An $\epsilon$-regularity result for generalized harmonic maps into spheres *

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Abstract

For $m, n \geq 2$ and $1 < p < 2$, we prove that a map $u \in W^{1,p}_{\text{loc}}(\Omega, S^{n-1})$ from an open domain $\Omega \subset \mathbb{R}^m$ into the unit $(n-1)$-sphere, which solves a generalized version of the harmonic map equation, is smooth, provided that $2 - p$ and $[u]_{\text{BMO}(\Omega)}$ are both sufficiently small. This extends a result of Almeida [1]. The proof is based on an inverse Hölder inequality technique.

1 Introduction

For integers $m, n \geq 2$, let $\Omega \subset \mathbb{R}^m$ be an open domain, and let $S^{n-1} \subset \mathbb{R}^n$ denote the $(n-1)$-dimensional unit sphere. Define the space

$$H^1(\Omega, S^{n-1}) = \{ v \in H^1(\Omega, \mathbb{R}^n) : |v| = 1 \text{ almost everywhere} \},$$

and consider the functional

$$E(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx, \quad u \in H^1(\Omega, S^{n-1}).$$

A map $u \in H^1(\Omega, S^{n-1})$ is called a weakly harmonic map, if it is a critical point of $E$, i.e.

$$\frac{d}{dt} \bigg|_{t=0} E\left( \frac{u + t\phi}{|u + t\phi|} \right) = 0$$

for all $\phi \in C_0^\infty(\Omega, \mathbb{R}^n)$. The Euler-Lagrange equation for this variational problem is

$$\Delta u + |\nabla u|^2 u = 0 \quad \text{in } \Omega$$

(in the distributions sense). Denote by $\wedge$ the exterior product $\wedge : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \Lambda_2 \mathbb{R}^n$, then (1.1) is equivalent to

$$\text{div}(u \wedge \nabla u) = 0 \quad \text{in } \Omega.$$
This form of the equation provides a natural extension of the notion of weakly harmonic maps into spheres. Whereas we need a map in $H^1_{\text{loc}}(\Omega, S^{n-1})$ to make any sense of (1.1), the equation (1.2) only requires

$$u \in W^{1,1}_{\text{loc}}(\Omega, S^{n-1}) = \{v \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n): |v| = 1 \text{ almost everywhere}\}.$$  

A map in this space satisfying (1.2) is called a generalized harmonic map.

For $m = 2$, it was proven by Hélein [8, 9], that any weakly harmonic map is smooth (also for more general target manifolds than spheres). For higher dimensions, this is no longer true. Indeed Rivière [13] constructed a weakly harmonic map in three dimensions which is discontinuous everywhere. But there exists an $\epsilon$-regularity result, due to Evans [4] (and to Bethuel [2] for more general targets), which can be stated as follows.

**Theorem 1.1** There exists a number $\epsilon > 0$, depending only on $m$ and $n$, such that any weakly harmonic map $u \in H^1(\Omega, S^{n-1})$ with the property $[u]_{\text{BMO}(\Omega)} \leq \epsilon$ is smooth in $\Omega$.

Here we use the notation

$$[u]_{\text{BMO}(\Omega)} = \sup_{B_r(x_0) \subset \Omega} \int_{B_r(x_0)} |u - \bar{u}_{B_r(x_0)}| \, dx,$$  

where $B_r(x_0)$ denotes the ball in $\mathbb{R}^m$ with centre $x_0$ and radius $r$, and

$$\bar{u}_{B_r(x_0)} = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u \, dx.$$  

Together with the well-known monotonicity formula for so-called stationary weakly harmonic maps, e.g. weakly harmonic maps which satisfy $\frac{d}{dt}|_{t=0} E(u(x+t\psi(x))) = 0$ for all $\psi \in C_c^\infty(\Omega, \mathbb{R}^m)$ (see Price [12]), one concludes that weakly harmonic maps with this property are smooth away from a closed singular set of vanishing $(m-2)$-dimensional Hausdorff measure.

Generalized harmonic maps on the other hand may have singularities even in two dimensions. A typical example is the map $u(x) = x/|x|$ in $\mathbb{R}^2$. For $m = 2$ and for any $p \in [1, 2)$, Almeida [1] even constructed generalized harmonic maps in $W^{1,2}(\Omega, S^1)$ which are nowhere continuous. Nevertheless, there is an $\epsilon$-regularity result for generalized harmonic maps in two dimensions, due to Almeida [1]. (Another proof was given by Ge [6].)

**Theorem 1.2** For $m = 2$, there exists $\epsilon > 0$, depending only on $n$, such that any weakly harmonic map $u \in W^{1,1}_{\text{loc}}(\Omega, S^{n-1})$ with the property $\|\nabla u\|_{L^2(\Omega)} \leq \epsilon$ is smooth in $\Omega$.

Here $\|\cdot\|_{L^2(\Omega)}$ is the norm of the Lorentz space $L^{2,\infty}(\Omega, \mathbb{R}^{m \times n})$. (For a definition and properties of Lorentz spaces, see e.g. [14], Chapter V.)
2 Results

The aim of this note is to extend and improve this result. We replace the smallness in the $L^{2,\infty}$-norm by a weaker condition (reminding of Theorem 1.1), and we prove the result for all dimensions. More precisely, we have the following theorem.

**Theorem 2.1** There exist $p < 2$ and $\epsilon > 0$, depending only on $m$ and $n$, such that any generalized harmonic map $u \in W^{1,p}_{\text{loc}}(\Omega, S^{n-1})$ with the property $[u]_{\text{BMO}(\Omega)} \leq \epsilon$ is in $C^\infty(\Omega, S^{n-1})$.

To prove this theorem, it suffices to show that under these conditions, the generalized harmonic map $u$ is in $H^1_{\text{loc}}(\Omega, S^{n-1})$. Higher regularity is then implied by Theorem 1.1 (provided that $\epsilon$ is chosen accordingly). For this first step on the other hand, we can also admit a non-vanishing right hand side in (1.2).

**Theorem 2.2** For any $q > 2$, there exist $p < 2$ and $\epsilon > 0$, depending only on $m$, $n$, and $q$, with the following property. Suppose that $u \in W^{1,p}_{\text{loc}}(\Omega, S^{n-1})$ is a distributional solution of
\[
\text{div}(u \wedge \nabla u) = F + \text{div} G,
\]
where $F \in L^{mq/(m+q)}_{\text{loc}}(\Omega, A_2 R^n)$ and $G \in L^q_{\text{loc}}(\Omega, R^m \otimes A_2 R^n)$. If $[u]_{\text{BMO}(\Omega)} \leq \epsilon$, then $u \in W^{1,p/(p-1)}_{\text{loc}}(\Omega, S^{n-1})$.

As mentioned above, Theorem 2.1 is an immediate consequence of Theorem 1.1 and Theorem 2.2. The proof of the latter is inspired by the inverse Hölder inequality technique used by Iwaniec–Sbordone [11] to prove regularity for solutions of equations of the form
\[
\text{div} A(x, \nabla u) = F + \text{div} G,
\]
where $A(x, \xi) = \frac{\partial F}{\partial \xi}(x, \xi)$ for a quasi-convex function $F$ (satisfying certain conditions). We combine these methods with arguments from the regularity theory for weakly harmonic maps.

We will use the following well-known results. The first one is due to Giaquinta–Modica [7].

**Proposition 2.3** For $1 < a < b$, and for some ball $B_R(x_0) \subset \mathbb{R}^m$, suppose that $g \in L^a(B_R(x_0))$ and $f \in L^b(B_R(x_0))$ are non-negative functions which satisfy
\[
\int_{B_{r/2}(x_1)} g^a dx \leq A \left[ \left( \int_{B_r(x_1)} g dx \right)^a + \int_{B_r(x_1)} f^a dx \right] + \theta \int_{B_r(x_1)} g^a dx
\]
for every ball $B_r(x_1) \subset B_R(x_0)$ and for certain constants $A, \theta > 0$. There exists a constant $\theta_0 = \theta_0(m, a, b) > 0$, such that whenever $\theta < \theta_0$, then $g \in L^c(B_{R/2}(x_0))$ with
\[
\left( \int_{B_{R/2}(x_0)} g^c dx \right)^{1/c} \leq B \left[ \left( \int_{B_R(x_0)} g^a dx \right)^{1/a} + \left( \int_{B_R(x_0)} f^c dx \right)^{1/c} \right]
\]
for certain numbers \( c > a \) and \( B > 0 \), both depending on \( m, A, \theta, a, \) and \( b \).

The following is a combination of the compensated compactness results of Coifman–Lions–Meyer–Semmes [3], and the duality of the space \( \text{BMO}(\mathbb{R}^m) = \{ f \in L^1_{\text{loc}}(\mathbb{R}^m) : [f]_{\text{BMO}(\mathbb{R}^m)} < \infty \} \) with the Hardy space \( \mathcal{H}^1(\mathbb{R}^m) \). The latter is due to Fefferman–Stein [5].

**Proposition 2.4** For \( 1 < p < \infty \), suppose that a function \( f \in W^{1,p}_{\text{loc}}(\mathbb{R}^m) \) with \( \| \nabla f \|_{L^p(\mathbb{R}^m)} < \infty \), a vector field \( g \in L^{p/(p-1)}(\mathbb{R}^m, \mathbb{R}^m) \) with \( \text{div} g = 0 \) in the distribution sense, and a function \( h \in \text{BMO}(\mathbb{R}^m) \) are given. Then

\[
\left| \int_{\mathbb{R}^m} \nabla f \cdot g \, h \, dx \right| \leq C \| \nabla f \|_{L^p(\mathbb{R}^m)} \| g \|_{L^{p/(p-1)}(\mathbb{R}^m)} [h]_{\text{BMO}(\mathbb{R}^m)}
\]

for a constant \( C \) which depends only on \( m \) and \( p \).

Having the ingredients ready, we can now prove Theorem 2.2.

**Proof of Theorem 2.2.** Suppose \( q > 2, F \in L^{mq/(m+q)}_{\text{loc}}(\Omega, \Lambda_2 \mathbb{R}^n) \), and \( G \in L^q_{\text{loc}}(\Omega, \mathbb{R}^m \otimes \Lambda_2 \mathbb{R}^n) \). Let for the moment \( p \) be any number in \((1,2)\), and suppose that \( u \in W^{1,p}_{\text{loc}}(\Omega, \mathcal{S}^{n-1}) \) is a solution of (2.1).

Let \( \psi \in W^{2,mq/(m+q)}_{\text{loc}}(\Omega, \Lambda_2 \mathbb{R}^n) \) be a solution of

\[
\Delta \psi = F \quad \text{in} \quad \Omega.
\]

Then \( \nabla \psi \in W^{1,q}_{\text{loc}}(\Omega, \mathbb{R}^m \otimes \Lambda_2 \mathbb{R}^n) \), and \( u \) satisfies

\[
\text{div}(u \wedge \nabla u) = \text{div}(G + \nabla \psi).
\]

Hence we may assume without loss of generality that \( F = 0 \). Choose a ball \( B_r(x_0) \subset \Omega \) and a cut-off function \( \zeta \in C_0^\infty(B_r(x_0)) \) with \( \zeta \equiv 1 \) in \( B_{r/2}(x_0) \), such that \( |\nabla \zeta| \leq 4r^{-1} \). Consider the Hodge decomposition

\[
|\nabla(\zeta(u - \bar{u}_{B_r(x_0)}))|^{p-2} u \wedge \nabla(\zeta(u - \bar{u}_{B_r(x_0)})) = \nabla \phi + \Phi,
\]

where \( \phi \in W^{1,p/(p-1)}_{\text{loc}}(\mathbb{R}^m, \Lambda_2 \mathbb{R}^n) \) and \( \Phi \in L^{p/(p-1)}(\mathbb{R}^m, \mathbb{R}^m \otimes \Lambda_2 \mathbb{R}^n) \) have the properties \( \text{div} \Phi = 0 \) and

\[
\| \nabla \phi \|_{L^p(\mathbb{R}^m)} + \| \Phi \|_{L^{p/(p-1)}(\mathbb{R}^m)} \leq C_1 \| \nabla(\zeta(u - \bar{u}_{B_r(x_0)})) \|_{L^{p/(p-1)}(B_r(x_0))}^{p-1}
\]

for any \( s \in (\frac{1}{p-1}, \frac{p}{p-1}] \) and for a constant \( C_1 = C_1(m,n,s) \). The existence of such a decomposition is due to Iwaniec–Martin [10]. In particular, we have

\[
\int_{B_r(x_0)} |\nabla \phi|^s dx \leq C_2 \left( \int_{B_r(x_0)} |\nabla u|^s dx \right)^{p-1} \tag{2.2}
\]
for a constant $C_2 = C_2(m, n, s)$, owing to the Poincaré and the Hölder inequality. Observe that

$$2^{-m} \int_{B_{r/2}(x_0)} |\nabla u|^p \, dx \leq \int_{B_r(x_0)} \left< u \wedge (\zeta (u - \bar{u}_{B_r(x_0)})), \nabla \phi + \Phi \right> \, dx$$

$$= \int_{B_r(x_0)} \left< u \wedge (\zeta (u - \bar{u}_{B_r(x_0)})), \Phi \right> \, dx$$

$$+ \int_{B_r(x_0)} \left< \nabla \zeta \cdot (u - \bar{u}_{B_r(x_0)}) \cdot \nabla \phi \right> \, dx$$

$$- \int_{B_r(x_0)} \left< \nabla \zeta \cdot (u - \nabla u) \cdot (\phi - \bar{\phi}_{B_r(x_0)}) \right> \, dx$$

$$+ \int_{B_r(x_0)} \left< G, \nabla (\zeta (\phi - \bar{\phi}_{B_r(x_0)})) \right> \, dx,$$

where we denote the standard scalar product in $\mathbb{R}^m$ and in $\mathbb{R}^m \otimes \Lambda_2 \mathbb{R}^n$ by $\langle \cdot, \cdot \rangle$, whereas we use a dot in $\mathbb{R}^n$ to avoid confusion. We have the estimates

$$\int_{B_r(x_0)} \left< \nabla \zeta \cdot (u - \bar{u}_{B_r(x_0)}) \cdot \nabla \phi \right> \, dx$$

$$\leq \frac{4}{r} \left( \int_{B_r(x_0)} |\nabla \phi|^\frac{2m}{m+1} \, dx \right)^{\frac{m+1}{2m}} \left( \int_{B_r(x_0)} |u - \bar{u}_{B_r(x_0)}|^{\frac{2m}{m+1}} \, dx \right)^{\frac{m-1}{2m}}$$

$$\leq C_3 \left( \int_{B_r(x_0)} |\nabla u|^\frac{2m}{m+1} \, dx \right)^{\frac{m+1}{2m}},$$

by (2.2) and the Sobolev inequality, and similarly

$$\int_{B_r(x_0)} \left< \nabla \zeta \cdot (u - \nabla u) \cdot (\phi - \bar{\phi}_{B_r(x_0)}) \right> \, dx \leq C_4 \left( \int_{B_r(x_0)} |\nabla u|^\frac{2m}{m+1} \, dx \right)^{\frac{m+1}{2m}},$$

for certain constants $C_3, C_4$ which depend only on $m$ and $n$.

Note that $\|u - \bar{u}_{B_r(x_0)}\|_{\text{BMO}(\mathbb{R}^m)} \leq C_5 \|u\|_{\text{BMO}(B_r(x_0))}$ for a constant $C_5 = C_5(m, n)$. (This is proven in [4].) Extending $\nabla u$ to $\mathbb{R}^m$ and applying Proposition 2.4, we thus find

$$\int_{B_r(x_0)} \left< u \wedge (\zeta (u - \bar{u}_{B_r(x_0)})), \Phi \right> \, dx$$

$$= -\int_{B_r(x_0)} \zeta \left< \nabla u \cdot (u - \bar{u}_{B_r(x_0)}), \Phi \right> \, dx$$

$$\leq C_6 \|u\|_{\text{BMO}(\Omega)} \int_{B_r(x_0)} |\nabla u|^p \, dx$$

for a constant $C_6 = C_6(m, n, p)$. 
Finally, choose a number $\sigma \in (2, q)$. We have
\[
\int_{B_r(x_0)} (G, \nabla (\zeta (\phi - \bar{\phi}_{B_r(x_0)}))) \, dx \\
\leq C_7 \left( \int_{B_r(x_0)} |G|^\sigma \, dx \right)^{1/\sigma} \left( \int_{B_r(x_0)} |\nabla \phi|^{\sigma/(\sigma - 1)} \, dx \right)^{\frac{\sigma - 1}{\sigma}} \\
\leq C_8 \left( \int_{B_r(x_0)} |G|^\sigma \, dx \right)^{1/\sigma} \left( \int_{B_r(x_0)} |\nabla u|^{\sigma/(\sigma - 1)} \, dx \right)^{\frac{1-(\sigma-1)}{\sigma}} \\
\leq C_8 \left( \int_{B_r(x_0)} |G|^\sigma \, dx + \left( \int_{B_r(x_0)} |\nabla u|^{\sigma/(\sigma - 1)} \, dx \right)^{\frac{\sigma(\sigma-1)}{\sigma-1}} + 1 \right)
\]
(for constants $C_7, C_8$ which depend on $m, n, \text{ and } \sigma$) by the H"older inequality, the Poincaré inequality, the estimate (2.2), and Young’s inequality.

Now choose $a \in (1, \min\{\frac{m+1}{m}, \frac{2(\sigma-1)}{\sigma}\})$, and set $b = \frac{2a}{\sigma}$. Let $\theta_0$ be the constant from Proposition 2.3 (belonging to $a$ and $b$), and choose a number $\theta \in (0, \theta_0)$. Then the conditions of Proposition 2.3 are satisfied for any ball $B_R(x_0) \subset \subset \Omega$, for the functions
\[
g = |\nabla u|^{p/a}, \quad f = |G|^{\sigma/a} + 1,
\]
and for a constant $A$ which depends only on $m, n, \text{ and } \sigma$, provided that $p \geq a \max\{\frac{2m}{m+1}, \frac{\sigma}{\sigma-1}\}$ (which is strictly less than 2) and $\|u\|_{\text{BMO}(\Omega)} \leq C^{-1}_c \theta$. Hence under these conditions, there exists a number $\epsilon > a$, not depending on $p$, such that $|\nabla u| \in L^{pc/a}_\text{loc}(\Omega)$. If $2 - p$ is sufficiently small, then $\frac{p}{a} \geq \frac{p}{p-1}$, and therefore $u \in W^{1,p/(p-1)}_\text{loc}(\Omega, S^{n-1})$. This concludes the proof. \hfill \Box

References


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