STRONG SOLUTIONS OF QUASILINEAR
INTEGRO-DIFFERENTIAL EQUATIONS WITH SINGULAR
KERNELS IN SEVERAL SPACE DIMENSIONS

HANS ENGLER

Abstract. For quasilinear integro-differential equations of the form
\( u_t - a \cdot A(u) = f \), where \( a \) is a scalar singular integral kernel that behaves like \( t^{-\alpha} \), \( \frac{1}{2} \leq \alpha < 1 \) and \( A \) is a second order quasilinear elliptic operator in divergence form, solutions are found
for which \( A(u) \) is integrable over space and time.

1. Introduction

The purpose of this note is a study of quasilinear integro-differential equations
of the form
\[
\frac{\partial}{\partial t} u(x,t) - \int_0^t a(t-s) \text{div} \tilde{g}(\nabla u(x,s)) \, ds = f(x,t)
\]
in cylindrical domains \( \Omega \times [0,T] \subset \mathbb{R}^{N+1} \), together with zero boundary data for
\( u \) on \( \partial \Omega \times [0,T] \) and initial data on \( \Omega \times \{0\} \). Here \( a(\cdot) \) is a scalar integral kernel
that behaves like \( t^{-\alpha} \), \( 0 < \alpha < 1 \) for \( t \) near zero. The equation occurs in the
description of simple shear motions (\( N = 1 \)) and of antiplane shear (\( N = 2 \)) of
certain idealized viscoelastic materials ([14]), and “power law” kernels \( a(t) = t^{-\alpha} \)
have been proposed elsewhere for such models ([13]). We are interested in large
time solutions without restrictions on the size of the data.

Recently, for \( \Omega = [0,1] \subset \mathbb{R} \), “strong solutions” with \( u_{xx}^2 \) integrable over \( \Omega \times [0,T] \)
have been found under two different sets of assumptions, namely when \( \alpha \geq \frac{1}{2} \) and
\( 0 < c \leq \tilde{g}^\prime(r) \leq c' < \infty \) ([9]) and when the Laplace transform of \( \hat{a} \)
maps the right half plane into a sector in the right half plane, under a restriction for \( \tilde{g} \)
that depends on the angle of aperture of this sector ([8]). Also, weaker solutions with
only \( u_x^2 \) integrable can be found without restrictions for \( \alpha, c, c' \) ([11]), and smooth
solutions (as smooth as the data permit) can be found if \( \frac{2}{3} < \alpha < 1 \) and only
\( 0 < \tilde{g}^\prime(r) \leq C < \infty \) ([3]). In other work, large time solutions for small smooth data
have been constructed, weak solutions for more regular kernels have been found,
and the break-down of smooth solutions for regular kernels has been established;
see the references in [8] and [9].

1991 Mathematics Subject Classification: 45K05.
Key words and phrases: Integro-differential equation, strong solution, singular kernel, quasilinear

©1995 Southwest Texas State University and University of North Texas.
Supported by the National Science Foundation under grant # DMS - 9003543

1
We want to extend the results of [9] in several ways, namely by treating also higher space dimensions and by allowing also nonlinear terms \( g \) that grow polynomially in their arguments, corresponding to “shear-thickening” behavior for viscoelastic materials. The function \( g : \mathbb{R}^N \to \mathbb{R}^N \) will be assumed to be “isotropic”, i.e. to have the form \( g(\xi) = g(|\xi|^2)\xi \). We also use a slightly different class of integral kernels and modified assumptions for the data. The main restriction, namely \( \frac{1}{2} < \alpha < 1 \), however is not weakened in the present paper. We also give linear examples that show that more must be assumed for the data of the problem if strong solutions are to be found for smaller values of \( \alpha \).

Throughout, \( \Omega \subset \mathbb{R}^N \) is a bounded domain with a \( C^2 \) smooth boundary \( \partial \Omega \), with exterior unit normal vector field \( n : \partial \Omega \to S^{N-1} \). Subscripts \( i \) denote partial differentiation with respect to \( x_i \), \( 1 \leq i \leq N \), and a subscript \( t \) is used to denote partial differentiation with respect to \( t \). We shall use the common symbols \( \nabla \) for the gradient of a scalar function and for the derivative matrix of a vector field and the symbol \( \text{div} \) for the divergence of a vector field: If \( \phi = (\phi_i)_{1 \leq i \leq M} \) and \( \psi = (\psi_i)_{1 \leq i \leq N} \) are vector fields, then

\[
\nabla \phi = (\phi_{i,j})_{1 \leq i \leq M, 1 \leq j \leq N}, \quad \text{div} \psi = \sum_i \psi_{i,i}.
\]

Also, \( \Delta u = \text{div} \nabla u \) is the Laplacian of a scalar function \( u \), and \( \nabla^2 u \) is its Hessian matrix.

The common notation for Sobolev spaces will be employed, and we write \( H^k \) for \( W^{k,2} \) and \( H_0^k \) for \( W_0^{k,2} \). We also use spaces of vector-valued \( H^s([0,T],V) \), for non-integer \( s = k + \gamma > 0 \), \( 0 < \gamma < 1 \), with a separable Hilbert space \( V \). Such spaces can be defined as complex interpolation spaces between \( H^k([0,T],V) \) and \( H^{k+1}([0,T],V) \) and can alternatively be characterized by the conditions that \( v \in H^s([0,T],V) \) if and only if \( v^{(k)} \in L^2(0,T;V) \) and

\[
\int_0^T \int_0^T \frac{\|v^{(k)}(t) - v^{(k)}(\tau)\|^2_V}{|t-\tau|^{1+2\gamma}} d\tau dt < \infty,
\]

see [1]. The norms of \( L^2(\Omega,\mathbb{R}) \) and \( L^2(\Omega,\mathbb{R}^N) \) are denoted by \( \| \cdot \| \), and the scalar products on these spaces are denoted by \( (\cdot,\cdot) \). Other norms on function spaces are identified by subscripts. Euclidean norms of vectors and matrices are denoted by \( | \cdot | \).

The convolution of two functions \( u, v \) that are supported on the positive half axis is written as \( u \ast v(t) = \int_0^t u(t-s)v(s) \, ds \). The letter \( C \) is used for constants whose values may be different in each occurrence and which can in principle be estimated explicitly in terms of known quantities.

2. Main Result

Consider the quasilinear integro-differential equation

\[
u_t(x,t) - \int_0^t a(t-s) \text{div} \left( g(|\nabla u(x,s)|^2) \nabla u(x,s) \right) \, ds = f(x,t) \tag{2.1}
\]
in $\Omega \times [0, T]$, with initial and boundary data
\[ u(\cdot, 0) = u_0, \quad u |_{\partial \Omega \times [0, T]} = 0. \] (2.2)

The kernel $a : [0, T] \rightarrow \mathbb{R}$ is assumed to be of the form
\[ a(t) = a_{\alpha}(t) + c \ast a_{\alpha}(t) \] (2.3)
with
\[ a_{\alpha}(t) = \frac{t^{-\alpha}}{\Gamma(1 - \alpha)}, \quad c \in W^{1,1}([0, T]; \mathbb{R}) \] (2.4)
and
\[ \frac{1}{2} < \alpha < 1. \] (2.5)

The case $\alpha = \frac{1}{2}$ will be discussed in section 3. We assume that the function $g : [0, \infty) \rightarrow \mathbb{R}$ is continuously differentiable and satisfies
\[ g(r) + (2 + \epsilon)rg'(r) \geq 0, \quad rg'(r) \leq \beta g(r) \] (2.6)
for some $\epsilon, \beta$, for all $r \geq 0$. Note that (2.6) implies that
\[ \epsilon \leq g(r) \leq g(1)r^\beta \] (2.7)
for all $r \geq 1$. We define
\[ G(r) = \frac{1}{2} \int_0^r g(s) \, ds; \] (2.8)
then the function $r \rightarrow G(r^2)$ is strictly convex on $\mathbb{R}$ by (2.6). For the data, we assume that
\begin{align*}
\|f(\cdot, 0)\| &+ \int_0^T \|f_t(\cdot, t)\| \, dt < \infty \\
\int_\Omega G(|\nabla u_0(x)|^2) &< \infty \\
\int_\Omega |\nabla (g(|\nabla u_0|^2)\nabla u_0)|^2 &< \infty.
\end{align*} (2.9)

The main result of this note is the following.

**Theorem 1.** Under these assumptions, there exists a function $u$ in
\[ W^{1, \infty}([0, T], L^2(\Omega)) \cap L^2(0, T; H^2(\Omega)) \cap H^{1,2}([0, T], H^1_0(\Omega)) \]
that satisfies (2.1) in the sense of distributions and for which $\lim_{t \rightarrow 0} u(\cdot, t) = u_0$ in $H^1(\Omega)$. Moreover, for some $q > 1$, $\nabla (g(|\nabla u|^2)\nabla u) \in L^q(\Omega \times [0, T])$, and thus (2.1) holds in this space.

**Proof.** We construct Galerkin approximations, establish suitable a priori estimates, and pass to the limit. This will yield a solution in the desired solution class. Thus let
\[ \{\varphi_i \mid i \geq 1\} \subset L^2(\Omega) \]
be a complete orthonormal system in $L^2(\Omega)$ of eigenfunctions of the negative Laplacian with zero boundary data. These functions are known to be in $W^{2,p}(\Omega) \cap C^\infty(\Omega)$ for all $p < \infty$. Let $V_n = \text{span}(\varphi_1, \ldots, \varphi_n)$ for $n = 1, 2, 3, \ldots$. For technical reasons, we shall construct solutions on $\Omega \times [0, T]$. Thus let $T' = T + 1$; we can extend $f$ and $a$ such that all their properties still hold on the larger interval. Let $f^n$ be a sequence in $C^1([0, T'], L^2(\Omega))$ that converges to $f$ in $W^{1,1}[0, T', L^2(\Omega))$, and let $u^n \in C^1([0, t_n), V_n)$ be the unique maximal solution of the system

$$
\langle u^n_v, v \rangle + \langle a * g(|\nabla u^n|^2) \nabla u^n, \nabla v \rangle = \langle f^n, v \rangle
$$

for all $v \in V_n$, with initial data defined by

$$
\langle u^n(0) - u_0, v \rangle = 0
$$

for all $v \in V_n$. It follows from standard results on functional differential equations ([10]) that $u^n$ exists and is unique for each $n$; possibly, $t_n < T'$, in which case

$$
\|u^n(t)\| \to \infty \quad \text{as} \quad t \to t_n.
$$

It is also easy to see (by differentiating the equation) that in fact $u^n \in C^2((0, t_n), V_n)$.

A Priori Estimate I. Let $d \in W^{1,1}([0, T'], \mathbb{R})$ be the resolvent kernel of $c$ ([7]), i.e. the function satisfying

$$
c + d + c * d = 0.
$$

We use the notation

$$
g^n(t) = g(|\nabla u^n(\cdot, t)|^2) \nabla u^n(\cdot, t), \quad g^n(0) = g^n_0.
$$

Then (2.10) is equivalent to

$$
\langle u^n_t + d * u^n, v \rangle + \langle a_n * g^n, \nabla v \rangle = \langle f^n + d * f^n, v \rangle
$$

for all $v \in V_n$. We differentiate with respect to $t$; after some manipulations, the result is the equation

$$
\langle u^n_{tt} + d(0)u^n_t + d' * u^n_t, v \rangle + \langle a_n(g^n - g^n_0), \nabla v \rangle
$$

$$
+ \left( \int_0^t (-a_n'(t - s))(g^n(t) - g^n(s)) \, ds, \nabla v \right) = \langle F^n(\cdot, t), v \rangle,
$$

where

$$
F^n(\cdot, t) = f^n(\cdot, t) + d(0)f^n(\cdot, t) + d' * f^n(\cdot, t) - a_n(t)\text{div} g^n_0.
$$

Note that the integral involving $a_n'$ is defined for all $t$, since $|g^n(t) - g^n(s)| = O(|t - s|)$, due to the differentiability of $u^n$. We now use the $t$-dependent test function $v = u^n(\cdot, t) \in V_n$. The result is the identity

$$
\frac{1}{2} \frac{d}{dt} \|u^n(\cdot, t)\|^2 + d(0)\|u^n(\cdot, t)\|^2 + \langle d' * u^n(\cdot, t), u^n(\cdot, t) \rangle
$$

$$
+ a_n(t)\langle g^n(t) - g^n_0, \nabla u^n(\cdot, t) \rangle + \int_0^t (-a_n'(t - s))(g^n(t) - g^n(s), \nabla u^n(\cdot, t)) \, ds
$$

$$
= \langle F^n(\cdot, t), u^n(\cdot, t) \rangle.
$$

(2.15)
Now
\[ \langle g^n(t) - g^n(s), \nabla u^n_i(\cdot, t) \rangle = \frac{d}{dt} \Phi^n(t, s) \tag{2.16} \]
where
\[ \Phi^n(t, s) = \int_\Omega H(\nabla u^n(\cdot, t), \nabla u^n(\cdot, s)) \, ds \]
\[ H(\xi, \zeta) = G(|\xi|^2) - G(|\zeta|^2) - (\xi - \zeta)^T g(|\zeta|^2) \zeta. \tag{2.17} \]
Thus
\[ \int_0^t (-a'_\alpha(t-s)) \langle g^n(t) - g^n(s), \nabla u^n_i(\cdot, t) \rangle \, ds \]
\[ = \frac{d}{dt} \int_0^t (-a'_\alpha(t-s)) \Phi^n(t, s) \, ds + \int_0^t a''_\alpha(t-s) \Phi^n(t, s) \, ds \tag{2.18} \]
and
\[ a_\alpha(t) \langle g^n(t) - g^n_0, \nabla u^n_i(\cdot, t) \rangle = \frac{d}{dt} (a_\alpha(t) \Phi^n(t, 0)) - a'_\alpha(t) \Phi^n(t, 0). \tag{2.19} \]

Note that \(|\Phi^n(t, s)| = O(|t-s|^2)\); thus the integral involving \(a''_\alpha(t) = \frac{t^{-\alpha-2}}{t(-\alpha-1)}\) converges. We insert these formulae into (2.15) and integrate with respect to \(t\). After a trivial estimate, the result is
\[ \frac{1}{2} \|u^n_i(\cdot, t)\|^2 - \frac{1}{2} \|u^n_i(\cdot, 0)\|^2 + \int_0^t (-a'_\alpha(t-s)) \Phi^n(t, s) \, ds + a_\alpha(t) \Phi^n(t, 0) \]
\[ + \int_0^t \left( \int_0^s a''_\alpha(s-\tau) \Phi^n(s, \tau) \, d\tau - a'_\alpha(s) \Phi^n(s, 0) \right) \, ds \tag{2.20} \]
\[ \leq \int_0^t \left( \|F^n(\cdot, s)\| + \|d' * u^n_i(\cdot, s)\| + |d(0)||\|u^n_i(\cdot, s)\|\|u^n_i(\cdot, s)\| \right) \, ds. \]

Next we note that for any \(\xi, \zeta \in \mathbb{R}^n\)
\[ H(\xi, \zeta) = \int_0^1 (1-s)(\xi - \zeta)^T K(\zeta + s(\xi - \zeta))(\xi - \zeta) \, ds. \]
where
\[ K(\lambda) = g(|\lambda|^2)\mathbf{I} + 2g'(|\lambda|^2)\lambda\lambda^T \]
is the derivative matrix of the function \(\lambda \to g(|\lambda|^2)\lambda\). By assumption (2.6),
\[ \mu^T K(\lambda) \mu \geq \epsilon |\mu|^2 \]
for all \(\mu \in \mathbb{R}^n\), and thus for all \(\xi, \zeta\)
\[ H(\xi, \zeta) \geq \frac{\epsilon}{2} |\xi - \zeta|^2. \tag{2.21} \]
In particular,
\[ \Phi^n(t, s) \geq \frac{\epsilon}{2} \|\nabla u^n(\cdot, t) - \nabla u^n(\cdot, s)\|^2 \geq 0 \tag{2.22} \]
for all \( 0 \leq s, t \leq T' \). Dropping now first all terms involving \( \Phi^n \) on the left hand side of (2.20) and applying Gronwall’s inequality shows that

\[
\sup_{0 \leq t \leq T_n} \|u^n(t)\| \leq C \tag{2.23}
\]

for some constant \( C \) that does not depend on \( n \). Consequently, \( t_n = T' \) for all \( n \).

Next we return to (2.20) and infer that also

\[
\begin{align*}
&\sup_{0 \leq t \leq T'} \left( \int_0^t (-a'_\alpha(t-s)) \Phi^n(t, s) \, ds + a_\alpha(t) \Phi^n(t, 0) \right) \leq C, \\
&\int_0^{T'} \left( \int_0^s a'_{\alpha}(s-\tau) \Phi^n(s, \tau) \, d\tau + (-a'_\alpha(s)) \Phi^n(s, 0) \right) \, ds \leq C
\end{align*} \tag{2.24}
\]

and therefore in particular

\[
\int_0^{T'} \int_0^s (s-\tau)^{-\alpha-2} \|\nabla u^n(\cdot, s) - \nabla u^n(\cdot, \tau)\|^2 \, ds \, d\tau + \sup_{0 \leq t \leq T'} \|\nabla u^n(\cdot, t)\| \leq C. \tag{2.25}
\]

The first estimate in (2.25) is equivalent to an a priori estimate in \( H^{1/2}([0,T'],H^1(\Omega)) \) for the approximate solutions \( u^n \). Finally, from (2.25), (2.24), and (2.9) we deduce that

\[
\sup_{0 \leq t \leq T'} \int_\Omega G(|\nabla u^n(\cdot, t)|^2) \leq C. \tag{2.26}
\]

A Priori Estimate II. As in (2.11), let \( d \) be the resolvent kernel of \( c \), and set

\[
b = b_\alpha + d * b_\alpha \quad \text{with} \quad b_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}.
\]

Then \( a_\alpha * b_\alpha(t) = 1 \) and therefore also \( a * b(t) = 1 \) for all \( t > 0 \). We form the convolution of (2.10) with \( b \) and differentiate with respect to \( t \). The result is the identity

\[
\frac{d}{dt} \langle b * u^n_t(\cdot, t), v \rangle + \langle g^n(t), \nabla v \rangle = \langle b(t)f^n(\cdot, 0) + b * f^n_t(\cdot, t), v \rangle \tag{2.27}
\]

for all \( v \in V_n \). We now use the test function \( v = (t-T')\Delta u^n(\cdot, t) \in V_n \) (recall that \( \Delta \) maps \( V_n \) into itself) and integrate from 0 to \( T' \). After some integrations by parts, we obtain

\[
\begin{align*}
&\int_0^{T'} \langle b * \nabla u^n_t(\cdot, t), (t-T')\Delta u^n(\cdot, t) + \nabla u^n(\cdot, t) \rangle \, dt + \int_0^{T'} (t-T') \langle g^n(t), \nabla \Delta u^n(\cdot, t) \rangle \, dt \\
&= \int_0^{T'} \langle b(t)f^n(\cdot, 0) + b * f^n_t(\cdot, t), (t-T')\Delta u^n(\cdot, t) \rangle \, dt. \tag{2.28}
\end{align*}
\]
The absolute value of the integral on the right hand side of this identity can be estimated from above by

\[
\int_0^{T'} (\|b(t)\|_2 f^n(\cdot, 0) + \|b \ast f^n_1(\cdot, t)\|_2) (T' - t) \|\Delta u^n(\cdot, t)\| dt
\]

\[
\leq C(\epsilon, T') \|b\|_{L^1(0, T')} \|f^n(\cdot, 0)\|^2 + \|f^n_1(\cdot, 0)\|^2 \epsilon \int_0^{T'} (T' - t) \|\Delta u^n(\cdot, t)\|^2 dt
\]

for any \( \epsilon > 0 \), since \( b \in L^2(0, T') \), due to the restriction \( \alpha > \frac{1}{2} \). To estimate the first integral on the left hand side of (2.28) in absolute value, we define

\[
e_\alpha(t) = \frac{t^{\alpha/2 - 1}}{\Gamma(\frac{\alpha}{2})}
\]

such that \( e_\alpha \ast e_\alpha = b_\alpha \) and thus \( b = e_\alpha \ast e_\alpha + d \ast e_\alpha \ast e_\alpha \). Then the integral can be rewritten as

\[
\cdots = \int_0^{T'} (e_\alpha + d \ast e_\alpha) \ast \nabla u^n(\cdot, t), \int_t^{T'} e_\alpha(\tau - t) ((\tau - T') \nabla u^n(\cdot, \tau) + \nabla u^n(\cdot, \tau)) d\tau dt.
\]

Now \( (e_\alpha + d \ast e_\alpha) \ast \nabla u^n(\cdot, t) = \frac{d}{dt}(e_\alpha + d \ast e_\alpha) \ast u^n(\cdot, t) \)

(2.30)

with \( u^n(\cdot, t) = \nabla u^n(\cdot, t) - \nabla u^n(\cdot, 0) \), and since \( \nabla u^n \in H^{1+\frac{\alpha}{2}}(0, T', L^2(\Omega)) \), due to (2.25), with an estimate in this space that does not depend on \( n \), it follows that \( (e_\alpha + d \ast e_\alpha) \ast w^n(\cdot, t) \) is in \( H^{\alpha+\frac{1}{2}}(0, T', L^2(\Omega)) \). Its derivative is therefore in \( H^{\alpha-\frac{1}{2}}(0, T', L^2(\Omega)) \), with an estimate that does not depend on \( n \) ([6]). Similarly,

\[
\int_t^{T'} e_\alpha(\tau - t) ((\tau - T') \nabla u^n(\cdot, \tau) + \nabla u^n(\cdot, \tau)) d\tau
\]

\[
= \frac{d}{dt} \left( \int_0^{T' - t} e_\alpha(\sigma)((t - T' + \sigma) \nabla u^n(\cdot, t + \sigma)) d\sigma \right)
\]

(2.31)

\[
= \frac{d}{dt} \left( (t - T') \int_0^{T' - t} e_\alpha(\sigma) \nabla u^n(\cdot, t + \sigma) d\sigma \right) + \frac{d}{dt} \int_0^{T' - t} \sigma e_\alpha(\sigma) \nabla u^n(\cdot, t + \sigma) d\sigma.
\]

As above, both integrals are in \( H^{\alpha+\frac{1}{2}}(0, T', L^2(\Omega)) \) as function of \( t \), and the entire expression is therefore bounded a priori in \( H^{\alpha-\frac{1}{2}}(0, T', L^2(\Omega)) \). The first integral on the left hand side of (2.28) can now be bounded independently of \( n \).

Finally consider the second integral on the left hand side of (2.28). Fix \( t \) and set \( w = u^n(\cdot, t) \), then we want to estimate

\[
\int_{\Omega} g(\nabla w)^2 \nabla w \cdot \nabla (\Delta w) = \int_{\Omega} \sum_{i,j} g(|\nabla w|^2) w_{;i} w_{;i,j,j}.
\]
The following arguments are taken from [2], where more details can be found. After an integration by parts, the integral equals

\[
- \int_{\Omega} \sum_{i,j} w_{i,j}(g(|\nabla w|^2)w_{i,j}) + \int_{\partial \Omega} \sum_{i,j} g(|\nabla w|^2)w_{i,j}w_{i,j}\nu_j .
\]

(2.32)

Since the function \( w \) vanishes on the hypersurface \( \partial \Omega \), its gradient is perpendicular to it and therefore parallel to the unit normal \( \nu \), i.e. \( \nabla w = \nu \partial_{\nu} w \), where \( \partial_{\nu} w = \nabla w \cdot \nu \). Also, at any point of this hypersurface, \( \Delta w = \Delta_{\partial \Omega} w + (N-1)H \partial_{\nu} w + \partial_{\nu}^2 w \), where \( \Delta_{\partial \Omega} \) is the induced Laplacian, \( H \) is the mean curvature of \( \partial \Omega \) with respect to \( \nu \) and \( \partial_{\nu}^2 w = \nu^T \nabla^2 w \nu \) ([15]). Since \( w \) vanishes on \( \partial \Omega \), this means that \( \Delta w = (N-1)H \partial_{\nu} w + \partial_{\nu}^2 w \). Since also \( \Delta w \in V_n \) vanishes on \( \partial \Omega \), we obtain at each boundary point

\[
\sum_{i,j} w_{i,j} w_{i,j} \nu_j = \partial_{\nu} w \partial_{\nu}^2 w = -(N-1)H|\partial_{\nu} w|^2 = -(N-1)H|\nabla w|^2 .
\]

Therefore the boundary integral reduces to a term containing only first derivatives that can be estimated from above by means of a trace theorem:

\[
\int_{\partial \Omega} \sum_{i,j} g(|\nabla w|^2)w_{i,j}w_{i,j}\nu_j = -\int_{\partial \Omega} (N-1)Hg(|\nabla w|^2)\nabla w \cdot \nu
\]

\[
\leq \epsilon ||\nabla \left( \sqrt{g(|\nabla w|^2)} \nabla w \right)||^2 + C ||g(|\nabla w|^2)\nabla w||_{L^2}
\]

\[
\leq \epsilon ||\nabla \left( \sqrt{g(|\nabla w|^2)} \nabla w \right)||^2 + C_1(1 + \int_{\Omega} G(|\nabla w|^2))
\]

(2.33)

for any \( \epsilon > 0 \) with suitable \( C, C_1 > 0 \). Here Lemma 2 was used to arrive at the last line.

The integral over \( \Omega \) in (2.32) is

\[
- \int_{\Omega} \sum_{i,j} w_{i,j}(g(|\nabla w|^2)w_{i,j})
\]

\[
= -\int_{\Omega} \sum_{i,j} \left( g(|\nabla w|^2)w_{i,j}w_{i,j} + 2\sum_k g'(|\nabla w|^2)w_{i,j}w_{i,k}w_{j,k} \right)
\]

\[
= -\int_{\Omega} \left( g(r^2)D^2 + 2g'(r^2)d^2 \right)
\]

with the abbreviations \( r^2 = |\nabla w|^2 \), \( D^2 = |\nabla^2 w|^2 = \sum_{i,j} w_{i,j}w_{i,j}, d^2 = |\nabla^2 w \nabla w|^2 = \sum_{i,j,k} w_{i,j}w_{i,j}w_{j,k}w_{k} \). On the other hand,

\[
|\nabla \left( \sqrt{g(|\nabla w|^2)} \nabla w \right)|^2 = g(r^2)D^2 + 2g'(r^2)d^2 + \frac{(g'(r^2))^2r^2d^2}{g(r^2)} .
\]
Using (2.6), this term can be estimated from above by a constant multiple of 
\( g(r^2)D^2 + 2g'(r^2)d^2 \), and therefore

\[-\int \sum_{i,j}(g(|\nabla w|^2)w_{i,j})w_{i,j} \leq -\delta \| \nabla \left( \sqrt{g(|\nabla w|^2)} \nabla w \right) \|_{L^2}^2 \]

with some fixed \( \delta > 0 \). Combining this last estimate and (2.33) with \( \epsilon = \delta/2 \), one finally obtains

\[
\int \Omega g(|\nabla w|^2)\nabla w \cdot \nabla (\Delta w) \leq -\epsilon \int \Omega |\nabla \left( \sqrt{g(|\nabla w|^2)} \nabla w \right) |^2 + C(1 + \int \Omega G(|\nabla w|^2)),
\]

for some \( \epsilon, C > 0 \). Using estimate (2.26), rearranging terms in (2.28), and combining all estimates, we thus deduce that

\[
\int_0^T (T - t) \| \nabla \left( \sqrt{g(|\nabla u^n|^2)} \nabla u^n(\cdot, t) \right) \|_{L^2}^2 \, dt \leq C
\]

independent of \( n \). In particular,

\[
\int_0^T \| \nabla^2 u^n(\cdot, t) \|_{L^2}^2 \, dt \leq C.
\]

**Passage to the Limit.** Estimates (2.23) and (2.36) imply that the set of all \( \nabla u^n \) is in a relatively compact subset of \( L^2(0, T; L^2(\Omega)) \) ([12]). We can therefore extract a subsequence \( u^{M(n)} \) such that

\[
\nabla u^{M(n)} \to \nabla u \quad \text{strongly in } L^2(0, T; L^2(\Omega))
\]

and pointwise a.e. on \( \Omega \times (0, T) \)

\[
\nabla^2 u^{M(n)} \to \nabla^2 u \quad \text{weakly in } L^2(0, T; L^2(\Omega))
\]

\[
u^{M(n)} \to u_t \quad \text{weakly-* in } L^\infty(0, T; L^2(\Omega))
\]

Also, \( u \in H^{1,\infty}([0, T], H^1(\Omega)) \) by (2.25). Estimate (2.26) and Lemma 2 below imply that the \( g^n = g(|\nabla u^n|^2)\nabla u^n \) are in a bounded subset of \( L^\infty(0, T; L^p(\Omega)) \) for some \( p > 1 \). Therefore the subsequence can be chosen such that also

\[
g(|\nabla u^{M(n)}|^2)\nabla u^{M(n)} \to \xi \quad \text{weakly-* in } L^\infty(0, T; L^p(\Omega))
\]

for some \( \xi \in L^\infty(0, T; L^p(\Omega)) \). Since also \( g(|\nabla u^{M(n)}|^2)\nabla u^{M(n)} \to g(|\nabla u|^2)\nabla u \) pointwise a.e., Lemma 3 implies that \( \xi = g(|\nabla u|^2)\nabla u \). Passing now to the limit in equation (2.10), we see that

\[
\langle u_t, v \rangle + \langle a * g(|\nabla u|^2)\nabla u, \nabla v \rangle = \langle f, v \rangle
\]

for all \( v \in \cup_n V_n \) and therefore by density for all \( v \in C_0^\infty(\Omega) \). Therefore, \( u \) is a distributional solution of (1.1) on \( \Omega \times [0, T] \).
Additional Regularity. For any differentiable function \( h : [0, \infty) \to \mathbb{R} \) and any sufficiently smooth function \( w : \Omega \to \mathbb{R} \), we compute

\[
\nabla (h(|\nabla w|^2)\nabla w) = \left( h(|\nabla w|^2) w_{i,j} + 2h'(|\nabla w|^2) \sum_k w_{k,j} w_{i,k} \right)_{i,j}
\]

Using again the notation \( r^2 = |\nabla w|^2 \), \( D^2 = |\nabla^2 w|^2 \), \( d^2 = |\nabla^2 w\nabla w|^2 \) and setting \( g = h \) we obtain after estimating

\[
| \nabla (g(|\nabla w|^2)\nabla w) | \leq C g(r^2) D. \tag{2.37}
\]

With \( h = \sqrt{g} \), we obtain at points where \( g'(|\nabla w|^2) < 0 \)

\[
| \nabla \left( \sqrt{g(|\nabla u|^2)} \nabla u \right) |^2 = g(r^2) D^2 + 2g'(r^2)d^2 + \frac{(g'(r^2))^2}{g(r^2)} r^2 d^2
\]

\[
\geq g(r^2) D^2 + 2g'(r^2)d^2
\]

\[
\geq g(r^2) D^2 + 2g'(r^2) D^2 r^2
\]

\[
\geq \frac{\epsilon}{2 + \epsilon} g(r^2) D^2 \tag{2.38}
\]

from (2.6). The same estimate holds trivially at points where \( g'(|\nabla w|^2) \geq 0 \). Now let \( 1 < q < 2 \), to be fixed later. Then by (2.37), (2.38), and Hölder’s inequality

\[
\| \nabla (g(|\nabla w|^2)\nabla w) \|^q_{L^q} \leq C \int_{\Omega} \left( g(r^2) D \right)^q
\]

\[
= C \int_{\Omega} g(r^2)^{q/2} D^{q/2} g(r^2)^{q/2}
\]

\[
\leq C \left( \int_{\Omega} g(r^2)^{2q/2} \right)^{q/2} \left( \int_{\Omega} g(r^2)^{q/(2-q)} \right)^{1-q/2}
\]

\[
\leq C \left( \int_{\Omega} |\nabla \left( \sqrt{g(|\nabla w|^2)}\nabla w \right) |^2 \right)^{q/2} \left( \int_{\Omega} g(r^2)^{q/(2-q)} \right)^{1-q/2}. \tag{2.39}
\]

If \( q = 1 + \frac{4}{1+2\delta} \in (1, 2) \) with \( \delta \) as in Lemma 2, we can estimate further

\[
\int_{\Omega} g(r^2)^{q/(2-q)} \leq C \left( 1 + \int_{\Omega} G(r^2) \right).
\]

We use these estimates for \( w = u(\cdot, t) \) for any fixed \( t \) and infer that

\[
\| \nabla (g(|\nabla u|^2)\nabla u(\cdot, t)) \|^2_{L^q} \leq C \| \nabla \left( \sqrt{g(|\nabla u|^2)}\nabla u(\cdot, t) \right) \|^2
\]

The right hand side is integrable over \([0, T]\), and therefore \( \nabla (g(|\nabla u|^2)\nabla u) \in L^2(0, T; L^4(\Omega)) \) with this \( q \).
Lemma 2. If $g : [0, \infty) \to \mathbb{R}$ satisfies (2.6), then there exist $\delta, C > 0$ such that for all $r \geq 0$

$$|g(r^2)r^{1+\delta}| \leq C(1 + G(r^2)),
$$
$$g(r^2)r^2 \leq C(1 + G(r^2)).$$

Proof. A constant $C$ can be found such that both inequalities are true for all $r \leq 1$.

For $r \geq 1$, we have with $\beta$ as in (2.6) and $\delta = \frac{1}{2\beta + 1}$

$$\frac{d}{dr} (g(r^2)r^{1+\delta}) = (1 + \delta)g^\delta(r^2)r^\delta (g(r^2) + 2r^2 g'(r^2))$$

$$\leq Cg^{1+\delta}(r^2)r^\delta$$

$$\leq Cg(r^2)r^{2\delta + \delta}$$

$$\leq Cg(r^2)r = C \frac{d}{dr} G(r^2).$$

An integration implies the first estimate. The second estimate follows from (2.6) in the form

$$g'(r^2)r^3 + g(r^2)r \leq (\beta + 1)g(r^2)r$$

by integrating.

Lemma 3. Let $\Omega \subset \mathbb{R}^N$ be measurable and bounded and let $z_n$ be a sequence in $L^1(\Omega)$ satisfying

$$z_n(x) \to \xi(x) \quad \text{for a.e. } x \in \Omega$$

$$z_n \to \zeta \quad \text{weakly in } L^1(\Omega).$$

Then $\zeta = \xi$ almost everywhere.

Proof. The function $\xi$ must be finite almost everywhere, otherwise the $z_n$ cannot converge weakly. Let $\epsilon > 0$. By Egorov’s Theorem, there exists a subset $\Omega_{\epsilon} \subset \Omega$ such that $z_n \to \xi$ uniformly on $\Omega_{\epsilon}$, $|\xi| \leq \epsilon^{-1}$ on $\Omega_{\epsilon}$, and $\text{meas}(\Omega - \Omega_{\epsilon}) < \epsilon$. Set $\varphi(x) = \text{sign}(\xi(x) - \zeta(x)) \cdot I_{\Omega_{\epsilon}}(x)$, then

$$\int_{\Omega_x} |\xi - \zeta| = \int_{\Omega_x} (\xi - z_n) \varphi_M - \int_{\Omega_x} (\zeta - z_n) \varphi_M.$$

As $n \to \infty$, both terms go to zero by assumption and because of the Dominated Convergence Theorem. Therefore $\xi = \zeta$ a.e. on $\Omega_{\epsilon}$. As $\epsilon \to 0$, the conclusion follows.

3. Extensions and Counterexamples

The question arises whether a result like Theorem 1 is still true if $0 < \alpha \leq \frac{1}{2}$. The short answer is that the theorem is no longer correct for this range of $\alpha$’s. Below, we give some counterexamples to show this. However, these counterexamples are essentially related to regularity properties for linear equations. A suitable modification can still be expected to hold for $0 < \alpha \leq \frac{1}{2}$. For $\alpha = \frac{1}{2}$, this is the content of the next corollary. For nonlinear problems with $0 < \alpha < \frac{1}{2}$, the question of existence of solutions $u$ for which $u(\cdot, t) \in H^2(\Omega)$ for $t > 0$ remains open, except for the result in [8].
Corollary 4. Let $\alpha = \frac{1}{2}$, let the assumptions of Theorem 1 hold, and assume that for some $r > 1$
\[ \int_0^T \| f_t(\cdot,t) \|^r \, dt < \infty. \]  
(3.1)
a) If $f(\cdot,0) \in H_0^1(\Omega)$, then the statement of Theorem 1 remains true.
b) If $f(\cdot,0) \in L^2(\Omega)$, then a solution $u$ of (2.1) exists for which
\[ \int_0^T t \| u(\cdot,t) \|^2_{H^2} \, dt < \infty \]  
(3.2)

and $u \in W^{1,\infty}([0,T], L^2(\Omega)) \cap H^{\frac{3}{2}}([0,T], H_0^1(\Omega))$.

Outline of Proof. We use the same approximation as in the proof of Theorem 1 and derive estimates (2.23) – (2.26) in the same way. In case a), we can assume that the $f^n(\cdot,0)$ are all in $H_0^1(\Omega)$. We can then carry out a priori estimate II by integrating by parts in the right hand side of (2.28), obtaining
\[ \int_0^{T'} (\langle -b(t) \nabla f^n(\cdot,0), (t-T') \nabla u^n(\cdot,t) \rangle + \langle b * f^n(\cdot,t), (t-T') \Delta u^n(\cdot,t) \rangle) \, dt. \]

Since $b \in L^q(0,T')$ for all $q < 2$ and due to (2.25), this can be estimated by
\[ C(b, \| \nabla f^n(\cdot,0) \|) + C(\epsilon, T') \| b \|_{L^q(0,T')} \| f^n_t \|^2_{L^2(0,T'; L^2)} + \epsilon \int_0^{T'} (T' - t) \| \Delta u^n(\cdot,t) \|^2 \, dt \]
with $q = \frac{2r}{3r - 2}$. No change is required for the remaining arguments, and the conclusion follows as before.

In case b), we proceed as before and obtain approximating solutions that satisfy (2.23) – (2.26). We then use the test function $v = t(t-T') \Delta u^n(\cdot,t)$ in the arguments that follow (2.27). Instead of (2.29), we then obtain an estimate with the right hand side
\[ C_0(\epsilon, T', b) \| f^n(\cdot,0) \|^2 + C_1(\epsilon, T') \| b \|_{L^q(0,T')} \| f^n_t \|^2_{L^2(0,T'; L^2)} + \epsilon \int_0^{T'} t(T' - t) \| \Delta u^n(\cdot,t) \|^2 \, dt. \]

The other terms in the identity that is analogous to (2.28) can be treated as before. The additional factor $t$ in the test function $v$ leads to some longer version of (2.31); we leave the details to the interested reader. In the passage to the limit, we now only assert that
\[ \nabla^2 u_{M(n)}^\epsilon \to \nabla^2 u \text{ weakly in } L^2(\delta,T; L^2(\Omega)) \]
for all $\delta > 0$. This is sufficient to conclude the proof.

To construct counterexamples that show that the conditions in these results are sharp, several facts about scalar integro-differential equations will be used.
Lemma 5. For $0 < \alpha < 1$ fixed and for arbitrary $\lambda > 0$, let $u_\lambda, v_\lambda : [0, 1] \to \mathbb{R}$ be the solutions of

\[
u'_\lambda(t) + \lambda a_\alpha * u_\lambda = 0 \quad (0 \leq t \leq 1), \quad u_\lambda(0) = 1
\]

\[
u'_\lambda(t) + \lambda a_\alpha * v_\lambda = 1 \quad (0 \leq t \leq 1), \quad v_\lambda(0) = 0
\]

where $a_\alpha$ is as in (2.4). Then

\[
sup_{\lambda} \int_0^1 \lambda^2 t^\gamma u_\lambda^2(t) \, dt < \infty \quad \text{iff} \quad 2\alpha + \gamma > 3 \quad \text{(3.3)}
\]

\[
sup_{\lambda} \int_0^1 \lambda^2 t^\gamma v_\lambda^2(t) \, dt < \infty \quad \text{iff} \quad 2\alpha + \gamma > 1. \quad \text{(3.4)}
\]

Proof. It is known that

\[
u_\lambda(t) = E_{2-\alpha}(-\lambda t^{2-\alpha}), \quad v_\lambda(t) = \int_0^t E_{2-\alpha}(-\lambda s^{2-\alpha}) \, ds
\]

where

\[
E_\sigma(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\sigma k)}
\]

is the Mittag-Leffler function ([4], [5]). Therefore the estimates (3.3) and (3.4) follow immediately for, say, $0 \leq \lambda \leq K$, where $K$ is any fixed finite value, and we only have to prove them for large values of $\lambda$. If $0 < \sigma < 2$, then

\[
E_\sigma(-z) = \frac{z^{-1}}{\Gamma(1-\sigma)} + O(z^{-2})
\]

as $z \to \infty$ ([4]). Therefore there exist constants $c_1, c_2, c_3, c_4 > 0$ such that

\[
|u_\lambda(t)| \leq c_1 \quad (\lambda t^{2-\alpha} \leq c_2)
\]

\[
c_3 \lambda^{-1} t^{\alpha-2} \leq |u_\lambda(t)| \leq c_4 \lambda^{-1} t^{\alpha-2} \quad (\lambda t^{2-\alpha} \geq c_2).
\]

The equation for $u_\lambda$ shows that the Laplace transform of $u_\lambda$ is given by

\[
u_\lambda(s) = \frac{1}{s + \lambda s^{\alpha-1}}
\]

which implies $\int_0^\infty u_\lambda(t) \, dt = 0$. Therefore, with the same constants $c_1, \ldots, c_4$ we can also assume that

\[
|v_\lambda(t)| \leq c_1 \quad (\lambda t^{2-\alpha} \leq c_2)
\]

\[
c_3 \lambda^{-1} t^{\alpha-1} \leq |v_\lambda(t)| \leq c_4 \lambda^{-1} t^{\alpha-1} \quad (\lambda t^{2-\alpha} \geq c_2).
\]
Suppose now that $2\alpha + \gamma > 3$. Then

$$
\int_0^1 \lambda^{2\gamma} u^2_\lambda(t) \, dt = \int_0^1 \lambda^{2\gamma} u^2_\lambda(t) \, dt + \int_0^1 \lambda^{2\gamma} u^2_\lambda(t) \, dt
\leq \int_0^1 \lambda^{2c_\lambda^{2\gamma} 2\alpha - 4} dt
\leq C\lambda^{2-\frac{2\alpha}{2-\alpha}} + C
\leq C < \infty
$$

independently of $\lambda$. The same argument, using (3.6) instead of (3.5), shows that

$$
\sup_{\lambda} \int_0^1 \lambda^{2\gamma} u^2_\lambda(t) \, dt < \infty
$$

if $2\alpha + \gamma > 1$. Suppose next that $2\alpha + \gamma \leq 3$. Then

$$
\int_0^1 \lambda^{2\gamma} u^2_\lambda(t) \, dt \geq \int_0^1 \lambda^{2\gamma} u^2_\lambda(t) \, dt \geq \int_0^1 c_\lambda^{2\gamma + 2\alpha - 4} dt
$$

and the last integral does not remain bounded as $\lambda \to \infty$. Thus the supremum in (3.3) is infinite if $2\alpha + \gamma \leq 3$. Using (3.6), it follows that the supremum in (3.4) is infinite if $2\alpha + \gamma \leq 1$.

It is also possible to show that

$$
\sup_{\lambda} \int_0^1 \lambda^{2\delta} u^2_\lambda(t) \, dt = \infty
$$

if $\delta > 1$ and that this supremum is finite for $\delta < 1$ iff in addition $2\delta(2 - \alpha) \leq \gamma + 1$. Similar statements hold for the supremum that can be formed with the $v_\lambda$. Note that the supremum is finite if $2\delta(2 - \alpha) = \gamma + 1$ for $\delta < 1$, but infinite in the case $\delta = 1$.

**Counterexamples.** Consider the problems

$$
\frac{\partial u}{\partial t}(x, t) - \int_0^t a_\alpha(t - s) \Delta u(x, s) \, ds = F(x, t) \quad (0 < t \leq 1, \ x \in \Omega) \quad (3.7)
$$

with

$$
F(x, t) = f_0(x) + \int_0^t f_1(x, s) \, ds
$$

and initial data $u(x, 0) = 0$ in $\Omega$. As before, let $(\varphi_i)_i$ be the set of eigenfunctions of $-\Delta$ with zero Dirichlet boundary conditions, with eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$.

Suppose first that $f_0 = \sum_i c_i \varphi_i \in L^2(\Omega)$ and $f_1 = 0$. The solution of (3.7) then is given by

$$
u(\cdot, t) = \sum_{i=1}^\infty c_i v_{\lambda_i}(t) \varphi_i$$
and thus
\[ \|u(\cdot, t)\|_{H^2}^2 \geq C \|\Delta u(\cdot, t)\|^2 = C \sum_{i=1}^{\infty} c_i^2 \lambda_i^2 |v_{\lambda_i}(t)|^2. \]  
(3.8)

If \( \alpha \leq \frac{1}{2} \), then there exist \( c_i \) such that \( f_0 \in L^2(\Omega) \) but \( \int_0^1 \|u(\cdot, t)\|_{H^2}^2 \, dt = \infty \), according to Lemma 5. However, if \( \gamma > 1 - 2\alpha \geq 0 \), then \( \int_0^1 t^\gamma \|u(\cdot, t)\|_{H^2}^2 \, dt < \infty \) for any such \( f_0 \).

Suppose next that \( f_0 = 0 \) and \( f_1(\cdot, t) = h_i(t) \varphi_i \). Then \( u(\cdot, t) = \int_0^1 h_i(t - s)v_{\lambda_i}(s) \, ds \cdot \varphi_i \). Pick \( h_i(t) = k_i h(k_i t) \), where \( h \geq 0 \) is smooth, supported on \([0, 1] \), \( \int_0^1 h(t) \, dt = 1 \), and \( k_i > 1 \) is so large that
\[ \int_0^1 |v_{\lambda_i} \ast h_i(t)|^2 \, dt \geq \frac{1}{2} \int_0^1 |v_{\lambda_i}(t)|^2 \, dt. \]

It follows that
\[ \|u(\cdot, t)\|_{H^2}^2 \geq C \|\Delta u(\cdot, t)\|^2 = C \lambda_i^2 \int_0^1 |v_{\lambda_i} \ast h_i(t)|^2 \, dt \geq C \lambda_i^2 \int_0^1 |v_{\lambda_i}(t)|^2 \, dt = M_i. \]
(3.9)

If \( \alpha \leq \frac{1}{2} \), then \( \lim \sup M_i = \infty \). On the other hand,
\[ \int_0^1 \|f_1(\cdot, t)\| \, dt = \int_0^1 h_i(t) \, dt = 1. \]

Choose \( c_i \) such that \( \sum_{i=1}^{\infty} |c_i| < \infty \) and \( \sum_{i=1}^{\infty} c_i^2 M_i = \infty \) and set \( f_1(\cdot, t) = \sum_{i=1}^{\infty} c_i h_i(t) \varphi_i \), then
\[ \int_0^1 \|f_1(\cdot, t)\| \, dt < \infty \quad \text{and} \quad \int_0^T \|u(\cdot, t)\|_{H^2}^2 \, dt = \infty. \]
(3.10)

These arguments show that Theorem 1 does not extend to the case \( \alpha \leq \frac{1}{2} \) without strengthening some assumptions or weakening some conclusions, as in Corollary 4. Similar constructions can be used to show that the assumption \( u_0 \in H^2(\Omega) \) can only be weakened slightly if solutions are still to belong to \( L^2(0, T; H^2(\Omega)) \).

**Acknowledgment.** The author would like to thank the referee for several useful comments.

**References**


**Department of Mathematics**
**Georgetown University**
**Washington, D.C. 20057**

*E-mail address: engler@guvax.acc.georgetown.edu*