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ON THE POINTWISE DIMENSION OF HYPERBOLIC MEASURES: A PROOF OF THE ECKMANN–RUELLE CONJECTURE

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Abstract. We prove the long-standing Eckmann–Ruelle conjecture in dimension theory of smooth dynamical systems. We show that the pointwise dimension exists almost everywhere with respect to a compactly supported Borel probability measure with non-zero Lyapunov exponents, invariant under a $C^{1+\alpha}$ diffeomorphism of a smooth Riemannian manifold.

Let M be a smooth Riemannian manifold without boundary, and $f\colon M\to M$ a $C^{1+\alpha}$ diffeomorphism of M for some $\alpha>0$. Also let μ be an f-invariant Borel probability measure on M with a compact support. Given a set $Z\subset M$, we denote respectively by $\dim_H Z$, $\underline{\dim}_B Z$, and $\overline{\dim}_B Z$ the *Hausdorff dimension of* Z and the *lower* and *upper box dimensions* of Z (see for example [F]). We will be mostly interested in subsets of positive measure invariant under f. To characterize their structure we use the notions of *Hausdorff dimension of* μ and *lower* and *upper box dimensions of* μ. We denote them by $\dim_H\mu$, $\underline{\dim}_B\mu$, and $\overline{\dim}_B\mu$, respectively. We have

$$\dim_H\mu=\inf\{\dim_H Z\mid \mu(Z)=1\},$$

$$\underline{\dim}_B\mu=\lim_{\delta\to 0}\inf\{\underline{\dim}_B Z\mid \mu(Z)\ge 1-\delta\},$$

$$\overline{\dim}_B\mu=\lim_{\delta\to 0}\inf\{\overline{\dim}_B Z\mid \mu(Z)\ge 1-\delta\}.$$

It follows from the definitions that

$$\dim_H\mu\le\underline{\dim}_B\mu\le\overline{\dim}_B\mu.$$

In [Y], Young found a criterion that guarantees the coincidence of the Hausdorff dimension and lower and upper box dimensions of measures. We define the *lower*

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and *upper pointwise dimensions* of μ at a point $x \in M$ by

$$\underline{d}_\mu(x) = \varliminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

$$\overline{d}_\mu(x) = \varlimsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

where $B(x,r)$ denotes the ball of radius r centered at x.

Proposition 1. *Let X be a complete metric space of finite topological dimension, and μ a Borel probability measure on X. Assume that*

$$\underline{d}_\mu(x) = \overline{d}_\mu(x) = d \tag{1}$$

for μ-almost every $x \in X$. Then

$$\dim_H \mu = \underline{\dim}_B \mu = \overline{\dim}_B \mu = d.$$

A measure μ which satisfies (1) is called *exact dimensional*. Since this result was established, it has become one of the most challenging problems in the interface of dimension theory and dynamical systems to describe measures which are exact dimensional.

In [Y], Young proved that an ergodic measure with non-zero Lyapunov exponents (such measures are called *hyperbolic* measures), invariant under a $C^{1+\alpha}$ diffeomorphism of a smooth compact surface, is exact dimensional. Ledrappier [L] proved exact dimensionality of hyperbolic Bowen–Ruelle–Sinai measures. In [PY], Pesin and Yue extended his result to hyperbolic measures satisfying the so-called semi-local product structure (this class includes Gibbs measures on locally maximal hyperbolic sets).

In [ER], Eckmann and Ruelle considered general hyperbolic measures and conjectured that they are exact dimensional. This statement has later become known as the Eckmann–Ruelle conjecture. In this paper we announce an affirmative solution of this conjecture.

We need some preliminary information on smooth dynamical systems with non-zero Lyapunov exponents. Given a point $x \in M$ and a vector $v \in T_xM$, the *Lyapunov exponent of v at x* is defined by the formula

$$\lambda(x,v) = \varlimsup_{n\to\infty} \frac{1}{n} \log \|d_x f^n v\|.$$

If x is fixed, then the function $\lambda(x,\cdot)$ can take only finitely many values $\lambda_1(x) \ge \cdots \ge \lambda_p(x)$, where $p = \dim M$. The functions $\lambda_i(x)$ are measurable and invariant under f. The measure μ is said to be *hyperbolic* if $\lambda_i(x) \ne 0$ for every $i = 1, \ldots, p$, and for μ-almost every $x \in M$.

There exists a measurable function $r(x) > 0$ such that for μ-almost every $x \in M$ the sets

$$W^s(x) = \left\{ y \in B(x, r(x)) \mid \varlimsup_{n\to+\infty} \frac{1}{n} \log d(f^n x, f^n y) < 0 \right\},$$

$$W^u(x) = \left\{ y \in B(x, r(x)) \mid \varliminf_{n\to-\infty} \frac{1}{n} \log d(f^n x, f^n y) > 0 \right\}$$

are immersed local manifolds called *stable* and *unstable local manifolds* at x. For each $0 < r < r(x)$ we consider the balls $B^s(x,r) \subset W^s(x)$ and $B^u(x,r) \subset W^u(x)$

centered at x with respect to the induced distances on $W^s(x)$ and $W^u(x)$, respectively.

In [LY], Ledrappier and Young constructed two measurable partitions ξ^s and ξ^u of M such that for μ-almost every $x \in M$:

1. $\xi^s(x) \subset W^s(x)$ and $\xi^u(x) \subset W^u(x)$;
2. $\xi^s(x)$ and $\xi^u(x)$ contain the intersection of an open neighborhood of x with $W^s(x)$ and $W^u(x)$, respectively.

We denote the system of conditional measures of μ with respect to the partitions ξ^s and ξ^u, respectively, by μ_x^s and μ_x^u, and for any measurable set $A \subset M$ we write $\mu_x^s(A) = \mu_x^s(A \cap \xi^s(x))$ and $\mu_x^u(A) = \mu_x^u(A \cap \xi^u(x))$.

In [LY], Ledrappier and Young introduced the quantities

$$d_\mu^s(x) \stackrel{\text{def}}{=} \lim_{r\to 0} \frac{\log \mu_x^s(B^s(x,r))}{\log r},$$

$$d_\mu^u(x) \stackrel{\text{def}}{=} \lim_{r\to 0} \frac{\log \mu_x^u(B^u(x,r))}{\log r}$$

provided that the corresponding limits exist at $x \in M$. We call them, respectively, the *stable* and *unstable pointwise dimensions* of μ at x.

Proposition 2 [LY]. 1. *For μ-almost every $x \in M$ the limits $d_\mu^s(x)$ and $d_\mu^u(x)$ exist μ-almost everywhere.*

2. *If μ is a hyperbolic measure, then for μ-almost every $x \in M$,*

$$\overline{d}_\mu(x) \le d_\mu^s(x) + d_\mu^u(x).$$

In this paper we announce the following statement.

Theorem. *Let f be a $C^{1+\alpha}$ diffeomorphism for some $\alpha > 0$ on a smooth Riemannian manifold without boundary, and μ an f-invariant compactly supported Borel probability measure. If μ is hyperbolic, then it is exact dimensional and its pointwise dimension is equal to the sum of the stable and unstable pointwise dimensions, i.e., for μ-almost every $x \in M$,*

$$\underline{d}_\mu(x) = \overline{d}_\mu(x) = d_\mu^s(x) + d_\mu^u(x).$$

We notice that using the ergodic decomposition of the measure one can reduce the proof to the case when μ is an ergodic measure. In this case the proof of the theorem is based upon a special countable partition $\mathfrak{P}$ of M constructed by Ledrappier and Young in [LY]. This partition simulates Markov partitions for hyperbolic sets. Let $\mathfrak{P}(x)$ be the element of the partition $\mathfrak{P}$ containing $x \in M$. The elements of the "shift" partition $\mathfrak{P}_n = \bigvee_{k=-n}^{n} f^{-k}\mathfrak{P}$ decompose $\mathfrak{P}(x)$ into finitely many subsets called rectangles, since they have the "direct product structure": for any $z, y \in \mathfrak{P}_n(x)$ the intersection $W^s(y) \cap W^u(z)$ consists of a unique point which belongs to $\mathfrak{P}_n(x)$. The number of rectangles which have positive measure is asymptotically equal to $\exp(nh)$, where h is the measure-theoretic entropy of f with respect to μ. We thoroughly study the distribution of these rectangles inside $\mathfrak{P}(x)$. In general, it does not reproduce the direct product structure. For "typical" points $y \in \mathfrak{P}(x)$ the number of rectangles intersecting $W^s(y)$ (respectively, $W^u(y)$) is "asymptotically" the same up to a subexponential factor, but their distribution along $W^s(y)$ (respectively, $W^u(y)$) may differ from point to point, causing a deviation from direct product structure. A simple combinatorial argument is used to show that the deviation can grow at most with a subexponential rate in n. We then estimate the measure of a ball by the product of its stable and unstable measures.

Remarks. 1. Let us point out that neither of the assumptions of the theorem can be omitted. Ledrappier and Misiurewicz [LM] constructed an example of a smooth map of a circle preserving an ergodic measure with zero Lyapunov exponent which is not exact dimensional. In [PW], Pesin and Weiss presented an example of a Hölder homeomorphism whose measure of maximal entropy is not exact dimensional.

2. It follows immediately from the theorem that the pointwise dimension of an ergodic invariant measure supported on a (uniformly) hyperbolic locally maximal set is exact dimensional. This result has not been known before. We emphasize that in this situation the stable and unstable foliations need not be Lipschitz, and in general the measure need not have a local product structure despite the fact that the set itself does. This illustrates that the theorem is non-trivial even for measures supported on hyperbolic locally maximal sets.

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