Some Pairwise Balanced Designs

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Abstract

A pairwise balanced design, $B(K; v)$, is a block design on v points, with block sizes taken from K, and with every pair of points occurring in a unique block; for a fixed K, $B(K)$ is the set of all v for which a $B(K; v)$ exists.

A set, S, is a PBD-basis for the set, T, if $T = B(S)$. Let $N_{a(m)} = \{n : n \equiv a \bmod m\}$, and $N_{\geq m} = \{n : n \geq m\}$; with Q the corresponding restriction of N to prime powers.

This paper addresses the existence of three PBD-basis sets.

1. It is shown that $Q_{1(8)}$ is a basis for $N_{1(8)} \setminus E$, where E is a set of 5 definite and 117 possible exceptions.
2. We construct a 78 element basis for $N_{1(8)}$ with, at most, 64 inessential elements.
3. Bennett and Zhu have shown that $Q_{\geq 8}$ is a basis for $N_{\geq 8} \setminus E'$, where E' is a set of 43 definite and 606 possible exceptions. Their result is improved to 48 definite and 470 possible exceptions. (Constructions for 35 of these possible exceptions are known.)

Finally, we provide brief details of some improvements and corrections to the generating/exception sets published in *The CRC Handbook of Combinatorial Designs*.

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1 Introduction

A pairwise balanced design, $B(K; v)$, is a block design on v points, with block sizes taken from K, and with every pair of points occurring in a unique block; for a fixed K, $B(K)$ is the set of all v for which a $B(K; v)$ exists.

A set, S, is a PBD-basis for the set, T, if $T = B(S)$. Let $N_{a(m)} = \{n : n \equiv a \bmod m\}$, and $N_{\geq m} = \{n : n \geq m\}$; with Q the corresponding restriction of N to prime powers.

This paper addresses the existence of three PBD-basis sets. The opening sections deal with useful known results and more general constructions. In Section 5, we give constructions that are particularly useful for the first of our two problems, then in Section 6, we show that $Q_{1(8)}$ is a basis for $N_{1(8)} \setminus E$, where E is a set of 5 definite and 117 possible exceptions; an application of this result is given in [5]. In Section 7, we construct a 78 element basis for $N_{1(8)}$ with, at least, 14 essential elements. In Section 8, we look at what $Q_{\geq 8}$ is a basis for. Bennett and Zhu have already tackled the last problem, and give some applications [9, 13]; they have shown that $Q_{\geq 8}$ is a basis for $N_{\geq 8} \setminus E'$, where E' is a set of 43 definite and 606 possible exceptions. Their result is improved to 48 definite and 470 possible exceptions. Results (without proofs) from this paper were incorporated into [12, Tables III.3.18–19].

Finally, in Section 9, we provide brief details of some improvements and corrections to the generating/exception sets published in *The CRC Handbook of Combinatorial Designs*, specifically [12, Tables III.3.17–19].

Although we sometimes give some non-existence results (phrased as *definite* exceptions and *essential* elements above) we will not establish these here; they require techniques quite different from the ones we are using here. We will merely note that most of them can be established by some results of Wilson's [41]: the integrality conditions, and the bound on the size of a flat; these conditions are also given in [12, Lemma 3.2, Theorem 3.1.2].

Remark 1.1 Wilson's integrality conditions imply that if $S \subset N_{1(m)}$, and if a $B(S; v)$ exists, then it follows that $v \in N_{1(m)}$.

2 Notation

The notation we use is fairly straightforward. A K-GDD is a group divisible design with block sizes taken from K, and with a group size vector of $\{G_1, G_2, \ldots, G_n\}$; we will usually write the group size vector in exponential notation. We will only be concerned with the $\lambda = 1$ case; i.e., a pair of elements from different groups occurs in one block of the design, whilst a pair of elements from the same group appears in no block.

A pairwise balanced block design is a GDD with a group size vector of 1^v; it is denoted by $B(K; v)$. If the block size is uniform, with $K = \{k\}$ then this block design is referred to as a balanced incomplete block design (a BIBD), and denoted by $B(k; v)$.

If a $B(k; v)$ exists, then the quantity $r = (v - 1)/(k - 1)$ is called the replication number of the BIBD, and we say that $r \in RN(k)$.

A transversal design, $TD(k;t)$, is a $\{k\}$-GDD of group type t^k.

The set of all v for which a $B(K;v)$ exists is denoted by $B(K)$, and a similar definition is used for $TD(k)$.

The notation $B(K \cup \{k^*\};v)$ indicates there is exactly one block of size k in the design in addition to those of sizes from K (further blocks of size k can occur only if $k \in K$).

Resolvable designs are designs that admit a partition of the block set into subsets of blocks that contain every point exactly once. These designs are denoted by the prefix R.

If a $RB(k;v)$ exists, then we say that $r \in RRN(k)$, where $RRN(k) \subset RN(k)$.

Some notation that is specific to a section will be introduced in that section.

One final notational convention we have adopted should be mentioned. We have labelled several lemmas by a one or two letter code. This is summarized in the key in the appendix, which gives the symbolic parameters used in the lemma where the code is introduced, but numeric values are substituted in the table of constructions. This device allows considerable compression of the table, yet retains all the information needed to verify the constructions.

3 Direct Constructions

We start this section by summarizing some well-known results in the first lemma.

Lemma 3.1 *If q is a prime or prime power, then*

1. $q^2 \in RB(q)$, *and*
2. $q^2 + q + 1 \in B(q+1)$, *and*
3. $q^3 + 1 \in RB(q+1)$, *and*
4. $q^3 + q^2 + q + 1 \in RB(q+1)$.

Proof: All these are standard results. The first two amount to the existence of the affine and projective geometries, $AG(2,q)$ and $PG(2,q)$. The resolvability of the unital design was shown by Bose (see [15, 26]), and the resolvability of the last design can be found in Lorimer [33]. ∎

Lemma 3.2 *Embedded in $PG(2, 2^N)$ is a set of $v(n) = 2^n(2^n - 1)2^{N-n} + 2^n$ points whose incidences with the lines of the plane are all either 0 or 2^n. Furthermore, if $m < n$, the set of $v(n)$ points contains a subset of $v(m)$ points whose incidence with the lines of the plane is either 0 or 2^m.*

Proof: The first part of this result was shown by Denniston [19]. Denniston considers an irreducible second order curve, $Q(x,y) = ax^2 + bxy + cy^2$ over $GF(2^N)$ in the non-homogeneous coordinates (x,y) of the plane. Let G be any additive sub-group of order

2^n of the additive group $GF(2^N)$. Then the set of points satisfying $Q(x,y) \subset G$ forms a Denniston arc, $\{v(n); \{0, 2^n\}\}$, in $PG(2, 2^N)$ with $v(n) = 2^{N+n} - 2^N + 2^n$; note if we consider a sub-group H of order 2^m and take $H \subset G$, this yields a $\{v(m); \{0, 2^m\}\}$-arc whose points are a subset of the $v(n)$ points, so that Denniston arcs can be taken to be nested. ∎

Corollary 3.3 *If k is a power of 2, and n is a non-negative integer, then $k(k-1)2^n+k \in RB(k)$.*

Proof: The resolvability was demonstrated by Seiden in the case $n = 1$, (see [39]), and her proof carries over to Denniston's arcs. The device used is to consider a line that has zero incidence with the arc; the incidence of its points with the non-zero lines of the arc identifies the resolution classes. ∎

Corollary 3.4 *The following designs exist:*

1. $B(\{9, 17^*\}; 120 + 17 = 137 = 8 * 17 + 1)$;
2. $B(\{9, 33^*\}; 232 + 33 = 265 = 8 * 33 + 1)$;
3. $B(\{9, 65^*\}; 456 + 65 = 521 = 8 * 65 + 1)$;
4. $B(\{17, 33^*\}; 496 + 33 = 529 = 8 * 66 + 1)$;
5. $B(\{9, 17\}; 273 - 120 = 153 = 8 * 19 + 1)$;
6. $B(\{17, 33\}; 1057 - 496 = 561 = 8 * 70 + 1)$;
7. $B(\{9, 17, 33^*\}; 496 - 232 + 33 = 297 = 8 * 37 + 1)$.

Proof: The first four examples are formed by adding a complete external line to the appropriate Denniston arc in planes of order 16, 32, 64 and 32. The next two designs are formed by taking the complements of the Denniston arcs used in parts 1. and 4. above. The final result comes from taking the difference of the two arcs of parts 2. and 4., in the plane of order 32, again adding a complete external line. ∎

Corollary 3.5 *The following designs exist:*

1. $B(\{8, 9, 16, 17, 27^*\}; 291)$;
2. $B(\{8, 9, 16, 17, 31^*\}; 295)$;

Proof: Again we use the 264 point difference of two Denniston arcs that we used in Corollary 3.4.7, and now we add an incomplete external line. ▮

If we have a projective plane, $PG(2,q)$, then we may generate a $TD(q+1;q)$ from it by deleting a point, and using the lines through that point to generate groups. Since all the Desarguesian planes have ovals, we may consider how these oval points are distributed amongst the groups. Recall that if q is odd, then there are $q+1$ points in the oval; (if D is the Singer difference set, then $-D$ is such an oval). If q is even, with $D = 2D$, then we may augment this set, $-D$, of $q+1$ points by the point $\{0\}$ to get a set of $q+2$ hyperoval points, which we will also term an oval.

Lemma 3.6 *If $q = 2t+1$ is an odd prime power, then we may form a $TD(q+1;q)$ such that the distribution of the oval points amongst the groups is of type $0^1 1^q$ or $0^t 1^2 2^t$ or $0^{t+1} 2^{t+1}$. If $q = 2t$ is an even prime power, then we may form a $TD(q+1;q)$ such that the distribution of the oval points amongst the groups is of type 1^{q+1} or $0^t 2^{t+1}$.*

Proof: To demonstrate this in the odd case, we either delete an oval point, or a non-oval tangent point, or a non-tangent point, and note the number of external lines, tangents and secants that the deleted point lay on. For the plane of even order, we delete an oval point, or a non-oval point; there are no tangents here. ▮

Lemma 3.7 $\{81, 585\} \subset RB(9)$, *and* $\{73, 433, 577\} \subset B(9)$.

Proof: The first three values follow from Lemma 3.1 using $q = 8$ or $q = 9$. The values 433 and 577 result from constructions by Abel [1] and Buratti [17]. ▮

Lemma 3.8 *If $t \in \{1, 6, 7\}$ or t is a power of 2, then $56t + 8 \in RB(8)$.*

Proof: This follows from Lemma 3.1 using $q = 8$ or $q = 7$, or from Corollary 3.3 using $k = 8$. ▮

Lemma 3.9 *If q is a prime or prime power, and $q \equiv 1 \bmod 8$, and $q \notin \{25, 89\}$ and $q < 4096$, then $7q + 1 \in RB(8)$.*

Proof: The results for $q \in \{9, 17\}$ are given above. The remaining constructions were given by Greig [20]. It can be shown that the restriction $q < 4096$ is unneccessary, although we do not need that improvement here. ▮

Lemma 3.10 *The $\{9\}$-RGDD of type 3^{33} exists.*

Proof: This design was constructed by Mathon; see [28]. ▮

The remaining basic designs we need are transversals. We will briefly summarize some well known results.

Lemma 3.11 *If q is a prime or prime power, then $q \in TD(q+1)$.*

Lemma 3.12 *If $m \in TD(k+1)$, then $m \in TD(k)$.*

Lemma 3.13 *If $\{m,n\} \subset TD(k)$, then $mn \in TD(k)$.*

These last three lemmas can be combined to give a weaker version of MacNeish's result [34].

Theorem 3.14 *If m has no prime divisors less than k, then $m \in TD(k+1)$.*

Lemma 3.15 $TD(k+1) = RTD(k)$.

Lemma 3.16 *If $t \neq 3$, then $8t \in TD(9)$, and if $t \neq 4$, then $8t+1 \in TD(9)$.*

Proof: See [3]; for Wojtas' $TD(9;48)$ see [18]. ∎

Unless otherwise noted, all the TDs we need are from [3]. The remaining designs we need are incomplete transversal designs (ITDs). Loosely speaking, an ITD, written as $TD(k;m) - TD(k;a)$, is a design that could be completed to a $TD(k;m)$ by adding the blocks of a $TD(k;a)$ to the ITD; we do not actually need to have a $TD(k;a)$ to have an ITD. We have chosen to provide constructions for the ITDs we need, even though better values can sometimes be obtained from [4].

Lemma 3.17 *If $m \in TD(k)$, then the incomplete transversal $TD(k;m) - TD(k;1)$ exists.*

Proof: Delete one block of the TD to get the ITD. ∎

The other two constructions of ITDs that we use are corollaries of Wilson's basic construction [40].

Lemma 3.18 *If $m \in TD(10)$ and $0 \leq n \leq m$, and $\{k, k+1\} \subset TD(9)$, then the incomplete transversal $TD(9;km+n) - TD(9;n)$ exists.*

Lemma 3.19 *If $m \in TD(n)$ and $8 \leq n$, then the incomplete transversal $TD(9;8m+n-8) - TD(9;n)$ exists.*

4 General Constructions

Lemma 4.1 (Direct Product) *If $m \in TD(n)$, then $mn \in B(\{m,n\})$.*

Proof: We just fill in the groups of the TD. ∎

Theorem 4.2 *Suppose we have a K-GDD with v points and a group size vector of $\{|G_j| : j = 1, \dots, g\}$, and, for the first $g-1$ groups, we have a $B(K \cup \{w^*\}; |G_j| + w)$ with $0 \leq w$, and, for the last group we have a $B(K; |G_g| + w)$; then there exists a $B(K; v + w)$.*

Proof: This is a standard result; we add w infinite points, and when we fill the first $g-1$ groups, we align the w block with the infinite points, and refrain from using it for these groups. Note that the resulting design does not contain a w block, unless, possibly, when $w \in K$. ∎

Corollary 4.3 (Indirect Product) *If $(m - w) \in TD(n)$, and a $B(K; m)$ containing a $B(K; w)$ sub-design exists, (or a $B(K; m)$ exists with $w = 1$), then $n(m - w) + w \in B(K \cup \{n\})$.*

Theorem 4.4 (Wilson's Fundamental Construction) *Suppose there is a "master" $K - GDD$ with g groups and a group size vector of $\{|G_j| : j = 1, \dots, g\}$, and a weighting that assigns a positive weight of $w(x)$ to each point x. Let $W(B_i)$ be the weight vector of the i-th block. If, for every block B_i, we have a $K' - GDD$ with a group size vector of $W(B_i)$, then there exists a $K' - GDD$ with a group size vector of $\{\sum_{x \in G_j} w(x) : j = 1, \dots, g\}$.*

Proof: See [41]. Also note that the blocks in the final design have cardinalities in K', rather than K. ∎

Theorem 4.5 *If $r \in RN(k)$, then a design with block size of k and a group vector of $(k-1)^r$ exists.*

Proof: Deleting a point, and using its blocks as groups in a $B(k; (k-1)r + 1)$, gives the GDD; note that this construction can be reversed; (see [25]). ∎

5 Specific Constructions

In this section, we start applying the results of the previous section to produce the tools for the second of the problems, that of constructing designs whose block sizes are in $Q_{1(8)}$, where $Q_{1(8)}$ is the set of prime-powers congruent to 1 modulo 8. It will be convenient to define $U_{1(8)}$ by:

$$t \in U_{1(8)} \iff 8t + 1 \in B(Q_{1(8)}).$$

Lemma 5.1 *There exist $\{9\}$-GDDs of type 8^9 and type 8^{10}.*

Proof: Apply Theorem 4.5 to the 73 and 81 point designs of Lemma 3.7. ∎

Lemma 5.2 *If $q = 8t + 1 \in RRN(8)$, then $56t + 8 \in RB(8)$, and $64t + 9 \in B(\{9, q^*\})$, and there exists a $\{9\}$-GDD of type $8^{7t+1}(q-1)^1$, and there exists a $\{9, q^*\}$-GDD of type 8^q.*

Proof: Complete the RBIBD with $q = 8t + 1$ infinite points, and then delete either an infinite point, or a finite point, and use its lines to indicate the groups. ∎

Theorem 5.3 *If there exists a K-GDD on v points, with group sizes contained in M, and $K \subset RN(9) \cup \left(Q_{1(8)} \setminus \{25, 89\}\right)$, then $M \subset U_{1(8)}$ implies $v \in U_{1(8)}$.*

Proof: Give each point a weight of 8, and apply Wilson's fundamental construction. The needed components come from Theorem 4.5 for $RN(9)$, and Lemma 3.9 via Lemma 5.2 for the other block sizes. We fill in the groups with a point at infinity to obtain the required result. ∎

Corollary 5.4 *If there exists a $\{9, 10\}$-GDD on v points, with group sizes chosen from M, and $M \subset U_{1(8)}$, then $v \in U_{1(8)}$.*

Lemma 5.5 (Code T) *If $m \in TD(10)$ and $0 \leq n \leq m$ and $\{m, n\} \subset U_{1(8)}$, then $9m + n \in U_{1(8)}$.*

Proof: Truncate one group of the transversal to size n, then use Corollary 5.4. ∎

Concentrating on the points in the last block yields the "last spike" and "block deletion" constructions given below.

Lemma 5.6 (Code Ls) *If $m \in TD(n)$ and $m \in U_{1(8)}$, and $n \geq 9$ and $n \in Q_{1(8)} \setminus \{25, 89\}$, then $9(m-1) + n \in U_{1(8)}$.*

Proof: Remove $m-1$ points from $n-9$ groups of the transversal, retaining all the points of the last block, to give a $\{9, 10, n^*\}$-GDD of type $m^9 1^{n-9}$. Now use Theorem 5.3. ∎

Lemma 5.7 (Code L) *If $m \in TD(n)$ and $m \in Q_{1(8)} \setminus \{25, 89\}$, and $n \geq 9$ and $n \in U_{1(8)}$, then $9(m-1) + n \in U_{1(8)}$.*

Proof: As in Lemma 5.7, we construct a $\{9, 10, n^*\}$-GDD of type $m^9 1^{n-9}$. Now fill in the groups and delete the big block to give a $\{9, 10, m\}$-GDD of type $1^{9m-9} n^1$, and use Theorem 5.3. ∎

Lemma 5.8 (Code BD) *If $m \in TD(10)$ and $9 \leq n \leq 10$ and $\{m, m-1\} \subset U_{1(8)}$, then $10m - n \in U_{1(8)}$.*

Proof: Remove n points from one block of the transversal, then use Corollary 5.4. ∎

Remark 5.9 Note that if $n = 10$, then $m \in U_{1(8)}$ is not needed.

Lemma 5.10 (Code R) *If $v \in RB(9)$ and $8n+1 \leq v$ and $n \in U_{1(8)}$, then $v+n \in U_{1(8)}$.*

Proof: Add n new points to the blocks of n resolution sets, then use Corollary 5.4. ∎

Lemma 5.11 *If $m \in TD(k+1)$, then:*

1. *there exists a $\{k+1, m+1\}$-GDD of type $k^m m^1$;*
2. *there exists a $\{k+1, m\}$-GDD of type $k^m (m-1)^1$.*

Proof: Take the transversal, and fill in groups with 1 or 0 points at infinity and then delete a finite point and use its blocks as groups to give the result. ∎

Lemma 5.12 *If k is a power of 2, then there exists a $\{k+1, 2k+1\}$-GDD on $k(2k+3)$ points, and this design has a group vector of $k^{2k-1}(2k)^2$.*

Proof: In some ways, this lemma is a corollary of Lemma 3.2. The Seiden design can be embedded in $PG(2, 2k)$. We now consider the non-Seiden points in $PG(2, 2k)$, and delete one of these and use its blocks to define the groups. ∎

Lemma 5.13 *There exist $\{9, 17\}$-GDDs with group vectors of $8^{15}16^1$, 8^{17}, $8^{16}16^1$, $8^{17}16^1$ and $8^{15}16^2$.*

Proof: This follows from Lemma 5.2 with $t = 2$ (twice), from Lemma 5.11 with $m = 16$ and $m = 17$, and from Lemma 5.12. ∎

Mullin et al. [38, Lemma 6.12], used a construction similar to Lemma 5.14. We could also adapt their Lemma 6.20, but this yields nothing new here, as we have a rich set of constructions in the next six lemmas, covering essentially the same ranges. Five of these constructions of these lemmas were used previously by Greig and Abel [22]; Lemma 5.16 (code B) is new.

Lemma 5.14 (Code A) *If $m \in TD(18)$ and $0 \leq s \leq t \leq m$ and $n = s + t$, and $\{m, 2m, s, t\} \subset U_{1(8)}$, then $17m + n \in U_{1(8)}$.*

Proof: Truncate two groups to sizes s and t, and give the points of these truncated groups a weight of 8. Give the points of one other group weight 16 and give all the other points weight 8. The needed component group vectors are $8^{15}16^1$, $8^{16}16^1$, and $8^{17}16^1$, and are obtained from Lemma 5.13, so we may apply Wilson's fundamental construction to give a design with a group vector of $(8m)^{15}(16m)^1(8s)^1(8t)^1$. We fill these groups in with the aid of a point at infinity. ∎

Remark 5.15 Note that, provided we took $s = 0$, we could relax the condition $m \in TD(18)$ to $m \in TD(17)$.

Lemma 5.16 (Code B) *If m is a prime power with $m \geq 17$, $0 \leq t \leq 17$, $n = 2t \leq m + 1$, and $\{m, m+2\} \subset U_{1(8)}$, then $17m + n \in U_{1(8)}$.*

Proof: Using Lemma 3.6, we may construct a $TD(17; m)$ with at least t groups containing a pair of oval points; give these t pairs of points a weight of 16, and give all the other points a weight of 8. The needed component group vectors are 8^{17}, $8^{16}16^1$, and $8^{15}16^2$, and are obtained from Lemma 5.13, so we may apply Wilson's fundamental construction to give a design with a group vector of $(8m)^{17-t}(8(m+2))^t$. We fill these groups in with the aid of a point at infinity. ∎

Lemma 5.17 (Code C) *If $m \in TD(17)$ and $0 \leq s \leq t \leq m$ and $n = 2m + s + t$, and $\{m, m+s, m+t\} \subset U_{1(8)}$, then $15m + n \in U_{1(8)}$.*

Proof: Give s points from one group a weight of 16, give t points from another group a weight of 16, and give all the other points a weight of 8. The needed component group vectors are 8^{17}, $8^{16}16^1$, and $8^{15}16^2$, and are obtained from Lemma 5.13, so we may apply Wilson's fundamental construction to give a design with a group vector of $(8m)^{15}(8m+8s)^1(8m+8t)^1$. We fill these groups in with the aid of a point at infinity. ∎

Lemma 5.18 (Code D) *If $m \in TD(17)$, $0 \leq s+t \leq m$, $n = s+2t$, and $\{m, 2m, n\} \subset U_{1(8)}$, then $17m + n \in U_{1(8)}$.*

Proof: Give s points from one group a weight of 8, give t points from the same group a weight of 16, give the points of another group a weight of 16, and give the points of all the other 15 groups a weight of 8. The needed component group vectors are $8^{15}16^1$, $8^{16}16^1$, and $8^{15}16^2$, and are obtained from Lemma 5.13, so we may apply Wilson's fundamental construction to give a design with a group vector of $(8m)^{15}(16m)^1(8s+16t)^1$. We fill these groups in with the aid of a point at infinity. ∎

Lemma 5.19 (Code E) *If $m \in TD(17)$ and $1 \leq s \leq m$ and $1 \leq t \leq m$ and $15(s-1)+t \leq m$ and $n = m+t$, and $\{m+1, m+s, m+t\} \subset U_{1(8)}$, then $16(m+1)+(s-1)+n \in U_{1(8)}$.*

Proof: Give the points from the last block a weight of 16, give $s-1$ other points from the first group a weight of 16, give $t-1$ other points from the second group a weight of 16, and give all the other points a weight of 8. There are $15(s-1)$ lines containing a pair of points with weight 16, one from group 1 and the other from groups 3 through 17; when assigning the extra $t-1$ weight 16 points in the second group, we must avoid these lines; this is possible by hypothesis. The needed component group vectors are 8^{17}, $8^{16}16^1$, $8^{15}16^2$, and 16^{17}, and are obtained from Lemma 5.13, or from Lemma 3.8 with $q = 16$, so we may apply Wilson's fundamental construction to give a design with a group vector of $(8m+8)^{15}(8m+8s)^1(8m+8t)^1$. We fill these groups in with the aid of a point at infinity. ∎

Remark 5.20 In all instances, we will take $s = 1$.

Lemma 5.21 (Code F) *If* $m \in TD(18)$*,* $0 \leq n < m$*, and* $\{m-1, 2m, n\} \subset U_{1(8)}$*, then* $18m - 16 + n \in U_{1(8)}$.

Proof: Give the points from the last block a weight of 0, except for the first group, where all points get a weight of 16; also give $m - n - 1$ other points from the last group a weight of 0; and give all the other points a weight of 8. The needed component group vectors are $8^{15}16^1$, $8^{16}16^1$, and $8^{17}16^1$, and are obtained from Lemma 5.13, so we may apply Wilson's fundamental construction to give a design with a group vector of $(8m-8)^{16}(16m)^1(8n)^1$. We fill these groups in with the aid of a point at infinity. ∎

Remark 5.22 In all instances of the preceeding six lemmas, rather than filling the groups with the aid of a single infinite point, we could use $w > 1$ infinite points, as allowed by Theorem 4.2; we shall have occassion to do this.

The *Singular Indirect Product* construction was introduced in [36]. The following is a particular instance.

Lemma 5.23 (Code Ir) *If* $56m + 8 \in RB(8)$*, and* $t \leq 8m + 1$*, and* $m + t = n \in U_{1(8)}$*, and the incomplete transversal* $TD(9; 56m + 8 + t) - TD(9; t)$ *exists, then* $63m + 9 + n \in U_{1(8)}$.

Proof: Since $56m + 8 \in RB(8)$, we may adjoin $8m + 1$ points to the resolution sets to produce a $B(\{9, 8m + 1^*\})$, We use this design to fill in the groups of the incomplete transversal design with aid of $8m + 1 - t$ points at infinity, with the points of the design labelled so that the single filling block of size $8m + 1$, (which is omitted), coincides with the missing subgroup and the infinite points. Filling in the infinite points and the missing subgroups with a single block of size $8(m + t) + 1 = 8n + 1$ completes the construction. ∎

Remark 5.24 For application in Appendix Table A.1, we will obtain the ITDs we need from Lemma 3.19, using $m = 37$, or $m = 107$.

6 The Spectrum

The objective of this section is to establish Theorem 6.1.

Theorem 6.1 *If* m *is not one of the values listed in Table 6.1, then* $m \in U_{1(8)}$*, and* $8m + 1 \in B(Q_{1(8)})$.

Table 6.1.

Values of m for which $m \in U_{1(8)}$ is unknown.

4	7	8	13	16	20	22	23	25	26
27	31	33	37	38	40	43	47	48	52
58	59	60	61	62	63	67	68	69	70
76	79	85	88	89	92	93	94	98	103
106	112	114	115	118	123	124	125	130	132
133	134	139	140	142	143	148	151	157	160
166	175	178	184	187	191	192	195	196	197
199	202	203	204	205	208	209	211	213	214
218	220	221	223	224	226	227	229	232	238
247	250	256	259	265	268	364	367	368	373
376	382	385	391	400	424	427	438	463	607
623	628	637	643	646	670	823	832	1024	1039
1048	1084								

We give, in appendix Table A.1, our constructions for values up to 288 together with further constructions covering portions of the range up to 1648. We begin this section by concentrating on the first 288 values, and in exhibiting subdesigns for this range.

Lemma 6.2 (Code RB) *If $m \in E$, then $8m+1 \in B(Q_{1(8)})$ with E given by: $E = \{9, 17, 41, 49, 97, 113, 121, 169, 193, 233, 241\} \cup \{361, 409, 1033\}$.*

Proof: This lemma is essentially a corollary of Lemma 5.2. ∎

Lemma 6.3 (Code ZZ) *If $m \in \{111, 198, 805, 814, 826\}$, then $m \in U_{1(8)}$.*

Proof: For 111; take Mathon's design of Lemma 3.10, and add 12 infinite points, to give a $\{10\}$-GDD of type $3^{33}12^1$; apply Lemma 5.3. For 198; take Mathon's design of Lemma 3.10, and give each point a weight of 16, and apply Wilson's Fundamental Construction of Theorem 4.4, using a $TD(9;16)$. For 805; take a $TD(10;81)$ and truncate one block to size 5; note that the short block only contains points from the full groups; fill the full groups, and remove the short block to give a $\{9,10,81\}$-GDD with group sizes from $\{1,80,5^*\}$, and thence a $\{9,10\}$-GDD with group sizes from $\{1,80,5^*\}$; the needed designs for Wilson's FC come from Lemmas 5.1. For $\{814, 826\}$; truncate one group of a $TD(18;23)$ to size 16 or 22, and give each point a weight of 16, and apply Wilson's FC; we may obtain $\{17\}$-GDDs of types 16^{17} and 16^{18} for this application by using Theorem 4.5 with $k = 17$ on Lemma 3.1 with $q = 16$ or 17. ∎

We next summarize some of the smaller designs constructions.

Table 6.2.

m	$8m+1$			Authority	Remarks
9	73	$=$	$PG(2,8)$	Lemma 3.1	
10	81	$=$	$AG(2,9)$	Lemma 3.1	
17	137	$=$	$17+RB(8;120)$	Lemma 6.2	
18	145	$=$	$9*(17-1)+1$	Lemma 4.3	
19	153	$=$	$9*17$	Lemma 4.1	
28	225	$=$	$9*25$	Lemma 4.1	
34	273	$=$	$PG(2,16)$	Lemma 3.1	
36	289	$=$	$AG(2,17)$	Lemma 3.1	
45	361	$=$	$9*(41-1)+1$	Lemma 4.3	[3] for $TD(9;40)$
46	369	$=$	$9*41$	Lemma 4.1	
53	425	$=$	$17*25$	Lemma 4.1	
54	433	$=$	$B(9;433)$	Lemma 3.7	
55	441	$=$	$9*49$	Lemma 4.1	
64	513	$=$	$RB(9;513)$	Lemma 3.1	
73	585	$=$	$9*(73-9)+9$	Lemma 4.3	
74	593	$=$	$9*(73-9)+17$	Lemma 5.23	ITD(9;65)-(9;1)
78	625	$=$	$AG(2,25)$	Lemma 3.1	
91	729	$=$	$9*81$	Lemma 4.1	
127	1017	$=$	$9*113$	Lemma 4.1	
128	1025	$=$	$25*41$	Lemma 4.1	
135	1081	$=$	$9*(121-1)+1$	Lemma 4.3	
136	1089	$=$	$9*121$	Lemma 4.1	
137	1097	$=$	$9*(137-17)+17$	Lemma 4.3	
138	1105	$=$	$9*(137-17)+25$	Lemma 5.23	ITD(9;121)-(9;1)
145	1161	$=$	$9*(137-9)+9$	Lemma 4.3	
146	1169	$=$	$9*(145-17)+17$	Lemma 4.3	
201	1609	$=$	$25*(73-9)+9$	Lemma 4.3	
206	1649	$=$	$17*97$	Lemma 4.1	
210	1681	$=$	$AG(2,41)$	Lemma 3.1	
216	1729	$=$	$9*(193-1)+1$	Lemma 4.3	
217	1737	$=$	$9*193$	Lemma 4.1	
228	1825	$=$	$25*73$	Lemma 4.1	
240	1921	$=$	$17*113$	Lemma 4.1	
251	2009	$=$	$41*49$	Lemma 4.1	

We next turn to the range 289–3439; constructions A and C, as detailed below in Tables 6.3 and 6.4 and Remark 6.4, cover large portions of this range. In both Tables 6.3 and 6.4, we get valid constructions for the range $17m$ through $19m$ with the exception of the noted failures.

Table 6.3.

m	Invalid n	$17m$	$19m$	Failures
17	33	289	323	322
32	55,63	544	608	599,607
41	79,81	697	779	776,778
64	116,122–127	1088	1216	1204,1210–1215
73	133–4,140–1	1241	1387	1374–5,1381–2
81		1377	1539	
97	189	1649	1843	1838
121		2057	2299	
127	250,251	2159	2413	2409,2410
137	269	2329	2603	2598
149	297	2533	2831	2830
163		2771	3097	
167	333	2839	3173	3172
181		3077	3439	

Remark 6.4 Using the overlapping ranges, or applying construction C of Lemma 5.17 with the same value of m to the exceptions in Table 6.3, we are able to remove all of them except 322, 607 and 1211.

Table 6.4 presents some cases of construction C. Again, apart from the noted failures, the inclusive range $17m$ through $19m$ has a valid construction available.

Table 6.4.

m	Invalid n	$15m$	$17m$	$19m$	Failures
19	39,41,44,46, 73–76	285	323	361	324,326,329,331, 358–361
29	115,116	435	493	551	550,551
49	189,195,196	735	833	931	924,930,931
53	115,116,211,212	795	901	1007	910,911,1006,1007
71	280,281,283,284	1065	1207	1349	None thru 1344
83	331,332	1245	1411	1577	1576,1577
101	392,397,403,404	1515	1717	1919	None thru 1911
107	409,415,421,423, etc.	1605	1819	2033	None thru 2013
113	227–8,231,443, etc.	1695	1921	2147	None in 1927–2137

Lemma 6.5 *If $289 \le t \le 3439$, and $t \notin E$, then $t \in U_{1(8)}$, where E is given by:*

$$\{322, 324, 326, 329, 331\} \cup N_{358}^{492} \cup \{607\} \cup N_{609}^{696} \cup N_{780}^{832} \cup N_{1006}^{1087} \cup N_{1576}^{1648}.$$

Proof: This follows by the application of constructions A and C as detailed in Tables 6.3 and 6.4, noting Remark 6.4. ∎

Most of the exception set E of Lemma 6.5 can be dealt with by applying our constructions exactly as given. However, for some constructions we will use Theorem 4.2 with more than one infinite point.

Lemma 6.6 *A* $B(Q_{1(8)} \cup \{8w+1^*\}; 8(m+w)+1)$ *exists when* (m,w) *is one of the following pairs:*

$$\begin{array}{ccccccc} (8,1) & (16,1) & (16,2) & (17,1) & (25,3) & (27,1) & (32,2) \\ (43,6) & (50,3) & (54,1) & (81,6) & (83,1) & (89,1) & \end{array}$$

Proof: For $m = 43$, this follows from Lemma 5.2; for other $m < 81$, this follows from Table 6.2; for $m \geq 81$, this follows from Lemma 5.5, truncating a $TD(10,9)$. ∎

We use these subdesigns in the constructions given in Table 6.5.

Table 6.5.

Code	m	w	Needed subdesigns	Result	n range	Successes
T	16	1	(16,1)	$144+n$	$1 \leq n \leq 17$	Several
T	25	3	(25,3)	$225+n$	$3 \leq n \leq 28$	Several
T	27	1	(27,1)	$243+n$	$1 \leq n \leq 28$	Several
A	16	2	(16,2) (32,2) (0,2)	$272+n$	$2 \leq n \leq 18$	274,277, 281,283,286
T	32	2	(32,2)	$288+n$	$2 \leq n \leq 34$	322
A	17	1	(17,1)	$289+n$	$n = 2m+w = 35$	324
T	43	6	(43,6)	$387+n$	$6 \leq n \leq 49$	Several
D	25	3	(25,3) (50,3)	$425+n$	$3 \leq n \leq 53$	454,457
A	27	1	(27,1) (54,1) (8,1)	$467+n$	$1 \leq n \leq 28$	472
C	43	6	(43,6) (81,6)	$726+n$	$49 \leq n \leq 92$	808,817
T	83	1	(83,1)	$747+n$	$1 \leq n \leq 84$	831
C	89	1	(89,1)	$1424+n$	$90 \leq n \leq 189$	1582

Lemma 6.7 *If* $t \leq 3439$, *and* $t \notin T$, *then* $t \in U_{1(8)}$, *where* T *is the list of unconstructed cases given in Table 6.1.*

Proof: Appendix Table A.1, in conjunction with Table 6.2, covers the values through 288 and most of the exception set E. ∎

To deal with the values exceeding 3439, we need three preliminary lemmas.

Lemma 6.8 *Any sequence of 10 consecutive integers contains at least one value from* $TD(10)$.

Proof: Consider only the 5 odd numbers of the sequence, and note that at most two are divisible by 3, and at most one by 5 or 7. Hence there is (at least) one value with no factors less than 11; the result then follows by Theorem 3.14. ∎

Lemma 6.9 *Let $E' = \{367, 373, 391, 463, 607, 643, 823, 1039\}$. If $m \notin U_{1(8)}$, and $288 \leq m \leq 3439$ and $m \notin E'$, then m is divisible by 2 or 7.*

Proof: This is simply a matter of checking Table 6.1. ∎

Lemma 6.10 *$G \subset U_{1(8)} \cap TD(10)$, where $G = \{331, 337, 343, 349, 353, 359, 361, 369, 379, 389, 397, 461, 467, 601, 611, 641, 647, 821, 827, 1033, 1040\}$.*

Proof: For $G \subset U_{1(8)}$, we check Table 6.1. For $G \subset TD(10)$, we note that all the numbers given are prime powers, except $369 = 9 * 41$, $611 = 13 * 47$, and $1040 = 16 * 64 + 16$. ∎

Lemma 6.11 *If $m > 3439$, then $m \in U_{1(8)}$.*

Proof: Note that $U_{1(8)}$ contains an unbroken sequence of values from 269 through 363; the range 269 through 331 contains $63 > 6 * 9$ values. We may use the truncated transversal construction of Lemma 5.5, with $331 \leq m \leq 361$ to construct the range $3248 = 9 * 331 + 269$ through 3610, with m being chosen from the set G of Lemma 6.10; note that the m values used do not differ by more than 6. Next, note the range 269 through 361 contains $93 > 10 * 9$ values. Again we use the truncated transversal construction of Lemma 5.5. By Lemma 6.8, we are assured that we may find a sequence of $m \in TD(10)$ close enough together that the ranges of the successive constructions overlap. The only problem is that some of the needed m may not be in $U_{1(8)}$. The set E' of Lemma 6.9 gives the only possible problem values in the values that Lemma 6.8 is providing. These problem values can be overcome by use of values from the set G of Lemma 6.10; the remaining values of G (i.e., those exceeding 361), were all chosen to straddle values in E', with the straddling values differing by at most 10. ∎

Theorem 6.12 *If v is not in Table 6.6, and $v \equiv 1 \bmod 8$, then $v \in B(Q_{1(8)})$.*

Proof: This is really a restatement of Theorem 6.1. ∎

Table 6.6.

Values of v for which $v \in B(Q_{1(8)})$ is unknown.

33	57	65	105	129	161	177	185	201	209
217	249	265	297	305	321	345	377	385	417
465	473	481	489	497	505	537	545	553	561
609	633	681	705	713	737	745	753	785	825
849	897	913	921	945	985	993	1001	1041	1057

1065	1073	1113	1121	1137	1145	1185	1209	1257	1281
1329	1401	1425	1473	1497	1529	1537	1561	1569	1577
1593	1617	1625	1633	1641	1665	1673	1689	1705	1713
1745	1761	1769	1785	1793	1809	1817	1833	1857	1905
1977	2001	2049	2073	2121	2145	2913	2937	2945	2985
3009	3057	3081	3129	3201	3393	3417	3505	3705	4857
4985	5025	5097	5145	5169	5361	6585	6657	8193	8313
8385	8673								

7 The Basis for 1 mod 8

Since we already have the machinery and most of the results already in place, we will take the opportunity to treat the PBD basis for the 1 mod 8 case. Let $N_{1(m)} = \{v : v \equiv 1 \bmod m\}$; we wish to find a set $H_{1(m)}$ containing as few elements as possible, such that $N_{1(m)} = B(H_{1(m)})$. This problem was considered by Wilson for $2 \leq m \leq 4$, and subsequently by Mullin for $5 \leq m \leq 7$, and by Greig [41, 35, 21].

Lemma 7.1 *Let a be a positive integer, and suppose that there exists a positive integer u such that $u \equiv 1 \bmod a$, and that there exists both a $TD(a+1; u-1)$ and a $TD(a+1; u)$. If there exists a $TD(u+1; m)$, then $m(u-1)(a+1) + at + 1$ is inessential in $H_{1(a)}$ for $0 \leq t \leq m$.*

Proof: See [35, Lemma 2.2]; we truncate one group of the $TD(u+1; m)$, and apply Wilson's FC, with group types found from the first two TDs, and then fill with an infinite point. ∎

Let $P_k = \{q : q \leq k, q \text{ prime}\}$; we define $U(P_k)$ to be the smallest integer δ such that, for any positive n, we can always find some s satisfying $n \leq s < n + \delta$ with the property that s is relatively prime to every element of P_k. The main consequence of this, via MacNeish [34], is the following lemma.

Lemma 7.2 *Let k be a positive integer. Then given any positive integer n, there exists an integer s such that $n \leq s < n + U(P_k)$ and there exists a $TD(k+2; s)$.*

Proof: See [35, Lemma 2.1]. ∎

Let $V(a, b) = \{v : a + 1 \leq v \leq b, \quad v \equiv 1 \bmod a\}$.

Lemma 7.3 *Let a be a positive integer, and let u be an integer such that $u \equiv 1 \bmod a$ and there exists a $TD(a+1; u-1)$ and a $TD(a+1; u)$. Let $\delta = U(P_k)$ and let w be an integer such that there exists a $TD(u+1; w)$ and $w \geq \delta(u-1)(a+1)/a - 1$. Then the set $V(a, w(u-1)(a+1) - a + 1)$ is a finite basis for $N_{1(a)}$.*

Proof: See [35, Theorem 2.3]; we give a slightly improved version, with the improvement resulting from the observation that consecutive intervals need only abut each other, not overlap; the hypothesized bound on w is decreased by unity. ∎

Corollary 7.4 *$V(8, 6401)$ is a basis for $N_{1(8)}$.*

Proof: Take $a = 8$, $u = 9$, $10 = \delta = U(P_9) = U(P_7)$, and $w = 89$. It follows from Lemma 6.8 that we can take $\delta = 10$. ∎

Actually, since the prime powers below 89 are within 6 of each other, there is more overlap of consecutive intervals than is provided for in Lemma 7.3, with its allowed gap of 10, so the result of Corollary 7.4 can be improved on considerably.

Lemma 7.5 *Let $S = N_1^{80} \cup N_{91}^{98} \cup N_{111}^{116} \cup N_{131}^{143} \cup N_{191}^{206}$. If $m \notin S \cup N_{321}^{332}$, then $8m + 1$ is inessential in $H_{1(8)}$.*

Proof: In Lemma 7.1, take $a = 8$, $u = 9$, and let m range through the prime powers from 9 through 83. ∎

Lemma 7.6 *If $m \in N_{321}^{332}$, then $8m + 1$ is inessential in $H_{1(8)}$.*

Proof: In Lemma 5.14, take $m = 17$ or $m = 19$. ∎

Lemma 7.7 *If $E = \{9, 17, 41, 49, 97, 113, 193\} \cup \{33, 57, 65\}$, then $8q + 1$, with $q \in E$, is inessential in $H_{1(8)}$.*

Proof: This lemma is essentially a corollary of Lemma 5.2, and Lemma 3.8. ∎

Lemma 7.8 *We have $125 \in B(\{5^*, 9, 11, 13\})$, and also the incomplete transversal design $TD(9; 125) - TD(9; 5)$ exists.*

Proof: The PBD was constructed in [21] from an $RTD(10; 13)$ in $PG(2, 13)$ and parallels the construction of the 158 point PBD in Lemma 8.2. The ITD result follows from this PBD. ∎

Table 7.1.

m	$8m+1$		Authority	Remarks
63	505	$= 9 * (57 - 1) + 1$	Lemma 4.3	$K = \{9, 57\}$
68	545	$= 17 * (33 - 1) + 1$	Lemma 4.3	$K = \{17, 33\}$
72	577	$= 9 * (65 - 1) + 1$	Lemma 4.3	$K = \{9, 65\}$
132	1057	$= PG(2, 32)$	Lemma 3.1	$K = \{33\}$
142	1137	$= 9 * (137 - 17) + 57$	Lemma 5.23	$K = \{9, 57\}$
			Lemma 7.8	ITD(9;125)-(9;5)
199	1593	$= 9 * 177$	Lemma 4.1	$K = \{9, 177\}$
200	1601	$= 25 * (65 - 1) + 1$	Lemma 4.3	$K = \{25, 65\}$
202	1617	$= 33 * 49$	Lemma 4.1	$K = \{33, 49\}$

Theorem 7.9 *If m is not listed in Table 7.2, then $8m+1$ is inessential in $H_{1(8)}$.*

Proof: We only have to deal with elements of S from Lemma 7.5. We use Corollary 3.4 to remove $m \in \{19, 37, 66, 70\}$; we use Lemma 7.7 to remove $m \in \{9, 17, 33, 41, 49, 57, 65, 97, 113, 193\}$; we use Lemma 3.7 to remove $m \in \{10, 54, 73\}$; we use Lemma 6.3 to remove $m \in \{111, 198\}$; the remaining constructions are to be found in Table 6.2, or Table 7.1. ∎

Table 7.2.

Values of m for which $8m+1 \in B(H_{1(8)} \setminus \{8m+1\})$ is unknown.

1	2	3	4	5	6	7	8	11	12
13	14	15	16	20	21	22	23	24	25
26	27	29	30	31	32	35	38	39	40
42	43	44	47	48	50	51	52	56	58
59	60	61	62	67	69	71	75	76	77
79	80	92	93	94	95	96	98	112	114
115	116	131	133	134	139	140	141	143	191
192	194	195	196	197	203	204	205		

Theorem 7.10 *If v is not listed in Table 7.3, then v is inessential in $H_{1(8)}$.*

Proof: This is just another version of Theorem 7.9. The first 14 values are essential. ∎

Table 7.3.

Values of v for which $v \in B(H_{1(8)} \setminus \{v\})$ is unknown.

9	17	25	33	41	49	57	65	89	97
105	113	121	129	161	169	177	185	193	201
209	217	233	241	249	257	281	305	313	321
337	345	353	377	385	401	409	417	449	465
473	481	489	497	537	553	569	601	609	617
633	641	737	745	753	761	769	785	897	913
921	929	1049	1065	1073	1113	1121	1129	1145	1529
1537	1553	1561	1569	1577	1625	1633	1641		

8 Prime Powers Exceeding Seven

The main aim of this section is to improve the spectrum of $B(Q_{\geq 8})$, where $Q_{\geq 8}$ is the set of all prime powers exceeding 7. This problem was introduced by Bennett [9] and further discussed in [13]. Direct constructions for the values 123, 156, 158, 254, 316, and 1262 are given in [21], and for 291 and 295 were given in Corollary 3.5. The 123 point design takes the 120 point Denniston arc used in Corollary 3.4.1 and adds three extra points on an 8-line. The 158 point design uses the $q - x$ construction from [21] (with $q = 16$ and $x = 6$): start with $PG(2, q)$, and remove a point and $x+1$ lines through it to produce a $RTD(q - x; q)$ and then take a $(q - x)$-line (in the plane) and flip the status of its $q+1$ points, now deleting $q - x$ and restoring $x + 1$, and finally add a non-planar point to the $(x + 1)$-line and the $q - x$ $(q - 1)$-groups. The remaining constructions are based on Brouwer [16], who considers the cyclic difference set for $PG(2, q)$, and looks at the points belonging to a residue set modulo w, for w a divisor of $q^2 + q + 1$. Here however, we take more than one residue set; we take twelve modulo 21 sets in $PG(2, 16)$, two modulo 3 sets in $PG(2, 19)$, four modulo 7 sets in $PG(2, 23)$, or two modulo 3 sets in $PG(2, 43)$ to get our designs; if $w = 3$, it doesn't matter which sets we pick; in the other two cases we used the eleven used sets plus 0, and the four least used sets, considering the use in the difference set given by Baumert [8].

Theorem 8.1 $\{123, 291, 295, 316, 574, 814, 1262\} \subset B(Q_{\geq 8})$.

Proof: See [21] or Corollary 3.4, or apply the $q-x$ construction with $q = 32$ and $x = 14$ or 6. ∎

We high-light two particularly useful values:

Lemma 8.2 $156 \in B(\{8, 9, 11, 17\})$ *and* $158 \in B(\{8^*, 9, 11, 16\})$.

Corollary 8.3 $158 \in TD(9)$.

Lemma 8.4

1. *If $TD(k; m)$ exists, then $km \in B(\{k, m\})$, and all block sizes occur.*
2. *If $TD(k + 1; m)$ exists, and $0 < t < m$, then $km + t \in B(\{k, k + 1, m, t\})$, and all block sizes occur.*

Proof: See Bennett [9, Lemma 2.8]. ∎

Lemma 8.5 *If $TD(t; m)$ exists, and $k + 1 \leq t$, then $km + t - k \in B(\{k, k + 1, m, t\})$, and all block sizes occur.*

Proof: See Bennett [9, Lemma 2.11]. ∎

Using the notation $TD^*(k; m)$ to denote a $TD(k; m)$ with at least one parallel class of k-lines, we have the following lemma which was exploited by Bennett and Zhu [13].

Lemma 8.6 *If a $TD(k; p)$ exists, and a $TD^*(p; q)$ exists, and a $TD^*(k; q)$ exists, then a $\{k, p, q\}$-GDD of type $(kp)^q$ exists.*

Proof: Filling in the groups of the third TD, and removing a parallel class produces a $\{k, q\}$-GDD of type k^q. Now a similar construction with the second TD shows we have a $\{p, q\}$-GDD of type p^q. Applying Wilson's FC with weight of p for all points in our first constructed design, using the second constructed design to give the ingredients for the blocks of size q, and using the first hypothesized design to give the ingredients for the blocks of size k, shows the required result. ∎

Lemma 8.7 *If $m \neq 4$, then $8m+1 \in B(Q_{\geq 8})$.*

Proof: See Bennett [9, Theorem 4.1]. ∎

Lemma 8.8 *1. If $m \notin \{3,5,6,7\}$, then $8m \in B(Q_{\geq 8})$;*

2. If $m \geq 8$ and $m \neq 22$, then $8m \in B(Q_{\geq 8} \cup \{8^*\})$;

Proof: For the first part, see Bennett [9, Theorem 4.2]. Actually a careful analysis of Bennett's proof establishes most of the second part. It $n \geq 8$ is odd and $n \notin E_9$, where E_9 is a set of 30 numbers, then a $TD(9;n)$ is known to exist, and Bennett also shows that $n \in B(Q_{\geq 8})$, [9, Theorem 4.1]. Hence, if $n \notin E_9$, then we have the result for $m = n$ by Lemma 8.4.1, and for $m = n+1$ by Lemma 8.4.2 with $t = 8$. Lemma 8.4.1 also deals with the case that $m \geq 8$ is a power of 2. Lemma 8.4.2 with $t = 16, 32$ also gives the cases $m = n+2$ and $m = n+4$ if n is not too small. Lemma 8.5 with $t = 32$ gives the cases $m = n+3$ for $n \geq 31$, and with $t = 72$ gives $m = 78$. We can use Lemma 8.4.2 with $k = 16$, $t = 8$ to deal with $m = 35, 95$. We also note that an $RB(8;v)$ exists for $v = 120, 288, 680$; (the latter deals with $m = 94$). This abbreviated proof establishes the second part of the lemma; Bennett's result now follows by noting that the missing smaller m are all powers of 2, and $8*22 = 11*16$, and we can again apply Lemma 8.4.1. ∎

Lemma 8.9 (Bennett [10])

1. *If $n = 10$ or $12 \leq n \leq 19$, then $8n+12 \in B(Q_{\geq 8} \cup \{w^*\})$ for $8 \leq w \leq 12$;*
2. *If $n = 12$ or $14 \leq n \leq 19$, then $8n+14 \in B(Q_{\geq 8} \cup \{w^*\})$ for $8 \leq w \leq 14$;*
3. *If $n \in \{11,13,19\}$ or $15 \leq n \leq 17$, then $8n+w \in B(Q_{\geq 8} \cup \{w^*\})$ for $w \leq 11$.*

Proof: If $w \leq n$, we can truncate one group of a $TD(9;n)$ to size w whenever n is a prime power, and if $n = 15$, we can add w points to a $RB(8;120)$. If $w \leq n+2$ and $n+1$ is a prime power, we can apply Lemma 8.5 to a $TD(w;n+1)$. Finally, for $n = 14$, the $RB(8;120)$ is embedded as a $\{120;\{0,8\}\}$-arc in $PG(2,16)$; we can extend an 8-secant to size w. ∎

Lemma 8.10 (Bennett [10])

1. *If $m = 136$ or $138 \leq m \leq 146$ or $m \in \{148,149\}$, then $8m+156 \in B(Q_{\geq 8})$;*
2. *If $m = 138$ or $140 \leq m \leq 146$ or $m \in \{148,149\}$, then $8m+158 \in B(Q_{\geq 8})$;*

Proof: Let $u \in \{156, 158\}$; for $m \leq 144$, set $w = u - 144$ and truncate one group of a $TD(9; 144)$; now add w infinite points, and fill the groups using Lemma 8.9 for all but the last group (on $144+w$ points) which we fill with a $B(Q_{\geq 8}; u)$ design from Lemma 8.2. For $m = 145$, set $w = u - 144$ and remove 8 collinear points from a $TD(9; 19)$, and use this as the master design in Wilson's Fundamental Construction of Theorem 4.4, giving all points weight 8, to get a $\{8, 9\}$-GDD of type $144^8 152^1$; now add w infinite points, and fill the groups using Lemma 8.9 for all but the last group (on $144 + w$ points) which we fill with a $B(Q_{\geq 8}; u)$ design from Lemma 8.2. For $m > 145$, set $w = u - 152$ and truncate one group of a $TD(9; 152)$; now add w infinite points, and fill the groups using Lemma 8.9 for all but the last group (on $152 + w$ points) which we fill with a $B(Q_{\geq 8}; u)$ design from Lemma 8.2. ∎

Lemma 8.11 *If $TD(k; v - u)$ exists, and $v \in B(Q_{\geq 8} \cup \{u\}^*)$, and $v \in B(Q_{\geq 8})$, then $k(v - u) + u \in B(Q_{\geq 8})$.*

Proof: See Bennett [9, Lemma 2.12]. ∎

Theorem 8.12 *Let $0 \leq y \leq u - x$ and $0 \leq a - y \leq v - (u - x)$ and suppose:*

1. $\{k, k + 1\} \subset B(Q_{\geq 8})$,
2. *the incomplete transversal $TD(k + 1; v) - TD(k + 1; a)$ exists,*
3. $v + x \in B(Q_{\geq 8} \cup \{a + x\}^*)$.

   Further suppose either (A):

4. $u \in B(Q_{\geq 8})$,
5. $ka + x + y \in B(Q_{\geq 8} \cup \{x + y\}^*)$;

   or else (B):

4′. $u \in B(Q_{\geq 8} \cup \{x + y\}^*)$,

5′. $ka + x + y \in B(Q_{\geq 8})$;

then $kv + u \in B(Q_{\geq 8})$.

Proof: This is a variation of the usual singular indirect product construction. We add x infinite points to the design, then truncate one of the groups of the ITD, leaving $u - x$ points with y of them in the missing part. The initial conditions on x and y ensure that we actually remove a non-negative number from both the missing and non-missing parts of the groups of the ITD, and also leave a non-negative number of points in each part of the group. We then use D_3 to fill the k untruncated groups, omitting the flat which we align with the infinite points and the hole. For version B, we then use $D_{4'}$ to fill the last, truncated, group, and finally use $D_{5'}$ to fill the holes and infinite points. For version A, we reverse this order, and use D_5 first, then D_4. ∎

Corollary 8.13 *If* $160 \le m \le 170$*, or* $192 \le m \le 279$*, and* $u \in \{156, 158\}$*, then* $8m + u \in B(Q_{\ge 8})$*.*

Proof: We will apply version A of Theorem 8.12, with $k = 8$ and $x = 0$. Express m as $m = v = 8q + a$ with q taken from 19, 23, 25, 27, 29, 31, and with $8 \le a \le q$; this form provides us with the ITD, and also with D_3 since $x = 0$. Take $y = 8$ for $q = 19$ or $a = 22$, and $y = 0$ otherwise. Lemma 8.2 gives us D_4; and since $ka + x + y$ is a multiple of 8, Lemma 8.8 gives us D_5 for both values of y, noting in the case $y = 0$ the subdesign is trivial. ∎

Corollary 8.14 *Let* $144 \le m \le 159$*, or* $180 \le m \le 187$*.*

1. *If* $m \ne 146$*, then* $8m + 156 \in B(Q_{\ge 8})$*.*
2. *If* $m \notin \{145, 147, 148, 149\}$*, then* $8m + 158 \in B(Q_{\ge 8})$*.*

Proof: We will apply version B of Theorem 8.12, with $k = 8$. For $144 \le m \le 159$, let $m = v = 144 + a$ and $y = a$; the ITD follows from Wilson's MOLS construction [40] and $v = 16 * 9 + a$ for $a \le 9$, or from $v = 8 * 19 + (a - 8)$ and for $8 < a \le 15$. For $u = 156$, let $x = 17 - a$, and for $u = 158$, let $x = 16 - a$; D_3 follows from $v + x = 8 * 19 + (x + a - 8)$ and Lemma 8.5, and $D_{4'}$ follows from Lemma 8.2, and $D_{5'}$ is on $8a + 17$ or $8a + 16$ points, which is alright by Lemma 8.7, unless $a = 2$ for $u = 156$, or else is alright by Lemma 8.8, unless $a \in \{1, 3, 4, 5\}$ for $u = 158$.

For $u \in \{156, 158\}$, if $180 \le m \le 187$, let $m = v = 176 + a$, let $x = 11 - a$, and $y = a - 2$; the ITD follows from $v = 16 * 11 + a$, and D_3 follows from $v + x = 11 * 17$ and Lemma 8.4.1, and $D_{4'}$ follows from Lemma 8.2, and $D_{5'}$ is on $8a + 9$ points, which is alright, by Lemma 8.7, for $4 \le a \le 11$. ∎

Lemma 8.15 *If* $171 \le m \le 191$*, and* $m \notin \{174, 175, 188\}$*, then* $8m + u \in B(Q_{\ge 8})$ *for* $u \in \{156, 158\}$*.*

Proof: For the odd values not covered by Corollary 8.14, and for 176, we may apply Lemma 8.4.2 with $t = u$ and $k = 8$. For the remaining m, we note that $m + 1$ is a prime, and apply Lemma 8.5 with $t = u$ and $k = 8$. ∎

Lemma 8.16 *If a* $TD(14; m)$ *exists,* $0 \le s, t, w, x, y, z \le m$ *and* $x + y + z \le m$*, then there is a* $Q_{\ge 8}$*-GDD of type* $(8m)^{10}(8x + 9y + 10z)^1(8s)^1(8t)^1(8w)^1$*.*

Proof: See Bennett [9, Lemma 3.11]. ∎

Corollary 8.17 $1558 \in B(Q_{\ge 8})$

Proof: Apply Lemma 8.16 with $m = 17$, $s = x = 1$, $t = 4$, $z = 15$ and $w = y = 0$ to give a GDD of type $136^{10}158^1 8^1 32^1$. ∎

Theorem 8.18 *If* $138 \le m \le 279$, *then* $8m + 156 \in B(Q_{\ge 8})$.

Proof: Combining the preceeding corollaries and lemmas, we only have to deal with the three exceptions of Lemma 8.15, and these are covered by Lemma 8.4 with k=31, using the prime powers 49 and 53. ∎

Theorem 8.19 *If* $140 \le m \le 279$, *then* $8m + 158 \in B(Q_{\ge 8})$.

Proof: The proof is essentially the same as for Theorem 8.18. The use of Lemma 8.4 with $k = 31$ and the prime powers 43, 49, 53, resolves $m = 147$, and two of the three exceptions of Lemma 8.15, and Corollary 8.17 resolves the third exception. ∎

We turn now to the construction of some miscellaneous values. In addition to the values mentioned in the beginning of this section, we also have:

Theorem 8.20 $1102 \in B(Q_{\ge 8})$.

Proof: Apply Lemma 8.6 with $k = 9$, $p = q = 11$ to give a $\{9, 11\}$-GDD of type 99^{11}, and fill with 13 infinite points using a $B(\{8, 9, 13\}; 112 = 99 + 13)$ provided by $112 = 8 * 13 + 8$. ∎

Remark 8.21 A construction similar to this one was used by Bennett and Zhu to remove this and two other values in [13]. In the introduction to this section, we mentioned the construction of adding $t - 8$ extra points from an 8-line through a 120 point Denniston arc in $PG(2, 16)$; if we use the lines through a further point on this line to generate the groups, we have a $\{8, 9\}$-GDD of type $8^{14}t^1$ for $8 \le t \le 16$ (and in particular $t = 15$); taking this in conjunction with the method used in the proof of [9, Lemma 3.9] and with $254 \in B(Q_{\ge 8})$ accounts for the other constructions in [13]. Actually, This particular GDD, in conjunction with Bennett [9, Lemma 3.8], enables us to extend Lemma 8.16 to allow six further groups receiving weights of some mix of 0 and 8, if we wish, Provided, of course we were weighting a $TD(20; m)$).

Theorem 8.22 $\{1182, 1220, 1222, 1244, 1260, 1268, 1946\} \subset B(Q_{\ge 8})$.

Proof: For 1182, use Brouwer's stacking Baer sub-planes construction of [16] to give 7 $PG(2, 4)$s in $PG(2, 16)$, i.e., a $B(\{7, 11\}; 147)$, and to this add 6 infinite points for the resolutions on the short blocks to give a $B(\{8, 11, 6^*\}; 153)$; use this design to fill 7 groups of a $TD(8; 147)$ with 6 infinite points, and then a $B(\{9, 17\}; 153)$ from $153 = 9 * 17$ to fill the last group. For the next two values, let $u \in \{156, 158\}$ and let $a = u - 152$; now use $u = 8 * 19 + a \in B(\{8, 9, 19, a^*\})$ to fill in all but the last group of a $TD(8; 152)$ with a added infinite points, and use Lemma 8.2 for the last group. The next two values are the two smallest values given by Lemma 8.10. For the last two values, we use Lemma 8.11, noting that $1268 = 9 * (156 - 17) + 17$ and $1946 = 13 * (158 - 9) + 9$. ∎

Remark 8.23 For the applications discussed in [9, 13], the following equations are worth noting:

1. $1578 = 11 * (158 - 16) + 16$, and
2. $1626 = 11 * (156 - 9) + 9$.

We may summarize the main results of this section:

Theorem 8.24 *If v has the value 4 or 6 modulo 8, and $v \geq 1276$, then $v \in B(Q_{\geq 8})$.*

Proof: Given in Theorems 8.18 and 8.19. ∎

Remark 8.25 We have now substantiated the majority of the improvements made in [12, Table 3.18] over the results in [9]. We have no construction for 1578 or 1626, the two values noted in Remark 8.23; it would appear that these were erroneously omitted from [12, Table 3.18]. In addition, we have noted constructions for the values 316, 574, 814, 1244, 1260 and 1318 which further improve [12, Table 3.18].

9 Update

In this final section, we wish to provide some updates and corrections to the three tables of generating/exception sets given in [12, Tables III.3.17–19].

9.1 Table III.3.17

Most of the changes here involve designs with $4 \in K$, and result from improvements to [37] during its later drafts. That paper has a couple of weak entries in the summary table [37, pp. 74–75]: ($55 \in B(\{4,6,8,9\})$, $30 \notin B(\{4,7,8,9\})$). It is known that $44, 45, 47 \notin B(\{4,7,8\})$ and also that $48 \notin B(\{4,7,9\})$. The remaining improvements with $4 \in K$ from [37] are in constructing designs; another results from Ling [29, p. 57] who gives a $B(\{4,5\}; 104)$ with 21 parallel classes of 4-blocks, so adding 11 points gives a $B(\{4,5,11^*\}; 115)$. For $4 \in K$, we have chosen to list the current possible exceptions for all such listed K, several of which seem close to solution.

K	Possible exceptions
4, 6	33, 34, 39, 45, 46, 51, 75, 87
4, 8	48, 53, 60, 65, 69, 77, 89, 101, 161, 164, 173
4, 9	60, 69, 84, 93, 96, 192
4, 5, 7	30, 39, 42, 51, 54
4, 5, 11	38, 42, 43, 46, 50, 54, 58, 62, 66, 67, 70, 74 78, 82, 90, 94, 98, 102, 106, 114, 118, 126
4, 6, 7	33, 45, 87
4, 6, 8	33, 34, 35, 39, 41, 47, 50, 51, 53, 59, 62, 65, 71, 75, 77, 87, 89, 95, 101, 110, 131, 161, 170
4, 6, 9	34, 75, 87
4, 7, 8	48, 51, 54, 59, 62, 65, 66, 69, 74, 75, 77, 78, 83, 87, 89, 90, 101, 102, 110, 111, 114, 123, 126, 131, 135, 143, 150, 159, 161, 162, 164, 167, 170, 173, 174, 186, 195
4, 7, 9	51, 54, 60, 66, 69, 75, 78, 84, 87, 93, 96, 102, 111, 114, 138, 147, 150, 159, 174, 183, 186, 195, 210
4, 8, 9	48, 53, 60, 69, 77, 101, 161, 164, 173
4, 5, 7, 8	30, 42, 51
4, 5, 7, 9	30, 51, 54
4, 5, 7, 11	30, 42, 54
4, 6, 7, 8	33, 35, 41, 65, 77, 131
4, 6, 8, 9	34, 35, 41, 47, 50, 53, 59, 62, 71, 75, 77, 87, 95, 101, 131, 161, 170
4, 7, 8, 9	35, 38, 39, 41, 42, 44, 47, 48, 51, 54, 59, 62, 110, 143, 150, 159, 161, 164, 167, 173, 174
4, 5, 7, 8, 9	30, 51
4, 6, 7, 8, 9	35, 41
4, 5, 7, 9, 10, 11	30
4, 5, 9, 11, 19, 31	38, 42, 43, 46, 50, 54, 62, 66, 74, 78, 90, 98, 102, 114, 126

There were three changes involving designs with $5 \in K$ resulting from improvements to [11] during its final draft. The remaining improvements to [12, Table III.3.17] come from other results in the table or from three designs: Bierbrauer [14] exhibits a pentario in $PG(2,8)$ (i.e., a set of 3 quintuples, any pair of which form a hyperoval), and the pentario forms a $\{15; \{0,2,3\}\}$-arc in the plane, so deleting the pentario and an external line gives a $B(\{5,6,8\}; 49)$; there are over 10^6 such configurations, and using the Singer difference set given by powers of 2 in Z_{73}, an example set of deleted points is:

$$\{1,2,11,27,60\}\{0,9,29,45,58\}\{7,30,38,46,71\}\{3,4,6,10,18,34,39,57,66\}$$

The two other designs are 7-GDDs of types 7^{25} and 7^{37} constructed by Janko and Tonchev [27] and by Abel [2], and modifying these designs give the remaining new PBDs.

K	Change
5, 7	$259, 709 \in B(K)$
5, 6, 8	$49, 307 \in B(K)$
5, 6, 8, 9	$174 \in B(K)$
5, 7, 8, 9	$163, 243 \in B(K)$
6, 7, 8	$170, 174, 175, 176 \in B(K)$
6, 7, 8, 9	$137, 170, 174, 175, 176 \in B(K)$
7, 8, 9	$175, 176, 259, 260 \in B(K)$

Finally, we note that [29, p. 21] contains an arithmetic error; this affects part of [30, Lemma 4.7], for (94, 95). More specifically, Ling gives a (lower) bound up to which we safely can add points to a certain configuration in a projective plane, although he mis-evaluated the bound here; using essentially Ling's plan, one can exceed this bound with carefully chosen points, and this was demonstrated in [7, Lemma 4.18], correcting the lacuna.

9.2 Table III.3.18

First, we note the errors in this table. For $K = Q_{\geq 4}$ clearly the prime powers 11, 19, 23, 27 are not exceptions, nor is 51 a possible exception (truncate one group of a $TD(5;11)$ to size 7); incidentally, the exception set can be deduced from the $K = \{4,5,7,8,9\}$ exception set. Earlier, we commented on the other possible errors in this table for $K = Q_{\geq 8}$ in Remark 8.25, (1578, 1626 omitted), as well as the improvements of constructions for 316, 574, 814, 1244, 1260, 1318; Bennett [10] has also obtained constructions for 1196, 1198, 1212, 1214, 1236, 1238, 1246, 1254, 1270, 1706, 1770, 1778, 1794 and all values ≥ 1810.

In [6], it is shown that:

$$\{161, 231, 266, 276, 296, 376, 561, 741, 946\} \subset B(Q_{1 \bmod 5} \cup \{6\}).$$

The Janko-Tonchev and Abel designs [27, 2] also show

$$\{175, 176, 259, 260\} \subset B(Q_{0,1 \bmod 7}).$$

9.3 Table III.3.19

There is one error: clearly 15 is an essential element of $H_{0,1(5)}$.

In [6], for $H_{1(5)}$, it is shown that the following are inessential:

$$\{131, 146, 166, 196, 221, 226, 231, 251, 261, 266, 296, 316, 326, 351, 356\}.$$

9.4 Updated References

Finally we note that the references in [37, 11] are more accurate than those in [12, Table III.3.19], and give the following updates to the reference section of [12].

Citation in [12]	Citation in this paper
[4]	[11]
[12]	This paper itself
[13]	[21]
[14]	[24]
[15]	[23]
[19]	[30]
[20]	[31]
[21]	[32]
[25]	[37]

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A Appendix

The object of this appendix is to provide the constructions for the first 288 values of $B(Q_{1(8)})$ plus portions of the range for the first 1648 values, as annonunced in Section 6.

Key to Table A.1.

Code		Result	Authority	Code		Result	Authority
A/a	m.n	$17m+n$	Lemma 5.14	P			Prime
B	m.n	$17m+n$	Lemma 5.16	PP			Prime power
BD	m.n	$10m-n$	Lemma 5.8	R	v.n	$v+n$	Lemma 5.10
C/c	m.n	$15m+n$	Lemma 5.17	RB	m	m	Lemma 6.2
D/d	m.n	$17m+n$	Lemma 5.18	T/t	m.n	$9m+n$	Lemma 5.5
E	m.n	$16m+16+n$	Lemma 5.19	Table			Table 6.2
F	m.n	$18m-16+n$	Lemma 5.21	X			Tables 6.3, 6.4
Ir	m.n	$63m+9+n$	Lemma 5.23	ZZ			Lemma 6.3
L	m.n	$9m+n-9$	Lemma 5.7				
Ls	m.n	$9m+n-9$	Lemma 5.6	** n **		Unconstructed	

Lower case versions of construction codes (and their m and n) are from Table 6.5.

Table A.1.

Table of Constructions for $U_{1(8)}$.

		0	1	2	3	4	5	6	7	8
0	0		PP	P	PP	***4**	P	PP	***7**	***8**
9	1	P	PP	P	P	**13**	P	PP	**16**	P
18	2	Table	Table	**20**	PP	**22**	**23**	P	**25**	**26**
27	3	**27**	Table	P	P	**31**	P	**33**	Table	P
36	4	PP	**37**	**38**	P	**40**	RB 41	P	**43**	P
45	5	PP	Table	**47**	**48**	RB 49	P	P	**52**	Table
54	6	P	Table	P	P	**58**	**59**	**60**	**61**	**62**
63	7	**63**	Table	P	PP	**67**	**68**	**69**	**70**	P
72	8	P	Table	P	P	**76**	P	PP	**79**	P
81	9	T 9.0	T 9.1	T 9.2	P	**85**	T 9.5	T 9.6	**88**	**89**
90	10	T 9.9	PP	**92**	**93**	**94**	P	P	RB 97	**98**
99	11	T 11.0	T 11.1	P	T 11.3	*103**	T 11.5	PP	*106**	P
108	12	T 11.9	T 11.10	P	ZZ	*112**	RB 113	*114**	*115**	P
117	13	P	*118**	P	PP	RB 121	P	*123**	*124**	*125**

126	14	P	Table	Table	P	*130**	P	*132**	*133**	*134**
135	15	Table	Table	P	Table	*139**	*140**	P	*142**	*143**
144	16	P	t 16.1	t 16.2	t 16.3	*148**	P	P	*151**	P
153	17	T 17.0	T 17.1	T 17.2	P	*157**	T 17.5	T 17.6	*160**	P
162	18	P	T 17.10	T 17.11	P	*166**	T 17.14	T 17.15	RB 169	P
171	19	PP	T 19.1	T 19.2	T 19.3	*175**	P	T 19.6	*178**	P
180	20	T 19.9	T 19.10	T 19.11	T 19.12	*184**	P	P	*187**	T 19.17
189	21	T 19.18	T 19.19	*191**	*192**	RB 193	P	*195**	*196**	*197**
198	22	ZZ	*199**	P	P	*202**	*203**	*204**	*205**	Table
207	23	P	*208**	*209**	PP	*211**	P	*213**	*214**	P
216	24	Table	Table	*218**	P	*220**	*221**	P	*223**	*224**
225	25	P	*226**	*227**	t 25.3	*229**	t 25.5	PP	*232**	RB 233
234	26	P	t 25.10	P	t 25.12	*238**	P	t 25.15	RB 241	t 25.17
243	27	t 25.18	t 27.1	t 27.2	t 27.3	*247**	t 27.5	P	*250**	Table
252	28	P	t 27.10	t 27.11	t 27.12	*256**	t 27.14	t 27.15	*259**	P
261	29	P	T 29.1	T 29.2	P	*265**	P	P	*268**	P
270	30	P	T 29.10	T 29.11	T 29.12	a 16.2	T 29.14	PP	a 16.5	T 29.17
279	31	T 29.18	T 29.19	a 16.9	T 29.21	a 16.11	P	P	a 16.14	P
288	32	T 32.0	X	X	X	X	X	X	X	X

For 289 – 321; use construction A with $m = 17$.

315	35	X	X	X	X	X	X	X	t 32.34	X
324	36	a 17.35	X	P	X	X	B 19.6	X	B 19.8	X

For 323 – 357 (with gaps); use construction C with $m = 19$.

351	39	X	X	X	X	X	X	X	Ir 5.34	Ir 5.35
360	40	Ir 5.36	RB 361	P	Ir 5.39	*364**	Ir 5.41	Ir 5.42	*367**	*368**
369	41	T 41.0	T 41.1	T 41.2	T 41.3	*373**	T 41.5	T 41.6	*376**	L 41.17
378	42	T 41.9	T 41.10	T 41.11	T 41.12	*382**	T 41.14	T 41.15	*385**	T 41.17
387	43	T 41.18	T 41.19	L 41.29	T 41.21	*391**	L 41.32	T 41.24	L 41.34	L 41.35
396	44	t 43.9	T 41.28	T 41.29	T 41.30	*400**	T 41.32	t 43.15	T 41.34	T 41.35
405	45	T 41.36	t 43.19	P	T 41.39	RB 409	T 41.41	t 43.24	E 23.28	E 23.29
414	46	E 23.30	t 43.28	t 43.29	t 43.30	E 23.34	t 43.32	E 23.36	t 43.34	t 43.35
423	47	t 43.36	*424**	E 23.41	t 43.39	*427**	t 43.41	t 43.42	E 23.46	t 43.44
432	48	t 43.45	t 43.46	F 25.0	F 25.1	t 43.49	F 25.3	*438**	F 25.5	F 25.6
441	49	T 49.0	T 49.1	T 49.2	T 49.3	F 25.11	T 49.5	T 49.6	F 25.14	F 25.15
450	50	T 49.9	T 49.10	T 49.11	T 49.12	d 25.29	T 49.14	T 49.15	d 25.32	T 49.17
459	51	T 49.18	T 49.19	L 49.29	T 49.21	*463**	L 49.32	T 49.24	L 49.34	L 49.35
468	52	L 49.36	T 49.28	T 49.29	T 49.30	a 27.5	T 49.32	L 49.42	T 49.34	T 49.35
477	53	T 53.0	T 53.1	T 53.2	T 53.3	L 49.49	T 53.5	T 53.6	E 27.36	T 49.44
486	54	T 53.9	T 53.10	T 53.11	T 53.12	T 49.49	T 53.14	T 53.15	X	X

For 493 – 606, 608; use constructions A, C with $m = 29, 32$.

603	67	X	X	X	X	*607**	X	R585.24	T 64.34	T 64.35
612	68	T 64.36	R585.28	R585.29	T 64.39	Ls64.49	T 64.41	T 64.42	R585.34	T 64.44
621	69	T 64.45	T 64.46	*623**	R585.39	T 64.49	T 64.50	T 64.51	*628**	T 64.53
630	70	T 64.54	T 64.55	T 64.56	T 64.57	R585.49	R585.50	R585.51	*637**	R585.53
639	71	T 71.0	T 71.1	T 71.2	T 71.3	*643**	T 71.5	T 71.6	*646**	Ls71.17
648	72	T 71.9	T 71.10	T 71.11	T 71.12	F 37.2	T 71.14	T 71.15	F 37.5	T 71.17
657	73	T 71.18	T 71.19	T 73.2	T 71.21	F 37.11	T 73.5	T 71.24	F 37.14	F 37.15
666	74	T 73.9	T 71.28	T 71.29	T 71.30	*670**	T 71.32	T 73.15	T 71.34	T 71.35
675	75	T 71.36	T 73.19	L 73.29	T 71.39	F 37.29	T 71.41	T 71.42	F 37.32	T 71.44
684	76	T 71.45	T 71.46	T 73.29	T 73.30	T 71.49	T 71.50	T 71.51	T 73.34	T 71.53

693	77	T 71.54	T 71.55	T 71.56	T 71.57	X	X	X	X	X

For 697 – 779; use constructions A, C with $m = 41$.

774	86	X	X	X	X	X	X	T 81.51	T 83.34	T 82.44
783	87	T 81.54	T 81.55	T 81.56	T 81.57	T 82.49	T 82.50	T 82.51	F 43.32	T 83.44
792	88	T 82.54	T 81.64	T 81.65	T 81.66	T 83.49	T 83.50	T 83.51	F 43.41	T 81.71
801	89	T 81.72	T 81.73	T 81.74	T 81.75	ZZ	T 81.77	T 81.78	c 43.82	T 81.80
810	90	T 81.81	T 82.73	T 82.74	T 82.75	ZZ	T 82.77	T 82.78	c 43.91	T 82.80
819	91	T 82.81	T 82.82	T 83.74	T 83.75	*823**	T 83.77	T 83.78	ZZ	T 83.80
828	92	T 83.81	T 83.82	T 83.83	t 83.84	*832**	X	X	X	X

For 833 – 1006; use construction C with $m = 49, 53$.

999	111	X	X	X	X	X	X	X	T101.97	T107.44
1008	112	T107.45	T107.46	T109.29	T109.30	T107.49	T107.50	T107.51	T109.34	T109.35
1017	113	T 113.0	T 113.1	T 113.2	T109.39	Ls109.49	T 113.5	T 113.6	*1024*	T109.44
1026	114	T 113.9	T113.10	T107.65	T113.12	T109.49	T113.14	T113.15	RB 1033	T113.17
1035	115	T113.18	T113.19	T109.56	T113.21	*1039*	T107.77	T113.24	L113.34	T107.80
1044	116	T107.81	T109.64	T113.29	T113.30	*1048*	T113.32	T107.87	T113.34	T113.35
1053	117	T113.36	T109.73	T109.74	T113.39	L113.49	T113.41	T113.42	T107.97	T113.44
1062	118	T113.45	T113.46	T109.83	T109.84	T113.49	T113.50	T113.51	Ls109.97	T113.53
1071	119	T113.54	T113.55	T113.56	T113.57	Ir15.121	T109.95	T109.96	T109.97	L113.71
1080	120	T109.99	T113.64	T113.65	T113.66	*1084*	T117.32	T109.105	T117.34	X

For 1088 – 1575; use constructions A, C with $m = 64, 71, 73, 81, 83$.

1575	175	X	T173.19	T169.56	T173.21	Ls163.121	T171.41	T173.24	c89.158	T171.44
1584	176	T 176.0	T 176.1	T 176.2	T 176.3	T171.49	T 176.5	T 176.6	T173.34	T173.35
1593	177	T 177.0	T 177.1	T 177.2	T 177.3	Ls173.49	T 177.5	T 177.6	T167.97	T176.17
1602	178	T 177.9	T177.10	T177.11	T177.12	T173.49	T177.14	T177.15	L169.97	T177.17
1611	179	T 179.0	T 179.1	T 179.2	T 179.3	Ls167.121	T 179.5	T 179.6	T176.34	T176.35
1620	180	T 179.9	T179.10	T179.11	T179.12	T167.121	T179.14	T179.15	T177.34	T179.17
1629	181	T 181.0	T 181.1	T 181.2	T 181.3	T176.49	T 181.5	T 181.6	T171.97	T177.44
1638	182	T 181.9	T181.10	T181.11	T181.12	T177.49	T181.14	T181.15	T179.34	T181.17
1647	183	T181.18	T181.19	X	X	X	X	X	X	X
		0	1	2	3	4	5	6	7	8