The Nonexistence of Certain Projective Planes of Order 15

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CONTENTS

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If a projective plane Π of order 15 contains a line-extended (15,3)-Kirkman design Σ^*, no collineation of Π of order 7 fixes (setwise) the point set of Σ^*.

INTRODUCTION

A (v, k, λ)-*design* is a set of v elements (known as *points*) and a collection of distinguished subsets of cardinality k (called *blocks*) with the property that each pair of points lies in precisely λ common blocks. A (v, k)-*Kirkman design* is a $(v, k, 1)$-design Σ whose blocks (called *lines*) are partitioned into "parallel" classes, each of which in turn partitions the points of Σ. An *extended* (v, k)-*Kirkman design* Σ^* is obtained from Σ by adjoining one new point $P(\gamma)$ for each parallel class γ, enlarging each line of γ to include $P(\gamma)$, and introducing new lines, consisting only of new points, in such a way that each pair of new points is joined by exactly one new line. We call Σ^* *line-extended* if there is a single new line, and (*projective*) *plane-extended* if the new lines induce a projective plane on the new points.

In the language of the preceding paragraph, a *projective plane* (of *order* n) is just an $(n^2 + n + 1, n + 1, 1)$-design, and an *affine plane* (of *order* n) is just an $(n^2 + n, n)$-Kirkman design. Every affine plane Σ of order n is contained in a projective plane Σ^* of order n, which is the line extension of Σ. A *collineation* of an affine or projective plane Π is a permutation of the points of Π that induces a permutation of the lines of Π. In this note, we report on an exhaustive computer search that produced an improvement of the following theorem.

Theorem 1. [Drake and Ho 1988] *Let* Π *be a projective plane of order* 15 *that contains a line-extended*

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$(15,3)$*-Kirkman design* Σ^**. If* Π *has a collineation* σ *of order* 7 *that leaves invariant the point set of* Σ^**, then* $\langle\sigma\rangle$ *is the full collineation group of* Π.

Several published papers contain nonexistence results for planes of order 15 with special properties. These include [Cigić 1983] and [Woodcock 1986], in addition to [Drake and Ho 1988]. The result obtained in this article is this:

Theorem 2. *Let* Π *be a projective plane of order* 15 *that contains a line-extended* $(15,3)$*-Kirkman design* Σ^**. Then* Π *has no collineation of order* 7 *that fixes (setwise) the point set of* Σ^**.*

Theorem 2 is similar in flavor to the following result.

Theorem 3. [Janko and van Trung 1980] *Suppose that* Π *is a projective plane of order* 12 *that contains a plane-extended* $(27,3)$*-Kirkman design* Σ^**. Then* Π *has no collineation of order* 13 *that fixes (setwise) the point set of* Σ^**.*

The desire to construct a projective plane of nonprime power order is the principal motivation behind many investigations of possible substructures of such planes. No doubt, the hopes of Janko and van Trung were raised by the following two facts: (1) By Lemma 3.1 of [Bruck 1955], a projective plane of order m is a proper subplane of a plane of order n only if $n = m^2$ or $n \geq m^2 + m$ (thus, for a subplane of order 3, the limit case is a putative plane of order 12). (2) If a plane Π of order 12 has a collineation of order 13 that induces a collineation of order 13 on a subplane of order 3, then Π also contains a plane-extended $(27,3)$-Kirkman design.

The hopes of Drake and Ho were also raised by the difficulty of raising the bound for a certain inequality. A *blocking set* is a subset of the point set of a design that contains a point of every block, but that contains no complete block. Bruen's classical result [Bruen 1970] guarantees that all blocking sets in a plane of order n have cardinality at least $n + \sqrt{n} + 1$, a bound that is attained if and only if the blocking set is itself a projective plane of order $\sqrt{n}$. Thus, in a putative plane of order 15, every blocking set has cardinality at least 20. Bruen's bound can be raised to 22 [Drake and Ho 1988], but further improvement seems to be very difficult. A line-extended $(15,3)$-Kirkman design in a plane of order 15 would be a blocking set of minimum cardinality 22.

A more successful engagement with putative blocking sets of small cardinality was carried out by Lam, Thiel and Swiercz [Lam et al. 1989]. These researchers completed the proof of the nonexistence of projective planes of order 10 by carrying out computer searches that demonstrated that such planes could not contain certain blocking sets of size 19. Their searches completed a massive program that began with independent computer proofs in [Denniston 1969] and [MacWilliams et al. 1973] of the nonexistence of certain blocking sets of size 15.

1. FRAMEWORK OF PROOF OF THEOREM 2

Assume, by way of contradiction to Theorem 2, that Π is a projective plane of order 15, that Π contains the line extension Σ^* of a $(15,3)$-Kirkman design Σ, and that Π has a collineation σ of order 7 that fixes (setwise) the point set of Σ^*. We write ℓ^* for the line of Σ^* that consists of the seven ideal points, ℓ for the line of Π that contains ℓ^*.

Step 1. [Drake and Ho 1988, Lemma 5.2] *The fixed point set of* σ *consists of one point* P_0 *of* Σ *and two points* P_1, P_2 *of* $\ell \setminus \ell^*$*. The lines fixed by* σ *are just the three lines determined by* P_0, P_1 *and* P_2*.*

Step 2. (This is a special case of Proposition 2.1 of [Drake and Ho 1988]). *The points of* Σ^* *constitute a blocking set of* Π*. Each of the* 210 *points of* Π *not in* ℓ *or* Σ *lies in two secants and fourteen tangents to* Σ^**. Each point of* $\ell \setminus \ell^*$ *lies in fifteen tangents to* Σ^**.*

Lemma 3. [Cole 1922; Mathon et al. 1983] *There are precisely three* $(15,3)$*-Kirkman designs with an automorphism of order* 7*. Each of the three has a nonabelian automorphism group* G *of order* 21*.*

Step 4. *It is possible to label the point orbits of* σ *in* Σ *by* $\{P_0\}, O_1^p, O_2^p$*, and the line orbits of* σ *in* Σ *by* O_j^ℓ*,* $0 \leq j \leq 4$*, so that the following conditions hold:*

(*i*) *Each line of* O_0^ℓ *contains the point* $P_0 = 0$ *and intersects each of* O_1^p *and* O_2^p *in a single point;*
(*ii*) *the lines of* O_1^ℓ *meet* O_1^p *in three points each (and they constitute a projective plane of order* 2 *on* O_1^p*);*

(*iii*) *every line of* O^ℓ_j *meets* O^p_i *in* i *points for* $2 \leq j \leq 4$ *and* $i = 1, 2$.

Proof: The design Σ consists of 15 points and 35 lines. Denote the orbit of lines that are incident with P_0 by O^ℓ_0, the other orbits by O^ℓ_j, $1 \leq j \leq 4$. Conclusion (i) follows from the fact that P_0 is joined to each point of $O^p_1 \cup O^p_2$. Say that a σ-orbit of lines of Σ is of type $[x, y]$ if each of its lines contains x points of O^p_1 and y points of O^p_2. Since the seven points of O^p_1 must be joined to the seven points of O^p_2, we get $49-7 = \sum 7xy$, where the sum is taken over all four line orbits O^ℓ_j with $j \neq 0$. For each of the four, the set $\{x, y\}$ is $\{0, 3\}$ or $\{1, 2\}$. To yield a sum of 42, three of the four must be $\{1, 2\}$. At most, a permutation of labels of point and line orbits is required to yield conclusions (ii) and (iii). □

A σ-orbit of points not in $\ell \cup \Sigma$ is said to be of *type* (j, m) if the two secants through any one of its points are in the secant orbits O^ℓ_j and O^ℓ_m with $j \leq m$.

Step 5. *If* $\{x, y\} \subset \{2, 3, 4\}$ *with* $x < y$, *there is exactly one point orbit of type* (x, y).

Proof: Each orbit O^ℓ_j contains one line from each of the seven parallel classes of Σ. In particular, each line g of O^ℓ_x is parallel, in Σ, to one line h of O^ℓ_y (and g and h meet in a point of ℓ^*.) By Step 4, g intersects one line of O^ℓ_y in a point of O^p_1, and four lines of O^ℓ_y in points of O^p_2. Altogether, g meets six of the seven lines of O^ℓ_y in points of Σ^* and, thus, must meet exactly one in a point of an orbit of type (x, y). □

Step 6. *Without loss of generality, one may assume that* P_0P_1 *and* P_0P_2 *contain the unique point orbits of types* $(2, 4)$ *and* $(2, 3)$, *respectively. Also,* P_0P_1 *contains one of the three point orbits of type* $(1, 3)$*; and* P_0P_2*, one of the three of type* $(1, 4)$.

Proof: The line P_0P_1 consists of points P_1 and P_0 and two σ-orbits of size 7, which are, say, of types (x, y) and (z, w). Since P_0P_1 meets all lines of O^ℓ_0 in $P_0 = 0$, we must have $\{x, y, z, w\} = \{1, 2, 3, 4\}$. Thus, one of the types must be one of $(2, 3)$, $(2, 4)$ or $(3, 4)$. The line P_0P_2 must also contain an orbit of one of these three types.

By Lemma 3, Σ has a nonabelian group G of order 21. If τ is an element of G of order 3, τ normalizes $\langle\sigma\rangle$ and therefore permutes σ-orbits. Since Σ does not induce isomorphic incidence structures on O^p_1 and O^p_2, τ fixes the point orbits O^p_1 and O^p_2 as well as $\{0\}$ and the line orbit O^ℓ_0. Since each line of O^ℓ_0 meets each point orbit O^p_1 and O^p_2 in a single point, every line of O^ℓ_0 fixed by τ must be pointwise fixed. Since $| O^\ell_0 |= 7, \tau$ fixes one, four or seven lines of O^ℓ_0. If τ fixed four or more lines of O^ℓ_0, it would fix at least four and, hence, all seven points of the Fano plane O^p_1; it thus would fix all lines of O^ℓ_0 and, hence, all points of O^p_2. Then τ would be the identity map on Σ. By the contradiction, τ fixes precisely one line of O^ℓ_0 and thus exactly one point of O^p_2, say P. Thus, the six lines of Σ that meet O^p_2 in P and a second point constitute two τ-orbits of size 3. It follows that τ permutes the line orbits O^ℓ_i with $i = 2, 3, 4$ in a τ-orbit of size 3.

In view of Step 2, the action of the group G extends naturally to an action as a group of automorphisms of the substructure Π' of Π that consists of all points of Π and all 36 secants to Σ^*. The extended automorphism τ permutes the point orbits of types $(2, 3), (2, 4), (3, 4)$ in a τ-orbit of size 3. Thus, one may assume that $P_0P_1 \cup P_0P_2$ contains the orbits of types $(2, 3), (2, 4)$. It requires no more than an interchange of the labels P_1, P_2 to insure that the orbit of type $(2, 4)$ is the one contained in P_0P_1.

It remains only to observe that the number of σ-point orbits of type $(1, i)$ is three for each $i = 2, 3, 4$. By Step 4, each line of O^ℓ_1 meets one line of O^ℓ_i in a point of ℓ^*, three lines of O^ℓ_i in points of O^p_1, and none in points of $O^p_2 \cup \{0\}$. Then, it must meet the remaining three lines of O^ℓ_i in three point orbits of type $(1, i)$. □

2. CONCLUSION OF PROOF: COMPUTATIONAL DETAILS

In view of Step 2 and Lemma 3, the 36 secants to Σ^* are determined as lines of Π for each of the three possible Σ^*. By Step 6, there are, for each of the three Σ^*, only nine possible definitions of the pair of lines P_0P_1, P_0P_2 in Π. For each of these $3 \cdot 9 = 27$ possible sets of 38 lines of Π, we have verified, by exhaustive enumeration, that there is no way to define the remaining seven lines through P_0 and the remaining fourteen lines through each of P_1 and P_2 in a manner that is consistent with

(i)	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 1 & 5 & 6 & 7 & 3 & 4 & 2 & 8 & 0 & 13 & 12 & 10 & 14 & 9 & 11 \end{pmatrix}$	$\{0,1,8\}$ $\{2,3,5\}$ $\{4,10,13\}$ $\{6,9,14\}$ $\{7,11,12\}$	$28 \to 40 \to 49 \to 5 \to 47 \to 14 \to 17$
(ii)	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 1 & 8 & 11 & 0 & 10 & 6 & 2 & 4 & 3 & 9 & 14 & 13 & 12 & 7 & 5 \end{pmatrix}$	$\{0,1,10\}$ $\{4,6,7\}$ $\{2,9,12\}$ $\{3,8,14\}$ $\{5,11,13\}$	$11 \to 53 \to 21 \to 40 \to 28 \to 42 \to 5$
(iii)	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 1 & 2 & 6 & 7 & 5 & 3 & 4 & 8 & 10 & 12 & 13 & 0 & 9 & 14 & 11 \end{pmatrix}$	$\{0,1,9\}$ $\{4,6,7\}$ $\{2,13,14\}$ $\{3,10,12\}$ $\{5,8,11\}$	$6 \to 2 \to 1 \to 7 \to 3 \to 4 \to 5$

TABLE 1. Relation between our notation and that of [Mathon et al. 1983] (abbreviated [MPR]). Each row corresponds to a Kirkman design: rows (i) and (ii) refer to Kirkman systems 1 and 2 of Steiner system 1 on page 18 of [MPR], while row (iii) refers to the unique Kirkman system of Steiner system 61 on page 80 of [MPR]. Within each row are given: the relabeling of points; the lines of the "basic" parallel class (after relabeling); the number of the basic parallel class in [MPR], and of its successive images under σ.

the assumption that Π is a projective plane. This contradiction yields Theorem 2.

By Lemma 3, Σ has a group G of order 21. As a first step toward simplifying the programming, it was useful to label the points of Σ in such a way that an element of G of order 7 has the following cycle representation on the points of Σ:

$$\sigma = (0)(1,2,3,4,5,6,7)(8,9,10,11,12,13,14).$$

Table 1 produces such labelings from the labelings in [Mathon et al. 1983] (henceforth abbreviated [MPR]). For example, for Kirkman system 1 of Steiner system 1 on page 18 of [MPR], we relabeled the points by the bijection displayed in row (i), left. After relabeling, [MPR]'s parallel class 28 for this Kirkman design consists of the five lines shown in row (i), middle. Successive applications of σ map this class into MPR's classes 40, 49, 5, 47, 14 and 17, as shown in row (i), right.

Initially, we labeled the lines of the "basic" parallel classes shown in the middle column of Table 1 by $[0,0], [0,1], [0,2], [0,3], [0,4]$, and we denoted $[0,j]\sigma^i$ by $[i,j]$. We wrote O_1^p and O_2^p, respectively, for the point orbits $\{1, 2, \ldots, 7\}$ and $\{8, 9, \ldots, 14\}$, and O_j^ℓ for the line orbit $\{\,[i, j] : 0 \le i \le 6\}$ for $0 \le j \le 4$. Clearly, this labeling satisfies the conclusions of Step 4 (with $P_0 = 0$). In view of Step 2, we labeled a point P that is not in $\ell \cup \Sigma$ by (i,j,k,m), where P is the intersection of secants $[i,j]$ and $[k,m]$. Then

$$(i,j,k,m)\sigma = (i+1,j,k+1,m),$$

where addition is performed modulo 7.

In the actual computer program, we referred to the points and lines by the numbers from 0 to 240. We labeled point P_i by i for $i = 0,1,2$. Point i of Σ was relabeled $i+2$ for $1 \le i \le 14$. By construction, the collineation σ fixes points 0, 1 and 2, and otherwise has orbits of seven consecutive integers.

For each of the 27 initial sets of data, we assigned numbers to all the points and entered the 38 known lines. We numbered the points as uniformly as possible: first the fixed points, then O_1^p and O_2^p, followed by the ideal points and the other 7-orbit of points on P_1P_2, the two 7-orbits of P_0P_1 followed by the two 7-orbits of P_0P_2, the orbits of points on secants through P_0, and last, the orbits of points on tangents through P_0.

We used a crude form of parallel processing by running our program in the background on 24 Sun 350s with 27 starts at different times. The program was basically a tree search. Taking advantage of the assumed collineation, new lines were adjoined, an orbit of seven at a time.

The number of possible choices for the remaining orbit of lines through P_0 ranged from a low of 161 to a high of 252 for the 27 various starts.

For such an orbit, there were typically about 2000 compatible orbits of lines through P_1. We collected information on the number of partial successes, where a *partial success* is defined to consist of four compatible orbits of lines: the remaining orbit of lines through P_0, the two remaining orbits of lines through P_1, and one of the two remaining orbits of lines through P_2. For all but two of the 27 starts, there were partial successes, typically a few hundred. It was impossible to extend any partial success by adjoining a final compatible orbit of lines through P_2.

Both exceptional starts were associated with the Kirkman system (iii) of Table 1; the full automorphism group of this Kirkman system has order 21, whereas the other two Kirkman systems have groups of order 168. For the two exceptional starts, it was not even possible to obtain compatible complete sets of lines through P_0 and P_1.

Since long running times were expected, the program was designed to be easy to start and stop. The quickest running time was about two weeks, whereas one set of data was restarted twelve times and ran for nearly five months.

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