can be seen as follows: consider the plane of symmetry of the button that intersects it in two large approximate circles and a smaller one in between. Clearly there is a fractional linear transformation of the plane that maps this configuration to one consisting of three equal plane circles centered at the vertices of an equilateral triangle. The latter approximates the intersection of $\xi_{1,2}$ with the plane, and the fractional linear transformation extends uniquely to all of three-space to give the desired Möbius transformation.

Our experiments also suggest that $\xi_{1,2}$ is stable for W. Figure 10 shows a surface obtained by jiggling $\xi_{1,2}$, and the corresponding evolved surface, which has $w \approx 21.89$ and is evidently conformally equivalent to $\xi_{1,2}$.

Genus-Three Minimizers

We carried out similar experiments that indicate that Lawson's genus-three surface $\xi_{1,3}$ is W-stable, with $w \approx 22.82$ (Figure 11 and Table 9).

We also studied the evolution of a genus-three surface produced by Genus, which fuses three cubical tori together (Table 10). The minimizing surface (Figure 12, bottom right) in this case has w very close to that of $\xi_{1,3}$. One can again see that this surface is conformally equivalent to $\xi_{1,3}$, using an argument like the one for genus two.

Another interesting W-critical surface of genus three, different from Lawson's example, is the stereographic projection of the tetrahedral minimal surface τ in $\mathbf{S}^3$ discovered by Karcher, Pinkall and Sterling [Karcher et al. 1988]. We were able to evolve under the w-flow to τ by starting from the polyhedron with the same symmetry (Figure 13).

number of facets	fineness	approx. max. angle	$\log_{10}$ iterations	w
144				
2240	83	46°	2	22.69
4968	201	43°	2	22.69
10920	314	13°	4	22.80
15550	602	13°	4	22.81
27118	993	13°	4	22.82

TABLE 9. The evolution toward the genus-three Lawson surface. See also Figure 11.

number of facets	fineness	approx. max. angle	$\log_{10}$ iterations	w
176				
4920	149	46°	2	27.73
9086	154	43°	2	23.01
12310	448	17°	4	22.90
18280	618	15°	4	22.81
26858	976	13°	4	22.83

TABLE 10. The evolution from a cubical surface of genus three to a button surface, conformally equivalent to the Lawson surface. See also Figure 12.

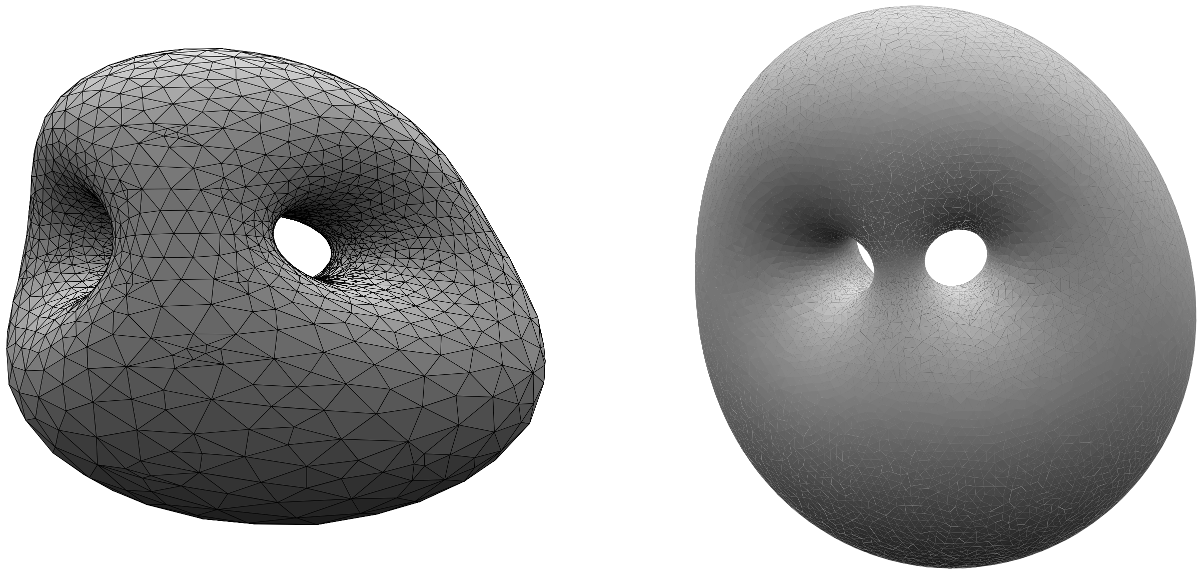

FIGURE 10. A deformation of the Lawson genus-two surface $\xi_{1,2}$ (left) evolves toward a surface (right) conformally equivalent to $\xi_{1,2}$.

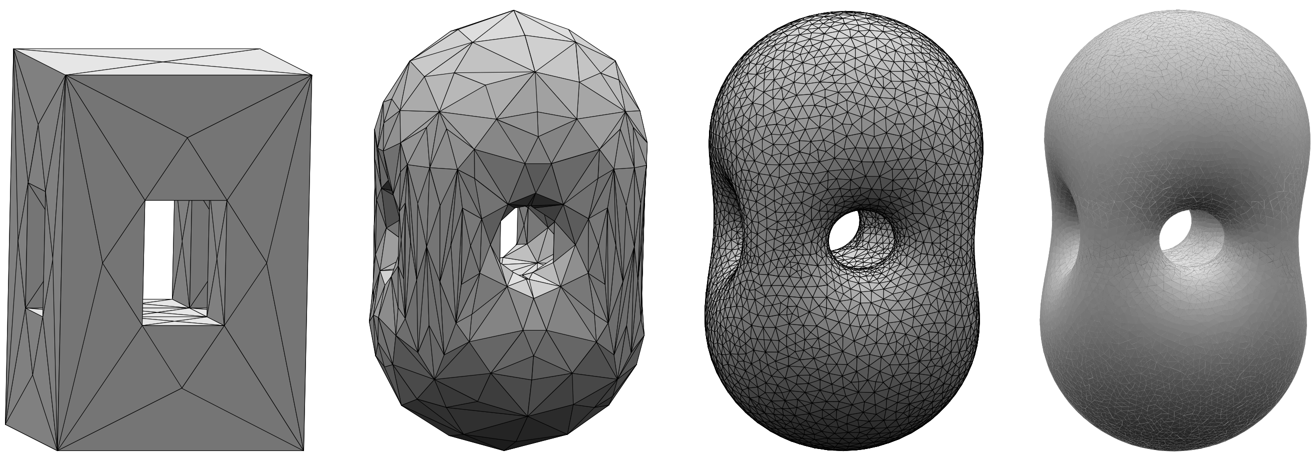

FIGURE 11. A polyhedral surface (left) with the symmetry of the genus-three Lawson surface $\xi_{1,3}$ evolves toward a close approximation to $\xi_{1,3}$. See also Table 9.

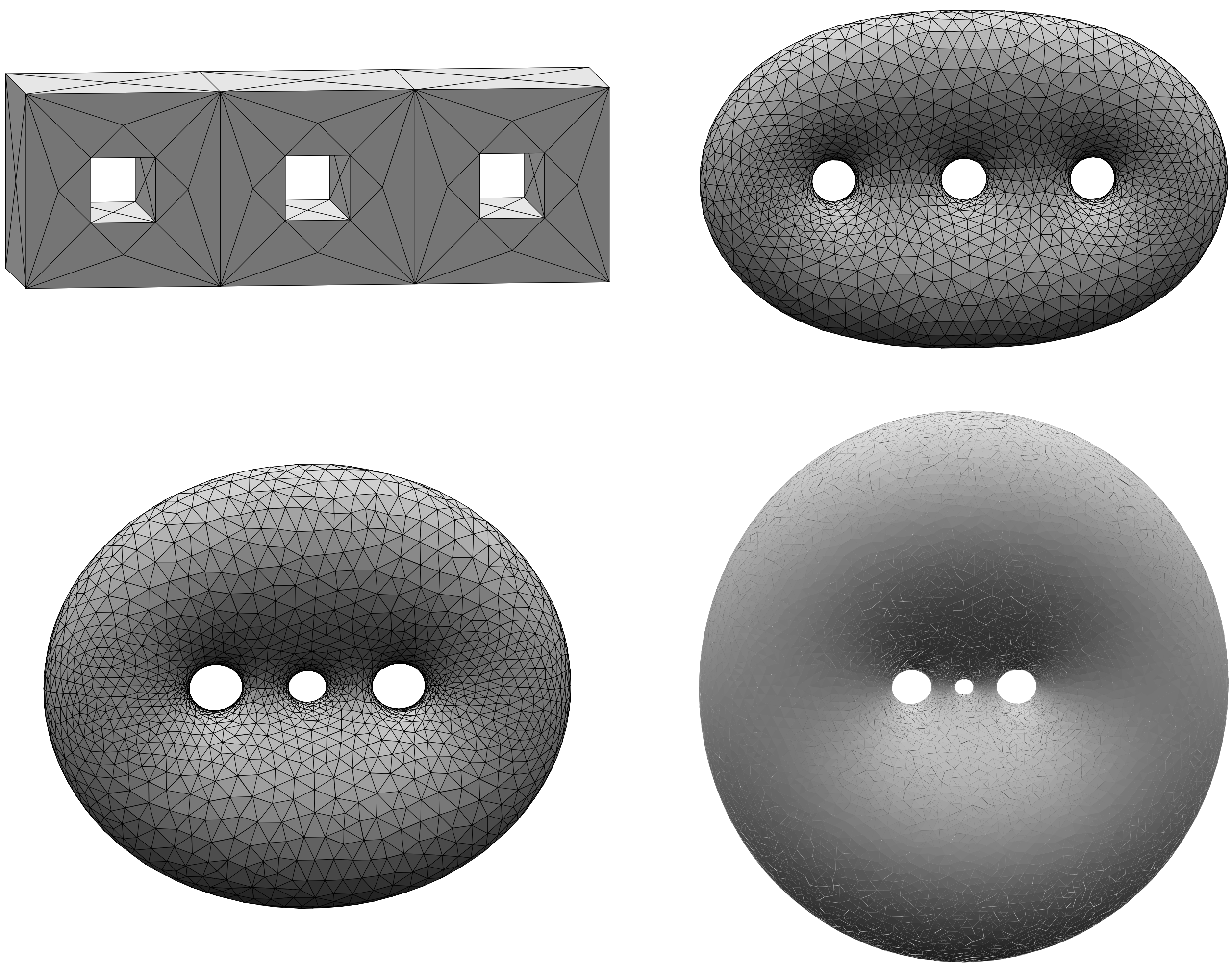

FIGURE 12. The program Genus builds a genus-three surface out of three cubes (top left). A smoothed version (top right), corresponding to the first full line of Table 10, still has energy $w = 28$. Under the w-flow, the surface evolves toward the button surface shown bottom right, which is conformally equivalent to $\xi_{1,3}$.

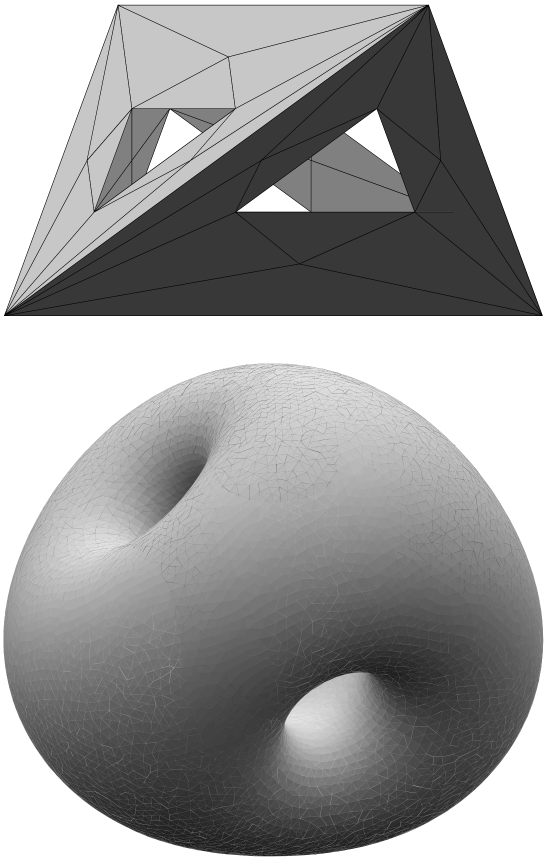

FIGURE 13. A polyhedral surface (top) of genus three with tetrahedral symmetry evolves toward a W-stable surface τ (bottom) with the same symmetry. See also Table 11.

Our experiments (summarized in Table 11) suggest that this surface τ is a local minimum, but *not* the global minimum for W among genus-three surfaces. (Compare the values of w for τ and for $\xi_{1,3}$.)

Figure 14 shows the evolution of two surfaces obtained from perturbations of τ. Apparently, each evolves toward a surface conformally equivalent to τ with $w \approx 23.36$.

It is clear, however, that some relatively large deformations of τ result in surfaces that do *not* evolve to τ. An interesting example of this is depicted in Figure 15. This surface comes from "stretching" the initial tetrahedral polyhedron to be much taller. From Figure 15 (middle left to right) and Table 12, it appears that this surface is evolving toward $\xi_{1,3}$. This suggests an interesting problem: to prove that there is an unstable W-critical surface "between" the W-stable surfaces τ and $\xi_{1,3}$, and to find such a surface.

number of facets	fineness	approx. max. angle	$\log_{10}$ iterations	w
76				
2240	14	80°	2	23.03
7344	100	15°	3	23.31
10416	321	15°	4	23.33
15142	569	13°	4	23.35
26898	1002	12°	4	23.36

TABLE 11. The evolution to the tetrahedral surface τ of genus three. See also Figure 13.

number of facets	fineness	approx. max. angle	$\log_{10}$ iterations	w
76				
2904	61	46°	3	22.55
8024	130	18°	3	22.79
11388	150	13°	4	22.78
19672	657	13°	4	22.81
26692	981	13°	4	22.82

TABLE 12. The evolution from a stretched tetrahedral surface to a genus-three Lawson surface. See also Figure 15.

Genus-Four and Genus-Five Minimizers

We now record some preliminary experimental results concerning W-minimizing surfaces of genus four and genus five. We can obtain initial surfaces by fusing several cubical tori together, using the program Genus. The corresponding evolved surfaces of genus four and five are shown in Figure 16 (top). Alternatively, we can start with initial surfaces close to Lawson's surfaces $\xi_{1,4}$ and $\xi_{1,5}$. Evolving these leads to the surfaces shown in Figure 16 (bottom), which have w near 23.31 and 23.66, respectively, quite a bit under $8\pi \approx 25.13$. The button surfaces seem to be conformally equivalent to the Lawson surfaces, and all four appear to be W-stable.

6. FURTHER WORK

We recently implemented a computer program to generate initial surfaces geometrically close to any one of Lawson's minimal surfaces $\xi_{m,n}$ of genus mn. Evolution of any of these initial surfaces seems to lead to the Lawson surface; all those we have tested seem to be W-stable.