ lie in the main diagonal from upper left corner to lower right corner.)

(2) For any $r \in R_i$ $(0 \leq i \leq u-1)$, the blocks in row r form a partial k-parallel class partitioning $X \setminus G_i$, that is, every point of $X \setminus G_i$ occurs in exactly k blocks in row r, while any point of G_i does not occur in any block in row r.

(3) For any $c \in C_j$ $(0 \leq j \leq u-1)$, the blocks in column c form a partial parallel class partitioning $X \backslash G_j$.

Recall that a (k, λ)-frame of type g^u is a (k, λ)-GDD of type g^u in which the blocks of $\mathcal{A}$ can be partitioned into partial parallel classes each partitioning $X \setminus G$ for some group G. So, an FGDRP(k, g^u) is a $(k, k-1)$-frame of type g^u with the prescribed property. The following existence results were proved in [8].

Lemma 1.1 There exists an FGDRP$(3, 3^u)$ for any integer $u \geq 5$ and $u \notin \{16, 18, 20, 22, 24, 28, 32, 34\}$.

Lemma 1.2 There exists an FGDRP$(3, 9^u)$ for any integer $u \geq 5$ and $u \notin \{6, 18, 26, 28, 30, 32, 34, 38, 39, 42, 44, 51, 52\}$.

In this paper, we are interested in the existence of FGDRP$(3, g^u)'$s for arbitrary group size g. The obvious necessary conditions for the existence of an FGDRP$(3, g^u)$ are $g \equiv 0 \pmod 3$ and $u \geq 5$. We will employ both direct and recursive constructions to show that these conditions are also sufficient with 5 possible exceptions of $(g, u) \in \{(6, 15), (9, 18), (9, 28), (9, 34), (30, 15)\}$.

2 Starters and Adders for FGDRP$(k, g^u)'$s

In this section, we develop a number of direct constructions for FGDRPs. Our direct constructions use a variation of the known starter-adder method (see, for example, [3]) in two ways. A similar version for GDRPs and HGBTDs can be found in [9] and [10], respectively.

The first one is established for the construction of an FGDRP(k, g^u) which contains no infinite points. Since $k \mid g$ in an FGDRP(k, g^u) by definition, we can write $g = tk$. Let

G be an additive abelian group of order ug admitting a subgroup G_0 of order g. We fix a system of representatives of the cosets of G_0 in G and denote it by $(h_0 = 0, h_1, \cdots, h_{u-1})$. Write $G_i = h_i + G_0$ $(0 \le i \le u-1)$ for the cosets of G_0 in G. A starter S for an FGDRP(k, g^u) defined on G with groups G_i $(0 \le i \le u-1)$ consists of t sets of k-tuples (base blocks), S_1, $S_2, \cdots$, S_t, which satisfies the following properties.

(1) For any i $(1 \le i \le t)$, S_i contains exactly $u-1$ base blocks, B_{ij}, $j = 1, 2, \cdots, u-1$.

(2) The $t(u-1)$ base blocks form a partition of $G \setminus G_0$ and the difference list from these base blocks contains every element of $G \setminus G_0$ precisely $k-1$ times and no element in G_0.

A corresponding adder $A(S)$ for S consists of t permutations (not necessarily distinct), $A(S_i) = (a_{i1}, a_{i2}, \cdots, a_{i(u-1)})$ $(1 \le i \le t)$ of the $u-1$ representatives $h_1, h_2, \cdots, h_{u-1}$, such that for any i $(1 \le i \le t)$, $\bigcup_{j=1}^{u-1}(B_{ij} + a_{ij})$ contains exactly k elements (not necessary distinct) from any group G_r for $1 \le r \le u-1$, and no element of G_0.

Theorem 2.1 If there exists a starter-adder pair $(S, A(S))$ for an FGDRP(k, g^u) over G with groups G_i $(0 \le i \le u-1)$, then there exists an FGDRP(k, g^u).

Proof: We first use $(S_i, A(S_i))$ for any i $(1 \le i \le t)$ to construct a square K_i of side u whose rows and columns are indexed with the elements of $h_0, h_1, h_2, \cdots, h_{u-1}$. All the cells on the main diagonal of K_i are empty. For any $h_r \in \{h_1, \cdots, h_{u-1}\}$, we place the block B_{ij} in the cell $(-h_r, 0)$ if and only if the corresponding adder a_{ij} of this base block is h_r. Here we identify $-h_r$ with a certain h_j $(0 \le j \le u-1)$ whenever $-h_r \in h_j + G_0$. This can be done, as $A(S_i)$ is a permutation of the representatives $h_1, h_2, \cdots, h_{u-1}$ and $\{-h_0 = 0, -h_1, -h_2, \cdots, -h_{u-1}\}$ is obviously a system of representatives of the cosets of G_0 in G. Now for the remaining columns $h_c \in \{h_1, h_2, \cdots, h_{u-1}\}$, we assign $B + h_c$ to the cell (h_r, h_c) where $h_r \ne h_c$ and B is the block in the cell $(h_r - h_c, 0)$. Here $h_r - h_c = h_j$ if and only if $h_r - h_c \in h_j + G_0$ $(0 \le j \le u-1)$.

Next, we superpose the rows of these t squares K_i $(1 \le i \le t)$ of size u in such a way that their h_r-th row lies in consecutive positions for $h_r \in \{h_0, h_1, \cdots, h_{u-1}\}$. This yields a $tu \times u$ array M whose u subarrays of sides $t \times 1$ in the main diagonal are empty.

Finally, let $G_0 = \{g_0, g_1, \cdots, g_{tk-1}\}$. We form a $tu \times gu$ array $\widehat{M}$ from M by replacing each column L of M with $g = tk$ columns of the following structure:

$L + g_0$	$L + g_1$	$\cdots$	$L + g_{tk-1}$

It can be easily checked that $\widehat{M}$ is an FGDRP(k, g^u), as desired. $\square$

The second construction method is established for obtaining an FGDRP(k, g^u) which contains infinite points. To do this, write $g = tk$ and let $w \le \lfloor (u-1)/(k+1) \rfloor$ be a positive integer. Let G be an additive abelian group of order $g(u-w)$ admitting a subgroup G_0 of order g. As above, we fix a system of representatives of the cosets of G_0 in G and denote it by $(h_0 = 0, h_1, \cdots, h_{u-w-1})$. Write $G_i = h_i + G_0$ $(0 \le i \le u-w-1)$ for the cosets of G_0. Let $G_{u-w-1+j} = \{\infty^j\} \times G_0$ $(1 \le j \le w)$ be w sets of g infinite points labelled by the g elements of G_0 each. We then take the points of an FGDRP(k, g^u) to be $X = \left(\bigcup_{i=u-w}^{u-1} G_i\right) \bigcup G$. An intransitive starter S for an FGDRP(k, g^u) defined on X with groups G_i $(0 \le i \le u-1)$ is defined as a triple (S, R, C) which is of the following structure.

- S consists of t sets of k-tuples (base blocks), $S_1, S_2, \cdots, S_t$. For any i $(1 \le i \le t)$, S_i contains exactly $u-w-1$ base blocks, B_{ij} $(j = 1, 2, \cdots, u-w-1)$ in which there exist precisely kw base blocks containing one infinite point each from $\bigcup_{i=u-w}^{u-1} G_i$.
- R consists of t sets of k-tuples (base blocks) over G, $R_1, R_2, \cdots, R_t$ in which every R_i $(1 \le i \le t)$ consists of exactly w base blocks containing no infinite points from $\bigcup_{i=u-w}^{u-1} G_i$.
- C consists of t sets of k-tuples (base blocks) over G, $C_1, C_2, \cdots, C_t$ in which every C_i $(1 \le i \le t)$ consists of exactly w base blocks, C_{ij} $(j = 1, 2, \cdots, w)$. For any j $(1 \le j \le w)$, $\pi\left(\bigcup_{i=1}^{t} C_{ij}\right) = G_0$. Here $\pi : G \longrightarrow G_0$ is a surjection given by $\pi(x) = y$ if $x = h_i + y \in G_i$ $(0 \le i \le u-w-1)$ under the fixed representative system $(h_0 = 0, h_1, \cdots, h_{u-w-1})$.

(S, R, C) satisfies the following properties:

- $S \bigcup R$ forms a partition of $X \backslash G_0$;
- the difference list from the base blocks of $S \cup R \cup C$ contains every element of $G \backslash G_0$ precisely $k-1$ times, and no element in G_0.

The properties of (S, R, C) imply that each base block contains at most one infinite point, and every infinite point occurs in exactly one base block. For each i $(1 \le i \le t)$, we can assume that the first kw base blocks of S_i contains one infinite point each from $\bigcup_{i=u-w}^{u-1} G_i$, which can be written in the following form:

$$B_{ij} = \{(\infty^s, g_{ij})\} \cup T_{ij}.$$

Here, $j = k(s-1) + d$ with $1 \le s \le w$ and $1 \le d \le k$. For $1 \le i \le t$, $1 \le j \le kw$, $g_{ij} \in G_0$ and T_{ij} is a $(k-1)$-subset of G by the definition of S.

A corresponding adder $A(S)$ for S consists of t permutations (not necessarily distinct), $A(S_i) = (a_{i1}, a_{i2}, \cdots, a_{i(u-w-1)})$ $(1 \le i \le t)$ of the $u-w-1$ representatives $h_1, h_2, \cdots, h_{u-w-1}$. $A(S)$ has the property that for any i $(1 \le i \le t)$, the multiset

$$\left(\bigcup_{j=1}^{u-w-1} (B_{ij} + a_{ij})\right) \bigcup \left(\bigcup_{j=1}^{w} C_{ij}\right)$$

contains exactly k elements (not necessary distinct) from any group G_r for $1 \leq r \leq u-1$ and no element of G_0. The addition $B_{ij} + a_{ij}$ is performed in G with the infinite point in $\bigcup_{i=u-w}^{u-1} G_i$ fixed whenever it occurs in B_{ij}.

When $t \geq 2$, there is one more constraint to the starter (S, R, C) which is marked by $(*)$. For any s $(1 \leq s \leq w)$,

$$(*) \qquad \pi\left(\bigcup_{i=1}^{t}\left(\bigcup_{d=1}^{k}(T_{ij} - g_{ij})\right)\right) = (k-1)G_0,$$

where $j = k(s-1) + d$. In $(*)$, the notation $(k-1)G_0$ stands for the $(k-1)$ copies of G_0. The right side of $(*)$ denotes the image of $\bigcup_{i=1}^{t}(\bigcup_{d=1}^{k}(T_{ij} - g_{ij}))$ under the action of π. It is remarkable that in the case $t = 1$, that is, $g = k$, the property $(*)$ is not required to the starter.

Theorem 2.2 If there exists an intransitive starter (S, R, C) for an FGDRP(k, g^u) over X with groups G_i $(0 \leq i \leq u-1)$ defined above and a corresponding adder $A(S)$, then there exists an FGDRP(k, g^u) missing an FGDRP(k, g^w) as a subdesign. Furthermore, if there exists an FGDRP(k, g^w), then an FGDRP(k, g^u) exists.

Proof: As in the proof of Theorem 2.1, we first use the starter S and the corresponding adder $A(S)$ to construct a square K_i of side $u - w$ for $1 \leq i \leq t$.

Secondly, we use $R_i = \{R_{ij} : j = 1, 2, \cdots, w\}$ to generate a $w \times (u-w)$ array $K(R_i)$ for $1 \leq i \leq t$. It is of the following form

$$K(R_i) = \begin{array}{|c|c|c|c|}\hline R_{i1}+h_0 & R_{i1}+h_1 & \cdots & R_{i1}+h_{u-w-1} \\ \hline R_{i2}+h_0 & R_{i2}+h_1 & \cdots & R_{i2}+h_{u-w-1} \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline R_{iw}+h_0 & R_{iw}+h_1 & \cdots & R_{iw}+h_{u-w-1} \\ \hline \end{array}$$

Thirdly, we use $C_i = \{C_{ij} : j = 1, 2, \cdots, w\}$ to generate a $(u-w) \times w$ array $K(C_i)$ for $1 \leq i \leq t$. It is of the following form

$$K(C_i) = \begin{array}{|c|c|c|c|}\hline C_{i1}+h_0 & C_{i2}+h_0 & \cdots & C_{iw}+h_0 \\ \hline C_{i1}+h_1 & C_{i2}+h_1 & \cdots & C_{iw}+h_1 \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline C_{i1}+h_{u-w-1} & C_{i2}+h_{u-w-1} & \cdots & C_{iw}+h_{u-w-1} \\ \hline \end{array}$$

Finally, let $G_0 = \{g_0 = 0, g_1, \cdots, g_{tk-1}\}$. We form a $tu \times gu$ array K given by

$$K = \begin{array}{|c|c|c|c|c|c|c|c|}
\hline
K_1 & K_1 + g_1 & \cdots & K_1 + g_{tk-1} & K(C_1) & K(C_1) + g_1 & \cdots & K(C_1) + g_{tk-1} \\
\hline
K_2 & K_2 + g_1 & \cdots & K_2 + g_{tk-1} & K(C_2) & K(C_2) + g_1 & \cdots & K(C_2) + g_{tk-1} \\
\hline
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\hline
K_t & K_t + g_1 & \cdots & K_t + g_{tk-1} & K(C_t) & K(C_t) + g_1 & \cdots & K(C_t) + g_{tk-1} \\
\hline
K(R_1) & K(R_1) + g_1 & \cdots & K(R_1) + g_{tk-1} & & & & \\
\hline
K(R_2) & K(R_2) + g_1 & \cdots & K(R_2) + g_{tk-1} & & & & \\
\hline
\cdots & \cdots & \cdots & \cdots & & & & \\
\hline
K(R_t) & K(R_t) + g_1 & \cdots & K(R_t) + g_{tk-1} & & & & \\
\hline
\end{array}$$

The arithmetic $x + g_j$ is done in G if $x \in G$. However, if $x = (\infty^s, y) \in \bigcup_{i=u-w}^{u-1} G_i$, then we have to change the label y. We calculate the sum in the following rule:

$$x + g_j = \begin{cases} (\infty^s, y + g_j), & \text{if } t \geq 2; \\ x, & \text{if } t = 1. \end{cases}$$

This rule in conjunction with the property $(*)$ guarantees that every infinite point meets any element of G exactly $k-1$ times in blocks. By permutating rows and columns of K appropriately, we get the desired FGDRP(k, g^u) missing an FGDRP(k, g^w) as a subdesign. If an FGDRP(k, g^w) exists, then the empty $tw \times gw$ subarray of K can be filled in to form an FGDRP(k, g^u). $\square$

Now we apply Theorem 2.1 and Theorem 2.2 to construct FGDRPs with small parameters. Our constructions for starter-adder pairs are based on two methods. One is to use algebraic structure of G, the other is to use computer searches. Whenever Galois field GF(q) is used, the notation ω stands for an arbitrary primitive element. We also write C_0^e for the unique multiplicative subgroup of GF(q) spanned by ω^e, and write C_i^e $(1 \leq i \leq e-1)$ for the multiplicative cosets $\omega^i \cdot C_0^e$ of C_0^e.

Lemma 2.3 For any odd prime power $q \geq 5$, there exists an FGDRP$(3, 6^q)$.

Proof: Apply Theorem 2.1 with $t = 2$ and $k = 3$. Here, we take the group G to be the additive group of GF$(q) \oplus \mathbb{Z}_6$, and its subgroup $G_0 = \{0\} \oplus \mathbb{Z}_6$. The fixed representative system $(h_0, h_1, \cdots, h_{q-1}) = ((0,0), (1,0), (\omega, 0), \cdots, (\omega^{q-2}, 0))$. Using the notations in the proof of Theorem 2.1, define

$$\begin{aligned}
B_{11} &= \{(1,0), (\omega, 0), (\omega+1, 4)\}, \\
B_{12} &= \{(\omega, 3), (\omega^2, 3), (\omega(\omega+1), 4)\}, \\
B_{21} &= \{(1,1), (\omega, 2), (\omega+1, 5)\}, \\
B_{22} &= \{(\omega, 1), (\omega^2, 2), (\omega(\omega+1), 5)\}.
\end{aligned}$$

The required starter-adder pair (S, A) is then given by

$$\begin{aligned}
S &= \{(g,1) \cdot B_{11},\ (g,1) \cdot B_{12},\ (g,1) \cdot B_{21},\ (g,1) \cdot B_{22} :\ g \in C_0^2\}, \\
A &= \{(g,1) \cdot (b,0),\ (g,1) \cdot (b\omega, 0),\ (g,1) \cdot (b,0),\ (g,1) \cdot (b\omega, 0) :\ g \in C_0^2\},
\end{aligned}$$

where $b \in$GF$(q) \setminus \{0, -1, -\omega, -(\omega+1)\}$. $\square$

Lemma 2.4 There exists an FGDRP$(3, 18^5)$.

Proof: For this FGDRP we again apply Theorem 2.1 with the starter-adder pair $(S \cup (-S), A \cup (-A))$, where $-S = S \cdot (-1, 1)$ and S, A are listed below. Here, we take $G = \mathbb{Z}_5 \oplus \mathbb{Z}_{18}$, $G_0 = \{0\} \oplus \mathbb{Z}_{18}$ and the fixed system of representatives is taken as $((0,0), (1,0), (2,0), (3,0), (4,0))$.

S	{(2,10), (3,8), (4,15)}	A	(4,0)	S	{(3,9), (4,12), (2,7)}	A	(4,0)
	{(2,6), (3,3), (4,2)}		(4,0)		{(4,11), (2,15), (3,1)}		(4,0)
	{(4,9), (2,0), (3,14)}		(4,0)		{(4,17), (2,11), (3,17)}		(4,0)
	{(1,16), (4,5), (3,16)}		(3,0)		{(1,1), (4,8), (3,2)}		(3,0)
	{(1,3), (3,13), (4,0)}		(3,0)		{(3,4), (4,13), (1,10)}		(3,0)
	{(1,7), (3,5), (4,6)}		(3,0)		{(1,14), (3,12), (4,4)}		(3,0)

□

Lemma 2.5 For any $u \in \{14, 20, 32\}$, there exists an FGDRP$(3, 6^u)$.

Proof: For these FGDRPs, we apply Theorem 2.2 with $k = 3$, $t = 2$ and $w = 1$. Here, $G = \mathrm{GF}(u-1) \oplus \mathbb{Z}_6$, $G_0 = \{0\} \oplus \mathbb{Z}_6$ and the fixed representative system $(h_0, h_1, \cdots, h_{u-1}) = ((0,0), (1,0), \cdots, (\omega^{u-3}, 0))$. The required intransitive starter is taken as $(S_1 \cup S_2, R_1 \cup R_2, C_1 \cup C_2)$ and the corresponding adder $A(S) = A(S_1) \cup A(S_2)$ which are given in the following tables. Remark that in our constructions the six infinite points from $\{\infty^1\} \times G_0$ can be distributed to the 3 blocks in S_1 and the 3 blocks in S_2 in an arbitrary way. In the following tables, the symbol " $-$ " is used to denote an arbitrary infinite point from $\{\infty^1\} \times G_0$.

$q = u - 1 = 13,\ \omega = 2$

S_1	$A(S_1)$		R_1
$\{-, (1,0), (2,0)\} \cdot (h,1)$	$(3,0) \cdot (h,1)$	$h \in C_0^4$	$\{(1,4), (3,4), (9,4)\}$
$\{(4,2), (3,3), (12,5)\} \cdot (h,1)$	$(4,0) \cdot (h,1)$	$h \in C_0^4$	C_1
$\{(1,1), (2,2), (7,3)\} \cdot (g,1)$	$(7,0) \cdot (g,1)$	$g \in C_0^2$	$\{(2,0), (6,2), (5,4)\}$
S_2	$A(S_2)$		R_2
$\{-, (1,0), (2,0)\} \cdot (f,1)$	$(3,0) \cdot (f,1)$	$f \in C_2^4$	$\{(12,4), (10,4), (4,4)\}$
$\{(4,2), (3,3), (12,5)\} \cdot (f,1)$	$(4,0) \cdot (f,1)$	$f \in C_2^4$	C_2
$\{(2,1), (6,4), (8,5)\} \cdot (g,1)$	$(2,0) \cdot (g,1)$	$g \in C_0^2$	$\{(11,1), (7,3), (8,5)\}$

$q = u - 1 = 19,\ \omega = 2$

S_1	$A(S_1)$		R_1
$\{-, (9,3), (10,4)\} \cdot (f,1)$	$(2,0) \cdot (f,1)$	$f \in C_3^6$	$\{(1,0), (7,0), (11,0)\}$
$\{(3,5), (4,0), (5,2)\} \cdot (g,1)$	$(4,0) \cdot (g,1)$	$g \in C_0^3$	C_1
$\{(2,0), (4,1), (8,2)\} \cdot (g,1)$	$(1,0) \cdot (g,1)$	$g \in C_0^3$	$\{(4,0), (9,2), (6,4)\}$
$\{(11,1), (1,4), (3,4)\} \cdot (h,1)$	$(2,0) \cdot (h,1)$	$h \in C_0^6$	
S_2	$A(S_2)$		R_2
$\{-, (9,3), (10,4)\} \cdot (h,1)$	$(2,0) \cdot (h,1)$	$h \in C_0^6$	$\{(18,0), (12,0), (8,0)\}$
$\{(3,3), (15,5), (8,5)\} \cdot (g,1)$	$(15,0) \cdot (g,1)$	$g \in C_0^3$	C_2
$\{(2,1), (4,2), (8,3)\} \cdot (g,1)$	$(1,0) \cdot (g,1)$	$g \in C_0^3$	$\{(15,1), (10,3), (13,5)\}$
$\{(11,1), (1,4), (3,4)\} \cdot (f,1)$	$(2,0) \cdot (f,1)$	$f \in C_3^6$	

$q = u - 1 = 31,\ \omega = 3$

S_1	$A(S_1)$		R_1
$\{-, (28,0), (17,1)\} \cdot (h,1)$	$(30,0) \cdot (h,1)$	$h \in C_0^{10}$	$\{(1,5), (25,5), (5,5)\}$
$\{(8,0), (3,2), (11,2)\} \cdot (g,1)$	$(12,0) \cdot (g,1)$	$g \in C_0^5$	C_1
$\{(11,1), (5,2), (4,3)\} \cdot (g,1)$	$(28,0) \cdot (g,1)$	$g \in C_0^5$	$\{(27,0), (24,2), (11,4)\}$
$\{(3,3), (1,4), (9,5)\} \cdot (g,1)$	$(9,0) \cdot (g,1)$	$g \in C_0^5$	
$\{(1,0), (2,0), (4,0)\} \cdot (g,1)$	$(4,0) \cdot (g,1)$	$g \in C_0^5$	
$\{(10,2), (2,4), (15,5)\} \cdot (h,1)$	$(5,0) \cdot (h,1)$	$h \in C_0^{10}$	
S_2	$A(S_2)$		R_2
$\{-, (28,0), (17,1)\} \cdot (f,1)$	$(30,0) \cdot (f,1)$	$f \in C_5^{10}$	$\{(26,5), (30,5), (6,5)\}$
$\{(1,1), (3,4), (10,5)\} \cdot (g,1)$	$(9,0) \cdot (g,1)$	$g \in C_0^5$	C_2
$\{(2,1), (9,3), (11,4)\} \cdot (g,1)$	$(11,0) \cdot (g,1)$	$g \in C_0^5$	$\{(20,1), (4,3), (7,5)\}$
$\{(3,1), (5,3), (14,4)\} \cdot (g,1)$	$(12,0) \cdot (g,1)$	$g \in C_0^5$	
$\{(8,2), (21,3), (11,5)\} \cdot (g,1)$	$(15,0) \cdot (g,1)$	$g \in C_0^5$	
$\{(10,2), (2,4), (15,5)\} \cdot (f,1)$	$(5,0) \cdot (f,1)$	$f \in C_5^{10}$	

□

Throughout the remainder of this section, all the constructions of FGDRP$(k, g^u)'$s follow from applying Theorem 2.2. In each case, we take the group G to be the additive group of $\mathbb{Z}_{u-w} \oplus \mathbb{Z}_g$. Then $G_0 = \{0\} \oplus \mathbb{Z}_g$ is the subgroup of order g in G. The fixed system of representatives of the cosets of G_0 is taken as $((0,0),(1,0),\cdots,(u-w-1,0))$. For ease of notation, we identify $\{\infty^s\} \times G_0$ with $\{\infty^s\} \times \mathbb{Z}_g$ for $1 \leq s \leq w$. When $w = 1$, we further abbreviate the notation (∞^1, x) to (∞, x).

Lemma 2.6 For any $u \in \{6, 8, 10, 12, 16, 18\}$, there exists an FGDRP$(3, 6^u)$.

Proof: For each stated value of u, apply Theorem 2.2 with $k = 3$, $t = 2$ and $w = 1$. The desired intransitive starters (S, R, C) and the corresponding adders are given in Appendix 1. $\square$

Lemma 2.7 For any $u \in \{22, 24, 28, 34\}$, there exists an FGDRP$(3, 6^u)$.

Proof: Since an FGDRP$(3, 6^5)$ exists by Lemma 2.3, we can employ Theorem 2.2 with $k = 3$, $t = 2$ and $w = 5$ to obtain an FGDRP$(3, 6^u)$ for each stated value of u. We take the required intransitive starter as $(S_1 \cup S_2, R_1 \cup R_2, C_1 \cup C_2)$ and the corresponding adder as $A_1 \cup A_2$. Here, S_1, R_1, C_1 and A_1 are indicated in the following tables. S_2, R_2, C_2 and A_2 are given by $S_2 = \{\widehat{B} : B \in S_1\}$, $R_2 = \{\widehat{B} : B \in R_1\}$, $C_2 = \{\widehat{B} : B \in C_1\}$ and $A_2 = -A_1$. For any $B = \{(x_1, x_2), (y_1, y_2), (z_1, z_2)\} \in S_1 \cup R_1 \cup C_1$, $\widehat{B}$ is defined as follows. When $u \in \{22, 34\}$,

$$\widehat{B} = \begin{cases} \{(-x_1, x_2), (-y_1, y_2), (-z_1, z_2)\}, & \text{if } B \in R_1 \text{ or } B \in S_1 \text{ and } x_1 \in \mathbb{Z}_{u-5}; \\ \{(-x_1, x_2+3), (-y_1, y_2+3), (-z_1, z_2+3)\}, & \text{if } B \in C_1; \\ \{(x_1, x_2-3), (-y_1, y_2), (-z_1, z_2)\}, & \text{if } B \in S_1, x_1 \notin \mathbb{Z}_{u-5}, y_2 \neq z_2; \\ \{(x_1, x_2-1), (-y_1, y_2), (-z_1, z_2)\}, & \text{if } B \in S_1, x_1 \notin \mathbb{Z}_{u-5}, y_2 = z_2. \end{cases}$$

When $u \in \{24, 28\}$,

$$\widehat{B} = \begin{cases} \{(-x_1, x_2), (-y_1, y_2), (-z_1, z_2)\}, & \text{if } B \in R_1 \text{ or } B \in S_1 \text{ and } x_1 \in \mathbb{Z}_{u-5}; \\ \{(-x_1, x_2+3), (-y_1, y_2+3), (-z_1, z_2+3)\}, & \text{if } B \in C_1; \\ \{(\infty^5, x_2), (-y_1, y_2), (-z_1, z_2)\}, & \text{if } B \in S_1, x_1 = \infty^4; \\ \{(\infty^4, x_2), (-y_1, y_2), (-z_1, z_2)\}, & \text{if } B \in S_1, x_1 = \infty^5; \\ \{(x_1, x_2-3), (-y_1, y_2), (-z_1, z_2)\}, & \text{if } B \in S_1, x_1 \in \{\infty^1, \infty^2, \infty^3\}. \end{cases}$$

$u = 22$

S_1	$\{(\infty^5,2), (1,3), (2,5)\}$	A_1	(1,0)	S_1	$\{(\infty^1,0), (15,4), (10,4)\}$	A_1	(15,0)
	$\{(\infty^2,0), (8,5), (6,3)\}$		(14,0)		$\{(\infty^2,4), (16,5), (1,2)\}$		(13,0)
	$\{(\infty^1,2), (14,2), (10,2)\}$		(12,0)		$\{(\infty^3,0), (3,4), (7,0)\}$		(11,0)
	$\{(\infty^4,0), (12,1), (1,4)\}$		(10,0)		$\{(\infty^3,4), (8,0), (12,3)\}$		(9,0)
	$\{(\infty^4,5), (5,5), (9,4)\}$		(8,0)		$\{(\infty^2,5), (3,5), (16,1)\}$		(7,0)
	$\{(\infty^1,4), (5,0), (13,0)\}$		(6,0)		$\{(\infty^4,4), (2,1), (1,0)\}$		(5,0)
	$\{(\infty^3,2), (11,5), (2,3)\}$		(4,0)		$\{(\infty^5,0), (5,4), (8,2)\}$		(3,0)
	$\{(\infty^5,4), (10,3), (6,4)\}$		(2,0)		$\{(13,4), (4,1), (14,1)\}$		(16,0)
R_1	$\{(8,1), (7,5), (13,5)\}$			C_1	$\{(5,4), (11,2), (2,3)\}$		
	$\{(9,3), (11,1), (6,0)\}$				$\{(3,1), (13,2), (1,0)\}$		
	$\{(3,0), (6,2), (13,3)\}$				$\{(16,1), (2,0), (8,2)\}$		
	$\{(3,3), (10,1), (15,0)\}$				$\{(16,3), (8,5), (10,4)\}$		
	$\{(2,2), (4,2), (5,2)\}$				$\{(4,3), (10,4), (11,5)\}$		

$u=34$

S_1	$\{(\infty^4,1), (17,4), (14,1)\}$	A_1	(1,0)	S_1	$\{(\infty^1,0), (13,0), (16,1)\}$	A_1	(15,0)
	$\{(\infty^1,1), (23,1), (1,0)\}$		(14,0)		$\{(\infty^2,0), (5,2), (12,0)\}$		(13,0)
	$\{(\infty^2,2), (13,2), (22,3)\}$		(12,0)		$\{(\infty^1,2), (13,4), (8,3)\}$		(11,0)
	$\{(\infty^3,0), (28,4), (21,1)\}$		(10,0)		$\{(\infty^3,2), (23,2), (10,1)\}$		(9,0)
	$\{(\infty^4,0), (16,5), (11,2)\}$		(8,0)		$\{(\infty^3,1), (23,4), (28,3)\}$		(7,0)
	$\{(\infty^5,0), (5,1), (1,1)\}$		(6,0)		$\{(\infty^2,1), (23,5), (19,3)\}$		(5,0)
	$\{(\infty^4,2), (7,0), (13,3)\}$		(4,0)		$\{(\infty^5,2), (1,5), (15,5)\}$		(3,0)
	$\{(\infty^5,4), (12,3), (5,3)\}$		(2,0)		$\{(4,3), (3,1), (22,1)\}$		(16,0)
	$\{(5,0), (20,3), (28,2)\}$		(28,0)		$\{(8,4), (27,5), (24,5)\}$		(27,0)
	$\{(3,2), (24,4), (8,0)\}$		(26,0)		$\{(25,2), (14,2), (4,0)\}$		(25,0)
	$\{(12,2), (27,4), (10,5)\}$		(24,0)		$\{(14,3), (27,1), (18,1)\}$		(23,0)
	$\{(27,3), (18,0), (6,3)\}$		(22,0)		$\{(21,2), (11,3), (20,2)\}$		(21,0)
	$\{(18,4), (6,0), (17,1)\}$		(20,0)		$\{(26,3), (7,5), (12,5)\}$		(19,0)
	$\{(14,0), (20,0), (25,4)\}$		(18,0)		$\{(11,5), (19,2), (9,5)\}$		(17,0)
R_1	$\{(21,5), (4,5), (3,0)\}$			C_1	$\{(15,3), (16,5), (18,4)\}$		
	$\{(2,0), (14,4), (25,1)\}$				$\{(27,4), (1,2), (26,3)\}$		
	$\{(2,2), (22,4), (9,1)\}$				$\{(9,4), (5,3), (7,2)\}$		
	$\{(19,4), (10,0), (3,4)\}$				$\{(13,1), (25,2), (10,3)\}$		
	$\{(3,5), (7,2), (9,4)\}$				$\{(4,1), (10,3), (25,2)\}$		

$u=24$

S_1	$\{(\infty^2,1), (6,2), (2,3)\}$	A_1	(1,0)	S_1	$\{(\infty^1,0), (4,4), (14,2)\}$	A_1	(15,0)
	$\{(\infty^1,4), (14,1), (10,4)\}$		(14,0)		$\{(\infty^2,0), (9,5), (4,0)\}$		(13,0)
	$\{(\infty^4,0), (16,3), (9,3)\}$		(12,0)		$\{(\infty^4,1), (2,0), (1,0)\}$		(11,0)
	$\{(\infty^4,5), (13,5), (2,5)\}$		(10,0)		$\{(\infty^3,0), (1,1), (8,4)\}$		(9,0)
	$\{(\infty^3,5), (4,1), (14,4)\}$		(8,0)		$\{(\infty^3,4), (3,1), (2,4)\}$		(7,0)
	$\{(\infty^5,2), (18,4), (13,3)\}$		(6,0)		$\{(\infty^2,5), (16,2), (18,3)\}$		(5,0)
	$\{(\infty^1,2), (15,3), (7,1)\}$		(4,0)		$\{(\infty^5,3), (1,5), (2,1)\}$		(3,0)
	$\{(\infty^5,4), (5,5), (18,2)\}$		(2,0)		$\{(10,2), (9,1), (8,0)\}$		(16,0)
	$\{(2,2), (15,2), (11,2)\}$		(18,0)		$\{(14,0), (7,4), (12,0)\}$		(17,0)
R_1	$\{(4,5), (12,3), (10,0)\}$			C_1	$\{(17,3), (11,2), (1,4)\}$		
	$\{(8,3), (13,0), (5,3)\}$				$\{(4,3), (16,5), (17,1)\}$		
	$\{(7,2), (11,1), (3,0)\}$				$\{(18,4), (11,5), (2,0)\}$		
	$\{(12,5), (6,1), (3,4)\}$				$\{(2,0), (18,4), (4,2)\}$		
	$\{(3,5), (6,4), (8,5)\}$				$\{(2,0), (9,5), (17,4)\}$		

$u = 28$

S_1	$\{(\infty^3,2), (4,0), (17,1)\}$	A_1	(1,0)	S_1	$\{(\infty^1,0), (14,3), (4,4)\}$	A_1	(15,0)
	$\{(\infty^2,5), (5,4), (9,5)\}$		(14,0)		$\{(\infty^1,0), (4,5), (9,1)\}$		(13,0)
	$\{(\infty^1,0), (8,0), (15,1)\}$		(12,0)		$\{(\infty^4,0), (15,3),(10,3)\}$		(11,0)
	$\{(\infty^2,5), (7,2), (15,5)\}$		(10,0)		$\{(\infty^2,4), (10,0), (18,1)\}$		(9,0)
	$\{(\infty^3,4), (9,2), (5,0)\}$		(8,0)		$\{(\infty^3,1), (22,3), (19,2)\}$		(7,0)
	$\{(\infty^5,3), (16,5), (8,4)\}$		(6,0)		$\{(\infty^4,2), (15,2), (18,2)\}$		(5,0)
	$\{(\infty^4,1), (2,5), (18,5)\}$		(4,0)		$\{(\infty^5,4), (7,3), (17,0)\}$		(3,0)
	$\{(\infty^5,5), (2,0), (10,4)\}$		(2,0)		$\{(21,2), (19,1), (17,2)\}$		(16,0)
	$\{(7,4), (2,3), (14,0)\}$		(22,0)		$\{(12,0), (11,5), (1,5)\}$		(21,0)
	$\{(3,4), (11,4), (17,4)\}$		(20,0)		$\{(5,3), (12,1), (17,3)\}$		(19,0)
	$\{(16,1), (9,4), (12,3)\}$		(18,0)		$\{(10,1), (3,2), (16,0)\}$		(17,0)
R_1	$\{(12,2), (1,1), (3,5)\}$			C_1	$\{(8,0), (22,4), (19,2)\}$		
	$\{(13,5), (19,3), (10,2)\}$				$\{(8,1), (2,0), (20,5)\}$		
	$\{(2,4), (6,5), (1,2)\}$				$\{(10,4), (16,2), (2,0)\}$		
	$\{(3,3), (22,0), (20,0)\}$				$\{(15,5), (17,1), (14,3)\}$		
	$\{(1,4), (2,1), (3,1)\}$				$\{(15,1), (5,3), (14,2)\}$		

$\square$

Lemma 2.8 If $u = 6$ or 32, then an FGDRP$(3, 9^u)$ exists.

Proof: Apply Theorem 2.2 with $k = 3$, $t = 3$ and $w = 1$. For $u = 6$, the desired intransitive starter (S, R, C) and the corresponding adder are as follows:

S	$\{(\infty,0), (4,6), (2,0)\}$	A	(4,0)	S	$\{(\infty,8), (3,3), (4,0)\}$	A	(4,0)
	$\{(\infty,6), (3,1), (4,4)\}$		(4,0)		$\{(\infty,5), (4,8), (3,2)\}$		(3,0)
	$\{(\infty,7), (1,6), (3,6)\}$		(3,0)		$\{(\infty,2), (3,7), (1,2)\}$		(3,0)
	$\{(\infty,3), (2,5), (4,5)\}$		(2,0)		$\{(\infty,1), (4,2), (2,8)\}$		(2,0)
	$\{(\infty,4), (1,0), (4,7)\}$		(2,0)		$\{(2,1), (1,1), (3,0)\}$		(1,0)
	$\{(2,2), (3,4), (1,7)\}$		(1,0)		$\{(1,3), (2,6), (3,5)\}$		(1,0)
R	$\{(1,5), (3,8), (2,7)\}$			C	$\{(4,0), (3,2), (2,4)\}$		
	$\{(1,4), (2,4), (4,3)\}$				$\{(1,5), (3,6), (2,1)\}$		
	$\{(1,8), (2,3), (4,1)\}$				$\{(1,7), (4,3), (2,8)\}$		

For $u = 32$, the starter (S, R, C) and the corresponding adder A are given by $S = \{S_1 \cdot (3^{10i}, 1) : i = 0, 1, 2\}$, $R = \{R_1 \cdot (3^{10i}, 1) : i = 0, 1, 2\}$, $C = \{C_1 \cdot (3^{10i}, 1) : i = 0, 1, 2\}$ and $A = \{A_1 \cdot (3^{10i}, 1) : i = 0, 1, 2\}$. We indicate S_1, R_1, C_1 and A_1 in the following table, where, for $v \in \{7, 8, 0\}$, $(\infty, v) \cdot (3^{10i}, 1)$ is defined to be $(\infty, v + 3i)$ $(i = 0, 1, 2)$ and the sum is calculated in $\mathbb{Z}_9$.

S_1	{(∞,0), (9,6), (7,7)}	A_1	(1,0)	S_1	{(∞,7), (5,3), (14,2)}	A_1	(3,0)
	{(∞,8), (13,8), (6,7)}		(2,0)		{(27,6), (6,5), (22,2)}		(4,0)
	{(2,8), (3,7), (16,7)}		(30,0)		{(15,6), (11,0), (30,2)}		(29,0)
	{(29,6), (15,2), (26,4)}		(28,0)		{(20,8), (12,7), (27,2)}		(27,0)
	{(9,4), (4,1), (22,4)}		(26,0)		{(15,0), (11,1), (26,0)}		(25,0)
	{(17,0), (18,8), (20,5)}		(24,0)		{((9,7), (1,1), (26,3)}		(23,0)
	{(9,5), (3,3), (17,1)}		(22,0)		{(1,6), (12,4), (25,2)}		(21,0)
	{(10,1), (28,3), (30,8)}		(20,0)		{(20,2), (13,5), (4,3)}		(19,0)
	{(7,6), (17,3), (18,6)}		(18,0)		{(18,2), (20,0), (30,1)}		(17,0)
	{(2,6), (29,3), (12,1)}		(16,0)		{(29,8), (18,0), (5,7)}		(15,0)
	{(28,5), (11,5), (23,7)}		(14,0)		{(14,0), (6,6), (24,3)}		(13,0)
	{(9,3), (14,8), (5,8)}		(12,0)		{(3,1), (10,3), (25,0)}		(11,0)
	{(2,5), (1,5), (9,1)}		(10,0)		{(19,0), (20,4), (21,0)}		(9,0)
	{(21,2), (12,5), (2,2)}		(8,0)		{(11,4), (24,8), (5,4)}		(7,0)
	{(15,4), (23,6), (16,1)}		(6,0)		{(19,7), (10,4), (22,8)}		(5,0)
R_1	{(11,7), (16,4), (17,5)}			C_1	{(19,0), (21,2), (24,1)}		

□

Lemma 2.9 For any $u \in \{16, 18, 20, 22, 24, 28, 32, 34\}$, there exists an FGDRP$(3, 3^u)$.

Proof: For these FGDRPs, we apply Theorem 2.2 with $k = 3$ and $t = 1$, where $w = 1$ or is chosen so that an FGDRP$(3, 3^w)$ exists. The required intransitive starters (S, R, C) and the corresponding adders are given in Appendix 2. Note that the property $(*)$ for the starters and adders shown in Appendix 2 are not required, since we are dealing with the case $g = k = 3$ (i.e., $t = 1$) there. For convenience, we abbreviate the infinite point $(\infty^i, (0, x)) \in \{\infty^i\} \times (\{0\} \oplus \mathbb{Z}_3)$ to ∞_{3i-2+x} for $1 \le i \le w$. □

3 The Spectrum of FGDRP$(3, g^u)'$s

In this section, we establish our main result. For this purpose, we describe some recursive methods. These constructions are the variations of standard techniques for the construction of resolvable designs (see, for example, [4, 6, 7]) and can be found in [8].

Construction 3.1 Suppose that there exists a K-GDD of type g^u. If for each $h \in K$ an FGDRP$(3, m^h)$ exists, then an FGDRP$(3, (mg)^u)$ also exists.

Construction 3.2 Suppose that an FGDRP$(3, g^u)$ and a TD$(5, n)$ exist. Then there exists an FGDRP$(3, (ng)^u)$.

Construction 3.3 Suppose that an FGDRP$(3, (sg)^u)$ and an FGDRP$(3, g^{s+1})$ both exist. Then there exists an FGDRP$(3, g^{su+1})$.

The following is an immediate corollary of Construction 3.1, since a $(v, K, 1)$-PBD is a K-GDD of type 1^v.

Construction 3.4 Suppose that there exist a $(v, K, 1)$-PBD and an FGDRP$(3, g^h)$ for each $h \in K$, then an FGDRP$(3, g^v)$ exists.

To apply the above constructions we will use the following known results.

Lemma 3.5 [5] For any integer $v \ge 5$ and $v \notin Q = \{10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 27, 28, 29, 32, 33, 34\}$, there exists a $(v, \{5, 6, 7, 8, 9\}, 1)$-PBD.

Lemma 3.6 [2] For any integer $n \geq 4$ and $n \neq 6, 10$, a $\mathrm{TD}(5, n)$ exists.

Lemma 3.7 [2] For any integer $n \geq 6$ and $n \notin \{6, 10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60, 62\}$, a $\mathrm{TD}(7, n)$ exists.

Now we are in a position to establish the spectrum of $\mathrm{FGDRP}(3, g^u)'$s.

Lemma 3.8 For any $u \geq 5$, there exists an $\mathrm{FGDRP}(3, 3^u)$.

Proof: The conclusion follows from Lemma 1.1 and Lemma 2.9. □

Lemma 3.9 For any integer $u \geq 5$ and $u \notin \{18, 28, 34\}$, an $\mathrm{FGDRP}(3, 9^u)$ exists.

Proof: For any stated value $u \notin \{18, 26, 28, 30, 34, 38, 39, 42, 44, 51, 52\}$, an FGDRP $(3, 9^u)$ was provided in Lemma 1.2 and Lemma 2.8. For these outstanding values u, a $(u, \{5, 6, 7, 8, 9\}, 1)$-PBD exists by Lemma 3.5 except for $u \in \{18, 28, 34\}$. Hence, we can apply Construction 3.4 with $g = 9$ to obtain the desired FGDRPs since an $\mathrm{FGDRP}(3, 9^t)$ exists for any $t \in \{5, 6, 7, 8, 9\}$ from Lemma 1.2 and Lemma 2.8. □

Lemma 3.10 There exists an $\mathrm{FGDRP}(3, 6^u)$ for any integer $u \geq 5$ and $u \neq 15$.

Proof: Let Q be the set defined in Lemma 3.5. Then a $(u, \{5, 6, 7, 8, 9\}, 1)$-PBD exists for any integer $u \geq 5$ and $u \notin Q$ by Lemma 3.5. So, for any stated value of $u \notin Q$, an $\mathrm{FGDRP}(3, 6^u)$ can be constructed by applying Construciton 3.4 with $g = 6$, since an $\mathrm{FGDRP}(3, 6^t)$ exists for any $t \in \{5, 6, 7, 8, 9\}$ from Lemma 2.3 and 2.6. For $u \in Q \setminus \{15, 33\}$, the desired FGDRPs have been constructed in Lemma 2.3 and Lemmas 2.5-2.7.

For $u = 33$, we first apply Construction 3.2 to an $\mathrm{FGDRP}(3, 6^8)$, making use of a $\mathrm{TD}(5, 4)$ as an ingredient. This produces an $\mathrm{FGDRP}(3, 24^8)$. Then add six new points to this FGDRP and apply Construction 3.3 with $g = 6$, $s = 4$ and $u = 8$ to obtain an $\mathrm{FGDRP}(3, 6^{33})$, as desired. □

Lemma 3.11 There exists an $\mathrm{FGDRP}(3, 18^u)$ for $u \geq 5$.

Proof: An $\mathrm{FGDRP}(3, 18^5)$ was provided in Lemma 2.4. From a $\mathrm{TD}(7, v)$ with $v \geq 7$, we can easily use the truncating groups technique to create a $\{6, 7, v\}$-GDD of type 6^{v+1}, a $\{6, v\}$-GDD of type 6^v and a $\{5, 6, v-1\}$-GDD of type 6^{v-1}. Hence, Construction 3.1 with the previous existence results guarantee that an $\mathrm{FGDRP}(3, 18^{v-1})$, an $\mathrm{FGDRP}(3, 18^v)$ and an $\mathrm{FGDRP}(3, 18^{v+1})$ all exist. Doing this for all integers v for which a $\mathrm{TD}(7, v)$ exists from Lemma 3.7 gives us the conclusion. □

We now reach the following theorem.

Theorem 3.12 Let g and u be positive integers with $g \equiv 0 \pmod 3$ and $u \geq 5$. Then an FGDRP$(3, g^u)$ exists with at most 5 possible exceptions of $(g, u) \in \{(6, 15), (9, 18), (9, 28), (9, 34), (30, 15)\}$.

Proof: Write $g = 3m$. From Lemmas 3.8-3.11 we need only to show the theorem for the cases where $m \notin \{1, 2, 3, 6\}$. To do this, we apply Construction 3.2 with $n = m$ to an FGDRP$(3, 3^u)$ to obtain an FGDRP$(3, (3m)^u)$. This covers all the values $g = 3m$ but $g = 30$, since a TD$(5, m)$ exists for any integer $m \geq 4$ and $m \notin \{6, 10\}$ by Lemma 3.6. Again apply Construction 3.2 with $n = 5$ to an FGDRP$(3, 6^u)$ to yield an FGDRP$(3, 30^u)$. Therefore, the conclusion holds. $\square$

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Appendix 1

The intransitive starters and adders for FGDRP$(3,6^u)'$s with $u \in \{6,8,10,12,16,18\}$, required in the proof of Lemma 2.6.

$u = 6$

S	{(∞,0), (1,5), (3,4)}	A	(3,0)	S	{(∞,5), (3,0), (1,0)}	A	(3,0)
	{(∞,3), (2,1), (4,5)}		(2,0)		{(∞,1), (4,4), (1,1)}		(2,0)
	{(∞,2), (1,2), (2,4)}		(1,0)		{(∞,4), (1,3), (3,1)}		(1,0)
	{(3,2), (2,5), (4,0)}		(4,0)		{(2,3), (4,3), (3,3)}		(4,0)
R	{(3,5), (2,0), (4,2)}			C	{(4,5), (3,4), (2,3)}		
	{(1,4), (2,2), (4,1)}				{(2,0), (4,1), (3,2)}		

$u = 8$

S	{(∞,0), (3,3), (2,1)}	A	(3,0)	S	{(∞,5), (5,1), (2,0)}	A	(3,0)
	{(∞,1), (2,5), (3,5)}		(2,0)		{(∞,2), (3,4), (6,5)}		(2,0)
	{(∞,4), (1,4), (4,3)}		(1,0)		{(∞,3), (2,3), (5,2)}		(1,0)
	{(3,0), (4,1), (2,2)}		(6,0)		{(3,1), (5,4), (4,5)}		(6,0)
	{(6,0), (3,2), (1,1)}		(5,0)		{(5,3), (6,3), (1,3)}		(5,0)
	{(4,2), (6,2), (5,5)}		(4,0)		{(4,4), (5,0), (1,2)}		(4,0)
R	{(1,5), (2,4), (6,1)}			C	{(3,1), (4,2), (6,4)}		
	{(1,0), (4,0), (6,4)}				{(2,3), (4,0), (6,5)}		

$u = 10$

S	{(∞,0), (3,1), (1,5)}	A	(3,0)	S	{(∞,5), (3,3), (1,1)}	A	(3,0)
	{(∞,3), (5,4), (6,0)}		(2,0)		{(∞,2), (6,2), (8,2)}		(2,0)
	{(∞,1), (3,5), (7,3)}		(1,0)		{(∞,4), (1,3), (6,1)}		(1,0)
	{(7,2), (5,1), (4,2)}		(8,0)		{(2,4), (6,3), (5,2)}		(8,0)
	{(8,5), (7,5), (4,4)}		(7,0)		{(5,3), (8,0), (7,1)}		(7,0)
	{(4,0), (1,0), (5,0)}		(6,0)		{(6,4), (8,1), (4,1)}		(6,0)
	{(5,5), (2,3), (7,4)}		(5,0)		{(8,4), (2,0), (3,2)}		(5,0)
	{(1,2), (8,3), (6,5)}		(4,0)		{(2,1), (3,4), (7,0)}		(4,0)
R	{(2,2), (4,3), (3,0)}			C	{(2,3), (5,0), (8,4)}		
	{(4,5), (2,5), (1,4)}				{(8,2), (2,1), (3,5)}		

$u = 12$

S	{(∞,0), (2,4), (3,5)}	A	(3,0)	S	{(∞,2), (4,3), (9,1)}	A	(3,0)
	{(∞,4), (8,0), (10,0)}		(2,0)		{(∞,1), (8,5), (10,1)}		(2,0)
	{(∞,5), (2,2), (7,5)}		(1,0)		{(∞,3), (7,0), (8,4)}		(1,0)
	{(4,1), (3,3), (10,2)}		(10,0)		{(4,4), (5,3), (3,4)}		(10,0)
	{(8,3), (4,5), (6,5)}		(9,0)		{(7,4), (5,5), (1,4)}		(9,0)
	{(6,2), (9,0), (1,2)}		(8,0)		{(5,0), (8,1), (10,4)}		(8,0)
	{(9,4), (8,2), (3,0)}		(7,0)		{(7,3), (10,3), (6,0)}		(7,0)
	{(6,4), (1,5), (9,2)}		(6,0)		{(1,0), (9,3), (2,5)}		(6,0)
	{(7,2), (3,2), (2,1)}		(5,0)		{(1,1), (5,4), (10,5)}		(5,0)
	{(4,2), (9,5), (6,3)}		(4,0)		{(5,2), (2,3), (4,0)}		(4,0)
R	{(6,1), (7,1), (3,1)}			C	{(9,4), (5,5), (7,3)}		
	{(5,1), (2,0), (1,3)}				{(9,1), (1,0), (5,2)}		

$u = 16$

S	{(∞,0), (9,0), (14,3)}	A	(3,0)	S	{(∞,5), (9,5), (10,2)}	A	(3,0)
	{(∞,4), (2,2), (12,5)}		(2,0)		{(∞,1), (6,5), (10,3)}		(2,0)
	{(∞,3), (3,2), (11,2)}		(1,0)		{(∞,2), (2,3), (3,4)}		(1,0)
	{(14,4), (7,1), (9,3)}		(14,0)		{(12,2), (4,1), (6,1)}		(14,0)
	{(5,0), (8,5), (12,1)}		(13,0)		{(13,3), (7,5), (3,5)}		(13,0)
	{(14,2), (11,1), (8,3)}		(12,0)		{(5,2), (13,4), (4,3)}		(12,0)
	{(13,5), (9,4), (5,4)}		(11,0)		{(11,0), (6,2), (12,0)}		(11,0)
	{(9,1), (3,0), (6,3)}		(10,0)		{(1,4), (14,0), (11,4)}		(10,0)
	{(1,0), (7,0), (8,2)}		(9,0)		{(13,1), (8,1), (10,0)}		(9,0)
	{(2,4), (13,0), (1,5)}		(8,0)		{(13,2), (6,4), (11,3)}		(8,0)
	{(7,4), (5,1), (4,4)}		(7,0)		{(7,3), (2,5), (5,5)}		(7,0)
	{(1,1), (14,5), (12,4)}		(6,0)		{(12,3), (8,4), (7,2)}		(6,0)
	{(3,1), (2,0), (8,0)}		(5,0)		{(5,3), (1,2), (4,5)}		(5,0)
	{(10,4), (3,3), (14,1)}		(4,0)		{(9,2), (6,0), (4,0)}		(4,0)
R	{(1,3), (2,1), (10,1)}			C	{(2,2), (11,5), (9,4)}		
	{(4,2), (10,5), (11,5)}				{(1,1), (5,0), (7,3)}		

$u = 18$

S	{(∞,0), (2,2), (15,5)}	A	(3,0)	S	{(∞,3), (2,0), (4,1)}	A	(3,0)
	{(∞,5), (2,3), (5,0)}		(2,0)		{(∞,2), (10,2), (16,3)}		(2,0)
	{(∞,4), (8,3), (12,5)}		(1,0)		{(∞,1), (6,0), (15,2)}		(1,0)
	{(12,1), (6,1), (9,3)}		(16,0)		{(13,4), (12,2), (7,4)}		(16,0)
	{(7,1), (9,0), (1,2)}		(15,0)		{(13,1), (3,0), (10,5)}		(15,0)
	{(5,3), (4,5), (13,0)}		(14,0)		{(6,4), (16,1), (1,1)}		(14,0)
	{(14,5), (2,4), (1,5)}		(13,0)		{(8,4), (7,2), (15,4)}		(13,0)
	{(11,0), (1,4), (13,3)}		(12,0)		{(14,1), (10,0), (6,2)}		(12,0)
	{(13,2), (1,0), (16,0)}		(11,0)		{(10,1), (11,2), (3,5)}		(11,0)
	{(6,5), (3,4), (8,0)}		(10,0)		{(11,1), (13,5), (10,4)}		(10,0)
	{(14,2), (12,0), (10,3)}		(9,0)		{(1,3), (6,3), (5,5)}		(9,0)
	{(12,3), (3,3), (4,3)}		(8,0)		{(5,4), (2,1), (16,2)}		(8,0)
	{(8,2), (2,5), (7,5)}		(7,0)		{(9,4), (12,4), (16,4)}		(7,0)
	{(9,5), (14,0), (15,1)}		(6,0)		{(3,2), (9,1), (4,2)}		(6,0)
	{(7,3), (9,2), (14,4)}		(5,0)		{(14,3), (3,1), (11,3)}		(5,0)
	{(5,2), (7,0), (16,5)}		(4,0)		{(4,4), (8,1), (15,3)}		(4,0)
R	{(4,0), (8,5), (11,4)}			C	{(6,1), (16,4), (8,3)}		
	{(5,1), (11,5), (15,0)}				{(14,5), (13,0), (9,2)}		

Appendix 2

The intransitive starters and adders for FGDRP$(3, 3^u)'$s with $u \in \{16, 18, 20, 22, 24, 28, 32, 34\}$, required in the proof of Lemma 2.9.

$u = 16,\ X = \mathbb{Z}_{15} \times \mathbb{Z}_3 \cup \{\infty_1, \infty_2, \infty_3\}$

S	{(8,1),(11,2),(9,2)}	A	(14,0)	S	{(12,1),(1,0),(11,0)}	A	(13,0)
	{(9,0),(2,0),(11,1)}		(12,0)		{(12,0),(14,0),(7,1)}		(11,0)
	{(6,2),(7,2),(9,1)}		(10,0)		{(8,2),(4,2),(13,2)}		(9,0)
	{(3,0),(12,2),(5,1)}		(8,0)		{(2,2),(5,2),(14,2)}		(7,0)
	{(5,0),(6,0),(10,1)}		(6,0)		{(6,1),(8,0),(13,1)}		(5,0)
	{(3,1),(10,0),(14,1)}		(4,0)		{∞_1,(1,1),(13,0)}		(3,0)
	{∞_2,(4,0),(10,2)}		(2,0)		{∞_3,(3,2),(4,1)}		(1,0)
R	{(2,1),(7,0),(1,2)}			C	{(9,0),(5,1),(2,2)}		

$u = 18,\ X = \mathbb{Z}_{17} \times \mathbb{Z}_3 \cup \{\infty_1, \infty_2, \infty_3\}$

S	{(4,2),(9,1),(5,2)}	A	(16,0)	S	{(6,0),(12,2),(13,2)}	A	(15,0)
	{(14,0),(2,2),(16,2)}		(14,0)		{(10,0),(6,2),(13,1)}		(13,0)
	{(6,1),(16,1),(14,1)}		(12,0)		{(8,0),(11,0),(3,2)}		(11,0)
	{(15,2),(11,1),(10,2)}		(10,0)		{(3,1),(4,0),(5,1)}		(9,0)
	{(15,0),(7,0),(1,2)}		(8,0)		{(7,2),(9,0),(15,1)}		(7,0)
	{(1,0),(9,2),(12,1)}		(6,0)		{(2,0),(11,2),(13,0)}		(5,0)
	{(3,0),(4,1),(8,1)}		(4,0)		{∞_1,(2,1),(7,1)}		(3,0)
	{∞_2,(8,2),(10,1)}		(2,0)		{∞_3,(1,1),(14,2)}		(1,0)
R	{(5,0),(16,0),(12,0)}			C	{(6,0),(13,1),(3,2)}		

$u = 20,\ X = \mathbb{Z}_{19} \times \mathbb{Z}_3 \cup \{\infty_1, \infty_2, \infty_3\}$

S	{(6,1),(18,1),(7,2)}	A	(18,0)	S	{(3,1),(11,2),(5,0)}	A	(17,0)
	{(16,0),(1,1),(6,2)}		(16,0)		{(6,0),(12,1),(15,0)}		(15,0)
	{(2,0),(16,2),(9,0)}		(14,0)		{(14,0),(1,0),(16,1)}		(13,0)
	{(12,0),(13,0),(14,2)}		(12,0)		{(2,1),(18,0),(10,0)}		(11,0)
	{(2,2),(14,1),(17,1)}		(10,0)		{(8,1),(9,1),(4,2)}		(9,0)
	{(10,1),(4,0),(8,0)}		(8,0)		{(3,2),(8,2),(13,1)}		(7,0)
	{(1,2),(15,2),(17,2)}		(6,0)		{(9,2),(15,1),(18,2)}		(5,0)
	{(5,1),(7,1),(11,1)}		(4,0)		{∞_1,(4,1),(13,2)}		(3,0)
	{∞_2,(7,0),(10,2)}		(2,0)		{∞_3,(5,2),(17,0)}		(1,0)
R	{(3,0),(11,0),(12,2)}			C	{(15,1),(3,2),(14,0)}		

$u = 22,\ X = \mathbb{Z}_{17} \times \mathbb{Z}_3 \cup \{\infty_i : 1 \leq i \leq 15\}$

S	{(12,2),(6,2),(10,1)}	A	(16,0)	S	{∞_1,(5,0),(15,0)}	A	(15,0)
	{∞_2,(15,1),(1,0)}		(14,0)		{∞_3,(9,2),(7,2)}		(13,0)
	{∞_4,(14,0),(2,2)}		(12,0)		{∞_5,(10,2),(12,1)}		(11,0)
	{∞_6,(5,1),(1,1)}		(10,0)		{∞_7,(3,0),(7,1)}		(9,0)
	{∞_8,(3,2),(16,2)}		(8,0)		{∞_9,(2,0),(11,1)}		(7,0)
	{∞_{10},(8,1),(16,1)}		(6,0)		{∞_{11},(5,2),(13,1)}		(5,0)
	{∞_{12},(2,1),(11,0)}		(4,0)		{∞_{13},(13,2),(16,0)}		(3,0)
	{∞_{14},(8,2),(11,2)}		(2,0)		{∞_{15},(1,2),(3,1)}		(1,0)
R	{(8,0),(6,0),(7,0)}			C	{(16,1),(6,0),(2,2)}		
	{(13,0),(14,2),(4,0)}				{(10,1),(8,0),(1,2)}		
	{(9,0),(12,0),(6,1)}				{(8,1),(7,2),(12,0)}		
	{(10,0),(4,2),(15,2)}				{(13,0),(7,1),(8,2)}		
	{(4,1),(14,1),(9,1)}				{(3,1),(14,0),(4,2)}		

$u = 24,\ X = \mathbb{Z}_{19} \times \mathbb{Z}_3 \cup \{\infty_i : 1 \leq i \leq 15\}$

S	{(4,1),(18,1),(12,2)}	A	(18,0)	S	{(1,2),(4,2),(16,2)}	A	(17,0)
	{(7,2),(8,1),(11,1)}		(16,0)		{∞_1,(11,0),(17,2)}		(15,0)
	{∞_2,(1,0),(13,2)}		(14,0)		{∞_3,(2,2),(5,1)}		(13,0)
	{∞_4,(12,1),(8,2)}		(12,0)		{∞_5,(14,2),(17,0)}		(11,0)
	{∞_6,(6,1),(2,1)}		(10,0)		{∞_7,(11,2),(14,1)}		(9,0)
	{∞_8,(8,0),(9,0)}		(8,0)		{∞_9,(3,0),(15,1)}		(7,0)
	{∞_{10},(6,2),(17,1)}		(6,0)		{∞_{11},(9,2),(10,2)}		(5,0)
	{∞_{12},(6,0),(9,1)}		(4,0)		{∞_{13},(7,1),(18,2)}		(3,0)
	{∞_{14},(3,2),(15,2)}		(2,0)		{∞_{15},(5,2),(13,1)}		(1,0)
R	{(5,0),(13,0),(7,0)}			C	{(6,0),(16,1),(1,2)}		
	{(18,0),(12,0),(10,0)}				{(13,2),(12,1),(3,0)}		
	{(16,0),(2,0),(3,1)}				{(7,2),(8,1),(2,0)}		
	{(15,0),(10,1),(1,1)}				{(18,1),(9,2),(11,0)}		
	{(16,1),(4,0),(14,0)}				{(11,2),(7,1),(9,0)}		

$u = 28,\ X = \mathbb{Z}_{22} \times \mathbb{Z}_3 \cup \{\infty_i : 1 \leq i \leq 18\}$

S	$\{(12,2),(15,2),(21,2)\}$	A	(21,0)	S	$\{(15,0),(16,0),(18,2)\}$	A	(20,0)
	$\{(11,1),(18,1),(19,0)\}$		(19,0)		$\{\infty_1,(7,1),(14,0)\}$		(18,0)
	$\{\infty_2,(16,1),(10,1)\}$		(17,0)		$\{\infty_3,(14,1),(2,1)\}$		(16,0)
	$\{\infty_4,(10,0),(3,1)\}$		(15,0)		$\{\infty_5,(4,1),(9,1)\}$		(14,0)
	$\{\infty_6,(21,1),(18,0)\}$		(13,0)		$\{\infty_7,(17,2),(20,0)\}$		(12,0)
	$\{\infty_8,(10,2),(8,1)\}$		(11,0)		$\{\infty_9,(13,1),(20,1)\}$		(10,0)
	$\{\infty_{10},(19,1),(17,1)\}$		(9,0)		$\{\infty_{11},(5,1),(11,0)\}$		(8,0)
	$\{\infty_{12},(3,2),(8,2)\}$		(7,0)		$\{\infty_{13},(7,0),(9,0)\}$		(6,0)
	$\{\infty_{14},(2,0),(21,0)\}$		(5,0)		$\{\infty_{15},(3,0),(15,1)\}$		(4,0)
	$\{\infty_{16},(6,2),(8,0)\}$		(3,0)		$\{\infty_{17},(7,2),(19,2)\}$		(2,0)
	$\{\infty_{18},(4,0),(5,0)\}$		(1,0)				
R	$\{(1,2),(13,0),(9,2)\}$			C	$\{(17,2),(16,1),(2,0)\}$		
	$\{(12,1),(1,0),(13,2)\}$				$\{(5,0),(20,2),(2,1)\}$		
	$\{(16,2),(20,2),(2,2)\}$				$\{(12,2),(6,1),(17,0)\}$		
	$\{(5,2),(14,2),(1,1)\}$				$\{(3,1),(20,0),(12,2)\}$		
	$\{(11,2),(17,0),(6,0)\}$				$\{(17,2),(21,1),(4,0)\}$		
	$\{(4,2),(6,1),(12,0)\}$				$\{(1,1),(2,0),(14,2)\}$		

$u = 32,\ X = Z_{25} \times Z_3 \cup \{\infty_i : 1 \leq i \leq 21\}$

S	$\{(13,2),(24,2),(6,0)\}$	A	(24,0)	S	$\{(1,1),(11,1),(20,2)\}$	A	(23,0)
	$\{(24,0),(16,0),(4,0)\}$		(22,0)		$\{\infty_1,(12,0),(7,0)\}$		(21,0)
	$\{\infty_2,(20,1),(2,0)\}$		(20,0)		$\{\infty_3,(2,2),(23,0)\}$		(19,0)
	$\{\infty_4,(6,1),(12,2)\}$		(18,0)		$\{\infty_5,(9,1),(21,0)\}$		(17,0)
	$\{\infty_6,(23,1),(10,2)\}$		(16,0)		$\{\infty_7,(6,2),(2,1)\}$		(15,0)
	$\{\infty_8,(3,1),(18,1)\}$		(14,0)		$\{\infty_9,(16,2),(22,2)\}$		(13,0)
	$\{\infty_{10},(7,2),(24,1)\}$		(12,0)		$\{\infty_{11},(21,2),(9,2)\}$		(11,0)
	$\{\infty_{12},(4,1),(22,1)\}$		(10,0)		$\{\infty_{13},(7,1),(11,2)\}$		(9,0)
	$\{\infty_{14},(15,0),(23,2)\}$		(8,0)		$\{\infty_{15},(4,2),(8,1)\}$		(7,0)
	$\{\infty_{16},(17,1),(18,2)\}$		(6,0)		$\{\infty_{17},(1,0),(5,0)\}$		(5,0)
	$\{\infty_{18},(10,0),(15,1)\}$		(4,0)		$\{\infty_{19},(1,2),(8,0)\}$		(3,0)
	$\{\infty_{20},(16,1),(17,2)\}$		(2,0)		$\{\infty_{21},(14,0),(15,2)\}$		(1,0)
R	$\{(19,0),(3,0),(22,0)\}$			C	$\{(18,1),(4,0),(9,2)\}$		
	$\{(10,1),(14,1),(13,1)\}$				$\{(8,1),(3,2),(6,0)\}$		
	$\{(20,0),(3,2),(5,2)\}$				$\{(8,0),(2,2),(5,1)\}$		
	$\{(11,0),(19,1),(21,1)\}$				$\{(10,2),(12,1),(22,0)\}$		
	$\{(12,1),(14,2),(5,1)\}$				$\{(13,1),(16,2),(2,0)\}$		
	$\{(18,0),(17,0),(9,0)\}$				$\{(3,1),(12,0),(2,2)\}$		
	$\{(19,2),(8,2),(13,0)\}$				$\{(22,2),(9,1),(20,0)\}$		

$u = 34,\ X = \mathbb{Z}_{26} \times \mathbb{Z}_3 \cup \{\infty_i : 1 \leq i \leq 24\}$

S	$\{(19,1),(2,1),(18,1)\}$	A	(25,0)	S	$\{\infty_1,(10,1),(24,1)\}$	A	(24,0)
	$\{\infty_2,(2,0),(6,2)\}$		(23,0)		$\{\infty_3,(21,1),(6,0)\}$		(22,0)
	$\{\infty_4,(19,0),(14,1)\}$		(21,0)		$\{\infty_5,(20,2),(19,2)\}$		(20,0)
	$\{\infty_6,(12,0),(5,0)\}$		(19,0)		$\{\infty_7,(23,2),(18,2)\}$		(18,0)
	$\{\infty_8,(21,2),(20,1)\}$		(17,0)		$\{\infty_9,(22,1),(1,1)\}$		(16,0)
	$\{\infty_{10},(15,0),(7,1)\}$		(15,0)		$\{\infty_{11},(16,1),(8,2)\}$		(14,0)
	$\{\infty_{12},(6,1),(10,0)\}$		(13,0)		$\{\infty_{13},(15,1),(13,1)\}$		(12,0)
	$\{\infty_{14},(5,1),(14,2)\}$		(11,0)		$\{\infty_{15},(23,0),(11,1)\}$		(10,0)
	$\{\infty_{16},(10,2),(25,1)\}$		(9,0)		$\{\infty_{17},(3,1),(16,0)\}$		(8,0)
	$\{\infty_{18},(9,2),(11,2)\}$		(7,0)		$\{\infty_{19},(17,1),(25,2)\}$		(6,0)
	$\{\infty_{20},(5,2),(14,0)\}$		(5,0)		$\{\infty_{21},(3,2),(16,2)\}$		(4,0)
	$\{\infty_{22},(8,0),(17,2)\}$		(3,0)		$\{\infty_{23},(13,0),(25,0)\}$		(2,0)
	$\{\infty_{24},(15,2),(17,0)\}$		(1,0)				
R	$\{(24,2),(4,1),(2,2)\}$			C	$\{(20,2),(7,1),(13,0)\}$		
	$\{(21,0),(3,0),(9,1)\}$				$\{(13,2),(3,1),(10,0)\}$		
	$\{(22,0),(4,0),(11,0)\}$				$\{(9,1),(23,0),(21,2)\}$		
	$\{(24,0),(7,0),(18,0)\}$				$\{(6,2),(21,1),(14,0)\}$		
	$\{(22,2),(12,2),(1,0)\}$				$\{(9,0),(6,1),(4,2)\}$		
	$\{(4,2),(1,2),(7,2)\}$				$\{(8,0),(2,2),(12,1)\}$		
	$\{(20,0),(23,1),(13,2)\}$				$\{(6,0),(5,1),(15,2)\}$		
	$\{(9,0),(8,1),(12,1)\}$				$\{(3,1),(2,0),(24,2)\}$		