SOME METRIC-SINGULAR PROPERTIES OF THE GRAPH OF SOLUTIONS OF THE ONE-DIMENSIONAL P-LAPLACIAN

Mervan Pašić & Vesna Županović

Abstract. We study the asymptotic behaviour of $\varepsilon$-neighbourhood of the graph of a type of rapidly oscillating continuous functions. Next, we estimate necessary and sufficient conditions for rapid oscillations of solutions of the main equation. This enables us to verify some new singular properties of bounded continuous solutions of a class of nonlinear $p$-Laplacian by calculating lower and upper bounds for the Minkowski content and the $s$-dimensional density of the graph of each solution and its derivative.

1. Introduction

Let $-\infty < a < b < \infty$. Let $y$ be a real function defined on $[a, b]$ and let $y'$ denote the derivative of $y$ in the classical sense. The main subjects of the paper are the graph $G(y)$ of a function $y$ and its $\varepsilon$-neighbourhood $G_\varepsilon(y)$, that is

$$G(y) = \{(t, y(t)) : a \leq t \leq b\}$$

$$G_\varepsilon(y) = \{(t_1, t_2) \in \mathbb{R}^2 : d((t_1, t_2), G(y)) \leq \varepsilon\}.$$  

Here $\varepsilon > 0$ and $d((t_1, t_2), G(y))$ denotes the distance between $(t_1, t_2)$ and $G(y)$.

In the author’s paper [9] for arbitrarily given $s \in (1, 2)$ it is constructed a class of Carathéodory functions $f(t, \eta, \xi)$ such that the graph $G(y)$ of each smooth enough solution $y$ of the main equation

$$-(|y'|^{p-2}y')' = f(t, y, y') \quad \text{in} \quad (a, b),$$

$$y(a) = y(b) = 0,$$

$$y \in W^{1,p}_{\text{loc}}((a, b]) \cap C([a, b]),$$

satisfies

$$\dim_M G(y) = s \quad \text{and} \quad \dim_M G(y') > 1,$$

$$\dim_{M_{\text{loc}}}(G(y); a) = s \quad \text{and} \quad \dim_{M_{\text{loc}}}(G(y); t) = 1 \quad \text{for each} \quad t \in (a, b).$$ (1.2)

2000 Mathematics Subject Classification. 35J60, 34B15, 26A75.

Key words and phrases. Nonlinear p-Laplacian, bounded solutions, qualitative properties, graph, singularity, Minkowski content, s-dimensional density.

©2004 Texas State University - San Marcos.

Here \( \dim_M G(y) \) denotes the upper Minkowski-Bouligand (box-counting) dimension of the graph \( G(y) \) and \( \dim_{Mloc} G(y); t \) denotes the locally upper Minkowski-Bouligand dimension of \( G(y) \) at a point \( t \in [a, b] \), defined by

\[
\dim_M G(y) = \limsup_{\varepsilon \to 0} \left( 2 - \frac{\log |G_\varepsilon(y)|}{\log \varepsilon} \right),
\]

\[
\dim_{Mloc} G(y); t = \limsup_{\varepsilon \to 0} \dim_M (G(y) \cap B_\varepsilon(t, y(t))).
\]

Here \( |G_\varepsilon(y)| \) denotes the Lebesgue measure of \( G_\varepsilon(y) \) and \( B_\varepsilon(t_1, t_2) \) denotes a ball in \( \mathbb{R}^2 \) centered at the point \( (t_1, t_2) \) with radius \( \varepsilon > 0 \).

The orders of growth for the asymptotic behaviour of \( |G_\varepsilon(y)| \) and \( |G_\varepsilon(y')| \) are given in (1.2). That is, when \( \varepsilon \approx 0 \) we have

\[
|G_\varepsilon(y)| \approx \varepsilon^{2-s} \quad \text{and} \quad |G_\varepsilon(y')| \approx \varepsilon^{2-q} \quad \text{for some } q > 1,
\]

where \( y \) is any smooth enough solution of (1.1). In particular from (1.2) we also have

\[
y \notin W^{1,p}(a, b) \quad \text{and} \quad \text{length}(G(y)) = \text{length}(G(y')) = \infty,
\]

\[
y \in W^{1,p}(a + \varepsilon, b) \quad \text{and} \quad \text{length}(G(y|_{a + \varepsilon, b})) < \infty \quad \text{for any } \varepsilon > 0.
\]

In the present paper, we derive some new singular properties of the graphs \( G(y) \) and \( G(y') \), improving (1.2), (1.3) and (1.4). In this purpose, we need an equivalent way to define box-counting dimension

\[
\dim_M G(y) = \inf \{ \tau \geq 0 : M^\tau(G(y)) = 0 \} = \sup \{ \tau \geq 0 : M^\tau(G(y)) = \infty \},
\]

where \( M^\tau(G(y)) \) denotes the \( \tau \)-dimensional upper Minkowski content of the graph \( G(y) \) defined by

\[
M^\tau(G(y)) = \limsup_{\varepsilon \to 0} (2\varepsilon)^{\tau-2} |G_\varepsilon(y)|.
\]

According to (1.2) and (1.5) we may conclude that

\[
M^\tau(G(y)) = 0 \quad \text{for all } \tau > s \quad \text{and} \quad M^\tau(G(y)) = \infty \quad \text{for all } \tau < s.
\]

Thus, the following three cases are possible

(i) \( M^s(G(y)) = 0 \)

(ii) \( M^s(G(y)) = \infty \)

(iii) \( 0 < M^s(G(y)) < \infty \).

In Section 2 and Section 5, for arbitrarily given \( s \in (1, 2) \) and under related assumptions on the nonlinearity \( f(t, \eta, \xi) \) as in [9], we will prove that each solution \( y \) of (1.1) satisfies \( 0 < M^s(G(y)) < \infty \). That is, the graph \( G(y) \) can be called as an \( s \)-set in respect to Minkowski content. Moreover, we find lower and upper bounds for \( M^s(G(y)) \) such that

\[
0 < \frac{1}{2^p} (b-a)^s \leq M^s(G(y)) \leq m_s (b-a)^s < \infty,
\]

where the positive constant \( m_s \) only depends on \( s \). It is interesting in (1.6) that the \( s \)-dimensional "length" of \( G(y) \) depends on the \( s \)-power of the length of interval \([a, b] \), which improves \( \text{length}(G(y)) = \infty \) appearing in (1.4). Furthermore, we will give the existence of an \( \varepsilon_0 > 0 \) such that

\[
0 < \frac{1}{2^p} (b-a)^s \varepsilon^{2-s} \leq |G_\varepsilon(y)| \quad \text{for each } \varepsilon \in (0, \varepsilon_0),
\]
where $y$ is any solution of (1.1). This statement gives us a lower bound for asymptotic behaviour of $|G_\varepsilon(y)|$ as $\varepsilon \approx 0$. It is more precise than corresponding one in (1.3).

As the second, for arbitrarily given $s \in (1, 2)$ and under the same assumptions on the nonlinearity $f(t, \eta, \xi)$ as in getting of (1.6)–(1.7), we will prove in Section 3 that each smooth enough solution $y$ of (1.1) satisfies

$$0 < \frac{1}{2^s} (b - a)^{s/2} \leq M^{1+s/2}(G(y')).$$

(1.8)

Because of (1.5), the inequality (1.8) improves corresponding result from (1.2), where $\dim_M G(y') > 1$. Here, we have shown that $\dim_M G(y') \geq 1 + s/2$. Furthermore, we will show the existence of an $\varepsilon_0 > 0$ such that

$$0 < \frac{\sqrt{2}}{2^s} (b - a)^{s/2} \leq |G_\varepsilon(y')| \quad \text{for each } \varepsilon \in (0, \varepsilon_0).$$

(1.9)

It gives us a lower bound for the asymptotic behaviour of $|G_\varepsilon(y')|$ as $\varepsilon \approx 0$.

Next, by means of (1.4) we have that the graph $G(y)$ of each solution $y$ of (1.1) is high concentrated (in some sense) at the boundary point $t = a$. How much of the graph $G(y)$ is concentrated near $t = a$ in the sense of Minkowski content, we will consider in Section 6. In this purpose, we define the $s$-dimensional upper (Minkowski) density of $G(y)$ at a point $t \in [a, b]$ as follows

$$D_s(G(y); t) = \limsup_{r \to 0} \frac{M^s(G(y) \cap B_r(t, y(t)))}{(2r)^s}.$$

Let us remark that by means of $y \in W^{1,p}(a + \varepsilon, b)$, where $\varepsilon > 0$, it is clear that

$$D_s(G(y); t) = 0 \quad \text{for any } t \in (a, b] \quad \text{and} \quad s \in (1, 2).$$

In Section 6, for arbitrarily given $s \in (1, 2)$ and under the same assumptions on the nonlinearity $f(t, \eta, \xi)$ as in getting of (1.6)–(1.7), we will find a constant $d_s$ such that each solution $y$ of (1.1) satisfies

$$0 < d_s \leq D_s(G(y); t = a).$$

(1.10)

Moreover, we will prove that

$$d_s(2r)^s \leq M^s(G(y) \cap B_r(a, y(a))) \quad \text{for each } r \in (0, b - a),$$

(1.11)

where $y$ is any smooth enough solution of (1.1). This inequality complete the statement (1.6).

To derive the statements (1.6)–(1.9), in Section 2, Section 3 and Section 5, we consider some metric properties of two types of rapid oscillations of real continuous functions: the first one is a kind of oscillations where the function is rapidly jumping over given obstacles, and the second one is a kind of oscillations where the convex and concave properties of the function are rapidly changing. In Section 4, some necessary conditions on the nonlinearity $f(t, \eta, \xi)$ will be given such that each solution of (1.1) is rapidly oscillating. These conditions on $f(t, \eta, \xi)$ will be very close to corresponding sufficient conditions used in Section 2. In appendix of the paper, we will give some technical results which play an important role in the proofs of the main results.
Further Remarks. A. It seems there is no known article dealing with this kind of problems. But, in the proofs of the main results we use some methods recently introduced in the author’s paper [9].

B. The existence result for the equation \([1.1]\) where the Caratheodory function \(f(t, \eta, \xi)\) satisfies the assumptions as in the paper, was considered in the appendix of [9]. It was based on some known results about the existence of continuous solutions for the equations with singular nonlinearity, explored via the sub- and super-solutions technique. See O’Regan’s book [8, Chapter 14].

C. Even the Minkowski content \(M^s\) is not a measure in the axiomatic sense, see for instance [3] and [7], the main results of the paper give some additional informations about the singular-metric boundary behaviour of the graph of any sufficiently smooth bounded solutions of the equation \([1.1]\).

D. To make a comparison between singularity and regularity of bounded solutions, we refer to [12] and references therein, where regularity of solutions of quasilinear elliptic equations associated with \(p\)-Laplacian are studied.

E. About the properties and calculations of fractal dimensions and Minkowski content of several types of sets in \(\mathbb{R}^n\), we refer to [1, 2, 3, 5, 7, 11, 14, 15].

F. Our main results can be generalized to the case of nonlinear variational inequalities and quasilinear elliptic systems associated with the one-dimensional \(p\)-Laplacian. See [10].

2. Rapid oscillations and lower bounds for \(|G_\varepsilon(y)|\) and \(M^s(G(y))\)

For a function \(y : [a, b] \to \mathbb{R}\) let us introduce a type of very rapid oscillations of \(y\) near the boundary point \(t = a\). A classical example for such type of oscillations is the function \(y(t) = t^\alpha \sin 1/t^\beta\) near \(t = 0\), where \(0 < \alpha < \beta\).

**Definition 2.1.** Let \(a_k\) be a decreasing sequence of real numbers from interval \((a, b)\) satisfying

\[
\begin{align*}
\text{there is a } k(\varepsilon) \in \mathbb{N} \text{ such that } a_{j-1} - a_j &\leq \varepsilon/2 \text{ for each } j \geq k(\varepsilon). \tag{2.1}
\end{align*}
\]

Let \(\theta\) and \(\omega\) be two measurable and bounded functions, both defined on \([a, b]\), such that \(\theta(t) \leq \omega(t)\) for each \(t \in [a, b]\). We say that a function \(y\) defined on \([a, b]\) is \((\theta, \omega, a_k)-rapidly oscillating\) if there is a sequence \(\sigma_k \in (a_k, a_{k-1}), k > 1\) such that

\[
\begin{align*}
y(\sigma_{2k}) &\geq \text{ess sup}_{(a_{2k}, a_{2k-1})} \omega \quad \text{and} \quad y(\sigma_{2k+1}) \leq \text{ess inf}_{(a_{2k+1}, a_{2k})} \theta, \quad k \geq 1. \tag{2.2}
\end{align*}
\]

For the record, the number \(k(\varepsilon)\) could be called as the index of \(\varepsilon\)-density of a sequence \(a_k\). It is interesting to show how to calculate the number \(k(\varepsilon)\) for a given sequence \(a_k\) which satisfies (2.1).

**Example 2.2.** Let \(a_k\) be a sequence of real numbers from interval \((a, b)\) given by

\[
a_k = a + \frac{b-a}{2} \left(\frac{1}{k^{1/\beta}}\right), \quad k \geq 1 \text{ and } 0 < \beta < \infty. \tag{2.3}
\]

Such a type of the sequence \(a_k\) is appearing in oscillations of the function \(y(t) = t^\alpha \sin 1/t^\beta\) near \(t = 0\), where \(a = 0, b = 1\) and \(0 < \alpha < \beta\). One can take for \(k(\varepsilon)\) to be any natural number which satisfies

\[
2\left(\frac{\beta \varepsilon}{b-a}\right)^{-\frac{1}{\beta}} \leq k(\varepsilon) \quad \text{for each } \varepsilon \in (0, \varepsilon_0) \text{ and for any } \varepsilon_0 > 0. \tag{2.4}
\]
Indeed, using an elementary inequality
\[
\left(\frac{1}{j-1}\right)^{1/\beta} - \left(\frac{1}{j}\right)^{1/\beta} \leq \frac{2^{1+1/\beta}}{\beta} \left(\frac{1}{j}\right)^{1+1/\beta},
\]
where \(\beta > 0\) and \(j \geq 2\), it is easy to show that the sequence \(a_k\) defined in (2.3) satisfies the statement (2.1) in respect to \(k\) satisfies Lemma 2.3.

Let \(\theta(t)\) and \(\omega(t)\) be two measurable and bounded real functions on \([a, b]\), \(\theta(t) \leq \omega(t), t \in [a, b]\), such that
\[
\text{ess inf}_{a_k} \theta \geq \text{ess inf}_{a_k+1} \theta, \quad \text{ess sup}_{a_k} \omega \leq \text{ess sup}_{a_k+1} \omega, \quad k \geq 1.
\]
(2.5)

Let \(y\) be \((\theta, \omega, a_k)\)-rapidly oscillating function on \([a, b]\) and \(y \in C((a, b])\). Then we have
\[
|G_{\varepsilon}(y)| \geq \int_{a}^{a_k(\varepsilon)} (\omega(t) - \theta(t)) dt \quad \text{for each} \quad \varepsilon \in (0, \varepsilon_0),
\]
where \(k(\varepsilon)\) and \(\varepsilon_0\) are appearing in (2.1).

Proof of Lemma 2.3. Let \(\varepsilon\) be a fixed real number such that \(\varepsilon \in (0, \varepsilon_0]\), where \(\varepsilon_0\) is from (2.1). We use the notation

\[
A(\varepsilon, \theta, \omega) = \{ (t, y) \in \mathbb{R}^2 : t \in (a, a_k(\varepsilon)), \ y \in [\theta(t), \omega(t)] \},
\]

\[
B_{2k} = [a_{2k}, a_{2k-1}] \times \left[ \text{ess inf}_{a_{2k-1}} \theta, \text{ess sup}_{a_{2k-1}} \omega \right],
\]

\[
B_{2k+1} = [a_{2k+1}, a_{2k}] \times \left[ \text{ess inf}_{a_{2k}} \theta, \text{ess sup}_{a_{2k}} \omega \right].
\]

Let \(y\) be a \((\theta, \omega, a_k)\)-rapidly oscillating function on \([a, b]\) such that \(y \in C((a, b])\). By Definition 2.1 it implies the existence of a sequence \(\sigma_k \in (a_k, a_{k-1})\) such that
\[
y(\sigma_{2k}) \geq \text{ess sup}_{a_{2k}} \omega \quad \text{and} \quad y(\sigma_{2k+1}) \leq \text{ess inf}_{a_{2k}} \theta, \quad k \geq 1.
\]
(2.7)

Let \(k\) be a fixed natural number such that \(k > k(\varepsilon) + 1\). Let \((t_0, y_0)\) be an arbitrarily given point of \(B_k\). Let us remark that from (2.7) we get:

if \((t_0, y_0) \in B_{2j}\) then \(y(\sigma_{2j-1}) \leq y_{0} \leq y(\sigma_{2j})\),

if \((t_0, y_0) \in B_{2j+1}\) then \(y(\sigma_{2j+1}) \leq y_{0} \leq y(\sigma_{2j})\), \(j \geq 1\).

In particular, it implies that
\[
\{(t, y_0) : t \in (\sigma_k, \sigma_{k-1})\} \cap G(y|_{\sigma_{k-1}, \sigma_{k-1}}) \neq \emptyset,
\]
where \(G(y|_{\sigma_{k}, \sigma_{k-1}})\) denotes the graph of the function \(y|_{\sigma_{k}, \sigma_{k-1}}\) (here \(y|_I\) denotes the function-restriction of \(y\) on interval \(I\)). Hence, there is a point \(s \in (\sigma_k, \sigma_{k-1})\)
such that \((s, y_0) \in G(y\mid_{\sigma_k, \sigma_{k-1}})\). Now, by the help of (2.1) we get
\[
d((t_0, y_0), G(y\mid_{\sigma_k, \sigma_{k-1}})) \leq d((t_0, y_0), (s, y_0)) \leq d((a_k, y_0), (\sigma_{k-1}, y_0))
\]
\[
= \sigma_{k-1} - a_k \leq a_{k-2} - a_k
\]
\[
= a_{k-1} - a_k + a_{k-2} - a_{k-1} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
Thus, we have proved that
\[
B_k \subseteq G_\varepsilon(y\mid_{\sigma_k, \sigma_{k-1}})
\]
for any \(k \geq k(\varepsilon) + 1\) and \(\varepsilon \in (0, \varepsilon_0)\), where \(G_\varepsilon(y\mid_{\sigma_k, \sigma_{k-1}})\) denotes the \(\varepsilon\)-neighbourhood of the graph \(G(y\mid_{\sigma_k, \sigma_{k-1}})\). Let us remark that from (2.5) we may ease conclude that
\[
A(\varepsilon, \theta, \omega) \subseteq \bigcup_{k=k(\varepsilon)}^{k(\varepsilon)+1} B_k,
\]
where \(\varepsilon \in (0, \varepsilon_0)\). According to (2.8) and (2.9) we obtain:
\[
A(\varepsilon, \theta, \omega) \subseteq \bigcup_{k=k(\varepsilon)}^{k(\varepsilon)+1} G_\varepsilon(y\mid_{\sigma_k, \sigma_{k-1}}) \subseteq G_\varepsilon\big(\bigcup_{k=k(\varepsilon)}^{k(\varepsilon)+1} y\mid_{\sigma_k, \sigma_{k-1}}\big) \subseteq G_\varepsilon(y),
\]
where \(\varepsilon \in (0, \varepsilon_0)\). Taking the Lebesgue measure on the both sides in the previous statement, it proves the inequality (2.6). □

How to calculate the right hand side in (2.6) it is shown in the following example.

**Example 2.4.** Let \(y\) be a real continuous function on \([a, b]\) and let \(y\) be \((\theta, \omega, a_k)\)-rapidly oscillating, where \(\theta, \omega\) and \(a_k\) are given by:
\[
a_k = a + \frac{b-a}{2} \left(\frac{1}{k}\right)^{1/\beta}, \quad k \geq 1,
\]
\[
\theta(t) = -(t-a)^{\alpha} \quad \text{and} \quad \omega(t) = (t-a)^{\alpha}, \quad t \in [a, b],
\]
where \(\alpha\) and \(\beta\) satisfy \(0 < \alpha < \beta < \infty\).

We can take for \(k(\varepsilon)\) to be any number that satisfies
\[
c_0 \left(\frac{1}{\varepsilon}\right)^{1/\beta} \leq k(\varepsilon) \leq 2c_0 \left(\frac{1}{\varepsilon}\right)^{1/\beta}
\]
for each \(\varepsilon \in (0, \varepsilon_0)\),
\[
\text{where } c_0 = 2 \left(\frac{b-a}{\beta}\right)^{1/\beta} \text{ and } \varepsilon_0 = \frac{b-a}{\beta}.
\]
It is clear that (2.11) implies (2.4) and so the main conclusion of Example 2.2 is still valid. That is, the sequence \(a_k\) given in (2.10) satisfies the condition (2.1), where \(k(\varepsilon)\) and \(\varepsilon_0\) are taken to be as in (2.11).

In contrast to (2.4), where \(\varepsilon_0\) is an arbitrary positive number, we need in (2.11) an \(\varepsilon_0 = \frac{(b-a)}{\beta}\). This condition on \(\varepsilon_0\) is essentialy to ensure that \(k(\varepsilon) \in \mathbb{N}\) for all \(\varepsilon \in (0, \varepsilon_0)\).

Next, since \(\theta(t) = -(t-a)^{\alpha}\) and \(\omega(t) = (t-a)^{\alpha}\) are decreasing and increasing on \([a, b]\), respectively, it is clear that \(\theta\) and \(\omega\) satisfy the condition (2.5).

Thus, the \((\theta, \omega, a_k)\) defined by (2.10) satisfies the hypotheses of Lemma 2.3. Hence, the inequality (2.6) can be applied to our situation here. It gives us
\[
|G_\varepsilon(y)| \geq \int_a^{a_k(\varepsilon)} (\omega(t) - \theta(t)) dt = 2 \int_a^{a_k(\varepsilon)} (t-a)^{\alpha} dt
\]
\[
= \frac{2}{\alpha+1} (a_k(\varepsilon) - a)^{\alpha+1} = \frac{2-\alpha(b-a)^{\alpha+1}}{\alpha+1} \left(\frac{1}{k(\varepsilon)}\right)^{1/\beta} \quad \text{for each } \varepsilon \in (0, \varepsilon_0).
\]
From the right inequality in (2.11) we have in particular

$$\frac{1}{k(\varepsilon)} \geq \frac{1}{4} \left( \frac{\beta}{b-a} \right)^{\frac{\alpha+1}{\alpha}} \varepsilon \frac{1}{2^{\alpha+1}}$$

for each \( \varepsilon \in (0, \varepsilon_0 = \frac{b-a}{\beta}) \).

These inequalities give us

$$|G_\varepsilon(y)| \geq c \varepsilon^{\frac{\alpha+1}{2\alpha+1}}$$

for each \( \varepsilon \in (0, \varepsilon_0) \),

where

$$\varepsilon_0 = \frac{b-a}{\beta} \quad \text{and} \quad c = \frac{2^{\alpha} \alpha+1}{\alpha+1} \left( \frac{1}{2^{\alpha}} \right)^{\frac{\alpha+1}{2\alpha+1}} \beta \varepsilon^{\frac{\alpha+1}{2\alpha+1}} (b-a)^{\frac{\alpha+1}{2\alpha+1}}.$$

To derive a lower bound for \( M^*(G(y)) \) where \( y \) is any solution of (1.1) we need to impose on the function \( f(t, \eta, \xi) \) some sufficient conditions such that each solution of (1.1) is \((\theta, \omega, a_k)\)-rapidly oscillating. It is the subject of the following lemma which is a modification of [9, Lemma 4.1, p. 280] and [9, Lemma 4.2, p. 281].

**Lemma 2.5.** Let \( a_k \) be a decreasing sequence of real numbers from interval \((a, b)\) satisfying (2.1). Let \( \tilde{\theta}_0 \) and \( \tilde{\omega}_0 \) be two arbitrarily given real numbers, and let \( \theta(t) \) and \( \omega(t) \) be two measurable and bounded real functions on \([a, b] \), \( \theta(t) \leq \omega(t) \), \( t \in [a, b] \), which satisfy (2.5) and

$$\tilde{\theta}_0 \leq \text{ess inf}_{(a, b)} \theta < \text{ess inf}_{(a_{2k+1}, a_{2k})} \theta, \quad \text{ess sup}_{(a_{2k}, a_{2k-1})} \omega < \text{ess sup}_{(a, b)} \omega \leq \tilde{\omega}_0, \quad k \geq 1. \tag{2.12}$$

Let the sets \( J_k \) be defined by

$$J_{2k} = (\tilde{\theta}_0, \text{ess sup}_{(a_{2k}, a_{2k-1})} \omega) \quad \text{and} \quad J_{2k+1} = (\text{ess inf}_{(a_{2k+1}, a_{2k})} \theta, \tilde{\omega}_0), \quad k \geq 1.$$

Next, let for each \( k \geq 1 \) the Caratheodory function \( f(t, \eta, \xi) \) satisfy

$$f(t, \eta, \xi) \geq 0, \quad t \in (a_{2k}, a_{2k-1}), \quad \eta \in J_{2k}, \quad \xi \in \mathbb{R}, \tag{2.13}$$

$$\int_{A_{2k} \times R} \text{ess inf}_{(\eta, \xi)} f(t, \eta, \xi) dt > \frac{c(p)}{(a_{2k+1} - a_{2k})^{p-1}} \left( \tilde{\omega}_0 - \text{ess sup}_{(a_{2k}, a_{2k-1})} \omega \right)^p, \tag{2.14}$$

$$f(t, \eta, \xi) \leq 0, \quad t \in (a_{2k+1}, a_{2k}), \quad \eta \in J_{2k+1}, \quad \xi \in \mathbb{R}, \tag{2.15}$$

$$\int_{A_{2k+1} \times R} \text{ess sup}_{(\eta, \xi)} f(t, \eta, \xi) dt < -\frac{c(p)}{(a_{2k} - a_{2k-1})^{p-1}} \left( \text{ess inf}_{(a_{2k+1}, a_{2k})} \theta - \tilde{\theta}_0 \right)^p, \tag{2.16}$$

where \( c(p) = 2[4(p - 1)]^{p-1} \) and \( A_k \) is a family of sets

$$A_k = [a_k + \frac{1}{4}(a_{k-1} - a_k), a_{k-1} - \frac{1}{4}(a_{k+1} - a_k)], \quad k \geq 1.$$

Then each solution \( y \) of the equation (1.1) such that

$$\tilde{\theta}_0 \leq y(t) \leq \tilde{\omega}_0, \quad t \in (a, b) \tag{2.17}$$

is \((\theta, \omega, a_k)\)-rapidly oscillating on \([a, b]\).

The proof of this lemma will be sketched in Appendix of this paper. It is worth to mention that the condition (2.17) will be easy achieved in Theorem 2.7 below. An example for such a class of Caratheodory functions \( f(t, \eta, \xi) \) which satisfies the hypotheses (2.13)–(2.16) is given as follows.
Example 2.6. To simplify the notation, let $\tilde{\theta}_0$, $\tilde{\omega}_0$, $\theta_{2k+1}$, and $\omega_{2k}$ be defined as follows:

$$\tilde{\theta}_0 = \operatorname{ess}
\inf_{(a,b)} \theta \quad \text{and} \quad \tilde{\omega}_0 = \operatorname{ess}
\sup_{(a,b)} \omega;$$

$$\theta_{2k+1} = \operatorname{ess}
\inf_{(a_{2k+1}, a_{2k})} \theta \quad \text{and} \quad \omega_{2k} = \operatorname{ess}
\sup_{(a_{2k}, a_{2k-1})} \omega.$$ 

Let $g = g(t, \eta, \xi)$ be a Caratheodory function defined by

$$g = \frac{\pi c(p)(\tilde{\omega}_0 - \tilde{\theta}_0)p}{\sin \frac{\pi}{4}} \sum_{k=1}^\infty \left[ (\eta - \tilde{\omega}_0)^+ \sin \left( \frac{\pi}{a_{2k-1} - a_{2k}} (t - a_{2k}) \right) \right]$$

$$\times \left( \tilde{\omega}_0 - \omega_{2k} \right)^2 \frac{(a_{2k-1} - a_{2k})^p}{K[a_{2k}, a_{2k-1}](t)}$$

$$+ \left( \theta_{2k+1} - \tilde{\theta}_0 \right)^+ \sin \left( \frac{\pi}{a_{2k} - a_{2k+1}} (t - a_{2k+1}) \right) \right]$$

$$\times \left( \theta_{2k+1} - \tilde{\theta}_0 \right)^2 \frac{(a_{2k} - a_{2k+1})^p}{K[a_{2k+1}, a_{2k}](t)},$$

where $c(p)$ is appearing in (2.14) and (2.16), and where $K_A(t)$ denotes, as usual, the characteristic function of a set $A$. Also, $\eta^- = \max \{0, -\eta\}$ and $\eta^+ = \max \{0, \eta\}$.

Even $K_A(t)$ is not continuous in $t$, it is not difficult to check that $g(t, \eta, \xi)$ is continuous in all its variables. Next, for any fixed $k \in \mathbb{N}$, it is clear that

$$(\eta - \tilde{\omega}_0)^- \sin \left( \frac{\pi}{a_{2k-1} - a_{2k}} (t - a_{2k}) \right) \geq 0 \quad \text{for any} \ t \in [a_{2k}, a_{2k-1}], \ \eta \in J_{2k},$$

$$(\eta - \tilde{\omega}_0)^- \tilde{\omega}_0 - \omega_{2k} \geq 1 \quad \text{for any} \ \eta \in J_{2k}, \ \text{where} \ J_{2k} = (\tilde{\theta}_0, \tilde{\omega}_0).$$

Hence, for any $k \in \mathbb{N}$, $t \in [a_{2k}, a_{2k-1}]$, $\eta \in J_{2k}$, and $\xi \in \mathbb{R}$ we have

$$g(t, \eta, \xi) = \frac{\pi c(p)(\tilde{\omega}_0 - \tilde{\theta}_0)p}{\sin \frac{\pi}{4}} \frac{(\eta - \tilde{\omega}_0)^- \sin \left( \frac{\pi}{a_{2k-1} - a_{2k}} (t - a_{2k}) \right)}{(\tilde{\omega}_0 - \omega_{2k})^2} \frac{(a_{2k-1} - a_{2k})^p}{K[a_{2k}, a_{2k-1}](t)} \geq 0,$$

$$\operatorname{ess}
\inf_{(\eta, \xi) \in J_{2k} \times \mathbb{R}} g(t, \eta, \xi) \geq \frac{\pi c(p)}{(\sin \frac{\pi}{4})} \frac{(\tilde{\omega}_0 - \tilde{\theta}_0)p}{(a_{2k-1} - a_{2k})^p} \sin \left( \frac{\pi}{a_{2k} - a_{2k+1}} (t - a_{2k+1}) \right) \right]$$

From the first equality, it follows that $g(t, \eta, \xi)$ satisfies (2.13). Also, integrating the second inequality over the set

$$[a_{2k} + \frac{1}{4}(a_{2k-1} - a_{2k}), a_{2k-1} - \frac{1}{4}(a_{2k-1} - a_{2k})],$$

it immediately shows that $g(t, \eta, \xi)$ satisfies (2.14) too. On the same way the dual hypotheses (2.15) and (2.16) can be verified. Thus, the function $f(t, \eta, \xi) = g(t, \eta, \xi)$ satisfies the assumptions of Lemma 2.5.

Now, we are able to formulate the first main result of the paper.

Theorem 2.7. Let $a_k$ be a decreasing sequence of real numbers from interval $(a, b)$ satisfying (2.1). Let $\theta(t)$ and $\omega(t)$ be two measurable and bounded real functions on $[a, b]$, $\theta(t) \leq \omega(t)$, $t \in [a, b]$, which satisfy (2.5), (2.12) and

$$\operatorname{ess}
\inf_{(a,b)} \theta < 0 < \operatorname{ess}
\sup_{(a,b)} \omega. \quad (2.18)$$

Next, let for each $k \geq 1$ the Caratheodory function $f(t, \eta, \xi)$ satisfy (2.13)–(2.16), and

$$f(t, \eta, \xi) < 0, \quad t \in (a, b), \ \eta > \tilde{\omega}_0 \text{ and } \xi \in \mathbb{R},$$

$$f(t, \eta, \xi) > 0, \quad t \in (a, b), \ \eta < \tilde{\theta}_0 \text{ and } \xi \in \mathbb{R}, \quad (2.19)$$

...
where \( \tilde{\theta}_0 \) and \( \tilde{\omega}_0 \) are two arbitrarily given real numbers satisfying (2.12). Then for any \( s \in (1, 2) \) and for each solution \( y \) of (1.1) there holds true
\[
|G_\varepsilon(y)| \geq \int_a^\alpha_k(\omega(t) - \theta(t)) \, dt \quad \text{for each } \varepsilon \in (0, \varepsilon_0),
\]
\[
M^*(G(y)) \geq 2 \varepsilon^{s-2} \lim_{\varepsilon \to 0} \varepsilon^{s-2} \int_a^\alpha_k(\omega(t) - \theta(t)) \, dt,
\]
where \( \varepsilon_0 \) is appearing in (2.1).

**Proof.** Let \( y \) be a solution of the equation (1.1). Because of [9, Lemma 3.2, p. 279], from (2.12), (2.18) and (2.19), we get \( \tilde{\theta}_0 \leq y(t) \leq \tilde{\omega}_0 \) for each \( t \in [a, b] \). Therefore the assumptions of Lemma 2.5 are satisfied and so \( y \) is \((\theta, \omega, a_k)\)-rapidly oscillating function on \([a, b]\). Now, by means of Lemma 2.3 and the definition of \( M^*(G(y)) \), the desired statement (2.20), easy follows.

The following result is the main consequence of (2.20). We can derive also lower bounds appearing in (1.6) and (1.7).

**Corollary 2.8.** For arbitrarily given real number \( s \in (1, 2) \), let \( a_k, \theta \) and \( \omega \) be given by
\[
a_k = a + \frac{b-a}{2} \left( \frac{1}{k} \right)^{1/\beta}, \quad k \geq 1,
\]
\[
\theta(t) = -(t-a) \quad \text{and} \quad \omega(t) = t-a, \quad t \in (a, b),
\]
where \( \beta \) satisfies \( 1 < \beta < \infty \) and \( \frac{2\beta}{\beta+1} = s \).

Let the Caratheodory function \( f(t, \eta, \xi) \) satisfy (2.13)-(2.16), and (2.19) in respect to such given \((\theta, \omega, a_k)\), where \( \theta_0 \) and \( \omega_0 \) are two arbitrarily given real numbers satisfying (2.12). Then each solution \( y \) of the equation (1.1) satisfies
\[
|G_\varepsilon(y)| \geq \frac{1}{2\varepsilon^s}(b-a)^s \varepsilon^{2-s} \quad \text{for each } \varepsilon \in (0, \varepsilon_0 = \frac{b-a}{\beta}),
\]
\[
M^*(G(y)) \geq \frac{1}{2^s}(b-a)^s.
\]

**Proof.** First, we know, by Example 2.4, that the sequence \( a_k \) defined in (2.21) satisfies the condition (2.1) in respect to \( k(\varepsilon) \) determined by (2.11). Next, since \( \theta(t) = -(t-a) \) is decreasing and \( \omega(t) = t-a \) is increasing, it is clear that the functions \( \theta \) and \( \omega \) defined in (2.21) satisfy the conditions (2.5), (2.12), and (2.18).

Thus, the hypotheses of Theorem 2.7 are satisfied and so the statement (2.20) may be used here. Putting given data from (2.21) into (2.20) we can calculate
\[
|G_\varepsilon(y)| \geq \int_a^\alpha_k(2(t-a)) \, dt = \left( \frac{b-a}{2} \right)^2 \left( \frac{1}{k(\varepsilon)} \right)^{2/\beta}.
\]
Since \( \beta > 1 \) and \( 2\beta/(\beta+1) = s \), by the help of the right inequality in (2.11) we get
\[
\left( \frac{b-a}{2} \right)^2 \left( \frac{1}{k(\varepsilon)} \right)^{2/\beta} \geq \frac{1}{4} \left( \frac{1}{4} \right)^{2/\beta} \frac{\beta^2 \pi^2}{(b-a) \pi^2} \varepsilon^{2-s} \geq \frac{1}{2^s}(b-a)^s \varepsilon^{2-s}.
\]

Putting this inequality into (2.23), we have proved (2.22). \( \square \)

At the end of this section we give an example for such a class of Caratheodory functions \( f(t, \eta, \xi) \) that satisfies the assumptions of Theorem 2.7.
**Example 2.9.** Let $h_1(t)$ and $h_2(t)$ be two measurable functions on $(a, b)$ such that $h_1(t) > 0$ and $h_2(t) > 0$. Let $f(t, \eta, \xi)$ be a function defined by

$$f(t, \eta, \xi) = -h_1(t)(\eta - \text{ess sup}_{(a,b)} \omega)^+ + h_2(t)(\eta - \text{ess inf}_{(a,b)} \theta) - g(t, \eta, \xi),$$

where the function $g(t, \eta, \xi)$ is constructed in Example 2.6 above. Since $g(t, \eta, \xi)$ satisfies the assumptions of Lemma 2.5, it is clear that such given $f(t, \eta, \xi)$ satisfies the assumptions of Theorem 2.7.

### 3. Lower bounds for $|G_\varepsilon(y')|$ and $M^*(G(y'))$

We proceed with some observations from the previous section but they are pointed to the derivative $y'$ of any smooth enough real function $y$. It is started with a relation between the $(\theta, \omega, a_k)$-rapid oscillations of a function $y$ and the asymptotic behaviour of $|G_\varepsilon(y')|$ as $\varepsilon \to 0$.

**Lemma 3.1.** Let $a_k$ be a decreasing sequence of real numbers from interval $(a, b)$ satisfying (2.1). Let $\theta(t)$ and $\omega(t)$ be two measurable and bounded real functions on $[a, b]$, $\theta(t) \leq \omega(t)$, $t \in [a, b]$, which satisfy (2.5). Next, let $y$ be $(\theta, \omega, a_k)$-rapidly oscillating on $[a, b]$ and let $y \in C((a, b]) \cap C^1(a, b)$. Then we have

$$|G_\varepsilon(y')| \geq \sum_{k=k(\varepsilon/2)}^{\infty} \left( \text{ess sup}_{(a_{2k}, a_{2k-1})} \omega - \text{ess inf}_{(a_{2k+1}, a_{2k})} \theta \right) \text{ for each } \varepsilon \in (0, \varepsilon_0),$$

where $k(\varepsilon)$ and $\varepsilon_0$ are defined in (2.1).

**Proof.** Let $y$ be a $(\theta, \omega, a_k)$-rapidly oscillating function on $[a, b]$. By means of Definition 2.1, there is a sequence $\sigma_k \in (a_k, a_{k-1})$, $k > 1$ such that

$$y(\sigma_{2k}) \geq \text{ess sup}_{(a_{2k}, a_{2k-1})} \omega \text{ and } y(\sigma_{2k+1}) \leq \text{ess inf}_{(a_{2k+1}, a_{2k})} \theta, \quad k \geq 1. \quad (3.2)$$

Let $k(\varepsilon)$ and $\varepsilon_0$ be taken from (2.1). So, we have that $a_{j-1} - a_j \leq \varepsilon/4$ for each $j \geq k(\varepsilon)$. Hence, the sequence $\sigma_k$ also satisfies

$$\sigma_k \searrow a \text{ and } \sigma_{j-1} - \sigma_j \leq \varepsilon/2 \text{ for each } j \geq k(\frac{\varepsilon}{2}) \text{ and } \varepsilon \in (0, \varepsilon_0). \quad (3.3)$$

Applying Lagrange’s mean value theorem on $(\sigma_k, \sigma_{k-1})$ we get a sequence $s_k \in (\sigma_k, \sigma_{k-1})$, $k > 1$ such that

$$y'(s_{2k+1}) = \frac{y(\sigma_{2k}) - y(\sigma_{2k+1})}{\sigma_{2k} - \sigma_{2k+1}} \text{ and } y'(s_{2k}) = \frac{y(\sigma_{2k-1}) - y(\sigma_{2k})}{\sigma_{2k-1} - \sigma_{2k}}. \quad (3.4)$$

Let us use the following notation

$$z(t) = y'(t), \quad t \in (a, b),$$

$$\begin{align*}
\delta_{2k+1} &= \frac{\text{ess sup}_{(a_{2k}, a_{2k-1})} \omega - \text{ess inf}_{(a_{2k+1}, a_{2k})} \theta}{\sigma_{2k} - \sigma_{2k+1}}, \\
\delta_{2k} &= \frac{\text{ess inf}_{(a_{2k-1}, a_{2k-2})} \theta - \text{ess sup}_{(a_{2k}, a_{2k-1})} \omega}{\sigma_{2k-1} - \sigma_{2k}}.
\end{align*}$$

Hence, from (2.5), (3.2), and (3.4) follows

$$z(s_{2k+1}) \geq \delta_{2k+1} \geq 0 \text{ and } z(s_{2k}) \leq \delta_{2k} \leq 0, \quad k \geq 1, \quad (3.5)$$

where $s_k \in (\sigma_k, \sigma_{k-1})$. To prove (3.1) we need a modification of Lemma 2.3. \qed
Lemma 3.2 (A version of Lemma 2.3). Let \( \sigma_k \) be a decreasing sequence of real numbers from interval \((a, b)\) satisfying (3.3), where \( k(\varepsilon) \) and \( \varepsilon_0 \) are taken from (2.1). Let \( \delta_k \) be a sequence of real numbers such that \( \delta_{2k+1} \geq 0 \) and \( \delta_{2k} \leq 0 \), \( k \geq 1 \). Let \( z(t) \) be a continuous function in \((a, b)\) such that there is a sequence \( s_k \in (\sigma_k, \sigma_{k-1}) \) satisfying (3.5). Then there holds
\[
|G_\varepsilon(z)| \geq \sum_{k=k(\varepsilon/2)}^{\infty} \delta_{2k+1}(\sigma_{2k} - \sigma_{2k+1}) \quad \text{for each } \varepsilon \in (0, \varepsilon_0).
\] (3.6)

To prove (3.6), we suggest reader to use the same argumentation from the proof of Lemma 2.3 to prove that \( B_{2k+1} \subseteq G_\varepsilon(z)|_{[\sigma_{2k+1}, \sigma_{2k}]} \) for each \( k \geq k(\varepsilon) + 1 \) and \( \varepsilon \in (0, \varepsilon_0) \), where \( B_{2k+1} = [\sigma_{2k+1}, \sigma_{2k}] \times [0, \delta_{2k+1}] \).

Using the above notation for \( \delta_k \) and \( z(t) \), from (3.6) follows
\[
|G_\varepsilon(y')| \geq \sum_{k=k(\varepsilon/2)}^{\infty} \frac{\text{ess sup}_{(a_{2k}, a_{2k-1})} \omega - \text{ess inf}_{(a_{2k+1}, a_{2k})} \theta}{\sigma_{2k} - \sigma_{2k+1}} \left( \sigma_{2k} - \sigma_{2k+1} \right)
= \sum_{k=k(\varepsilon/2)}^{\infty} \left( \text{ess sup}_{(a_{2k}, a_{2k-1})} \omega - \text{ess inf}_{(a_{2k+1}, a_{2k})} \theta \right).
\]

Thus, Lemma 3.1 is verified.

Next, we consider an easy example for the calculation of the right hand side in (3.1).

Example 3.3. Let \( y \) be a real function smooth enough in \((a, b)\) and let \( y \) be \((\theta, \omega, a_k)\)-rapidly oscillating, where the data \( \theta \), \( \omega \) and \( a_k \) are given in (2.10) above. Regarding to Example 2.4 we know such given \( \theta \), \( \omega \) and \( a_k \) satisfies the condition (2.1) and (2.5), where \( k(\varepsilon) \) and \( \varepsilon_0 \) are given in (2.11). Hence, we may calculate the right hand side in (3.1) for such function \( y \). First, we have
\[
\text{ess sup}_{(a_{2k}, a_{2k-1})} \omega - \text{ess inf}_{(a_{2k+1}, a_{2k})} \theta = \omega(a_{2k-1}) - \theta(a_{2k})
= \left( \frac{b-a}{2} \right)^{a/\beta} + \left( \frac{1}{2k} \right)^{a/\beta}
\geq \left( \frac{b-a}{2} \right)^{a/\beta} \left( \frac{1}{2k} \right)^{a/\beta}.
\] (3.7)

In particular, from (2.11) we have
\[
\frac{1}{k(\varepsilon/2)} \geq \frac{\alpha}{4} \left( \frac{\beta}{2(b-a)} \right)^{a/\beta} \varepsilon^{\alpha/\beta} \quad \text{for each } \varepsilon \in (0, \varepsilon_0 = \frac{b-a}{\beta}).\] (3.8)

According to (3.7) and (3.8), from (3.1) follows
\[
|G_\varepsilon(y')| \geq \sum_{k=k(\varepsilon/2)}^{\infty} \left( \frac{b-a}{2} \right)^{a/\beta} \left( \frac{1}{2k} \right)^{a/\beta} \geq \left( \frac{b-a}{2a-1} \right)^{a/\beta} \left( \frac{1}{k(\varepsilon/2)} \right)^{a/\beta}
\geq \left( \frac{b-a}{2a-1} \right)^{a/\beta} \frac{1}{8^{a/\beta}} \left( \frac{\beta}{2(b-a)} \right)^{a/\beta} \varepsilon^{\alpha/\beta},
\]
that is \( |G_\varepsilon(y')| \geq C \varepsilon^{\alpha/(\beta+1)} \) for each \( \varepsilon \in (0, \varepsilon_0) \), where
\[
\varepsilon_0 = \frac{b-a}{\beta} \quad \text{and} \quad C \left( \frac{b-a}{2a-1} \right)^{a/\beta} \left( \frac{\beta}{2} \right)^{a/\beta}.
\]

The main result of this section is the following.
Theorem 3.4. Let the hypotheses of Theorem 2.7 be still valid, that is: let the sequence \( a_k \) satisfy (2.1), let the functions \( \theta(t) \) and \( \omega(t) \), \( \theta(t) \leq \omega(t) \), \( t \in [a,b] \), satisfy (2.5), (2.12), and (2.18), and let the function \( f(t,\eta,\xi) \) satisfy the assumptions (2.13), (2.16), and (2.19). Then each solution \( y \) of (1.1) such that \( y \in C^1(a,b) \) satisfies
\[
|G_\varepsilon(y')| \geq h(\varepsilon) \quad \text{for each} \quad \varepsilon \in (0,\varepsilon_0),
\]
\[
M^s(G(y')) \geq 2^{-s-2} \limsup_{\varepsilon \to 0} \varepsilon^{s-2}h(\varepsilon) \quad \text{for any} \quad s \in (1,2),
\]
where \( \varepsilon_0 \) is appearing in (2.1) and
\[
h(\varepsilon) = \sum_{k=k(\varepsilon/2)}^{\infty} (\text{ess sup}_{(a_{2k},a_{2k+1})} \omega - \text{ess inf}_{(a_{2k+1},a_{2k})} \theta).
\]

Proof. As a consequence of the assumptions (2.13), (2.16), and (2.18)–(2.19) we may use Lemma 2.5. Hence, each solution \( y \) of (1.1) is \((\theta,\omega,a_k)\)-rapidly oscillating and so we may use Lemma 3.1 too. It gives us the inequality (3.1) from which immediately follows (3.9). \( \square \)

Now, we are able to prove the inequalities (1.8) and (1.9).

Corollary 3.5. Let the hypotheses of Corollary 2.8 be still valid, that is: for arbitrarily given real number \( s \in (1,2) \), let \( a_k, \theta \) and \( \omega \) be given by (2.21), and let the function \( f(t,\eta,\xi) \) satisfies (2.13), (2.16), and (2.19) in respect to such given \((\theta,\omega,a_k)\). Then each solution \( y \) of (1.1) such that \( y \in C^1(a,b) \) satisfies
\[
|G_\varepsilon(y')| \geq \frac{\sqrt{2}}{2^4} (b-a)^{s/2} \varepsilon^{1-s/2} \quad \text{for each} \quad \varepsilon \in (0,\varepsilon_0),
\]
\[
M^{1+s/2}(G(y')) \geq \frac{1}{2^4} (b-a)^{s/2}.
\]

Proof. From the proof of Corollary 2.8, we know that the sequence \( a_k \) given in (2.21) satisfies the condition (2.1), where \( k(\varepsilon) \) and \( \varepsilon_0 \) are determined in (2.11). Also, the functions \( \theta \) and \( \omega \) given in (2.21) satisfy the condition (2.5), (2.12), and (2.18). Therefore, we may apply Theorem 3.4 and to calculate the right hand side in (3.9), where \( \theta, \omega \) and \( a_k \) are given in (2.21). In this direction, by the help of (3.7) for \( \alpha = 1 \) and (3.8) we obtain
\[
\sum_{k=k(\varepsilon/2)}^{\infty} (\text{ess sup}_{(a_{2k},a_{2k+1})} \omega - \text{ess inf}_{(a_{2k+1},a_{2k})} \theta) \geq (b-a) \sum_{k=k(\varepsilon/2)}^{\infty} \left( \frac{1}{2^k} \right)^{1/\beta}
\]
\[
\geq (b-a) \left( \frac{1}{2^k(\varepsilon/2)} \right)^{1/\beta}
\]
\[
\geq \left( \frac{\beta}{2} \right)^{1/\beta} \left( \frac{1}{8} \right)^{1/\beta} (b-a)^{\beta/2} \varepsilon^{1-\frac{s}{\beta}}
\]
\[
\geq \frac{\sqrt{2}}{2^4} (b-a)^{\frac{s}{2}} \varepsilon^{1-s/2},
\]
where (2.21) is used, that is, \( \beta > 1 \) and \( 2\beta/(\beta+1) = s \). Now from (3.9) immediately follows
\[
|G_\varepsilon(y')| \geq \frac{\sqrt{2}}{2^4} (b-a)^{s/2} \varepsilon^{1-s/2} \quad \text{for each} \quad \varepsilon \in (0,\varepsilon_0),
\]
Also, we have

\[
M^{1+s/2}(G(y')) \geq 2^{(1+s/2) - 2} \lim_{\varepsilon \to 0} \sup_{\varepsilon} \left[ \frac{1}{24} (b - a)^{s/2} \right] \\
= \frac{2^{s/2+1/2}}{2^s} (b - a)^{s/2} \lim_{\varepsilon \to 0} \sup_{\varepsilon} (\varepsilon^{s/2-1} \varepsilon^{1-s/2}) \geq \frac{1}{24} (b - a)^{s/2}.
\]

This proves the desired statement (3.10). □

4. Necessary conditions for rapid oscillations

Let us mention that the inequality (2.6), stated in Lemma 2.3, was useful in the calculation of a lower bound of \( |G_\varepsilon(y)| \), where \( y \) is a \((\theta, \omega, a_k)\)-rapidly oscillating function. Next, in Lemma 2.5 some (sufficient) conditions on the nonlinearity \( f(t, \eta, \xi) \) were given such that each solution \( y \) of (1.1) is \((\theta, \omega, a_k)\)-rapidly oscillating. In this section, we consider an inverse of Lemma 2.5. Precisely, supposing that there is at least one \((\theta, \omega, a_k)\)-rapidly oscillating and smooth enough solution of (1.1) it is shown what type of (necessary) conditions on the nonlinearity \( f(t, \eta, \xi) \) must be satisfied. Let us remark that in both cases we are working with solutions \( y \) of (1.1)

which satisfy a basic condition

\[
\tilde{\theta}_0 \leq y(t) \leq \tilde{\omega}_0 \quad \text{for each} \ t \in [a, b], \quad \text{where} \ \tilde{\theta}_0 \leq \inf_{(a, b)} \theta \quad \text{and} \quad \tilde{\omega}_0 \geq \sup_{(a, b)} \omega. \tag{4.1}
\]

As we have seen in Theorem 2.7, the condition (4.1) can be easily verified if the assumption (2.19) is imposed on the nonlinear term \( f(t, \eta, \xi) \).

Theorem 4.1. Let \( a_k \) be a decreasing sequence of real numbers from interval \((a, b)\) satisfying (2.1). Let \( \theta(t) \) and \( \omega(t) \) be two measurable and bounded real functions on \([a, b], \theta(t) \leq \omega(t), t \in [a, b], \) which satisfy (2.5) and

\[
\sup_{(a, b)} \omega > \inf_{(a, b)} \theta, \quad \sup_{(a, b)} \omega > \inf_{(a_k, a_{k-1})} \theta, \quad k \geq 1.
\]

If there is at least one solution \( y \in C^2(a, b) \) of (1.1) which is \((\theta, \omega, a_k)\)-rapidly oscillating and satisfies (4.1), then for each \( k \geq 1 \), the Caratheodory function \( f(t, \eta, \xi) \) needs to satisfy the following inequalities:

\[
\int_{a_k+1}^{a_k+2} \omega \sup_{(\eta, \xi) \in I \times R} f^+(t, \eta, \xi) dt \\
\geq \frac{1}{(a_{k-2} - a_{k+1})^{p-1}} \left[ \left( \sup_{(a_{k+1}, a_{k-1})} \omega - \inf_{(a_{k+1}, a_{k-1})} \theta \right) - \left( \sup_{(a_{k}, a_{k-2})} \omega - \inf_{(a_{k}, a_{k-2})} \theta \right) \right], \tag{4.2}
\]

and

\[
\int_{a_{k+2}}^{a_{k-1}} \omega \inf_{(\eta, \xi) \in I \times R} f^-(t, \eta, \xi) dt \\
\leq -\frac{1}{(a_{k-1} - a_{k+2})^{p-1}} \left[ \left( \inf_{(a_{k+2}, a_{k})} \omega - \sup_{(a_{k+2}, a_{k})} \theta \right) - \left( \inf_{(a_{k}, a_{k-2})} \omega - \sup_{(a_{k}, a_{k-2})} \theta \right) \right], \tag{4.3}
\]

where \( I = (\inf_{(a, b)} \theta, \sup_{(a, b)} \omega) \), and \( f^- = \min\{f, 0\} \leq 0 \), and \( f^+ = \max\{f, 0\} \geq 0 \).
Taking in (2.12) that $\hat{\theta}_0 = \text{ess inf}_{(a,b)} \theta$ and $\text{ess sup}_{(a,b)} \omega = \hat{\omega}_0$ we see that both types of conditions, the sufficient conditions from Lemma 2.5 and the necessary conditions from Theorem 4.1 are similar each other, especially in the variable $t$.

**Proof.** Let $y$ be a solution of the equation (1.1) which is $(\theta, \omega, a_k)$-rapidly oscillating and satisfies the condition (4.1). According to (2.2) and (2.3) it is easy to check that there are two numbers $s^* \in (a_{2k+1}, a_{2k-1})$ and $t^* \in (a_{2k}, a_{2k-2})$, $s^* < t^*$, such that

$$y'(s^*) \geq 0 \quad \text{and} \quad y(s^*) \geq \text{ess sup}_{(a_{2k}, a_{2k-1})} \theta \geq \text{ess inf}_{(a_{2k}, a_{2k-1})} \theta,$$  
(4.4)

$$y(t^*) = \text{ess inf}_{(a_{2k}, a_{2k-1})} \theta \quad \text{and} \quad y(t) \geq \text{ess inf}_{(a_{2k}, a_{2k-1})} \theta, \quad t \in (s^*, t^*),$$  
(4.5)

where $k$ is a fixed natural number. It yields

$$\text{ess sup}_{(a_{2k}, a_{2k-1})} \omega - \text{ess inf}_{(a_{2k}, a_{2k-1})} \theta \leq y(s^*) - y(t^*) \leq \int_{s^*}^{t^*} |y'|dt$$

$$\leq (t^* - s^*)^{1/p'} (\int_{s^*}^{t^*} |y'|^{p'} dt)^{1/p}$$

$$\leq (a_{2k-2} - a_{2k+1})^{1/p'} (\int_{s^*}^{t^*} |y'|^{p'} dt)^{1/p},$$

where $1/p + 1/p' = 1$. Thus, we have

$$\frac{\left(\text{ess sup}_{(a_{2k}, a_{2k-1})} \omega - \text{ess inf}_{(a_{2k}, a_{2k-1})} \theta\right)^p}{(a_{2k-2} - a_{2k+1})^{p-1}} \leq \int_{s^*}^{t^*} |y'|^p dt.$$  
(4.6)

On the other hand, by means of (4.4) and (4.5) and multiplying the equation (1.1) by the test function $\varphi = y - \text{ess inf}_{(a_{2k}, a_{2k-1})} \theta$ and integrating both sides over $[s^*, t^*]$ we obtain

$$\int_{s^*}^{t^*} |y'|^p dt$$

$$= \left(|y|^{p-2}y(y - \text{ess inf}_{(a_{2k}, a_{2k-1})} \theta)\right)_{s^*}^{t^*} + \int_{s^*}^{t^*} f(t, y, y')(y - \text{ess inf}_{(a_{2k}, a_{2k-1})} \theta)dt$$

$$\leq \int_{s^*}^{t^*} f(t, y, y')(y - \text{ess inf}_{(a_{2k}, a_{2k-1})} \theta)dt$$

$$\leq \left(\text{ess sup}_{(a,b)} \omega - \text{ess inf}_{(a,b)} \theta\right) \int_{a_{2k+1}}^{a_{2k-2}} \text{ess sup}_{(\eta, \xi) \in I \times R} f^+(t, \eta, \xi) dt,$$

where $I = (\text{ess inf}_{(a,b)} \theta, \text{ess sup}_{(a,b)} \omega)$ and $f^+ = \max\{f, 0\} \geq 0$. Now, combining previous inequality with (4.6) we immediately derive the inequality (4.2).

Similar to (4.4) and (4.5) and by the help of (2.2) and (2.5) we get two numbers $s^* \in (a_{2k+2}, a_{2k})$ and $t^* \in (a_{2k+1}, a_{2k-1})$, $s^* < t^*$, such that

$$y'(s^*) \leq 0 \quad \text{and} \quad y(s^*) \leq \text{ess inf}_{(a_{2k+1}, a_{2k})} \theta \leq \text{ess sup}_{(a_{2k+1}, a_{2k})} \omega,$$  
(4.7)

$$y(t^*) = \text{ess sup}_{(a_{2k+1}, a_{2k})} \omega \quad \text{and} \quad y(t) \leq \text{ess sup}_{(a_{2k+1}, a_{2k})} \omega, \quad t \in (s^*, t^*).$$  
(4.8)

Using the same observation as in the proof of (4.2), the inequalities (4.7) and (4.8) verify (4.3).
5. Upper bounds for $|G_\varepsilon(y)|$ and $M^*(G(y))$

In this section, we will derive an upper bound for the behaviour of $|G_\varepsilon(y)|$ as $\varepsilon \approx 0$, where $y$ is any smooth enough solution of (1.1). In this direction, we introduce the second type of rapid oscillations near the boundary point $t = a$. For the record, the function $y(t) = t^\alpha \cos(1/t^\beta)$, where $0 < \alpha < \beta < \infty$ has such type of rapid oscillations near $t = 0$.

**Definition 5.1.** Let $a_k$ be a decreasing sequence of real numbers from interval $(a, b)$ satisfying

\[
a_k \searrow a \quad \text{and there is an } \varepsilon_1 > 0 \quad \text{such that for each } \varepsilon \in (0, \varepsilon_1)
\]

there is an $m(\varepsilon) \in \mathbb{N}$ such that $a_{j-1} - a_j > 4\varepsilon$, for each $j \leq m(\varepsilon)$. \hfill (5.1)

Let $\tilde{\theta}(t)$ and $\tilde{\omega}(t)$ be two measurable and bounded functions, both defined on $[a, b]$, such that $\tilde{\theta}(t) \leq \tilde{\omega}(t)$, for each $t \in [a, b]$. We say that a real function $y$ defined on $[a, b]$ has convex-concave rapid oscillations in respect to $(\tilde{\theta}, \tilde{\omega}, a_k)$ if there hold true:

- $y$ is concave in $(a_{2k}, a_{2k-1})$ and $y$ is convex in $(a_{2k+1}, a_{2k})$, \hfill (5.2)
- for each $k \geq 1$ and
- $\tilde{\theta}(t) \leq y(t) \leq \tilde{\omega}(t)$ \quad for each $t \in [a, b]$. \hfill (5.3)

In contrast to (2.1) where the number $k(\varepsilon)$ was appearing like an index of $\varepsilon$-density of $a_k$, here in (5.1) the number $m(\varepsilon)$ could be taken as an index of $\varepsilon$-separation of the most finite numbers of $a_k$. Let us remark that there is a sequence $a_k$ which satisfies both conditions (2.1) and (5.1), such that $\tilde{\theta}(t) \leq \tilde{\omega}(t)$, for each $t \in [a, b]$. It will be the case of the following example, in which we show a calculation of the number $m(\varepsilon)$.

**Example 5.2.** Let $a_k$ be a sequence of real numbers defined as in Example 2.2 above, that is

\[
a_k = a + \frac{b - a}{2} \left( \frac{1}{k} \right)^{1/\beta}, \quad k \geq 1, \quad 0 < \beta < \infty. \hfill (5.4)
\]

Let us take for $m(\varepsilon)$ any natural number which satisfies

\[
m(\varepsilon) \leq 2 \left( \frac{b - a}{\beta^{2+2/\beta}} \right)^{\beta/\beta+2/\beta} \varepsilon^{-\beta/\beta+2/\beta} \quad \text{for each } \varepsilon \in (0, \varepsilon_1) \text{ and } \varepsilon_1 > 0. \hfill (5.5)
\]

Using an elementary inequality

\[
\frac{1}{\beta} \left( \frac{1}{j} \right)^{1+1/\beta} \leq \left( \frac{1}{j-1} \right)^{1/\beta} - \left( \frac{1}{j} \right)^{1/\beta},
\]

where $\beta > 0$ and $j \geq 2$, it is easy to show that the sequence $a_k$ defined in (5.4) satisfies the condition (5.1) in respect to $m(\varepsilon)$ determined in (5.5).

It is clear now that by the help of Example 2.2 and Example 5.2 we have in (5.4) a sequence of real numbers which satisfies both conditions (2.1) and (5.1).

Now, we are interested to relate the convex-concave rapid oscillation of a function $y$ with the asymptotic behaviour of $|G_\varepsilon(y)|$ as $\varepsilon \approx 0$.

**Lemma 5.3.** Let $a_k$ be a decreasing sequence of real numbers from $(a, b)$ satisfying (5.1). Let $\tilde{\theta}(t)$ and $\tilde{\omega}(t)$ be two continuous functions on $[a, b]$ satisfying $\tilde{\theta}(a) = \tilde{\omega}(a) = 0$ and

\[
\tilde{\theta} \quad \text{is decreasing, and } \tilde{\omega} \quad \text{is increasing in } [a, b]. \hfill (5.6)
\]
Let $y \in C^2([a,b]) \cap C([a,b])$ have convex-concave rapid oscillations in respect to $(\hat{\theta}, \hat{\omega}, a_k)$. Then for each $\varepsilon \in (0, \varepsilon_1)$ we have
\[
|G_\varepsilon(y|a,a_1)| \leq (a_{m(\varepsilon)} - a + 2\varepsilon)(\hat{\omega}(a_{m(\varepsilon)}) - \hat{\theta}(a_{m(\varepsilon)}) + 2\varepsilon) + \varepsilon \sum_{j=2}^{m(\varepsilon)} [(6 + \pi)(\hat{\omega}(a_{j-1}) - \hat{\theta}(a_{j-1})) + 2(a_{j-1} - a_j)] + 2(\pi + 4)\varepsilon^2 m(\varepsilon),
\]
where \(\varepsilon_1\) and \(m(\varepsilon)\) are defined in the assumption \((5.1)\) and where \(a_1\) is the first member of \(a_k\).

The proof of this lemma is omitted because it is very similar to the proof of \([9, \text{Lemma } 2.2, \text{p. } 273-277]\).

Next, we give an example for the calculation of the right hand side in \((5.7)\).

**Example 5.4.** Let \(y \in C^2((0,1)) \cap C([0,1])\) be a real continuous function on \([0,1]\) and let \(y\) have convex-concave rapid oscillations in respect to \((\hat{\theta}, \hat{\omega}, a_k)\), where \(\hat{\theta}, \hat{\omega}\) and \(a_k\) are given by
\[
a_k = \frac{1}{2} (\frac{1}{k})^{1/\beta}, \quad k \geq 1,
\]
\[
\hat{\theta}(t) = -2t^\alpha \quad \text{and} \quad \hat{\omega}(t) = 2t^\beta, \quad t \in [0,1],
\]
where \(\alpha\) and \(\beta\) satisfy \(0 < \alpha < \beta < \infty\).

We take for \(m(\varepsilon)\) any number which satisfies
\[
(\beta 2^{4+2/\beta} \varepsilon)^{-\frac{\beta}{\beta + 1}} \leq m(\varepsilon) \leq 2(\beta 2^{4+2/\beta} \varepsilon)^{-\frac{\beta}{\beta + 1}} \quad \text{for each } \varepsilon \in (0, \varepsilon_1),
\]
where \(\varepsilon_1 = 1/(\beta 2^{4+2/\beta})\). It is easy to check that \(m(\varepsilon) \in \mathbb{N}\) for each \(\varepsilon \in (0, \varepsilon_1)\).

Since from \((5.9)\) follows \((5.5)\) we know that the sequence \(a_k\) defined in \((5.8)\) satisfies the condition \((5.1)\), where \(m(\varepsilon)\) is determined by \((5.9)\). Also, it is clear that the functions \(\hat{\theta}\) and \(\hat{\omega}\) defined in \((5.8)\) satisfy the condition \((5.6)\). Thus, the data \(\hat{\theta}, \hat{\omega}\) and \(a_k\) from \((5.8)\) satisfy the assumptions of Lemma \((5.3)\) and therefore we may calculate \((5.7)\), where \(a = 0\).

Let us remark that, in particular, from \((5.9)\) we have
\[
\frac{1}{m(\varepsilon)} \leq (\beta 2^{4+2/\beta} \varepsilon)^{\frac{-\beta}{\beta + 1}} \quad \text{for each } \varepsilon \in (0, \varepsilon_1).
\]

From \((5.8)\), and using the inequality \((5.10)\), for each \(\varepsilon \in (0, \varepsilon_1)\) we get
\[
(a_{m(\varepsilon)} + 2\varepsilon)(\hat{\omega}(a_{m(\varepsilon)}) - \hat{\theta}(a_{m(\varepsilon)}) + 2\varepsilon) \leq c_1 \varepsilon^{\frac{\alpha + 1}{\beta + 1}} + c_2 \varepsilon^{\frac{\beta + 1}{\beta + 1}} + c_3 \varepsilon^{\frac{\alpha + 1}{\beta + 1}} + 4\varepsilon^2,
\]
where
\[
c_1 = 2^{1-\alpha + (4/\beta) \frac{\alpha + 1}{\beta + 1}} \beta^{\frac{\alpha + 1}{\beta + 1}}, \quad c_2 = 2^{\alpha - (4 + \frac{1}{\beta}) \frac{\alpha + 1}{\beta + 1}} \beta^{\frac{\alpha + 1}{\beta + 1}}, \quad c_3 = 2^{(4 + \frac{1}{\beta}) \frac{\alpha + 1}{\beta + 1}} \beta^{\frac{\alpha + 1}{\beta + 1}}.
\]

From \((5.8)\), \((5.9)\), and using the inequality
\[
\sum_{j=1}^{n} \left(\frac{1}{j}\right)^H \leq \frac{2}{1 - H} n^{1-H}
\]
for each \(n \in \mathbb{N}\) and \(H \in (0,1)\), we obtain
\[
(6 + \pi)\varepsilon \sum_{j=2}^{m(\varepsilon)} \left[\hat{\omega}(a_{j-1}) - \hat{\theta}(a_{j-1})\right] \leq c_4 \varepsilon^{\frac{\alpha + 1}{\beta + 1}}
\]
where \(\varepsilon_1\) and \(m(\varepsilon)\) are defined in the assumption \((5.1)\) and where \(a_1\) is the first member of \(a_k\).
for each $\varepsilon \in (0, \varepsilon_1)$, where
\[
c_4 = \frac{6 + \pi}{\beta - \alpha} 2^{4 - \alpha - \frac{\beta}{\pi} - (4 + \frac{\beta}{\pi}) \frac{\alpha - \beta}{\pi + 1}}.
\]

From (5.8), and using the inequality
\[
\left(\frac{1}{j - 1}\right)^{1/\beta} - \left(\frac{1}{j}\right)^{1/\beta} \leq \frac{1}{\beta} \left(\frac{1}{j - 1}\right)^{1 + 1/\beta}
\]
for each $j \in \mathbb{N}$ and $\beta > 0$, we get
\[
2\varepsilon \sum_{j=2}^{m(\varepsilon)} (a_{j-1} - a_j) \leq c_5 \varepsilon \quad \text{for each } \varepsilon \in (0, \varepsilon_1),
\]
where
\[
c_5 = \frac{1}{\beta} \sum_{j=1}^{\infty} \left(\frac{1}{j}\right)^{1+1/\beta}.
\]

From (5.8), and using the right inequality from (5.9) we have
\[
2(\pi + 4)\varepsilon^2 m(\varepsilon) \leq c_6 \varepsilon^\frac{\alpha + 2}{\pi + 2} \quad \text{for each } \varepsilon \in (0, \varepsilon_1),
\]
where
\[
c_6 = (\pi + 4)2^{2 - (4 + \frac{\beta}{\pi})} \frac{\alpha + 1}{\pi + 1} \beta^{-\frac{\beta}{\pi + 1}}.
\]

Putting the inequalities (5.11), (5.13), (5.15), and (5.16) into (5.7) we obtain that for each $\varepsilon \in (0, \varepsilon_1)$ there holds
\[
|G_{\varepsilon}(y|_{[0, a_1]})| \leq (c_1 + c_4)\varepsilon^\frac{\alpha + 1}{\pi + 1} + c_5 \varepsilon + (c_3 + c_6)\varepsilon^\frac{\alpha + 2}{\pi + 2} + c_2 \varepsilon^\frac{\alpha + 3}{\pi + 1} + 4\varepsilon^2,
\]
where the constants $c_1, c_2, c_3, c_4, c_5, c_6$ are determined in the process above.

Next, we give some sufficient conditions on the nonlinearity $f(t, \eta, \xi)$ such that each smooth enough solution can have convex-concave rapid oscillations in the sense of Definition 5.1

**Lemma 5.5.** Let $a_k$ be a decreasing sequence of real numbers from $(a, b)$ which satisfies (5.1). Let $\theta(t)$ and $\tilde{\omega}(t)$ be two continuous functions on $[a, b]$ satisfying $\theta(a) = \tilde{\omega}(a) = 0$ and
\[
\tilde{\theta} \text{ is decreasing and convex on } [a, b], \quad \tilde{\omega} \text{ is increasing and concave on } [a, b].
\]

Let the Carathéodory function $f(t, \eta, \xi)$ satisfy
\[
f(t, \eta, \xi) < 0, \quad t \in (a, b), \quad \eta > \tilde{\omega}(t), \quad \xi \in \mathbb{R},
\]
\[
f(t, \eta, \xi) > 0, \quad t \in (a, b), \quad \eta < \tilde{\theta}(t), \quad \xi \in \mathbb{R},
\]
and let for each $k \in \mathbb{N},$
\[
f(t, \eta, \xi) > 0, \quad t \in (a_{2k}, a_{2k-1}), \quad \eta \in (\tilde{\theta}_0, \tilde{\omega}(t)), \quad \xi \in \mathbb{R},
\]
\[
f(t, \eta, \xi) < 0, \quad t \in (a_{2k+1}, a_{2k}), \quad \eta \in (\tilde{\theta}(t), \tilde{\omega}_0), \quad \xi \in \mathbb{R},
\]
where $\tilde{\theta}_0$ and $\tilde{\omega}_0$ be two arbitrary given real numbers such that $\tilde{\theta}_0 \leq \tilde{\theta}(b) < 0$ and $0 < \tilde{\omega}(b) \leq \tilde{\omega}_0$. Then each solution $y$ of (1.1) such that $y \in C^2((a, b]) \cap C([a, b])$ has convex-concave rapid oscillations in respect to $(\tilde{\theta}, \tilde{\omega}, a_k)$.

The assumption (5.17) can be avoided in a particular case as follows.
Lemma 5.6. Let \( a_k \) be a decreasing sequence of real numbers from \((a, b)\) which satisfies \([5.1]\). Let \( \tilde{\theta}(t) \) and \( \tilde{\omega}(t) \) be two real functions on \([a, b]\) defined by
\[
\tilde{\theta}(t) = -2(t - a) \quad \text{and} \quad \tilde{\omega}(t) = 2(t - a) \quad \text{for each} \ t \in [a, b].
\]
Let the Carathéodory function \( f(t, \eta, \xi) \) satisfy the hypotheses \([5.18] \) and \([5.19]\) in respect to \( \tilde{\theta} \) and \( \tilde{\omega} \) defined in \([5.20]\). Then each solution \( y \) of \((1.1)\) such that \( y \in C^2((a, b)) \cap C([a, b]) \) has convex-concave rapid oscillations in respect to \((\tilde{\theta}, \tilde{\omega}, a_k)\).

Proof. From Lemma 5.5 or Lemma 5.6 we have that each solution \( y \) of \((1.1)\) such that \( y \in C^2((a, b)) \cap C([a, b]) \) has convex-concave rapid oscillations in respect to \((\tilde{\theta}, \tilde{\omega}, a_k)\). Now, from Lemma 5.3 we have that such solutions \( y \) satisfy the inequality \([5.7]\). Multiplying both side in \([5.7]\) by \((2\varepsilon)^{s-2}\) and taking \( \lim \sup \) as \( \varepsilon \to 0 \) we immediately obtain the inequality \([5.21]\). Note that it is not necessary to consider the part \( y|_{[a_1,b]} \), because \( y \in W^{1,p}(a_1,b) \).

Theorem 5.7. Let \( a_k \) be a decreasing sequence of real numbers from \((a, b)\) which satisfies \([5.1]\). Let \( \tilde{\theta}(t) \) and \( \tilde{\omega}(t) \) be two continuous functions on \([a, b]\), \( \tilde{\theta}(a) = \tilde{\omega}(a) = 0 \) which satisfy \([5.6]\) and either \([5.17]\) or \([5.20]\). Next, let the Carathéodory function \( f(t, \eta, \xi) \) satisfy the hypotheses \([5.18] \) and \([5.19]\). Then each solution \( y \) of \((1.1)\) such that \( y \in C^2((a, b)) \cap C([a, b]) \) satisfies
\[
M^s(G(y)) \leq 2s^{-2} \limsup_{\varepsilon \to 0} \varepsilon^{s-2} \{ (a_m(\varepsilon) - a) (\tilde{\omega}(a_m(\varepsilon)) - \tilde{\theta}(a_m(\varepsilon))) + \varepsilon \sum_{j=2}^{m(\varepsilon)} [ (6 + \pi) (\tilde{\omega}(a_{j-1}) - \tilde{\theta}(a_{j-1})) + 2(a_{j-1} - a_j) ] + 2(\pi + 4)\varepsilon^2 m(\varepsilon) \},
\]
where \( s \in (1, 2) \).

Proof. From Lemma 5.5 or Lemma 5.6 we have that each solution \( y \) of \((1.1)\) such that \( y \in C^2((a, b)) \cap C([a, b]) \) has convex-concave rapid oscillations in respect to \((\tilde{\theta}, \tilde{\omega}, a_k)\). Now, from Lemma 5.3 we have that such solutions \( y \) satisfy the inequality \([5.7]\). Multiplying both side in \([5.7]\) by \((2\varepsilon)^{s-2}\) and taking \( \lim \sup \) as \( \varepsilon \to 0 \) we immediately obtain the inequality \([5.21]\). Note that it is not necessary to consider the part \( y|_{[a_1,b]} \), because \( y \in W^{1,p}(a_1,b) \).

At the end of this section, we are able to derive the constant \( m_s \) appearing in \([1.6]\).

Corollary 5.8. For arbitrarily given \( s \in (1, 2) \) let \( a_k, \tilde{\theta}(t), \) and \( \tilde{\omega}(t) \) be defined by
\[
\begin{align*}
a_k &= a + \frac{b - a}{2} (\frac{1}{k})^{1/\beta}, \quad k \geq 1, \\
\tilde{\theta}(t) &= -\tilde{\omega}(t) \quad \text{and} \quad \tilde{\omega}(t) = 2(t - a), \quad t \in (a, b),
\end{align*}
\]
where \( 1 < \beta < \infty \) and \( \frac{2\beta}{\beta + 1} = s \).
Using inequality (5.12), from (5.22) and (5.25) we have
\[ M^*(G(y)) \leq m_s(b-a)^s, \] (5.23)
where
\[ m_s = 2^6 + 2^2(6 + \pi) \frac{2 - s}{s - 1}. \] (5.24)

**Proof.** As in Example 5.4 we take for \( m(\varepsilon) \) any number that satisfies
\[ \left( \frac{b - a}{\beta 2^{4 + 2/\beta}} \right) \frac{\pi^{1/\beta}}{\varepsilon^{1/\beta}} \leq m(\varepsilon) \leq 2 \left( \frac{b - a}{\beta 2^{4 + 2/\beta}} \right) \frac{\pi^{1/\beta}}{\varepsilon^{1/\beta}}, \] (5.25)
where \( \varepsilon \in (0, \varepsilon_1 = \frac{b - a}{\beta 2^{4 + 2/\beta}}) \). From (5.22) and (5.25) we have
\[
(a_{m(\varepsilon)} - a)(\tilde{\omega}(a_{m(\varepsilon)}) - \tilde{\theta}(a_{m(\varepsilon)})) = 4 \left( \frac{b - a}{2} \right)^2 \left( \frac{1}{m(\varepsilon)} \right)^{2/\beta} \\
\leq 2^6(b - a)^{2\varepsilon^2 - s} \quad \text{for each} \quad \varepsilon \in (0, \varepsilon_1),
\]
where \( s \in (1, 2) \). This implies
\[ 2^{s-2} \limsup_{\varepsilon \to 0} \varepsilon^{s-2} \left[ (a_{m(\varepsilon)} - a)(\tilde{\omega}(a_{m(\varepsilon)}) - \tilde{\theta}(a_{m(\varepsilon)})) \right] \leq 2^6(b - a)^s. \] (5.26)

Using inequality (5.12), from (5.22) and (5.25) we have
\[
\varepsilon \sum_{j=2}^{m(\varepsilon)} (6 + \pi)(\tilde{\omega}(a_{j-1}) - \tilde{\theta}(a_{j-1})) = \frac{b - a}{2} (6 + \pi) \varepsilon \sum_{j=2}^{m(\varepsilon)} \left( \frac{1}{j - 1} \right)^{1/\beta} \\
\leq 2(6 + \pi)(b - a) \frac{2}{1 - 1/\beta} \varepsilon (m(\varepsilon))^{1-1/\beta} \\
\leq (6 + \pi)^{1 - \frac{1}{\beta} - (4 + \frac{1}{\beta}) \frac{1}{2^{4 + 2/\beta}} \frac{\beta \pi^{1/\beta}}{\beta - 1} (b - a)^{\frac{2\beta}{\beta + 1}} \varepsilon \frac{2}{\pi^{1/\beta}} \\
\leq 2^2(6 + \pi) \frac{2 - s}{s - 1} (b - a)^s \varepsilon^{2-s} \quad \text{for each} \quad \varepsilon \in (0, \varepsilon_1).
\]

For any \( s \in (1, 2) \), this implies
\[ 2^{s-2} \limsup_{\varepsilon \to 0} \varepsilon^{s-2} \left[ \varepsilon \sum_{j=2}^{m(\varepsilon)} (6 + \pi)(\tilde{\omega}(a_{j-1}) - \tilde{\theta}(a_{j-1})) \right] \leq \frac{2^2(6 + \pi)(2 - s)}{s - 1} (b - a)^s. \] (5.27)

Using inequality (5.14), from (5.22) and (5.25) we have
\[
2 \varepsilon \sum_{j=2}^{m(\varepsilon)} (a_{j-1} - a_j) = (b - a) \varepsilon \sum_{j=2}^{m(\varepsilon)} \left( \frac{1}{j - 1} \right)^{1/\beta} - \left( \frac{1}{j} \right)^{1/\beta} \\
\leq \varepsilon \frac{b - a}{\beta} \sum_{j=2}^{\infty} \left( \frac{1}{j - 1} \right)^{1+1/\beta} \quad \text{for each} \quad \varepsilon \in (0, \varepsilon_1).
\]
For any \( s \in (1, 2) \), this implies
\[
2^{s-2} \limsup_{\varepsilon \to 0} \varepsilon^{s-2} \left[ 2\varepsilon \sum_{j=2}^{m(\varepsilon)} (a_{j-1} - a_j) \right] = 0. \tag{5.28}
\]

Finally, from (5.22) and (5.25), we have
\[
2(\pi + 4)\varepsilon^2 m(\varepsilon) \leq 4(\pi + 4) \left( \frac{b - a}{\beta^{2+\gamma/\beta}} \right)^{\frac{\gamma}{\beta}} \varepsilon^{2-\frac{\gamma}{\beta}}. \tag{5.22}
\]

For any \( s \in (1, 2) \), this implies
\[
2^{s-2} \limsup_{\varepsilon \to 0} \varepsilon^{s-2} \left[ 2(\pi + 4)\varepsilon^2 m(\varepsilon) \right] = 0. \tag{5.29}
\]

Putting (5.26), (5.27), (5.28), and (5.29) into (5.21) we obtain (5.23) and (5.24). \( \square \)

At the end of this section we consider both inequalities in (1.6). In this direction, let us remark that both types of hypotheses on the nonlinear term \( f(t, \eta, \xi) \) appearing in Corollary 2.8 and Corollary 5.8 are completely harmonized. It is because the data \( \theta, \omega, \tilde{\theta}, \tilde{\omega} \) defined in (2.21) and (5.22) satisfy
\[
\tilde{\theta}(t) < \theta(t) < 0 < \omega(t) < \tilde{\omega}(t),
\]
\[
\text{ess inf}_{(a_{2k-1}, a_{2k})} \theta \geq \text{ess sup}_{(a_{2k+1}, a_{2k})} \tilde{\theta},
\]
\[
\text{ess sup}_{(a_{2k}, a_{2k-1})} \omega \leq \text{ess inf}_{(a_{2k-1}, a_{2k})} \tilde{\omega}. \tag{5.30}
\]

Therefore, we may combine Corollary 2.8 and Corollary 5.8 to obtain the following consequence.

**Corollary 5.9.** For arbitrarily \( s \in (1, 2) \) let \( a_k, \theta, \omega, \tilde{\theta}(t), \text{ and } \tilde{\omega}(t) \) be given by (2.21) and (5.22). Let the Carathéodory function \( f(t, \eta, \xi) \) satisfy the hypotheses (2.13), (2.16), and (5.18), \( \text{and } (5.19) \) in respect to such given \( \theta, \omega, \tilde{\theta}, \tilde{\omega}, a_k \). Then each solution \( y \) of (1.1), such that \( y \in C^2(a, b) \) satisfies
\[
\frac{1}{2}(b - a)^s \leq M^s(G(y)) \leq m_s(b - a)^s, \tag{5.31}
\]
where the constant \( m_s \) is defined in (5.24).

Thus, because of (5.31), we have proved the desired statement (1.6).

6. **Lower Bound for the \( s \)-Dimensional Upper Density**

In this section, we derive the inequalities (1.10) and (1.11). In this direction, we need some preliminaries. The first one is a version of Lemma 2.3 above.

**Lemma 6.1.** Let \( a_k \) be a decreasing sequence of real numbers from interval \((a, b)\) satisfying (2.1). Let \( \theta(t) \) and \( \omega(t) \) be two measurable and bounded real functions on \([a, b], \theta(t) \leq \omega(t), t \in [a, b], \) which satisfy (2.5). Let \( y \) be \((\theta, \omega, a_k)\)-rapidly oscillating function on \([a, b]\) and let \( y \in C([a, b]) \). Let \( k(\varepsilon) \) and \( \varepsilon_0 \) be from (2.1). Then for any \( c \in (a, b) \) such that
\[
\text{there exists } \varepsilon_c \in (0, \varepsilon_0) \text{ satisfying } a_{k(\varepsilon)} - 1 \in (a, c) \text{ for each } \varepsilon \in (0, \varepsilon_c), \tag{6.1}
\]
we have
\[
|G_c(y|_{[a, c]})| \geq \int_{a}^{a_{k(\varepsilon)}} (\omega(t) - \theta(t)) dt \text{ for each } \varepsilon \in (0, \varepsilon_c). \tag{6.2}
\]
Proof. Regarding to the last line in the proof of Lemma 2.3 above and using the same notations, we have already proved that
\[ A(\varepsilon, \theta, \omega) \subseteq G_{\varepsilon} \left( \bigcup_{k=1}^{\infty} y|_{[\sigma_k, \sigma_{k+1}]} \right) \quad \text{for each } \varepsilon \in (0, \varepsilon_0), \]
where \( k(\varepsilon) \) and \( \varepsilon_0 \) are appearing in (2.1). Let \( c \in (a, b) \) be a real number which satisfies (6.1). It is clear that
\[ G_{\varepsilon}(y|_{[a, a_0(c)]}) \subseteq G_{\varepsilon}(y|_{[a, c]}) \quad \text{for each } \varepsilon \in (0, \varepsilon_c). \] (6.4)

Since \( \varepsilon_c \in (0, \varepsilon_0) \) and \( \sigma_k \in (a_k, a_{k-1}) \), combining (6.3) and (6.4) we obtain:
\[ A(\varepsilon, \theta, \omega) \subseteq G_{\varepsilon} \left( \bigcup_{k=1}^{\infty} y|_{[\sigma_k, \sigma_{k+1}]} \right) \]
\[ = G_{\varepsilon}(y|_{[a, a_{k(c)}]}) \subseteq G_{\varepsilon}(y|_{[a, c]}); \]
that is,
\[ A(\varepsilon, \theta, \omega) \subseteq G_{\varepsilon}(y|_{[a, c]}) \quad \text{for each } \varepsilon \in (0, \varepsilon_c). \]
(6.5)

Taking the Lebesgue measure of the both sides in (6.5) we get (6.2). □

A choice for the number \( \varepsilon_c \) appearing in (6.1) and (6.2) will be given in Corollary 6.3 below. Analogously to Theorem 6.2 we can state the following result.

**Theorem 6.2.** Let \( a_k \) be a decreasing sequence of real numbers from interval \( (a, b) \) satisfying (2.1). Let \( \theta(t) \) and \( \omega(t) \) be two measurable and bounded real functions on \( (a, b] \), \( \theta(t) \leq \omega(t) \), \( t \in (a, b] \), which satisfy (2.5), (2.12), and (2.18). Next, let the Carathéodory function \( f(t, \eta, \xi) \) satisfy (2.13), (2.16), and (2.19). Let \( c \in (a, b) \) and \( \varepsilon_c \) be numbers satisfying (6.1). Then each solution \( y \) of the equation (1.1) satisfies
\[ |G_{\varepsilon}(y|_{[a, c]})| \geq \int_a^{a_k(c)} (\omega(t) - \theta(t))dt \quad \text{for each } \varepsilon \in (0, \varepsilon_c), \]
\[ M^s(G(y|_{[a, c]})) \geq 2s^{-2} \limsup_{\varepsilon \to 0} \varepsilon^{s-2} \int_a^{a_k(c)} (\omega(t) - \theta(t))dt, \]
for any \( s \in (1, 2) \).

The proof of this theorem is the same as the proof of Theorem 6.2 but using Lemma 6.1 instead of Lemma 2.3. Now, from Theorem 6.2 we obtain the following results.

**Corollary 6.3.** For an arbitrarily real number \( s \in (1, 2) \), let \( a_k \), \( \theta \) and \( \omega \) be given by (2.21). Let the Carathéodory function \( f(t, \eta, \xi) \) satisfy (2.13), (2.16), and (2.19) in respect to such given \( (\theta, \omega, a_k) \). Then for each \( c \in (a, b) \) and for each solution \( y \) of the equation (1.1) there holds
\[ |G_{\varepsilon}(y|_{[a, c]})| \geq \frac{1}{26} (c - a)^s \varepsilon^{2-s} \quad \text{for each } \varepsilon \in (0, \varepsilon_c), \]
(6.6)
\[ M^s(G(y|_{[a, c]})) \geq \frac{1}{27} (c - a)^s, \]
where
\[ \varepsilon_c = \min\{\varepsilon_0, 1 \beta (b - a)^{\beta+1} \} \quad \text{and } \varepsilon_0 = \frac{b - a}{\beta}. \] (6.7)

Proof. It is simple to check that every \( c \in (a, b) \) satisfies the condition (6.1) in respect to \( a_k, k(\varepsilon) \) and \( \varepsilon_c \) given in (2.21), (2.11) and (6.7), respectively. Next, using the same calculation as in the proof of Corollary 2.8 the statement (6.6) immediately follows from Theorem 6.2. □
To derive the inequalities (1.10) and (1.11) we need a comparison result for solutions of (1.1) which is a modification of [9, Lemma 3.1, p. 278].

**Lemma 6.4.** Let $\tilde{\theta}(t) = -2(t - a)$ and $\tilde{\omega}(t) = 2(t - a)$. Let the Caratheodory function $f(t, \eta, \xi)$ satisfy

$$f(t, \eta, \xi) < 0, \quad t \in (a, b), \quad \eta > \tilde{\omega}(t), \quad \xi \in \mathbb{R},$$

$$f(t, \eta, \xi) > 0, \quad t \in (a, b), \quad \eta < \tilde{\theta}(t), \quad \xi \in \mathbb{R}.$$

Then each solution $y$ of (1.1) such that $y \in C^2(a, b)$ satisfies

$$\tilde{\theta}(t) \leq y(t) \leq \tilde{\omega}(t) \quad \text{for each } t \in [a, b]. \quad (6.8)$$

The proof of this lemma is omitted because it is the same as the proof of [9, Lemma 3.1, p. 278]. Finally, according to Corollary 6.3 and Lemma 6.4 we are able to state the main result of this section.

**Corollary 6.5.** For an arbitrarily $s \in (1, 2)$, let the hypotheses of Corollary 6.3 and Lemma 6.4 be still valid. Then each solution $y$ of (1.1) such that $y \in C^2(a, b)$ satisfies

$$M^*(G(y) \cap B_r(a, 0)) \geq \frac{1}{2^7} \left(\frac{r}{\sqrt{5}}\right)^s \quad \text{for each } r \in (0, b - a), \quad (6.9)$$

where $G(y)$ satisfies

$$\tilde{\omega}(t) \leq y(t) \leq \tilde{\theta}(t) \quad \text{for each } t \in [a, b].$$

That is, the constant $d_s$ appearing in (1.10) and (1.11) satisfies

$$d_s = \frac{1}{2^7} \left(\frac{1}{2\sqrt{5}}\right)^s.$$

**Proof.** Let us remark that because of (5.30) the assumptions of Corollary 6.3 and Lemma 6.4 are completely harmonized, where $a_k, \theta$ and $\omega$ be given by (2.21) and where $\tilde{\theta}(t) = -2(t - a)$ and $\tilde{\omega}(t) = 2(t - a)$. Therefore, the main conclusions of Corollary 6.3 and Lemma 6.4 may be used together.

By (6.8) and making intersections of $\tilde{\theta}(t)$ and $\tilde{\omega}(t)$ with $B_r(a, 0)$, it is clear that for any $r \in (0, \sqrt{5}(b - a))$ we have

$$G(y|_{[a, a + \sqrt{5}r]}) \subseteq G(y) \cap B_r(a, 0),$$

where $y$ is any smooth enough solution of (1.1). Since $M^*$ is a monotone set function, it yields

$$M^*(G(y|_{[a, a + \sqrt{5}r]}) \subseteq M^*(G(y) \cap B_r(a, 0)). \quad (6.10)$$

Next, we apply Corollary 6.3 to $y|_{[a, a + \sqrt{5}r]}$ and from (6.6) we derive

$$M^*(G(y|_{[a, a + \sqrt{5}r]}) \geq \frac{1}{2^7} \left(\frac{r}{\sqrt{5}}\right)^s \quad \text{for any } r \in (0, b - a). \quad (6.11)$$

Combining (6.10) and (6.11) we conclude that

$$M^*(G(y) \cap B_r(a, 0)) \geq \frac{1}{2^7} \left(\frac{r}{\sqrt{5}}\right)^s \quad \text{for any } r \in (0, b - a).$$

Multiplying both inequalities by $1/(2r)^s$ and taking lim sup as $r \to 0$ we immediately derive the desired statement (6.9).
7. Appendix

In this appendix, we sketch the proof of Lemma 2.5. Under the assumptions of Lemma 2.5 we verify that each solution \( y \) of the equation (1.1) which satisfies (2.17) is \((\theta, \omega, a_k)\)-rapidly oscillating on \([a, b]\). In this proof, a simple method of the localisation of integration in (1.1) is exploited, often used in analysis of local regular properties of solutions of PDE’s (see for instance [3, 6, 12, 13]).

Regarding Definition 2.1, it is sufficient to prove that for any fixed \( k \in \mathbb{N} \) the hypotheses (2.13), (2.14) and (2.17) verify that

\[
\exists \sigma_{2k} \in (a_{2k}, a_{2k-1}) \text{ such that } y(\sigma_{2k}) \geq \text{ess sup}_{(a_{2k}, a_{2k-1})} \omega, \tag{7.1}
\]

and on the other hand, that the hypotheses (2.15), (2.16) and (2.17) verify that

\[
\exists \sigma_{2k+1} \in (a_{2k+1}, a_{2k}) \text{ such that } y(\sigma_{2k+1}) \leq \text{ess inf}_{(a_{2k+1}, a_{2k})} \theta. \tag{7.2}
\]

Since (7.2) is the dual statement of (7.1), in order to simplify the proof, we will prove only the statement (7.1) for a fixed \( k \in \mathbb{N} \). In this direction, let \( \sigma \) and \( r \) be two real numbers defined by

\[
\sigma = \frac{a_{2k} + a_{2k-1}}{2} \quad \text{and} \quad r = \frac{1}{4}(a_{2k-1} - a_{2k}).
\]

Let \( B_r = B_r(\sigma) \) denote a ball with radius \( r > 0 \) centered at the point \( \sigma \). Then we have:

\[
B_{2r} = B_{2r}(\sigma) = (a_{2k}, a_{2k-1}),
\]

\[
B_r = B_r(\sigma) = (a_{2k} + \frac{1}{4}(a_{2k-1} - a_{2k}), a_{2k-1} - \frac{1}{4}(a_{2k-1} - a_{2k})).
\]

Also, let \( \tilde{\theta}_0 = \text{ess inf}_{(a, b)} \theta \), and \( \tilde{\omega}_0 = \text{ess sup}_{(a, b)} \omega \), and \( \omega_r = \text{ess sup}_{B_r} \omega \), and \( J_{2r} = (\tilde{\theta}_0, \omega_{2r}) \). Because of (2.5) and (2.12) we have that \( \tilde{\theta}_0 < \omega_{2r} < \tilde{\omega}_0 \). Using the preceding notation, we can rewrite the main assumptions (2.13) and (2.14) in the form

\[
f(t, \eta, \xi) \geq 0, \quad t \in B_{2r}, \ \eta \in J_{2r}, \ \xi \in \mathbb{R}, \tag{7.3}
\]

\[
\int_{B_r(\eta, \xi) \subset J_{2r} \times \mathbb{R}} \text{ess inf} f(t, \eta, \xi) dt \geq \frac{c(p) - 1}{4^{p-1} r^{p-1}} \frac{\tilde{\omega}_0 - \tilde{\theta}_0}{\tilde{\omega}_0 - \omega_{2r}} \tag{7.4}
\]

\[
= (p-1)^{p-1} \frac{\tilde{\omega}_0 - \tilde{\theta}_0}{\tilde{\omega}_0 - \omega_{2r}} \frac{|B_r|}{r^p},
\]

where we have used \( |B_r| = 2r \) and \( c(p) = 2[4(p-1)]^{p-1} \). Next, let \( y \) be a solution of (1.1) which satisfies (1.14) and (2.17). Let us suppose the contrary statement to (7.1), that is

\[
y(t) < \omega_{2r} = \text{ess sup}_{B_{2r}} \omega \quad \text{for each} \quad t \in B_{2r}. \tag{7.5}
\]

According to (2.17) and (7.3)–(7.5) we have

\[
f(t, y, y') \geq 0 \quad \text{in} \quad B_{2r}, \tag{7.6}
\]

\[
\int_{B_r} f(t, y, y') dt > (p-1)^{p-1} \frac{\tilde{\omega}_0 - \tilde{\theta}_0}{\tilde{\omega}_0 - \omega_{2r}} \frac{|B_r|}{r^p} \tag{7.7}
\]

Now we can repeat a similar argument as in [3, Theorem 5, p. 256] or [9, Lemma 4.1, p. 280]. It is known that for any \( c_0 > 1 \) there exists a function \( \Phi \in C_0^\infty(\mathbb{R}) \),
0 \leq \Phi \leq 1 \text{ in } \mathbb{R} \text{ such that the following properties are fulfilled, see [4] Lemma 5, pp. 267],}

\[ \Phi(t) = 1 \text{ for } t \in B_r \quad \text{and} \quad \Phi(t) = 0 \text{ for } t \in \mathbb{R} \setminus B_{2r}, \]

\[ \Phi(t) > 0 \text{ for } t \in B_{2r} \quad \text{and} \quad |\Phi'(t)| \leq \frac{c_0}{r}, \quad t \in \mathbb{R}. \quad (7.8) \]

For any \( c_0 > 1 \), we take a test function \( \varphi \) defined by

\[ \varphi(t) = \begin{cases} (y(t) - \tilde{\omega}_0)\Phi^p(t) & \text{if } t \in B_{2r}, \\ 0 & \text{otherwise}. \end{cases} \]

It is clear that \( \varphi \in W_0^{1,p}(B_{2r}) \cap L^\infty(B_{2r}) \). Putting in (1.1) this test function we obtain

\[ \int_{B_{2r}} |y'|^p \Phi^p dt \leq p \int_{B_{2r}} |y'|^{p-1} \Phi^{p-1}(\omega_0 - y(t))|\Phi'| dt - \int_{B_{2r}} f(t, y, y')(\omega_0 - y(t))\Phi^p dt. \quad (7.9) \]

Next, using that \( \omega_0 - y(t) \leq \omega_0 - \tilde{\theta}_0 \), and \( (p-1)p' = p \), and \( \delta_1 (p\delta_2) \leq \delta_1^{p'} + (\frac{p}{p'})^{p-1}\delta_2^p \)
in particular for

\[ \delta_1 = |y'|^{p-1}\Phi^{p-1} \quad \text{and} \quad \delta_2 = (\omega_0 - y(t))|\Phi'|, \]

from (7.9) we obtain

\[ 0 = [1 - \frac{p'}{p}] \int_{B_{2r}} |y'|^p \Phi^p dt \leq \left( \frac{p}{p'} \right)^{p-1}(\omega_0 - \tilde{\theta}_0)^p \int_{B_{2r}} |\Phi'|^p dt - \int_{B_{2r}} f(t, y, y')(\omega_0 - y(t))\Phi^p dt. \]

Now, by (7.5), (7.6) and (7.8), we have

\[ 0 \leq \left( \frac{p}{p'} \right)^{p-1}(\omega_0 - \tilde{\theta}_0)^p |B_{2r} \setminus B_r| \left( \frac{c_0}{r} \right)^p - (\omega_0 - \omega_{2r}) \int_{B_r} f(t, y, y') dt. \]

Since \( |B_{2r} \setminus B_r| = |B_r| \) and passing to the limit as \( c_0 \to 1 \) we obtain

\[ \int_{B_r} f(t, y, y') dt \leq (p-1)^{p-1} \frac{|B_r|}{p'} (\omega_0 - \tilde{\theta}_0)^p \frac{1}{r^p} (\omega_0 - \omega_{2r}). \]

This inequality contradicts (7.7) and so the assumption (7.3) is not possible. Thus, the statement (7.1) is proved.

REFERENCES

[12] J. M. Rakotoson, *Equivalence between the growth of \( \int_{B(x,r)} |\nabla u|^p \, dy \) and \( T \) in the equation \( P(u) = T \)*, J. Differential Equations, **86** (1990), 102-122.

Mervan Pašić
Department of Mathematics, Faculty of Electrical Engineering and Computing, University of Zagreb, Unska 3, 10000 Zagreb, Croatia

E-mail address: mervan.pasic@fer.hr

Vesna Županović
Department of Mathematics, Faculty of Electrical Engineering and Computing, University of Zagreb, Unska 3, 10000 Zagreb, Croatia

E-mail address: vesna.zupanovic@fer.hr