SOME NATURAL BIGRADED S_n-MODULES
and
q,t-KOSTKA COEFFICIENTS.

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This work is gratefully dedicated to Dominique Foata for his inspiring and pioneering work in algebraic combinatorics. We hope that he will find it to be in harmony with the Lotharingian spirit which he has nurtured for so many years.

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ABSTRACT. We construct for each $\mu \vdash n$ a bigraded S_n-module $\mathbf{H}_\mu$ and conjecture that its Frobenius characteristic $C_\mu(x;q,t)$ yields the Macdonald coefficients $K_{\lambda\mu}(q,t)$. To be precise, we conjecture that the expansion of $C_\mu(x;q,t)$ in terms of the Schur basis yields coefficients $C_{\lambda\mu}(q,t)$ which are related to the $K_{\lambda\mu}(q,t)$ by the identity $C_{\lambda\mu}(q,t) = K_{\lambda\mu}(q,1/t)t^{n(\mu)}$. The validity of this would give a representation theoretical setting for the Macdonald basis $\{P_\mu(x;q,t)\}_\mu$ and establish the Macdonald conjecture that the $K_{\lambda\mu}(q,t)$ are polynomials with positive integer coefficients. The space $\mathbf{H}_\mu$ is defined as the linear span of derivatives of a certain bihomogeneous polynomial $\Delta_\mu(x,y)$ in the variables $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$. On the validity of our conjecture $\mathbf{H}_\mu$ would necessarily have $n!$ dimension. We refer to the latter assertion as the $n!$-conjecture. Several equivalent forms of this conjecture will be discussed here together with some of their consequences. In particular, we derive that the polynomials $C_{\lambda\mu}(q,t)$ have a number of basic properties in common with the coefficients $\tilde{K}_{\lambda\mu}(q,t) = K_{\lambda\mu}(q,1/t)t^{n(\mu)}$. For instance, we show that $C_{\lambda\mu}(0,t) = \tilde{K}_{\lambda\mu}(0,t)$, $C_{\lambda\mu}(q,0) = \tilde{K}_{\lambda\mu}(q,0)$ and show that on the $n!$ conjecture we must also have the equalities $C_{\lambda\mu}(1,t) = \tilde{K}_{\lambda\mu}(1,t)$ and $C_{\lambda\mu}(q,1) = \tilde{K}_{\lambda\mu}(q,1)$. The conjectured equality $C_{\lambda\mu}(q,t) = K_{\lambda\mu}(q,1/t)t^{n(\mu)}$ will be shown here to hold true when λ or μ is a hook. It has also been shown (see [9]) when μ is a 2-row or 2-column partition and in [18] when μ is an augmented hook.

Introduction

Throughout this writing μ will be a partition of n and μ' will denote its conjugate. We shall also identify μ with its Ferrers' diagram. As customary, for $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0)$, we let

$$n(\mu) \;=\; \sum_{i=1}^{k} (i-1)\,\mu_i \;. \qquad I.1$$

We draw μ as a set of successive left justified rows of lattice squares by the French convention, with row lengths decreasing from bottom to top. We also let the coordinates (i,j) of a cell $s \in \mu$ respectively measure the height of s and the position of s in its row. We shall set

$$B_\mu(q,t) = \sum_{(i,j)\in\mu} t^{i-1}q^{j-1} \;. \qquad I.2$$

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The pairs $(i-1,j-1)$ occurring in this sum will be briefly referred to as the *biexponents* of μ and $B_\mu(q,t)$ itself will be called the *biexponent generator* of μ. Now let $(p_1,q_1),\ldots,(p_n,q_n)$ denote the set of biexponents arranged in lexicographic order and set

$$\Delta_\mu(x,y) \;=\; \Delta_\mu(x_1,\ldots,x_n;y_1,\ldots,y_n) \;=\; det\,\|x_i^{p_j}y_i^{q_j}\|_{i,j=1..n} \;. \qquad I.3$$

This given we let $\mathbf{H}_\mu$ be the collection of polynomials in the variables $x_1,\ldots,x_n;y_1,\ldots,y_n$obtained by taking the linear span of all the partial derivatives of Δ_μ. In symbols we may write

$$\mathbf{H}_\mu \;=\; \mathcal{L}\{\partial_x^p\partial_y^q\,\Delta_\mu(x;y)\} \qquad I.4$$

where $\partial_x^p = \partial_{x_1}^{p_1}\cdots\partial_{x_n}^{p_n}$ and $\partial_y^q = \partial_{y_1}^{q_1}\cdots\partial_{y_n}^{q_n}$. Sometimes we may write $\mathbf{H}_\mu[X_n,Y_n]$ for $\mathbf{H}_\mu$ to distinguish it from the singly graded module $\mathbf{H}_\mu[X_n]$ studied in [12]. Here X_n and Y_n denote the two alphabets $X_n=\{x_1,\ldots,x_n\}$, $Y_n=\{y_1,\ldots,y_n\}$.

The natural action of a permutation $\sigma=(\sigma_1,\ldots,\sigma_n)$ on a polynomial $P(x_1,\ldots,x_n;y_1,\ldots,y_n)$ is the so called *diagonal action* which is defined by setting

$$\sigma\; P(x_1,\ldots,x_n;y_1,\ldots,y_n) \;=\; P(x_{\sigma_1},\ldots,x_{\sigma_n};y_{\sigma_1},\ldots,y_{\sigma_n}) \;.$$

Since $\sigma\Delta_\mu \;=\; \pm\Delta_\mu$ according to the sign of σ, the space $\mathbf{H}_\mu[X_n;Y_n]$ necessarily remains invariant under this action. Our main object of study here is the character of the corresponding S_n representation. Note that, since Δ_μ is bihomogeneous of degree $n(\mu)$ in x and $n(\mu')$ in y, we also have the direct sum decomposition

$$\mathbf{H}_\mu[X_n;Y_n] \;=\; \bigoplus_{h=0}^{n(\mu)}\bigoplus_{k=0}^{n(\mu')} \mathcal{H}_{h,k}(\mathbf{H}_\mu) \;,$$

where $\mathcal{H}_{h,k}(\mathbf{H}_\mu)$ denotes the subspace of $\mathbf{H}_\mu$ spanned by its bihomogeneous elements of degree h in x and degree k in y. Since the diagonal action clearly preserves bidegree, each of the subspaces $\mathcal{H}_{h,k}(\mathbf{H}_\mu)$ is also S_n-invariant. Thus we see that $\mathbf{H}_\mu$ has the structure of a bigraded module. The generating function of the characters of its bihomogeneous components, which we shall refer to as the *bigraded character* of $\mathbf{H}_\mu$, may be written in the form

$$\pi^\mu(q,t) \;=\; \sum_{h=0}^{n(\mu)}\sum_{k=0}^{n(\mu')} t^h q^k\; char\,\mathcal{H}_{h,k}(\mathbf{H}_\mu) \;. \qquad I.5$$

We then let $C_\mu(x;q,t)$ denote the Frobenius characteristic of $\mathbf{H}_\mu$ that is

$$C_\mu(x;q,t) \;=\; \frac{1}{n!}\sum_{\sigma\in S_n} \pi^\mu(\sigma;q,t)p_{\lambda(\sigma)}(x) \;, \qquad I.6$$

where $\pi^\mu(\sigma;q,t)$ denotes the value of $\pi^\mu(q,t)$ at σ and $p_{\lambda(\sigma)}(x)$ is the power symmetric function indexed by the shape of σ. Expanding $C_\mu(x;q,t)$ in terms of the Schur basis we get

$$C_\mu(x;q,t) \;=\; \sum_{\mu\vdash n} S_\lambda(x)\,C_{\lambda\mu}(q,t) \;. \qquad I.7$$

where the coefficients $C_{\lambda\mu}(q,t)$ are themselves polynomials with positive integer coefficients. Indeed, the coefficient of $t^h q^k$ in $C_{\lambda\mu}(q,t)$ gives the multiplicity of the irreducible S_n character χ^λ in the character of the submodule $\mathcal{H}_{h,k}(\mathbf{H}_\mu)$.

There is no question that the intimate relationship of the polynomial Δ_μ and the corresponding module $\mathbf{H}_\mu$ to the partition μ makes the character $\pi_\mu(q,t)$ a very natural object of study. However, some deep developments are actually involved here which may not be apparent from the simplicity of our construction of $\mathbf{H}_\mu$. Indeed, in all the cases that we have been able to compute them, the polynomials $C_{\lambda\mu}(q,t)$ are in remarkable agreement with the available tables of the Macdonald q,t-Kostka coefficients. To make this connection precise we recall that Macdonald in [17] established the existence of a symmetric function basis $\{P_\mu(x;q,t)\}_\mu$ uniquely characterized by the following conditions

$$\begin{aligned} a)\quad & P_\lambda = S_\lambda + \sum_{\mu<\lambda} S_\mu\, \xi_{\mu\lambda}(q,t) \\ b)\quad & \langle P_\lambda\,,\, P_\mu\rangle_{q,t} \;=\; 0 \quad for \quad \lambda\neq\mu \end{aligned} \qquad I.8$$

Where $\langle\ ,\ \rangle_{q,t}$ denotes the scalar product of symmetric polynomials defined by setting for the power basis $\{p_\rho\}$

$$\langle p_{\rho_1}\,,\, p_{\rho_2}\rangle_{q,t} \;=\; \begin{cases} z_\rho\; p_\rho\left[\frac{1-q}{1-t}\right] & \text{if } \rho_1=\rho_2=\rho \text{ and} \\ 0 & \text{otherwise} \end{cases} \qquad I.9$$

Here we use λ-ring notation and z_ρ is the integer that makes $n!/z_\rho$ the number of permutations with cycle structure ρ. There are a number of outstanding conjectures concerning these polynomials (see [17]). We are dealing here with those involving the so called *integral forms* $J_\mu(x;q,t)$ and their associated Macdonald-Kostka coefficients $K_{\lambda\mu}(q,t)$. We shall use the same notation as in [17]. In particular $\{Q_\lambda(x;q,t)\}$ denotes the basis dual to $\{P_\lambda(x;q,t)\}$ with respect to the scalar product $\langle\ ,\ \rangle_{q,t}$. Clearly, I.8 b) gives

$$Q_\lambda(x;q,t) \;=\; d_\lambda(q,t)\, P_\lambda(x;q,t)\ , \qquad I.10$$

for a suitable rational function $d_\lambda(q,t)$. However in [17] it is shown that

$$d_\lambda(q,t) \;=\; \frac{h_\lambda(q,t)}{h'_\lambda(q,t)}$$

with

$$h_\lambda(q,t) \;=\; \prod_{s\in\lambda}(1-q^{a_\lambda(s)}t^{l_\lambda(s)+1}) \quad , \quad h'_\lambda(q,t) \;=\; \prod_{s\in\lambda}(1-q^{a_\lambda(s)+1}t^{l_\lambda(s)})$$

where s denotes a generic lattice square and $a_\lambda(s)$, $l_\lambda(s)$ respectively denote the *arm* and the *leg* of s in the Ferrers' diagram of λ.

We recall from [17] that

$$J_\mu(x;q,t) \;=\; h_\mu(q,t)P_\mu(x;q,t) \;=\; h'_\mu(q,t)Q_\mu(x;q,t)\ , \qquad I.11$$

and the coefficients $K_{\lambda\mu}(q,t)$ are defined through an expansion which in λ-ring notation may be written as

$$J_\mu(x;q,t) \;=\; \sum_\lambda S_\lambda[X(1-t)]\, K_{\lambda\mu}(q,t) \;, \qquad I.12$$

where for an alphabet of n letters we set $X = x_1 + x_2 + \cdots + x_n$. Macdonald conjectures that these coefficients are polynomials in q and t with non-negative integer coefficients. We shall refer to this here and after as the MPK conjecture. Now, as we shall see, we can calculate by hand our coefficients $C_{\lambda\mu}(q,t)$ without difficulty for all partitions of $n \leq 6$ and for most partitions of 7. In all these cases we invariably verify that

$$C_{\lambda\mu}(q,t) \;=\; K_{\lambda\mu}(q,1/t)\, t^{n(\mu)} \;. \qquad I.13$$

Our investigations have yielded overwhelming evidence of the general validity of this identity. In terms of $C_\mu(x;q,t)$, I.13 can be written in the form

$$C_\mu(x;q,t) \;=\; \tilde{H}_\mu(x;q,t) \;, \qquad I.14$$

where we have set

$$\tilde{H}_\mu(x;q,t) \;=\; H_\mu(x;q,1/t)\, t^{n(\mu)} \qquad I.15$$

with

$$H_\mu(x;q,t) \;=\; \sum_\lambda S_\lambda\, K_{\lambda\mu}(q,t) \;=\; J_\mu[X/(1-t);q,t] \;. \qquad I.16$$

We shall here and after refer to I.14 as the $C = \tilde{H}$ conjecture. Macdonald in [17] derives a number of properties of the $K_{\lambda\mu}(q,t)$, in particular he shows that for any partition μ

$$K_{\lambda\mu}(1,1) \;=\; f_\lambda \;, \qquad I.17$$

where as customary f_λ denotes the number of standard tableaux of shape λ. Thus the validity of I.14 requires that our module $\mathbf{H}_\mu$ should yield a bigraded version of the left regular representation. Then a fortiori $\mathbf{H}_\mu$ should have $n!$ dimension. It is easy to see that for $\mu = 1^n$ and $\mu = (n)$ Δ_μ reduces to the Vandermonde determinant in x and y respectively. In these cases it is a classical result (see [21]) that $dim\, \mathbf{H}_\mu = n!$. Surprisingly, so far even this simpler identity has been difficult to establish in full generality. We shall here and after refer to it as the *$n!$ conjecture*. This conjecture has been verified (by computer) for all $\mu \vdash n \leq 8$. Here we shall deal only with the case when μ is a hook. A proof of $C = \tilde{H}$ for the case that μ is a two-column or a two-row partition may be found in [9].

The contents of this paper are divided into five sections. In the first section we show that $dim\, \mathbf{H}_\mu \leq n!$ and show that when equality holds, $\mathbf{H}_\mu$ is, as desired, a bigraded version of the left regular representation. We also derive there the identities $C_{\lambda\mu}(0,t) = \tilde{K}_{\lambda\mu}(0,t)$ and $C_{\lambda\mu}(q,0) = \tilde{K}_{\lambda\mu}(q,0)$ mentioned above. In the second section we show that on the $n!$ conjecture, we must necessarily have also the equalities $C_{\lambda\mu}(1,t) = \tilde{K}_{\lambda\mu}(1,t)$ and $C_{\lambda\mu}(q,1) = \tilde{K}_{\lambda\mu}(q,1)$. We do this by introducing two additional S_n-modules $\mathbf{M}^x_\mu$ and $\mathbf{M}^y_\mu$ whose x-graded and y-graded characters respectively have the expansions

$$char_t\, \mathbf{M}^x_\mu \;=\; \sum_{\lambda\vdash n} \chi^\lambda\, \tilde K_{\lambda\mu}(1,t)$$

and

$$char_q\, \mathbf{M}^y_\mu \;=\; \sum_{\lambda\vdash n} \chi^\lambda\, \tilde K_{\lambda\mu}(q,1) \ .$$

Then we show that on the $n!$ conjecture both of them must be equal to $\mathbf{H}_\mu$. In the third section, we show that conversely the equality of $\mathbf{M}^x_\mu$ and $\mathbf{M}^y_\mu$ implies the $n!$ conjecture. The developments in this section should throw some light on the true nature of the $n!$ conjecture.

In the fourth section we present some further remarkable conjectures regarding our space $\mathbf{H}_\mu$. We show there that their validity not only implies the $n!$-conjecture but reveals that several properties of the coefficients $\tilde K_{\lambda\mu}$ (which may be observed from the computer constructed tables) are due to a very surprising algebraic combinatorial mechanism. Finally in the fifth section we give our original (1992) proof of the $C = \tilde H$ conjecture for μ a hook. We should mention that since then E. Allen (see [1]) found a completely different proof which uses Rota's straightening in biletter algebras.

Our crucial tools in this paper are some general results on orbit harmonics which are developed in [9]. The reader will also find in [9] a leisurely presentation of all the identities on Macdonald polynomials we shall have to take for granted in the present treatment. We should also mention that in [10] we have shown that the polynomials $C_\mu(x;q,t)$ satisfy Pieri rules which are consistent with $C = \tilde H$. Further evidence supporting our conjectures may be found in [11].

1. The singly graded versions.

To proceed we need some material from the theory of orbit harmonics. We shall only review the definitions and state the basic results to the extent needed to make the presentation here self-contained. The reader is referred to [9] for a fully detailed, elementary treatment of the topic and further information.

Let $\mathbf{R} = \mathbf{Q}[x]$ denote the ring of polynomials in $x_1, \ldots, x_m$ with rational coefficients. We can introduce a scalar product in $\mathbf{R}$ by setting for any pair $P, Q \in \mathbf{R}$

$$\langle P \ , \ Q \rangle \;=\; L_o P(\partial_x) Q(x) \tag{1.1}$$

where $P(\partial_x)$ denotes the differential operator obtained by replacing the variable x_i in P by the corresponding partial derivative ∂_{x_i}, and L_o represents evaluation at the origin. If A is an $m \times m$ matrix we can make A act on m-dimensional row vectors by right multiplication and set for each polynomial $P(x) = P(x_1, \ldots, x_m)$

$$T_A\, P(x) \;=\; P(xA) \ . \tag{1.2}$$

Note that this action preserves degree as well homogeneity. It is easy to show that, if A is orthogonal, then this action also preserves our scalar product. We shall need a more general grading of $\mathbf{R}$ defined by setting for any monomial $x^p = x_1^{p_1} \cdots x_m^{p_m}$

$$d_w(x^p) \;=\; \sum_{i=1}^{m} w_i p_i$$

where $w_1, \ldots, w_m$ are given positive weights. For any polynomial $P = \sum_p c_p\, x^p$ and any integer k we shall set

$$\pi_k^w\, P \;=\; \sum_{d_w(x^p)=k} c_p\, x^p$$

and call it the w-homogeneous component of w-degree k.(*) This given we shall say that a subspace $\mathbf{V}$ of $\mathbf{R}$ is w-homogeneous if $\pi_k^w \mathbf{V} \subseteq \mathbf{V}$ for all k. Since our scalar product makes the monomials into an orthogonal set, we necessarily have that components of different w-degree will also be orthogonal. In particular the orthogonal complement of a homogeneous subspace of $\mathbf{R}$ is also homogeneous. Here and after we shall use the symbol $\perp$ to denote orthogonal complementation with respect to the scalar product in 1.1. We should note that as long as it is applied to homogeneous spaces, this operation behaves like orthogonal complementation in a finite dimensional space. To be precise we have

Proposition 1.1

If $\mathbf{V}$ is a w-homogeneous subspace then so is $\mathbf{V}^\perp$ and

$$(\mathbf{V}^\perp)^\perp \;=\; \mathbf{V}\ . \qquad 1.3$$

If $J \subseteq \mathbf{R}$ is an ideal then

$$J^\perp \;=\; \{\ P \in \mathbf{Q}[X]\ :\ f(\partial_x)P(x) = 0\ \ \forall\ \ f \in J\ \}\ , \qquad 1.4$$

in particular $J^\perp$ is closed under differentiation.

Proof

The first assertion and identity 1.3 are immediate since $\perp$ acts separately on each w-homogeneous component of $\mathbf{R}$, and each component is finite-dimensional. As for 1.4 we note that if $P \in \mathbf{R}$ is orthogonal to J then, since J is an ideal, we must necessarily have

$$L_o \partial_x^p f(\partial_x) P(X) \;=\; 0 \qquad (\ \forall f \in J \quad \text{and} \quad \forall p\ )\ .$$

However, this means that $f(\partial_x)P(x)$ and all its derivatives vanish at the origin. Taylor's theorem then gives that this polynomial must vanish identically. Thus 1.4 must hold true as asserted. The last assertion is a trivial consequence of 1.4. Q.E.D.

We should note that if J is not a homogeneous ideal the collection in 1.4 may consist only of $\{0\}$ (for example let J be the ideal $(x-a)$ for any non vanishing scalar a). Thus 1.3 fails when J is not homogeneous.

(*) In a context when the grading is kept fixed we will omit the prefix w.

Now let $J \subseteq \mathbf{R}$ be any ideal and set $\mathbf{R}_J = \mathbf{R}/J$. The *associated w-graded ideal* of J is by definition the ideal

$$gr_w\ J \;=\; \big(\ h_w(P)\ :\ P \in J\ \big)\ ,$$

where for each $P \in \mathbf{R}$ we let $h_w(P)$ denote the w-homogeneous component of P of highest w-degree. This given, the quotient ring $\mathbf{R}/gr_w\ J$ is referred to as the *w-graded version of* $\mathbf{R}_J$ and denoted by $gr_w \mathbf{R}_J$.

Let now G be a finite group of orthogonal matrices. In the present context there is no loss in assuming that the matrices in G have rational entries. Given a point $\rho = (\rho_1, \ldots, \rho_m)$ its *G-orbit* is the set

$$[\rho]_G \;=\; \{\ \rho A\ :\ A \in G\ \}.$$

We also set

$$\mathbf{R}_{[\rho]_G} \;=\; \mathbf{R}/J_{[\rho]_G}$$

where

$$J_{[\rho]_G} \;=\; \{\ P \in \mathbf{R}\ :\ P(x) = 0\ \forall\ \ x \in [\rho]_G\ \ \}$$

We may view $\mathbf{R}_{[\rho]_G}$ as the *coordinate ring* of $[\rho]_G$ (considered as an algebraic variety). Since the ideal $J_{[\rho]_G}$ is clearly G-invariant, it follows that G acts on $\mathbf{R}_{[\rho]_G}$. In fact, it is easy to see that the corresponding representation is equivalent to the action of G on left cosets of the stabilizer of ρ. In particular, when ρ is a regular point (i.e. when its stabilizer is trivial), then $\mathbf{R}_{[\rho]_G}$ is simply a version of the left regular representation of G. If, for each $A \in G$, the action in 1.2 preserves the given w-degree then we can associate to $[\rho]_G$ two further G-modules. Namely, $gr_w\ \mathbf{R}_{[\rho]_G}$ and the orthogonal complement

$$\mathbf{H}_{[\rho]_G} \;=\; \big(gr_w\ J_{[\rho]_G}\big)^{\perp}\ . \qquad 1.5$$

Note that if $f(x)$ is any G-invariant polynomial then $f(x) - f(\rho)$ belongs to the ideal $J_{[\rho]_G}$ and hence if $f(x)$ is also w-homogeneous then $f(x)$ itself must belong to $gr_w J_{[\rho]_G}$. This implies that any element $P \in \mathbf{H}_{[\rho]_G}$ must satisfy the differential equation

$$f(\partial_x)\ P \;=\; 0\ .$$

Since G is assumed to consist of orthogonal matrices, so that the polynomial $x_1^2 + \cdots + x_m^2$ is G-invariant, we see that the elements of $\mathbf{H}_{[\rho]_G}$ must be harmonic polynomials. For this reason we shall refer to them as the *harmonics* of the orbit $[\rho]_G$. Clearly, $gr_w\ \mathbf{R}_{[\rho]_G}$ and $\mathbf{H}_{[\rho]_G}$ are equivalent w-graded modules since the elements of $\mathbf{H}_{[\rho]_G}$ may be taken as representatives of the classes of $gr_w\ \mathbf{R}_{[\rho]_G}$. More importantly, it develops that these two spaces realize a graded version of the representation corresponding to $\mathbf{R}_{[\rho]_G}$. This is not difficult to show and we refer the reader to [9] for a proof. We see then that in this manner we have a mechanism for producing a variety of graded versions of the regular representation of G. We shall see shortly that the S_n-module $\mathbf{H}_\mu$ defined in the introduction can be constructed by means of this mechanism.

To do this we need to introduce a few more ingredients . Given a k-part partition $\mu = (\mu_1 \geq \mu_2, \ldots \geq \mu_k > 0)$ let $h = \mu_1$ denote the number of parts of its conjugate μ' and let $\{\alpha_1, \ldots, \alpha_k\}$,

$\{\beta_1,\ldots,\beta_h\}$ be distinct rational numbers. Alternatively, the α's and β's may be taken to be additional indeterminates. Recall that an *injective tableau* T of shape $\mu \vdash n$ is a labelling of the cells of μ by the numbers $\{1,2,\ldots,n\}$. The collection of all such tableaux will be denoted by $\mathcal{IT}(\mu)$. For each $T \in \mathcal{IT}(\mu)$ and $l \in \{1,\ldots,n\}$ we let $s_T(l) = (i_T(l), j_T(l))$ denote the cell of T which contains the label l. This given we construct a point $\rho(T) = (a(T), b(T))$ in $2n$-dimensional space by setting

$$a_l(T) = \alpha_{i_T(l)} \quad , \quad b_l(T) = \beta_{j_T(l)} \qquad (for \quad l = 1,\ldots,n\ ) \ . \tag{1.6}$$

Note that the collection

$$\{\ \rho(T)\ :\ T \in \mathcal{IT}(\mu)\ \} \tag{1.7}$$

consists of $n!$ distinct points. Indeed, since the $\alpha\,'s$ and the $\beta\,'s$ are assumed to be all distinct, we can reconstruct the position of any label l in T by simply looking at the coordinates a_l and b_l of $\rho(T)$. Note that the collection in 1.7 is simply the S_n-orbit under the diagonal action of any one its elements, where the *diagonal action* of S_n on $2n$-dimensional space is defined by setting for $\sigma \in S_n$

$$\sigma(x_1,\ldots,x_n;y_1,\ldots,y_n) = (x_{\sigma_1},\ldots,x_{\sigma_n};y_{\sigma_1},\ldots,y_{\sigma_n}) \ .$$

We can thus construct graded versions of the left regular representation of S_n by the mechanism described above following Proposition 1.1. In this instance we take G to be the group of permutation matrices giving the diagonal action of S_n on the vectors $(x_1,\ldots,x_n;y_1,\ldots,y_n)$. The number of variables m is here equal to $2n$ and $\mathbf{R}$ is the ring of polynomials

$$\mathbf{R} \ = \ \mathbf{R}[X,Y] \ = \ \mathbf{Q}[x_1,\ldots,x_n;y_1,\ldots,y_n] \ .$$

As for the w-degree, we still have some freedom since any vector $w = (w_1,\ldots,w_{2n})$ which gives a weight w_x to all the $x_i's$ and another weight w_y to all the $y_j's$ will produce a grading that is invariant under the diagonal action. To be consistent with the notation introduced earlier we shall denote the orbit in 1.7 by $[\rho_\mu]$. We shall also denote by $\mathbf{R}_{[\rho_\mu]}$, $gr_w\mathbf{R}_{[\rho_\mu]}$ and $\mathbf{H}_{[\rho_\mu]}$ the corresponding coordinate ring, its graded version and the corresponding space of harmonics. Here we omit the subscript G, as the group will remain fixed throughout the rest of our treatment. The definition of these spaces suggests that they might depend on our choice of the α_i's and β_j's. This is certainly so for the coordinate ring $\mathbf{R}_{[\rho_\mu]}$. Nevertheless, the next two results give strong evidence that the space of harmonics $\mathbf{H}_{[\rho_\mu]}$ as well as the ideal $gr_w J_{[\rho_\mu]}$ and the quotient ring $gr_w\,\mathbf{R}_{[\rho_\mu]}$ only depend on the choice of the partition μ.

The point of departure is the following remarkable fact.

Proposition 1.2

If (i,j) is a cell outside μ then for any $s \in \{1,\ldots,n\}$ the monomial $x_s^{i-1}y_s^{j-1}$ belongs to the ideal $gr_w\ J_{[\rho_\mu]}$. In particular if a monomial $x^p y^q = x_1^{p_1}\cdots x_n^{p_n}y_1^{q_1}\cdots y_n^{q_n}$ does not vanish in $gr_w\ \mathbf{R}_{[\rho_\mu]}$ then all the pairs (p_s,q_s) must be biexponents of μ, and every polynomial in $\mathbf{H}_{[\rho_\mu]}$ must be a linear combination of monomials satisfying the same condition.

Proof

We claim that the polynomial

$$f(x,y) = \prod_{i'=1}^{i-1}(x_s - \alpha_{i'}) \prod_{j'=1}^{j-1}(y_s - \beta_{j'})$$

must necessarily vanish on all of $[\rho_\mu]$. Indeed, if f were non-zero at a point

$$(a,b) = (a_1, \ldots, a_n, b_1, \ldots, b_n)$$

then the a_s could not take any of the values $\alpha_1, \alpha_2, \ldots, \alpha_{i-1}$. If T is the injective tableau giving (a,b) via the construction in 1.6, then the label s could not fall in any of the first $i-1$ rows of T. Similarly, the non-vanishing of f at (a,b) would force the label s out of the first $j-1$ columns of T. However, this would force s out of the diagram of μ altogether, an impossibility. We must therefore conclude that $f \in J_{[\rho_\mu]}$ and a fortiori its highest w-homogeneous component, $x_s^{i-1}y_s^{j-1}$, must lie in $gr_w\ J_{[\rho_\mu]}$. This verifies our first assertion.

We deduce that a monomial $x^p y^q = x_1^{p_1} \cdots x_n^{p_n} y_1^{q_1} \cdots y_n^{q_n}$ is in $gr_w\ J_{[\rho_\mu]}$ if it contains any $x_s^{i-1}y_s^{j-1}$ as a factor. If the monomial does not vanish in $gr_w\ \mathbf{R}_{[\rho_\mu]}$, this forces all the pairs $(p_s + 1, q_s + 1)$ inside the diagram of μ. But this is the same as saying that all the pairs (p_s, q_s) must be biexponents of μ. Finally, note that using Proposition 1.1 (equation 1.4) we deduce that every polynomial $P \in \mathbf{H}_{[\rho_\mu]}$ must satisfy the differential equations

$$\partial_{x_s}^{i-1}\partial_{y_s}^{j-1}P(x,y) \;=\; 0\ .$$

Therefore any monomial which contains $x_s^{i-1}y_s^{j-1}$ as a factor must have zero coefficient in P. This completes the proof.

Theorem 1.1

For any choice of the α_i's and β_j's and any diagonally invariant w-grading we have the containment

$$\mathbf{H}_\mu \subseteq \mathbf{H}_{[\rho_\mu]} \tag{1.8}$$

Proof

Since $\mathbf{H}_{[\rho_\mu]}$ as an S_n-module is, for any choice of our parameters, a version of the left regular representation of S_n, it must contain the alternating representation with multiplicity 1. This means that $\mathbf{H}_{[\rho_\mu]}$ contains a polynomial $\Delta(x,y)$, unique up to a scalar factor, which alternates under the diagonal action. Such a polynomial is necessarily a combination of monomials $x^p y^q = x_1^{p_1} \cdots x_n^{p_n} y_1^{q_1} \cdots y_n^{q_n}$ with distinct biexponent pairs $(p_1,q_1),(p_2,q_2),\ldots,(p_n,q_n)$. On the other hand, by the last assertion of Proposition 1.2 all these pairs must also be biexponents of μ. Since μ has altogether only n distinct biexponents, we must conclude that these pairs must be a permutation of the biexponents of μ. This forces $\Delta(x,y)$ to be a scalar multiple of the diagonal alternation of a *tableau monomial*

$$m_T(x,y) \;=\; \prod_{l=1}^{n} x_l^{i_T(l)-1} y_l^{j_T(l)-1}\ . \tag{1.9}$$

In other words $\Delta(x,y)$ must be a multiple of the polynomial $\Delta_\mu(x,y)$ defined in I.3. This places $\Delta_\mu(x,y)$ in $\mathbf{H}_\mu$. Since, by Proposition 1.1, $\mathbf{H}_{[\rho_\mu]}$ is closed under differentiation, the entire space $\mathbf{H}_\mu$ must be contained in $\mathbf{H}_{[\rho_\mu]}$. Q.E.D.

We are thus led to the following remarkable corollary.

Theorem 1.2

On the n! conjecture, *for any S_n-invariant choice of w and any specialization of the α_i's and β_j's, we must have the equalities*

$$\mathbf{H}_{[\rho_\mu]} \;=\; \mathbf{H}_\mu \ , \tag{1.10}$$

and

$$gr_w J_{[\rho_\mu]} \;=\; \mathcal{I}_\mu \ , \tag{1.11}$$

where $\mathcal{I}_\mu$ is the ideal of polynomials $f(x,y)$ which kill Δ_μ, that is

$$\mathcal{I}_\mu \;=\; \Big(\, f\in \mathbf{R} \,:\, f(\partial_x,\partial_y)\,\Delta_\mu(x,y)=0 \,\Big)\ . \tag{1.12}$$

In particular, $\mathbf{H}_\mu$ must be a bigraded version of the left regular representation of S_n.

Proof

We only need to verify 1.11. To this end note that since $gr_w J_{[\rho_\mu]}$ is a w-homogeneous ideal we can use Proposition 1.1 and deduce from 1.10 that

$$gr_w J_{[\rho_\mu]} \;=\; \left(\, \mathbf{H}_\mu \,\right)^{\perp}\ . \tag{1.13}$$

Now for a polynomial $f(x,y)$ to be orthogonal to $\mathbf{H}_\mu$ it is necessary and sufficient that for all p amd q we have

$$L_o\; f(\partial_x,\partial_y)\;\partial_x^p\partial_y^q\Delta_\mu \;=\; 0\ .$$

this means that the polynomial $f(\partial_x,\partial_y)\Delta_\mu$ and all its derivatives must vanish at the origin. Taylor's theorem then yields that it must vanish identically. Conversely, any element of $\mathcal{I}_\mu$ is trivially orthogonal to $\mathbf{H}_\mu$. Thus 1.11 must hold true as asserted. The last assertion is an immediate consequence of 1.10. Q.E.D.

The module $\mathbf{H}_\mu$ has a linear automorphism that complements bidegree and conjugates representations, which we shall refer to as the **flip**. To define it we simply set for any $\phi\in\mathbf{H}_\mu$

$$\mathbf{flip}\,\phi \;=\; \phi(\partial_x,\partial_y)\;\Delta_\mu(x,y)\ . \tag{1.14}$$

We can easily see that if ϕ is of bidegree (h,k) then $\mathbf{flip}\,\phi$ is of bidegree (h',k') with

$$h+h' \;=\; n(\mu) \quad \text{and} \quad k+k'=n(\mu') \tag{1.15}$$

Moreover, because of the alternating property of Δ_μ the isotypic component of $\mathcal{H}_{h,k}(\mathbf{H}_\mu)$ corresponding to character χ^λ is mapped by **flip** into the isotypic component of $\mathcal{H}_{h',k'}(\mathbf{H}_\mu)$ corresponding to $\chi^{\lambda'}$. This simple observation translates into the following basic identities.

Theorem 1.3

For any two partitions λ, μ we have

$$C_{\lambda\mu}(q,t) \;=\; C_{\lambda'\mu}(1/q,1/t)q^{n(\mu')}t^{n(\mu)} \;. \qquad 1.16$$

Equivalently, for all μ

$$C_{\mu}(x;q,t) \;=\; \omega\, C_{\mu}(x;1/q,1/t)q^{n(\mu')}t^{n(\mu)} \qquad 1.17$$

where ω denotes the involution which sends the homogeneous basis of symmetric polynomials into the elementary basis.

The corresponding identity to 1.16 for the Macdonald coefficients $K_{\lambda\mu}(q,t)$ is among the properties derived by Macdonald in [17]; thus Theorem 1.3 is our first evidence for the $C = \tilde{H}$ conjecture. Further evidence comes from a relationship between the $C_{\lambda\mu}(q,t)$, specialized to $q = 0$, and the Kostka-Foulkes coefficients $K_{\lambda\mu}(t)$.

To obtain this relationship by the shortest and most elementary path we shall have to rely on results from [2] and [12]. To this end we need to review some definitions and recall some basic theorems.

If $A = \{a_1, a_2, \ldots, a_m\}$ is an ordered subset of $\{1, 2, \ldots, n\}$ let $\Delta(X_A)$ denote the Vandermonde determinant in the subalphabet $X_A = \{x_{a_1}, x_{a_2}, \ldots, x_{a_m}\}$. To be definite (since order matters), we set

$$\Delta(X_A) \;=\; det\|x_{a_i}^{j-1}\|_{i,j=1}^{m} \;,$$

Let $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k)$ be a partition of n and $\mu' = (\mu'_1 \geq \mu'_2 \geq \cdots \geq \mu'_h > 0)$ denote its conjugate. Let T be a tableau of shape μ with entries $1, 2, \ldots, n$ and let $C_1, C_2, \ldots, C_h$ denote the columns of T ordered from left to right. Note, that with this notation, column C_i is necessarily of length μ'_i. We shall also agree to order the elements of C_i as they occur in T from bottom to top. This given, we set

$$\Delta_T(x) \;=\; \Delta(X_{C_1})\Delta(X_{C_2})\ldots\Delta(X_{C_h}) \;, \qquad 1.18$$

and refer to it as the *Garnir polynomial* corresponding to T. We shall also say that $\Delta_T(x)$ is *standard, of shape* μ, *etc.*, if the same holds true for T itself.

For example if

$$T_1 \;=\; \begin{matrix} 1 & & \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{matrix} \qquad 1.19$$

Then

$$\Delta_{T_1}(x) \;=\; det\begin{pmatrix} 1 & x_5 & x_5^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_1 & x_1^2 \end{pmatrix} \times det\begin{pmatrix} 1 & x_6 \\ 1 & x_3 \end{pmatrix} \times det\begin{pmatrix} 1 & x_7 \\ 1 & x_4 \end{pmatrix} \;. \qquad 1.20$$

These polynomials were used by Garnir in [14], in his reconstruction of Young's natural representation (see also [19]). For convenience let $\mathcal{CI}(\mu)$ denote the collection of all *column increasing* injective tableaux of shape μ with entries $1, 2, \ldots, n$. This given it is not difficult to show (see [2]) that the space

$$\Gamma_\mu \;=\; \mathcal{L}\{\Delta_T \;:\; T \in \mathcal{CI}(\mu)\,\} \;=\; \mathcal{L}\{\Delta_T \;:\; T \in \mathcal{IT}(\mu)\,\}\ , \qquad 1.21$$

the linear span of the Garnir polynomials of shape μ, is an irreducible S_n-module with character given by χ^μ in the Young indexing.

Let us split ρ_μ into its x and y parts by writing $\rho_\mu = (a_\mu, b_\mu)$. Recall that $a_\mu = a(T_o)$ and $b_\mu = b(T_o)$ according to 1.6 applied to T_o (any fixed tableau of shape μ). We may then apply the orbit harmonics construction separately to each of the points a_μ and b_μ. Of course here we should take the number of variables m to be n, and G to be the group of permutation matrices giving the standard action of S_n on n-component vectors $(x_1, \ldots, x_n)$. If we do this we are led to two additional spaces of orbit harmonics. To distinguish them from each other and from the space $\mathbf{H}_\mu$ defined by I.4, we shall denote the latter by $\mathbf{H}_\mu[X_n, Y_n]$, $\mathbf{H}_{[a_\mu]}$ by $\mathbf{H}_\mu[X_n]$ and $\mathbf{H}_{[b_\mu]}$ by $\mathbf{H}_{\mu'}[Y_n]$. The reasons for this last piece of notation are clear. We can easily see that b_μ is precisely what $a_{\mu'}$ would be if we replaced the α_i's by the β_j's. Moreover, to study how these three spaces fit together we need to consider the elements of $\mathbf{H}_{[b_\mu]}$ as polynomials in the variables $y_1, \ldots, y_n$.

Contrarily to the present situation concerning our space $\mathbf{H}_\mu[X_n, Y_n]$, these last two spaces of harmonics are well understood. Indeed, there are very natural monomial bases for the corresponding coordinate rings, and very useful sets of generators are available for the graded versions of the corresponding defining ideals (see [12]). We shall review the definitions of all these ingredients as our developments will require. For the moment we only need to recall the following basic facts.

Theorem 1.4

For any choices of the α_i's we have

$$\mathbf{H}_\mu[X_n] \;=\; \mathcal{L}\{\partial_x^p \Delta_T(x) \;\;:\;\; T \in \mathcal{CI}(\mu)\,\}\ . \qquad 1.22$$

Moreover, if $\pi^\mu(t)$ denotes the (singly) graded character of $\mathbf{H}_\mu[X_n]$, we have the expansion

$$\pi^\mu(t) \;=\; \sum_{\lambda \vdash n} \chi^\lambda\, \tilde{K}_{\lambda\mu}(t) \qquad 1.23$$

where the coefficients $\tilde{K}_{\lambda\mu}(t)$ are related to the Kostka-Foulkes polynomials by the identity

$$\tilde{K}_{\lambda\mu}(t) = K_{\lambda\mu}(1/t)\, t^{n(\mu)}\ . \qquad 1.24$$

.

Elementary proofs of 1.22 and 1.23 can be found in [2]. Here we shall only add a few comments. Essentially, 1.22 and 1.23 may be interpreted as stating that the analogues of the $n!$ and the $C = \tilde{H}$ conjectures hold true hold true for $[a_\mu]$. The space $\mathbf{H}_{[a_\mu]}$ is not a cone (*i.e.*, the space of derivatives of a single polynomial) for the simple reason that a_μ is not a regular point with respect to the standard action of S_n. Nevertheless, it is the linear span of the derivatives of a very natural collection of polynomials. Clearly, similar relations hold for the space $\mathbf{H}_{[b_\mu]}$ as well. We need only replace μ by μ' and $x_1,\ldots,x_n$ by $y_1,\ldots,y_n$ in 1.22 and 1.23.

An immediate corollary of Theorem 1.4 is an identification of the marginal values of our coefficients $C_{\lambda\mu}(q,t)$. Namely,

Theorem 1.5

The homogeneous components of 0 *y-degree and* 0 *x-degree in* $\mathbf{H}_\mu[X_n,Y_n]$ *are respectively equal to the spaces* $\mathbf{H}_\mu[X_n]$ *and* $\mathbf{H}_{\mu'}[Y_n]$. *In particular we derive that*

$$C_{\lambda\mu}(0,t)=\tilde{K}_{\lambda\mu}(t)=\tilde{K}_{\lambda\mu}(0,t) \quad , \quad C_{\lambda\mu}(q,0)=\tilde{K}_{\lambda\mu'}(q)=\tilde{K}_{\lambda\mu}(q,0)\ , \qquad 1.25$$

and thus also

$$C_{\lambda\mu}(q,t)\big|_{q^{n(\mu')}} \;=\; \tilde{K}_{\lambda\mu}(q,t)\big|_{q^{n(\mu')}} \quad , \quad C_{\lambda\mu}(q,t)\big|_{t^{n(\mu)}} \;=\; \tilde{K}_{\lambda\mu}(q,t)\big|_{t^{n(\mu)}}\ , \qquad 1.26$$

Proof

We can obtain our polynomial $\Delta_\mu(x,y)$ by alternating the monomial in 1.9 corresponding to the superstandard tableau T_o of shape μ, that is, the tableau obtained by numberging the cells of μ from left to right within each row, one row at a time, from the bottom row to the top. Thus we have

$$\Delta_\mu(x,y) \;=\; \sum_{\sigma\in S_n}\epsilon(\sigma)\,\sigma m_{T_o}(x,y)\ . \qquad 1.27$$

where $\epsilon(\sigma)$ denotes the the sign of σ. Using notation which goes back to A. Young (see [23]) we let $N(T)$ denote the formal *signed* sum of the elements of the column group of the tableau T. By breaking up the summation according to the left cosets of the column group of T_o we can rewrite 1.27 in the form

$$\Delta_\mu(x,y) \;=\; \sum_{\tau\in S_n/N(T_o)}\epsilon(\tau)\,\tau\prod_{l=1}^n y_l^{j_{T_o}(l)-1}\;N(T_o)\prod_{l=1}^n x_l^{i_{T_o}(l)-1}\ , \qquad 1.28$$

where with some abuse of notation $S_n/N(T_o)$ is to denote a complete set of distinct representatives of the left cosets of the column group of T_o. It is easy to see that the expression $N(T_o)\prod_{l=1}^n x_l^{i_{T_o}(l)-1}$ is none other than the Garnir polynomial corresponding to T_o. Now, for a given $T\in\mathcal{CI}_(\mu)$, let $\tau(T)$ denote the permutation that sends $\Delta_{T_o}(x)$ into $\Delta_T(x)$ and let $\epsilon(T)$ denote its sign. This given, by chosing $S_n/N(T_o)$, to be the collection $\{\tau(T)\ :\ T\in\mathcal{CI}(\mu)\ \}$, 1.28 simplifies to the rather suggestive form

$$\Delta_\mu(x,y) \;=\; \sum_{T\in\mathcal{CI}(\mu)}\epsilon(T)\prod_{l=1}^n y_l^{j_T(l)-1}\;\Delta_T(x)\ . \qquad 1.29$$

Note that the elements of $\mathbf{H}_\mu[X_n,Y_n]$ which have no dependence on y may all be written in the form $f(\partial_x,\partial_y)\Delta_\mu(x,y)$ with $f(x,y)$ a polynomial which is homogeneous of degree $n(\mu')$ in $y_1,\ldots,y_n$. In view of 1.29 the resulting polynomial is then necessarily a linear combination of derivatives of Garnir polynomials for column increasing tableaux of shape μ. This immediately yields that the submodule of $\mathbf{H}_\mu[X_n, Y_n]$ consisting of elements of degree zero in y is precisely given by the space $\mathbf{H}_\mu[X_n]$ as asserted. By reversing the roles of x and y in the above argument we get that the submodule of elements of degree zero in x is given by $\mathbf{H}_{\mu'}[Y_n]$. Equation 1.23 then yields the equalities

$$C_{\lambda\mu}(0,t) = \tilde{K}_{\lambda\mu}(t) \quad , \quad C_{\lambda\mu}(q,0) = \tilde{K}_{\lambda\mu'}(q) \ ,$$

The proof of 1.25 is completed by verifying that indeed we have

$$\tilde{K}_{\lambda\mu}(t) = \tilde{K}_{\lambda\mu}(0,t) \quad , \quad \tilde{K}_{\lambda\mu'}(q) = \tilde{K}_{\lambda\mu}(q,0) \ .$$

However, this follows easily from the fact (proved by Macdonald in [17]) that the Macdonald polynomials reduce to the Hall-Littlewood polynomials when $q = 0$. The equalities in 1.26 then follow from those in 1.25 by a simple application of 1.16 and the corresponding identity

$$\tilde{K}_{\lambda\mu}(q,t) \ = \ \tilde{K}_{\lambda'\mu}(1/q, 1/t) q^{n(\mu')} t^{n(\mu)} \ . \qquad 1.30$$

for Macdonald coefficients, which may be straightforwardly derived from identities given in [17]. This completes our proof.

As an immediate application of 1.25 and 1.26 we obtain the following special cases of the $C = \tilde{H}$ conjecture.

Theorem 1.6

The identities

$$C_{\lambda,\mu}(q,t) \ = \ \tilde{K}_{\lambda,\mu}(q,t) \qquad 1.31$$

hold true for all λ whenever μ is of the form $(2, 1^{n-2})$ or $(n-1, 1)$.

Proof

When $\mu = (2, 1^{n-2})$ then $n(\mu') = 1$ so both $C_{\lambda,\mu}(q,t)$ and $\tilde{K}_{\lambda,\mu}(q,t)$ are linear in q and 1.31 is a consequence of the first equalities in 1.25 and 1.26. On the other hand if $\mu = (n-1, 1)$ then $n(\mu) = 1$ and 1.31 is a consequence of the last equalities in 1.25 and 1.26. Q.E.D.

We terminate the section with two further identities matching what is known about the coefficients $\tilde{K}_{\lambda\mu}(q,t)$.

Theorem 1.7

For any pair λ, μ we have

$$C_{\lambda\mu'}(q,t) \ = \ C_{\lambda\mu}(t,q) \qquad 1.32$$

Proof

The identity

$$\Delta_{\mu'}(x,y) \ = \ \Delta_\mu(y,x)$$

is an immediate consequence of the definition I.3. Thus the elements of $\mathbf{H}_{\mu'}[X_n, Y_n]$ are obtained by interchanging $x_i's$ with the $y_i's$ in each of the elements of $\mathbf{H}_\mu[X_n, Y_n]$. This clearly accounts for

1.32 since such an interchange does not affect the character of any of the isotypic components, but exchange x-degree with y-degree. Q.E.D.

In [17] Macdonald was able to determine the coefficients $K_{\lambda\mu}(q,t)$ explicitly for arbitrary μ when λ is a hook. In the present notation, his result may be rewritten in the form

$$\sum_{s=0}^{n} u^s \ \tilde{K}_{(1^s n-s),\mu}(q,t) \ = \ \prod_{(i,j)\in\mu}^{(o)} (1 - u\, t^{i-1} q^{j-1}) \ , \tag{1.33}$$

where the (o) superscript is to indicate that the product omits the $(0,0)$ biexponent of μ. We can show that our coefficients $C_{(1^s n-s),\mu}(q,t)$ are given by the same formula.

Our proof is based on a remark of A. Young who, with surprising historical premonition, admonished in [23], that some other approaches to the construction of the irreducible representations of S_n, might only be superficially different from his. Young's observation is relevant not only to Garnir's modules and the oft-cited Specht modules [16], [19] but it also provides precisely what is needed in our developments.

To see how this comes about, we shall construct irreducible components of $\mathbf{H}_\mu$ corresponding to hook shapes by differentiating Δ_μ by certain polynomials $\Delta_{S,A}$ patterned upon Young's idempotents. Then Young's observation yields us that our procedure produces complete decompostions of the isotypic components indexed by hook shapes. The omitted proofs and details in the necessarily short treatment given here can be found in [9] or [13].

To prove that our coefficients $C_{(1^s,n-s),\mu}(q,t)$ satisfy 1.33, we shall need to construct submodules of H_μ indexed by $(s+1)$-element subsets of the biexponents, in which the first element is the biexponent $(0,0)$. It will be convenient thereore to let $(p_1,q_1)=(0,0),(p_2,q_2),\ldots,(p_n,q_n)$ denote the biexponents of μ, and to consider index sets of the form

$$S = \{1 = i_1 < i_2 < \cdots < i_{s+1} \leq n\}. \tag{1.34}$$

Let also A be any $(s+1)$-element set

$$A = \{1 \leq a_1 < a_2 < \cdots < a_{s+1} \leq n\} \tag{1.35}$$

To the pair S,A we assign the $(s+1)\times(s+1)$ determinant

$$\Delta_{S,A}(x,y) \ = \ det\| x_{a_h}^{p_{i_k}} y_{a_h}^{q_{i_k}} \|_{h,k=1}^{s+1} \ . \tag{1.36}$$

Our submodule $\mathbf{H}_{S,\mu}[X,Y]$ will now be the linear span of the derivatives

$$\Delta_{S,A}(\partial_x,\partial_y)\Delta_\mu, \tag{1.37}$$

as A ranges over the sets of labels in the first column of all standard tableaux of shape $(1^s, n-s)$.

Young's observation alluded to above can be stated in full as follows.

Lemma 1.1

Let T be a tableau of shape λ, λ a parition of n. As in 1.28, let $N(T)$ denote the signed sum of the elements in the column group of T. Let m be an element of an S_n-module V which satisfies $\sigma m = m$ for all σ in the row group of T. Let $\Delta = N(T)m$, and assume $\Delta \neq 0$. Then

(1) The linear span $\mathcal{L}\{\sigma\Delta : \sigma \in S_n\}$ is an irreducible submodule of V, with character χ^λ;

(2) The set $\{\sigma\Delta : \sigma \in S_n, \sigma(T) \text{ standard}\}$ is a basis of the submodule in (1); and

(3) Denoting the elements of the basis in (2) by $b_1, b_2, \ldots, b_{f_\lambda}$, there exists for each i, j an element E_{ij} of the group algebra of S_n such that $E_{ij}b_i = b_j$ and $E_{ij}b_k = 0$ for all $k \neq i$.

Now let $T_1, \ldots, T_N$, for $N = \binom{n-1}{s}$, be all the standard tableaux of shape $(1^s, n-s)$, and let $A_1, \ldots, A_N$ be their first columns. The determinant Δ_{S,A_1} is equal to $N(T_1)m_{T_1}$, where

$$m_{T_1} = \prod_{j=1}^{s+1} x_{a_j}^{p_{i_j}} y_{b_j}^{q_{i_j}}. \qquad 1.38$$

Note that because we have set $i_1 = 1$ in S and $(p_1, q_1) = (0, 0)$, the monomial m_{T_1} involves only the variables x_{a_j} and y_{a_j} for $j = 2, \ldots, s+1$, and so is invariant for the row group of T_1.

We can apply the lemma with $\Delta = \Delta_{S,A_1}$, and we see that the polynomials $\Delta_{S,A_1}, \ldots, \Delta_{S,A_N}$ form the basis in (2). It also follows, since Δ_μ is S_n-alternating, that either

$$\{\Delta_{S,A_i}(\partial_x, \partial_y)\Delta_\mu : i = 1, \ldots, N\} \qquad 1.39$$

is a basis of an irreducible submodule of $\mathbf{H}_\mu$ with character $\chi^{\lambda'}$, or else all the polynomials $\Delta_{S,A}$, for any A whatsoever, vanish identically. Now we are in position to prove our desired theorem.

Theorem 1.8

The subspaces

$$\mathbf{H}_{S,\mu}[X,Y] \;=\; \mathcal{L}\{\Delta_{S,A}(\partial_x, \partial_y)\Delta_\mu(x,y) \;:\; A = \{1 \leq a_1 < a_2 < \cdots < a_{s+1} \leq n\}\,\} \qquad 1.40$$

obtained as S varies among the $\binom{n-1}{s}$ subsets given by 1.34 are independent, irreducible submodules yielding a complete decomposition of the isotypic component of shape $(1^{n-s-1}, s+1)$ in $\mathbf{H}_\mu[X,Y]$.

Proof

We must first show that the polynomials $\Delta_{S,A}(\partial_x, \partial_y)\Delta_\mu(x,y)$ do not all vanish identically. Suppose the contrary. The vanishing of these polynomials means the determinants $\Delta_{S,A}(x,y)$ lie in the ideal $\mathcal{I}_\mu$ defined in 1.12. However with our choice of biexponents this cannot occur, for a very simple reason. Computing $\Delta_\mu(x,y)$ by the Laplace expansion with respect to the columns indexed by the elements of S we derive that

$$\Delta_\mu(x,y) \;=\; \sum_{A=\{1\leq a_1<a_2<\cdots<a_{s+1}\leq n\}} \Delta_{S,A}(x,y)\ {}^c\Delta_{S,A}(x,y)\ , \qquad 1.41$$

where ${}^c\Delta_{S,A}(x,y)$ denotes the complementary minor to $\Delta_{S,A}(x,y)$ in the matrix $\|x_i^{p_j} y_i^{q_j}\|_{i,j=1..n}$. Thus if each $\Delta_{S,A}(x,y)$ lies in $\mathcal{I}_\mu$ so must $\Delta_\mu(x,y)$ as well. But then Δ_μ would be orthogonal to itself, a patent impossibility.

We are left to show that the constructed submodules are independent. Again suppose the contrary, so there is a linear dependence

$$\sum_{i,j=1}^{\binom{n-1}{s}} c_{ij} \Delta_{S_j,A_i}(\partial_x, \partial_y) \Delta_\mu = 0, \tag{1.42}$$

or equivalently,

$$\sum_{i,j=1}^{\binom{n-1}{s}} c_{ij} \Delta_{S_j,A_i} \in \mathcal{I}_\mu. \tag{1.43}$$

Applying the operator $E_{i,1}$ from part (3) of the lemma, we obtain for each i

$$\sum_{j=1}^{\binom{n-1}{s}} c_{ij} \Delta_{S_j,A_1} \in \mathcal{I}_\mu. \tag{1.44}$$

Finally, acting on this equation by $\sigma \in S_n$, we may replace A_1 here by any A at all. However, we have again by the Laplace expansion that

$$\sum_{A=\{1\le a_1<a_2<\cdots<a_{s+1}\le n\}} \Delta_{S,A}(x,y)\ {}^c\Delta_{S_j,A}(x,y) \;=\; \begin{cases} \Delta_\mu(x,y) & \text{if } S_j = S \text{ and} \\ 0 & \text{otherwise} \end{cases} . \tag{1.45}$$

Indeed, the first case is again 1.41 and in the second case the sum on the left hand side is the Laplace expansion of a determinant with repeated rows. This given, multiplying 1.44 (with A in place of A_1) by ${}^c\Delta_{S_j,A}(x,y)$ and summing with respect to A we finally derive that

$$c_{ij}\ \Delta_\mu(x,y) \in \mathcal{I}_\mu \ . \tag{1.46}$$

Since as we have seen $\Delta_\mu(x,y)$ cannot lie in $\mathcal{I}_\mu$ the inevitable conclusion is that all the coefficients c_{ij} must be equal to zero.

To prove the last assertion of the theorem we need only observe that, in view of 1.8, the maximum number of irreducible constituents with character $\chi^{(n-s-1,s+1)}$ in $\mathbf{H}_\mu[X,Y]$ is $\binom{n-1}{s}$. Thus our submodules necessarily must give a complete set. QED

As an immediate corollary of this theorem we obtain the equality of $\tilde{K}_{\lambda\mu}(q,t)$ and $C_{\lambda\mu}(q,t)$ when λ is a hook.

Theorem 1.9

Setting

$$B^o_\mu(q,t) \;=\; B_\mu(q,t) - 1 \ ,$$

for all $\mu \vdash n$ we have (in λ-ring notation)

$$C_{(1^s,n-s),\mu}\,(q,t) \;=\; e_s[B^o_\mu(q,t)] \tag{1.47}$$

Proof

Since all of the elements of $\mathbf{H}_{S,\mu}[X,Y]$ are bihomogeneous polynomials of bidegree

$$(\ n(\mu) - \sum_{i\in S} p_i \ , \ n(\mu') - \sum_{i\in S} q_i \ )$$

this submodule must contribute the term

$$t^{n(\mu)} q^{n(\mu')} \prod_{i\in S} t^{-p_i} q^{-q_i} \quad = \quad \prod_{i\in S'} t^{p_i} q^{q_i} \qquad 1.48$$

to our coefficient $C_{(1^{n-s-1},s+1),\mu}\,(q,t)$. Here S' denotes the complement of S in $\{1,2,\ldots,n\}$. Theorem 1.8 gives that this coefficient may be obtained by summing these monomials over all possible choices of S. This gives

$$C_{(1^{n-s-1},s+1),\mu)}\,(q,t) \quad = \quad \sum_{\substack{|S'|=n-s-1 \\ S'\subseteq\{2,3,\ldots,n\}}} \ \prod_{i\in S'} t^{p_i} q^{q_i} \ , \qquad 1.49$$

and this is another way of stating 1.47. Note that 1.33 is likewise just another way of stating the same equality for the Macdonald coefficients $\tilde{K}_{(1^s,n-s),\mu}$.

2. Calculation of the x-graded and y-graded characters of $\mathbf{H}_\mu[X,Y]$

The main object of this section is to show that, on the $n!$ conjecture, we have the identities

$$a) \quad C_{\lambda\mu}(q,1) \ = \ \tilde{K}_{\lambda\mu}(q,1) \ , \qquad b) \quad C_{\lambda\mu}(1,t) \ = \ \tilde{K}_{\lambda\mu}(1,t) \ . \qquad 2.1$$

Our basic ingredients here are formal power series

$$f(x,y) \ = \ \sum_{p,q} c_{pq}\, x^p y^q \ , \qquad 2.2$$

in the two sets of variables, $x_1,\ldots,x_n, y_1,\ldots,y_n$ with the diagonal action of S_n defined (as was done for polynomials) by setting for $\sigma\in S_n$

$$\sigma\, f(x_1,\ldots,x_n;y_1,\ldots,y_n) \ = \ f(x_{\sigma_1},\ldots,x_{\sigma_n};y_{\sigma_1},\ldots,y_{\sigma_n}) \ .$$

All standard operations such as linear combinations, products, differentiations by any of the variables are defined for formal power series as they are for polynomials. This will also be the case for any operations which, likewise, permit the computation of each of the coefficients of the resulting series in a finite number of steps and as such involve no convergence questions.

In this section we will use only the customary grading of polynomials and we shall denote total degree simply by *deg*. In particular, if f is given by 2.2 then the polynomial

$$f^{(m)}(x,y) \ = \ \sum_{|p|+|q|=m} c_{pq}\, x^p y^q \ , \qquad 2.3$$

will be referred to as the homogeneous component of degree m in f or briefly the m^{th} component of f. The homogeneous component of f of minimum degree which does not vanish identically will be denoted by $m(f)$. Note that the operation $f \to m(f)$ is not linear. Indeed, for $m(f)$ to be homogeneous of degree m_o we need that

$$f^{(m)} = 0 \quad \text{for} \quad m < m_o \quad \text{and} \quad f^{(m_o)} \neq 0 \ . \tag{2.4}$$

Then and only then can we conclude that $m(f) = f^{(m_o)}$. This defines $m(f)$ as a non-zero polynomial for all but the zero formal power series. It will be convenient to set $m(0) = 0$. Clearly, if f_1 and f_2 are two formal power series, we shall have

$$m(f_1 + f_2) = m(f_1) + m(f_2) \quad \Leftrightarrow \quad \begin{cases} a) & deg\big(m(f_1)\big) = deg\big(m(f_2)\big) \\ b) & m(f_1) + m(f_2) \neq 0 \end{cases} \ . \tag{2.5}$$

We shall use the symbol (,) to denote the ordinary scalar product of n-vectors. Thus for $x = (x_1, \ldots, x_n)$ and $a = (a_1, \ldots, a_n)$ we set

$$(x, a) \ = \ x_1 a_1 + x_2 a_2 + \cdots + x_n a_n \ .$$

All the formal power series that occur in our treatment will be obtained by carrying out the above mentioned operations on polynomial multiples of the exponential series

$$e^{(x,a)} \ = \ \sum_{p_1 \geq 0} \cdots \sum_{p_n \geq 0} \frac{x_1^{p_1} a_1^{p_1}}{p_1!} \cdots \frac{x_n^{p_n} a_n^{p_n}}{p_n!} \ = \ \sum_p \frac{x^p a^p}{p!} \ .$$

This given, for a partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0)$ we set

$$\Phi_\mu(x, y) \ = \ \sum_{\sigma \in S_n} \epsilon(\sigma) \, \sigma \left(e^{\left(x, a(T_o)\right)} \, m_{T_o'}(y) \right) \ , \tag{2.6}$$

where, as in section 1, T_o denotes the superstandard tableau of shape μ, $a(T_o)$ is the vector whose coordinates are given by the first equalities in 1.6 and $m_{T_o'}(y)$ is the y-monomial corresponding to the conjugate of the tableau T_o. More precisely, using again the same notation as in 1.6 we have

$$m_{T_o'}(y) \ = \ \prod_{l=1}^{n} y_l^{j_{T_o}(l) - 1} \ . \tag{2.7}$$

With these conventions and notations we are in a position to introduce two spaces which play a crucial role in our proof of the identities in 2.1. Namely, we define $\mathbf{\Gamma}_\mu$ as the linear span of the derivatives of Φ_μ and $m(\mathbf{\Gamma}_\mu)$ as the space spanned by the least non-vanishing homogeneous components of elements of $\mathbf{\Gamma}_\mu$. In symbols

$$\mathbf{\Gamma}_\mu \ = \ \mathcal{L}\{ \, \partial_x^p \partial_y^q \, \Phi_\mu \, \} \tag{2.8}$$

and

$$m(\mathbf{\Gamma}_\mu) \;=\; \mathcal{L}\{\, m(f) \;:\; f \in \mathbf{\Gamma}_\mu \,\} \;. \qquad 2.9$$

Our reasoning will proceed as follows. Without any additional assumptions we can compute the character of $\mathbf{\Gamma}_\mu$ as a y-graded S_n-module. We next show that, on the $n!$ conjecture, $m(\mathbf{\Gamma}_\mu)$ and our module $\mathbf{H}_\mu[X,Y]$ are one and the same. This result permits us to derive that $\mathbf{\Gamma}_\mu$ and $\mathbf{H}_\mu[X,Y]$ are equivalent as y-graded modules and thereby complete our identification of the y-graded character of $\mathbf{H}_\mu[X,Y]$. To carry this out in detail we need to establish a few auxiliary results.

Let $\mathcal{RI}_\mu$ denote the collection of row increasing injective tableaux of shape μ. For a given $T \in \mathcal{RI}_\mu$ let $\epsilon(T)$ denote the sign of the permutation that sends T_o into T. For any biexponent sequence $(p_1,q_1),(p_2,q_2),\ldots(p_n,q_n)$ set

$$\Delta_{pq}(x,y) \;=\; det\|x_j^{p_i} y_j^{q_i}\| \;.$$

Lemma 2.1

The formal power series Φ_μ has also the following two equivalent definitions

$$\Phi_\mu \;=\; \sum_{T \in \mathcal{RI}_\mu} \epsilon(T)\, e^{(x,a(T))}\, \Delta_{T'}(y) \;, \qquad 2.10$$

and

$$\Phi_\mu \;=\; \sum_p \frac{a(T_o)^p}{p!}\, \Delta_{p,q_o}(x,y) \qquad 2.11$$

where q_o is the exponent sequence of the monomial $m_{T_o'}(y)$, i.e., $m_{T_o'}(y) = y^{q_o}$.

Proof

Let $T_1, T_2, \ldots, T_N$ ($N = n!/\mu!$) be the elements of $\mathcal{RI}_\mu$ in any total order that makes $T_1 = T_o$. Let τ_i be the permutation that sends T_o into T_i. Let $P(T_o)$ denote the row group of T_o. This given we have the left coset decomposition

$$S_n \;=\; \tau_1 P(T_o) + \tau_2 P(T_o) + \cdots + \tau_N P(T_o) \;. \qquad 2.12$$

In other words every $\sigma \in S_n$ has a unique factorization of the form $\sigma = \tau_i \alpha$ with $\alpha \in P(T_o)$. Since for any $\alpha \in P(T_o)$ we have

$$\tau_i \alpha e^{(x,a(T_o))} \;=\; e^{(x,a(T_i))} \;,$$

we may rewrite 2.6 in the form

$$\Phi_\mu(x,y) \;=\; \sum_{i=1}^N \epsilon(\tau_i) e^{(x,a(T_i))}\, \tau_i \sum_{\alpha \in P(T_o)} \epsilon(\alpha)\, \alpha\, m_{T_o'}(y) \;. \qquad 2.13$$

Now we can easily see that

$$\sum_{\alpha \in P(T_o)} \epsilon(\alpha)\, \alpha\, m_{T_o'}(y) \;=\; \Delta_{T_o'}(y) \;,$$

the latter being the Garnir polynomial (in the y variables) corresponding to the conjugate of the tableau T_o. Substituting in 2.13 and noting that

$$\tau_i \Delta_{T_o'}(y) \;=\; \Delta_{T_i'}(y)$$

we finally get

$$\Phi_\mu(x,y) \;=\; \sum_{i=1}^{N} \epsilon(\tau_i) e^{(x,a(T_i))}\, \Delta_{T_i'}(y) \qquad 2.14$$

which is another way of writing 2.10.

To verify 2.11 we simply note that if we expand the exponential in 2.6, set $m_{T_o'}(y) = y^{q_o}$ and change order of summation we obtain

$$\Phi_\mu(x,y) \;=\; \sum_p \frac{a(T_o)^p}{p!} \sum_{\sigma \in S_n} \epsilon(\sigma)\, \sigma x^p y^{q_o} \;\;.$$

Since the inner summation yields the polynomial $\Delta_{pq_o}(x,y)$ this expression reduces to 2.11 as desired.

The expansion in 2.14 immediately yields us the following useful fact:

Lemma 2.2

The collection of formal power series

$$\mathcal{C}_\mu \;=\; \{\, e^{(x,a(T_j))}\;\; \partial_y^q \Delta_{T_j'}(y) \;:\; j = 1,2,\ldots,n!/\mu!\,;\; q=(q_1,\ldots,q_n)\,\}$$

spans $\mathbf{\Gamma}_\mu$.

Proof

Note that if, for any $T \in \mathcal{RI}_\mu$, we set (using the notation of section 1)

$$\phi_T(x) \;=\; \prod_{l=1}^{n} \frac{(x_l - \alpha_1)}{(\alpha_{i_T(l)} - \alpha_1)} \cdots \frac{(x_l - \alpha_{i_T(l)-1})}{(\alpha_{i_T(l)} - \alpha_{i_T(l)-1})} \;\;.$$

Then we easily verify that

$$\phi_{T_j}(x) \;=\; \begin{cases} 1 & \text{if } x = a(T_j) \\ 0 & \text{if } x = a(T_i) \text{ with } i \neq j \end{cases} \qquad 2.15$$

This given, applying the differential operator $\phi_{T_j}(\partial_x)\partial_y^q$ to both sides of 2.14 we get

$$\phi_{T_j}(\partial_x)\partial_y^q \Phi_\mu(x,y) \;=\; e^{(x,a(T_j))}\;\; \partial_y^q \Delta_{T_j'}(y) \;\;. \qquad 2.16$$

We thus obtain that all the elements of $\mathcal{C}_\mu$ belong to $\mathbf{\Gamma}_\mu$. On the other hand, 2.14 gives that for any exponents p,q we have

$$\partial_x^p \partial_y^q\, \Phi_\mu(x,y) \;=\; \sum_{i=1}^{N} \epsilon(T_i)\, a(T_i)^p\, e^{(x,a(T_i))}\, \partial_y^q \Delta_{T_i'}(y) \;\;,$$

and this shows that every element of $\mathbf{\Gamma}_\mu$ is a linear combination of the elements of $\mathcal{C}_\mu$.

Q.E.D.

Let us recall that if G is a finite group of $n \times n$ matrices, then the polynomials in $x_1, \ldots, x_n$ which are killed by all non-trivial homogeneous G-invariant differential operators are referred to as the *harmonics* of G. They form a finite dimensional subspace which we shall denote here by $\mathbf{H}_G[X]$. It is well known (see [21] or [9]) that, when G is generated by reflections, $\mathbf{H}_G[X]$ is the linear span of the derivatives of a single polynomial $\Delta_G(x)$. In symbols,

$$\mathbf{H}_G \;=\; \mathcal{L}\{\partial_x^p \Delta_G(X)\} \;. \tag{2.17}$$

It is also shown in [21] that

$$dim\, \mathbf{H}_G \;=\; |G| \tag{2.18}$$

These two properties are, in fact, characteristic [21] of finite groups generated by reflections. It is well known that when G is the group of permutation matrices corresponding to the natural action of S_n on polynomials then $\Delta_G(x)$ reduces to the Vandermonde determinant in $x_1, \ldots, x_n$. In the general case $\Delta_G(x)$ can be obtained by taking the product of the linear forms giving the equations of the reflecting hyperplanes. The latter is usually referred to as the discriminant of G. This given, it is not difficult to see that the Garnir polynomial $\Delta_{T'}(y)$ is also the discriminant of the row group $P(T)$. This observation leads us immediately to the following basic fact.

Proposition 2.1

The space $\mathbf{\Gamma}_\mu$ decomposes into the direct sum

$$\mathbf{\Gamma}_\mu \;=\; \bigoplus_{T \in \mathcal{RI}_\mu} e^{(x,a(T))}\, \mathbf{H}_{P(T)}[Y] \;. \tag{2.19}$$

In particular, we have

$$dim\, \mathbf{\Gamma}_\mu \;=\; n! \;. \tag{2.20}$$

Proof

The identity 2.17, for $G = P(T)$, gives that for any fixed $T \in \mathcal{RI}_\mu$, the elements $e^{(x,a(T))}\partial_y^p \Delta_{T'}(y)$ must necessarily span the subspace obtained by multiplying the harmonics of $P(T)$ (in the variables $y_1, \ldots, y_n$) by the exponential $e^{(x,a(T))}$. And this is what is meant by $e^{(x,a(T))}\, \mathbf{H}_{P(T)}[Y]$ in 2.19. The fact that the sum in 2.19 is *direct* then follows from the fact that exponentials $e^{(x,a)}$ with different exponents are independent. The dimension of $\mathbf{\Gamma}_\mu$ must then be $n!$ since (by 2.18) each of the direct summands has dimension $\mu!$ and the collection $\mathcal{RI}_\mu$ has precisely $n!/\mu!$ elements.

Q.E.D.

Note further that the linear span in $\mathbf{\Gamma}_\mu$ of those elements $e^{(x,a(T))}\partial_y^q \Delta_{T'}$ which are of degree m in y is invariant under the diagonal action. Thus we have also a direct sum decomposition of $\mathbf{\Gamma}_\mu$ as a y-graded S_n-module:

$$\mathbf{\Gamma}_\mu \;=\; \bigoplus_{m=0}^{n(\mu')} \mathcal{H}_m(\mathbf{\Gamma}_\mu) \;, \tag{2.21}$$

where

$$\mathcal{H}_m(\mathbf{\Gamma}_\mu) \;=\; \mathcal{L}\{\, e^{(x,a(T))}\,\partial_y^q \Delta_{T'}(y) \;:\; T\in\mathcal{RI}_\mu;\;\; |q| = n(\mu') - m\} = \bigoplus_{T\in\mathcal{RI}_\mu} e^{(x,a(T))}\mathcal{H}_m(H_{P(T)}[Y]). \tag{2.22}$$

In particular, the generating function

$$\pi_y^\mu(q) \;=\; \sum_{m=0}^{n(\mu')} q^m \; char\, \mathcal{H}_m(\mathbf{\Gamma}_\mu) \;\; .$$

may be viewed as the y-graded character of $\mathbf{\Gamma}_\mu$.

From 2.19 and the left coset decomposition 2.12 it is plain that the character in question is nothing but the induced character from $P(T_o)$ to S_n of the character of $H_{P(T_o)}[Y]$. By 2.22, this also holds separately in each y-degree.

Now $P(T_o)$ is the Young subgroup $S_{\mu_1}\times S_{\mu_2}\times\cdots\times S_{\mu_k}$. The multiplicative property of the usual Frobenius characteristic map F, namely $F\ \mathrm{Ind}_{S_l\times S_m}^{S_{l+m}}(\chi_1\otimes\chi_2) = F\chi_1\ F\chi_2$, immediately implies the same for the Frobenius characterstic of a graded character. Finally, it is well known that the Frobenius characteristic of $H_{S_m}[Y]$ is $(1-q)(1-q^2)\cdots(1-q^m)h_m[X/(1-q)]$ in λ-ring notation, where h_m is the complete homogeneous symmetric function of degree m.

These observations demonstrate the following crucial result.

Theorem 2.1

When $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0)$, the Frobenius image of the y-graded character of $\mathbf{\Gamma}_\mu$ is given by the identity

$$F\ \pi_y^\mu(q) \;=\; \tilde H_{\mu_1}(x;q)\tilde H_{\mu_2}(x;q)\cdots\tilde H_{\mu_k}(x;q) \;=\; \tilde H_\mu(x;q,1)\;\;, \tag{2.23}$$

Where for an integer m we have (in λ-ring notation)

$$\tilde H_m(x,q) \;=\; (1-q)(1-q^2)\cdots(1-q^m)\; h_m[X/(1-q)]\;\; . \tag{2.24}$$

In particular, the q-multiplicities of the irreducible character χ^λ in $\mathbf{\Gamma}_\mu$ are given by the Macdonald coefficients, that is,

$$\pi_y^\mu(q) \;=\; \sum_{\lambda\vdash n} \chi^\lambda\ \tilde K_{\lambda\mu}(q,1)\;\; . \tag{2.25}$$

Proof

Only 2.25 remains to be verified. But Macdonald in [17] obtained explicit expressions for his polynomials when q or t are set equal to 1, and these expressions readily yield the equivalence of 2.23 and 2.25 (see [9]).

Before we can use this result to calculate the y-graded character of $\mathbf{H}_\mu[X,Y]$ we must look a bit closely at the relationship between the $m(\mathbf{\Gamma}_\mu)$ and $\mathbf{\Gamma}_\mu$, which is expressed by the following result.

Proposition 2.2

We have

$$dim\, m(\mathbf{\Gamma}_\mu) \;=\; dim\, \mathbf{\Gamma}_\mu \;=\; n!\ . \qquad 2.26$$

Moreover the two spaces $m(\mathbf{\Gamma}_\mu)$ and $\mathbf{\Gamma}_\mu$ have the same y-graded character.

Proof

To start, we show that the decomposition of $m(\mathbf{\Gamma}_\mu)$ into y-homogeneous subspaces is given by

$$m(\mathbf{\Gamma}_\mu) = \bigoplus_k m(\mathcal{H}_k(\mathbf{\Gamma}_\mu)),$$

where $\mathcal{H}_k(\mathbf{\Gamma}_\mu)$ denotes the subspace of elements in $\mathbf{\Gamma}_\mu$ homogeneous of degree k in y. What is clear is that $m(\mathcal{H}_k(\mathbf{\Gamma}_\mu))$ is homogeneous of degree k in y, so that the sum above is indeed direct, and that $m(\mathcal{H}_k(\mathbf{\Gamma}_\mu)) \subseteq m(\mathbf{\Gamma}_\mu)$. Therefore we need to prove the reverse inclusion $m(\mathbf{\Gamma}_\mu) \subseteq \bigoplus_k m(\mathcal{H}_k(\mathbf{\Gamma}_\mu))$. To this end we let f be an arbitrary element of $m(\mathbf{\Gamma}_\mu)$ and write $m(f)$ as $m(f) = g_0+g_1+\cdots+g_{m_o}$, where g_k is homogeneous of degree k in y (and therefore also of degree $m_o - k$ in x, where $m_o = deg\ m(f)$). Also let us write f as the sum of its y-homogeneous componenents $f = f_0 + f_1 + \cdots f_{n(\mu')}$, with $f_k \in \mathcal{H}_k(\mathbf{\Gamma}_\mu)$. Now clearly for each k, either $g_k = m(f_k)$ or $g_k = 0$, so $m(f) \in \bigoplus_k m(\mathcal{H}_k(\mathbf{\Gamma}_\mu))$.

To complete the proof we must establish that for each k, the two spaces $m(\mathcal{H}_k(\mathbf{\Gamma}_\mu))$ and $\mathcal{H}_k(\mathbf{\Gamma}_\mu)$ have the same dimension and character. To simplify notation, set $V = \mathcal{H}_k(\mathbf{\Gamma}_\mu)$. For each d, the set $V_{\leq d} = \{f \in V : deg\ m(f) \geq d\}$ is an S_n-invariant subspace of V, provided we employ the convention $deg\ m(0) = \infty$. The linear function ϕ_d: $\phi_d(f) = f^{(d)}$ maps $V_{\geq d}$ surjectively on $\mathcal{H}_d(m(V))$, with kernel $V_{\geq d+1}$, and commutes with the S_n action. (Here the symbol $\mathcal{H}_d$ refers to the customary total degree, not the y-degree). As S_n modules we therefore have $\mathcal{H}_d(m(V)) \cong V_{\geq d}/V_{\geq d+1}$ and hence $char\ m(V) = \sum_d char\ \mathcal{H}_d(m(V)) = \sum_d char\ V_{\geq d} - char\ V_{\geq d+1} = char\ V_{\geq 0} = char\ V$. Q.E.D.

The following result also holds without assuming the $n!$ conjecture.

Proposition 2.3

Our determinant $\Delta_\mu(x,y)$ is an element of $m(\mathbf{\Gamma}_\mu)$. Indeed, we have the identity

$$\Delta_\mu(x,y) \;=\; \frac{1}{\Delta_{T_o}(a(T_o))}\ m(\Phi_\mu) \qquad 2.27$$

Proof

To show 2.27 we resort to the identity in 2.11. Note that by definition

$$\Delta_{p,q_o}(x,y) \;=\; \sum_{\sigma \in S_n} \epsilon(\sigma)\ \sigma\ x^p y^{q_o}\ .$$

Moreover, we have observed (in the proof of Theorem 1.1) that

$$\Delta_\mu(x,y) \;=\; \Delta_{p_o,q_o}(x,y)\ , \qquad 2.28$$

where p_o and q_o are the exponent sequences giving

$$m_{T_o}(x,y) \;=\; x^{p_o} y^{q_o}$$

(see 1.9). This given, we need only verify that the surviving terms of minimum degree in 2.11 add up to a non vanishing multiple of $\Delta_{p_o,q_o}(x,y)$. However, we can easily see that $\Delta_{p,q_o}(x,y)$ does not vanish only if the exponent vector p is not stabilized by any of the elements of the stabilizer of q_o. Since the latter is the column group of T_o, this condition forces the components p_i to take different values as i varies in a column of T_o. A moment's reflection should reveal that, under this condition, the smallest possible value of $|p|$ is $n(\mu)$. Indeed, such a minimum value of $|p|$ can be achieved only when for some β in the column group of T_o we have $p = p_o\beta$. But in that case

$$\Delta_{p,q_o}(x,y) \;=\; \epsilon(\beta)\, \Delta_\mu(x,y) \;\; .$$

This gives that all the homogeneous components of $\Phi_\mu(x,y)$ of degree less than $n(\mu)+n(\mu')$ vanish identically and the component of degree $n(\mu)+n(\mu')$ reduces to the expression

$$\sum_{\beta\in N(T_o)} \epsilon(\beta)\; a(T_o)^{p_o\beta}\; \Delta_\mu(x,y) \;\; .$$

From our Remark 1.1 we see that this sum evaluates to

$$\Delta_{T_o}(a(T_o)) \;\; \Delta_\mu(x,y) \;\; .$$

where $\Delta_{T_o}(x)$ denotes the corresponding Garnir polynomial. Now by definition (see 1.18), $\Delta_{T_o}(x)$ factors into a product of the Vandermonde determinants corresponding to the columns of T_o. Thus $\Delta_{T_o}(x)$ does not vanish at $x = (x_1,\ldots,x_n)$ if and only if the components x_i take different values when i varies in a column of T_o. The very construction of $a(T_0)$ guarantees this property, as our $\alpha_1,\alpha_2,\ldots,\alpha_k$ are all distinct. Thus the constant $\Delta_{T_o}(a(T_o))$ is different from zero and 2.27 must hold true.

Q.E.D.

As immediate corollary we obtain our desired connection between $m(\mathbf{\Gamma}_\mu)$ and $\mathbf{H}_\mu[X,Y]$.

Theorem 2.2

We always have the containment

$$\mathbf{H}_\mu[X,Y] \;\subseteq\; m(\mathbf{\Gamma}_\mu) \;\; . \qquad 2.29$$

In particular, on the $n!$ conjecture *these two spaces must be one and the same.*

Proof

Note that if $Q = m(f)$ for some $f \in \mathbf{\Gamma}_\mu$ and $\partial_{x_i} Q \neq 0$ then $\partial_{x_i} Q = m(\partial_{x_i} f)$. Clearly, the same must be true for ∂_{y_i}. In other words $m(\mathbf{\Gamma}_\mu)$ is closed under differentiation. Since we have just shown that $\Delta_\mu(x,y) \in m(\mathbf{\Gamma}_\mu)$, the whole of $\mathbf{H}_\mu[X,Y]$ must lie in $m(\mathbf{\Gamma}_\mu)$ as well. The last assertion follows from the dimension equality in 2.26.

We are finally in a position to verify the equalities 2.1 (a) and (b).

Theorem 2.3

On the $n!$ conjecture $\mathbf{\Gamma}_\mu$ and $\mathbf{H}_\mu$ have the same y-graded character. In particular we must have

$$C_\mu(q,1) \;=\; \tilde{H}_\mu(x;q,1) \;=\; \tilde{H}_{\mu_1}(x;q)\tilde{H}_{\mu_2}(x;q)\cdots\tilde{H}_{\mu_k}(x;q)\ . \tag{2.30}$$

Proof

This follows directly from Proposition 2.2 and Theorem 2.2. Q.E.D.

Theorems 2.1 and 2.3 combined yield the validity of 2.1 a) on the $n!$ conjecture. Clearly, the dual identity 2.1 b) can be established in an analogous manner by simply reversing the roles of x and y. We may be tempted to leave it at that and go on with our study of $\mathbf{H}_\mu$. However, we shall see that it is worth delving a bit further into the nature of the additional modules we have introduced in this section. To do this in a precise manner it will be convenient, (also in anticipation of the developments of the next section), to make some slight changes of notation. First of all we should distinguish the y-graded nature of the ingredients used in our proof of 2.1 a) from the x-graded nature of those needed to study 2.1 b). We shall do this by denoting the power series in 2.6 by Φ^y_μ rather than Φ_μ and the module in 2.8 by $\mathbf{\Gamma}^y_\mu$. We shall then set

$$\Phi^x_\mu(x,y) \;=\; \sum_{\sigma\in S_n} \epsilon(\sigma)\,\sigma\,\left(\,e^{(y,b(T_o))}\,m_{T_o}(x)\right)\ , \tag{2.31}$$

and let

$$\mathbf{\Gamma}^x_\mu \;=\; \mathcal{L}\{\partial^p_x\partial^q_y\,\Phi^x_\mu\,\} \tag{2.32}$$

This given, the reader should have no difficulty obtaining the analogue of theorem 2.1 for $\mathbf{\Gamma}^x_\mu$ and then translating the arguments that yielded 2.1 a) into arguments establishing 2.1 b). On the $n!$ conjecture we would then deduce that the two modules $m(\mathbf{\Gamma}^y_\mu)$ and $m(\mathbf{\Gamma}^x_\mu)$, in spite of their apparently different definitions, must be identical. In fact, Theorem 2.2 and its analogue concerning $m(\mathbf{\Gamma}^x_\mu)$ yield that they are then both equal to $\mathbf{H}_\mu$. These are the modules $\mathbf{M}^x_\mu$ and $\mathbf{M}^y_\mu$ we referred to in the introduction. In the next section we shall study their relation to the coordinate rings of the orbits studied in section 1 and as a by-product obtain that the equality $m(\mathbf{\Gamma}^x_\mu) = m(\mathbf{\Gamma}^y_\mu)$ is in fact equivalent to the $n!$-conjecture.

3. Gradings and the $n!$ conjecture.

We shall have to deal in this section with a number of related spaces, some of which we have already introduced. Before we prove anything it will be good to get across at least a general view of the relationships between these various ingredients. Let us recall that in Section 1 we have defined the ideal

$$J_{[\rho_\mu]} = \{p(x,y) \in \mathbf{Q}[X,Y] : p(a,b) = 0\ \ \forall\ \ (a,b) \in [\rho_\mu]_{S_n}\}.$$

In words, $J_{[\rho_\mu]}$ is the ideal of polynomials in x, y which vanish at all points in the S_n orbit of $\rho_\mu = (a(T_o), b(T_o))$. We may therefore identify the quotient ring

$$R_{[\rho_\mu]} = \mathbf{Q}[X, Y]/J_{[\rho_\mu]}$$

with the ring of rational-valued functions on the orbit in question.

Given an S_n-invariant choice of weights w assigning weight w_x to each variable $x_1, \ldots, x_n$ and w_y to each $y_1, \ldots, y_n$, we have formed the associated graded ideal

$$gr_w\ J_{[\rho_\mu]} = \mathcal{L}\{h_w(p) : p \in J_{[\rho_\mu]}\}$$

and graded ring

$$gr_w\ R_{[\rho_\mu]} = \mathbf{Q}[X, Y]/(gr_w\ J_{[\rho_\mu]}),$$

where $p \to h_w(p)$ denotes the operation of extracting the w-homogeneous component of highest degree in the polynomial $p(x, y)$.

For our present purpose, it will be useful to allow either or both components w_x, w_y of the weight w to be zero. Then, for example, taking $w = (1, 0)$ makes the w-degree simply the x-degree, so that the operation gr_w is a grading with respect to x only. When this particular choice of w is in force, we write gr_x in place of gr_w. Similarly we write gr_y for gr_w when $w = (0, 1)$.

Another important special grading occurs when we take both w_x and w_y positive, but w_x extremely large compared to w_y. If w_x is sufficiently large then the highest w-homogeneous component of a given polynomial will be the highest y-degree component of its highest x-degree component, so that the operation gr_w in this case reduces to $gr_y\ gr_x$, and we shall so denote it. In similar fashion we can realize $gr_x\ gr_y$ as gr_w for a suitable w.

Extending the inner product defined in 1.1 to allow one of P or Q to be a formal power series, it is still well-defined, and in fact makes the formal power series ring into the dual space of the polynomial ring. The second part of Proposition 1.1 (see I.4) continues to hold: allowing power series solutions, $J^\perp$ is the space of solutions to the differential equations $p(\partial_x, \partial_y)f = 0$, for all $p(x, y) \in J$, even if J is a non-homogeneous ideal. In our context J will always be S_n-invariant and $\mathbf{Q}[X, Y]/J$ finite-dimensional. Then we can prove that $J^{\perp\perp} = J$ and that $J^\perp$ is equivalent to $\mathbf{Q}[X, Y]/J$ as an S_n module, and indeed as a w-graded S_n module when J is w-homogeneous.

What we shall establish is that all the spaces

$$\Gamma^x_\mu,\ \Gamma^y_\mu,\ m(\Gamma^x_\mu) \text{ and } m(\Gamma^y_\mu) \qquad 3.1$$

have the form $(gr_w\ J_{[\rho_\mu]})^\perp$ for suitable choices of w. To achieve this we need to introduce another space which has been a missing piece from our considerations to this point. To this end, by analogy to 2.6–2.8 and 2.31–2.32, we set

$$\Phi_\mu = \sum_{\sigma \in S_n} \epsilon(\sigma)\sigma(e^{(x, a(T_o))}e^{(y, b(T_o))}) \qquad 3.2$$

and let

$$\Gamma_\mu = \mathcal{L}\{\partial_x^p \partial_y^q \Phi_\mu\}. \qquad 3.3$$

It then develops that $\Gamma_\mu = J^\perp_{[\rho_\mu]}$, and that the various spaces in 3.1 all occur as $m_w(\Gamma_\mu)$ for suitable weights w, where $m_w(\Gamma_\mu) = \{m_w(f) : f \in \Gamma_\mu\}$, and $f \to m_w(f)$ denotes the operation of extracting the w-homogeneous component of *lowest* degree from a formal power series f.

This given, our characterization of the spaces in 3.1 follows from the fact, which we can prove for any of the ideals J occuring in our picture, that $m_w(J^\perp) = (gr_w\ J)^\perp$.

Our first goal is now to prove the statements made above. Having completed that, and in particular having shown that $m(\Gamma_\mu^x) = (gr_y\ gr_x\ J_{[\rho_\mu]})^\perp$ and $m(\Gamma_\mu^y) = (gr_x\ gr_y\ J_{[\rho_\mu]})^\perp$, we shall then turn to the question of deriving the $n!$ conjecture on the assumption that $gr_y\ gr_x\ J_{[\rho_\mu]} = gr_x\ gr_y\ J_{[\rho_\mu]}$.

Proposition 3.1

Let J be an S_n invariant ideal in $\mathbf{Q}[X,Y]$, w-homogeneous for a weighted degree deg_w. Assume that $\mathbf{Q}[X,Y]/J$ is finite-dimensional. Then

(1) $J^\perp = \{f : p(\partial_x, \partial_y)f = 0\ \forall\ p(x,y) \in J\}$, $J^\perp$ is a w-homogeneous space of formal power series, and in particular $J^\perp$ is closed under differentiation;

(2) $J^{\perp\perp} = J$; and

(3) $J^\perp$ and $\mathbf{Q}[X,Y]/J$ are equivalent as w-graded S_n modules.

Proof

The identity

$$J^\perp = \{f : p(\partial_x, \partial_y)f = 0\ \forall\ p(x,y) \in J\}$$

holds just as in Proposition 1.1, without essential change to the proof given there. Since w-homogeneous polynomials and power series of different w-degrees are mutually orthogonal (indeed, distinct monomials are mutually orthogonal), it is clear that if f is in $J^\perp$ so is each w-homogeneous component of f. That is what it means for $J^\perp$ to be a w-homogeneous space.

For (2) we only have to prove $J^{\perp\perp} \subseteq J$ since the reverse is obvious. Let $x^{r_1}y^{s_1}, \ldots, x^{r_N}y^{s_N}$ be a collection of monomials whose images modulo J form a basis of $\mathbf{Q}[X,Y]/J$. For $i = 1, \ldots, N$, let $l_i(p)$ denote the coefficient of $x^{r_i}y^{s_i}$ in the expansion of p on this basis, modulo J, for any polynomial $p(x,y)$. Then there is a formal power series f_i such that the linear functional l_i is given by $l_i(p) = \langle f_i, p\rangle$ for all p. Namely, for monomials we have

$$\langle x^p y^q, x^r y^s\rangle = \begin{cases} p!q! & \text{if } (r,s) = (p,q), \text{ or} \\ 0 & \text{otherwise.} \end{cases}$$

From this it is plain that

$$f_i = \sum_{p,q} l_i(x^p y^q) x^p y^q / p!q!$$

will do. Since by definition $l_i(p) = 0$ for $p \in J$ we clearly have $f_i \in J^\perp$. Now if q is a polynomial in $J^{\perp\perp}$ then $\langle f_i, q\rangle = 0$ for all i and therefore $q \equiv 0$ modulo J, that is, $q \in J$.

For (3) note that the scalar product $\langle f, p\rangle$ is well-defined for $f \in J^\perp$ and $p \in \mathbf{Q}[X,Y]/J$, since if p' and p are the same modulo J then $p - p'$ is orthogonal to f. This pairing makes $J^\perp$ the dual space to $\mathbf{Q}[X,Y]/J$, as we can easily see: if f is orthogonal to every p, then clearly $f = 0$, while if $p \in \mathbf{Q}[X,Y]$ is orthogonal to every $f \in J^\perp$ then $p \in J$ by (2). Likewise corresponding w-homogeneous components of $J^\perp$ and $\mathbf{Q}[X,Y]/J$ are dually paired, and of course the pairing is S_n invariant. But a finite-dimensional rational representation of S_n is always equivalent to its dual, since they have the same character. Q.E.D.

The operation of grading an ideal is best understood in terms of a filtration of the corresponding quotient ring. We shall use this concept heavily later on to prove the main result of this section, but introduce it now because it will aid in proving our next proposition. Thus, given an ideal J in $R = \mathbf{Q}[X,Y]$ (or any polynomial ring), we define $R_J = R/J$ and $gr_w\ R_J = R/(gr_w\ J)$. Our object is to make a direct construction of $gr_w\ R_J$ from R_J. To this end we define the w-filtration of R_J to be the sequence of subspaces

$$F_{\leq 0} \subseteq F_{\leq 1} \subseteq F_{\leq 2} \subseteq \cdots \subseteq R_J, \tag{3.4}$$

where

$$F_{\leq k} = \{p + J : p \in R,\ deg_w(p) \leq k\} \tag{3.5}$$

is the subspace spanned modulo J by all polynomials of w-degree at most k. To say that the spaces $F_{\leq k}$ form a *filtration of the ring* R_J means that they satisfy the evident property

$$F_{\leq k} F_{\leq l} \subseteq F_{\leq k+l} \quad \text{for all } k \text{ and } l. \tag{3.6}$$

Owing to this property, there is a well-defined multiplication

$$(F_{\leq k}/F_{\leq k-1}) \times (F_{\leq l}/F_{\leq l-1}) \to F_{\leq k+l}/F_{\leq k+l-1},$$

which we use to make the *a priori* purely formal space

$$F_{\leq 0} \oplus (F_{\leq 1}/F_{\leq 0}) \oplus (F_{\leq 2}/F_{\leq 1}) \oplus \cdots \tag{3.7}$$

into a graded ring. This construction is related to $gr_w\ R_J$ by the following lemma.

Lemma 3.1

The graded ring defined in 3.7 is isomorphic in a natural way to the ring $gr_w\ R_J$, defined as $R/(gr_w\ J)$.

Proof

Let $p \in R$ be a w-homogeneous polynomial, say of degree k. Modulo $gr_w\ J$, p represents an element $\tilde{p}$ of $gr_w\ R_J$. Also, modulo J, p represents an element of $F_{\leq k}$ and therefore also an element q of $F_{\leq k}/F_{\leq k-1}$. Our procedure is to define the desired isomorphism α by the rule $\alpha(\tilde{p}) = q$. Of course we must check that this is first well-defined, and then an isomorphism of graded rings.

To see that α is well-defined we have only to note that if $\tilde{p} = \tilde{p'}$ then $p - p'$ belongs to $gr_w\ J$ and is homogeneous of degree k, so we must have $p - p' = h_w(s)$ for some polynomial $s \in J$. Then decomposing s as $h_w(s) + r$, we have $p - p' + r \in J$, where r has only terms of degree less than k. Now letting q' denote the image of p' in $F_{\leq k}/F_{\leq k-1}$ we see that $q - q' \equiv -r \equiv 0$.

That α is a ring homomorphism is immediate, since the multiplication in 3.7 is induced from that in R_J. It is also clear that α is surjective. It remains only to show that the kernel of α is zero. But if $\alpha(\tilde{p}) = 0$, it means that the image of p modulo J belongs to $F_{\leq k-1}$, that is, $p = t + j$ for some $j \in J$ and some t of w-degree less than k. But then p, being homogeneous of w-degree k, must be $h_w(j)$. In other words $p \in gr_w(J)$, so $\tilde{p} = 0$. Q.E.D.

Proposition 3.2

Let J be an S_n invariant ideal in $\mathbf{Q}[X,Y]$, and assume $\mathbf{Q}[X,Y]/J$ is finite-dimesional. Then

$$(gr_w\ J)^{\perp} = m_w(J^{\perp}). \tag{3.8}$$

Moreover, $\mathbf{Q}[X,Y]/J$ and $\mathbf{Q}[X,Y]/(gr_w\ J)$ are equivalent as S_n-modules, and even as x- or y-graded S_n-modules in the event that J is x- or y-homogeneous to begin with.

Proof

One containment of the equality in 3.8 is easily derived. Namely, if $p \in J$ and $f \in J^{\perp}$, we may write

$$p = h_w(p) + r, \quad f = m_w(f) + R,$$

where r has only terms of degree less than $deg_w\ h_w(p)$ and R likewise has only terms of degree greater than $deg_w\ m_w(f)$. Then we have

$$p(\partial_x, \partial_y)f = h_w(p)(\partial_x, \partial_y)m_w(f) + S,$$

in which S has only terms of degree strictly greater than the degree of the first term. Since the left hand side is zero, so must be $h_w(p)(\partial_x, \partial_y)m_w(f)$, which shows that $m_w(J^{\perp}) \subseteq (gr_w\ J)^{\perp}$.

The argument used to prove Proposition 2.2 shows, without essential changes, that $dim\ m_w(J^{\perp}) = dim\ J^{\perp}$, and even that these two spaces are equivalent as (possibly graded) S_n modules. In turn we have $dim\ J^{\perp} = dim\ \mathbf{Q}[X,Y]/J$ and $dim\ (gr_w\ J)^{\perp} = dim\ \mathbf{Q}[X,Y]/(gr_w\ J)$ by Proposition 3.1. Hence the equality in 3.8 follows once we prove the second part of the proposition, that $\mathbf{Q}[X,Y]/J$ and $\mathbf{Q}[X,Y]/(gr_w\ J)$ are equivalent modules.

But this is a completely general fact, as is clear from the previous Lemma and 3.7, since a filtered S_n module is always equivalent to the direct sum of the successive quotients in the filtration. For the additional assertions in the x- or y- graded case, we need only note that when J is already x- or y- homogeneous, so are the filtering spaces $F_{\leq k}$ in 3.4. Q.E.D.

Now we have the tools at hand to establish the relations between the various spaces reviewed at the outset.

Theorem 3.1

We have the equalities

$$
\begin{aligned}
a)\ \Gamma_\mu &= J_{[\rho_\mu]}^{\perp} \\
b)\ \Gamma_\mu^x &= (gr_x\ J_{[\rho_\mu]})^{\perp} \\
c)\ \Gamma_\mu^y &= (gr_y\ J_{[\rho_\mu]})^{\perp} \\
d)\ m(\Gamma_\mu^x) &= (gr_y\ gr_x\ J_{[\rho_\mu]})^{\perp} \\
e)\ m(\Gamma_\mu^y) &= (gr_x\ gr_y\ J_{[\rho_\mu]})^{\perp}
\end{aligned}
\qquad 3.9
$$

In particular, all these spaces and the corresponding quotient rings are equivalent S_n modules, affording the left regular representation.

Proof

Our strategy here is to first prove (a). It then follows by Proposition 3.2 that each space on the right hand side will be $m_w(\Gamma_\mu)$ for the appropriate w, and that all these spaces have dimension $n!$. In Section 2 we have already shown that the spaces on the left hand side in (b)–(e) have dimension $n!$, so it will be enough to show that each is contained in $m_w(\Gamma_\mu)$ for its relevant w. This we can do by investigating the 'kernels' Φ_μ^x, Φ_μ^y, and Φ_μ that generate them.

To prove (a), we take note of the straightforward identity

$$p(\partial_x, \partial_y)e^{(x,a)}e^{(y,b)} = p(a,b)e^{(x,a)}e^{(y,b)}, \qquad 3.10$$

which holds for any polynomial $p(x,y)$. In particular, if we choose p so that $p(a(T), b(T)) = 0$ for every injective tableau T except T_o, while $p(a(T_o), b(T_o)) = 1$ then we shall have

$$p(\partial_x, \partial_y)\Phi_\mu = e^{(x,a(T_o))}e^{(y,b(T_o))}.$$

By S_n invariance this shows that the collection

$$\{e^{(x,a(T))}e^{(y,a(T))} : T \in \mathcal{IT}_\mu\} \qquad 3.11$$

is contained in Γ_μ. Since its linear span is closed under differentiation and contains Φ_μ, this collection is evidently a basis of Γ_μ, and therefore $dim\ \Gamma_\mu = n!$. From 3.10 we see that the collection in 3.11, and hence also the space Γ_μ, is contained in $J_{[\rho_\mu]}^{\perp}$. Both spaces have the same dimension so they are equal.

Identity (d) is implied by (b), for given a power series f that is already x-homogeneous, its lowest customary total degree component $m(f)$ is also its lowest y-degree component $m_y(f)$, and consequently $m(\Gamma_\mu^x)$ is the same as $m_y(\Gamma_\mu^x)$. In the same way, exchanging x and y, identity (e) is implied by (c).

It remains to prove (b) and (c). Appealing to symmetry for (c), we only treat (b). All we need show is that Φ_μ^x, as defined by 2.31, belongs to the space $m_x(\Gamma_\mu)$, which is the content of the lemma below.

Lemma 3.2

Up to a scalar factor, Φ^x_μ is the x-homogeneous component of smallest x-degree in Φ_μ. Likewise Φ^y_μ is the y-homogeneous component of smallest y-degree. More precisely we have

$$\begin{aligned} a)\quad m_x(\Phi_\mu) &= \frac{1}{p_o!}\Delta_{T_o}\big(a(T_o)\big)\ \Phi^x_\mu \\ b)\quad m_y(\Phi_\mu) &= \frac{1}{q_o!}\Delta_{T'_o}\big(b(T_o)\big)\ \Phi^y_\mu \end{aligned} \qquad 3.12$$

Proof

Expanding only the second exponential in 3.2 we may write

$$\Phi_\mu \;=\; \sum_q \frac{b^q(T_o)}{q!}\ \Phi_q(y) \qquad 3.13$$

where we have set

$$\Phi_q \;=\; \sum_{\sigma\in S_n} \epsilon(\sigma)\ \sigma\big(\ e^{(x,a(T_o))}\ y^q\ \big) \qquad 3.14$$

Note that if we rearrange the components of q by a permutation $\gamma\in S_n$, Φ_q changes according to the identity

$$\Phi_{q\gamma} \;=\; \epsilon(\gamma) \sum_{\sigma\in S_n} \epsilon(\sigma)\ \sigma\big(\ e^{(x,a(\gamma T_o))}\ y^q\ \big) \qquad 3.15$$

Thus, if γ is a permutation in the row group of T_o, 3.15 yields that

$$\Phi_{q\gamma} \;=\; \epsilon(\gamma)\ \Phi_q \qquad 3.16$$

and therefore Φ_q must vanish if γ is a simple transpostion and $q = q\gamma$. So the only summands that survive in 3.13 are those corresponding to exponent sequences q whose components take different values as their index varies in a row of T_o. However, we can easily see that the minimum value of $|q|$ for such an exponent sequence is obtained when $q = q_o\alpha$ where q_o is the exponent sequence that gives

$$m_{T'_o}(y) \;=\; y^{q_o} \qquad 3.17$$

and α is any element of the row group of T_o. Since Φ_q is y-homogeneous of y-degree $|q|$, we see from 3.13 that the homogeneous component of smallest y-degree in Φ_μ must be given by the expression

$$m_y(\Phi_\mu) \;=\; \frac{1}{q_o!} \sum_{\alpha\in P(T_o)} b^{q_o\alpha}(T_o)\ \Phi_{q_o\alpha}\ .$$

On the other hand, since α is in the row group of T_o we can use 3.16 and obtain the factorization

$$m_y(\Phi_\mu) \;=\; \frac{1}{q_o!}\ \Phi_{q_o} \sum_{\alpha\in P(T_o)} \epsilon(\alpha)\ b^{q_o\alpha}(T_o) \quad .$$

Recalling the definition 1.18 of a Garnir polynomial, we finally obtain that

$$m_y(\Phi_\mu) \;=\; \frac{1}{q_o!}\,\Phi_{q_o}\Delta_{T_o'}(b(T_o)) \;\; .$$

However, for $q = q_o$, Φ_{q_o} reduces to Φ^x_μ by Lemma 2.1, equation 2.10, which establishes 3.12 (b). The proof of 3.12 (a) is analogous. Q.E.D.

We come now to the main technical device available to us for establishing the $n!$ conjecture in any particular case.

Proposition 3.3

Let G be a finite group acting linearly on $\mathbf{Q}^m$, and let $x_1, \ldots, x_m$ be the coordinate functions on $\mathbf{Q}^m$, so that G also acts on the polynomial ring $R = \mathbf{Q}[x_1, \ldots, x_m]$. Let ρ be a point in $\mathbf{Q}^m$, let $[\rho]$ denote its G-orbit, and let $J_{[\rho]}$ be the ideal in R consisting of polynomials that vanish at all points of $[\rho]$. Finally let w be a G-invariant system of weights making $deg_w\ x_i$ strictly positive for each variable x_i, and let the Hilbert series of $gr_w\ R_{J_{[\rho]}} = R/(gr_w\ J_{[\rho]})$ be

$$H(q) = \sum_{k=0}^{m_o} h_k q^k,$$

where $h_{m_o} \neq 0$, that is, m_o is the largest degree k for which $\mathcal{H}_k(gr_w\ R_{J_{[\rho]}})$ does not reduce to zero. Then the following conditions are equivalent:

a) $h_0 + h_1 + \cdots h_k \geq h_{m_o-k} + h_{m_o-k+1} + \cdots + h_{m_o}$ for all $0 \leq k \leq m_0$;

b) $h_k = h_{m_o-k}$ for all $0 \leq k \leq m_0$;

c) $gr_w\ R_{J_{[\rho]}}$ is a Gorenstein graded algebra; and

d) the space $(gr_w\ J_{[\rho]})^\perp$ is the linear span

$$\mathcal{L}\{\partial_x^p \Delta\}$$

of all derivatives of a single polynomial Δ. (We describe this by saying that $(gr_w\ J_{[\rho]})^\perp$ is a cone*, and Δ is its* summit*.)*

Proof

The equivalence of (c) and (d) is well-known (see [3]) as is the implication (c) $\Rightarrow$ (b). Alternatively, one can obtain (d) $\Rightarrow$ (b) without recourse to (c) by using the **flip** operator of 1.14 for Δ.

So the only new feature, and indeed the only one that depends upon the set $[\rho]$ being a G-orbit, is the implication (a) $\Rightarrow$ (c). Given (a), we have *a fortiori* $h_{m_o} = h_0 = 1$. Thus up to a scalar multiple $gr_w\ R_{J_{[\rho]}}$ contains a unique homogeneous element δ of w-degree m_o, and correspondingly $(gr_w\ J_{[\rho]})^\perp$ contains a unique w-homogeneous polynomial Δ of the same degree. The 1-dimensional space $\mathcal{L}\{\delta\}$ affords a 1-dimensional representation of G, with character ϵ, let us say.

Since $[\rho]$ is a G-orbit, the representation of G afforded by $R_{J_{[\rho]}}$ and $gr_w\ R_{J_{[\rho]}}$ is equivalent to a quotient of the regular representation, and thus contains the character ϵ with multiplicity one. In particular this character cannot occur at all in any degree less than m_o; in terms of the filtration 3.4 of $R_{J_{[\rho]}}$ this means that ϵ does not occur in $F_{\leq m_o-1}$. Let us define on $R_{J_{[\rho]}}$ a scalar product $\langle , \rangle_\epsilon$ by the rule

$$\sum_{\sigma\in G} \epsilon(\sigma^{-1})\sigma(pq) = \langle p, q\rangle_\epsilon \delta. \qquad 3.18$$

This definition makes sense because the element of $R_{J_{[\rho]}}$ represented by δ is non-zero (otherwise δ, being homogeneous, would belong to $gr_w\ J_{[\rho]}$), and its scalar multiples are the only elements transforming under G according to the character ϵ.

The scalar product in 3.18 is non-degenerate, as we can see by the following argument. Suppose p were such that $\langle p, q\rangle_\epsilon = 0$ for every q. Unless p vanishes at every point of $[\rho]$, we can choose q so that $pq = 1$ at one point $\gamma\rho$ and $pq = 0$ at all other points of $[\rho]$. But if the sum in 3.18 were zero for this function pq, then by construction it would also have to be zero for $\alpha(pq)$, for every $\alpha \in G$, and hence it would be zero with any function f in place of pq, that is, the scalar product $\langle , \rangle_\epsilon$ would be identically zero. And this is clearly not the case, for $\langle 1, \delta\rangle_\epsilon = |G|$.

As we have already noted, $F_{\leq m_o-1}$ does not contain the character ϵ, which implies by the filtration property 3.6 that $F_{\leq k}$ and $F_{\leq m_o-1-k}$ are orthogonal to each other for the scalar product in 3.18. However the inequalities in (a) mean precisely that $dim\ F_{\leq k} + dim\ F_{\leq m_o-1-k} \geq dim\ R_{J_{[\rho]}}$. Therefore these inequalities must be equalities, and we must have

$$(F_{\leq k})^{\perp} = F_{\leq m_o-1-k} \qquad 3.19$$

for each k. Then the scalar product $\langle , \rangle_\epsilon$ induces a non-degenerate pairing of $F_{\leq k}/F_{\leq k-1}$ with $F_{\leq m_o-k}/F_{\leq m_o-1-k}$, which by 3.7 and Lemma 3.1 is simply the multiplication pairing $\mathcal{H}_k(gr_w\ R_{J_{[\rho]}}) \times \mathcal{H}_{m_o-k}(gr_w\ R_{J_{[\rho]}}) \to \mathcal{H}_{m_o}(gr_w\ R_{J_{[\rho]}})$. By definition, the non-degeneracy of this pairing means that $(gr_w\ R_{J_{[\rho]}})$ is Gorenstein. Q.E.D.

For an alternative elementary proof of this proposition, not using the theory of Gorenstein rings, we refer the reader once again to [9].

We are now ready to proceed in earnest toward our main result. Throughout the rest of this section, we fix the partition μ and the S_n orbit $[\rho_\mu]$ of $\rho_\mu = (a(T_o), b(T_o))$, and set

$$R = \mathbf{Q}[X,Y], \quad J = J_{[\rho_\mu]}, \quad R_J = R/J.$$

Associated with $gr_x\ R_J$ and $gr_y\ R_J$ are two filtrations

$$F^x_{\leq 0} \subseteq F^x_{\leq 1} \subseteq \cdots \subseteq F^x_{\leq n(\mu)} = R_J \qquad 3.20$$

and

$$F^y_{\leq 0} \subseteq F^y_{\leq 1} \subseteq \cdots \subseteq F^x_{\leq n(\mu')} = R_J, \qquad 3.21$$

where $F^x_{\leq k}$ consists of all elements in R_J representable by a polynomial $p(x,y)$ of degree at most k in x; and similarly for y. Note that the termination of these filtrations at degrees $n(\mu)$ and $n(\mu')$, respectively, follows from Theorem 2.2, together with the identifications 3.9 (b) and (c) of Γ^x_μ and Γ^y_μ as $(gr_x\ J)^\perp$ and $(gr_y\ J)^\perp$.

The x-graded ring $gr_x\ R_J$ comes equipped with *two* possible y-filtrations, either of which may be used to produce from it a doubly graded ring. The first is the normal y-filtration of $gr_x\ R_J = R/(gr_x\ J)$ defined by 3.5, which is the filtration associated to $gr_y\ gr_x\ R_J$. We may denote this filtration

$$G^y_{\leq 0} \subseteq G^y_{\leq 1} \subseteq \cdots \subseteq gr_x\ R_J. \tag{3.22}$$

The second and more remarkable filtration is induced by the filtration F^y of R_J. Specifically, for each k the k-th graded component of $gr_x\ R_J$, namely $F^x_{\leq k}/F^x_{\leq k-1}$, is itself filtered by its subspaces $(F^x_{\leq k}\cap F^y_{\leq l})/(F^x_{\leq k-1}\cap F^y_{\leq l})$, and we therefore set

$$G'^y_{\leq l} = \bigoplus_{k=0}^{n(\mu)} (F^x_{\leq k}\cap F^y_{\leq l})/(F^x_{\leq k-1}\cap F^y_{\leq l}) \tag{3.23}$$

Of course the filtration property 3.6 for F^y immediately implies the same for G'^y, and so we obtain a doubly graded ring

$$\begin{aligned} bigr\ R_J &= \bigoplus_{k=0}^{n(\mu)}\bigoplus_{l=0}^{n(\mu')} \mathcal{H}^x_k(G'^y_{\leq l}/G'^y_{\leq l-1}) \\ &= \bigoplus_{k=0}^{n(\mu)}\bigoplus_{l=0}^{n(\mu')} (F^x_{\leq k}\cap F^y_{\leq l})/(F^x_{\leq k-1}\cap F^y_{\leq l} + F^x_{\leq k}\cap F^y_{\leq l-1}). \end{aligned} \tag{3.24}$$

From the second equality in 3.24 we see that in the construction of $bigr\ R_J$, x and y actually play perfectly symmetrical roles.

Proposition 3.4

The ring $bigr\ R_J$ defined in 3.24 is a Gorenstein graded algebra.

Proof

We are going to imitate the proof of Proposition 3.3, with some refinements to take account of the double grading. To begin, on R_J we define a scalar product $\langle,\rangle_\epsilon$ just as in 3.18 by

$$\sum_{\sigma\in S_n} \epsilon(\sigma)\sigma(pq) = \langle p,q\rangle_\epsilon \Delta_\mu. \tag{3.25}$$

In the present case, of course, ϵ is the sign character of S_n, and we know that the bihomogeneous polynomial Δ_μ represents a non-zero element of R_J, since otherwise it would belong to $gr_w\ J_\mu$ for any w, contrary to Theorem 1.1. Then 3.25 is well-defined since every S_n-alternating element of R_J must be a scalar multiple of Δ_μ.

From the equivalence of $gr_x\ R_J$ with Γ_μ^x as x-graded S_n-modules, we see that the sole instance of the alternating representation in $gr_x\ R_J$ occurs in the highest x-degree, $n(\mu)$, which is to say that $F^x_{\leq n(\mu)-1}$ contains no alternating elements. It follows that the spaces $F^x_{\leq k}$ and $F^x_{\leq n(\mu)-1-k}$ are mutually orthogonal for the scalar product $\langle , \rangle_\epsilon$. Now we already know that each space $F^x_{\leq k}$ has dimension

$$dim\ F^x_{\leq k} = \sum_{k'\leq k} dim\ \mathcal{H}_k(\Gamma_\mu^x)$$

and further, as a consequence of Theorem 2.2, that

$$dim\ \mathcal{H}_k(\Gamma_\mu^x) = \binom{n}{\mu'} \prod_i \frac{1-q^{\mu'_i}}{1-q}\Bigg|_{q^k} ,$$

where the vertical bar denotes extracting the coefficient of q^k. In particular,

$$dim\ \mathcal{H}_k(\Gamma_\mu^x) = dim\ \mathcal{H}_{n(\mu)-k}(\Gamma_\mu^x)$$

and therefore $dim\ F^x_{\leq k} + dim\ F^x_{\leq n(\mu)-1-k} = n! = dim\ R_J$. Hence, with respect to our scalar product $\langle , \rangle_\epsilon$, we have for all k

$$(F^x_{\leq k})^\perp = F^x_{\leq n(\mu)-1-k}. \qquad 3.26$$

By analogous reasoning with y in place of x we also have for all l

$$(F^y_{\leq l})^\perp = F^y_{\leq n(\mu')-1-l} \qquad 3.27$$

Given these orthogonalities, we see that our scalar product induces for each k and l a well-defined pairing of the space

$$\mathcal{H}_{k,l}(bigr\ R_J) = (F^x_{\leq k} \cap F^y_{\leq l})/(F^x_{\leq k-1} \cap F^y_{\leq l} + F^x_{\leq k} \cap F^y_{\leq l-1})$$

with

$$\mathcal{H}_{h,j}(bigr\ R_J) = (F^x_{\leq h} \cap F^y_{\leq j})/(F^x_{\leq h-1} \cap F^y_{\leq j} + F^x_{\leq h} \cap F^y_{\leq j-1}),$$

where we have set $h = n(\mu) - k$ and $j = n(\mu') - l$. Moreover this pairing is clearly the same as that given by multiplication

$$\mathcal{H}_{k,l}(bigr\ R_J) \times \mathcal{H}_{h,k}(bigr\ R_J) \to \mathcal{H}_{n(\mu),n(\mu')}(bigr\ R_J),$$

followed by extracting the coefficient of Δ_μ. We are to show that the pairing is non-degenerate, which is to say, that the orthogonal complement of $F^x_{\leq k} \cap F^y_{\leq l}$ in $F^x_{\leq h} \cap F^y_{\leq j}$ is $(F^x_{\leq h-1} \cap F^y_{\leq j} + F^x_{\leq h} \cap F^y_{\leq j-1})$. Now the orthogonal complement in question is

$$F^x_{\leq h} \cap F^y_{\leq j} \cap (F^x_{\leq k} \cap F^y_{\leq l})^\perp = F^x_{\leq h} \cap F^y_{\leq j} \cap (F^x_{h-1} + F^y_{j-1}),$$

by 3.26–3.27, so we are reduced to proving

$$F^x_{\leq h} \cap F^y_{\leq j} \cap (F^x_{h-1} + F^y_{j-1}) = (F^x_{\leq h-1} \cap F^y_{\leq j} + F^x_{\leq h} \cap F^y_{\leq j-1}). \qquad 3.28$$

But 3.28 is an instance of a completely general and easy to prove identity holding for any subspaces of a vector space:

$$\text{if } X_1 \subseteq X \text{ and } Y_1 \subseteq Y, \text{ then } X \cap Y \cap (X_1 + Y_1) = X_1 \cap Y + X \cap Y_1.$$

Q.E.D.

Remark 3.1

One might at first suppose that Proposition 3.4 combined with the 'cone' condition (d) of Proposition 3.3 could be used to prove the $n!$ conjecture outright. It is quite interesting to explore just what goes wrong with this supposition. The difficulty is that $bigr\ R_J$ need not be a quotient of $R = \mathbf{Q}[X, Y]$ at all. The reason for this is that while every element a of $F^x_{\leq k} \cap F^y_{\leq l}$ is representable modulo J by a polynomial $p(x, y)$ of x-degree at most k, and also by a polynomial $p'(x, y)$ of y-degree at most l, it may not be possible to represent it so simultaneously by the *same* polynomial. In that case, a can represent an element of $\mathcal{H}_{k,l}(bigr\ R_J)$ which is not the image of any polynomial in R.

In fact, this situation actually occurs: in place of a Ferrers diagram μ, we may take *any* diagram D consisting of n lattice squares in the plane, and construct $[\rho_D]$, $J_{[\rho_D]}$, and $bigr\ R/J_{[\rho_D]}$ all by exact analogy to what we have done for partitions. When D is not a partition the ring $bigr\ R_J$ usually does fail to be generated by X and Y. Nevertheless it is a very interesting ring in its own right, which we expect not to depend on the parameters α and β introduced into the construction by 1.6. Assuming this and the $C = \tilde{H}$ conjecture, it is appropriate to think of the Frobenius characteristic of $bigr\ R/J_{[\rho_D]}$ as a kind of Macdonald polynomial $\tilde{H}_D(x; q, t)$ indexed by an arbitrary diagram D.

At present we can prove very little about $\tilde{H}_D$, although it does possess the pleasant property that when $D = D_1 \oplus D_2$, meaning that D_1 and D_2 occupy disjoint sets of rows and columns, and D is their union, then $\tilde{H}_D = \tilde{H}_{D_1}\tilde{H}_{D_2}$. There is also reason to suspect further relations, especially when D is a skew-Ferrers diagram. In [10], we managed to define the polynomials $\tilde{H}_D$ for all two-rowed skew diagrams based solely on their expected properties, without having to prove anything about the underlying rings. Everything done there was however necessarily motivated by hidden considerations involving $bigr\ R/J_{[\rho_D]}$.

Returning from this digression, we arrive at our main theorem.

Theorem 3.2

The condition $m(\Gamma^x_\mu) = m(\Gamma^y_\mu)$ is equivalent to the $n!$ conjecture for the partition μ. Indeed, both spaces are then equal to H_μ, and moreover for $J = J_{[\rho_\mu]}$, we have

$$bigr\ R_J = gr_x\ gr_y\ R_J = gr_y\ gr_x\ R_J. \tag{3.29}$$

Proof

It is actually 3.29 that we are going to prove. For then the ring $gr_y\ gr_x\ R_J$ is Gorenstein, and by Proposition 3.3 the space $(gr_y\ gr_x\ J)^\perp$ is a cone, necessarily generated by its element Δ of bidegree $(n(\mu), n(\mu'))$. However by Theorem 1.1, this element Δ must be a scalar multiple of Δ_μ, which shows that $H_\mu = (gr_y\ gr_x\ J)^\perp = m(\Gamma^x_\mu) = m(\Gamma^y_\mu)$.

What we must show is that in $gr_x\ R_J$ the two filtrations G^y and G'^y given by 3.22 and 3.23 are the same. Now an element of $\mathcal{H}^x_k(G^y_{\leq l})$ is by definition represented modulo $gr_x\ J$ by a polynomial $p(x,y)$ homogeneous of bidegree k, l. Hence it also belongs to $\mathcal{H}^x_k(G'^y_{\leq l})$, that is, we always have

$$G^y_{\leq l} \subseteq G'^y_{\leq l}. \qquad 3.30$$

On the assumption that $gr_x\ gr_y\ R_J = gr_y\ gr_x\ R_J$, the dimension of $G^y_{\leq l}$ is equal to

$$\sum_{l' \leq l} \sum_k dim\ \mathcal{H}_{k,l'}(gr_x\ gr_y\ R_J) = \sum_{l' \leq l} dim\ \mathcal{H}^y_{l'}(gr_y\ R_J) = dim\ F^y_{\leq l}. \qquad 3.31$$

On the other hand G'^y_l is the direct sum of the successive quotients in the filtration of the space $F^y_{\leq l}$ by its intersections with the spaces $F^x_{\leq k}$. Therefore we always have $dim\ G'^y_l = dim\ F^y_{\leq l}$, and this with 3.30 and 3.31 establishes the result. Q.E.D.

4. Our bigraded modules $\mathbf{H}_\mu$ under Raising Operators.

There is an elementary approach to proving the $n!$ conjecture which can be successfully used in various special cases. More importantly, this approach leads to the discovery of some truly remarkable properties of the modules $\mathbf{H}_\mu$. The basic idea can be best understood by working out some examples. Note that Theorem 1.5 guarantees that the 0 y-degree portion of $\mathbf{H}_\mu$ has dimension $\binom{n}{\mu}$. In the particular case that $\mu = (1^{n-2}, 2)$ this dimension is $n!/2$. Since **flip** (see 1.14) yields a sign-twisted S_n-module isomorphism of the 0 y-degree portion into the $n(\mu')$ y-degree portion, we see that for all these partitions (and their conjugates) the $n!$ conjecture must necessarily hold true. Moreover the case of $\mu = 1^n$ (single columns) or $\mu = (n)$ (single rows) is classical since Δ_μ then reduces to the Vandermonde determinant in the $x's$ and the $y's$ respectively. In particular, for $n = 4$ we are only left with $\mu = (2,2)$. We shall start by dealing with this case.

Expanding with respect to the last column we get

$$\Delta_{2,2} \;=\; det \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 \end{pmatrix} \;=\; \Phi_{oo} + x_4\ \Phi_{1o} + y_4\ \Phi_{o1} + x_4y_4\Phi_{11}\ . \qquad 4.1$$

So we may write

$$\partial_{x_4}\Delta_{2,2} = \Phi_{1o} + y_4\Phi_{11}\ ,\quad \partial_{y_4}\Delta_{2,2} = \Phi_{o1} + x_4\Phi_{11}\ ,\quad \partial_{x_4}\partial_{y_4}\Delta_{2,2} = \Phi_{11}\ , \qquad 4.2$$

and note that

$$\Phi_{11} \;=\; \Delta_{1,2} \;=\; det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \qquad 4.3$$

If P is a polynomial in $x_1,..,x_n;y_1,..,y_n$, the differential operator $P(\partial_{x_1},..,\partial_{x_n};\partial_{y_1},..,\partial_{y_n})$ here and after will be denoted by $P(\partial)$. This given, let $\mathcal{B}$ be a collection of monomials in $x_1,x_2,x_3;y_1,y_2,y_3$ chosen so that $\{b(\partial)\Delta_{1,2}\}_{b\in\mathcal{B}}$ is a basis for $\mathbf{H}_{21}$. We claim that the polynomials in the union

$$\mathcal{B}_{2,2}=\{b(\partial)\Delta_{2,2}\}_{b\in\mathcal{B}}+\{b(\partial)\partial_{x_4}\Delta_{2,2}\}_{b\in\mathcal{B}}+\{b(\partial)\partial_{y_4}\Delta_{2,2}\}_{b\in\mathcal{B}}+\{b(\partial)\partial_{x_4}\partial_{y_4}\Delta_{2,2}\}_{b\in\mathcal{B}} \qquad 4.4$$

are linearly independent. Note that since we know in full generality that $dim\mathcal{B}_\mu\leq n!$, we can thus conclude that $\mathcal{B}_{2,2}$ is a basis for $\mathbf{H}_{2,2}$ and that the conjecture is true for $\mu=(2,2)$. To this end, let there be $b_{oo},b_{1o},b_{o1},b_{11}$ in the linear span of $\mathcal{B}$ giving

$$b_{oo}(\partial)\Delta_{2,2}\ +\ b_{1o}(\partial)\partial_{x_4}\Delta_{2,2}\ +\ b_{o1}(\partial)\partial_{y_4}\Delta_{2,2}\ +\ b_{11}(\partial)\partial_{x_4}\partial_{y_4}\Delta_{2,2}\ =\ 0\ .$$

Using 4.1, 4.2 and 4.3 and extracting the coefficients of $1,x_4,y_4,x_4y_4$ we derive that we must have the simultaneous equations

$$\begin{array}{ccccccccc}
b_{oo}(\partial)\Phi_{oo} & + & b_{1o}(\partial)\Phi_{1o} & + & b_{o1}(\partial)\Phi_{o1} & + & b_{11}(\partial)\Delta_{1,2} & = & 0\\
 & & b_{oo}(\partial)\Phi_{1o} & & & + & b_{o1}(\partial)\Delta_{1,2} & = & 0\\
 & & & & b_{oo}(\partial)\Phi_{o1} & + & b_{1o}(\partial)\Delta_{1,2} & = & 0\\
 & & & & & & b_{oo}(\partial)\Delta_{1,2} & = & 0
\end{array}$$

Note now that if $b_{oo}=\sum_{b\in\mathcal{B}}c_b\,b$ then the last equation may be written as

$$\sum_{b\in\mathcal{B}}c_b\,b(\partial)\Delta_{1,2}\ =\ 0\ ,$$

but this contraddicts the choice of $\mathcal{B}$ unless the coefficients c_b are all equal to zero. Substituting $b_{oo}=0$ in the third equation reduces it to $b_{1o}(\partial)\Delta_{1,2}=0$ which again implies that the coefficients of b_{1o} must all vanish as well. Substituting $b_{oo}=0$ in the second equation yields the same for the coefficients of b_{o1}. Finally, setting $b_{oo}=b_{1o}=b_{o1}=0$ in the first equation forces the vanishing of the coefficients of b_{11}. This proves that $\mathcal{B}_{2,2}$ is an independent set as asserted. We should note that our argument here establishes a bit more that the validity of the $n!$ conjecture for $\mu=(2,2)$.

Let us recall that if θ is the Frobenius image of an S_n character χ then the partial derivative $\partial_{p_1}\theta$ (*) yields the Frobenius image of the restriction of χ to S_{n-1}. In particular, if $C_\mu(x;q,t)$ (as in section 1.) denotes the Frobenius characteristic of $\mathbf{H}_\mu$ then the polynomial

$$G_\mu(q,t)=\partial_{p_1}^n\,C_\mu(x;q,t)\ =\ \sum_{\lambda\vdash n}f_\lambda C_{\lambda\mu}(q,t) \qquad 4.5$$

gives the bigraded Hilbert series of $\mathbf{H}_\mu$. Of course under the $C=\tilde H$ conjecture this Hilbert series is given by the polynomial

$$\tilde F_\mu(q,t)\ =\ \partial_{p_1}^n\,\tilde H_\mu(x;q,t)\ =\ \sum_{\lambda\vdash n}f_\lambda\tilde K_{\lambda\mu}(q,t)\ . \qquad 4.6$$

(*) Here p_1 denotes the first power symmetric polynomial and the symbol $\partial_{p_1}\theta$ is to mean the partial derivative of θ as a polynomial in the power symmetric functions.

We should also keep in mind (see [9]) that the "duality" result for Macdonald polynomials in our notation becomes

$$\omega \tilde{H}_\mu(x; 1/q, 1/t) t^{n(\mu)} q^{n(\mu')} \;=\; \tilde{H}_\mu(x; q, t) \tag{4.7}$$

where ω is the involution which sends S_λ into $S_{\lambda'}$. Thus 4.7 implies that we must also have

$$\tilde{F}_\mu(1/q, 1/t) t^{n(\mu)} q^{n(\mu')} \;=\; \tilde{F}_\mu(q, t) \ . \tag{4.8}$$

The analogous identities

$$\omega C_\mu(x; 1/q, 1/t) t^{n(\mu)} q^{n(\mu')} \;=\; C_\mu(x; q, t) \tag{4.9}$$

$$G_\mu(1/q, 1/t) t^{n(\mu)} q^{n(\mu')} \;=\; G_\mu(q, t) \ . \tag{4.10}$$

follow from the fact that **flip** is a sign-twisting, degree-complementing S_n-module automorphism of $\mathbf{H}_\mu$.

It is easy to see that if $\beta(q,t)$ gives the bidegree distribution of the monomials in $\mathcal{B}$ then $qt\beta(1/q,1/t)$ must give the Hilbert series of $\mathbf{H}_{1,2}$. Thus from 4.10 we get that $\beta = G_{1,2}(t,q)$. From our construction 4.4 of the basis $\mathcal{B}_{2,2}$ we can then immediately derive that the Hilbert series of $\mathbf{H}_{2,2}$ must be given by formula

$$G_{2,2}(q,t) = \beta(1/q, 1/t)(1 + 1/t + 1/q + 1/tq)q^2t^2 = (1 + t + q + tq)G_{1,2}(q,t) \ .$$

A slightly more refined argument which takes account of the action of S_3 on the basis $\mathcal{B}_{2,2}$, yields that we must also have

$$\partial_{p_1} C_{2,2}(x; q, t) \;=\; (1 + t + q + tq) C_{1,2}(x; q, t) \ .$$

It is not difficult to see that the argument we have given here can be generalized to the case of arbitrary rectangular partitions $\mu = r^s$ and obtain that

Theorem 4.1

If the $n!$ conjecture holds for the partition $(r-1, r^{s-1})$ then it holds for r^s, Moreover,

$$\partial_{p_1} C_{r^s}(x; q, t) \;=\; B_{r^s}(q,t)\, C_{r-1, r^{s-1}}(x; q, t) \tag{4.11}$$

$$G_{r^s}(q,t) \;=\; B_{r^s}(q,t)\, G_{r-1, r^{s-1}}(q,t) \tag{4.12}$$

Continuing with our examples we see that for $n = 5$ we need to deal with the partitions $(1,2,2)$, $(1,1,3)$, $(3,2)$. In view of the identities

$$C_{\mu'}(x; q, t) = C_\mu(x; t, q) \qquad\qquad \tilde{H}_{\mu'}(x; q, t) = \tilde{H}_\mu(x; t, q) \ , \tag{4.13}$$

proving the $n!$ conjecture for μ yields it also for μ'. So we are left with $(1,2,2)$ and $(1,1,3)$. We start with $(1,2,2)$. Expanding again with respect to the last column we get

$$\Delta_{1,2,2} \;=\; det\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 & x_5y_5 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 \end{pmatrix} \;=\; \Phi_{oo} + x_5\,\Phi_{1o} + y_5\,\Phi_{o1} + x_5y_5\Phi_{11} + x_5^2\Phi_{2o}\;. \tag{4.14}$$

So we may write

$$\begin{array}{rclllllllll} \Delta_{1,2,2} & = & \Phi_{oo} & + & y_5\Phi_{o1} & + & x_5\Phi_{1o} & + & x_5y_5\Delta_{1,1,2} & + & x_5^2\Delta_{2,2} \\ \partial_{y_5}\Delta_{1,2,2} & = & & & \Phi_{o1} & & & + & x_5\Delta_{1,1,2} & & \\ \partial_{x_5}\Delta_{1,2,2} & = & & & & & \Phi_{1o} & + & y_5\Delta_{1,1,2} & + & 2x_5\Delta_{2,2} \\ \partial_{x_5}\partial_{y_5}\Delta_{1,2,2} & = & & & & & & & \Delta_{1,1,2} & & \\ \partial_{x_5}^2\Delta_{1,2,2} & = & & & & & & & & & 2\Delta_{2,2} \end{array} \tag{4.15}$$

Proceeding as we did for $\mu=(2,2)$ we are to construct 5 collections $\mathcal{B}_{oo},\mathcal{B}_{1o},\mathcal{B}_{o1},\mathcal{B}_{11},\mathcal{B}_{20}$ of polynomials in $x_1,x_2,x_3,x_4;y_1,y_2,y_3,y_4$ such that the polynomials in the union

$$\begin{aligned}\mathcal{B}_{1,2,2} = \left\{b(\partial)\Delta_{1,2,2}\right\}_{b\in\mathcal{B}_{oo}} + \left\{b(\partial)\partial_{y_4}\Delta_{1,2,2}\right\}_{b\in\mathcal{B}_{o1}} + \left\{b(\partial)\partial_{x_4}\Delta_{1,2,2}\right\}_{b\in\mathcal{B}_{1o}} + \\ + \left\{b(\partial)\partial_{x_4}\partial_{y_4}\Delta_{1,2,2}\right\}_{b\in\mathcal{B}_{11}} \left\{b(\partial)\partial_{x_4}^2\Delta_{1,2,2}\right\}_{b\in\mathcal{B}_{2o}}\end{aligned}$$

are independent. If we succeed in choosing them so that $\mathcal{B}_{1,2,2}$ has altogether 120 elements we will have proved the $n!$ conjecture for $\mu=(1,2,2)$. In this case we shall not venture a guess but rather determine the $\mathcal{B}_{ij}$ from the equations that would have to hold if there was a dependence between the elements of $\mathcal{B}_{1,2,2}$. So let b_{ij} be in the linear span of $\mathcal{B}_{ij}$ and assume that we have

$$b_{oo}(\partial)\Delta_{1,2,2} + b_{o1}(\partial)\partial_{y_5}\Delta_{1,2,2} + b_{1o}(\partial)\partial_{x_5}\Delta_{1,2,2} + b_{11}(\partial)\partial_{x_5}\partial_{y_5}\Delta_{1,2,2} + b_{2o}(\partial)\partial_{x_5}^2\Delta_{1,2,2} \;=\; 0$$

Using the expansions in 4.12 and 4.13 and equating coefficients of $1, x_5, y_5, x_5y_5, x_5^2$ we get the system of equations

$$\begin{array}{lllllllll} b_{oo}(\partial)\Phi_{oo} & + & b_{o1}(\partial)\Phi_{o1} & + & b_{1o}(\partial)\Phi_{1o} & + & b_{11}(\partial)\Delta_{1,1,2} & + & 2b_{2o}(\partial)\Delta_{2,2} & =0 \\ & & b_{oo}(\partial)\Phi_{o1} & & & + & b_{1o}(\partial)\Delta_{1,1,2} & & & =0 \\ & & & & b_{oo}(\partial)\Phi_{1o} & + & b_{o1}(\partial)\Delta_{1,1,2} & + & 2b_{1o}(\partial)\Delta_{2,2} & =0 \\ & & & & & & b_{oo}(\partial)\Delta_{1,1,2} & & & =0 \\ & & & & & & & & b_{oo}(\partial)\Delta_{2,2} & =0 \end{array} \tag{4.16}$$

Recall that in section 1 (see 1.12) we denoted by $\mathcal{I}_\mu$ the ideal of polynomials which kill Δ_μ. With this notation the last two equations yield us that $b_{oo}\in\mathcal{I}_{1,1,2}\cap\mathcal{I}_{2,2}$ and since $(\mathcal{I}_{1,1,2}\cap\mathcal{I}_{2,2})^\perp = \mathbf{H}_{1,1,2}+\mathbf{H}_{2,2}$, we see that we can choose $\mathcal{B}_{oo}$ to be any bihomogeenous basis of $\mathbf{H}_{1,1,2}+\mathbf{H}_{2,2}$. Indeed, if we do so any linear combination b_{oo} of elements of $\mathcal{B}_{oo}$ which satisfies the last two equations in 4.13 would have to be orthogonal to itself and therefore must have all its coefficients equal to zero. We shall visualize this choice by writing

$\mathcal{B}_{oo} \;=\;$  $\vee$  4.17

This done the second equation reduces to $b_{1o}(\partial)\Delta_{1,1,2} = 0$ which gives $b_{1o} \in \mathcal{I}_{1,1,2}$. Thus using the same imagery we can set

$$\mathcal{B}_{1o} \;=\; \ydiagram{1,1,2} \tag{4.18}$$

This reduces the third equation to $b_{o1}(\partial)\Delta_{1,1,2} = 0$. Thus we can set

$$\mathcal{B}_{o1} \;=\; \ydiagram{1,1,2} \tag{4.19}$$

Substituting $b_{oo} = b_{1o} = b_{o1} = 0$ in the first equation reduces it to $b_{11}(\partial)\Delta_{1,1,2} + b_{2o}(\partial)\Delta_{2,2} = 0$. Here we have two possible choices. We take

$$\mathcal{B}_{2o} \;=\; \ydiagram{2,2} \tag{4.20}$$

Next, we want $b_{11}(\partial)\Delta_{1,1,2} \in \mathbf{H}_{2,2}$ to imply $b_{11} = 0$. We can assure this if $b_{11}(\partial)\Delta_{1,1,2}$ lies in the orthogonal complement of $\mathbf{H}_{1,1,2} \cap \mathbf{H}_{2,2}$ in $\mathbf{H}_{1,1,2}$. In other words we want $\mathbf{flip}_{1,1,2} b_{11}$ (*) to lie in $\mathcal{I}_{2,2} \cap \mathbf{H}_{1,1,2}$. So our final choice is

$$\mathcal{B}_{11} \;=\; \mathbf{flip}_{1,1,2}^{-1}\left[\left(\ydiagram{1,1,2} \wedge \ydiagram{2,2}\right)^{\perp} \wedge \ydiagram{1,1,2}\right]$$

These choices assure that $\mathcal{B}_{1,1,2}$ is an independent set. Let us now count how many elements it has. Now, 4.18, 4.19 and 4.20 yield 3×24 elements and 4.17 yields $2 \times 24 - dim\, \mathbf{H}_{1,1,2} \cap \mathbf{H}_{2,2}$ elements. Since $\mathbf{H}_{1,2,2} \leq 120$ the independence of $\mathcal{B}_{1,2,2}$ yields the inequality

$$5 \times 24 \;-\; dim\, \ydiagram{1,1,2} \wedge \ydiagram{2,2} \;+\; dim\, \left(\ydiagram{1,1,2} \wedge \ydiagram{2,2}\right)^{\perp} \wedge \ydiagram{1,1,2} \;\leq 120$$

or better

$$dim\, \left(\ydiagram{1,1,2} \wedge \ydiagram{2,2}\right)^{\perp} \wedge \ydiagram{1,1,2} \;\leq\; dim\, \ydiagram{1,1,2} \wedge \ydiagram{2,2} \tag{4.21}$$

In particular, $\mathcal{B}_{1,2,2}$ would be a basis for $\mathbf{H}_{1,2,2}$ and the $n!$ conjecture would then be established for $\Delta_{1,2,2}$ if we prove that equality holds in 4.21. Amazingly it develops that this equality holds true in the strongest possible sense. More precisely, we can prove that

$$\mathbf{flip}_{1,1,2}\, \ydiagram{1,1,2} \wedge \ydiagram{2,2} \;=\; \left(\ydiagram{1,1,2} \wedge \ydiagram{2,2}\right)^{\perp} \wedge \ydiagram{1,1,2}\;. \tag{4.22}$$

Let us postpone the proof of this identity and proceed to examine all its implications. To begin with it shows that we can take $\mathcal{B}_{11}$ to be any bihomogeneous basis of $\mathbf{H}_{1,1,2} \cap \mathbf{H}_{2,2}$. This done we can visualise the Hilbert series of $\mathbf{H}_{1,2,2}$ by writing

$$G_{1,2,2} \;=\; \ydiagram{1,1,2} \;+\; \ydiagram{2,2} \;-\; \ydiagram{1,1,2} \wedge \ydiagram{2,2} \;+\; q\, \ydiagram{1,1,2} \;+\; t\, \ydiagram{1,1,2} \;+\; t^2\, \ydiagram{1,1,2} \;+\; tq\, \ydiagram{1,1,2} \vee \ydiagram{2,2}\;.$$

To be precise, the left hand side of this relation is $G_{1,2,2}(1/q, 1/t)t^4q^2$, however we can write it that way because of 4.10. It will be convenient to extend the definition of $\mathbf{flip}_\mu$ to act on a Frobenius characteristic $H(x;q,t)$ and on a Hilbert series $F(q,t)$ by setting

$$\mathbf{flip}_\mu\, H(x;q,t) = \omega H(x;1/q,1/t)\, t^{n(\mu)} q^{n(\mu')} \;, \qquad \mathbf{flip}_\mu\, F(q,t) = F(1/q,1/t)\, t^{n(\mu)} q^{n(\mu')} \tag{4.23}$$

(*) Here and after $\mathbf{flip}_\mu$ will denote flipping with respect to Δ_μ.

This given, the more refined argument which takes account of the action of S_4 on the basis $\mathcal{B}_{1,2,2}$ yields the relation

$$\partial_{p_1} C_{1,2,2}(x;q,t) = (1+t+q)C_{1,1,2}(x;q,t) + (1+t^2)C_{2,2}(x;q,t) + (tq-1)\ \phi \qquad 4.24$$

where for convenience we have denoted by ϕ the Frobenius characteristic of $\yng(1,1,2) \wedge \yng(2,2)$. Here again the left-hand side should have been $\mathbf{flip}_{1,2,2}\partial_{p_1} C_{1,2,2}$, but 4.9 makes it right the way it is.

Note that if we follow the other alternative and choose $\mathcal{B}_{11} = \yng(1,1,2)$ instead of 4.20, then we must take

$$\mathcal{B}_{2o} \;=\; \mathbf{flip}_{2,2}^{-1}\left[\left(\yng(1,1,2) \wedge \yng(2,2)\right)^{\perp} \wedge \yng(2,2)\right]$$

However, the analogue of 4.22

$$\mathbf{flip}_{2,2}\, \yng(1,1,2) \wedge \yng(2,2) \;=\; \left(\yng(1,1,2) \wedge \yng(2,2)\right)^{\perp} \wedge \yng(2,2)\ . \qquad 4.25$$

yields 4.24 back again. Note that formulas 4.22 and 4.25 give us the decompositions

$$C_{1,1,2}(x;q,t) = \phi \ + \ \mathbf{flip}_{1,1,2}\,\phi\ , \qquad C_{2,2}(x;q,t) = \phi \ + \ \mathbf{flip}_{2,2}\,\phi\ , \qquad 4.26$$

from which we derive that

$$\phi \;=\; \frac{q\, C_{1,1,2} - t\, C_{2,2}}{q-t} \qquad 4.27$$

It goes without saying that all these relations can be verified by computer. In particular in this manner we can establish 4.22 and 4.25 and also check that the $C = \tilde{H}$ conjecture is in fact true for $(1,2,2)$, $(1,1,2)$ and $(2,2)$. Nevertheless it is more illuminating to verify these identities using representation theory. Recalling that

$$\Delta_{2,2} = det \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 \end{pmatrix}\ , \qquad \Delta_{1,1,2} \;=\; det \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \end{pmatrix}\ ,$$

we get

$$\partial_{x_4}\partial_{y_4}\Delta_{2,2} \;=\; \frac{1}{2}\,\partial_{x_4}^2\Delta_{1,1,2} \;=\; det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$$

Now (by Theorem 1.8) the action of S_4 on this element generates a 3-dimensional irreducible representation with character $\chi^{1,1,2}$ and weight tq which is shared by $\mathbf{H}_{1,1,2}$ and $\mathbf{H}_{2,2}$. Similarly, we see that its partial derivatives

$$\partial_{x_3} det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \;=\; det \begin{pmatrix} 1 & 1 \\ y_1 & y_2 \end{pmatrix}\ , \qquad \partial_{y_3} det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \;=\; det \begin{pmatrix} 1 & 1 \\ x_1 & x_2 \end{pmatrix}\ ,$$

generate common 3-dimensional representations with character $\chi^{1,3}$ and weights q and t respectively. Similar calculations yields that

$$(\partial_{x_3}\partial_{y_4}+\partial_{x_4}\partial_{y_3})\Delta_{1,1,2} \;=\; 2\partial_{y_3}\partial_{y_4}\Delta_{2,2} \;=\; 2det\begin{pmatrix}1 & 1\\ x_1 & x_2\end{pmatrix} det\begin{pmatrix}1 & 1\\ x_3 & x_4\end{pmatrix} \ ,$$

and this generates a common 2 dimensional representation with character $\chi^{2,2}$ and weight t^2. Including the trivial, we have thus identified a submodule of $\mathbf{H}_{1,1,2}\cap\mathbf{H}_{2,2}$ with bigraded Frobenius chracteristic

$$\Pi \;=\; S_4+(t+q)\,S_{1,3} \;+\; tq\,S_{1,1,2} \;+\; t^2\,S_{2,2} \ .$$

Using 4.9 and the definition in 4.23 we then immediately derive that $\mathbf{H}_{1,1,2}$ and $\mathbf{H}_{2,2}$ contain submodules with respective Frobenius characteristics

$$\begin{aligned}\mathbf{flip}_{1,1,2}\,\Pi \;&=\; t^3q\,S_{1^4}+(t^2q+t^3)\,S_{1,1,2} \;+\; t^2\,S_{1,3} \;+\; tq\,S_{2,2} \ ,\\ \mathbf{flip}_{2,2}\,\Pi \;&=\; t^2q^2S_{1^4}+(tq^2+t^2q)\,S_{1,1,2} \;+\; tq\,S_{1,3} \;+\; q^2\,S_{2,2} \ .\end{aligned} \qquad 4.28$$

Since Π and $\mathbf{flip}_{1,1,2}\,\Pi$ have no common terms and each accounts for 12 dimensions we conclude that together they must give the Frobenius characteristic of $\mathbf{H}_{1,1,2}$. A similar reasoning applies to Π and $\mathbf{flip}_{2,2}\,\Pi$ and we can write

$$C_{1,1,2} \;=\; \Pi \;+\; \mathbf{flip}_{1,1,2}\,\Pi \ , \qquad\qquad C_{2,2} \;=\; \Pi \;+\; \mathbf{flip}_{2,2}\,\Pi \ ,$$

Comparing the expressions in 4.28, we see that the possibility still remains that $\mathbf{H}_{1,1,2}$ and $\mathbf{H}_{2,2}$ may have in common an irreducible submodule with character $\chi^{1,1,2}$ and weight t^2q. However, this submodule is generated in $\mathbf{H}_{2,2}$ by the action of S_4 on the polynomial

$$P \;=\; \begin{pmatrix}1 & 1\\ \partial_{y_1} & \partial_{y_2}\end{pmatrix}\,\Delta_{2,2}$$

If this polynomial were to belong to $\mathbf{H}_{1,1,2}$ it would be killed by any element that kills $\Delta_{1,1,2}$. In particular we should have $\partial_{x_4}\partial_{y_4}P=0$. But we see that

$$\partial_{x_4}\partial_{y_4}P \;=\; \begin{pmatrix}1 & 1\\ \partial_{y_1} & \partial_{y_2}\end{pmatrix} det\begin{pmatrix}1 & 1 & 1\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{pmatrix} \;=\; x_2-x_1\neq 0 \ .$$

This completes the proof of 4.22 and 4.25 and shows that Π is none other than the Frobenius characteristic ϕ of the intersection of $\mathbf{H}_{112}$ and $\mathbf{H}_{2,2}$.

Remark 4.1

It is interesting to see what 4.24 reduces to when we express its right hand side entirely in terms of ϕ. To this end we use the identities in 4.26 to replace $C_{1,1,2}$ and $C_{2,2}$ and obtain

$$\partial_{p_1}\,C_{1,2,2} \;=\; (1+t+q)(\phi+t^3q\omega\downarrow\phi) \;+\; (1+t^2)(\phi+t^2q^2\omega\downarrow\phi) \;+\; (tq-1)\phi \ ,$$

where for convenience we have let $\downarrow$ denote the operator on rational functions of t,q which replaces t by $1/t$ and q by $1/q$. Collecting terms in ϕ and $\omega\downarrow\phi$ gives

$$\begin{aligned}\partial_{p_1} C_{1,2,2} &= (1+t+q+tq+t^2)\,\phi \;+\; (t^4q^2+t^3q^2+t^4q+t^3q+t^2q^2)\omega\downarrow\phi \\ &= (1+t+q+tq+t^2)\,\phi \;+\; t^4q^2(1+1/t+1/q+1/tq+1/t^2)\omega\downarrow\phi\end{aligned},$$

and this may be rewritten in the very suggestive form

$$\partial_{p_1} C_{1,2,2} \;=\; B_{1,2,2}(q,t)\,\phi \;+\; \mathbf{flip}_{1,2,2}\big(\,B_{1,2,2}(q,t)\,\phi\,\big)\ . \qquad 4.29$$

What we have discovered in this particular example appears to hold true in full generality. We are thus lead to a number of results and conjectures of various strengths.

Proposition 4.1

Let λ be a two-corner partition of $n+1$ and let μ and ν be the partitions obtained by removing one of the corners. Assume the $n!$ conjecture holds for μ and ν. Then

$$dim\,\Big(\mathbf{H}_\mu\cap\mathbf{H}_\nu\Big)^{\perp}\cap\mathbf{H}_\mu \;\le\; dim\,\mathbf{H}_\mu\cap\mathbf{H}_\nu\ . \qquad 4.30$$

Moreover, if equality holds here then the $n!$ conjecture holds for λ as well.

This result can be established by following very closely what we did for the partition $1,2,2$.

Let μ and ν be two partitions of the same number n. Following A. Young it is customary to say that μ is obtained from ν by a "Raising Operator" if μ can be obtained by lifting a number of cells of ν from lower to upper rows.(*) A raising operator will be called minimal if it lifts a single cell up a single row or to the right a single column. It can be shown (see [17]) that the dominance partial order is the transitive closure of the relation $\mu=R\nu$ with R a minimal raising operator. L. Butler [4] observed that the Macdonald coefficients $\tilde K_{\lambda\mu}(q,t)$ change in a remarkably simple way when μ is obtained from ν by a minimal raising operator. Her observation which was originally made from tables computed by hand by Macdonald [17] (up to $n\le 6$) is now confirmed by more extensive tables obtained by computer (up to $n\le 8$). Butler noticed that for any such pair of partitions μ,ν and any λ there are two polynomials ϕ_λ and θ_λ with non-negative integer coefficients such that

$$\begin{aligned}\tilde K_{\lambda\mu}(q,t) &= \phi_\lambda \;+\; \theta_\lambda\\ \tilde K_{\lambda\nu}(q,t) &= \phi_\lambda \;+\; \theta_\lambda\,T_\nu/T_\mu,\end{aligned} \qquad 4.31$$

where for convenience we have set $T_\mu=q^{n(\mu')}t^{n(\mu)}$, $T_\nu=q^{n(\nu')}t^{n(\nu)}$. This has come to be referred to as the *Butler conjecture*. To explore the full implications of this conjecture let us set for a moment $\psi_\lambda=\theta_\lambda/T_\mu$ and write these two identities in the more symmetric form

$$\begin{aligned}\tilde K_{\lambda\mu}(q,t) &= \phi_\lambda \;+\; \psi_\lambda\,T_\mu\\ \tilde K_{\lambda\nu}(q,t) &= \phi_\lambda \;+\; \psi_\lambda\,T_\nu,\end{aligned}$$

(*) This definition requires that diagrams of partitions be drawn by the English convention

Equivalently we may write

$$\phi_\lambda = \frac{T_\nu\, \tilde{K}_{\lambda\mu} \ - \ T_\mu\, \tilde{K}_{\lambda\nu}}{T_\nu \ - \ T_\mu} \ , \qquad \psi_\lambda = \frac{\tilde{K}_{\lambda\mu} \ - \ \tilde{K}_{\lambda\nu}}{T_\mu \ - \ T_\nu} \ . \tag{4.32}$$

Now the identity in 4.7 (written for μ and ν) gives

$$\tilde{K}_{\lambda\mu}(1/q, 1/t)\ T_\mu = \tilde{K}_{\lambda'\mu}(q,t) \ , \qquad \tilde{K}_{\lambda\nu}(1/q, 1/t)\ T_\nu = \tilde{K}_{\lambda'\nu}(q,t) \ .$$

These relations show that ψ_λ is superfluous here. In fact, combinining them with 4.31 we can easily derive that

$$\psi_\lambda(q,t) \ = \ \phi_{\lambda'}(1/q, 1/t) \ .$$

Setting $\phi(x;q,t) = \sum_\lambda \ \phi_\lambda(q,t)\ S_\lambda(x)$ the identites in 4.31 become

$$\tilde{H}_\mu(x;q,t) = \phi(x;q,t) + \mathbf{flip}_\mu\ \phi(x;q,t) \ , \qquad \tilde{H}_\nu(x;q,t) = \phi(x;q,t) + \mathbf{flip}_\nu\ \phi(x;q,t) \ . \tag{4.33}$$

Now we see that our findings in the case of the pair $(1,1,2),(2,2)$ reveal that the phenomenon observed by Butler may be the result of a beautiful mechanism which we venture to state as follows.

Conjecture 4.1

If $\mu = R\nu$ with R a minimal raising operator then the identities in 4.33 hold true with $\phi(x;q,t)$ the bigraded Frobenius characteristic of $\mathbf{H}_\mu \cap \mathbf{H}_\nu$

On the validity of our $C = \tilde{H}$ conjecture, this property of the Macdonald polynomials would be an immediate consequence of

Conjecture 4.2

If $\mu = R\nu$ with R a minimal raising operator then

$$\begin{aligned} \mathbf{H}_\mu \ &= \ \mathbf{H}_\mu \cap \mathbf{H}_\nu \ \oplus \ \mathbf{flip}_\mu\ \mathbf{H}_\mu \cap \mathbf{H}_\nu \ , \\ \mathbf{H}_\nu \ &= \ \mathbf{H}_\mu \cap \mathbf{H}_\nu \ \oplus \ \mathbf{flip}_\nu\ \mathbf{H}_\nu \cap \mathbf{H}_\nu \ . \end{aligned} \tag{4.34}$$

Remark 4.2

There are even stronger forms of this conjecture. Indeed, we expect it to hold true that

$$\mathbf{H}_\mu \cap \mathbf{H}_\nu \ = \ \mathbf{flip}_\mu\ \mathbf{H}_\mu \cap \mathcal{I}_\nu \ = \ \mathbf{flip}_\nu\ \mathbf{H}_\nu \cap \mathcal{I}_\mu \tag{4.35}$$

which would imply 4.34. Our evidence suggests that all these equalities hold true whenever $\mu = R\nu$ with R any raising operator, not necessarily minimal, which lifts a single cell.

Remark 4.3

We should note that the equalities

$$dim \left(\mathbf{H}_\mu \cap \mathbf{H}_\nu\right)^\perp \cap \mathbf{H}_\mu \ = \ dim\ \mathbf{H}_\mu \cap \mathbf{H}_\nu \ = \ dim \left(\mathbf{H}_\mu \cap \mathbf{H}_\nu\right)^\perp \cap \mathbf{H}_\nu \ . \tag{4.36}$$

for all pairs $\mu = R\nu$ with R minimal do imply the $n!$ conjecture. This is because every partition of n can be reached from $\mu = 1^n$ by a sequence of minimal raisings. At the start 4.36 would give $dim\, \mathbf{H}_{1^n} \cap \mathbf{H}_\nu = n!/2$ which recursively would yield

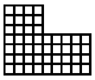
$$dim\ \mathbf{H}_\mu \cap \mathbf{H}_\nu \;=\; n!/2 \qquad 4.37$$

for all pairs $\mu = R\nu$ with R minimal. For this reason we have come to refer to 4.36 as the $n!/2$ *conjecture.*

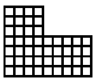
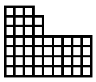
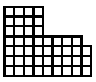
Remark 4.4

Let ρ be any two-corner partition and μ and ν be obtained by removing one of its corner cells. Given 4.36, the same mechanism we applied to the partition $(1,2,2)$ yields the $n!$ conjecture for ρ whenever it is known to be true for both μ and ν. By taking account of the action of S_{n-1} on the resulting basis of $\mathbf{H}_\rho$ we can obtain a general result relating the Frobenius characteristics of $\mathbf{H}_\rho$, $\mathbf{H}_\mu$ and $\mathbf{H}_\nu$ which specializes to 4.29 when $\rho = (1,2,2)$.

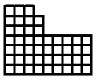
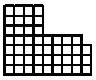
To state it we need some notation. Let ρ be the partition which has L_2 rows of length L_1 and A_2 rows of length $L_1 + A_1$. To visualize our identities, we shall act as if ρ is the Ferrers diagram of the partition obtained by setting $L_1 = A_2 = 4$, $L_2 = 3$ and $A_1 = 5$. Pictorially, this is setting

$\rho \;=$  .

Now let μ be the partition obtained by removing the upper corner cell from ρ and ν be obtained by removing the lower corner cell. Following the same conventions we adopted in [10] we shall represent a Frobenius characteristic by the Ferrers diagram of the partition that indexes the corresponing module. This given, the Frobenius characteristics of $\mathbf{H}_\rho, \mathbf{H}_\mu$ and $\mathbf{H}_\nu$ are depicted as follows:

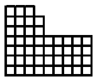
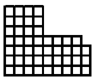
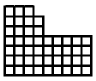
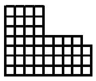
$C_\rho(x;q,t) =$ , $\quad C_\mu(x;q,t) =$ , $\quad C_\nu(x;q,t) =$  .

Letting ϕ, U and V denote the Frobenius characteristics of $\mathbf{H}_\mu \cap \mathbf{H}_\nu$, $\big(\mathbf{H}_\mu \cap \mathbf{H}_\nu\big)^\perp \cap \mathbf{H}_\mu$ and $\big(\mathbf{H}_\mu \cap \mathbf{H}_\nu\big)^\perp \cap \mathbf{H}_\nu$ respectively and assuming that the identities in 4.34 hold true even in this more general situation, we get

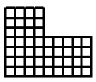
 $= \phi + \mathbf{flip}_\mu \phi$,  $= \phi + \mathbf{flip}_\nu \phi$, 4.38

from which we derive that

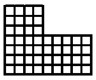
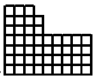
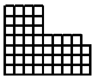
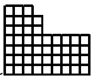
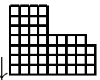
$\phi \;=\; \dfrac{T_\nu\,\square - T_\mu\,\square}{T_\mu - T_\nu} \;=\; \dfrac{t^{L_2}\,\square - q^{A_1}\,\square}{t^{L_2} - q^{A_1}}$. 4.39

(numerator diagrams, in order:    )

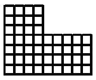
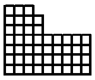
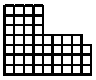
The mechanism of proof we have illustrated in the case $L_1 = L_2 = A_1 = 1$ and $A_2 = 2$ can be extended to the general case and (upon the validity of 4.38) it yields the identity

∂_{p_1}  $\;=\; B_\rho(q,t)\,\phi \;+\; \mathbf{flip}_\rho\,\big(\,B_\rho(q,t)\,\phi\,\big)$. 4.40

This given, we can eliminate ϕ from this relation by means of 4.39 and obtain

∂_{p_1}  $= B_\rho \dfrac{T_\nu}{}$  $- \, T_\mu$  $\Big/ \, (T_\mu - T_\nu) \; + \; T_\rho \downarrow B_\rho \, \dfrac{T_\mu \omega \downarrow}{}$  $- \, T_\nu \omega \downarrow$  $\Big/ \, (T_\nu - T_\mu)$

Using 4.9 we can collect terms and rewrite this on the form

∂_{p_1}  $= \dfrac{P}{t^{L_2} - q^{A_1}}$  $+ \dfrac{Q}{t^{L_2} - q^{A_1}}$  4.41

with

$$P = B_\rho t^{L_2} - \downarrow B_\rho q^{L_1 - 1} t^{A_2 + L_2 - 1} \qquad \text{and} \qquad Q = -B_\rho q^{A_1} + \downarrow B_\rho t^{A_2 - 1} q^{A_1 + L_1 - 1}$$

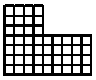
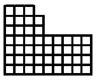
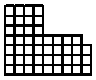
Since with our choice of parameters we can write

$$B_\rho(q,t) \;=\; [L_1]_q [L_2]_t \;+\; [L_1 + A_1]_q [A_2]_t$$

a simple calculation yields that

$$P = (t^{L_2 + A_2} - q^{A_1})\, [L_1]_q [L_2]_t \qquad \text{and} \qquad Q = (t^{L_2} - q^{L_1 + A_1})\, [A_1]_q [A_2]_t$$

Now it was shown in [10] (see 4.40 there) that the Macdonald-Stanley Pieri rules yield the identity in 4.41 with precisely the same P and Q when

 $= \tilde{H}_\rho(x;q,t)$,  $= \tilde{H}_\mu(x;q,t)$,  $= \tilde{H}_\nu(x;q,t)$.

This not only supports the $C = \tilde{H}$ conjecture and the validity of the identities in 4.34 (even when R is not minimal) but it reveals the combinatorial mechanism which underlies the mysterious intricacy of the coefficients appearing in this particular instance of the Pieri rules.

5. Hooks.

In this section we shall show how the various ingredients we have put together allow us to prove the $C = \tilde{H}$ conjecture for μ a hook. To guide us and to help our understanding of the proof it is best to begin with a close study of the symmetric function $\tilde{H}_\mu(x;q,t)$ and the polynomial $\tilde{F}_\mu(q,t) = \partial_{p_1}^n \tilde{H}_\mu(x;q,t)$ for μ a hook. To this end it will be convenient to use a notation that is consistent with that introduced in [10] so we may freely refer to [10] for the proof of some of the auxiliary identities. To begin with we need to imbed the collection of hook indexed $\tilde{H}_\mu$*'s* into a larger family which includes $\tilde{H}$*'s* indexed by *broken hooks*.(*) More precisely we set

$$\begin{cases} a) & \tilde{H}[l,0,a] \;=\; \tilde{H}_{1^l}(x;q,t)\, \tilde{H}_a(x;q,t) \\ b) & \tilde{H}[l,1,a] \;=\; \tilde{H}_{1^a b+1}(x;q,t) \end{cases} \quad . \qquad 5.1$$

(*) Diagrams which consist of the disjoint union of a row and a column.

Since $[l,1,a]$ represents a hook with leg l and arm a we may visualize the symbol $[l,0,a]$ as the diagram obtained by removing the corner cell from $[l,1,a]$.

Remark 5.1

We should note at this point that the symmetric function $\tilde{H}[l,0,a]$, for $l+a+1=n$, is the Frobenius characteristic of a module which yields a bigraded version of the regular representation of S_{n-1}. The easiest way to obtain such a module is to induce from $S_{[1,\ldots,l]}\times S_{[l+1,\ldots,n-1]}$ to S_{n-1} the tensor product of the $S_{[1,\ldots,l]}$-Harmonics in $x_1,\ldots,x_l$ by the $S_{[l+1,\ldots,n-1]}$- Harmonics in $y_{l+1},\ldots,y_{n-1}$. In the present context, this representation arises in the following manner. Set for a moment

$$\Delta_{[l,0,a]}(x;y) \;=\; \prod_{1\le i<j\le l}(x_j-x_i)\prod_{l+1\le i<j\le n-1}(y_j-y_i)\ , \tag{5.2}$$

and let

$$b^{\epsilon,\eta}(x,y)=\partial_{x_1}^{\epsilon_1}\cdots\partial_{x_l}^{\epsilon_l}\partial_{y_{l+1}}^{\eta_1}\cdots\partial_{y_{n-1}}^{\eta_{n-1-l}}y_{l+1}\cdots y_{n-1}\Delta_{[l,0,a]}(x;y)\ .$$
$$(\ with\quad 0\le\epsilon_i,\eta_i\le i-1\ )$$

Now it is easy to see that the lexicographically first monomial in $b^{\epsilon,\eta}(x,y)$ is

$$y_{l+1}\cdots y_{n-1}\prod_{i=1}^{l}x_i^{i-1-\epsilon_i}\prod_{j=1}^{n-1-l}y_{l+j}^{j-1-\eta_j}\ .$$

Thus the $b^{\epsilon,\eta}\,'s$ form an independent set of polynomials. Now for a given decomposition of the indices $\{1,2,\ldots,n-1\}$ into two disjoint subsets $S=\{i_1<\cdots<i_l\}$, $T={}^cS=\{j_1<\cdots,j_a\}$ let $b_S^{\epsilon,\eta}(x,y)$ denote the polynomial obtained by substituting in $b^{\epsilon,\eta}(x,y)$ x_s by x_{i_s} and y_j by y_{j_t}. It is not difficult to see that not only the $b_S^{\epsilon,\eta}(x,y)$, for fixed S, are independent but the whole collection

$$\mathcal{B}_{[l,o,a]}\;=\;\{\ b_S^{\epsilon,\eta}(x,y)\ \}_{\substack{S=\{1\le i_1<\cdots<i_l\le n-1\}\\ 0\le\epsilon_i,\eta_i\le i-1}} \tag{5.3}$$

forms a basis for the module obtained by letting S_{n-1} act diagonally on the linear span of the $b^{\epsilon,\eta}\,'s$. Moreover, this action induces a representation with bigraded Frobenius Chacteristic precisely given by the product

$$q^a\tilde{H}[l,0,a]\;=\;q^a\tilde{H}_{1^l}(x;q,t)\,\tilde{H}_a(x;q,t)\ .$$

It will be good to keep this observation in mind as we continue our proof of the $C=\tilde{H}$ conjecture for hooks.

Note next that using the theory of Macdonald polynomial we can derive the following basic identities.

Proposition 5.1

$$\begin{aligned} a)\quad & \tilde{H}_{[l,0,a]}\;=\;\frac{t^l-1}{t^l-q^a}\,\tilde{H}_{[l-1,1,a]}\;+\;\frac{1-q^a}{t^l-q^a}\,\tilde{H}_{[l,1,a-1]}\,,\\ b)\quad & \partial_{p_1}\tilde{H}_{[l,1,a]}\;=\;[l]_t\,\frac{t^{l+1}-q^a}{t^l-q^a}\,\tilde{H}_{[l-1,1,a]}\;+\;[a]_q\,\frac{t^l-q^{a+1}}{t^l-q^a}\,\tilde{H}_{[l,1,a-1]}\ . \end{aligned} \tag{5.4}$$

The proof of a) is a straightforward translation in the present notation of one of the Stanley-Macdonald Pieri rules ([20],[17]). As for 5.4 b), we need only observe that differentiation by p_1 is Cauchy-dual to multiplication by e_1. This given, we may obtain it as another consequence of the same Pieri rules. Details can be found in [9].

Simple manipulations yield that 5.4 (a) & (b) are equivalent to the following two different ways of expressing $\partial_{p_1} \tilde{H}_{[l,1,a]}$.

Proposition 5.2

$$\begin{aligned} a) \quad \partial_{p_1} \tilde{H}_{[l,1,a]} &= [l]_t \tilde{H}_{[l-1,1,a]} + t^l \tilde{H}_{[l,0,a]} + q[a]_q \tilde{H}_{[l,1,a-1]}\,, \\ b) \quad \partial_{p_1} \tilde{H}_{[l,1,a]} &= t[l]_t \tilde{H}_{[l-1,1,a]} + q^a \tilde{H}_{[l,0,a]} + [a]_q \tilde{H}_{[l,1,a-1]}\,. \end{aligned} \tag{5.5}$$

On the validity of the $C = \tilde{H}$ conjectures the Hilbert series of our module $\mathbf{H}_{1^l,a+1}$ should be given by the expression

$$\tilde{F}_{1^l,a+1}(q,t) = \partial_{p_1}^n \tilde{H}_{[l,1,a]}\ , \tag{5.6}$$

Note that the original theory [17] only defines this to be a rational function of q and t. However, we can easily show here that it is a polynomial. To this end note that $\tilde{F}_{1^l,a+1}(q,t)$ satisfies either of the following two recursions.

Proposition 5.3

$$\begin{aligned} a) \quad \tilde{F}_{[l,1,a]} &= [l]_t \tilde{F}_{[l-1,1,a]} + \binom{l+a}{a} t^l\ [l]_t!\, [a]_q! + q[a]_q \tilde{F}_{[l,1,a-1]}\,, \\ b) \quad \tilde{F}_{[l,1,a]} &= t[l]_t \tilde{F}_{[l-1,1,a]} + \binom{l+a}{a} q^a\ [l]_t!\, [a]_q! + [a]_q \tilde{F}_{[l,1,a-1]}\,. \end{aligned} \tag{5.7}$$

Proof

A simple use of Leibnitz formula gives

$$\partial_{p_1}^{n-1} \tilde{H}_{[l,0,a]} = \partial_{p_1}^{n-1} \tilde{H}_{1^l}(x;q,t)\, \tilde{H}_a(x;q,t) = \binom{l+a}{a} [l]_t!\, [a]_q!$$

Thus 5.7 a) and b) are obtained by applying $\partial_{p_1}^{n-1}$ to both sides of 5.5 a) and b).

This yields

Proposition 5.4

$\tilde{F}_{[l,1,a]}$ is a polynomial in q,t whose component of highest degree is bihomogeneous of degree $\binom{l+1}{2}$ in t and degree $\binom{a+1}{2}$ in q. Moreover, it is bisymmetric in t and q, that is we have

$$t^{\binom{l+1}{2}} q^{\binom{a+1}{2}}\, \tilde{F}_{1^l,a+1}(1/q,1/t) = \tilde{F}_{1^l,a+1}(q,t) \tag{5.8}$$

Proof

The polynomiality follows from the fact that $\tilde{F}_{[l,1,a]}$ may be recursively computed by using either of 5.7 a) or b) with the initial condition $\tilde{F}_{1^l,a+1}(q,t)|_{l=a=0} = 1$. On the other hand we see from either recursion that each step increases the degree in t by l and the degree in q by a, this establishes the degree assertion. This given, 5.8 follows immediately from the fact that replacing t by $1/t$ and q by $1/q$ in 5.7 and multiplying both equations by $t^{\binom{l+1}{2}}q^{\binom{a+1}{2}}$ amounts to switching the right-hand sides of 5.7 a) and b). Thus the left-hand side of 5.8 satisfies the same recursions and initial conditions as the right-hand side and thus they must necessarily be identical.

A corollary of the identity in 5.8 is that the specialization $\tilde{F}_{1^l,a+1}(t,t)$ has symmetric coefficient sequence. Our plan is to derive the $n!$ conjecture for $\mu = (1^l, a+1)$ by means of Proposition 3.3. More precisely, we shall construct a basis $\mathcal{B}_{1^l,a+1}$ for the module $\mathbf{R}_{[\rho]}$ (with $\rho = \rho_{1^l,a+1}$) with the property that

$$\sum_{b\in\mathcal{B}_{1^l,a+1}} t^{degree(b)} \;=\; \tilde{F}_{1^l,a+1}(t,t)\ . \tag{5.9}$$

The symmetry of $\tilde{F}_{1^l,a+1}(t,t)$ then guarantees the validity of condition (a) in Proposition 3.3, it follows that $\mathbf{H}_{[\rho]}$ is a cone, necessarily with summit Δ_μ.

The recursions in 5.7 have a natural combinatorial interpretation which we shall use as a guide to the construction of our basis $\mathcal{B}_{1^l,a+1}$. For $n = l+a+1$ let us say that π is a *labelled hook* if π is a filling of the shape $[l,1,a]$ by the entries $\sigma_1,\sigma_2,\ldots,\sigma_n$ of a permutation $\sigma\in S_n$ by placing $\sigma_1,\sigma_2,\ldots,\sigma_a$ from right to left in its arm and $\sigma_{a+1},\sigma_{a+2},\ldots,\sigma_n$ from bottom to top in its leg. For instance the figure below gives the labelled hook corresponding to the filling of $[3,1,5]$ by 143679258.

$$\pi \;=\; \begin{array}{llllll} 8 \\ 5 \\ 2 \\ 9 & 7 & 6 & 3 & 4 & 1 \end{array}$$

Setting $A(\pi) = \sigma_1,\sigma_2,\ldots,\sigma_a$ and $L(\pi) = \sigma_a,\sigma_{a+1},\ldots,\sigma_n$ we can easily show that we have

Proposition 5.5

$$\tilde{F}_{1^l,a+1}(q,t) \;=\; \sum_{\pi} t^{inv(L(\pi))}q^{inv(A(\pi))}, \tag{5.10}$$

where π runs over all labelled hooks of shape $[l,1,a]$ and "inv" denotes the number of inversions in a word.

Proof

Let $l+a+1 = n$ and let, for a moment, $\Sigma_{1^l,a+1}(q,t)$ denote the right hand side of 5.10. Since the initial condition $\Sigma_{[l-1,1,a]}|_{a=l=0} = 1$ may be taken as the definition of our sum for these parameter values, it is sufficient to show that $\Sigma_{1^l,a+1}$ satisfies one of the recursions in 5.7. We can derive the first recursion by grouping summands according to the position of n. To this end note that if n is in the leg of π it will produce an inversion in the word $L(\pi) = \sigma_a\cdots\sigma_n$ for each label that lies above it in π. A moment's reflection should reveal that the terms where there are $0\le i\le l-1$ labels above n must add up to $t^i\,\Sigma_{1^{l-1},a+1}(q,t)$. So the contribution of all the terms where n is in the leg is $[l]_t\Sigma_{1^{l-1},a+1}(q,t)$. On the other hand, if n is in the arm of π it will produce a number $1\le i\le a$ inversions in the word $A(\pi)$ according to the number of labels that are to the left of it in π. Thus all of these terms will add up to $q[a]_q\Sigma_{1^l,a}(q,t)$. Note next that we can construct each labelled hook with n in the corner cell by first choosing a labels out of $1,\ldots,n-1$ and place them in the arm in one of the $a!$ possible ways. Then place n in the corner cell and finally place the remaining labels in the leg in one of the $l!$ possible ways. Let π be the resulting labelled hook. By our construction n will be the last entry in $A(\pi)$ and the first entry in $L(\pi)$ thus n will contribute no inversions to $A(\pi)$ and l inversions to $L(\pi)$. This given we see that the sum of $t^{inv(L(\pi))}q^{inv(A(\pi))}$ over all these terms must be none other than

$$\binom{l+a}{a} t^l \; [l]_t![a]_q! \ .$$

Putting all this together we see that we must have

$$\Sigma_{[l,1,a]} \;=\; [l]_t \, \Sigma_{[l-1,1,a]} \;+\; \binom{l+a}{a} t^l \;\; [l]_t! \, [a]_q! \;+\; q[a]_q \, \Sigma_{[l,1,a-1]}$$

as desired.

We should note that a similar reasoning based on the position of 1 yields that $\Sigma_{[l,1,a]}$ satisfies the second recursion as well.

We shall next carry out the construction of our module $\mathbf{R}_{[\rho_\mu]}$ by the process outlined in Section 1, specializing the "injective tableaux of shape μ" we used there to the labelled hooks of shape $[l,1,a]$. Having chosen the parameters $\alpha_1, \alpha_2, \ldots, \alpha_{l+1}; \beta_1, \beta_2, \ldots, \beta_{a+1}$ to label the rows and columns of $[l,1,a]$, we shall total order the elements of the corresponding orbit according to the lexicographic order of the labeling permutation $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_n$. For instance, for $a = l = 1$ the labeled hooks in this order are

$$\begin{matrix} 3 & \\ 2 & 1 \end{matrix}\ , \quad \begin{matrix} 2 & \\ 3 & 1 \end{matrix}\ , \quad \begin{matrix} 3 & \\ 1 & 2 \end{matrix}\ , \quad \begin{matrix} 1 & \\ 3 & 2 \end{matrix}\ , \quad \begin{matrix} 2 & \\ 1 & 3 \end{matrix}\ , \quad \begin{matrix} 1 & \\ 2 & 3 \end{matrix}\ ,$$

and the corresponding orbit points in this total order are

$$(\alpha_1,\alpha_1,\alpha_2;\beta_2,\beta_1,\beta_1)\ ,\quad (\alpha_1,\alpha_2,\alpha_1;\beta_2,\beta_1,\beta_1)\ ,\quad (\alpha_1,\alpha_1,\alpha_2;\beta_1,\beta_2,\beta_1)\ ,$$
$$(\alpha_2,\alpha_1,\alpha_1;\beta_1,\beta_2,\beta_1)\ ,\quad (\alpha_1,\alpha_2,\alpha_1;\beta_1,\beta_1,\beta_2)\ ,\quad (\alpha_2,\alpha_1,\alpha_1;\beta_1,\beta_1,\beta_2)\ .$$

The idea in our construction of the basis $\mathcal{B}_{1^l,a+1}$ is to produce, for each labelled hook π a polynomial $\phi_\pi(x;y)$ such that

$$\phi_\pi(a_{\pi'};b_{\pi'}) \;=\; \begin{cases} 1 & \text{if } \pi' = \pi \\ 0 & \text{if } \pi' <_T \pi \end{cases}\ , \tag{5.11}$$

where $(a_{\pi'};b_{\pi'})$ is the orbit point corresponding to the labeled hook π' and $\pi' <_T \pi$ means that π' precedes π in the given total order. This given, we let

$$\mathcal{B}_{1^l,a+1} \;=\; \{\ \phi_\pi(x;y)\ \}_\pi \tag{5.12}$$

Clearly, the equalities in 5.11 assure that $\mathcal{B}_{1^l,a+1}$ is a basis. To satisfy the additional condition 5.9 we shall construct $\phi_\pi(x;y)$ so that its component of highest degree is bihomogeneous of degree $inv(L(\pi))$ in t and of degree $inv(A(\pi))$ in q. To better get across how we do this we shall use a jargon that helps visualize the successive steps of the construction. Note that the linear factor

$x_i - \alpha_j$ is different from zero at (a_π, b_π) if and only if i is not in the j^{th} row of π. We shall express this property by saying that $x_i - \alpha_j$ *kicks the label "i" out of the* j^{th} *row of* $[l, 1, a]$. We shall put together our polynomial $\phi_\pi(x; y)$ as the product of linear factors which together *kick* all the labels into the exact position they occupy in π. By this we mean that (a_π, b_π) will be the lexicographically first orbit point at which none of the factors vanish. We describe this construction in general terms first then make it precise by carrying it out in a special case.

Let i_o be the label that occupies the corner cell of π and let $L_{<i_o}$ and $A_{<i_o}$ respectively denote the sets of labels less than i_o that fall in the leg and the arm of π. Similarly let $L_{>i_o}$ and $A_{>i_o}$ denote the sets of labels greater than i_o that fall in the leg and the arm of π. Our first step is to kick every label in $L_{<i_o}$ out of the first row and every label in $A_{>i_o}$ out of the first column. This can be achieved by means of the polynomial

$$\psi(x; y) \ = \ \prod_{i \in L_{<i_o}} (x_i - \alpha_1) \prod_{i \in A_{>i_o}} (y_i - \beta_1) \ . \tag{5.13}$$

It is easy to see that if $(a_{\pi'}, b_{\pi'})$ is the lexicographically first orbit element at which ψ doesn't vanish then all the labels in $L_{<i_o} \cup L_{>i_o}$ fall in increasing order up the leg of π' and all the labels in $A_{<i_o} \cup A_{>i_o}$ fall in the arm of π' in increasing order from right to left. To force the labels in $L_{<i_o} \cup L_{>i_o}$ precisely into the cells they occupy in π, we need to throw in a few additional kicks. We shall illustrate how this is done in a special case The reader should have no difficulty seing that how this last step is to be carried out in the general case. Note that if

$$\pi \ = \ \begin{array}{lllll} 2 & & & & \\ 5 & & & & \\ 6 & 4 & 1 & 3 & 7 \end{array}$$

Then $\psi = (x_2 - \alpha_1)(x_5 - \alpha_1)(y_7 - \beta_1)$ and the lexicographically first labelled hook at which ψ doesn't vanish is

$$\pi' \ = \ \begin{array}{lllll} 5 & & & & \\ 2 & & & & \\ 6 & 7 & 4 & 3 & 1 \end{array}$$

Now we can kick 5 out of the third row by means of $x_5 - \alpha_3$ and kick 7 into the 5^{th} column by means of $(y_7 - \beta_2)(y_7 - \beta_3)(y_7 - \beta_4)$. We see that the lexicographically first labelled hook at which the polynomial

$$\psi'(x; y) = (x_5 - \alpha_3)(y_7 - \beta_2)(y_7 - \beta_3)(y_7 - \beta_4)\psi$$

doesn't vanish is

$$\pi'' \ = \ \begin{array}{lllll} 2 & & & & \\ 5 & & & & \\ 6 & 4 & 3 & 1 & 7 \end{array}$$

Now to force 3 into its desired position we need only kick it out of the 3^{rd} column. This makes π the lexicographically first labelled hook at which the polynomial

$$\psi''(x; y) = (y_3 - \beta_3)\psi''$$

doesn't vanish. So we may set

$$\phi_\pi(x;y) \;=\; \frac{\psi''(x;y)}{\psi''(a_\pi;b_\pi)}\ .$$

A moment's reflection should reveal that, in the general case, a label i, in the arm of π that lies in the j^{th} column can be forced there by kicking out of the j^{th} column all the labels that are greater than i and are to the right of i in π. But that means that the contribution of the label i to the y-degree of $\phi_\pi(x;y)$ is equal to the number of inversions that i causes in the word $A(\pi)$. A similar reasoning applies to the labels in the leg of π. Since the x and y degrees of the polynomial ψ in 5.13 are respectively equal to the number of inversions that i_o makes in $L(\pi)$ and $A(\pi)$ we see that the homogeneous component of highest degree in $\phi_\pi(x;y)$ will be bihomogeneous of of bidegree $\big(inv(L(\pi)), inv(A(\pi))\big)$. This assures that our basis $\mathcal{B}_{1^l,a+1}$ satisfies 5.9 and via Proposition 1.4 proves the $n!$ conjecture for all hook shapes.

Our next and final task is to verify that the Frobenius characteristic of the module $\mathbf{H}_{1^l,a+1}$ is indeed given by the polynomial $\tilde H_{1^l,a+1}(x;q,t)$. There are several ways to do this, we shall follow a path that appears most amenable to extensions to more general shapes. Our plan is to show that the Frobenius characterstic $C_{1^l,a+1}(x;q,t)$ satisfies the recursions in 5.5.To be precise and to conform with the notation in 5.5 we shall set $C_{[l,1,a]} = C_{1^l,a+1}(x;q,t)$ (for all choices of $l,a\geq 0$) and show that we must have

$$\begin{aligned} a)\quad \partial_{p_1} C_{[l,1,a]} \;&=\; [l]_t\, C_{[l-1,1,a]} \;+\; t^l\, \tilde H_{[l,0,a]} \;+\; q[a]_q\, C_{[l,1,a-1]}\ ,\\ b)\quad \partial_{p_1} C_{[l,1,a]} \;&=\; t[l]_t\, C_{[l-1,1,a]} \;+\; q^a\, \tilde H_{[l,0,a]} \;+\; [a]_q\, C_{[l,1,a-1]}\ . \end{aligned} \qquad 5.14$$

This given, subtracting 5.14 b) from 5.14 a) and 5.5 b) from 5.5 a) we can write the difference of the resulting equations in the form

$$(1-t^l)\big(\, C_{[l-1,1,a]} \;-\; \tilde H_{[l-1,1,a]}\big) \;=\; (q^a-1)\big(\, C_{[l,1,a-1]} \;-\; \tilde H_{[l,1,a-1]}\big)\ . \qquad 5.15$$

Since we trivially have the equality $\tilde H_{[l-1,1,a]} \;=\; C_{[l-1,1,a]}$ for $a \;=\; 0$ (*) the desired equality $\tilde H_{[l-1,1,a]} = C_{[l-1,1,a]}$ immediately follows for each $n = l+a+1$ by induction on a.

For $n = l+1+a$, the equalities in 5.14 may be obtained by a suitable decomposition of the action of S_{n-1} on $\mathbf{H}_{1^l,a+1}$. Having proved the $n!$ conjecture in this case, we know that $\mathbf{H}_{1^l,a+1}$ affords the left regular representation of S_n. Thus, its restriction to S_{n-1} must be decomposable into a direct sum of n copies of the left regular representation of S_{n-1}. The equations in 5.14 result from two different ways of carrying out such a decomposition. We shall prove them by studying the action of S_{n-1} on some special bases of $\mathbf{H}_{1^l,a+1}$. To this end we shall pick once and for all two bases $\mathcal{B}_{1^{l-1},a+1}$ and $\mathcal{B}_{1^l,a}$ for $\mathbf{H}_{1^{l-1},a+1}$ and $\mathbf{H}_{1^l,a}$ respectively, then let

$$\begin{aligned} \mathcal{B}_{1^l,a+1} \;=\; \sum_{i=1}^{l}\ \{f_i(\partial)\partial_{x_n}^{i-1}\Delta_{1^l,a+1}\,\}_{f_i\in\mathcal{B}_{1^{l-1},a+1}} \;+\; &\\ \{\, m_S^{\epsilon,\eta}(\partial)\partial_{x_{i_1}}\cdots\partial_{x_{i_l}}\ \Delta_{1^l,a+1}\,\}_{\substack{S=\{1\leq i_1<\cdots<i_l\leq n-1\}\\ 0\leq\epsilon_i,\eta_i\leq i-1}} \;+\; & \\ +\ \sum_{j=1}^{a}\ \{g_j(\partial)\partial_{y_n}^{j}\Delta_{1^l,a+1}\,\}_{g_j\in\mathcal{B}_{1^{l-1},a+1}}\ , & \end{aligned} \qquad 5.16$$

(*) in this case Δ_μ reduces to the Vandermonde determinant in the $x_i\,'s$

where, for a given decomposition of the indices $\{1,\ldots,n-1\}$ into two disjoint subsets,

$$S=\{i_1<\cdots<i_l\} \quad \text{and} \quad T={}^cS=\{j_1<\cdots<j_a\}$$

we let

$$m_S^{\epsilon,\eta}(x,y) \;=\; x_{i_1}^{\epsilon_1}\cdots x_{i_l}^{\epsilon_l}\; y_{j_1}^{\eta_1}\cdots y_{j_a}^{\eta_a}\;. \tag{5.17}$$

We claim that $\mathcal{B}_{1^l,a+1}$ is a basis for $\mathbf{H}_{1^l,a+1}$ and the action of S_{n-1} on this basis induces precisely the decomposition in 5.14 a). The proof of this is best understood by carrying it out on a special case. We shall work with the case $l=3$, $a=4$. We start by showing that $\mathcal{B}_{1^35}$ is a basis. So let f_1,f_2,f_3 be elements in the linear span of $\mathcal{B}_{1^25}$ and let g_1,g_2,g_3,g_4 be in the linear span of $\mathcal{B}_{1^34}$. Note that our construction requires that these are polynomials in $x_1,\ldots,x_7;y_1,\ldots,y_7$. Furthermore, let $h_{i_1,i_2,i_3}(x,y)$ be an element in the linear span of the monomials in 5.17 for fixed $S=\{1\le i_1<i_2<i_3\le 7\}$. We claim that the equality

$$\begin{aligned} &f_1(\partial)\Delta_{1^35}+f_2(\partial)\partial_{x_8}\Delta_{1^35}+f_3(\partial)\partial_{x_8}^2\Delta_{1^35}\;+\\ &\quad+\sum_{S=\{1\le i_1<i_2<i_3\le 7\}} h_{i_1,i_2,i_3}(\partial)\partial_{x_{i_1}}\partial_{x_{i_2}}\partial_{x_{i_3}}\Delta_{1^35}\;+\\ &\qquad+\;g_1(\partial)\partial_{y_8}\Delta_{1^35}+g_2(\partial)\partial_{y_8}^2\Delta_{1^35}+g_3(\partial)\partial_{y_8}^3\Delta_{1^35}+g_4(\partial)\partial_{y_8}^4\Delta_{1^35}\;=\;0 \end{aligned} \tag{5.18}$$

implies that $f_1,f_2,f_3,g_1,g_2,g_3,g_4$ and all h_{i_1,i_2,i_3} are equal to zero. To start with note that from the definition I.3 we have that

$$\Delta_{1^35}(x;y) \;=\; det\begin{pmatrix} 1&1&1&1&1&1&1&1\\ x_1&x_2&x_3&x_4&x_5&x_6&x_7&x_8\\ x_1^2&x_2^2&x_3^2&x_4^2&x_5^2&x_6^2&x_7^2&x_8^2\\ x_1^3&x_2^3&x_3^3&x_4^3&x_5^3&x_6^3&x_7^3&x_8^3\\ y_1&y_2&y_3&y_4&y_5&y_6&y_7&y_8\\ y_1^2&y_2^2&y_3^2&y_4^2&y_5^2&y_6^2&y_7^2&y_8^2\\ y_1^3&y_2^3&y_3^3&y_4^3&y_5^3&y_6^3&y_7^3&y_8^3\\ y_1^4&y_2^4&y_3^4&y_4^4&y_5^4&y_6^4&y_7^4&y_8^4 \end{pmatrix} \tag{5.19}$$

and we see that are no terms in x_8^4, x_8y_8 or y_8^5 in the expansion of Δ_{1^35}. This implies that we have the equalities

$$\partial_{x_8}^4\Delta_{1^35}\;=\;0 \quad,\quad \partial_{x_8}\partial_{y_8}\Delta_{1^35}\;=\;0 \quad\text{and}\quad \partial_{y_8}^5\Delta_{1^35}\;=\;0 \tag{5.20}$$

Moreover, since for $i_1<i_2<i_3<8$, the polynomial

$$(x_{i_1}-\alpha_1)(x_{i_2}-\alpha_1)(x_{i_3}-\alpha_1)(x_8-\alpha_1) \tag{5.21}$$

vanishes identically on the orbit of ρ_{1^35} we deduce that the monomial $x_{i_1}x_{i_2}x_{i_3}x_8$ lies in the ideal $gr\ J_{[\rho_{135}]}$ and therefore we must also have

$$\partial_{x_{i_1}}\partial_{x_{i_2}}\partial_{x_{i_3}}\partial_{x_8}\Delta_{1^35}\;=\;0\;. \tag{5.22}$$

This given, we see that differentiating 5.18 by $\partial_{x_8}^3$ and using 5.20 and 5.22 we get that

$$0 \;=\; f_1(\partial)\partial_{x_8}^3\Delta_{1^35} \;=\; 6f_1(\partial)\Delta_{1^25}\ .$$

However, this cannot happen for a harmonic f_1 in the linear span of $\mathcal{B}_{1^25}$ without it being identically zero. Replacing f_1 by zero in 5.18 and differentiating by $\partial_{x_8}^2$ now gives

$$0 \;=\; f_2(\partial)\partial_{x_8}^3\Delta_{1^35} \;=\; 6f_2(\partial)\Delta_{1^25}\ ,$$

which for the same reason yields that f_2 must be a zero linear combination of the elements of $\mathcal{B}_{1^25}$. Finally, replacing $f_1\&f_2$ by zero and differentiating by $\partial_{x_8}^2$ 5.18 gives that $f_3=0$.

Next we must see what becomes of 5.18 if we replace f_1, f_2, f_3 by zero and set $y_8=0$. To this end observe that

$$\partial_{x_{i_1}}\partial_{x_{i_2}}\partial_{x_{i_3}}\Delta_{1^35}\ |_{y_8=0} \;=\; y_{j_1}y_{j_2}y_{j_3}y_{j_4}\,det\begin{pmatrix}1 & 1 & 1\\ x_{i_1} & x_{i_2} & x_{i_3}\\ x_{i_1}^2 & x_{i_2}^2 & x_{i_3}^2\end{pmatrix} det\begin{pmatrix}1 & 1 & 1 & 1\\ y_{j_1} & y_{j_2} & y_{j_3} & y_{j_4}\\ y_{j_1}^2 & y_{j_2}^2 & y_{j_3}^2 & y_{j_4}^2\\ y_{j_1}^3 & y_{j_2}^3 & y_{j_3}^3 & y_{j_4}^3\end{pmatrix}\ ,$$

where $\{j_1,j_2,j_3,j_4\}$ is the complement of $\{i_1,i_2,,i_3\}$ in $\{1,2,3,\ldots,7\}$. Furthermore it is easy to check that by contrast none of the determinants

$$g_1(\partial)\partial_{y_8}\Delta_{1^35}\ |_{y_8=0}\ ,\ g_2(\partial)\partial_{y_8}^2\Delta_{1^35}\ |_{y_8=0}\ ,\ g_3(\partial)\partial_{y_8}^3\Delta_{1^35}\ |_{y_8=0}\ ,\ g_4(\partial)\partial_{y_8}^4\Delta_{1^35}\ |_{y_8=0}$$

have in their expansions monomials with any of the products $y_{j_1}y_{j_2}y_{j_3}y_{j_4}$ as a factor. These observations allow us to conclude that the relation

$$\begin{aligned}&\sum_{S=\{1\le i_1<i_2<i_3\le 7\}} h_{i_1,i_2,i_3}(\partial)\partial_{x_{i_1}}\partial_{x_{i_2}}\partial_{x_{i_3}}\Delta_{1^35}|_{y_8=0}\ +\\ &+\ g_1(\partial)\partial_{y_8}\Delta_{1^35}|_{y_8=0}+g_2(\partial)\partial_{y_8}^2\Delta_{1^35|_{y_8=0}}+g_3(\partial)\partial_{y_8}^3\Delta_{1^35}|_{y_8=0}+g_4(\partial)\partial_{y_8}^4\Delta_{1^35}|_{y_8=0}\\ &=\ 0\end{aligned}$$

can hold true only if each of the individual summands in the first sum vanishes separately. Note that the term

$$h_{i_1,i_2,i_3}(\partial)\partial_{x_{i_1}}\partial_{x_{i_2}}\partial_{x_{i_3}}\Delta_{1^35}|_{y_8=0}$$

is, by construction, a linear combination of the polynomials

$$b^{\epsilon,\eta}_{\{i_1,i_2,i_3\}}(x_{i_1},x_{i_2},x_{i_3};y_{j_1},y_{j_2},y_{j_3},y_{j_4})\qquad(\text{with}\quad 0\le\epsilon_i,\eta_i\le i-1\ )$$

which by our Remark 5.1 form an independent set. This gives that each of the h_{i_1,i_2,i_3} must be the zero linear combination of the monomials in 5.17. We can now set $f_1=f_2=f_3=0$ and $h_{i_1,i_2,i_3}=0$ for all subsets $\{i_1,i_2,i_3\}$ of $\{1,2,\ldots,7\}$ in 5.18 and reduce it to

$$g_1(\partial)\partial_{y_8}\Delta_{1^35}+g_2(\partial)\partial_{y_8}^2\Delta_{1^35}+g_3(\partial)\partial_{y_8}^3\Delta_{1^35}+g_4(\partial)\partial_{y_8}^4\Delta_{1^35}\ =\ 0$$

Differentiating by $\partial_{y_8}^3$ and using the last of 5.20 we derive that

$$0 \;=\; g_1(\partial)\partial_{y_8}^4 \Delta_{1^3 5} \;=\; 24 g_1(\partial)\Delta_{1^3 4}$$

which forces $g_1 = 0$ since a non trivial linear combination harmonics in $\mathcal{B}_{1^3 4}$ cannot kill $\Delta_{1^3 4}$. We should easily see that the process which yielded the vanishing of f_2 and f_3 can be repeated here to deliver in the same manner the vanishing of g_2, g_3 and g_4. This shows that the collection $\mathcal{B}_{1^3 5}$ defined in 5.16 is an independent set of harmonics in the linear span of derivatives of $\Delta_{1^3 5}$ and since this collection has exactly 8! elements it must be a basis of our module $\mathbf{H}_{1^3 5}$. The reader should have no difficulty seeing that the argument we carried out for $l = 3$ and $a = 4$ can be extended to the general case.

We observe next that each of the summands in the definition 5.16 of $\mathcal{B}_{1^l, a+1}$ is a basis of a subspace of $\mathbf{H}_{1^l, a+1}$ that is invariant under the action of S_{n-1}. However it is not difficult to derive that the bigraded Frobenius characteristic of the action of S_{n-1} on the linear span of the basis $\{ f_i(\partial)\partial_{x_n}^{i-1} \Delta_{1^l, a+1} \}_{f_i \in \mathcal{B}_{1^{l-1}, a+1}}$ is given by the symmetric polynonial

$$t^{\binom{l}{2} - i + 1} q^{\binom{a}{2}} C_{[l-1,1,a]}(x; 1/q, 1/t) \ .$$

Similarly we derive that the bigraded Frobenius characteristic of the action of S_{n-1} on the linear span of the basis $\{ g_j(\partial)\partial_{y_n}^{j} \Delta_{1^l, a+1} \}_{g_j \in \mathcal{B}_{1^l, a}}$ is

$$t^{\binom{l}{2}} q^{\binom{a}{2} - j} C_{[l,1,a-1]}(x; 1/q, 1/t) \ .$$

Finally, using Remark 5.1, we obtain that the bigraded Frobenius characteristic of the action of S_{n-1} on the linear span of $\{ \, m_S^{\epsilon,\eta}(\partial)\partial_{x_{i_1}} \cdots \partial_{x_{i_l}} \, \Delta_{1^l, a+1} \, \}_{\substack{S = \{1 \leq i_1 < \cdots < i_l \leq n-1\} \\ 0 \leq \epsilon_i, \eta_i \leq i-1}}$ is

$$q^a \, \tilde{H}_{1^l}(x; q, t) \tilde{H}_a(x; q, t) \ .$$

Putting all this together, gives the recursion

$$\begin{aligned} \partial_{p_1} C_{[l,1,a]}(x; q, t) \;=\; & \sum_{i=1}^{l} t^{\binom{l}{2} - i + 1} q^{\binom{a}{2}} C_{[l-1,1,a]}(x; 1/q, 1/t) \;+ \\ & + \; q^a \, \tilde{H}_{1^l}(x; q, t) \tilde{H}_a(x; q, t) \;+ \\ & + \; \sum_{j=1}^{a} t^{\binom{l}{2}} q^{\binom{a}{2} - j} C_{[l,1,a-1]}(x; 1/q, 1/t) \end{aligned} \qquad 5.23$$

Since we do have the identities

$$\begin{aligned} \partial_{p_1} C_{[l,1,a]}(x; q, t) \;&=\; t^{\binom{l}{2}} q^{\binom{a}{2}} \, \partial_{p_1} C_{[l,1,a]}(x; 1/q, 1/t) \\ C_{[l-1,1,a]}(x; q, t) &= t^{\binom{l-1}{2}} q^{\binom{a}{2}} \, \partial_{p_1} C_{[l-1,1,a]}(x; 1/q, 1/t) \\ C_{[l,1,a-1]}(x; q, t) &= t^{\binom{l}{2}} q^{\binom{a-1}{2}} \, \partial_{p_1} C_{[l,1,a-1]}(x; 1/q, 1/t) \\ \tilde{H}_{1^l}(x; q, t) \tilde{H}_a(x; q, t) &= t^{\binom{l-1}{2}} q^{\binom{a-1}{2}} \tilde{H}_{1^l}(x; 1/q, 1/t) \tilde{H}_a(x; 1/q, 1/t) \end{aligned} ,$$

it is easy to see that 5.23 is just another way of writing the identity in 5.14 a).

To establish 5.14 b) we can proceed in two ways. We can follow the argument which gave 5.14 a) reversing the roles of x and y. Or, (which is basically equivalent) note that it follows from the fact that $C_{[l,1,a]}(x;t,q) = C_{[a,1,l]}(x;q,t)$. In other words 5.14 b) follows from 5.14 a) by interchanging arm with leg and q with t. This completes our proof of the $C = \tilde{H}$ conjecture for hooks.

Remark 5.2

The above proof of $C = \tilde{H}$ for hooks also establishes the $n!$ conjecture for hooks, theoretically making the 'kicking' proof superfluous. It can be shown, however, that on the validity of the $n!$ conjecture, the kicking method which we followed in the hook case is bound to work in general. Nevertheless, the complexity of the polynomials that accomplish the desired kicking increases considerably as we pass to more general shapes. For instance, as we pass from hooks to extended hooks (three-corner shapes obtained by adding a corner square to a hook) we can no longer achieve the needed kicking by a product of linear factors. E. Reiner [18] discovered that in this case some of the kicking has to be achieved by means of polynomials which can be recognized as *skew flag Schur functions*. This additional ingredient permitted Reiner to establish the $n!$ conjecture for all extended hooks. In view of the formidable complexity of Reiner's proof it is apparent that some new ingredients will have to be added in order to extend this method of proof to general shapes. We should also mention that completely different approaches to the $n!$ conjecture have been developed by E. Allen [1] and Dunkl & Hanlon [5]. Allen's aproach, which uses Rota's biletter algebras yielded a combinatorial interpretation of the $\tilde{K}_{\lambda\mu}(q,t)$ for μ a Hook. An equivalent interpretation had been also been previously obtained by Stembridge [22]. The work of Dunkl & Hanlon uses Dunkl operators and although falls short of proving the $n!$ conjecture it does reduce it to remarkable new conjectures.

Remark 5.3

We should mention that the spaces $\mathbf{H}_\mu$ are all contained in the space $\mathbf{DH}_n[X;Y]$ of harmonics corresponding to the diagonal action of S_n. It can be shown that $\mathbf{DH}_n[X;Y]$ consists of all solutions $P(x;y)$ of the differential equations

$$\sum_{i=1}^{n} \partial_{x_i}^h \partial_{y_i}^k \, P(x;y) \;=\; 0 \;. \qquad\qquad (\, h+k \geq 1 \,)$$

Our discovery of the relation between the Frobenius characteristics of the $\mathbf{H}_\mu$ *'s* and the Macdonald polynomials naturally brought particular attention to the space $\mathbf{DH}_n[X;Y]$ itself. Computer data revealed remarkable properties of $\mathbf{DH}_n[X;Y]$ which have been translated by several observers (see [15]) into a collection of outstanding conjectures. An algebraic geometrical study of this space led M. Haiman to conjecture an explicit expansion of the bigraded Frobenius characteristic of $\mathbf{DH}_n[X;Y]$ in terms of our polynomials $\tilde{H}_\mu[x;q,t]$. The consequences of this expansion in the theory of symmetric functions and its combinatorial implications are given in [11].

REFERENCES

[1] E. Allen, *The Decomposition of a Bigraded Left Regular Representation of the Diagonal Action of* S_n, to appear in the Journal of Comb. Theory Series A.

[2] N. Bergeron and A. M. Garsia, *On Certain Spaces of Harmonic Polynomials*, Hypergeometric Functions on domains of Positivity, Jack polynomials and Applications, Contemporary Mathematics, **138** (1992) 51-86.

[3] D. Bernstein and A. Iarrobino, *A Non-unimodal Graded Gorenstein Artin Algebra in Codimension Five*, Communications in Algebra, **20**, No. 8 (1992) 2323-2336.

[4] L. Butler, Personal communication.

[5] C. Dunkl and P. Hanlon, *Integrals of Polynomials associated with Tableaux and the Garsia-Haiman Coinjecture*, to appear in Mathematische Zeitschrift.

[6] Y. M. Chen, A. M. Garsia and J. Remmel, *Algorithms for Plethysm*, Contemporary Math. #34. Combinatorics and Algebra, Curtis Greene ed. (1984) pp. 109-153

[7] A. Garsia, *Recent Progress on the Macdonald q,t-Kostka conjecture*, Actes du 4^e Colloque sur les Séries Formelles et Combinatoire Algébrique, U.Q.A.M. , Pub. L.A.C.I.M. Montréal, (1992) 249-255.

[8] A. M. Garsia and M. Haiman, *A Graded Representation Model for Macdonald's Polynomials*, Proceedings of The National Academy (1993) 3607-3610.

[9] A. M. Garsia and M. Haiman, *Orbit Harmonics and Graded Representations* (Research Monograph to appear as part of the Collection Published by the Lab. de Comb. et Informatique Mathématique, edited by S. Brlek, U. du Québec à Montréal)

[10] A. M. Garsia and M. Haiman, *Factorizations of Pieri rules for Macdonald polynomials*, To appear in the Proceedings of the IV^{th} Colloquium on Algebraic Combinatorics

[11] A. Garsia and M. Haiman, *A Remarkable q,t-Catalan Sequence and q-Lagrange inversion*, To appear in the Journal of Algebraic Combinatorics.

[12] A. M. Garsia and C. Procesi, *On certain graded* S_n*-modules and the q-Kostka polynomials*, Advances in Mathematics **94** (1992) 82-138.

[13] A. Garsia and M. Wachs, *Combinatorial Aspects of Skew Representations of the Symmetric Group*, J. Comb. Theory, **A** V. 50 (1989) 47-81.

[14] H. Garnir, *Théorie de la représentation lineaire des groupes symmétriques*, Mémoires de la Soc. Royale de Liège, (4), 10 (1950).

[15] M. Haiman, *Conjectures on the Quotient Ring by Diagonal Invariants*, J. Alg. Combinatorics **3** (1994) 17–76.

[16] G. D. James, *The representation theory of the symmetric group*, Addison-Wesley, Reading, Mass. (1981).

[17] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Clarendon Press, Oxford (1979).

[17] I. G. Macdonald, *A new class of symmetric functions*, Actes du 20^e Séminaire Lotharingien, Publ. I.R.M.A. Strasbourg, (1988) 131-171. (To appear also in the the new edition of [17] above)

[18] E. Reiner, *A proof of the $n!$ conjecture for extended hooks*, to appear in the Journal of Comb. Theory Series A. See also *Some applications of the theory of orbit harmonics*, Doctoral dissertation UCSD (1993)

[19] W. Specht, *Die irreduciblen Darstellungen der Symmetrischen Gruppen*, Math. Zeit. **39** (1935) 696-711.

[20] R.Stanley, *Some combinatorial properties of Jack Symmetric functions*, Adv. Math. 77 (1989) 76-117.

[21] R. Steinberg, *Differential equations invariant under finite reflection groups*, A. M. S. Transactions 112 (1964) 392-400

[22] J. Stembridge, *Some particular entries of the two-parameter Kostka matrix*, Proc. Amer. Math. Soc. 121 (1994), 367–373.

[23] A. Young, *On quantitative substitutional analysis* (sixth paper), The collected papers of A. Young, University of Toronto Press (1977) pp. 434-435.