LATTICE TILINGS BY CUBES: WHOLE, NOTCHED AND EXTENDED

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Abstract

We discuss some problems of lattice tiling via Harmonic Analysis methods. We consider lattice tilings of $\mathbf{R}^d$ by the unit cube in relation to the Minkowski Conjecture (now a theorem of Hajós) and give a new equivalent form of Hajós's theorem. We also consider "notched cubes" (a cube from which a reactangle has been removed from one of the corners) and show that they admit lattice tilings. This has also been been proved by S. Stein by a direct geometric method. Finally, we exhibit a new class of simple shapes that admit lattice tilings, the "extended cubes", which are unions of two axis-aligned rectangles that share a vertex and have intersection of odd codimension.

In our approach we consider the Fourier Transform of the indicator function of the tile and try to exhibit a lattice of appropriate volume in its zero-set.

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§0. Introduction

0.1 Results.

We obtain some results about translational tilings of $\mathbf{R}^d$ with some simple classes of of polyhedra as tiles (cubes as well as "notched" and "extended" cubes–see §2 for a definition of the latter shapes). The approach we use is to study the zero-set of the Fourier Transform (FT) of the indicator function of the tile. If that set contains a lattice except 0 then the set tiles $\mathbf{R}^d$ when translated at the locations of the dual lattice. This means that the translated copies of the tile cover (almost) every point in $\mathbf{R}^d$ a constant number of times–see Theorem 2.

In §1 we use our harmonic analysis approach to derive a new equivalent form of the Minkowski conjecture (every lattice tiling of $\mathbf{R}^d$ with the unit cube contains two cubes which share an entire $(d-1)$-dimensional face) which was proved by Hajós [**Haj**] in 1941. This new form of the Minkowski conjecture (Theorem 6) is an elementary number-theoretic statement that involves no inequalities and could conceivably lead to a new, elementary proof of the conjecture.

In §2 we prove that certain classes of polyhedra tile $\mathbf{R}^d$ if translated by an appropriate lattice. The notched cube (see Figure 1) has already been shown by Stein [**St**] to tile $\mathbf{R}^d$ by a lattice (Conlan [**Con**] has done this in some cases). Stein's method was a direct geometric one. We give a new proof that the notched cube is a tile using our approach. That is, we find lattices in the zero-set of the FT of the indicator function of the notched cube, which is a very explicit function (see (11)). We find all the tilings discovered by Stein, which, by a deeper theorem of Schmerl [**Sch**], is the complete list of possible translational tilings (lattice or not) of the notched cube.

However, our approach for the notched cube leads us to the discovery of a whole class of simple tiles of $\mathbf{R}^d$ (the "extended cubes"–see Figure 2), for which we know of no geometric proof of the fact that they tile. These tiles consist of two axis-aligned rectangles which share a vertex and have intersection of odd codimension, and the lengths of their sides can be completely arbitrary. The tiling lattices for each of these tiles are very simple to describe. Furthermore, the proof that the notched cube tiles essentially proves that the extended cubes tile as well, as the FT of the two indicator functions (that of the notched cube and of that of the extended cube) have the same form and differ only at the values of some parameters.

0.2 Translational tiling in $\mathbf{R}^d$.

Let $f \in L^1(\mathbf{R}^d)$ and $A \subset \mathbf{R}^d$ be a discrete point set. We say that (the *tile*) f tiles $\mathbf{R}^d$ with (the *tile set*) A and with weight w if for almost all (Lebesgue) $x \in \mathbf{R}^d$ we have

$$\sum_{a \in A} f(x-a) = w, \tag{1}$$

where the series above converges absolutely. If f is the indicator function of a (measurable) set $T \subset \mathbf{R}^d$ then we also say that T (the *tile*) tiles $\mathbf{R}^d$ with weight w, which then has to be a nonnegative integer. When $w = 1$ we sometimes write

$$\mathbf{R}^d = T \oplus A. \tag{2}$$

We restrict our attention to tile sets A of *bounded density.* That is, we demand that

$$\#\left(A \cap (x + [0,1]^d)\right) \le C, \tag{3}$$

for all $x \in \mathbf{R}^d$ and for some constant C, a requirement which is automatically fulfilled whenever $f \ge 0$.

0.3 A spectral condition for tiling.

In [**KLa**] a necessary and sufficient condition was given for (1) to hold. It was proved for dimension $d = 1$ only. Here we state it for arbitrary d. We ommit the proof as it is identical to the one-dimensional case.

For a tempered distribution μ we denote by $\widehat{\mu}$ its Fourier Transform (see for example [**Str**]).

Theorem 1 *Assume that $f \in L^1(\mathbf{R}^d)$ has Fourier Transform $\widehat{f} \in C^\infty(\mathbf{R}^d)$ and that the discrete set $A \subset \mathbf{R}^d$ is of bounded density. Write*

$$\mu = \sum_{a \in A} \delta_a, \tag{4}$$

where δ_a is a point mass at a.

(i) *If f tiles $\mathbf{R}^d$ with the tile set A and some weight w then*

$$\operatorname{supp} \widehat{\mu} \ \subseteq\ Z := \{0\} \cup \left\{\xi \in \mathbf{R}^d :\ \widehat{f}(\xi) = 0\right\}. \tag{5}$$

(ii) *If $\widehat{\mu}$ is locally a finite measure and* $\operatorname{supp} \widehat{\mu} \subseteq Z$*, then f tiles $\mathbf{R}^d$ with the tile set A and weight $w = \widehat{\mu}(\{0\}) \int_{\mathbf{R}^d} f$.*

Note that the requirement that $\widehat{f} \in C^\infty(\mathbf{R})$ is true for all f of compact support.

0.4 Fourier Transform and the Poisson Summation Formula.

The definition of Fourier Transform we use throughout this paper is

$$\widehat{f}(\xi) = \int_{\mathbf{R}^d} e^{-2\pi i \langle \xi, x \rangle} f(x)\ dx,$$

for $f \in L^1(\mathbf{R}^d)$.

Let $\Lambda = A\mathbf{Z}^d$, A a non-singular $n \times n$ real matrix, be a lattice in $\mathbf{R}^d$ and write

$$\Lambda^* = \left\{x \in \mathbf{R}^d :\ \forall \lambda \in \Lambda\ \langle x, \lambda \rangle \in \mathbf{Z}\right\}.$$

It turns out that $\Lambda^* = A^{-\top}\mathbf{Z}^d$ is a lattice which we call the *dual* lattice of Λ.

The Poisson Summation Formula (PSF)

$$\sum_{\lambda \in \Lambda} \varphi(x - \lambda) = \left|\det A\right|^{-1} \sum_{\lambda^* \in \Lambda^*} \widehat{\varphi}(x - \lambda^*),$$

valid for all smooth φ of compact support, can be written as a distribution identity as follows:

$$\left(\sum_{\lambda\in\Lambda}\delta_\lambda\right)^\wedge = |\det A|^{-1}\sum_{\lambda^*\in\Lambda^*}\delta_{\lambda^*}. \tag{6}$$

Our spectral criterion for tiling (Theorem 1) then takes the following simpler form for lattice tilings.

Theorem 2 *Assume that $f \in L^1(\mathbf{R}^d)$. Then f tiles $\mathbf{R}^d$ with a lattice Λ and some weight w if and only if*

$$\Lambda^* \setminus \{0\} \subseteq \left\{\xi \in \mathbf{R}^d :\ \widehat{f}(\xi) = 0\right\}. \tag{7}$$

In this case we have $w = |\det\Lambda|^{-1}\displaystyle\int_{\mathbf{R}^d} f$.

Proof. (without using Theorem 1) Let D be a fundamental parallelepiped of Λ. The function

$$g(x) = \sum_{\lambda\in\Lambda} f(x-\lambda)$$

is defined as an absolutely convergent series for almost all $x \in \mathbf{R}^d$ (since $f \in L^1(\mathbf{R}^d)$), is Λ-periodic, and $g \in L^1(D)$.

The dual group of $D = \mathbf{R}^d/\Lambda$ is identified with Λ^*. That is the continuous characters of $\mathbf{R}^d/\Lambda$ are the functions

$$\phi_{\lambda^*}(x) = \exp(2\pi i\langle\lambda^*, x\rangle), \quad \lambda^* \in \Lambda^*,$$

and g is constant (i.e., f tiles with Λ) if and only if

$$\widehat{f}(\lambda^*) = \langle g, \phi_{\lambda^*}\rangle = 0, \quad \forall\lambda^* \in \Lambda^*\setminus\{0\}.$$

■

An alternative would be to prove Theorem 2 for $\Lambda = \mathbf{Z}^d$ using ordinary multiple Fourier series and then use a linear transformation to get the general form of the theorem.

Thus, a measurable $T \subseteq \mathbf{R}^d$ tiles with Λ (in the ordinary sense of weight 1) if and only if $\widehat{\mathbf{1}_T}$ vanishes on $\Lambda^*\setminus\{0\}$ and the volume of Λ is equal to that of T.

All tilings in §1 and §2 are tilings of weight 1.

§1. The Minkowski Conjecture

1.1 Two equivalent forms of the conjecture.

Minkowski's theorem on linear forms is the following statement.

Theorem 3 (Minkowski) *If $A \in M_d(\mathbf{R})$ has $\det A = 1$ then there is $x \in \mathbf{Z}^d\setminus\{0\}$ such that*

$$\|Ax\|_\infty \leq 1.$$

Minkowski conjectured around 1900 that one can always get $\|Ax\|_\infty < 1$ except when A has an integral row. This was proved by Hajós [**Haj**] in 1941.

Theorem 4 (Hajós) *If $A \in M_d(\mathbf{R})$ has $\det A = 1$ then there is $x \in \mathbf{Z}^d$ such that*

$$\|Ax\|_\infty < 1,$$

unless A has an integral row.

Hajós actually worked on the following equivalent form of the Minkowski conjecture, which involves lattice tilings by a cube. This form was already known to Minkowski and most results on Minkowski's conjecture leading up to Hajós's eventual proof have used this form.

Theorem 5 *If $Q = [-1/2, 1/2]^d$ is a cube of unit volume in $\mathbf{R}^d$, $\Lambda \subset \mathbf{R}^d$ is a lattice, and*

$$\mathbf{R}^d = Q \oplus \Lambda$$

is a lattice tiling of $\mathbf{R}^d$ then there are two cubes in the tiling that share a $(d-1)$-dimensional face. In other words, for some $i = 1, \ldots, d$, the standard basis vector $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^\top \in \Lambda$.

Before going on to describe a new equivalent form of the Minkowski conjecture (Theorem 6) we sketch a proof of the equivalence of Theorems 4 and 5.

Theorem 4 $\Longrightarrow$ Theorem 5.

Let $\Lambda = A\mathbf{Z}^d$ with $\det A = 1$, $Q \oplus \Lambda = \mathbf{R}^d$. Then, either there is a non-zero Λ-point in the interior of $2Q$ or A has an integral row. The first cannot happen because of the tiling assumption. Therefore $a_{ij} \in \mathbf{Z}$ for some i and for all j. Again because of tiling it follows that $\gcd(a_{i1}, \ldots, a_{i,d}) = 1$. Let $\mathbf{R}^{d-1}$ be the subspace spanned by all e_j, $j \neq i$, and define $\Lambda' = \Lambda \cap \mathbf{R}^{d-1}$ and $Q' = Q \cap \mathbf{R}^{d-1}$. It follows that $\mathbf{R}^{d-1} = \Lambda' \oplus Q'$ is a tiling of $\mathbf{R}^{d-1}$. By induction then Λ' contains some vector of the standard basis and so does Λ.

■

Theorem 5 $\Longrightarrow$ Theorem 4.

Theorem 5 easily implies the seemingly stronger statement that, if $A\mathbf{Z}^d \oplus Q = \mathbf{R}^d$ is a tiling then, after a permutation of the coordinate axes, the matrix A takes the form

$$\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ a_{2,1} & 1 & 0 & & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{d,1} & \ldots & \ldots & \ldots & 1 \end{pmatrix} \tag{8}$$

Using this remark, if $A\mathbf{Z}^d \cap (-1,1)^d = \{0\}$ we get, since $\det A = 1$, that $A\mathbf{Z}^d \oplus Q = \mathbf{R}^d$ and, therefore, A is (after permutation of the coordinate axes) of the type (8), and thus has an integral row (which property is preserved under permutation similarity).

■

1.2 A new equivalent form.

In this section we prove that the following is equivalent to the Minkowski conjecture (Theorems 4 and 5).

Theorem 6 *Let $B \in M_d(\mathbf{R})$ have $\det B = 1$ and the property that for all $x \in \mathbf{Z}^d \setminus \{0\}$ some coordinate of the vector Bx is a non-zero integer. Then B has an integral row.*

Remark. One might think that Theorem 6 can be proved equivalent directly to Theorem 4, which it resembles most. It is, indeed, clear that Theorem 4 implies Theorem 6. However, the proof that is given here is that of the equivalence of Theorems 6 and 5 using our spectral criterion for tilings (Theorem 2) and I do not know of a more direct proof that Theorem 6 implies Theorem 4.

We shall need the following simple lemma.

Lemma 1 *Let $A \in M_d(\mathbf{R})$ be a non-singular matrix. The lattice $A^{-\top}\mathbf{Z}^d$ contains the basis vector e_i if and only if the i-th row of A is integral.*

Proof. Without loss of generality assume $i = 1$.

If $e_1 \in A^{-\top}\mathbf{Z}^d$ then $e_1 = A^{-\top}x$ for some $x \in \mathbf{Z}^d$. Therefore, for all $y \in \mathbf{Z}^d$ we have

$$(Ay)_1 = e_1^\top Ay = x^\top A^{-1}Ay = x^\top y \in \mathbf{Z}.$$

It follows that $(Ay)_1 \in \mathbf{Z}$ for all $y \in \mathbf{Z}^d$ and the first row of A is integral.

Conversely, if the first row of A is integral, then, for all $y \in \mathbf{Z}^d$

$$\mathbf{Z} \ni (Ay)_1 = x^\top y,$$

where $A^{-\top}x = e_1$ ($x \in \mathbf{R}^d$). It follows that $x \in \mathbf{Z}^d$ and $e_1 \in A^{-\top}\mathbf{Z}^d$.

∎

Proof of the equivalence of Theorems 5 and 6.

Let $f(x) = \mathbf{1}(x \in Q)$ be the indicator function of the unit-volume cube $Q = [-1/2, 1/2]^d$. A simple calculation shows that

$$\widehat{f}(\xi) = \prod_{j=1}^{d} \frac{\sin \pi\xi_j}{\pi\xi_j}, \tag{9}$$

so that

$$Z = \left\{\widehat{f} = 0\right\} = \left\{\xi \in \mathbf{R}^d : \text{ some } \xi_j \text{ is a non-zero integer}\right\}. \tag{10}$$

Therefore, if $\Lambda = B^{-\top}\mathbf{Z}^d$ then (since Λ has volume 1)

$$Q \oplus \Lambda = \mathbf{R}^d \Longleftrightarrow \Lambda^* \setminus \{0\} \subseteq Z,$$

where $\Lambda^* = B\mathbf{Z}^d$, by Theorem 2. In words, Q tiles with Λ if and only if for every $x \in \mathbf{Z}^d \setminus \{0\}$ the vector Bx has some non-zero integral coordinate.

Theorem 5 $\Longrightarrow$ Theorem 6.

Suppose $x \in \mathbf{Z}^d \setminus \{0\}$ implies some $(Bx)_i \in \mathbf{Z} \setminus \{0\}$. Then $Q \oplus \Lambda = \mathbf{R}^d$ and from Theorem 5, say, $e_1 \in \Lambda$, which, from Lemma 1, implies that the first row of B is integral.

Theorem 6 $\Longrightarrow$ Theorem 5.

Assume $Q \oplus \Lambda = \mathbf{R}^d$. It follows that for every $x \in \mathbf{Z}^d \setminus \{0\}$ the vector Bx has some non-zero integral coordinate. By Theorem 6 B must have an integral row, which, by Lemma 1, implies that some $e_i \in \Lambda$.

∎

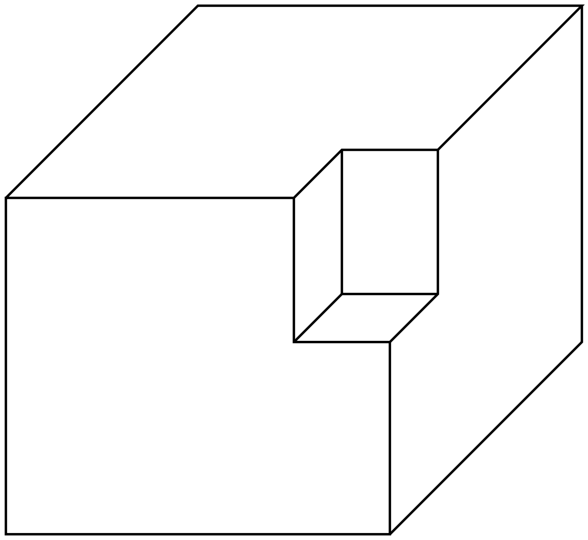

Figure 1: A notched cube in $\mathbf{R}^3$.

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§2. The notched and the extended cube

In this section we prove that some simple shapes (like those in Figures 1 and 2) admit lattice tilings. That the "extended cubes" (Fig. 2–see Theorem 8) admit lattice tilings has not been shown before.

2.1 The notched cube

We now consider the unit cube

$$Q = \left[-\frac{1}{2}, \frac{1}{2}\right]^d$$

from whose corner (say in the positive orthant) a rectangle R has been removed with sides-lengths $\delta_1, \ldots, \delta_d$ $(0 \le \delta_j \le 1)$. That is, we consider the "notched cube":

$$N = Q \setminus R$$

where

$$R = \prod_{j=1}^{d} \left[\frac{1}{2} - \delta_j, \frac{1}{2}\right].$$

It is shown in Figure 1.

We give a new proof of the following result of Stein [**St**].

Theorem 7 *The notched cube N admits a lattice tiling of $\mathbf{R}^d$.*

After a simple calculation we obtain

$$\widehat{\mathbf{1}_N}(\xi) = \prod_{j=1}^{d} \frac{\sin \pi\xi_j}{\pi\xi_j} - F(\xi) \prod_{j=1}^{d} \frac{\sin \pi\delta_j\xi_j}{\pi\xi_j}, \tag{11}$$

where $F(\xi) = \exp(\pi i K(\xi))$ with

$$K(\xi) = \sum_{j=1}^{d} (\delta_j - 1)\xi_j. \tag{12}$$

Using Theorem 2 it is enough to exhibit a lattice $\Lambda \subset \mathbf{R}^d$, of volume equal to

$$|N| = 1 - \delta_1 \cdots \delta_d,$$

such that $\widehat{\mathbf{1}_N}$ vanishes on $\Lambda^* \setminus \{0\}$.

2.2 Lattices in the zero-set

We define the lattice Λ^* as those points ξ for which

$$\begin{aligned} \xi_1 - \delta_2\xi_2 &= n_1, \\ \xi_2 - \delta_3\xi_3 &= n_2, \\ &\cdots \\ \xi_d - \delta_1\xi_1 &= n_d, \end{aligned} \tag{13}$$

for some $n_1, \ldots, n_d \in \mathbf{Z}$. That is, $\Lambda^* = A^{-1}\mathbf{Z}^d$, where

$$A = \begin{pmatrix} 1 & -\delta_2 & & & \\ & 1 & -\delta_3 & & \\ & & \ddots & & \\ & & & 1 & -\delta_d \\ -\delta_1 & & & & 1 \end{pmatrix}. \tag{14}$$

Therefore $\Lambda = A^\top \mathbf{Z}^d$ and the volume of Λ is equal to $|\det A|$. Expanding A along the first column we get easily that $\det A = 1 - \delta_1 \cdots \delta_d$, which is the required volume.

We now verify that $\widehat{\mathbf{1}_N}$ vanishes on $\Lambda^* \setminus \{0\}$.

Assume that $0 \neq \xi \in \Lambda^*$. Adding up the equations in (13) we get

$$K = K(\xi) = -(n_1 + \cdots + n_d).$$

If all the coordinates of ξ are non-zero we can write

$$\widehat{\mathbf{1}_N}(\xi) = \frac{1}{\pi^d \xi_1 \cdots \xi_d} \left(\prod_{j=1}^{d} \sin \pi\xi_j - (-1)^K \prod_{j=1}^{d} \sin \pi\delta_j\xi_j \right). \tag{15}$$

Observe from (13) that

$$\sin \pi\xi_j = (-1)^{n_j} \sin \pi\delta_{j+1}\xi_{j+1},$$

where the subscript arithmetic is done modulo d, from which we get $\widehat{\mathbf{1}_N}(\xi) = 0$, since the factors in the two terms of (15) match one by one.

It remains to show that $\widehat{\mathbf{1}_N}(\xi) = 0$ even when ξ has some coordinate equal to 0, say $\xi_1 = 0$.

Consider the numbers $\xi_1, \ldots, \xi_d$ arranged in a cycle and let

$$I = \{\xi_m, \xi_{m+1}, \ldots, \xi_1, \ldots, \xi_{k-1}, \xi_k\}$$

be an interval around ξ_1 which is maximal with the property that all its elements are 0. Then $\xi_{m-1} \neq 0$ and $\xi_{k+1} \neq 0$ and from (13) we get

$$\xi_{m-1} - \delta_m \xi_m = n_m \quad \text{and} \quad \xi_k - \delta_{k+1}\xi_{k+1} = n_k. \tag{16}$$

We deduce that n_m and n_k are both non-zero and therefore that ξ_{m-1} and $\delta_{k+1}\xi_{k+1}$ are both non-zero integers and $\sin \pi\xi_{m-1} = \sin \pi\delta_{k+1}\xi_{k+1} = 0$. This means that both terms in (11) vanish and so does $\widehat{\mathbf{1}_N}(\xi)$.

■

So we proved that for the lattice $\Lambda = A^\top \mathbf{Z}^d$, where A is defined in (14), we have $N \oplus \Lambda = \mathbf{R}^d$. Clearly, if σ is a cyclic permutation of $\{1, \ldots, d\}$ and if instead of the matrix A we have the matrix A' whose i-th row has 1 on the diagonal, $-\delta_{\sigma i}$ at column σi and 0 elsewhere, we get again a lattice tiling with the lattice $(A')^\top \mathbf{Z}^d$. Stein [**St**] as well as Schmerl [**Sch**] have shown that these $(d-1)!$ lattice tilings of the notched cube (one for each cyclic permutation of $\{1., \ldots, d\}$) are all non-isometric when the side-lengths δ_j are all distinct.

A deeper result of Schmerl [**Sch**] is that there are no other translational tilings of the notched cube, lattice or not. This is something that I cannot prove with the harmonic analysis approach.

2.3 Extended cubes

Let us now allow the parameters $\delta_1, \ldots, \delta_d$ to take on any non-zero real value subject only to the restriction

$$\delta_1 \cdots \delta_d \neq 1, \tag{17}$$

and let the function $\varphi(\xi)$ be equal to the right-hand side of (11). Let again the matrix A be defined by (14) and $\Lambda = A^\top \mathbf{Z}^d$ as before. We have again $\det A = 1 - \delta_1 \cdots \delta_d$.

The calculations we did in §2.2 show that φ vanishes on $\Lambda^* \setminus \{0\}$, hence, if $\check{\varphi}$ is the inverse FT of φ, $\check{\varphi}$ tiles $\mathbf{R}^d$ with Λ and weight

$$\frac{\varphi(0)}{|1 - \delta_1 \cdots \delta_d|} = \mathrm{sgn}(1 - \delta_1 \cdots \delta_d), \tag{18}$$

where $\mathrm{sgn}(x) = \pm 1$ is the sign of x.

The function $\check{\varphi}$ is given by

$$\check{\varphi}(x) = \mathbf{1}_Q(x) - \mathrm{sgn}(\delta_1 \cdots \delta_d)\psi(x), \tag{19}$$

where

$$\psi(x) = \mathbf{1}_Q\left(\frac{x_1 - (1-\delta_1)/2}{|\delta_1|}, \ldots, \frac{x_d - (1-\delta_d)/2}{|\delta_d|}\right). \tag{20}$$

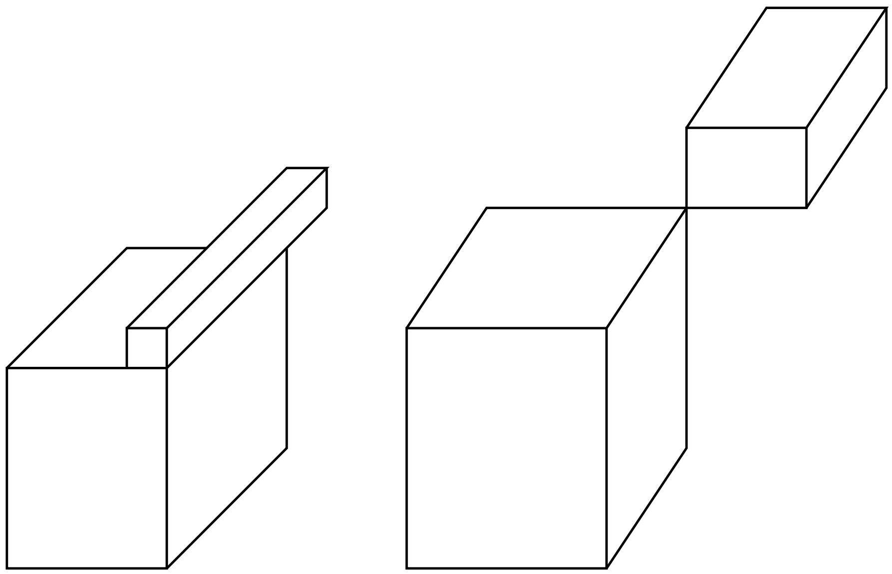

Figure 2: Some extended cubes in $\mathbf{R}^3$ that admit lattice tilings. The codimension of the intersection is 1 (left) and 3 (right).

Notice that $\psi(x)$ is the indicator function of a rectangle $R = R(\delta_1, \ldots, \delta_d)$ with side-lengths $|\delta_1|, \ldots, |\delta_d|$ centered at the point

$$P = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right) - \frac{1}{2}(\delta_1, \ldots, \delta_d). \tag{21}$$

The rectangle R intersects the interior of Q only in the case $\delta_1 > 0, \ldots, \delta_d > 0$ and when this happens $\check{\varphi}$ is an indicator function only if we also have $\delta_1 \le 1, \ldots, \delta_d \le 1$, which is the case of the notched cube that we examined in §2.2.

Otherwise (not all the δs are non-negative) $\check{\varphi}$ is an indicator function only when $\mathrm{sgn}(\delta_1 \cdots \delta_d) = -1$, i.e., the number of negative δs is odd. In this case we have that

$$\check{\varphi} = \mathbf{1}_{Q \cup R}$$

and from (18) we get that $Q \cup R$ tiles with Λ and weight 1. We can now prove the following.

Theorem 8 (Lattice tiling by extended cubes)
Let Q and R be two axis-aligned rectangles in $\mathbf{R}^d$ with sides of arbitrary length and disjoint interiors. Assume also that Q and R have a vertex K in common and intersection of odd codimension.

Then $Q \cup R$ admits a lattice tiling of $\mathbf{R}^d$ of weight 1.

For example, the extended cubes shown in Figure 2 admit lattice tilings of $\mathbf{R}^3$.
Proof. After a linear transformation we can assume that $Q = [-1/2, 1/2]^d$, that Q and R share the vertex $K = (1/2, \ldots, 1/2)$ and that $Q \cap R$ has codimension k (an

odd number) and

$$Q \cap R \subseteq \left\{ x \in Q : \ x_1 = \cdots = x_k = \frac{1}{2} \right\}.$$

Let the side-lengths of R be $\gamma_1, \ldots, \gamma_d > 0$. Define

$$\delta_j = \begin{cases} -\gamma_j, & \text{if } 1 \le j \le k, \\ \gamma_j, & \text{if } k+1 \le j \le d. \end{cases}$$

It follows that, with this assignment for the δ_j, the indicator function of R is equal to the function $-\mathrm{sgn}(\delta_1 \cdots \delta_d)\psi(x)$ of (19) and tiling follows from the previous discussion. ■

I believe that extended cubes with an intersection of even codimension do not tile, at least not for general side-lengths. This is clear in dimension two and it is conceivable that some combinatorial argument could easily show this in any dimension. The Harmonic Analysis approach does not seem to be very helpful when one tries to disprove that something is a (translational) tile.

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