Multiple positive solutions for equations involving critical Sobolev exponent in $\mathbb{R}^N$ *

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Abstract

This article concerns with the problem
\[-\text{div}(\nabla u^{m-2} \nabla u) = \lambda h u^q + u^{m^*-1}, \quad \text{in} \quad \mathbb{R}^N.\]

Using the Ekeland Variational Principle and the Mountain Pass Theorem, we show the existence of $\lambda^* > 0$ such that there are at least two non-negative solutions for each $\lambda \in (0, \lambda^*)$.

1 Introduction

In this work, we study the existence of solutions for the problem

\[
\begin{aligned}
-\Delta_m u &= \lambda h u^q + u^{m^*-1}, & & \text{in } \mathbb{R}^N \\
\end{aligned}
\]

where $\Delta_m u = \text{div}(\nabla u^{m-2} \nabla u)$, $\lambda > 0$, $N > m \geq 2$, $m^* = \frac{Nm}{N-m}$, $0 < q < m-1$, $h$ is a nonnegative function with $L^\Theta(\mathbb{R}^N)$ with $\Theta = \frac{Nm}{N-(q+1)(N-m)}$, and

\[D^{1,m}(\mathbb{R}^N) = \{ u \in L^{m^*}(\mathbb{R}^N) \mid \frac{\partial u}{\partial x_i} \in L^m(\mathbb{R}^N) \}\]

endowed with the norm $\| u \| = (\int |\nabla u|^m)^{1/m}$.

The case $q = 0$, $m = 2$ was studied by Tarantello [20], and a more general case with $m \geq 2$ by Cao, Li & Zhou [5]. In these two references, [5] and [20], it is proved that (P) has multiple solutions. In the case $m = 2$, $h \in L^p(\mathbb{R}^N)$ with $p_1 \leq p \leq p_2$ and $1 < q < 2^* - 1$, Pan [18] proved the existence of a positive solution for (P). In the more general case, $m \geq 2$, $h \in L^p(\mathbb{R}^N)$, Gonçalves & Alves [10] proved the existence of a positive solution for (P).


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By a solution to (P), we mean a function \( u \in D^{1,m}(\mathbb{R}^N) \), \( u \geq 0 \) and \( u \neq 0 \) satisfying
\[
\int |\nabla u|^{m-2} \nabla u \nabla \Phi = \lambda \int hu^q \Phi + \int u^{m^*-1} \Phi, \quad \forall \Phi \in D^{1,m}(\mathbb{R}^N).
\]
Hereafter, \( f, D^{1,m}, L^p \) and \( |.|_p \) stand for \( \int \mathbb{R}^N, D^{1,m}(\mathbb{R}^N), L^p(\mathbb{R}^N) \) and \( |.|_{L^p} \) respectively.

In the search of solutions we apply minimizing arguments to the energy functional
\[
I(u) = \frac{1}{m} \int |\nabla u|^m - \frac{\lambda}{q+1} \int h (u^+)^{q+1} - \frac{1}{m^*} \int (u^+)^{m^*}
\]
associated to (P), where \( u^+(x) = \max\{u(x), 0\} \). Note that the condition \( h \in L^\Theta \) implies that \( I \in C^1(D^{1,m}, \mathbb{R}) \).

To show the existence of at least two critical points of the energy functional, we shall use the Ekeland Variational Principle [8], and the Mountain Pass Theorem of Ambrosetti & Rabinowitz [2] without the Palais-Smale condition. Using the Ekeland Variational Principle, we obtain a solution \( u_1 \) with \( I(u_1) < 0 \), and by the Mountain Pass Theorem we prove the existence of a second solution \( u_2 \) with \( I(u_2) > 0 \). Techniques for finding the solutions \( u_1 \) and \( u_2 \) are borrowed from Cao, Li & Zhou [5]. Then we combine these techniques with arguments developed by Chabrowski [6], Noussair, Swanson & Jianfu [17], Jianfu & Xiping [12], Azorero & Alonzo [9], Gonçalves & Alves [10] and Alves, Gonçalves & Miyagaki [1] to obtain the following result

**Theorem 1** There exists a constant \( \lambda^* > 0 \), such that (P) has at least two solutions, \( u_1 \) and \( u_2 \), satisfying
\[
I(u_1) < 0 < I(u_2) \quad \forall \lambda \in (0, \lambda^*).
\]

## 2 Preliminary Results

In this section we establish some results needed for the proof of Theorem 1.

**Definition.** A sequence \( \{u_n\} \subset D^{1,m} \) is called a \((PS)_c\) sequence, if \( I(u_n) \to c \) and \( I'(u_n) \to 0 \).

**Lemma 1** If \( \{u_n\} \) is a \((PS)_c\) sequence, then \( \{u_n\} \) is bounded, and \( \{u_n^+\} \) is a \((PS)_c\) sequence.
Proof. Using the hypothesis that \( \{ u_n \} \) is a \((PS)_c\) sequence, there exist \( n_0 \) and \( M > 0 \) such that
\[
I(u_n) - \frac{1}{m^*} I'(u_n)u_n \leq M + \| u_n \| \quad \forall n \geq n_0 .
\]
(2)

Now, using (1) and the Hölder’s inequality, we have
\[
I(u_n) - \frac{1}{m^*} I'(u_n)u_n \geq \frac{1}{N} \| u_n \|^m + c_1 \| u_n \|^{q+1}
\]
(3)
where \( c_1 \) is a constant depending of \( N, m, q, \| h \|_\Theta \) and \( \Theta \). It follows from (2) and (3) that \( \{ u_n \} \) is bounded. Now, we shall show that \( \{ u_n \} \) is a also \((PS)_c\) sequence. Since \( \{ u_n \} \) is bounded, the sequence \( u_n^- = u_n - u_n^+ \) is also bounded. Then
\[
I'(u_n)u_n^- \rightarrow 0
\]
and we conclude that
\[
\| u_n^- \| \rightarrow 0 .
\]
(4)

From (4) we achieve that
\[
\| u_n \| = \| u_n^+ \| + o_n(1) .
\]
(5)

Therefore, by (4) and (5)
\[
I(u_n) = I(u_n^+) + o_n(1)
\]
and
\[
I'(u_n) = I'(u_n^+) + o_n(1)
\]
which consequently implies that \( \{ u_n^+ \} \) is a \((PS)_c\) sequence. \( \square \)

From Lemma 1, it follows that any \((PS)_c\) sequence can be considered as a sequence of nonnegative functions.

Lemma 2 If \( \{ u_n \} \) is a \((PS)_c\) sequence with \( u_n \rightharpoonup u \) in \( D^{1,m} \), then \( I'(u) = 0 \), and there exists a constant \( M > 0 \) depending of \( N, m, q, \| h \|_\Theta \) and \( \Theta \), such that
\[
I(u) \geq -M \lambda^\Theta .
\]

Proof. If \( \{ u_n \} \) is a \((PS)_c\) sequence with \( u_n \rightharpoonup u \), using arguments similar to those found in [10], [12] and [17], we can obtain a subsequence, still denoted by \( u_n \), satisfying
\[
u_n(x) \rightharpoonup u(x) \text{ a.e. in } \mathbb{R}^N
\]
(6)
\[\nabla u_n(x) \rightharpoonup \nabla u(x) \text{ a.e. in } \mathbb{R}^N
\]
(7)
\[u(x) \geq 0 \text{ a.e. in } \mathbb{R}^N .
\]
(8)
From (6), (7) and using the hypothesis that \( \{u_n\} \) is bounded in \( D^{1,m} \), we get

\[
I'(u) = 0 ,
\]

which in implies \( I'(u)u = 0 \), and

\[
\|u\|^m = \lambda \int hu^{q+1} + \int u^{m^*}.
\]

Consequently

\[
I(u) = \lambda \left( \frac{1}{m} - \frac{1}{q+1} \right) \int hu^{q+1} + \frac{1}{N} \int u^{m^*}.
\]

Using Hölder and Young Inequalities, we obtain

\[
I(u) \geq -\frac{1}{N} |u|^{m^*} - M\lambda^\Theta + \frac{1}{N} |u|^{m^*} = -M\lambda^\Theta
\]

where \( M = M(N, m, q, \Theta, \|h\|_\Theta) \).

For the remaining of this article, we will denote by \( S \) the best Sobolev constant for the imbedding \( D^{1,m} \hookrightarrow L^{m^*} \).

**Lemma 3** Let \( \{u_n\} \subset D^{1,m} \) be a \( (PS)_c \) sequence with

\[
c < \frac{1}{N} S^{N/m} - M\lambda^\Theta,
\]

where \( M > 0 \) is the constant given in Lemma 2. Then, there exists a subsequence \( \{u_{n_j}\} \) that converges strongly in \( D^{1,m} \).

**Proof** By Lemmas 1 and 2, there is a subsequence, still denoted by \( \{u_n\} \) and a function \( u \in D^{1,m} \) such that \( u_n \rightharpoonup u \). Let \( w_n = u_n - u \). Then by a lemma in Brezis & Lieb [3], we have

\[
\|w_n\|^m = \|u_n\|^m - \|u\|^m + o_n(1) \quad \text{(10)}
\]

\[
\|w_n\|^{m^*} = |u_n|^{m^*} - |u|^{m^*} + o_n(1). \quad \text{(11)}
\]

Using the Lebesgue theorem (see Kavian [13]), it follows that

\[
\int hu_n^{q+1} \longrightarrow \int hu^{q+1}.
\]

From (10), (11) and (12), we obtain

\[
\|w_n\|^m = |w_n|^{m^*} + o_n(1) \quad \text{(13)}
\]

and

\[
\frac{1}{m} \|w_n\|^m - \frac{1}{m^*} |w_n|^{m^*} = c - I(u) + o_n(1). \quad \text{(14)}
\]
Using the hypothesis that \( \{w_n\} \) is bounded in \( D^{1,m} \), there exists \( l \geq 0 \) such that
\[
\|w_n\|^m \to l \geq 0. \tag{15}
\]
From (13) and (15), we have
\[
|w_n|^{m^*}_{m^*} \to l, \tag{16}
\]
and using the best Sobolev constant \( S \) and recalling that
\[
\|w_n\|^m \geq S \left( \int |w_n|^{m^*} \right)^{m/m^*}, \tag{17}
\]
we deduce from (15), (16) and (17) that
\[
l \geq Sl^{m/m^*}. \tag{18}
\]
Now, we claim that \( l = 0 \). Indeed, if \( l > 0 \), from (18)
\[
l \geq S^{N/m}. \tag{19}
\]
By (14), (15) and (16), we have
\[
\frac{1}{N} l = c - I(u). \tag{20}
\]
From (19), (20) and Lemma 2 we get
\[
c \geq \frac{1}{N} S^{N/m} - M \lambda^\theta,
\]
but this result contradicts the hypothesis. Thus, \( l = 0 \) and we conclude that
\[
u_n \to u \quad \text{in} \quad D^{1,m}.
\]

3 Existence of a first solution (Local Minimization)

**Theorem 2** There exists a constant \( \lambda_1^* > 0 \) such that for \( 0 < \lambda < \lambda_1^* \) Problem (P) has a weak solution \( u_1 \) with \( I(u_1) < 0 \).

**Proof.** Using arguments similar to those developed in [5], we have
\[
I(u) \geq \left( \frac{1}{m} - \epsilon \right) \|u\|^m + o(\|u\|^m) - C(\epsilon)\lambda^{m/(m-(q+1))},
\]
where \( C(\epsilon) \) is a constant depending on \( \epsilon > 0 \). The last inequality implies that for small \( \epsilon \), there exist constants \( \gamma, \rho \) and \( \lambda_1^* > 0 \) such that
\[
I(u) \geq \gamma > 0, \quad \|u\| = \rho, \quad \text{and} \quad 0 < \lambda < \lambda_1^*.
\]
Using the Ekeland Variational Principle, for the complete metric space $\overline{B}_\rho(0)$ with $d(u, v) = ||u - v||$, we can prove that there exists a $(PS)_{\gamma_0}$ sequence $\{u_n\} \subset \overline{B}_\rho(0)$ with

$$\gamma_0 = \inf \{ I(u) \mid u \in \overline{B}_\rho(0) \}.$$ 

Choosing a nonnegative function $\Phi \in D^{1,m} \setminus \{0\}$, we have that $I(t\Phi) < 0$ for small $t > 0$ and consequently $\gamma_0 < 0$.

Taking $\lambda_1^* > 0$, such that

$$0 < \frac{1}{N} S^{N/m} - M \lambda^\Theta \quad \forall \lambda \in (0, \lambda_1^*)$$

from Lemma 3, we obtain a subsequence $\{u_{n_j}\} \subset \{u_n\}$ and $u_1 \in D^{1,m}$, such that

$$u_{n_j} \rightarrow u \quad \text{in} \quad D^{1,m}.$$ 

Therefore,

$$I'(u_1) = 0 \quad \text{and} \quad I(u_1) = \gamma_o < 0,$$

which completes this proof. \qed

4 Existence of a second solution (Mountain Pass)

In this section, we shall use arguments similar to those explored by Cao, Li & Zhou [5], Chabrowski [6], Noussair, Swanson & Jianfu [17], Jianfu & Xiping [12] and Gonçalves & Alves [10] to obtain the following

**Theorem 3** There exists a constant $\lambda_2^* > 0$ such that for $0 < \lambda < \lambda_2^*$ Problem (P) has a weak solution $u_2$ with $I(u_2) > 0$.

**Proof.** By arguments found in [5] and [10], we can prove that there exists $\delta_1 > 0$ such that for all $\lambda \in (0, \delta_1)$, the functional $I$ has the Mountain Pass Geometry, that is:

(i) There exist positive constants $r, \rho$ with $I(u) \geq r > 0$ for $||u|| = \rho$

(ii) There exists $e \in D^{1,m}$ with $I(e) < 0$ and $||e|| > \rho$ .

Then by [16], there exists a $(PS)_{\gamma_1}$ sequence $\{v_n\}$ with

$$\gamma_1 = \inf_{g \in \Gamma} \max_{t \in [0,1]} I(g(t))$$

where

$$\Gamma = \{g \in C([0,1]; D^{1,m}) \mid g(0) = 0 \quad \text{and} \quad g(1) = e\}.$$ 

Using the next claim, which is a variant of a result found in [5], we can complete the proof of this theorem.
Claim. There exists $\lambda_2^* > 0$ such that for the constant $M$ given by Lemma 2,
\[ 0 < \gamma_1 < \frac{1}{N} S^{N/m} - M \lambda^\Theta \quad \forall \lambda \in (0, \lambda_2^*) . \]

Assuming this claim, by Lemma 3 there exists a subsequence $\{v_{n_j}\} \subset \{v_n\}$ and a function $u_2 \in D^{1,m}$ such that $v_{n_j} \to u_2$. Therefore,
\[ I'(u_2) = 0 \quad \text{and} \quad I(u_2) = \gamma_1 > 0 . \]
Which concludes the present proof. \hfill \Box

Verification of the above claim. For $x \in \mathbb{R}^N$, let
\[ \Psi(x) = \left[ \frac{N \left( \frac{N-m}{m-1} \right)^{m-1}}{1 + |x|^{m/(m-1)}} \right] \frac{N-m}{m} . \]
Then it is well known that (see [7] or [19])
\[ \|\Psi\|^m = \|\Psi\|_{m^*}^m = S^{N/m} . \] (21)
Let $\delta_2 > 0$ such that
\[ \frac{1}{N} S^{N/m} - M \lambda^\Theta > 0 \quad \forall \lambda \in (0, \delta_2) . \]
Then from (1) and (21), we have
\[ I(t\Psi) \leq \frac{t^m}{m} S^{N/m} , \]
and there exists $t_0 \in (0, 1)$ with
\[ \sup_{0 \leq t \leq t_0} I(t\Psi) < \frac{1}{N} S^{N/m} - M \lambda^\Theta \quad \forall \lambda \in (0, \delta_2) . \]
Moreover, from (1) and (21), we have
\[ I(t\Psi) = \left( \frac{t^m}{m} - \frac{t^{m^*}}{m^*} \right) S^{N/m} - \frac{\lambda t q + 1}{q + 1} \int h \Psi^{q+1} , \]
and remarking that
\[ \left( \frac{t^m}{m} - \frac{t^{m^*}}{m^*} \right) \leq \frac{1}{N} \quad \forall t \geq 0, \]
we obtain
\[ I(t\Psi) \leq \frac{1}{N} S^{N/m} - \frac{\lambda t q + 1}{q + 1} \int h \Psi^{q+1} . \]
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therefore,

\[ \sup_{t \geq t_0} I(t\psi) \leq \frac{1}{N} S^{N/m} - \frac{\lambda_0^{q+1}}{q+1} \int h\psi^{q+1}. \]

Now, taking \( \lambda > 0 \) such that

\[-\frac{\lambda^{q+1}}{q} \int h\psi^{q+1} < -M \lambda^{\Theta}\]

that is,

\[0 < \lambda < \left( \frac{t_0^{q+1} \int h\psi^{q+1}}{M(q+1)} \right)^{1/(\Theta-1)} = \delta_3\]

we deduce that

\[ \sup_{t \geq t_0} I(t\psi) < \frac{1}{N} S^{N/m} - M \lambda^{\Theta} \quad \forall \lambda \in (0, \delta_3). \]

Choosing \( \lambda_2^* = \min\{\delta_1, \delta_2, \delta_3\} \), we have

\[ \sup_{t \geq t_0} I(t\psi) < \frac{1}{N} S^{N/m} - M \lambda^{\Theta} \quad \forall \lambda \in (0, \lambda_2^*). \]

and consequently

\[0 < \gamma_1 < \frac{1}{N} S^{N/m} - M \lambda^{\Theta} \quad \forall \lambda \in (0, \lambda_2^*)\]

which proves the claim.

**Proof of Theorem 1.** Theorem 1 is an immediate consequence of Theorems 2 and 3.

**Remark.** Using Lemma 3 and the same arguments explored by Azorero & Alonzo, in the case \( 0 < q < p \) [9], we can easily show that for small \( \lambda \) the following problem has infinitely many solutions with negative energy levels.

\[(P)_* \quad -\Delta_m u = \lambda |u|^{q-1} u + |u|^{m^*-2} u, \quad \text{in} \quad \mathbb{R}^N,
\quad u \in D^{1,m}\]

This result is obtained using the concept and properties of genus, and working with a truncation of the energy functional associated with \((P)_*\).
References


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