# Chromatic number for a generalization of Cartesian product graphs

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**Abstract**

Let $\mathcal{G}$ be a class of graphs. A *$d$-fold grid over $\mathcal{G}$* is a graph obtained from a $d$-dimensional rectangular grid of vertices by placing a graph from $\mathcal{G}$ on each of the lines parallel to one of the axes. Thus each vertex belongs to $d$ of these subgraphs. The class of $d$-fold grids over $\mathcal{G}$ is denoted by $\mathcal{G}^d$.

Let $f(\mathcal{G}; d) = \max_{G\in\mathcal{G}^d} \chi(G)$. If each graph in $\mathcal{G}$ is $k$-colorable, then $f(\mathcal{G}; d) \le k^d$. We show that this bound is best possible by proving that $f(\mathcal{G}; d) = k^d$ when $\mathcal{G}$ is the class of all $k$-colorable graphs. We also show that $f(\mathcal{G}; d) \ge \left\lfloor\sqrt{\frac{d}{6\log d}}\right\rfloor$ when $\mathcal{G}$ is the class of graphs with at most one edge, and $f(\mathcal{G}; d) \ge \left\lfloor\frac{d}{6\log d}\right\rfloor$ when $\mathcal{G}$ is the class of graphs with maximum degree 1.

## 1 Introduction

The *Cartesian product* of graphs $G_1, \ldots, G_d$ is the graph with vertex set $V(G_1) \times \cdots \times V(G_d)$ in which two vertices $(v_1, \ldots, v_d)$ and $(v'_1, \ldots, v'_d)$ are adjacent if they agree in all but one coordinate, and in the coordinate where they differ the values are adjacent

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vertices in the corresponding graph. The product can be viewed as a rectangular grid with copies of $G_1, \dots, G_d$ placed on vertices forming lines parallel to the $d$ axes. It is well-known (and easy to show) that the chromatic number of the Cartesian product of $G_1, \dots, G_d$ is the maximum of the chromatic numbers of $G_1, \dots, G_d$ [12].

In this paper, we consider bounds on the chromatic number of graphs in a family resulting from a more general graph operation. Instead of placing copies of the same graph $G_i$ on all the lines parallel to the $i$-th axis, we may place different graphs from a fixed class. Let $[n]$ denote $\{1, \dots, n\}$. For a class $\mathcal{G}$ of graphs, a *$d$-fold grid over $\mathcal{G}$* is a graph with vertex set $[n_1] \times \cdots \times [n_d]$ such that each set of vertices where all but one coordinate is fixed induce a graph from $\mathcal{G}$. For example, a Cartesian product of graphs in $\mathcal{G}$ is a $d$-fold grid over $\mathcal{G}$. The family of all $d$-fold grids over $\mathcal{G}$ is denoted by $\mathcal{G}^d$.

The study of the chromatic number and independence number of graphs in $\mathcal{G}^d$ is related to similar problems appearing in computational geometry. Frequency assignment problems for transmitters in the plane are modeled by coloring and independence problems on certain graphs (see [5]). These graphs arise from sets of points using the Euclidean metric. Analogous problems for the Manhattan metric were addressed in [3]. Since $d$-fold grids over some classes of graphs can be represented by graphs appearing in this setting, Szegedy [13] posed the following open problem at the workshop "Combinatorial Challenges":

> *What is the maximum chromatic number of a graph $G \in \mathcal{G}^d$ when $\mathcal{G}$ is the class $\mathcal{B}$ of all bipartite graphs or the class $\mathcal{S}$ of graphs containing at most one edge?*

If each graph in $\mathcal{G}$ is $k$-colorable, then every graph in $\mathcal{G}^d$ has chromatic number at most $k^d$, since it is the union of $d$ subgraphs, each of which is $k$-colorable. In particular, all graphs in $\mathcal{B}^d$ are $2^d$-colorable. We show that this bound is sharp, which is somewhat surprising since Cartesian products of bipartite graphs are bipartite. More generally, let $f(\mathcal{G}; d) = \max_{G \in \mathcal{G}^d} \chi(G)$. We show that if $\mathcal{G}$ is the class of all $k$-colorable graphs, then $f(\mathcal{G}; d) = k^d$. We prove the existence of $k^d$-chromatic graphs in $\mathcal{G}^d$ probabilistically, but an explicit construction can then be obtained by building, for each $n$, a graph in $\mathcal{G}^d$ that is "universal" in the sense that it contains all graphs in $\mathcal{G}^d$ with vertex set $[n]^d$. This settles the first part of Szegedy's question.

Determining $f(\mathcal{S}; d)$ is more challenging. Since the maximum degree of a graph in $\mathcal{S}^d$ does not exceed $d$, and these graphs do not contain $K_{d+1}$, Brooks' Theorem [2] implies that each graph in $\mathcal{S}^d$ is $d$-colorable (when $d \geq 3$). Also graphs in $\mathcal{S}^2$ are bipartite, since cycles in such a graph alternate between horizontal and vertical edges. In general, graphs in $\mathcal{S}^d$ are triangle-free, since any two adjacent vertices differ in exactly one coordinate.

When $d$ is large, we can use a refinement of Brook's Theorem obtained by Reed et al. [6, 9, 10, 11] to improve the upper bound.

**Theorem 1** (Molloy and Reed [10]). *There exists a constant $D_0$ such that if $D \geq D_0$ and $k^2 + 2k < D$, then every graph $G$ with maximum degree $D$ and $\chi(G) > D - k$ has a subgraph $H$ with at most $D + 1$ vertices and $\chi(H) > D - k$.*

Theorem 1 implies that $f(\mathcal{S}; d) \leq d - \sqrt{d} + O(1)$. To see this, observe that any $(d+1)$-vertex subgraph $H$ of a $d$-fold grid over $\mathcal{S}$ has chromatic number at most $d/2 + 1$. If $H$ has no vertex with degree at least $d/2 + 1/2$, then $H$ is $(d/2 + 1)$-colorable. If $H$ has a vertex with degree at least $d/2 + 1$, then its neighbors form an independent set $A$; since $H - A$ has at most $d/2$ vertices, the graph $H$ has a proper coloring with $d/2 + 1$ colors.

A still better upper bound follows from another result.

**Theorem 2** (Johansson [8]). *The chromatic number of a triangle-free graph with maximum degree $D$ is at most $O(D/\log D)$.*

This result, which was further strengthened by Alon et al. [1], implies that $f(\mathcal{S}; d) \in O(d/\log d)$, since the neighborhood of every vertex of a $d$-fold grid over $\mathcal{S}$ is independent.

We show that though graphs in $\mathcal{S}^d$ are very sparse, and it is natural to expect that they can be colored properly using just a few colors, $f(\mathcal{S}; d) \geq \left\lfloor \sqrt{\frac{d}{6 \log d}} \right\rfloor$. Our argument is again probabilistic. A similar argument yields $f(\mathcal{M}; d) \geq \left\lfloor \frac{d}{6 \log d} \right\rfloor$, where $\mathcal{M}$ is the class of all matchings (i.e., graphs with maximum degree 1). This lower bound is asymptotically best possible, since the discussion above yields $f(\mathcal{M}; d) \in O(d/\log d)$.

## 2 Preliminaries

In this section, we make several observations used in the proofs of our subsequent lower bounds on $f(\mathcal{G}; d)$ for various $\mathcal{G}$. We start by recalling the Chernoff Bound, an upper bound on the probability that a sum of independent random variables deviates greatly from its expected value (see [7] for more details).

**Proposition 3.** *If $X$ is a random variable equal to the sum of $N$ independent identically distributed $0, 1$-random variables having probability $p$ of taking the value $1$, then the following holds for every $0 < \delta \leq 1$:*

$$\mathrm{Prob}(X \geq (1+\delta)pN) \leq e^{-\frac{\delta^2 pN}{3}} \quad \textit{and} \quad \mathrm{Prob}(X \leq (1-\delta)pN) \leq e^{-\frac{\delta^2 pN}{2}}.$$

Next, we establish two technical claims. We begin with a standard bound on the number of subsets of a certain size.

**Proposition 4.** *For $\ell, N \in \mathbb{N}$ with $\ell > 2$ and $N$ a multiple of $\ell$, the number of $N/\ell$-element subsets of an $N$-element set is at most $2^{\frac{N}{\ell}(1+\log \ell)}$.*

*Proof.* An $N$-element set has $\binom{N}{N/\ell}$ subsets of size $N/\ell$. It is well known that $\binom{N}{N/\ell} \leq 2^{N \cdot H(1/\ell)}$, where $H(p) = -p \log p - (1-p)\log(1-p)$ (see [4], for example). A simple calculation yields the upper bound:

$$\binom{N}{N/\ell} \leq 2^{N\cdot(\frac{1}{\ell}\log \ell + \frac{\ell-1}{\ell}\log\frac{\ell}{\ell-1})} \leq 2^{N\cdot(\frac{1}{\ell}\log \ell + \frac{\ell-1}{\ell}\cdot\frac{1}{\ell-1})} \leq 2^{\frac{N}{\ell}(1+\log \ell)}$$

□

The second claim is a straightforward upper bound on a certain type of product of expressions of the form $(1-\varepsilon)$:

**Proposition 5.** *If $a_1, \ldots, a_m$ are nonnegative integers with sum $n$, then*

$$\prod_{i=1}^{m}\left(1 - \binom{a_i}{2}\frac{1}{x}\right) \leq \left(1 - \frac{1}{x}\right)^{n-m}$$

*for any positive real $x$ such that $x \leq \binom{\max_i a_i}{2}$.*

*Proof.* Since $\sum a_i = n$, it suffices to show that

$$\left(1 - \binom{a}{2}\frac{1}{x}\right) \leq \left(1 - \frac{1}{x}\right)^{a-1} \tag{1}$$

for every nonnegative integer $a$. If $a \leq 1$, then the left side of (1) is 1 and the right side is at least 1. If $a \geq 2$, then (1) follows (by setting $k = a-1$) from the well-known inequality

$$1 - \frac{k}{x} \leq \left(1 - \frac{1}{x}\right)^{k},$$

which holds whenever $0 \leq k \leq x$. □

# 3 Grids over $k$-colorable graphs

In this section, we prove that $f(\mathcal{B}; d) = 2^d$. Note again that after the probabilistic proof of existence, we can construct such graphs explicitly as explained in Section 1. Even so, the argument that they are not $(2^d-1)$-colorable remains probabilistic. Theorem 6 can also be proved by bounding the probability that a random $(2^d-1)$-coloring of some $d$-fold grid over the class of complete bipartite graphs is proper, but we prefer giving a proof via a bound on the size of the largest independent set, since such a bound may be of independent interest.

We prove the result in the more general setting of $k$-colorable graphs.

**Theorem 6.** *For $d, k \in \mathbf{N}$, there exists a $d$-fold grid $G$ over the class of $k$-colorable graphs such that $\chi(G) = k^d$.*

*Proof.* The claim holds trivially if $d = 1$, so we assume $d \geq 2$. The integers $k$ and $d$ are fixed, and $N$ is a large integer to be chosen in terms of $k$ and $d$. Consider the set $[N]^d$. For each $v \in [N]^d$, define a random $d$-tuple $X(v)$ such that $X(v)_i$ takes each value in $[k]$ with probability $1/k$, and the $d$ coordinate variables are independent. Generate a graph $G$ with vertex set $[N]^d$ by making two vertices $u$ and $v$ adjacent if they differ in exactly one coordinate and $X(u)_\ell \neq X(v)_\ell$, where $\ell$ is the coordinate in which $u$ and $v$ differ.

By construction, any set of vertices in $G$ that all agree outside a fixed coordinate induce a complete multipartite graph with at most $k$ parts. Hence $G$ is a $d$-fold grid over the class of $k$-colorable graphs. It will suffice to show that almost surely (as $N$ tends to infinity) $G$ does not have an independent set with more than $\frac{N^d}{k^d - 0.5}$ vertices. This yields $\chi(G) \geq k^d$, since otherwise some color class is an independent set of size at least $\frac{N^d}{k^d - 1}$.

For an independent set $A$ in $G$, let the *shade* of $A$ be the function $\sigma\colon\ [d] \times [N]^{d-1} \to [k]$ defined as follows. For $z = (\ell; i_1, \ldots, i_{\ell-1}, i_{\ell+1}, \ldots, i_d) \in [d] \times [N]^{d-1}$, consider the vertices in $A$ of the form $(i_1, \ldots, i_{\ell-1}, j, i_{\ell+1}, \ldots, i_d)$. By the construction of $G$, the value of $X(v)_\ell$ is the same for each such vertex $v$, since vertices of $A$ are nonadjacent. Let this value be $\sigma(z)$. If there is no vertex of $A$ with this form, then let $\sigma(z) = 1$.

The union of independent sets with the same shade is an independent set. Hence for each function $\sigma$ there is a unique maximal independent set in $G$ with shade $\sigma$; denote it by $A_\sigma$. To have $v \in A_\sigma$, where $v = (i_1, \ldots, i_d)$, the random variables $X(v)_1, \ldots, X(v)_d$ must satisfy $X(v)_\ell = \sigma(\ell; i_1, \ldots, i_{\ell-1}, i_{\ell+1}, \ldots, i_d)$. Hence each $v$ lies in $A_\sigma$ with probability $k^{-d}$.

As a result, the expected size of $A_\sigma$ is $(N/k)^d$. Since the variables $X(v)_\ell$ are independent for all $v$ and $\ell$, we can bound the probability that $|A_\sigma| \geq \frac{N^d}{k^d - 0.5}$ using the Chernoff Bound (Proposition 3). Applied with $\delta = \frac{1}{2k^d - 1}$, this yields

$$\text{Prob}\left(|A_\sigma| \geq \frac{N^d}{k^d - 0.5}\right) \leq e^{-\frac{N^d}{3(2k^d-1)^2 k^d}} \leq e^{-\frac{N^d}{12k^{3d}}}.$$

Since there are $k^{dN^{d-1}}$ possible shades, the probability that some independent set has more than $\frac{N^d}{k^d - 0.5}$ vertices is at most $k^{dN^{d-1}} \cdot e^{-N^d/12k^{3d}}$, and we compute

$$k^{dN^{d-1}} \cdot e^{-N^d/12k^{3d}} = e^{\log k dN^{d-1} - N^d/12k^{3d}} \to 0\ .$$

If $N$ is sufficiently large in terms of $k$ and $d$, then the bound is less than 1, and there exists such a graph $G$ with no independent set of size at least $\frac{N^d}{k^d - 0.5}$. $\square$

# 4 Grids over single-edge graphs

In this section, we prove the lower bound for $d$-fold grids over the class of graphs with at most one edge.

**Theorem 7.** *For $d \geq 2$, there exists $G \in \mathcal{S}^d$ such that $\chi(G) \geq \left\lfloor \sqrt{\frac{d}{6 \log d}} \right\rfloor$.*

*Proof.* Let $k = \left\lfloor \sqrt{\frac{d}{6 \log d}} \right\rfloor$. For $k \leq 2$, the conclusion is immediate. Hence, we assume $k \geq 3$. We generate a graph $G$ with vertex set $[2k]^d$. For $(\ell; i_1, \ldots, i_{\ell-1}, i_{\ell+1}, \ldots, i_d) \in [d] \times [2k]^{d-1}$, choose a random pair of distinct integers $j$ and $j'$ from $[2k]$, and make the vertices $(i_1, \ldots, i_{\ell-1}, j, i_{\ell+1}, \ldots, i_d)$ and $(i_1, \ldots, i_{\ell-1}, j', i_{\ell+1}, \ldots, i_d)$ adjacent in $G$. The choices of $\{j, j'\}$ are independent for all elements of $[d] \times [2k]^{d-1}$.

Since $G \in \mathcal{S}^d$, its chromatic number is at most $d$. To show that the event $\chi(G) \geq k$ has positive probability, it suffices to show that with positive probability, $G$ has no independent set of size at least $(2k)^d/k$.

Consider a set $A$ in $V(G)$ with size $(2k)^d/k$; we bound the probability that $A$ is an independent set in $G$. Again we think of an element $z$ in $[d] \times [2k]^{d-1}$ as designating a line in $[2k]^d$ parallel to some axis. Let $A[z]$ be the intersection of $A$ with this line. By the construction of $G$, the probability that no two vertices in $A[z]$ are adjacent in $G$ is

$$1 - \binom{|A[z]|}{2} \Big/ \binom{2k}{2},$$

which is at most $1 - \binom{|A[z]|}{2} \frac{1}{2k^2}$. By applying Proposition 5 with $x = 1/2k^2$, $n = |A| \geq \frac{(2k)^d}{k} = 2(2k)^{d-1}$, and $m = (2k)^{d-1}$, we conclude that the probability of all subsets of $A$ lying along lines in a particular direction being independent in $G$ is at most

$$\prod_{z \in \{\ell\} \times [2k]^{d-1}} \left(1 - \binom{|A[z]|}{2} \frac{1}{2k^2}\right) \leq \left(1 - \frac{1}{2k^2}\right)^{(2k)^{d-1}}.$$

Let $p$ be the probability that $A$ is an independent set in $G$. Since the edges in each of the $d$ directions are added to $G$ independently,

$$p \leq \left(1 - \frac{1}{2k^2}\right)^{d(2k)^{d-1}} \leq e^{-\frac{d(2k)^{d-1}}{2k^2}} \leq e^{-2d(2k)^{d-3}} \leq 2^{-2d(2k)^{d-3}}.$$

We want to show that with positive probability, $G$ has no independent set of size $(2k)^d/k$. Let $M$ be the number of subsets of $V(G)$ with size $(2k)^d/k$. By Proposition 4,

$$M \leq 2^{\frac{(2k)^d}{k} \cdot (1 + \log k)} \leq 2^{2(2k)^{d-1} \cdot (1 + \log k)} \leq 2^{3(2k)^{d-1} \log k}.$$

Therefore, we bound the probability that $G$ has an independent set of size $(2k)^d/k$ by the following computation:

$$2^{3(2k)^{d-1}\log k} \cdot 2^{-2d(2k)^{d-3}} = 2^{2(2k)^{d-3}(6k^2\log k - d)} < 1 .$$

The last inequality uses the fact that $6k^2 \log k - d$ is negative, by the choice of $k$. We conclude that some such graph $G$ has no independent set of size at least $(2k)^d/k$. $\square$

# 5 Grids over matchings

Finally, we consider $d$-fold grids over the class $\mathcal{M}$ of matchings.

**Theorem 8.** *For $d \geq 2$, there exists $G \in \mathcal{M}^d$ such that $\chi(G) \geq \left\lfloor \frac{d}{6\log d} \right\rfloor$.*

*Proof.* As the proof is similar to the proof of Theorem 7, we will give less detail and focus on the differences between the proofs. Set $k = \left\lfloor \frac{d}{6\log d} \right\rfloor$ and assume $k \geq 3$. We randomly generate a graph $G$ with vertex set $[2k]^d$. As before an element $z$ in $[d] \times [2k]^{d-1}$ designates a line in $[2k]^d$ parallel to some axis. We place a random perfect matching on the $2k$ vertices in each such line. Hence, the resulting graph $G$ is $d$-regular. It suffices to show that with positive probability, $G$ has no independent set of size at least $(2k)^d/k$.

Consider a set $A$ in $V(G)$ with size $(2k)^d/k$; we bound the probability that $A$ is an independent set in $G$. Let $A[z]$ be the intersection of $A$ with a line designated by $z \in [d] \times [2k]^{d-1}$. Let $X$ be the random variable that is the number of edges in $G$ induced by $A[z]$. By the construction of $G$, we have $\mathrm{E}(X) = \frac{1}{2k-1}\binom{|A[z]|}{2}$. When $X$ is a nonnegative integer-valued random variable, $\mathrm{Prob}[X \geq 1] \geq \frac{\mathrm{E}(X)}{\max(X)}$. Since $A[z]$ cannot induce more than $|A[z]|/2$ edges, we obtain a lower bound on the probability $p$ that $A[z]$ contains an edge by computing

$$p \geq \frac{\frac{1}{2k-1}\binom{|A[z]|}{2}}{|A[z]|/2} = \frac{|A[z]|-1}{2k-1} \geq \frac{|A[z]|-1}{2k}.$$

Let $q_\ell$ denote the probability that all subsets of $A$ lying along lines in direction $\ell$ are independent (each such line consists of $d$-tuples that agree outside the $\ell$th coordinate). We compute

$$q_\ell \leq \prod_{z \in \{\ell\}\times[2k]^{d-1}} \left(1 - \frac{|A[z]|-1}{2k}\right) \leq \prod_{z \in \{\ell\}\times[2k]^{d-1}} e^{-\frac{|A[z]|-1}{2k}}$$

$$= e^{-\frac{(2k)^d/k-(2k)^{d-1}}{2k}} = e^{-(2k)^{d-2}}.$$

The probability $P$ that $A$ is independent can now be bounded as follows:

$$P \leq e^{-d(2k)^{d-2}} \leq 2^{-d(2k)^{d-2}} .$$

Finally, an upper bound on the probability that $G$ has an independent set of size $(2k)^d/k$ is obtained by multiplying the bound on $P$ and the bound on the number of $(2k)^d/k$-element subsets of vertices from Proposition 4.

$$2^{3(2k)^{d-1}\log k} \cdot 2^{-d(2k)^{d-2}} = 2^{(2k)^{d-2}(6k\log k - d)} < 1 \ .$$

$\square$

# 6 Open problem

We determined assymptotically the function $f(\mathcal{G};d)$ for the class $\mathcal{G}$ of $k$-colorable graphs and the class $\mathcal{M}$ of matchings. For the class $\mathcal{S}$ of graphs with at most one edge, we were not able to obtain matching lower and upper bounds. Our results imply only that

$$\left\lfloor \sqrt{\frac{d}{6\log d}} \right\rfloor \leq f(\mathcal{M};d) \leq O\left(\frac{d}{\log d}\right) \ .$$

Hence, it remains open to determine the assymptotic behavior of the function $f(\mathcal{M};d)$ in terms of $d$.

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