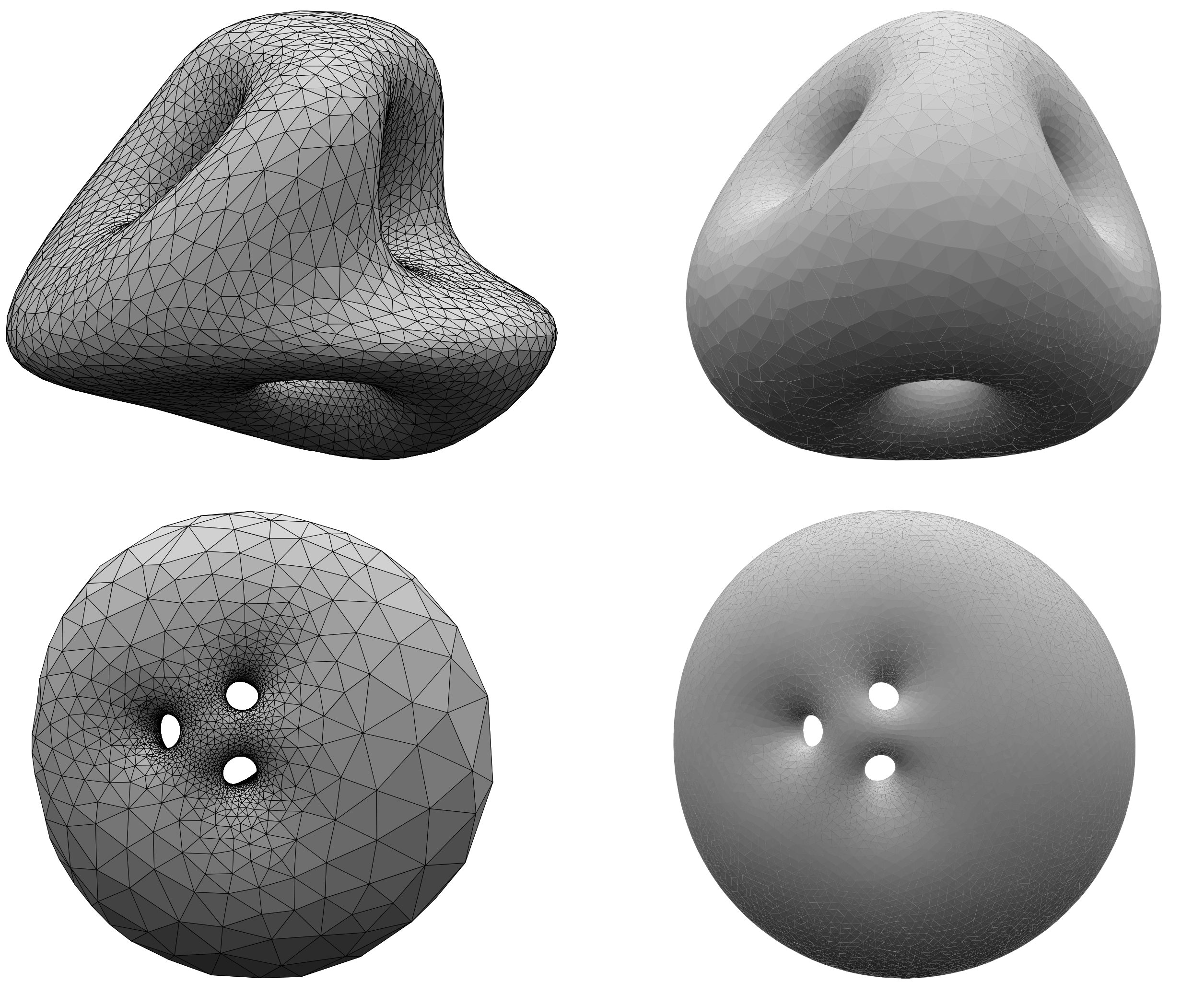

FIGURE 14. If we deform our approximation to τ with the jiggle commands, the result (top left) evolves toward a surface (top right) that is conformally equivalent to τ. Similarly, if we invert τ in a sphere (bottom left) and jiggle, the resulting surface again evolves to one that is conformally equivalent to τ (bottom right).

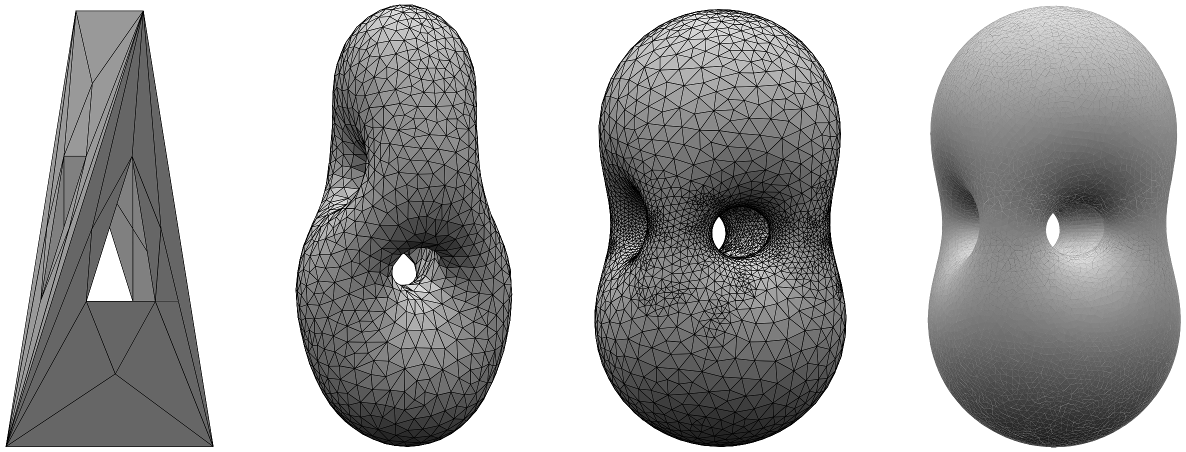

FIGURE 15. Stretching the initial surface of Figure 13 vertically gives a surface (left) that no longer evolves toward the surface τ. The smoothed version (middle left) still has just the initial symmetry, but as we evolve further (middle right) the symmetry increases, and we approach the Lawson surface (right). See also Table 12.

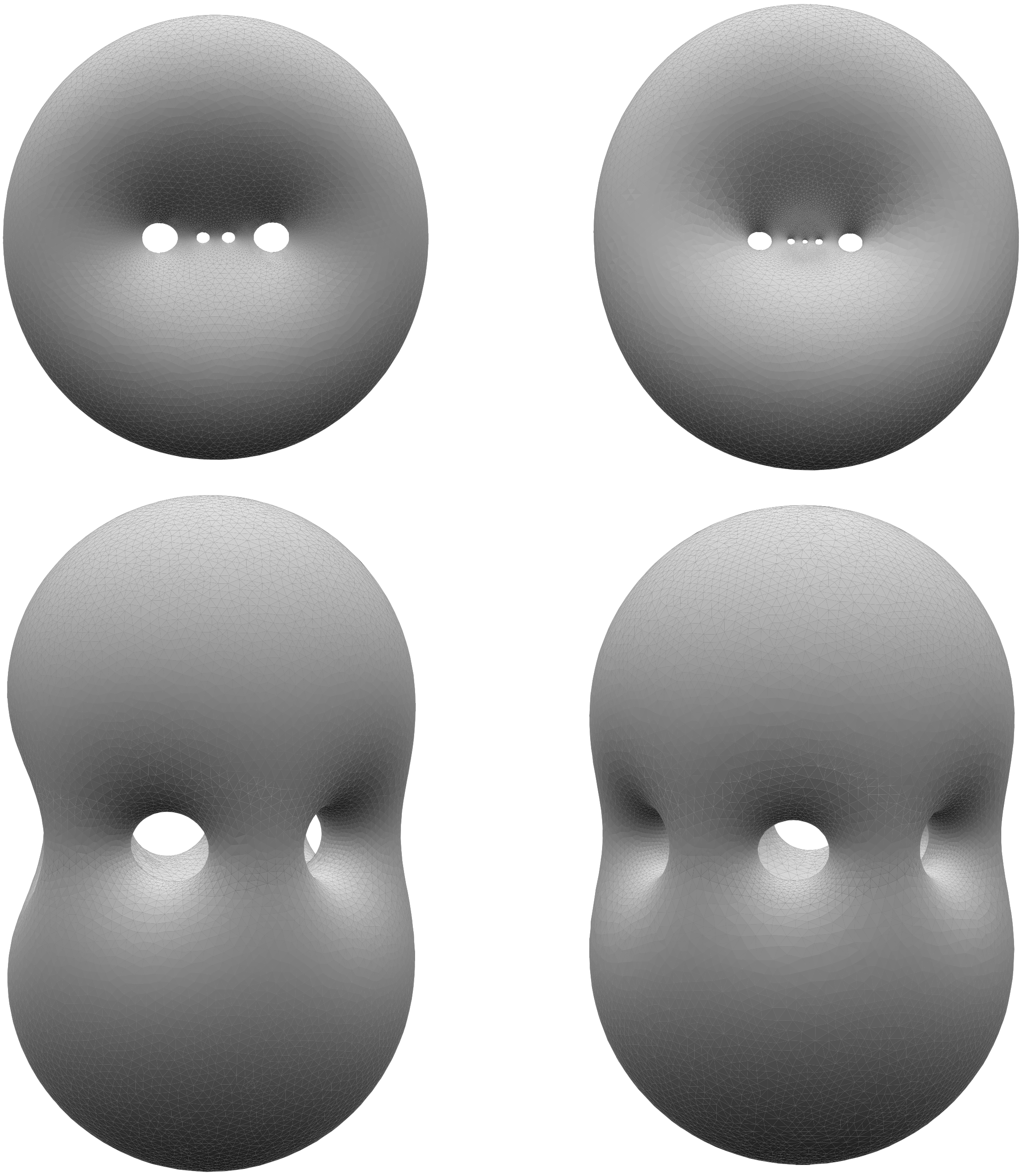

FIGURE 16. Surfaces of genus four (left) and five (right) minimizing the energy w. The initial configurations for surfaces in the top row were similar to the initial surface in Figure 12, and those for the bottom row were approximations to the Lawson surfaces.

The Evolver allows the evolution of a surface under symmetry constraints. We can specify three intersecting mirror planes, and the piece of a surface lying between them. The evolution of this piece will proceed as if we were evolving the whole surface, while enforcing the symmetry. In particular, the surface ends up meeting each mirror plane perpendicularly.

This technique has allowed us to study more complicated surfaces that have the same $k \times 2 \times 2$-fold Euclidean symmetry as the projection $\xi_{1,k-1}$ of the Lawson surface. One of these, a surface of genus k obtained by adding a single handle to Lawson's surface, seems to give a heretofore unknown minimal surface in $\mathbf{S}^3$. We expect that others will provide examples of surfaces that are

W-stable when the symmetries are enforced, but unstable when they are allowed to be broken.

We intend to give a fuller description of these surfaces, and of some new notions of polyhedral mean curvature h, in future reports.

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REFERENCES

[Brakke 1992] K. A. Brakke, "The Surface Evolver", *Experimental Mathematics* **1** (1992), 141–165.

[Bryant 1984] R. Bryant, "A duality theorem for Willmore surfaces", *J. Differential Geom.* **20** (1984), 23–53.

[Bryant 1988] R. Bryant, "Surfaces in conformal geometry", *Proc. Sympos. Pure Math.* **48** (1988), 227–240.

[Choi and Schoen 1985] H. I. Choi and R. Schoen, "The space of minimal embeddings of a surface into a three-manifold of positive Ricci curvature", *Invent. Math.* **81** (1985), 387–394.

[Darboux 1887] G. Darboux, "Les coordonnées pentasphériques", pp. 265–284 in *Leçons sur la Théorie Générale des Surfaces*, vol. 1, Gauthier-Villars, Paris, 1887.

[Karcher et al. 1988] H. Karcher, U. Pinkall and I. Sterling, "New minimal surfaces in S^3", *J. Differential Geom.* **28** (1988), 169–185.

[Kusner 1987] R. Kusner, "Conformal geometry and complete minimal surfaces", *Bull. Amer. Math. Soc.* **17** (1987), 291–295.

[Kusner 1989] R. Kusner, "Comparison surfaces for the Willmore problem", *Pacific J. Math.* **138** (1989), 317–345.

[Lawson 1970] H. B. Lawson, "Complete minimal surfaces in S^3", *Ann. Math.* **92** (1970), 335–374.

[Seifert 1991] U. Seifert, "Vesicles of toroidal topology", *Phys. Rev. Lett.* **66** (1991), 2404–2407.

[Simon 1986] L. Simon, "Existence of Willmore surfaces", *Proc. Cen. Math. Anal. Aust. Nat. Univ.* **10** (1986), 187–216.

[Thomsen 1924] G. Thomsen, "Grundlagen der konformen Flächentheorie", *Abh. Math. Sem. Univ. Hamburg* **3** (1924), 31–56.

[Underwood 1992] A. Underwood, "Polyhedral mean curvature and its relationship to smooth mean curvature", informal notes, Princeton University, 1992.

[Weiner 1978] J. Weiner, "On a problem of Chen, Willmore et al", *Indiana U. Math. J.* **27** (1978), 19–35.

[Willmore 1965] T. J. Willmore, "Note on embedded surfaces", *Analele Stiintifice ale Universitatii "Al. I. Cuza" din Iasi* (Sect. Ia) **11** (1965), 493–496.

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