A Sixteenth-order Polylogarithm Ladder

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CONTENTS

Using the LLL algorithm and the second author's "ladder" method, we find (conjectural) $\mathbf{Z}$-linear relations among polylogarithms of order up to 16 evaluated at powers of a single algebraic number. These relations are in accordance with theoretical predictions and are valid to an accuracy of 300 decimal digits, but we cannot prove them rigorously.

1. INTRODUCTION

Let m be a positive integer. The m-th polylogarithm function is defined for complex z with $|z| < 1$ by

$$\mathrm{Li}_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m}.$$

For $m = 1$, $\mathrm{Li}_1(z) = -\log(1-z)$, while for $m > 1$, the function $\mathrm{Li}_m(z)$ is a higher transcendental function. The functions $\mathrm{Li}_m(z)$ can be extended analytically to the whole complex plane if one makes a cut, for example, the real line from 1 to infinity (corresponding, for $m = 1$, to the principal branch of the logarithm).

The function $\mathrm{Li}_1(z)$ satisfies the functional equation

$$\mathrm{Li}_1(1-xy) = \mathrm{Li}_1(1-x) + \mathrm{Li}_1(1-y),$$

which implies that any linear combination with integral coefficients of Li_1 values of elements of a number field will again be an Li_1 value of an element of this number field. The corresponding statement for higher polylogarithms is not true. However, the Li_m satisfy "trivial" functional equations relating $\mathrm{Li}_m(x)$ to $\sum_{y^p=x} \mathrm{Li}_m(y)$ for all integers $p \neq 0$; and for small orders m, they also satisfy more complicated nontrivial functional equations. (The present record, due to H. Gangl, gives nontrivial functional equations for $m = 7$.) In addition, these functional equations give relations among polylogarithm values of elements of a given

1058-6458/92 \$0.50 per page

number field. It is conjectured that *all* relations among polylogarithms of algebraic arguments can be obtained by specializing functional equations, but this is not known for any $m \ge 2$.

The main goal of this paper is to provide an example of "ladder" relations (i.e., relations among polylogarithms of powers of a fixed number α) for the number $\alpha \cong 1.17628$ defined as the unique root outside the unit circle of the self-reciprocal polynomial

$$X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1\,.$$

Dilogarithm relations for this number were first investigated by G. Ray and are given in Chapter 7 of [Lewin 1991]. The ladder relations we will find go up to order 16, confirming the general theory (as described in [Zagier 1991a] and recalled in Section 3). This is quite possibly the highest order occurring for any algebraic number, a property related to the well-known conjecture of Lehmer [1933] that α has the smallest possible Mahler measure for an algebraic integer that is not a root of unity (the Mahler measure of an algebraic integer is equal to the product of the moduli of the number's conjugates that are greater than 1, hence here equal to α). The computation of the ladder relations requires both high-accuracy computation of polylogarithms (we needed more than 300 decimal digits) and careful use of linear algebra and lattice reduction techniques for fairly large matrices with nonexact entries, so both of these issues will have to be addressed in the paper.

2. NUMERICAL COMPUTATION OF POLYLOGARITHMS

To achieve our goal, it is necessary to compute polylogarithms to a reasonably high accuracy. To do this, we use the following very simple and pretty formula for the polylogarithm, which we could not find in the literature:

Proposition 1. *For m a natural number and $|z| < 2\pi$ we have*

$$\mathrm{Li}_m(e^z) = \sum_{\substack{n\ge 0\\ n\ne m-1}} \zeta(m-n)\,\frac{z^n}{n!} + \left(1+\frac{1}{2}+\cdots+\frac{1}{m-1} - \log(-z)\right)\frac{z^{m-1}}{(m-1)!}\,.$$

The proof by induction is straightforward. We observe, as an amusing remark, that the polylogarithm of complex order s (defined for $|z|<1$ like Li_m but with m replaced by s) can be analytically extended by the formula

$$\mathrm{Li}_s\,(e^z) = \sum_{n\ge 0}\zeta(s-n)\frac{z^n}{n!} + \Gamma(1-s)\,(-z)^{s-1}$$

for all $s \notin \mathbf{N}$; the limit of the right-hand side as $s \to m \in \mathbf{N}$ is the expression occurring in the proposition.

If $x = \rho e^{i\theta} \in \mathbf{C}$ and we write $x = e^z$, the condition $|z| < 2\pi$ is equivalent to

$$\exp(-\sqrt{4\pi^2-\theta^2}) < \rho < \exp(\sqrt{4\pi^2-\theta^2}).$$

Since we may assume that $|\theta| \le \pi$, this region contains the annulus $e^{-\pi\sqrt3} < \rho < e^{\pi\sqrt3}$ (note that $e^{-\pi\sqrt3} < 0.005$ and $e^{\pi\sqrt3} > 230$). So, to compute $\mathrm{Li}_m(x)$, we use the power series if $|x|$ is small (say $|x| \le \frac12$), the formula of proposition 1 if $|x|$ is near 1 (say $\frac12 < |x| < 2$), and the functional equation relating $\mathrm{Li}_m(x)$ to $\mathrm{Li}_m(x^{-1})$ if $|x|$ is large (say $|x| > 2$; in fact, as we will see, we will never have to use this).

There is still one more point to treat. We will need polylogarithm values to several hundred decimals, hence we also need the values of $\zeta(k)$ for integers k to such a high accuracy. When $k \le 0$ or $k > 0$ and even, $\zeta(k)$ is given explicitly in terms of Bernoulli numbers B_n. We must also compute $\zeta(k)$ for k odd, $k \ge 3$. We could use the standard Euler–McLaurin method for doing this, but there is a simpler and much faster method, coming from the theory of modular forms (more precisely, by integrating k times the Fourier expansion of the holomorphic Eisenstein series of weight $k+1$); these formulas were known to Ramanujan.

Proposition 2. *For k odd, $k > 1$, we have*

$$\zeta(k) = \frac{(2\pi)^k}{k-1}\sum_{n=0}^{(k+1)/2}(-1)^n(1-2n)\beta_{2n}\beta_{k+1-2n} - 2\sum_{n\ge1}\frac{n^{-k}}{e^{2\pi n}-1}\left(1+\frac{2\pi n\varepsilon_k}{1-e^{-2\pi n}}\right),$$

where β_n denotes $B_n/n!$ and $\varepsilon_k = 0$ for $k \equiv 3 \pmod 4$, $\varepsilon_k = 2/(k-1)$ for $k \equiv 1 \pmod 4$.

The above propositions give us rapidly convergent methods for computing polylogarithms in the whole complex plane, and in particular near or on the unit circle. (Of course, Proposition 2 is not really necessary, since the values of $\zeta(k)$ for k up to 16 could be computed just once by a slower method and then stored.)

We will actually be working with a modified version of the polylogarithm function which is defined by

$$P_m(x) = \mathrm{Re}_m\biggl(\sum_{r=0}^{m} \frac{2^r B_r}{r!}(\log|x|)^r \,\mathrm{Li}_{m-r}(x)\biggr),$$

where Re_m denotes real or imaginary part depending on whether m is odd or even, and Li_0 has been set equal to $-\frac{1}{2}$ [Zagier 1991a]. In contrast with the original polylogarithms, these modified functions satisfy "clean" functional equations (i.e., equations involving only polylogarithms of a given order, without lower-order correction terms). In particular, we have $P_m(1/x) = (-1)^{m-1}P_m(x)$.

3. LADDERS

In [Abouzahra and Lewin 1985] and [Lewin 1984], recipes to obtain linear relations among polylogarithm values of powers of a single algebraic integer α are given. These recipes, which are special cases of the theory describing linear relations among polylogarithm values of arbitrary algebraic arguments (as explained in [Zagier 1991a; 1991b]), are as follows.

Assume that one knows, for some algebraic number α, a certain number of relations of the form

$$\prod_{n=1}^{\infty}(\alpha^n - 1)^{c_n} = \zeta\alpha^N$$

for some integers c_1, c_2, … ($c_n = 0$ for almost all n) and N and some root of unity ζ. Such relations are called *cyclotomic relations*. For the special number defined in the introduction, since α is real and greater than 1, we must have $\zeta = 1$; moreover, since $1/\alpha$ is a conjugate of α, we deduce by conjugating the equation that N must be equal to $\frac{1}{2}\sum_n nc_n$. Since the modified unilogarithm function P_1 is given by $P_1(x) = -\log|1-x| + \frac{1}{2}\log|x|$, the cyclotomic equation can be rewritten

$$\sum_{n=1}^{\infty} c_n\, P_1(\alpha^n) = 0.$$

So, these are our unilogarithm relations, the first rung of the ladder. The main statement of the theory is that all relations among m-th polylogarithms are obtained by $(m-1)$-fold *pseudointegration*, that is, the formal replacement of $P_1(\alpha^n)$ by $n^{-(m-1)}P_m(\alpha^n)$. In other words, any ladder relation at the m-th rung should have the form

$$\sum_{n=1}^{\infty} \frac{c_n}{n^{m-1}}\, P_m(\alpha^n) = 0$$

for some cyclotomic relation $\prod(\alpha^n-1)^{c_n} = \alpha^N$.

However, not all relations obtained by pseudointegration will hold. The rest of the picture is as follows. First, observe that the set of cyclotomic equations forms a group under multiplication, so the set of possible collections of exponents $\{c_n\}$ forms a group under addition; obviously, the same is true for the linear relations among the polylogarithms of any given order. Suppose that we have found $I(1)$ multiplicatively independent cyclotomic relations. Then, starting with $m = 2$, suppose inductively that we have $I(m-1)$ linearly independent equations at order $m-1$, say

$$S_{m-1,i}(\alpha) = \sum_{n=1}^{\infty} \frac{c_n^{(i)}}{n^{m-2}}\, P_{m-1}(\alpha^n) = 0$$

for $i = 1, \dots, I(m-1)$. We suppose further that the same relation remains true if α is replaced by any of its conjugates α^σ. (In practice, and according to conjectures, this is automatically true, but in any case, we need it for the inductive setup.) We write this symbolically as $S_{m-1,i}(\alpha^\sigma) = 0$. We then consider the pseudointegrated quantities

$$S_{m,i}(\alpha^\sigma) = \sum_{n=1}^{\infty} \frac{c_n^{(i)}}{n^{m-1}}\, P_m((\alpha^\sigma)^n).$$

There are $r_1 + 2r_2$ possible embeddings σ of $\mathbf{Q}[\alpha]$ into the complex numbers, where r_1 and r_2 denote as usual the number of real and complex conjugates of α, so for each i the collection of numbers $\{S_{m,i}(\alpha^\sigma)\}_\sigma$ can be considered as a vector in $\mathbf{R}^{r_1+2r_2}$. However, because of the functional equations of the function $P_m(x)$, not all of the components of this vector are independent, so we can think of it as a vector in a smaller dimensional vector space $\mathbf{R}^{d(m)}$ of dimension $d(m)$ depending on the parity of m. Specifically, since $P_m(x)$ is invariant under $x \mapsto \bar{x}$ for m odd and anti-invariant for

m even, we need only half of the complex conjugates and can ignore the real conjugates when m is even, reducing the number of components from r_1+2r_2 to r_1+r_2 or r_2 (in our special case, 6 and 4), respectively. Moreover, for α like ours satisfying a self-reciprocal equation, the functional equation

$$P_m(1/x) = (-1)^{m-1} P_m(x)$$

implies that we also need only half of the real embeddings when m is odd, i.e., $d(m)$ is r_2 (in our case, 4) when m is even, and $r_1/2 + r_2$ (in our case, 5) when m is odd. Now the theory says that the vectors obtained by pseudointegrating the valid $(m-1)$-th order relations will lie in a *lattice* in $\mathbf{R}^{d(m)}$, or in other words, that any $d(m)+1$ of them will satisfy a linear relation over $\mathbf{Z}$ rather than merely over $\mathbf{R}$. (This assertion, given conjecturally in [Zagier 1991a], is actually now a theorem, to appear in forthcoming work by Beilinson and Deligne.) In particular, at least $I(m) = I(m-1)-d(m)$ of our $I(m-1)$ vectors in $\mathbf{R}^{d(m)}$ will be 0. Changing bases if necessary, we can suppose that these are simply the relations $S_{m,i}(\alpha^\sigma) = 0$ for $i = 1, \ldots, I(m)$, and this puts us back in the same situation as before and lets us continue the inductive process. Summarizing, in climbing the ladder, we lose (at most) $d(m)$ relations at each step and hence, starting with the $I(1)$ cyclotomic relations, can mount the ladder as long as the quantity $I(m) = I(1) - d(2) - \cdots - d(m)$ remains positive. Note that, although the theory tells us that such relations exist, it does not give any bound on the size of the coefficients. As a consequence, although the relations which we will give are correct to several hundred decimal places, and although the number and type of relations we find agrees with the number which we know must exist, the specific relations that we give are not rigorously proved.

A striking example of a "ladder" of the sort just described is given in [Abouzahra et al. 1987] and [Zagier 1991a], where one takes for α a root of $X^3 - X - 1 = 0$. In this case, one can climb the ladder up to level 8, and at level 9, there remain 2 pseudointegrated relations which are just sufficient to give a relation between nonalogarithms if one adds $P_9(\alpha^0) = P_9(1) = \zeta(9)$ to the list of admissible powers of α. In the present paper, we study the special Salem number mentioned in the introduction, an even more spectacular example.

The first step is to get the ladder started by finding cyclotomic relations for the number α as explained at the beginning of this section. We do this as follows. First notice that the polynomial $X^n - 1$ factors as a product of cyclotomic polynomials $\Phi_k(X)$ over all divisors k of n, so that finding multiplicative relations among α and the numbers $\alpha^n - 1$ is the same as finding multiplicative relations among α and the numbers $\Phi_k(\alpha)$. Any such relation will imply a relation among the norms, so we first compute the norm of $\Phi_k(\alpha)$ for many k. We find that this norm is ± 1 for 66 values of k less than 1000, namely,

$k = 1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 13, 16, 17, 18, 20, 21, 23, 24, 27, 28, 29, 30, 34, 37, 38, 40, 42, 44, 45, 47, 50, 52, 56, 57, 60, 63, 64, 65, 66, 70, 74, 75, 76, 78, 84, 86, 92, 96, 98, 105, 110, 118, 132, 138, 144, 154, 160, 165, 186, 195, 204, 212, 240, 270, 286, 360.$

But even when the norm of $\Phi_k(\alpha)$ is not a unit, we can hope to (multiplicatively) combine several $\Phi_k(\alpha)$ to obtain a unit, and this indeed happens, giving us the nine further units $\Phi_{k_1}(\alpha)/\Phi_{k_2}(\alpha)$ for

$(k_1, k_2) = (12,4), (36,4), (62,31), (108,4), (124,31), (130,26), (175,25), (182,14), (246,82).$

Together with α itself, this gives us 76 multiplicatively independent units and hence, since the unit rank of $K = \mathbf{Q}[\alpha]$ is equal to $r_1+r_2-1 = 5$, 71 linearly independent cyclotomic relations. (The five units α and $\Phi_k(\alpha)$ for $k = 1, 2, 3, 5$ form a generating system of independent units.)

Using Baker-type methods, it is possible to give an effective upper bound on the index of the cyclotomic polynomials which have to be examined. However, as often with this type of bound, it does not seem possible to decrease it to a computationally feasible value. Hence, although it is possible that some cyclotomic relation has escaped our search, this seems quite unlikely.

4. CLIMBING THE LADDER

By the construction sketched above, from our initial $I(1) = 71$ cyclotomic relations, we must obtain $I(2) = 67$ dilogarithm relations, $I(3) = 62$ trilogarithm relations, decreasing alternatively by 4 or 5, and finally we must obtain $I(16) = 4$ hexadecalogarithm relations. (We would need six such relations to be able to climb to the 17th level.) To obtain these relations is not a completely trivial task, and in this section we explain the method that has been used to obtain them. All the computations were done using the GP/PARI calculator developed by C. Batut, D. Bernardi, H. Cohen and M. Olivier.

There are exactly 111 different exponents of α which occur in the cyclotomic relations. Hence, we will represent the cyclotomic relations as a 111×71 matrix L_1 with integral entries, each column representing a relation. This matrix is, of course, not unique, and we can try to get the coefficients as small as possible by using the LLL algorithm [Lenstra et al. 1982]. We can give a rough measure of the size of the coefficients by computing the L^2-norm of L_1. In the present case, the minimal value found (and probably the absolute minimum) is 20.

Now we pseudointegrate these relations, which simply means that we divide each of the 111 *rows* j by its corresponding exponent (from 1 to 360), which we denote by $e(j)$. Let M_2 be the resulting matrix. We form the 4×111 matrix $V_2 = (v_{i,j})$ with $v_{i,j} = P_2(\alpha_i^{e(j)})$, where the α_i are four nonreal conjugates of α (one for each complex conjugate pair), and then form the 4×71 matrix $V_2 \cdot M_2$ of 4-tuples of dilogarithm values for the 71 pseudointegrated relations. This matrix turns out to have maximal rank (4), so we pick four independent columns (actually, the first four), and then form the 71×67 matrix

$$K_2 = \ker(V_2 \cdot M_2),$$

whose 67 columns are the relations obtained by writing each of the remaining (last) 67 columns of $V_2 \cdot M_2$ as a linear combination of the chosen four. (Notice that this procedure is numerically well-defined and stable even though the entries of our matrices are only approximate, because the invertibility of the 4×4 matrix depends on the nonvanishing of a number, and this is something which can be checked numerically.) The theory described in Section 3 tells us that K_2 should, in fact, have rational coefficients, and it is then easy to see that a possible choice for the matrix L_2 of relations for dilogarithms would then be $L_2 = M_2 \cdot K_2$.

The first problem that faces us is that K_2 has only been calculated approximately, so we have to somehow recognize its entries as rational numbers. Because M_2 has integer entries and we are only concerned with the product of this matrix with K_2, we may as well first compute

$$L_2' = M_2 \cdot K_2$$

and then try to approximate it by a matrix with rational entries. If we are given sufficient accuracy, we can simply use the continued fraction expansion of the coefficients and hope to find a reasonably simple denominator. In the present case, the coefficients are apparently all integers, so we have nothing more to do than rounding, but in higher levels, we will have to find a common denominator and define L_m by multiplying L_m' by this denominator and rounding. (Even here, if we had not LLL-reduced the matrix L_1, we would have found the rather small denominator 6.)

If L_2 is our rounded matrix, we can say with confidence (especially if we work with 300 decimal digits, which we will need for higher levels) that we have found the correct 67 relations at the dilogarithm level. This is however still not entirely satisfactory, since we may not have a $\mathbf{Z}$-basis of the lattice of relations, but only a $\mathbf{Q}$-basis.

To check whether we have a $\mathbf{Z}$-basis, we use the following easily proved lemma.

Lemma. *Let M be an $m \times n$ matrix with integer entries and rank n. Then the lattice spanned by the columns of M is equal to the intersection with $\mathbf{Z}^m$ of the $\mathbf{Q}$-vector space spanned by the columns of M if and only if the GCD of all $n \times n$ subdeterminants is equal to* 1.

We will say that a matrix satisfying the condition of the preceding lemma is *primitive*. To replace a matrix M by a primitive one whose columns generate the same $\mathbf{Q}$-vector space, we can, for instance, proceed as follows. We extract at random two $n \times n$ determinants and let d be their GCD. Then, for every prime p dividing d, we make M primitive at p by elementary linear algebra modulo p. Note that for this, we do not need to factor d completely, since once the small prime factors have been removed, we can treat the remaining part as a prime, and the only thing which can go wrong is that we obtain a nontrivial factor of this "prime," in which case we try again with that factor and its cofactor. Alternatively, we could apply Hermite reduction to M, after which it is obvious how to make it primitive.

Applying this procedure to L_2, we now obtain a $\mathbf{Z}$-basis of the space of relations, not just a $\mathbf{Q}$-basis. As before, we terminate that matrix by LLL-reducing. In our case, if we start from the LLL-reduced matrix L_1, the GCD d of subdeterminants is already equal to 1, so the reduction step is unnecessary. On the other hand, if we had started from a non–LLL-reduced matrix L_1 giving a denominator 6, we would have had to eliminate 6 (i.e., 2 and 3) from the GCD of the subdeterminants.

Since we LLL-reduce each matrix L_m as we proceed, the relations that we obtain are essentially optimal. Finding a kernel is no problem, since it involves inverting a 4×4 or 5×5 matrix followed by a matrix multiplication. Similarly, the process of making a matrix primitive is fairly straightforward. The crucial problem which arises now is that of explosion of denominators. Since the kernel is computed in an arbitrary way at first, and only corrected afterwards, it is not surprising that we have little control on the denominators which occur. In fact, working still with 300 decimals of accuracy, we are able to go not much further than level 9 or 10. So we must still improve our strategy.

A first improvement is as follows. It is clear that the pseudointegrated matrices M_m (after clearing denominators) may not be primitive in the above sense. In fact, the GCD of maximal subdeterminants will be divisible by all the primes dividing the exponents of the cyclotomic relations. In our case, this corresponds exactly to all the primes up to and including 59. We can make the matrices M_m primitive with respect to all of these primes, and experiment shows that no other primes need to be eliminated. With this improvement, we were able to reach level 12. However, once again, we were caught up by the explosion of the size of the denominators, i.e., after computing $L'_{13} = M_{13} \cdot \ker(V_{13} \cdot M_{13})$, we were unable to recognize the common denominator of the coefficients of L'_{13}.

At that point, several ideas were tried. First, we could simply have increased the precision of the polylogarithm values, but a rough estimate showed that we probably would have exceeded reasonable computer storage and time limitations. Second, since the coefficients of L'_{13} must have a *common* denominator, it might be possible to use the LLL algorithm to find it, by using a version which is essentially dual to the version used in finding linear relations between real numbers. However, we did not succeed with that either. The idea which finally worked is very simple: Before computing the kernel, do an LLL-reduction of the matrix M_m. It is not unreasonable to expect that this will lower not only the size of the coefficients of M_m, but also that of the denominators. This is, in fact, strikingly true (see next section), and enabled us to climb to the top of the ladder (i.e., $m = 16$) in a few more minutes and thus to obtain the desired four hexadecalogarithm relations between powers of α.

Remarks

(1) It would be of considerable interest to develop an algorithm which can compute directly an LLL-reduced $\mathbf{Z}$-basis of relations between linearly dependent vectors $v_1, \dots, v_n$ without computing first a $\mathbf{Q}$-basis of $\sum \mathbf{Q} v_i$, then transforming this into a $\mathbf{Z}$-basis, and finally LLL-reducing. The algorithm should be able to deal with exact or imprecise entries, i.e., we should not have to first find an $\mathbf{R}$-basis and then "recognize" the real entries as rational numbers. One way to do this is to apply LLL-reduction to the lattice $\mathbf{Z}^n$ equipped with the quadratic form

$$Q(a_1, \dots, a_n) = \|a_1 v_1 + \cdots + a_n v_n\|^2 + \varepsilon(a_1^2 + \cdots + a_n^2)$$

for $a_1, \dots, a_n \in \mathbf{Z}$, with a suitably small ε; then the very short vectors correspond to the kernel and automatically give an LLL-reduced basis of it. (When the Euclidean space in which the v_i lie is one-dimensional, this is the standard way of finding integer relations among real numbers.) This algorithm can be made to work but requires careful choice of ε.

(2) Initially, instead of using the conjugates of α, we simply looked (using LLL) for relations among the unmodified polylogarithm functions Li_m evaluated at powers of the real number α. However, the size of the denominators and of the coefficients which occur increases even more rapidly than in the method we finally used, and we were unable to climb beyond level 7.

5. NUMERICAL RESULTS

Let us summarize the procedure described above. Let L_1 be the 111×71 LLL-reduced matrix of cyclotomic relations. Consider some integer $m > 1$ and assume inductively that the matrix L_{m-1} has been computed. Let M_m be the matrix obtained after performing successively the following steps:

(a) pseudointegrate the matrix L_{m-1};
(b) reduce the columns to integer entries having GCD equal to 1;
(c) eliminate the primes up to 59;
(d) LLL-reduce.

Let V_m be the $d(m) \times 111$ ($d(m) = 4$ or 5) matrix of polylogarithm values of the powers of the relevant conjugates of α. Set $L'_m = M_m \cdot \ker(V_m \cdot M_m)$, where the kernel is computed by simple matrix inversion. Using the continued fraction expansion of a few coefficients of L'_m, determine a plausible denominator d_m for L'_m, and let L_m be the matrix obtained from L'_m by the following steps:

(a) Multiply L'_m by d_m and then round each coefficient to the nearest integer; let ε_m be the sum of the absolute values of the distances between each coefficient and its rounded value.
(b) Eliminate d_m from the resulting matrix. In practice, eliminate individually the small prime factors of d_m, and globally the remaining cofactor, even if it is not prime.
(c) LLL-reduce.

In Table 1, we give the results obtained by working with 305 decimals of accuracy. Note that since there is some freedom in the LLL-reduction procedure, another implementation may give slightly different results, although, of course, the lattices of polylogarithm relations will be the same. We give successively m, ε_m, d_m and l_m, the L^2 norm of the LLL-reduced matrix L_m.

A few remarks are in order concerning this table. First, thanks to the combination of all the tricks mentioned above, the explosion of denominators has been completely avoided for small levels, and in fact, the denominator is equal to 1 for all even levels up to 10. On the other hand, when the level is 13 or above, the denominators become very large. We believe that the reason for this is that the matrices M_m get narrower as m grows (for example, M_{16} has only eight columns), hence the LLL-reduction of these matrices which saved us in the end is unfortunately not of much help. A related observation is that up to level 12, the precision of the results stays excellent, but thereafter

m	ε_m	d_m	l_m
1	—	—	$2 \cdot 10^{1}$
2	10^{-305}	1	$4.4 \cdot 10^{1}$
3	10^{-304}	2	$1.4 \cdot 10^{2}$
4	10^{-305}	1	$6.4 \cdot 10^{2}$
5	10^{-301}	24	$8.8 \cdot 10^{3}$
6	10^{-305}	1	$2.2 \cdot 10^{4}$
7	10^{-297}	81	$1.0 \cdot 10^{6}$
8	10^{-305}	1	$2.2 \cdot 10^{6}$
9	10^{-294}	63057	$2.3 \cdot 10^{7}$
10	10^{-305}	1	$6.6 \cdot 10^{8}$
11	10^{-286}	234280024	$1.3 \cdot 10^{11}$
12	10^{-291}	12	$6.9 \cdot 10^{13}$
13	10^{-265}	$2.5 \cdot 10^{24}$	$1.5 \cdot 10^{18}$
14	10^{-259}	$1.0 \cdot 10^{25}$	$7.4 \cdot 10^{23}$
15	10^{-204}	$4.6 \cdot 10^{72}$	$7.4 \cdot 10^{37}$
16	10^{-153}	$1.1 \cdot 10^{106}$	$2.7 \cdot 10^{71}$

TABLE 1. Salem number ladder data

we suffer a loss of precision of about the same order as $d_m l_m$. In particular, if we had worked with only 250 decimals instead of 300, we would not have been able to obtain the hexadecalogarithm relations. By using the method sketched earlier in Remark (1) preceding this section, we could have avoided the denominator explosion altogether, but apparently the same precision, about 300 digits, would still have been needed to obtain the highest-order relation.

The third remark is that the relations have been LLL-reduced, hence the L^2-norms given in the last column are certainly close to best possible, up to maybe a variation of 1 or 2 in the exponent of 10. In particular, the four relations that we obtain at level 16 have 70-digit coefficients, and the LLL bounds tell us that no such relation exists whose coefficients have at most 69 digits. Just for fun, we have given one of these relations in the Appendix.

The last remark is the following. At even orders m, we have worked with four nonreal conjugates of α, since this is enough to obtain the relations, and also because the modified polylogarithm function P_m vanishes at real arguments. However, the theory tells us that for each relation $\sum c_n P_m((\alpha^\sigma)^n) = 0$, the corresponding combination $\sum c_n L_m(\alpha^n)$ of the modified *real* polylogarithm function

$$L_m(x) = \sum_{r=0}^{m-2} \frac{(-\log|x|)^r}{r!} \operatorname{Li}_{m-r}(x) - \frac{(-\log|x|)^{m-1}}{m(m-2)!} \log|1-x|$$

for m even and $-1 \le x \le 1$ should give a rational multiple of π^m. This is indeed what we find. Moreover, the calculation lets us answer a question that had arisen in earlier polylogarithm investigations but could not previously be answered because the orders of the polylogarithms occurring were too small. In all known relations among the values of the real polylogarithm function L_m at real arguments, the combination of polylogarithms which occur are not merely rational multiples of π^m, but are rational multiples with a small denominator. Conjecturally, a prime p can occur in the denominator only if some extension of F of degree m contains a p-th root of unity; in particular, for generic fields, not having especially large intersections with cyclotomic fields, one expects (at most) the same denominator as for $\mathbf{Q}$, i.e., at most D_m, if we write $\zeta(m) = \zeta_{\mathbf{Q}}(m)$ as $N_m \pi^m / D_m$ with N_m and D_m coprime integers. For instance, all known relations among values of the values of the function L_2 ("Rogers dilogarithm function") at arguments in real quadratic fields other than $\mathbf{Q}(\sqrt{5})$ give integral multiples of $\pi^2/6 = \zeta(2)$. The question was whether the right general conjecture should be that polylogarithm relations give simple rational multiples of π^m, or simple rational multiples of $\zeta(m)$, i.e., whether for generic F, they are integral multiples of $N_m \pi^m / D_m$ or merely of π^m / D_m. This could not be checked before because no relations beyond order 9 were known, and the first m with $N_m > 1$ is 12 ($N_{12} = 691$). It turned out that for $m = 12$ and $m = 16$, the rational multiples occurring did *not* have numerators divisible by N_m. For instance, the combination of modified hexadecalogarithms given in the Appendix is an integer multiple of $\pi^{16}/D_{16} = \pi^{16}/325641566250$, but not of $\zeta(16) = 3617\pi^{16}/325641566250$.

APPENDIX: A (CONJECTURAL) HEXADECALOGARITHM RELATION

As promised, we give one of the four relations among the hexadecalogarithm values of powers of α. We give the coefficients in tabular form, each line containing an integer n between 1 and 360, and a coefficient c_n of at most 71 digits such that the sum $\sum_n c_n P_{16}((\alpha^\sigma)^n)$ vanishes for each conjugate α^σ of α. As mentioned in Section 3, these relations are not rigorously proved, even though we know that relations of this form exist. The value of the sum $\sum_n c_n L_{16}(\alpha^n)$, which should be a multiple of π^{16}, turns out to be—at least, to 305 digit accuracy—equal to

$$\frac{904068896677319441471456383959007967274858676781011907976799888762570756}{1391630625}\,\pi^{16}.$$

n	c_n
1	31646773047454089991074874872115377409615412289257883821314849289011200
2	25091067534991111610431336422823261608598939493205807887933701844011125
3	6426516851004945630792401712640218924732096012292988922326557153689600
4	13013897237290091458885488420735508994896225052675065321587511815005700
5	25345322420083129704294138983088658472477961828569859945452256855687168
6	7654077937866991663920269850717110624730841668799203698754516215186400
7	−21386348881155389104805966211003822515378446012186901064890133970944000
8	−21266457895771812182877802673643067539198868643537409156120825599294550
9	−1608913679173101993450445560427931226772106165751903603469046557081600
10	−3006602534018037060249498048362745795863844063262283069357027575007928
11	−22235894431165401037596833795954124164938433066802035066390519945625600
12	−22388021786025993535779596394330877113951486019249754962484078194515225
13	41163804442546495646577273482500927058102389473207630788145055334400
14	12174993046884795857107894115241919739043395622527097206004912578483400
15	6844294516616620021500451124873691690268431574889793783745454924496896
16	8841135683976436138756874554221330573774436325888095357358355348722400

17	24829097569661524994885437517127853543701708402435233939574620268134400
18	28498255925832894617958079027165538540096843712257925301476024055353325
19	9166754527778720014326220044587296067183679250386045439949761784217600
20	−11580817676721524765641846866621533288082812848980579270656302811233794
21	−16245642940492771699066931364054744350799304973969834884361467289600000
22	−15748085766951359294881055072851790196558295332691594137452459843851200
23	−19496977777664007309259466929947777237584024647491075153218786682470400
24	−18468083522223918020497518383177026788374351085140391933539543804612050
25	−4314755729017687337460748836872559802235519309558587091661435090567168
26	11249447059018351361515061841298059010373082845210017340809155226388025
27	15574480717724962535771441206565522595335157033372363898543889893785600
28	11313602564500264241445701107299008848994067532467357431502793906232400
29	8329120904293952846317524093674042573441422235605490061915675232665600
30	5881501014664351408528126106754613171638191830768453338752077170311308
31	191988690513201672643226156999718508747812398819257361702468805427200
32	−6115696834618709572715929974058317496542201476850795855640204890331650
33	−6984083513726594321542993303029682093208835049867782759769999093760000
34	−3896514477700363180358570911410118549912874771249610267529568732036175
35	−1363416169238179500564473491601832074427698837002808780243998008999936
36	−903764737187791886343931740183989556263420012398505816271293386992500
37	−51195722004398505437105920189828358814685835219637545355574434201600
38	1163858882869007998810076740532670732825220728781566343586502555831225
39	1398945336511019013294751363659871061475779643795817999989787280998400
40	401900291391359658691015310335821114495744899780687502125176072970091
41	−18357497447021300346677766880194392253038676659250053967753609216000
42	209880652610016890846130073291719509897808136124915586249413635498075
43	412990576874941373394884781116178108644080222467096311650201895731200
44	147097619489559301363658769085991890779408701292435263778808443130875
45	26662567639841242164177967788044676520121768521780581605608586969088
46	28184479296211476501339430810207216945481802649216778423277040453425
47	34371308605186686784233939719327830943454035247612304383577681920000
48	−139034598108199572180998699853756044594232438214055322684489812764750
49	−154939072804941017664824356756544281283235003570131737980669213081600
50	−74362675620298525792761656043445961174236678839179357013950442867933
52	−29495684051643376094549597202160922698911489595477464365002174841550
54	16452242625559198968231473456491258203663003099675105036364981270425
55	18905707048913496529575418250969743894561871484696810372746491133952
56	−13209236265564285433229502274658002415493230454733393040344290144300
57	−9610371858322898586160077094828183617550288535157616732513751859200
59	11443329818363659341033908014497533737433694941430174192677899468800
60	−1193932707685224601260269373076166635681028991057046910394337393982
62	3991108023687448573845112533922056512921201331434304763379645714025
63	4699947234560986525969593455253779168536932503797443546952338636800
64	−711352582326217890935595563079394659279498801576913672868145475350
65	−481217733654795574991446635983309504421295585637628292706891857920
66	−894495639961104361350886089272720783794797054463528471161638817675
68	−199395616572195395131575401491160423300783980332658045871204603850
69	−6076633047125082595552897971541438049980798750085760172101249433600
70	−522812233806755930281418663796770354717983191617052591157520725149
72	38592369445189040089172813248529688077232236384716603967584502075
74	87900235836110072486516618886252756513330060427527726240803828700
75	−92296088852587011584265879238930165708734428314973179691065573376
76	92901138031960478944908373516705488635104747018011399399862844625
77	61759736128804947044786852084370129719837055961943460068032512000
78	−36756884725656885385346538935669669783355213490762341399834816925
80	46954418051568593610854764980666238482860005680106500237468038898
82	560226362518960581868828334966869880769002583595277525871387000
84	38880028529730786000654022153333960255700788217780274959909199725
86	−12603472194669841717373192783086490131960456007906991932684384025
90	−1772013084326244638329408775516950199159156645553313060217738625
91	1952516224546369458374277811771795588226096112561310304858112000
92	−8980606928663980602335450279457132621021364591928525016790968150
93	−2036505707118527674428009434244432177573411351052922439923302400
96	3443514710968399697082419611169526220190407281647120506902956850
98	4728365258939850392603282371720711709083099474186149230367102450

102	−4553514468968053599951175902152229957807998523887944111079424 00
105	44553215218036346993424705809807484023854712511854390960186982 4
106	12555507611135480495074548856463391238927539226970410118823936 00
108	−61709594443276795231396655653132467684224970739729141602908047 5
110	−46068542376769383557635836480242232693726306507310389348493242 0
118	−34922271174205503360088830610649211845195602238251264015740660 0
120	16271995378811785637604375219275388295759069926033850692305806 1
123	63968278026407517822360152171152800185542622372735616614400 0
124	−90330489950660250766032160181307416012932865571448141865191200
130	13422221884763830010346349052260427346011792262378618866661480
132	41348897223186853810603447102780351742930433494271919545526225
135	−40466723177732202396167224971309260089320562856453778923520 00
138	18544412375259651475613341416669862365136680605644536502113325
143	−74235259286282685743414533832916145080139499217807914508288 00
144	56047095282964557121285650059532947851634341539878622518761 00
154	−18847575722901900343257706324575845251415117175886065694590 00
160	−65488073813305687308272549726138437716413180049995320141938 0
165	−26552315677380199937244748698447687238349481032815544703385 6
175	39655925446974095909657618399957910474600616907463188653670 4
180	23863642665396372880484463596413525754989182479437718378905 6
182	−59586066422923872631050958611199816535220218278848581080875
186	62149222018998036939331342597791509325360453828519361569925
195	−63732363369085073158214770955336850477808355944208006184 96
204	13896223355005046386569750677954803338037104870263501315550
212	−38316368442185914596785122242625095333641171957307159786450
240	−17768103190315879804419957482758230443577155278324707054 94
246	−19521569221926122382312058157700439509748114737773320500
270	12349463860391907469533454886263812283117847551408013587 5
286	22654803248987636030094767405064741540569915532778294222 5
360	−72826057938831704347181590565226824203458198484612177 67

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Received September 16, 1991; corrected in proofs February 25, 1992